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Postulation of general quintuple fat point schemes in $\mathbb{P}^3$

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ABSTRACT

We study the postulation of a general union Y of double, triple, quartuple and quintuple points of $\mathbb{P}^3$. In characteristic 0, we prove that Y has good postulation in degree $d \geqslant 11$. The proof is based on the combination of the Horace differential lemma with a computer-assisted proof. We also classify the exceptions in degree 9 and 10.

1. Introduction

Let $\mathbb{K}$ be a field of characteristic 0, $n \in \mathbb{N}$ and $\mathbb{P}^n = \mathbb{P}^n(\mathbb{K})$. In this paper we study the postulation of general fat point schemes of $\mathbb{P}^3$ with multiplicity up to 5. A *fat point* mP is a zero-dimensional subscheme of $\mathbb{P}^3$ supported on a point P and with $(\mathcal{I}_{P,\mathbb{P}^3})^m$ as its ideal sheaf. A *general fat point scheme* $Y = m_1P_1 + \cdots + m_kP_k$, with $m_1 \geqslant \cdots \geqslant m_k \geqslant 1$ is a general zero-dimensional scheme such that its support Y_{red} is a union of k points and for each i the connected component of Y supported on P_i is the fat point m_iP_i. We say that the multiplicity of Y is the maximal multiplicity, m_1, of its components.

Studying the postulation of Y means to compute the dimension of the space of hypersurfaces of any degree containing the scheme Y. In other words this problem is equivalent to computing the

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dimension δ of the space of homogeneous polynomials of any degree vanishing at each point P_i and with all their derivatives, up to multiplicity $m_i - 1$, vanishing at P_i. We say that Y has good postulation if δ is the expected dimension, that is, either the difference between the dimension of the polynomial space and the number of imposed conditions or just the dimension of the polynomial space (when δ would exceed it).

This problem was investigated by many authors in the case of $\mathbb{P}^2$, where we have the important Harbourne–Hirschowitz conjecture (see [7] for a survey). In the case of $\mathbb{P}^n$, for $n \geqslant 2$, the celebrated Alexander–Hirschowitz theorem gives a complete answer in the case of double points, that is when $m_i = 2$ for any i ([1,2], for a survey see [5]). For arbitrary multiplicities and arbitrary projective varieties there is a beautiful asymptotic theorem by Alexander and Hirschowitz [3].

Here we focus on the case of general fat point schemes $Y \subset \mathbb{P}^3$. In this case a general conjecture which characterizes all the general fat point schemes not having good postulation was proposed by Laface and Ugaglia in [12]. The good postulation of general fat point schemes of multiplicity 4 was proved for degrees $d \geqslant 41$ in [4] by the first two authors. Then Dumnicki made a real breakthrough. In particular he showed, in [10], how to check the cases with degree $9 \leqslant d \leqslant 40$. Stimulated by his results, we consider now the case of fat point schemes of multiplicity 5 and we solve completely the problem of the good postulation. Indeed we prove the following theorem.

Theorem 1. *Let $\mathbb{P}^3 = \mathbb{P}^3(\mathbb{K})$, where $\mathbb{K}$ is a field of characteristic 0. Fix non-negative integers d, w, x, y, z such that $d \geqslant 11$. Let $Y \subset \mathbb{P}^3$ be a general union of w 5-points, x 4-points, y 3-points and z 2-points. Then Y has good postulation with respect to degree-d forms.*

The more natural way to prove our result would be to adopt a usual two-parts proof: we might prove the theorem for $d \geqslant 66$ with the same theoretical approach as in [4] and then we might prove the remaining finite cases with the computer. We do not follow this consolidated path, because the computer calculations at level $d \geqslant 60$ are infeasible with nowadays means. Instead, the proof of our result is an innovative combination of computer computation and theoretical argument, as in the following logical outline:

a) First, we prove Theorem 1 for degree $d = 11$ using our servers (Theorem 16).
b) Second, we improve the argument of [4] and so we are able to prove Theorem 1 for degrees $d \geqslant 53$, with a theoretical proof depending on *both* known results (Remark 13) in the case of fat points of $\mathbb{P}^2$ *and* on a) ($d = 11$).
This is presented in Section 3.
c) Then, we perform several computer calculations (Lemma 15).
d) Then, we give a theoretical proof that restricts the required computations for the remaining cases ($11 \leqslant d \leqslant 52$) to some feasible jobs. This proof depends on the previous computational results. The main point here is that an iterated use of some results by Dumnicki [9,10] allows us to greatly reduce the number of cases to be considered, by adding points of higher multiplicity. In particular we make use of points of multiplicity 10 and 13. Another tool we use is a result concerning low degrees and few quintuple points (see Proposition 23). This result is proved by a modification of the general proof contained in Section 3 and indeed allows us to exclude many cases from the explicit checking by computer. All this is reported in Section 4.
e) Finally, we perform direct computer checks for the surviving cases, as detailed in Section 5. Our computer calculations are deterministic and produce several digital certificates, that allow any other researcher to verify our results precisely. They rely on the efficient software package MAGMA [15], whose linear algebra over finite fields outperforms any other software that we tried. All our programmes and their digital certificates are accessible online as Supplementary material to this article (see section Supplementary material).

In the remainder of the paper we provide two sections, as follows.

In Section 6 we classify all the exceptions arising in degree 9 and 10 (relying again on a computer-aided proof). It turns out that, in these cases, the Laface–Ugaglia conjecture is true.

In Section 7 we collect several remarks on our results and their consequences.

2. Preliminaries

In this section we fix our notation (which is the same as in [4] whenever possible), prove several preliminary results and summarize our computational results.

Let $\mathbb{P}^n$ be the projective space on a field $\mathbb{K}$, with $\mathrm{char}(\mathbb{K}) = 0$ and $n \in \mathbb{N}$. Note that we do *not* assume that $\mathbb{K}$ is algebraically closed. However, some of the references which we will use assume that the base field is algebraically closed. In the next lemma we explain why we are allowed to use these results.

Lemma 2. *Let $\overline{\mathbb{K}}$ denote the algebraic closure of $\mathbb{K}$. Fix non-negative integers n, d, x, y, z, w, s such that $n \geqslant 1$. Assume that a general disjoint union of w quintuple points, x quartuple points, y triple points, z double points and s (simple) points in $\mathbb{P}^n(\overline{\mathbb{K}})$ has good postulation in degree d, i.e. either*

- $h^0(\mathbb{P}^n(\overline{\mathbb{K}}), \mathcal{I}_Z(d)) = 0$ *and* $\binom{n+4}{4}w + \binom{n+3}{3}x + \binom{n+2}{2}y + (n+1)z + s \geqslant \binom{n+d}{n}$
- *or* $h^1(\mathbb{P}^n(\overline{\mathbb{K}}), \mathcal{I}_Z(d)) = 0$ *and* $\binom{n+4}{4}w + \binom{n+3}{3}x + \binom{n+2}{2}y + (n+1)z + s \leqslant \binom{n+d}{n}$.

Then there is a disjoint union W of w quintuple points, x quartuple points, y triple points, z double points and s points in $\mathbb{P}^n(\mathbb{K})$ with good postulation in degree d.

Proof. Increasing s, if necessary, we reduce to the case $\binom{n+4}{4}w + \binom{n+3}{3}x + \binom{n+2}{2}y + (n+1)z + s \geqslant \binom{n+d}{n}$. Let $\nu = w + x + y + z + s$. Let E be the subset of $\mathbb{P}^n(\overline{\mathbb{K}})^\nu$ parameterizing all the ν-ples of distinct points of $\mathbb{P}^n(\overline{\mathbb{K}})$. For any $A \in E$, let $Z_A \subset \mathbb{P}^n(\overline{\mathbb{K}})$ be the fat point subcheme of $\mathbb{P}^n(\overline{\mathbb{K}})$ in which the first w (resp. x, resp. y, resp. z, resp. s) fat points share multiplicity 5 (resp. 4, resp. 3, resp. 2, resp. 1) and $(Z_A)_{red}$ is the set associated to A. By semicontinuity there is a non-empty open subset U of E such that for all $A \in U(\overline{\mathbb{K}})$ we have $\mathrm{h}^0(\mathbb{P}^n(\overline{\mathbb{K}}), \mathcal{I}_{Z_A}(d)) = 0$. Since $\mathbb{K}$ is infinite, $\mathbb{K}^{n\nu}$ is dense in $\overline{\mathbb{K}}^{n\nu}$. Hence $\mathbb{P}^n(\mathbb{K})^\nu$ is Zariski dense in $\mathbb{P}^n(\overline{\mathbb{K}})^\nu$. Thus, there is $B \in U(\mathbb{K})$ such that the scheme Z_B satisfies $\mathrm{h}^0(\mathbb{P}^n(\mathbb{K}), \mathcal{I}_{Z_B}(d)) = 0$ and it is defined over $\mathbb{K}$. $\square$

From now on, $\mathbb{K}$ is *any* field with $\mathrm{char}(\mathbb{K}) = 0$ and $\mathbb{P}^n = \mathbb{P}^n(\mathbb{K})$.

For any smooth n-dimensional connected variety A, any $P \in A$ and any integer $m > 0$, an *m-fat point of A* (or just m-point) $\{mP, A\}$ is defined to be the $(m-1)$-th infinitesimal neighborhood of P in A, i.e. the closed subscheme of A with $(\mathcal{I}_{P,A})^m$ as its ideal sheaf. As a consequence, $\{mP, A\}_{red} = \{P\}$ and the length of $\{mP, A\}$ is $\mathrm{length}(\{mP, A\}) = \binom{n+m-1}{n}$. To ease our notation, we will write mP instead of $\{mP, A\}$ when the space A is clear from the context, and mostly we will have $A = \mathbb{P}^n$ for $n = 2, 3$.

We call *general fat point scheme of A* (or *general union* for short) any union $Y = m_1P_1 + \cdots + m_kP_k$, with $m_1 \geqslant \cdots \geqslant m_k \geqslant 1$, and $P_1, \ldots, P_k$ general points of $\mathbb{P}^n$. We denote $\deg(Y) = \sum \mathrm{length}(m_iP_i)$.

Given a positive integer d, we will say that a zero-dimensional scheme Y of $\mathbb{P}^n$ has *good postulation in degree d* if the following conditions hold:

(a) if $\deg(Y) \leqslant \binom{n+d}{n}$, then $\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_Y(d)) = 0$,
(b) if $\deg(Y) \geqslant \binom{n+d}{n}$, then $\mathrm{h}^0(\mathbb{P}^n, \mathcal{I}_Y(d)) = 0$.

We will also use the notation $\mathcal{L}_n(d; m_1, \ldots, m_k)$ for the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a general union $Y = m_1P_1 + \cdots + m_kP_k$. The *virtual dimension* of $L = \mathcal{L}_n(d; m_1, \ldots, m_k)$ is

$$\mathrm{vdim}(L) = \binom{n+d}{n} - \deg(Y) - 1$$

and the dimension of the linear system always satisfies $\dim(L) \geqslant \mathrm{vdim}(L)$. We say that L is *special* if $\dim(L) > \max\{\mathrm{vdim}(L), -1\}$. It is easy to see that a linear system L is special if and only if the corresponding general union does not have good postulation in degree d. For more details we refer to [7].

Remark 3. Let $d_0 \geqslant 2$. Assume that Y is any general fat point scheme in $\mathbb{P}^n$ such that $\deg(Y) \geqslant \binom{n+d_0}{n}$. If we know that Y has good postulation in degree $d \geqslant d_0$, we can claim that Y has good postulation in any degree, as follows.

For $d \geqslant d_0$, there is nothing to prove.

Since, for any $d \geqslant 1$, there is an injective map

$$H^0\big(\mathbb{P}^n, \mathcal{I}_Y(d-1)\big) \hookrightarrow H^0\big(\mathbb{P}^n, \mathcal{I}_Y(d)\big),$$

then $\mathrm{h}^0(\mathbb{P}^n, \mathcal{I}_Y(d)) = 0$ implies $\mathrm{h}^0(\mathbb{P}^n, \mathcal{I}_Y(d-1)) = 0$. But $\mathrm{h}^0(\mathbb{P}^n, \mathcal{I}_Y(d_0)) = 0$ and so $\mathrm{h}^0(\mathbb{P}^n, \mathcal{I}_Y(d)) = 0$ for any $d < d_0$, which proves that Y has good postulation.

Similarly, if $\mathrm{h}^0(H, \mathcal{I}_{Y\cap H}(d_0)) = 0$, then $\mathrm{h}^0(H, \mathcal{I}_{Y\cap H}(d_0 - 1)) = 0$.

The following general lemma will be useful in the sequel.

Lemma 4. *Let Σ be an integral projective variety on $\overline{\mathbb{K}}$ and let $\mathcal{L}$ be a linear system (not necessarily complete) of divisors on Σ. Fix an integer $m \geqslant 1$ and a general point $P \in \Sigma$. Let $\mathcal{L}(-mP)$ be the sublinear system of $\mathcal{L}$ formed by all divisors with a point of multiplicity at least m at P. Then we have*

$$\dim\big(\mathcal{L}(-mP)\big) \leqslant \max\big\{\dim(\mathcal{L}) - m, -1\big\},$$

and, for any $1 \leqslant k \leqslant m$,

$$\dim\big(\mathcal{L}(-mP)\big) \leqslant \max\big\{\dim\big(\mathcal{L}(-kP)\big) - (m-k), -1\big\}.$$

Proof. The case $m = 1$ is obvious. We assume by induction that

$$\dim\big(\mathcal{L}\big(-(m-1)P\big)\big) \leqslant \max\big\{\dim(\mathcal{L}) - m + 1, -1\big\}.$$

By [6, Proposition 2.3] it follows that

$$\dim\big(\mathcal{L}(-mP)\big) \leqslant \max\big\{\dim\big(\mathcal{L}\big(-(m-1)P\big)\big) - 1, -1\big\},$$

and so we get the desired inequality. The proof of the second inequality is analogous. $\square$

In the following lemma we show that in order to prove Theorem 1 for all quadruples (w, x, y, z) of non-negative integers it is sufficient to prove it only for a small set of quadruples (w, x, y, z).

Lemma 5. *Fix an integer $d > 0$. For any quadruple of non-negative integers (w, x, y, z), let $Y(w, x, y, z) \subset \mathbb{P}^3$ denote a general union of w 5-points, x 4-points, y 3-points and z 2-points. If $Y(w, x, y, z)$ has good postulation in degree d for any quadruple (w, x, y, z) such that*

$$\binom{d+3}{3} - 3 \leqslant 35w + 20x + 10y + 4z \leqslant \binom{d+3}{3} + \Delta$$

where

$$\Delta = \begin{cases} 13 & \textit{if } w > 0 \textit{ and } x = y = z = 0, \\ 8 & \textit{if } x > 0 \textit{ and } y = z = 0, \\ 4 & \textit{if } y > 0 \textit{ and } z = 0, \\ 1 & \textit{if } z > 0 \end{cases} \tag{1}$$

then any general quintuple fat point scheme has good postulation in degree d.

Proof. If a quadruple (w, x, y, z) is such that $35w + 20x + 10y + 4z \leqslant \binom{d+3}{3} - 4$, then we want to prove that $\mathrm{h}^1(\mathbb{P}^n, \mathcal{I}_Y(d)) = 0$, where $Y = Y(w, x, y, z)$. Let $z' > 0$ be the integer such that $\binom{d+3}{3} - 3 \leqslant 35w + 20x + 10y + 4z + 4z' \leqslant \binom{d+3}{3}$. By hypothesis we know that $Y' = Y(w, x, y, z + z')$ has good postulation, that is, $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) = 0$. Since $Y \subset Y'$, then it is easy to see that $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) = 0$, and so Y has good postulation.

Now assume that $w > 0$, $x = y = z = 0$, and $35w \geqslant \binom{d+3}{3} + 14$. Let Y be the corresponding general union of w quintuple points. This time we want to prove that $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) = 0$. Let $w' > 0$ such that $\binom{d+3}{3} - 21 \leqslant 35(w - w') \leqslant \binom{d+3}{3} + 13$. Now we consider the following subcases:

- if $35(w - w') \geqslant \binom{d+3}{3}$, then we take the union $Y' = Y(w - w', 0, 0, 0)$ of $w - w'$ quintuple general points. Since we can assume $Y' \subset Y$, we immediately have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) = 0$, and so Y has good postulation.
- If $\binom{d+3}{3} - 5 \leqslant 35(w - w') \leqslant \binom{d+3}{3} - 1$, we take the union $Y' = Y(w - w', 0, 0, 0)$ of $w - w'$ quintuple general points. Since Y contains at least one further quintuple point, we can consider $Y'' = Y(w - w' + 1, 0, 0, 0)$ and we can assume that $Y' \subset Y'' \subseteq Y$. Note that Y' has good postulation by hypothesis, and $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) \leqslant 5$. Hence by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y''}(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 5, 0\} = 0$. Then we have that Y'' has good postulation, and consequently Y has good postulation.
- If $\binom{d+3}{3} - 12 \leqslant 35(w - w') \leqslant \binom{d+3}{3} - 6$, then we take $Y' = Y(w - w', 0, 1, 0)$, i.e. a general union of $w - w'$ quintuple points and one triple point. Now $\binom{d+3}{3} - 2 \leqslant \deg(Y') \leqslant \binom{d+3}{3} + 4$ and by hypothesis Y' has good postulation. Since we can assume $Y' \subset Y$, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 2, 0\} = 0$, and so Y has good postulation.
- If $\binom{d+3}{3} - 21 \leqslant 35(w - w') \leqslant \binom{d+3}{3} - 11$, then we take $Y' = Y(w - w', 1, 0, 0)$. Now $\binom{d+3}{3} - 2 \leqslant \deg(Y') \leqslant \binom{d+3}{3} + 9$ and by hypothesis Y' has good postulation. Since we can assume $Y' \subset Y$, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 1, 0\} = 0$, and so Y has good postulation.

Assume now $x > 0$, $y = z = 0$ and $35w + 20x \geqslant \binom{d+3}{3} + 9$. Let $Y = Y(w, x, 0, 0)$ be the corresponding general union and we want to prove that $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) = 0$.

If $35w \geqslant \binom{d+3}{3}$, then $Y' = Y(w, 0, 0, 0)$ has good postulation by the previous step and clearly it follows that Y has good postulation. Otherwise there exists $0 < x' < x$ such that $\binom{d+3}{3} - 11 \leqslant 35w - 20(x - x') \leqslant \binom{d+3}{3} + 8$.

Now we consider the following subcases:

- If $35w - 20(x - x') \geqslant \binom{d+3}{3} - 4$, then we take the union $Y' = Y(w, x - x', 0, 0)$. Since Y contains at least one further quartuple point, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 4, 0\} = 0$, and so Y has good postulation.
- If $\binom{d+3}{3} - 6 \leqslant 35w + 20(x - x') = \binom{d+3}{3} - 5$, then we take $Y' = Y(w, x - x', 0, 1)$ and we have $\binom{d+3}{3} - 2 \leqslant \deg(Y') \leqslant \binom{d+3}{3} - 1$ and by hypothesis Y' has good postulation. Since we can assume $Y' \subset Y$, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 2, 0\} = 0$, and so Y has good postulation.
- If $\binom{d+3}{3} - 11 \leqslant 35w + 20(x - x') \leqslant \binom{d+3}{3} - 7$, then we take $Y' = Y(w, x - x', 1, 0)$ and we have $\binom{d+3}{3} - 1 \leqslant \deg(Y') \leqslant \binom{d+3}{3} + 3$ and by hypothesis Y' has good postulation. Since we can assume

$Y' \subset Y$, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 1, 0\} = 0$, and so Y has good postulation.

Now assume $y > 0$, $z = 0$ and $35w + 20x + 10y \geqslant \binom{d+3}{3} + 6$. Let $Y = Y(w, x, y, 0)$ be the corresponding general union and we want to prove that $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) = 0$.

If $35w + 20x \geqslant \binom{d+3}{3}$, then $Y' = Y(w, x, 0, 0)$ has good postulation by the previous steps and clearly it follows that Y has good postulation. Otherwise there exists $0 < y' < y$ such that $\binom{d+3}{3} - 4 \leqslant 35w - 20x + 10(y - y') \leqslant \binom{d+3}{3} + 5$.

Now we consider the following subcases:

- If $35w - 20x + 10(y - y') \geqslant \binom{d+3}{3} - 3$, then we take the union $Y' = Y(w, x, y - y', 0)$. Since Y contains at least one further triple point, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 3, 0\} = 0$, and so Y has good postulation.
- If $35w - 20x + 10(y - y') = \binom{d+3}{3} - 4$, then we take $Y' = Y(w, x, y - y', 1)$ and we have that $\deg(Y') = \binom{d+3}{3}$ and by hypothesis Y' has good postulation. It immediately follows that $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) = 0$, and so Y has good postulation.

Finally assume that $z > 0$ and $35w + 20x + 10y + 4z \geqslant \binom{d+3}{3} + 2$. Let $Y = Y(w, x, y, z)$ be the corresponding general union and we want to prove that $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) = 0$.

If $35w + 20x + 10y \geqslant \binom{d+3}{3}$, then $Y' = Y(w, x, y, 0)$ has good postulation by the previous steps and clearly it follows that Y has good postulation. Otherwise there exists $0 < z' < z$ such that $\binom{d+3}{3} - 2 \leqslant 35w - 20x + 10y + 4(z - z') \leqslant \binom{d+3}{3} + 1$. Now we take the union $Y' = Y(w, x, y, z - z')$. Since Y contains at least one further double point, by Lemma 4 we have $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_Y(d)) \leqslant \max\{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{Y'}(d)) - 2, 0\} = 0$, and so Y has good postulation. □

Remark 6. Lemma 4 and Lemma 5 heavily use $\mathrm{char}(\mathbb{K}) = 0$, but they will be useful also in Section 5.

Given a general fat point scheme Y of $\mathbb{P}^n$ and a hyperplane $H \subset \mathbb{P}^n$, we will call *trace* of Y the subscheme $(Y \cap H) \subset H$ and *residual* of Y the scheme $\mathrm{Res}_H(Y) \subset \mathbb{P}^n$ with ideal sheaf $\mathcal{I}_Y : \mathcal{O}_{\mathbb{P}^n}(-H)$. Notice that if Y is an m-point supported on H, then its trace $Y \cap H$ is an m-point of H and its residual $\mathrm{Res}_H(Y)$ is an $(m-1)$-point of $\mathbb{P}^n$. We will often use the following form of the so-called *Horace lemma.*

Lemma 7. *Let $H \subset \mathbb{P}^n$ be a hyperplane and $X \subset \mathbb{P}^n$ a closed subscheme. Then*

$$\mathrm{h}^0\big(\mathbb{P}^n, \mathcal{I}_X(d)\big) \leqslant \mathrm{h}^0\big(\mathbb{P}^n, \mathcal{I}_{\mathrm{Res}_H(X)}(d-1)\big) + \mathrm{h}^0\big(H, \mathcal{I}_{X \cap H}(d)\big),$$
$$\mathrm{h}^1\big(\mathbb{P}^n, \mathcal{I}_X(d)\big) \leqslant \mathrm{h}^1\big(\mathbb{P}^n, \mathcal{I}_{\mathrm{Res}_H(X)}(d-1)\big) + \mathrm{h}^1\big(H, \mathcal{I}_{X \cap H}(d)\big).$$

Proof. The statement is a straightforward consequence of the well-known *Castelnuovo exact sequence*

$$0 \to \mathcal{I}_{\mathrm{Res}_H(X)}(d-1) \to \mathcal{I}_X(d) \to \mathcal{I}_{X \cap H}(d) \to 0.$$

For more details see e.g. [5, Section 4]. □

The basic tool we will need is the so-called *Horace differential lemma*. This technique allows us to take a *differential trace* and a *differential residual*, instead of the classical ones. For an explanation of the geometric intuition of the Horace differential lemma see [3, Section 2.1]. Here we give only an idea of how the lemma works. Let Y be an m-point of $\mathbb{P}^n$ supported on a hyperplane $H \subset \mathbb{P}^n$. Following the language of Alexander and Hirschowitz, we can describe Y as formed by infinitesimally piling up some subschemes of H, called *layers*. For example the layers of a 3-point $\{3P, \mathbb{P}^n\}$ are $\{3P, H\}$, $\{2P, H\}$, and

$\{P, H\}$. Then the differential trace can be any of these layers and the differential residual is a *virtual* zero-dimensional scheme formed by the remaining layers. In this paper we will apply several times the following result which is a particular case of the Horace differential lemma (see [3, Lemma 2.3]).

Lemma 8 *(Alexander–Hirschowitz). Fix an integer $m \geqslant 2$ and assume that* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > m$*. Let X be an m-point of $\mathbb{P}^n$ supported on P and $H \subset \mathbb{P}^n$ a hyperplane. Then for $i = 0, 1$ we have*

$$\mathrm{h}^i\big(\mathbb{P}^n, \mathcal{I}_X(d)\big) \leqslant \mathrm{h}^i\big(\mathbb{P}^n, \mathcal{I}_R(d-1)\big) + \mathrm{h}^i\big(H, \mathcal{I}_T(d)\big)$$

where the differential residual R *and the* differential trace T *are virtual schemes of the following type*:

m	T	R	
2	$\{P, H\}$	$\{2P, H\}$	$(1, 3)$
3	$\{P, H\}$	$(\{3P, H\}, \{2P, H\})$	$(1, 6, 3)$
3	$\{2P, H\}$	$(\{3P, H\}, \{P, H\})$	$(3, 6, 1)$
4	$\{P, H\}$	$(\{4P, H\}, \{3P, H\}, \{2P, H\})$	$(1, 10, 6, 3)$
4	$\{2P, H\}$	$(\{4P, H\}, \{3P, H\}, \{P, H\})$	$(3, 10, 6, 1)$
4	$\{3P, H\}$	$(\{4P, H\}, \{2P, H\}, \{P, H\})$	$(6, 10, 3, 1)$
5	$\{P, H\}$	$(\{5P, H\}, \{4P, H\}, \{3P, H\}, \{2P, H\})$	$(1, 15, 10, 6, 3)$
5	$\{2P, H\}$	$(\{5P, H\}, \{4P, H\}, \{3P, H\}, \{P, H\})$	$(3, 15, 10, 6, 1)$
5	$\{3P, H\}$	$(\{5P, H\}, \{4P, H\}, \{2P, H\}, \{P, H\})$	$(6, 15, 10, 3, 1)$
5	$\{4P, H\}$	$(\{5P, H\}, \{3P, H\}, \{2P, H\}, \{P, H\})$	$(10, 15, 6, 3, 1)$

In the previous lemma we described the possible differential residuals by writing the subsequent layers from which they are formed. These layers are obtained by intersecting with the hyperplane H many times. In particular the notation e.g. $R = (\{3P, H\}, \{2P, H\})$ means that $R \cap H = \{3P, H\}$ and $\mathrm{Res}_H(R) \cap H = \{2P, H\}$, and, finally, $\mathrm{Res}_H(\mathrm{Res}_H(R)) \cap H = \emptyset$, the latter equality being equivalent to $\mathrm{Res}_H(\mathrm{Res}_H(R)) = \emptyset$, because $R_{red} \subset H$. Moreover, for each case in the statement we write in the last column the list of the lengths of the fat points of H that we will obtain intersecting many times with H. Throughout the paper, when we will apply Lemma 8, we will specify which case we are considering by recalling this sequence of the lengths. For example, if we apply the first case of Lemma 8, we will say that we apply the lemma with respect to the sequence $(1, 3)$.

The next two arithmetic lemmas will be used in the sequel.

Lemma 9. *Let w, x, y, z be non-negative integers such that*

$$35w + 20x + 10y + 4z \leqslant \binom{d+3}{3} + 13.$$

Let $\alpha = \lfloor \frac{2x+y}{42} \rfloor$ and assume that $w \leqslant \alpha - 1$. Then $35w \leqslant \frac{1}{12}\binom{d+3}{3}$.

Proof. By hypothesis we have $20x + 10y \leqslant \binom{d+3}{3} + 13$, from which we have

$$w \leqslant \alpha - 1 \leqslant \frac{20x + 10y}{420} - 1 \leqslant \frac{1}{420}\binom{d+3}{3} + \frac{13}{420} - 1 \leqslant \frac{1}{420}\binom{d+3}{3}. \qquad \square$$

Lemma 10. *Fix non-negative integers t, a, b, c, u, v, e, f, g, h such that $t \geqslant 18$,*

$$15a + 10b + 6c + 3u + v + 10e + 6f + 3g + h \leqslant \binom{t+2}{2} \tag{2}$$

and (e, f, g, h) *is one of the following quadruples:* $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,0,2)$, $(0,0,1,0)$, $(0,0,1,1)$, $(0,0,1,2)$, $(0,1,0,0)$, $(0,1,0,1)$, $(0,1,0,2)$, $(0,1,1,0)$, $(1,0,0,0)$, $(1,0,0,1)$, $(1,0,0,2)$, $(1,0,1,0)$, $(1,0,1,1)$. *Then the following inequality holds*:

$$10a + 6b + 3c + u + 15e + 15f + 15g + 15h \leqslant \binom{t+1}{2}. \tag{3}$$

If $e + f + g + h \leqslant 2$*, then the statement holds for any* $t \geqslant 15$*. If* $e = f = g = h = 0$*, then the statement holds for any* $t \geqslant 4$*.*

Proof. By contradiction, let us assume that

$$10a + 6b + 3c + u + 15e + 15f + 15g + 15h > \binom{t+1}{2}, \tag{4}$$

which, together with (2), implies

$$5a + 4b + 3c + 2u + v - 5e - 9f - 12g - 14h \leqslant t + 1. \tag{5}$$

From (4) and (5) we get

$$\binom{t+1}{2} - 2t - 2 < -2b - 3c - 3u - 2v + 15(e + f + g + h) + 2(5e + 9f + 12g + 14h)$$

that is

$$\binom{t+1}{2} - 2t - 2 < -2b - 3c - 3u - 2v + 25e + 33f + 39g + 43h < 125,$$

which implies $t^2 - 3t - 254 < 0$, which is false as soon as $t \geqslant 18$.

If $e + f + g + h \leqslant 2$, the same steps give $t^2 - 3t - 176 < 0$ which is false as soon as $t \geqslant 15$.

If $e = f = g = h = 0$, the same steps give $t^2 - 3t - 4 < 0$ which is false as soon as $t \geqslant 4$. □

Remark 11. Let $Y \subset \mathbb{P}^3$ be a zero-dimensional scheme and H a hyperplane of $\mathbb{P}^3$. Fix non-negative integers c_2, c_3, c_4, c_5. Denoting by Y' the union of the connected components of Y intersecting H, the scheme $Y \setminus Y'$ is a general union of c_5 5-points, c_4 4-points, c_3 3-points, and c_2 2-points. Moreover, the subscheme Y' is supported on general points of H and it is given by a union of points of multiplicity 2, 3, 4, 5 or virtual schemes arising as residual in Lemma 8.

In the following basic lemma we show how to apply the Horace differential Lemma 8 in our situation.

Lemma 12. *Fix a plane* $H \subset \mathbb{P}^3$*. Let* Y *be a zero-dimensional scheme as in Remark* 11*, for some integers* c_2, c_3, c_4, c_5*. If the following condition holds for some positive integer* t*:*

$$\beta := \binom{t+2}{2} - \deg(Y \cap H) \geqslant 0, \tag{6}$$

then it is possible to degenerate Y *to a scheme* X *such that one of the following possibilities is verified*:

(I) $\deg(X \cap H) = \binom{t+2}{2}$,
(II) $\deg(X \cap H) < \binom{t+2}{2}$, *and all the irreducible components of* X *are supported on* H*. This is possible only if* $c_2 + c_3 + c_4 + c_5 < \beta$ *and* $c_2 + c_3 + c_4 + c_5 \leqslant 2$*.*

Moreover, if we assume $t \geqslant 18$ *in case* (I) *and* $t \geqslant 15$ *in case* (II), *we also have*

$$\deg\big(\mathrm{Res}_H(X) \cap H\big) \leqslant \binom{t+1}{2}. \tag{7}$$

Proof. By specializing some of the connected components of Y to isomorphic schemes supported on points of H we may assume that $\beta \geqslant 0$ is minimal. Let us denote now by Y' the union of the connected components of Y intersecting H.

By minimality of β it follows that if $c_2 > 0$ then $\beta < 3$, if $c_2 = 0$ and $c_3 > 0$ then $\beta < 6$, if $c_2 = c_3 = 0$ and $c_4 > 0$ then $\beta < 10$, if $c_2 = c_3 = c_4 = 0$ and $c_5 > 0$, then $\beta < 15$. If $c_2 = c_3 = c_4 = c_5 = 0$ and $\beta > 0$, we are obviously in case (II).

We degenerate now Y to a scheme X described as follows. The scheme X contains all the connected components of Y'. Write

$$\beta = 10e + 6f + 3g + h$$

for a unique quadruple of non-negative integers (e, f, g, h) in the following list: $(0,0,0,0)$, $(0,0,0,1)$, $(0,0,0,2)$, $(0,0,1,0)$, $(0,0,1,1)$, $(0,0,1,2)$, $(0,1,0,0)$, $(0,1,0,1)$, $(0,1,0,2)$, $(0,1,1,0)$, $(1,0,0,0)$, $(1,0,0,1)$, $(1,0,0,2)$, $(1,0,1,0)$, $(1,0,1,1)$ (i.e. in the list of Lemma 10). If $c_2 > 0$, then $e = f = g = 0$ and $h \leqslant 2$. If $c_2 = 0$ and $c_3 > 0$, then $e = f = 0$, $g \leqslant 1$ and $h \leqslant 2$. If $c_2 = c_3 = 0$ and $c_4 > 0$, then $e = 0$, $f \leqslant 1$, $g \leqslant 1$, $h \leqslant 2$ and $h = 0$ if $f = g = 1$.

Consider first the case $c_2 > 0$ and recall that in this case $e = f = g = 0$ and $h \leqslant 2$. Assume now $c_2 \geqslant h$. Take as X a general union of Y', c_5 5-points, c_4 4-points, c_3 3-points, $(c_2 - h)$ 2-points, h virtual schemes obtained by applying Lemma 8 at h general points of H with respect to the sequence $(1,3)$. Clearly we have $\deg(X \cap H) = \binom{t+2}{2}$. Let us see now how to specialize Y to X in the remaining cases with $c_2 > 0$. If $c_2 = 1 < h$ and $c_3 + c_4 + c_5 \geqslant 1$, then in the previous step we apply Lemma 8 using the unique 2-point and one 3-point or 4-point or 5-point with respect to the sequence $(1,6,3)$ or $(1,10,6,3)$ or $(1,15,10,6,3)$ (recall that we assumed $c_i > 0$ for at least one $i \in \{3,4,5\}$) and we conclude in the same way. If $c_2 = 1 < h$ and $c_3 = c_4 = c_5 = 0$, then we apply Lemma 8 to the unique double point with respect to the sequence $(1,3)$, and we are in case (II). Here and in all later instances of case (II) it is straightforward to check that the inequalities $c_2 + c_3 + c_4 + c_5 < \beta$ and $c_2 + c_3 + c_4 + c_5 \leqslant 2$ are verified.

Assume now $c_2 = 0$ and $c_3 > 0$. Recall that $e = f = 0$, $g \leqslant 1$ and $h \leqslant 2$. If $c_3 \geqslant g + h$ we take as X a general union of Y', c_5 5-points, c_4 4-points, $c_3 - g - h$ 3-points, g virtual schemes obtained applying Lemma 8 at f general points of H with respect to the sequence $(3,6,1)$ and g virtual schemes obtained applying Lemma 8 at g general points of H with respect to the sequence $(1,6,3)$. If $0 < c_3 < g + h$ and $c_4 + c_5 \geqslant g + h - c_3$, then in the previous step we apply Lemma 8 using c_3 3-points, and $(f + g - c_3)$ 4-points or 5-points, with respect to the sequences $(3,10,6,1)$ or $(1,10,6,3)$ or $(3,6,10,15,1)$ or $(1,15,10,6,3)$. In all these cases we clearly have $\deg(X \cap H) = \binom{t+2}{2}$. If $c_2 = 0$, $0 < c_3 < g + h$ and $c_4 + c_5 < g + h - c_3$, then we have either $c_3 = 1$ and $c_4 + c_5 \leqslant 1$, or $c_3 = 2$, $g = 1$, $h = 2$ and $c_4 = c_5 = 0$. In both cases $\beta > c_3 + c_4 + c_5$. In this cases we can specialize all the components on H, possibly applying Lemma 8 and we are in case (II).

Now, assume that $c_2 = c_3 = 0$ and $c_4 > 0$. Hence $e = 0$, $f \leqslant 1$, $g \leqslant 1$, $h \leqslant 2$ and $h = 0$ if $f = g = 1$. If $c_4 + c_5 \geqslant f + g + h$, then we take as X a general union of Y', $(c_4 + c_5 - f - g - h)$ 4-points or 5-points, f virtual schemes obtained applying Lemma 8 at f general points of H with respect to the sequence $(6,10,3,1)$ or $(6,15,10,3,1)$, g virtual schemes obtained applying Lemma 8 at g general points of H with respect to the sequence $(3,10,6,1)$ or $(3,15,10,6,1)$ and h virtual schemes obtained applying Lemma 8 at h general points of H with respect to the sequence $(1,10,6,3)$ or $(1,15,10,6,3)$. Thus we have again $\deg(X \cap H) = \binom{t+2}{2}$. If $c_2 = c_3 = 0$ and $0 < c_4 + c_5 < f + g + h$, then we are again in case (II), because we can specialize all the 4-points and 5-points on H (possibly applying Lemma 8), since $c_4 + c_5 \leqslant f + g + h \leqslant 3$ and $\beta = 6f + 3g + h$; in this case we may also assume that if $f \neq 0$ (i.e. $f = 1$), then either one of the 4-points is specialized with respect to the sequence $(6,10,3,1)$.

Finally assume that $c_2 = c_3 = c_4 = 0$ and $c_5 > 0$. If $c_5 \geqslant e+f+g+h$ we apply Lemma 8 at e general points of H with respect to the sequence $(10, 15, 6, 3, 1)$, at f general points of H with respect to the sequence $(6, 15, 10, 3, 1)$, at g general points of H with respect to the sequence $(3, 15, 10, 6, 1)$ and at h general points of H with respect to the sequence $(1, 15, 10, 6, 3)$. In this way we arrive to case (I). If $e + f + g + h > c_5$, then we start applying again Lemma 8 as in the previous step, but we have to stop at some point and we land in case (II).

Finally, we note that the property (7) follows immediately from the construction above and from Lemma 10. □

In order to prove the good postulation of schemes in $\mathbb{P}^3$ by applying induction, we need to know the good postulation of schemes in $\mathbb{P}^2$. In the next remark we point out the related results that we need.

Remark 13. When the general union has multiplicity up to 4 and $n = 2$, then we can use some results by Mignon (see [18, Theorem 1]). In particular we know that a general fat point scheme in $\mathbb{P}^2$ of multiplicity $1 \leqslant m \leqslant 4$ has good postulation in degree $d \geqslant 3m$. Interestingly, this result is valid for any characteristic of the ground field $\mathbb{K}$ (for a discussion about char($\mathbb{K}$) see Section 7).

For multiplicities up to 7 and when char($\mathbb{K}$) = 0, we can use some results by Yang (see [21, Theorem 1 and Lemma 7]), which imply that a general fat point scheme in $\mathbb{P}^2$ of multiplicity $m \leqslant 7$ has good postulation in degree $d \geqslant 3m$.

The case with no quintuple points has already been solved, as explained below.

Remark 14. When the general union Y has multiplicity up to 4 and $n = 3$, we know that Y must have good postulation in any degree $d \geqslant 9$, thanks to [4] and [10]. There is no self-contained theoretical proof for this, but we have a theoretical proof for $d \geqslant 41$ in [4], along with a computer check up to $d = 13$, and the missing computations can be found in [10].

2.1. Summary of our computational results

We list the results from Section 5 that we need in the following sections.

Lemma 15. *The following linear systems are non-special and have virtual dimension* -1:

(1) $\mathcal{L}_3(3; 2^5)$,
(2) $\mathcal{L}_3(9; 4^a, 3^b)$ *with* $2a + b = 22$,
(3) $\mathcal{L}_3(9; 5^4, 4^4)$,
(4) $\mathcal{L}_3(12; 5^a, 4^b, 3^c)$ *with* $7a + 4b + 2c = 91$.

Theorem 16. *Fix non-negative integers d, w, x, y, z such that $11 \leqslant d \leqslant 21$ and $0 \leqslant z \leqslant 4$. Let $N = \binom{d+3}{3}$. Let $Y \subset \mathbb{P}^3$ be a general union of w 5-points, x 4-points, y 3-points and z 2-points such that*

$$N - 3 \leqslant 35w + 20x + 10y + 4z \leqslant N + \Delta,$$

where Δ is as in Lemma 5.

Then Y has good postulation.

Theorem 17. *Fix non-negative integers d, q, w, x, y, z such that*:

- $22 \leqslant d \leqslant 37$,
- $0 \leqslant z \leqslant 4$,
- $0 \leqslant 2x + y \leqslant 21$,
- $0 \leqslant w \leqslant 3$ *or* $0 \leqslant x \leqslant 3$.

Let $N = \binom{d+3}{3}$. *Let* $Y \subset \mathbb{P}^3$ *be a general union of* w 5*-points,* x 4*-points,* y 3*-points and* z 2*-points such that*

$$N - 3 \leqslant 220q + 35w + 20x + 10y + 4z \leqslant N + \Delta,$$

where Δ *is as in Lemma* 5.

Then Y *has good postulation.*

Theorem 18. *Fix non-negative integers* d, r, w, x, y, z *such that*:

- $38 \leqslant d \leqslant 52$,
- $0 \leqslant z \leqslant 4$,
- $0 \leqslant 2x + y \leqslant 41$,
- $0 \leqslant w \leqslant 12$.

Let $N = \binom{d+3}{3}$. *Let* $Y \subset \mathbb{P}^3$ *be a general union of* w 5*-points,* x 4*-points,* y 3*-points and* z 2*-points such that*

$$N - 3 \leqslant 455r + 35w + 20x + 10y + 4z \leqslant N + \Delta,$$

where Δ *is as in Lemma* 5.

Then Y *has good postulation.*

3. The proof of Theorem 1 for high degrees

This section is devoted to the proof of Theorem 1 for high degrees, that is for $d \geqslant 53$. Throughout the section we fix a hyperplane $H \subset \mathbb{P}^3$. We recall that our ground field $\mathbb{K}$ has characteristic zero.

In the different steps of the proof we will work with zero-dimensional schemes that are slightly more general than a union of fat points. In particular, we will say that a zero-dimensional scheme Y is *of type* $(\star)$ if its irreducible components are of the following types:

- m-points with $2 \leqslant m \leqslant 5$, supported on general points of $\mathbb{P}^3$,
- m-points with $1 \leqslant m \leqslant 5$, or virtual schemes arising as residual in the list of Lemma 8, supported on general points of H.

Given a scheme Y of type $(\star)$ satisfying (6) for some integer t, we will say that Y is of type (I, t) if, when we apply Lemma 12 to Y, we are in case (I). Otherwise we say that Y is of type (II, t).

We fix now (and we will use throughout this section) the following notation, for any integer t: given a scheme Y_t of type $(\star)$ and satisfying (6) for t, we will denote by X_t the specialization described in Lemma 12. We write the residual $\mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}$, where Y_{t-1} is the union of all unreduced components of $\mathrm{Res}_H(X_t)$ and $Z_{t-1} = \mathrm{Res}_H(X_t) \setminus Y_{t-1}$. Clearly Z_{t-1} is the union of finitely many simple points of H. Thus, at each step $t \mapsto t-1$ we will have

$$Y_t \mapsto X_t \mapsto \mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}.$$

For any integer t, we set $z_t := |Z_t|$, $\alpha_t := \deg(Y_t) = \deg(X_t)$, and

$$\delta_t := \max\left(0, \binom{t+2}{3} - \deg(Y_{t-1} \cup Z_{t-1})\right).$$

We fix the following statements:

- $\mathsf{A}(t) = \{Y_t$ has good postulation in degree $t\}$,
- $\mathsf{B}(t) = \{\mathrm{Res}_H(X_t)$ has good postulation in degree $t-1\}$,
- $\mathsf{C}(t) = \{\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_H(Y_{t-1})}(t-2)) \leqslant \delta_t\}$.

Claim 19. *Fix $t \geqslant 16$. If Y_t is a zero-dimensional scheme of type (II, t), then it has good postulation, i.e.* $\mathsf{A}(t)$ *is true. Moreover if $t \geqslant 17$, also* $\mathsf{B}(t)$ *is true.*

Proof. Since Y_t is of type (II, t), when we apply Lemma 12 to Y_t, we obtain a specialization X_t such that all its irreducible components are supported on H and such that $\deg(X_t \cap H) \leqslant \binom{t+2}{2}$.

We prove now the vanishing $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{Y_t}(t)) = 0$. By semicontinuity, it is enough to prove the vanishing $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{X_t}(t)) = 0$. Notice that by taking the residual of X_t with respect to H for at most five times we get at the end the empty set.

Since $\deg(X_t \cap H) \leqslant \binom{t+2}{2}$ and $t \geqslant 15$, by Remark 13 it follows the vanishing $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{X_t \cap H}(t)) = 0$. Let R_{t-1} denote the residual $\mathrm{Res}_H(X_t)$ and recall that any component of R_{t-1} is supported on H. We check now that $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{R_{t-1}}(t-1)) = 0$.

In order to do this we take again the trace and the residual with respect to H. By (7) we know that $\deg(\mathrm{Res}_H(X_t) \cap H) \leqslant \binom{t+1}{2}$ then again by Remark 13, since $t-1 \geqslant 15$, we have $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-1} \cap H}(t-1)) = 0$.

We repeat this step taking $R_{t-2} := \mathrm{Res}_H(R_{t-1})$ and noting that the trace $R_{t-2} \cap H$ has degree less than or equal to $\binom{t}{2}$, by Lemma 10. Moreover this time the scheme $R_{t-2} \cap H$ cannot contain quintuple points, in fact it is a general union of quartuple, triple, double and simple points. Hence by Remark 13 we have $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-2} \cap H}(t-2)) = 0$, since $t-2 \geqslant 12$.

We repeat once again the same step and we obtain $R_{t-3} := \mathrm{Res}_H(R_{t-2})$. Now the trace $R_{t-3} \cap H$ contains only triple, double or simple points and so we have again the vanishing $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-3} \cap H}(t-3)) = 0$, by Remark 13, since $t-3 \geqslant 9$. Set $R_{t-4} := \mathrm{Res}_H(R_{t-3})$. The scheme $R_{t-4} \cap H$ is reduced and formed by less than $\binom{t-2}{2}$ general points of H. Hence $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{R_{t-4} \cap H}(t-4)) = 0$. Notice that this time the residual $\mathrm{Res}_H(R_{t-4})$ must be empty and so, since $\mathcal{I}_{\mathrm{Res}_H(R_{t-4})} = \mathcal{O}_{\mathbb{P}^3}$, we obviously have $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{\mathrm{Res}_H(R_{t-4})}(t-5)) = 0$. Hence thanks to Lemma 7 we obtain $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{Y_t}(t)) = 0$.

We also know that

$$\deg(Y_t) = \deg(X_t) \leqslant \binom{t+2}{2} + \binom{t+1}{2} + \binom{t}{2} + \binom{t-1}{2} + \binom{t-2}{2} \leqslant \binom{t+3}{3} \tag{8}$$

where the second inequality is equivalent to $\binom{t-2}{3} \geqslant 0$, which is true if $t \geqslant 4$. Hence Y_t has good postulation, that is, $\mathsf{A}(t)$ is true.

It is easy to see that also the scheme $\mathrm{Res}(X_t)$ must be of type $(\mathrm{II}, t-1)$. Hence $\mathsf{B}(t)$ follows from the first part of the proof. □

Claim 20. *Fix $t \geqslant 15$. If Y_t is a zero-dimensional scheme of type (I, t), then* $\mathsf{A}(t)$ *is true if* $\mathsf{B}(t)$ *is true.*

Proof. Since Y_t is of type (I, t), we can apply Lemma 12 and we obtain a specialization X_t such that $\deg(X_t \cap H) = \binom{t+2}{2}$. Thus, since $t \geqslant 15$, by Remark 13 it follows

$$\mathrm{h}^0\big(H, \mathcal{I}_{X_t \cap H}(t)\big) = \mathrm{h}^1\big(H, \mathcal{I}_{X_t \cap H}(t)\big) = 0.$$

Then, thanks to Lemma 7, it follows, for $i = 0, 1$,

$$\mathrm{h}^i\big(\mathbb{P}^3, \mathcal{I}_{X_t}(t)\big) = \mathrm{h}^i\big(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_H(X_t)}(t-1)\big).$$

Thus in order to prove that the scheme X_t has good postulation in degree t, it is sufficient to check the good postulation of $\mathrm{Res}_H(X_t)$ in degree $t-1$. □

Claim 21. *If* $\mathsf{A}(t-1)$ *and* $\mathsf{C}(t)$ *are true, then* $\mathsf{B}(t)$ *is true.*

Proof. Recall that we write $\mathrm{Res}_H(X_t) = Y_{t-1} \cup Z_{t-1}$,where Z_{t-1} is a union of simple points supported on H. By [4, Lemma 7], to check that the scheme $\mathrm{Res}_H(X_t)$ has good postulation in degree $t-1$ (i.e. $\mathsf{B}(t)$), it is sufficient to check the good postulation of Y_{t-1} in degree $t-1$ (i.e. $\mathsf{A}(t-1)$) and to prove that $\mathsf{C}(t)$ is true. □

Claim 22. *If Y_t is of type* (I, t)*, then* $\mathsf{B}(t-1)$ *implies* $\mathsf{C}(t)$*.*

Proof. The statement $\mathsf{C}(t)$ is true if $\mathrm{h}^0(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_H(Y_{t-1})}(t-2)) \leqslant \delta_t$. Note that since $\deg(X_t \cap H) = \binom{t+2}{2}$, we have

$$\deg\bigl(\mathrm{Res}_H(X_t)\bigr) = \deg(Y_{t-1} \cup Z_{t-1}) = \alpha_{t-1} + z_{t-1} = \alpha_t - \binom{t+2}{2},$$

and thus it follows

$$\delta_t := \max\left(0, \binom{t+2}{3} - \alpha_{t-1} - z_{t-1}\right) = \max\left(0, \binom{t+3}{3} - \alpha_t\right).$$

Notice that, by (7), we have $\deg(\mathrm{Res}_H(X_t) \cap H) \leqslant \binom{t+1}{2}$. Hence, it follows

$$\deg\bigl(\mathrm{Res}_H(Y_{t-1})\bigr) = \deg\bigl(\mathrm{Res}_H\bigl(\mathrm{Res}_H(X_t)\bigr)\bigr) \geqslant \alpha_t - \binom{t+2}{2} - \binom{t+1}{2}$$

and then, since $\binom{t+2}{2} + \binom{t+1}{2} = \binom{t+3}{3} - \binom{t+1}{3}$, we get

$$\deg\bigl(\mathrm{Res}_H(Y_{t-1})\bigr) \geqslant \binom{t+1}{3} - \binom{t+3}{3} + \alpha_t \geqslant \binom{t+1}{3} - \delta_t.$$

So in order to prove $\mathsf{C}(t)$ it is enough to prove that $\mathrm{Res}_H(Y_{t-1})$ has good postulation in degree $t-2$. □

Now we are in position to prove our main result.

Proof of Theorem 1 for $d \geqslant 53$. For all non-negative integers $d \geqslant 53$ and w, x, y, z, we set

$$\epsilon(d, w, x, y, z) := \binom{d+3}{3} - 35w - 20x - 10y - 4z.$$

We will often write ϵ instead of $\epsilon(d, w, x, y, z)$ in any single step of the proofs in which the parameters d, w, x, y, z are fixed.

By Lemma 5, in order to prove our statement for all quadruples (w, x, y, z) it is sufficient to check it for all quadruples (w, x, y, z) such that $-13 \leqslant \epsilon(d, w, x, y, z) \leqslant 3$. We fix any such quadruple and we consider a general union Y of w 5-points, x 4-points, y 3-points and z 2-points.

Notice that

$$w + x + y + z \geqslant \left\lceil \frac{1}{35}\left(\binom{d+3}{3} - 3\right)\right\rceil \geqslant \frac{1}{35}\binom{d+3}{3} - \frac{3}{35}, \tag{9}$$

i.e. the scheme Y has at least $\lceil \frac{1}{35}(\binom{d+3}{3} - 3)\rceil$ connected components.

The proof is by induction, based on Lemma 7, and it requires different steps.

Set $Y_d = Y$ and fix a hyperplane $H \subset \mathbb{P}^3$. We can assume by generality that $\deg(Y_d \cap H) \leqslant \binom{d+2}{2}$, hence we can apply Lemma 12, thus specializing the scheme Y_d to a scheme X_d. If Y_d is of type (II, d), then we conclude by Claim 19, since $d \geqslant 16$. Hence we can assume that Y_d is of type (I, d), and so, since $d \geqslant 15$, by Claim 20 it is enough to check that the scheme $\mathrm{Res}_H(X_d)$ has good postulation in degree $d-1$. Now we write $\mathrm{Res}_H(X_d) = Y_{d-1} \cup Z_{d-1}$, where Y_{d-1} is the union of all unreduced components of $\mathrm{Res}_H(X_d)$ and $Z_{d-1} = \mathrm{Res}_H(X_d) \setminus Y_{d-1}$. By Claim 21, it is enough to prove that $\mathsf{A}(d-1)$ and $\mathsf{C}(d)$ are true. Notice that, since $d \geqslant 18$, we get (7), i.e. $\deg(Y_{d-1} \cap H) \leqslant \deg(\mathrm{Res}_H(X_d) \cap H) \leqslant \binom{d+1}{2}$. Hence Y_{d-1} satisfies condition (6) in degree $d-1$, then we can apply again Lemma 12.

We have now two alternatives: either Y_{d-1} is of type $(\mathrm{I}, d-1)$, or of type $(\mathrm{II}, d-1)$. In both cases, we note that by Claim 22 the statement $\mathsf{C}(d)$ follows from $\mathsf{B}(d-1)$, since Y_d is of type (I, d). Now assume that Y_{d-1} is of type $(\mathrm{II}, d-1)$. Then by Claim 19, since $d-1 \geqslant 17$ we know that $\mathsf{B}(d-1)$ and $\mathsf{A}(d-1)$ are true and this concludes the proof. It remains to consider the case Y_{d-1} of type $(\mathrm{I}, d-1)$. We apply again Claim 21 and we go on iterating the same steps.

Now we have two cases: either in a finite number v of steps the procedure described above gives us a scheme X_{d-v} of type $(\mathrm{II}, d-v)$, for a degree $d-v \geqslant 18$, or the procedure goes on until we get X_{18}, a scheme of type $(\mathrm{I}, 18)$.

In the first case, the steps of the procedure above prove that the scheme X_d has good postulation, and the statement is proved.

Assume now that we are in the second case, i.e. X_{18} is of type $(\mathrm{I}, 18)$, that is, $\deg(X_{18} \cap H) = \binom{20}{2}$. Note that, since $\epsilon \geqslant -13$, we have

$$\deg(X_{18}) = \binom{21}{3} - \epsilon - \sum_{t=18}^{d-1} z_t \leqslant \binom{21}{3} + 13 - \sum_{t=18}^{d-1} z_t. \tag{10}$$

Now we want to estimate $\sum_{t=18}^{d-1} z_t$, which is the number of simple points we have removed in the steps above. Since we started from the scheme Y_d, in $d-18$ steps we arrived at the scheme X_{18} in such a way that the case (II) never occurred. Assume that in these $d-18$ steps we have applied γ times Lemma 8 with respect to sequences of type $(1, 15, 10, 6, 3)$, $(1, 10, 6, 3)$, $(1, 6, 3)$ or $(1, 3)$. As it is clear looking at the proof of Lemma 12, at each step the number of times we used a sequence giving as a trace a simple point is at most 2, hence we have $\gamma \leqslant 2(d-18)$. Let u_{18} denote the number of connected components of X_{18}. Hence it follows that

$$\sum_{t=18}^{d-1} z_t \geqslant w + x + y + z - 2(d-18) - u_{18}. \tag{11}$$

Now we need to estimate the number u_{18}. Let us denote by T the union of components of X_{18} of length 3. Then any component of the scheme $X_{18} \setminus T$ has length at least 4, and $\deg(T) \leqslant \binom{20}{2}$ since the scheme T is completely contained in the trace $X_{18} \cap H$. So we have

$$u_{18} \leqslant \frac{1}{3}\deg(T) + \frac{1}{4}\big(\deg(X_{18}) - \deg(T)\big) \leqslant \frac{1}{12}\binom{20}{2} + \frac{1}{4}\big(\deg(X_{18})\big),$$

and using (11) and (9), we get

$$\sum_{t=18}^{d-1} z_t \geqslant \frac{1}{35}\binom{d+3}{3} - \frac{3}{35} - 2(d-18) - \frac{1}{12}\binom{20}{2} - \frac{1}{4}\big(\deg(X_{18})\big). \tag{12}$$

By using (10) and (12) we get

$$\frac{3}{4}\deg(X_{18}) \leqslant \binom{21}{3} + 13 - \frac{1}{35}\binom{d+3}{3} + \frac{3}{35} + 2(d-18) + \frac{1}{12}\binom{20}{2}$$

and, since $d \geqslant 53$, it is easy to check that $\deg(X_{18}) \leqslant \binom{21}{3}$. Note that X_{18} depends implicitely on d, but we use the above inequality to show what happened for $d = 53$. Of course, for higher d's we can easily show that $\deg(X_{18})$ is actually even smaller, but we do not need it and we content ourselves with the claimed $\deg(X_{18}) \leqslant \binom{21}{3}$.

Hence we need to prove the vanishing $h^1(\mathbb{P}^3, \mathcal{I}_{X_{18}}(18)) = 0$, and by Lemma 12, it is enough to prove that $h^1(\mathbb{P}^3, \mathcal{I}_{\mathrm{Res}_H(X_{18})}(17)) = 0$.

Now we change the procedure. Denote $\mathrm{Res}_H(X_{18}) = R_{17}$.

Since $\deg(R_{17} \cap H) \leqslant \binom{19}{2}$, specializing some points on H we can degenerate (without applying the Horace differential lemma) the scheme R_{17} to a scheme $\widetilde{R}_{17}$ in such a way that one of the following cases happens:

(1a) either $\deg(\widetilde{R}_{17} \cap H) \leqslant \binom{19}{2} - 15$ and all the components of $\widetilde{R}_{17}$ are supported on H,
(1b) or $\binom{19}{2} - 14 \leqslant \deg(\widetilde{R}_{17} \cap H) \leqslant \binom{19}{2}$.

Denote now, in both cases, $\widetilde{R}_{17} = E_{17} \cup F_{17}$, where E_{17} is supported on H and F_{17} is supported outside H. Take now the trace $\widetilde{R}_{17} \cap H = E_{17} \cap H$ and the residual $\mathrm{Res}_H(\widetilde{R}_{17}) = R_{16}$.

Note that in case (1a) $F_{17} = \emptyset$, while in case (1b)

$$\deg(R_{16}) \leqslant \deg(X_{18}) - \binom{20}{2} - \left(\binom{19}{2} - 14\right).$$

By Lemma 7, to prove that $h^1(\mathbb{P}^3, \mathcal{I}_{R_{17}}(17)) = 0$, it is enough to prove that $h^1(\mathbb{P}^2, \mathcal{I}_{E_{17}\cap H}(17)) = 0$ (by Remark 13, since $17 \geqslant 15$) and $h^1(\mathbb{P}^3, \mathcal{I}_{R_{16}}(16)) = 0$.

Now we repeat the same step, that is, we specialize some points on H without applying the Horace differential lemma, degenerating R_{16} to $\widetilde{R}_{16}$ in such a way that one of the following cases happens:

(2a) either $\deg(\widetilde{R}_{16} \cap H) \leqslant \binom{18}{2} - 15$ and all the components of $\widetilde{R}_{16}$ are supported on H,
(2b) or $\binom{18}{2} - 14 \leqslant \deg(\widetilde{R}_{16} \cap H) \leqslant \binom{18}{2}$.

Denote again, in both cases, $\widetilde{R}_{16} = E_{16} \cup F_{16}$, where E_{16} is supported on H and F_{16} is supported outside H.

Note that E_{16} is given by quintuple, quartuple, triple, double and simple points or virtual schemes arisen by the application of Lemma 10. In any case taking the residual with respect to H five times we get that the last residual has no components supported on H.

Take now the trace $\widetilde{R}_{16} \cap H = E_{16} \cap H$ and the residual $\mathrm{Res}_H(\widetilde{R}_{16}) = R_{15}$.

Note that in case (2a) $F_{16} = \emptyset$, while in case (2b)

$$\deg(R_{15}) \leqslant \deg(X_{18}) - \binom{20}{2} - \left(\binom{19}{2} - 14\right) - \left(\binom{18}{2} - 14\right).$$

By Lemma 7, to prove $h^1(\mathbb{P}^3, \mathcal{I}_{R_{16}}(16)) = 0$, we only need $h^1(\mathbb{P}^2, \mathcal{I}_{E_{16}\cap H}(16)) = 0$ (which is true by Remark 13, since $16 \geqslant 15$) and $h^1(\mathbb{P}^3, \mathcal{I}_{R_{15}}(15)) = 0$.

Now, without specializing furtherly, we denote $R_{15} = E_{15} \cup F_{15}$, where E_{15} is supported on H and $F_{15} = F_{16}$ is supported outside H. Take now the trace $\widetilde{R}_{15} \cap H = E_{15} \cap H$ and the residual $\mathrm{Res}_H(\widetilde{R}_{15}) = R_{14}$.

By Lemma 7, to prove that $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{15}}(15)) = 0$, it is enough to prove that $\mathrm{h}^1(\mathbb{P}^2, \mathcal{I}_{E_{15}\cap H}(15)) = 0$ (which is true by Remark 13, since $15 \geqslant 12$ and the trace contains at most quartuple points) and $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{14}}(14)) = 0$.

We repeat again the same step and we get $R_{14} = E_{14} \cup F_{14}$, where E_{14} is supported on H and $F_{14} = F_{16}$ is supported outside H.

Take now the trace $\widetilde{R}_{14} \cap H = E_{14} \cap H$ and the residual $\mathrm{Res}_H(\widetilde{R}_{14}) = R_{13}$.

By Lemma 7, to prove that $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{14}}(14)) = 0$, it is enough to prove that $\mathrm{h}^1(\mathbb{P}^2, \mathcal{I}_{E_{14}\cap H}(14)) = 0$ (which is true by Remark 13, since the trace contains at most triple points) and $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{13}}(13)) = 0$.

We repeat again the same step and we get $R_{13} = E_{13} \cup F_{13}$, where E_{13} is supported on H and $F_{13} = F_{16}$ is supported outside H.

Take now the trace $\widetilde{R}_{13} \cap H = E_{13} \cap H$ and the residual $\mathrm{Res}_H(\widetilde{R}_{13}) = R_{12}$.

By Lemma 7, to prove that $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{13}}(13)) = 0$, it is enough to prove that $\mathrm{h}^1(\mathbb{P}^2, \mathcal{I}_{E_{13}\cap H}(13)) = 0$ (which is true by Remark 13, since the trace contains at most double points) and $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{12}}(12)) = 0$.

Now we take again for the last time the trace and the residual with respect to H. Denote $R_{12} = E_{12} \cup F_{12}$, where E_{12} is supported on H and $F_{12} = F_{16}$ is supported outside H. Taking the trace and the residual we have that $E_{12} \cap H$ is given by general simple points in H and obviously we have $\mathrm{h}^1(\mathbb{P}^2, \mathcal{I}_{E_{12}\cap H}(12)) = 0$.

So we need only to show that the residual $\mathrm{Res}_H(E_{12} \cup F_{12}) = R_{11}$ satisfies $\mathrm{h}^1(\mathbb{P}^3, \mathcal{I}_{R_{11}}(11)) = 0$.

Note that the residual does not have components supported on H. More precisely, $R_{11} = F_{12} = F_{16}$, that is, the residual is a general collection of double, triple, quartuple and quintuple points.

But Theorem 16 ensures that any general collection of double, triple, quartuple and quintuple points has good postulation in degree 11.

So in order to conclude the proof of the theorem it is enough to prove the following inequality:

$$\deg(R_{11}) = \deg(F_{16}) \leqslant \binom{14}{3}. \tag{13}$$

Let us check this condition in any of the previous cases: in cases (1a) and (2a) we have $F_{16} = \emptyset$ and so condition (13) is obviously satisfied. It remains to prove (13) in case (2b), where

$$\deg(R_{11}) \leqslant \deg(R_{15}) \leqslant \deg(X_{18}) - \binom{20}{2} - \left(\binom{19}{2} - 14\right) - \left(\binom{18}{2} - 14\right).$$

By (10) and (12) we have

$$\deg(X_{18}) \leqslant \binom{21}{3} + 13 - \left(\frac{1}{35}\binom{d+3}{3} - \frac{3}{35} - 2(d-18) - \frac{1}{12}\binom{20}{2} - \frac{1}{4}\bigl(\deg(X_{18})\bigr)\right)$$

from which we obtain

$$\deg(X_{18}) \leqslant \frac{4}{3}\left(\binom{21}{3} + 13 - \frac{1}{35}\binom{d+3}{3} + \frac{3}{35} + 2(d-18) + \frac{1}{12}\binom{20}{2}\right)$$

and so

$$\deg(R_{15}) \leqslant \deg(X_{18}) - \binom{20}{2} - \left(\binom{19}{2} - 14\right) - \left(\binom{18}{2} - 14\right).$$

It is easy to check that, for any $d \geqslant 53$ the inequality

$$\frac{4}{3}\left(\binom{21}{3} + 13 - \frac{1}{35}\binom{d+3}{3} + \frac{3}{35} + 2(d-18) + \frac{1}{12}\binom{20}{2}\right) - \binom{20}{2} - \left(\binom{19}{2} - 14\right) - \left(\binom{18}{2} - 14\right) \leqslant \binom{14}{3}$$

is verified, and this implies (13) and completes the proof. □

4. The proof of Theorem 1 for low degrees

In this section we discuss Theorem 1 in the remaining cases, that is, when the degree d satisfies $11 \leqslant d \leqslant 52$. In these cases the proof is based on computer calculations, which are described explicitly in Section 5. Although in principle it is possible to go through all cases in Lemma 5 for $11 \leqslant d \leqslant 52$, this is impractical with nowadays computers. In order to shorten the computational time we need some other auxiliary theoretical results, that we develop in this section. First we prove Theorem 1 for degrees $\geqslant 38$ in the special case when we have few quintuple points (more precisely when $35w \leqslant \frac{1}{12}\binom{d+3}{3}$). Then we will present how to apply a result by Dumnicki in order to greatly reduce the cases to be tested by our computers.

The proof of the following proposition is a modification of the argument in the previous section, where we proved Theorem 1 for $d \geqslant 53$.

Proposition 23. *Fix non-negative integers $d \geqslant 38$, w, x, y, z such that $35w \leqslant \frac{1}{12}\binom{d+3}{3}$ and*

$$\binom{d+3}{3} - 3 \leqslant 35w + 20x + 10y + 4z \leqslant \binom{d+3}{3} + 13.$$

Let $Y \subset \mathbb{P}^3$ be a general union of w 5-points, x 4-points, y 3-points and z 2-points. Then Y has good postulation in degree d.

Proof. We follow the same procedure as in the main proof of Section 3. The first difference is that every time we apply Lemma 12, we specialize on the plane H as many quintuple points as possible.

So starting with $Y_d = Y$, we obtain in a finite number of steps $d - d_0$ a scheme X_{d_0} which does not contain quintuple points. We prove that in particular $d_0 \geqslant 20$. Indeed since by assumption we have $35w \leqslant \frac{1}{12}\binom{d+3}{3}$ and $d \geqslant 20$, it easily follows that $w \leqslant \frac{1}{35}(\binom{d+3}{3} - \binom{22}{3})$.

Thus we have a general union X_{d_0} of quartuple, triple and double points, and of virtual schemes of the type listed in the table of Lemma 8, arisen by the application of Lemma 12.

If X_{d_0} is of type (II, d_0), then we conclude, as in the previous proof, that it has good postulation and this implies that Y has good postulation.

Let us assume that X_{d_0} is of type (I, d_0). Applying again Lemma 12 we can go on with our usual argument and we will obtain or a scheme of type (II, e), for some $e \geqslant 18$, which concludes the proof, or a scheme X_{18} of type $(\mathrm{I}, 18)$ and without quintuple points.

At this point we can apply the same argument used in the proof of [4, Theorem 1], regarding union of quartuple, triple and double points and virtual schemes of the type listed in [4, Lemma 4]. In particular we apply [4, Lemma 8] until we get a scheme X_e of type (II, e) in degree $e \geqslant 13$. In this case we conclude, as in [4], that our scheme has good postulation.

Now it remains to consider the case when we get a scheme X_{13} of type $(\mathrm{I}, 13)$. Notice that in this case we want to prove that $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{X_{13}}(13)) = 0$.

Indeed let us prove that $\deg(X_{13}) \leqslant \binom{16}{3}$. First of all, note that, since $35w \leqslant \frac{1}{12}\binom{d+3}{3}$,

$$w + x + y + z \geqslant x + y + z \geqslant \frac{1}{20} \cdot \frac{11}{12}\binom{d+3}{3} - \frac{3}{20} \tag{14}$$

and setting $\epsilon = \binom{d+3}{3} - 35w - 20x - 10y - 4z \geqslant -13$ we have:

$$\deg(X_{13}) = \binom{16}{3} - \epsilon - \sum_{t=13}^{d-1} z_t \leqslant \binom{16}{3} + 13 - \sum_{t=13}^{d-1} z_t, \tag{15}$$

where z_t denotes, as in Section 3, the number of simple points we have removed at the $(d-t)$-th step. As in (11) we have

$$\sum_{t=13}^{d-1} z_t \geqslant w + x + y + z - 2(d-13) - u_{13},$$

where u_{13} is the number of connected component of X_{13}. Since X_{13} does not contain simple points we have $u_{13} \leqslant \frac{1}{3}\deg(X_{13})$ and so by (14) we get

$$\sum_{t=13}^{d-1} z_t \geqslant \frac{11}{240}\binom{d+3}{3} - \frac{3}{20} - 2(d-13) - \frac{1}{3}\bigl(\deg(X_{13})\bigr) \tag{16}$$

and by (15) we get

$$\deg(X_{13}) \leqslant \frac{3}{2}\left(\binom{16}{3} + 13 - \frac{11}{240}\binom{d+3}{3} + \frac{3}{20} + 2(d-13)\right). \tag{17}$$

But now it is easy to check that

$$\frac{3}{2}\left(\binom{16}{3} + 13 - \frac{11}{240}\binom{d+3}{3} + \frac{3}{20} + 2(d-13)\right) \leqslant \binom{16}{3}$$

as soon as $d \geqslant 30$. Then it is enough to prove that $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{X_{13}}(13)) = 0$.

Now we apply the residual without specializing any further components on H. In other words we take $Y_{12} := \mathrm{Res}_H(X_{13})$, $Y_{11} := \mathrm{Res}_H(Y_{12})$, $Y_{10} := \mathrm{Res}_H(Y_{11})$ and $Y_9 := \mathrm{Res}_H(Y_{10})$. Notice that $\deg(Y_{12} \cap H) \leqslant \binom{14}{2}$, $\deg(Y_{11} \cap H) \leqslant \binom{13}{2}$, and $\deg(Y_{10} \cap H) \leqslant \binom{12}{2}$. So by Remark 13 all the vanishings $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{Y_{12}\cap H}(12)) = 0$, $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{Y_{11}\cap H}(11)) = 0$ and $\mathrm{h}^1(\mathbf{P}^2, \mathcal{I}_{Y_{10}\cap H}(10)) = 0$ are satisfied.

Hence by Lemma 7, it is sufficient to prove $\mathrm{h}^1(\mathbf{P}^3, \mathcal{I}_{Y_9}(9)) = 0$. Recall that for any integer $t \geqslant 9$ a general union of quadruple, triple and double points has good postulation in degree t by [4,10].

Thus it is sufficient to prove that $\deg(Y_9) \leqslant \binom{12}{3}$. Indeed obviously we have $\deg(Y_9) \leqslant \deg(X_{13})$. It is easy to check that

$$\frac{3}{2}\left(\binom{16}{3} + 13 - \frac{11}{240}\binom{d+3}{3} + \frac{3}{20} + 2(d-13)\right) \leqslant \binom{12}{3}$$

for any $d \geqslant 38$. Hence by (17) we have $\deg(Y_9) \leqslant \binom{12}{3}$ and this concludes our proof. $\square$

The crucial tool which allows us to perform our computation in a reasonable time is the following special case of [9, Theorem 1].

Theorem 24 *(Dumnicki). Let $d, k, m_1, \dots, m_s, m_{s+1}, \dots, m_r \in \mathbb{N}$. If*

- $L_1 = \mathcal{L}_3(k; m_1, \dots, m_s)$ *is non-special*;

- $L_2 = \mathcal{L}_3(d; m_{s+1}, \dots, m_r, k+1)$ *is non-special*;
- $\mathrm{vdim} L_1 = -1$

then the system $L = \mathcal{L}_3(d; m_1, \dots, m_r)$ *is non-special.*

Remark 25. To obtain Theorem 24 we have applied [9, Theorem 1] to the case $n = 3$ and $\mathrm{vdim}(L_1) = -1$, since the latter clearly guarantees $(\mathrm{vdim} L_1 + 1)(\mathrm{vdim} L_2 + 1) \geqslant 0$. Although this is apparently very restrictive, in practice it is very difficult to find different applications which perform efficiently.

The next three lemmas explain how to use Theorem 24 in order to reduce the computations.

Lemma 26. *Fix a positive integer d and let* $N = \binom{d+3}{3}$. *For any quadruple of non-negative integers* (w, x, y, z), *let* $Y(w, x, y, z) \subset \mathbb{P}^3$ *denote a general union of w 5-points, x 4-points, y 3-points and z 2-points. If* $Y(w, x, y, z)$ *has good postulation in degree d for any quadruple* (w, x, y, z) *such that*

$$0 \leqslant z \leqslant 4,$$

$$N - 3 \leqslant 35w + 20x + 10y + 4z \leqslant N + \Delta,$$

where Δ *is defined as in* (1), *then any general quintuple fat point scheme has good postulation in degree d.*

Proof. Let $Y = Y(w, x, y, z)$ be a general quintuple fat point scheme. Recall that by Lemma 5 it is enough to prove the good postulation of Y when $N - 3 \leqslant 35w + 20x + 10y + 4z \leqslant N + \Delta$. Now assume that $z \geqslant 5$. By Lemma 15 we know that $\mathcal{L}_3(3; 2^5)$ is non-special and $\mathrm{vdim}(\mathcal{L}_3(3; 2^5)) = -1$. Then by Theorem 24 in order to prove that Y has good postulation in degree d, it is enough to prove that $Y(w, x+1, y, z-5)$ has good postulation. Repeating this step, we reduce to the case when $z \leqslant 4$, and this proves our lemma. □

Lemma 27. *Fix a positive integer d and let* $N = \binom{d+3}{3}$. *Given non-negative integers q, w, x, y, z, let* $Y(w, x, y, z) \subset \mathbb{P}^3$ *denote a general union of w 5-points, x 4-points, y 3-points and z 2-points and let* $Y'(q, w, x, y, z)$ *denote the union of q general 10-fat points with* $Y(w, x, y, z)$. *If* $Y'(q, w, x, y, z)$ *has good postulation in degree d for any quintuple* (q, w, x, y, z) *such that*

$$0 \leqslant z \leqslant 4, \qquad 0 \leqslant 2x + y \leqslant 21, \qquad 0 \leqslant w \leqslant 3 \quad \text{or} \quad 0 \leqslant x \leqslant 3,$$

$$N - 3 \leqslant 220q + 35w + 20x + 10y + 4z \leqslant N + \Delta,$$

where Δ *is defined as in* (1), *then any general quintuple fat point scheme has good postulation in degree d.*

Proof. Let $Y = Y(w, x, y, z)$ be a general quintuple fat point scheme. As in the proof of Lemma 26 we can assume $N - 3 \leqslant 35w + 20x + 10y + 4z \leqslant N + \Delta$ and $z \leqslant 4$.

Now assume that $2x + y \geqslant 22$. Then there exist two integers a, b such that $a \leqslant x$ and $b \leqslant y$ and $2a + b = 22$. By Lemma 15 we know that the linear system $\mathcal{L}_3(9; 4^a, 3^b)$ is non-special and with virtual dimension -1. So by Theorem 24 in order to prove that Y has good postulation in degree d, it is enough to prove that $Y'(1, w, x-a, y-b, z)$ has good postulation. Repeating this step, we reduce to check all the general unions $Y'(q, w, x', y', z)$ such that $N - 3 \leqslant 220q + 35w + 20x + 10y + 4z \leqslant N + \Delta$ and $2x' + y' \leqslant 21$.

Now assume that $w \geqslant 4$ and $x' \geqslant 4$. By Lemma 15 we know that $\mathcal{L}_3(9; 5^4, 4^4)$ is non-special and with virtual dimension -1. Thus by Theorem 24 it is enough to prove that $Y'(q+1, w-4, x'-4, y', z)$ has good postulation. Repeating this step, we complete the proof of the lemma. □

Lemma 28. *Fix an integer $d \geqslant 38$ and let $N = \binom{d+3}{3}$. Given non-negative integers r, w, x, y, z, let $Y(w,x,y,z) \subset \mathbb{P}^3$ denote a general union of w 5-points, x 4-points, y 3-points and z 2-points and let $Y''(r,w,x,y,z)$ denote the union of r general 13-fat points with $Y(w,x,y,z)$. If $Y''(r,w,x,y,z)$ has good postulation in degree d for any quintuple (r,w,x,y,z) such that*

$$0 \leqslant z \leqslant 4, \qquad 0 \leqslant w \leqslant 12, \qquad 0 \leqslant 2x + y \leqslant 41,$$

$$N - 3 \leqslant 455r + 35w + 20x + 10y + 4z \leqslant N + \Delta,$$

where Δ is defined as in (1)*, then any general quintuple fat point scheme has good postulation in degree d.*

Proof. Let $Y = Y(w,x,y,z)$ be a general quintuple fat point scheme. As in the proof of Lemma 26 we can assume $N - 3 \leqslant 35w + 20x + 10y + 4z \leqslant N + \Delta$ and $z \leqslant 4$.

Let $\alpha = \lfloor \frac{2x+y}{42} \rfloor$. Now if $w \leqslant \alpha - 1$, then by Lemma 9 we also have $35w \leqslant \frac{1}{12}\binom{d+3}{3}$ and we can apply Proposition 23 which says that Y has good postulation.

Assume now that $w \geqslant \alpha$. For $1 \leqslant i \leqslant \alpha$, let a_i, b_i be such that $2a_i + b_i = 42$ for all i, $\sum_{i=1}^{\alpha} a_i \leqslant x$ and $\sum_{i=1}^{\alpha} b_i \leqslant y$. Note that by Lemma 15 all the linear systems $\mathcal{L}_3(12; 5, 4^{a_i}, 3^{b_i})$ are non-special and with virtual dimension -1, for $1 \leqslant i \leqslant \alpha$.

Then in order to prove that Y has good postulation in degree d, we apply α times Theorem 24 and we reduce to prove that $Y''(\alpha, w - \alpha, x - \sum a_i, y - b_i, z)$. So we have to check all the unions of the form $Y''(r, w', x', y', z)$, where $0 \leqslant 2x + y \leqslant 41$ and $N - 3 \leqslant 455r + 35w' + 20x' + 10y' + 4z \leqslant N + \Delta$.

Now assume that $w' \geqslant 13$ and recall that by Lemma 15 the linear system $\mathcal{L}_3(12; 5^{13})$ is non-special and with virtual dimension -1. Then applying Theorem 24 we reduce to the case when the number of quintuple points is less than or equal to 12, and this completes the proof. $\square$

We are now in position to complete the proof of Theorem 1.

Proof of Theorem 1 for $11 \leqslant d \leqslant 52$. Let d satisfy $11 \leqslant d \leqslant 21$ and let $N = \binom{d+3}{3}$. Lemma 26 says that to prove the good postulation of any general union it is enough to check all the general unions with $0 \leqslant z \leqslant 4$, and $N - 3 \leqslant 35w + 20x + 10y + 4z \leqslant N + \Delta$, where Δ is defined as in (1). This is precisely Theorem 16.

Now assume that $22 \leqslant d \leqslant 37$. By Lemma 27 it is enough to prove that a general union of q 10-points, w quintuple points, x quartuple points, y triple points and z double points has good postulation, when $0 \leqslant z \leqslant 4$, $0 \leqslant 2x + y \leqslant 21$, $0 \leqslant w \leqslant 3$ or $0 \leqslant x \leqslant 3$ and $N - 3 \leqslant 220q + 35w + 20x + 10y + 4z \leqslant N + \Delta$. This is Theorem 17.

Finally if $38 \leqslant d \leqslant 52$, Lemma 28 proves that it is enough to check all the general unions of r 13-points, w quintuple points, x quartuple points, y triple points and z double points have good postulation, when $0 \leqslant z \leqslant 4$, $0 \leqslant w \leqslant 12$, $0 \leqslant 2x + y \leqslant 41$ and $N - 3 \leqslant 455r + 35w + 20x + 10y + 4z \leqslant N + \Delta$. This is precisely Theorem 18.

This concludes the proof of Theorem 1. $\square$

5. A computational proof for the remaining cases

In this section we show how several computer calculations allow to prove Lemma 15, Theorems 16, 17, 18.

The core of our computation is a programme `exact_case.magma`, that can be found online, see section Supplementary material.

We can idealize the operations performed by `exact_case.magma` as in the following pseudo-code description of a routine called `exact`.

Exact

```
Input
(w,x,y,d).

N:=Binomial(d+3,3);
MonomialMatrices(MList);
L:=35*w+20*x+10*y+4*z; // Length
// We create the matrix and compute its rank
BigM:=EvaluationMatrix(MList,q,w,x,y,z);
r:=Rank(BigM);
// We check the speciality
if ((L lt N) and (r ne L)) then
 WriteToFile([q,w,x,y,z]);
end if;
if ((L ge N) and (r ne N)) then
 WriteToFile([q,w,x,y,z]);
end if;
WriteToFile(certificate);
```

The first function `MonomialMatrices` creates a list of matrices

$$\texttt{MList} = \{M_2, M_3, M_4, M_5\}$$

with monomials entries, where for all matrices the columns correspond to all degree-d monomials in four variables, and the rows of M_h correspond to the conditions (partial derivatives) of points with multiplicity h.

This list is passed to function `EvaluationMatrix` alongside with the number of points of given multiplicities.

The function `EvaluationMatrix` creates a set of corresponding random points with coordinates in the finite field $\mathbb{F}_p$. The matrices in `MList` are evaluated at this set.

The output matrix is stored into `BigM`, whose rank is computed immediately afterwards.

Depending on the rank and on the length, if the point configuration is special then a line is written, otherwise no extra output is needed (see later for a discussion on the certificate).

Several comments on the above algorithm and its implementation are in order:

- The algorithm as described is non-deterministic because it uses *random* points; we have limited ourselves to use pseudorandom sequences and so we need to choose a *seed* (and a *step*) whenever we launch an instance of the procedure, making the algorithm deterministic. In practice, we use the in-built MAGMA pseudo-random generator: MAGMA contains an implementation of the *Monster* random number generator by G. Marsaglia [16] combined with the MD5 hash function. The period of this generator is $2^{29\,430} - 2^{27\,382}$ and passes all tests in the Diehard test suite [17].
- The bottle-neck of the algorithm is the rank computation. Although in principle it is possible to check the matrix rank over $\mathbb{Q}$, in practice it is much more efficient to perform these computations over a finite field $\mathbb{F}_p$, with p a prime. This is lecit thanks to Remark 29. The smaller p is, the faster the rank computation is (and the smaller the memory requirement); however, a smaller prime is more likely to trigger a wrong rank (*failure*), because of the larger number of triggered linear relations; therefore, it is important to find a prime which is both small enough to use a reasonable memory amount and large enough to avoid failures, if possible. It turns out that $p = 31\,991$ works well up to the degrees that we needed. Its size is also very close to 2^{15}, and so the computer will allocate exactly 2 bytes to represent it, without losing an overhead.
- The rank computation itself is performed by the internal MAGMA rank routine for dense matrices over finite fields. By using several optimization techniques, it can compute the ranks also for large matrices in a reasonable time. We did some tests and MAGMA's rank routine not only outperforms by far any other software package we tried, but it also competes with ad hoc compiled programmed using specialized libraries, such as FFLAS–FFPACK [11] or M4RI [14], although the

matrices are not so large as to take advantage of sophisticated algorithms such as Strassen's [19] or Winograd's [20].
- The algorithm writes a *digital certificate*, i.e. a file containing vital information enabling a third party to check the correctness of the output. Our certificates vary slightly depending on the cases examined, but in each we need: the MAGMA's version, the input variables, the pair seed/step, the prime, the total computation time and a list of failures (if any).
  Anyone reading a certificate is able to run the corresponding procedure instance and verify the output (assuming that our same pseudorandom sequence is utilized).

Remark 29. Let $d \geqslant 11$ be an integer and p be a prime. As usual, let $\mathbb{K}$ be any field with characteristic zero. Given a quadruple of integers (w, x, y, z), the computer finds (in absence of failures) a union $Y(w, x, y, z) \subset \mathbb{P}^3(\mathbb{F}_p)$ that is not defective in degree d. By semicontinuity, this proves that a general union of w 5-points, x 4-points, y 3-points and z 2-points defined over $\mathbb{F}_p$ is not defective in degree d. By semicontinuity this is true for a general union $Y(w, x, y, z)$ defined over $\overline{\mathbb{F}}_p$. By semicontinuity this is also true for a general union $Y(w, x, y, z)$ defined first over $\overline{\mathbb{Q}}$ and then over $\overline{\mathbb{K}}$. Thanks to Lemma 2 this holds also over $\mathbb{K}$.

The first cases that we checked are the small-degree cases in Lemma 15. The programme and the digital certificates can be found online, see section Supplementary material.

Although cases a) and b) of Lemma 15 were already known in [10], we redid also them for completeness and check.

To check the cases in Theorem 16 we prepared a slightly more complex programme, `fat_points_brutal.magma`. We obviously reuse `exact` but we have to take into consideration the Δ values from Lemma 5. A pseudo-code description goes as follows.

Check of cases 11–21

```
Input: d.
N:=Binomial(d+3,3);
// We determine the maximum number of points
z1:=4;y1:=Ceiling(N/10);x1:=Ceiling(N/20);w1:=Ceiling(N/35);
// We set the maximum value of_D, but since the computations are fast
// we leave it except for z>0_D:=13;
for z in [0..z1] do
 if (z gt 0) then
  _D:=1;
 end if;
 for y in [0..y1] do
  for x in [0..x1] do
   for w in [1..w1] do // we start from w=1, because w=0 is already in [10]
    L:=35*w+20*x+10*y+4*z; // Length
    if ((L gt N-4) and (L lt N+_D+1)) then
     exact(w,x,y,d);
    end if;
   end for;
  end for;
 end for;
end for;
```

The programme and the digital certificates can be found online, see section Supplementary material.

We report the timings in Table 1.

We proved Theorem 17 similarly, using our programme `fat_points_10p.magma`. We do not give a pseudo-code, since now it is quite obvious how we proceed. We note only two key differences. First of all, we used fully the advantage offered by the tight determination of Δ. Second, we needed also 10-degree points, but this offered no difficulty, since a slight modification of `exact` can handle

Table 1
Timings in seconds for $d = 11\dots21$ from Theorem 17.

d	11	12	13	14	15	16	17	18	19	20	21
	54	137	309	683	1449	2879	5736	11 016	19 857	35 707	61 171

Table 2
Timings in seconds for $d = 21\dots37$ from Theorem 17.

d	21	22	23	24	25	26	27	28	
	3539	5137	7557	10 911	18 020	20 535	29 089	40 221	
d	29	30	31	32	33	34	35	36	37
	53 583	87 968	107 677	143 758	194 358	255 239	378 412	511 234	695 840

Table 3
Timings in seconds for $d = 38\dots52$ from Theorem 18.

d	38	39	40	41	42	43	44	45
	147 495	158 191	198 248	202 834	216 555	232 417	245 465	325 837
d	46	47	48	49	50	51	52	
	323 154	373 451	460 022	517 266	717 031	783 861	1 200 723	

them easily. The programme and the digital certificates can be found online, see section Supplementary material.

We report the timings in Table 2. We did also the defective case $d = 21$ as a sanity check.

Finally, we proved Theorem 18 in a similar manner, by using our programme `fat_points_13p.magma`. Again, a slight modification of `exact` was needed in order to handle 13-degree points. The programme and the digital certificates can be found online, see section Supplementary material.

The timings are reported in Table 3.

By observing the timings, we note an exponential behavior (in d) for Table 1, approximately of behavior 2^d. This is easily explained, because the cost of the rank computation grows as d^3, but the number of cases to be examined grows exponentially. A similar behavior can be seen in Table 2, where the times grow like $(1.4)^d$. Indeed the reason why these latter computations are feasible lies in the significant cut in the number of cases to be observed. However, the real case thinning happens in Table 3, where the grows is only *cubic* in d. This fall from an exponential behavior to a polynomial one can be explained only in a more-or-less constant number of cases to be considered (the cubic cost being unavoidable due to the cost of the rank computation). On the other hand, in Theorem 18 r can take only two values and the other integers are strictly bounded. As a further check, we computed the number of cases up to $d = 100$ and its maximum value is 405.

Remark 30. We have used four Dell servers, each with two processors Intel Xeon X5460 at 3.16 GHz (for a total of 32 processor cores) and with 32 GB's of RAM (for a total of 128 GB). The underlying operating system has been Linux, kernel version 2.6.18-6-amd64.

6. The exceptions in degree 9 and 10

Our main theorem states that a general fat point scheme in $\mathbb{P}^3$ of multiplicity 5 has good postulation in degree $d \geqslant 11$. Here we classify all the exceptional cases in degree 10 and 9.

Let us consider first the case of degree 10. Let Y be a general union of w quintuple points, x quartuple points, y triple points and z double points. Let $N = \binom{13}{3} = 286$. Then the linear system $L = \mathcal{L}_3(10; 5^w, 4^x, 3^y, 2^z)$ has virtual dimension $\mathrm{vdim}(L) = 286 - \deg(Y) - 1$ where $\deg(Y) = 35w + 20x + 10y + 4z$ and the expected dimension is $\max\{\mathrm{vdim}(L), -1\}$.

Table 4
Exceptions in degree 10.

w	x	y	z	$\min(\deg(Y), N)$	e	r	d
9	0	0	0	286	−1	285	0
8	1	0	0	286	−1	284	1
8	0	1	1	286	−1	285	0
8	0	1	0	286	−1	283	2
8	0	0	2	286	−1	284	1
8	0	0	1	284	1	282	3
7	2	0	1	286	−1	284	1
7	1	2	0	285	0	284	1
7	2	0	0	285	0	280	5

Our programme checked all the cases with:

- either $w \geqslant 1$ and $286 - 3 \leqslant \deg(Y) \leqslant 286 + 34$,
- $w = 7, 8$ and $\deg(Y) \leqslant 286 + 34$.

The programme found only nine cases of bad postulation, listed in Table 4. In this table, we denote by e the expected dimension of the corresponding linear system, by r the rank of the matrix given by our construction, and by d the dimension of the linear system.

From this computation we obtain the following classification:

Theorem 31. *In $\mathbb{P}^3$ a general union Y of w 5-points, x 4-points, y 3-points and z 2-points has good postulation in degree 10, except if the 4-tuple (w, x, y, z) is one of those listed in Table 4.*

Proof. If $w = 0$, then Y is a quartuple general fat point scheme and we already know by [4,10] that it has good postulation in degree 10. We can thus assume $w > 0$.

If Y is a general union of degree $283 \leqslant \deg(Y) \leqslant 320$, our programme checked that there are no other cases of bad postulation, except for the ones listed in the table.

Now if Y is a general union of degree $\deg(Y) \geqslant 321$, then it contains a subscheme Y' of degree $286 \leqslant \deg(Y') \leqslant 320$ which has good postulation, except if Y is the union of $w \geqslant 10$ quintuple points, where the only possible Y' is given by 9 quintuple points, which has bad postulation. On the other hand, by our computation we know that the dimension of the linear system $\mathcal{L}_3(10; 5^9)$ is 0. This means that as soon as we add a general simple point to Y' we immediately have an empty linear system. This implies that any union of $w \geqslant 10$ quintuple points has good postulation.

Now if Y has degree $\deg(Y) \leqslant 282$, then it is contained in a scheme Y' of degree $283 \leqslant \deg(Y') \leqslant 286$ which has good postulation, obtained by adding only general double points. The only case we need to study more carefully are $(w, x, y, z) = (8, 0, 0, 0), (7, 2, 0, 0), (8, 0, 1, 0)$, which correspond to subschemes of the exceptional cases with $z > 0$. We have checked directly that the first case (8 quintuple points) has good postulation, while the other two are exceptional cases. This completes the proof. □

Some of the exceptional cases we found were already known, see e.g. [12] and [8]. Note that all the exceptions we found satisfy the conjecture of Laface and Ugaglia (see [12] and [13, Conjecture 6.3]).

In the case of degree 9 we found many more exceptions, that we list in Table 5.

In this case we have tested all the configurations where $w \geqslant 1$ and $220 - 3 \leqslant \deg(Y) \leqslant 220 + 34$, and all the configurations with $1 \leqslant w \leqslant 6$ and $\deg(Y) \leqslant 220 - 4$.

From our computational experiments we can deduce the following complete classification:

Theorem 32. *In $\mathbb{P}^3$ a general union Y of w 5-points, x 4-points, y 3-points and z 2-points has good postulation in degree 9, except if the 4-tuple (w, x, y, z) is one of those listed in Table 5.*

Table 5
Exceptions in degree 9.

w	x	y	z	$\min(\deg(Y), N)$	e	r	d
6	0	1	1	220	−1	219	0
6	0	1	0	220	−1	216	3
6	0	0	3	220	−1	218	1
6	0	0	2	218	1	214	5
6	0	0	1	214	5	210	9
6	0	0	0	210	9	206	13
5	2	0	1	219	0	217	2
5	2	0	0	215	4	213	6
5	1	2	1	219	0	218	1
5	1	2	0	215	4	214	5
5	1	1	3	217	2	216	3
5	1	1	2	213	6	212	7
5	1	1	1	209	10	208	11
5	1	1	0	205	4	204	5
5	1	0	6	219	0	218	1
5	1	0	5	215	4	214	5
5	1	0	4	211	8	210	9
5	1	0	3	207	12	206	13
5	1	0	2	203	16	202	17
5	1	0	1	199	20	198	21
5	1	0	0	195	24	194	25
4	3	2	0	220	−1	218	1
3	6	0	0	220	−1	218	1
3	5	1	1	219	0	218	1
3	5	1	0	215	4	214	5

Proof. If $w = 0$, then Y is a quartuple general fat point scheme and it has good postulation in degree 10 by [4,10]. We can thus assume $w > 0$.

If Y is a general union of degree $\deg(Y) \leqslant 254$, our programme checked that there are no other cases of bad postulation, except for the ones listed in the table.

Now if Y is a general union of degree $\deg(Y) \geqslant 255$, then it is easy to check that Y contains a subscheme Y' of degree $286 \leqslant \deg(Y') \leqslant 320$ which has good postulation. □

Remark 33. Also in the case of degree 9 all the exceptions we found satisfy the conjecture of Laface and Ugaglia [13, Conjecture 6.3].

The relevant computations can be found online, see section Supplementary material.

7. Further remarks

In this final section we provide two remarks on the field characteristics and a direct consequence of Theorem 1.

Since the result by Yang (Remark 13) is proved only for characteristic zero, we assume in this paper that $\mathrm{char}(\mathbb{K}) = 0$. However we underline that our proof of Theorem 1 can easily be adapted to any $\mathrm{char}(\mathbb{K}) \neq 2, 3, 5$. Hence the statement of Theorem 1 could immediately be generalized to any characteristic different from $2, 3, 5$ as soon as we know that a general fat point scheme in $\mathbb{P}^2(\mathbb{F})$ of multiplicity 5 has good postulation in degree $d \geqslant 3m$, for any field $\mathbb{F}$ with that characteristic, provided the result holds again for $d = 11$ in $\mathbb{P}^3(\mathbb{F})$.

In positive characteristic the proof of Lemma 5 fails, since we cannot make use of Lemma 4. However, following the same outline as in the proof of Lemma 5 and recalling that a fat point always contains a simple point, it is not difficult to prove the following lemma.

Lemma 34. *Let $\mathbb{F}$ be an infinite field of any characteristic. Fix an integer $d > 0$. For any quadruple of non-negative integers (w, x, y, z), let $Y(w, x, y, z) \subset \mathbb{P}^3(\mathbb{F})$ denote a general union of w 5-points, x 4-points, y 3-points and z 2-points. If $Y(w, x, y, z)$ has good postulation in degree d for any quadruple (w, x, y, z) such that*

$$\binom{d+3}{3} - 3 \leqslant 35w + 20x + 10y + 4z \leqslant \binom{d+3}{3} + \Delta$$

where

$$\Delta = \begin{cases} 14 & \text{if } w > 0 \text{ and } x = y = z = 0, \\ 9 & \text{if } x > 0 \text{ and } y = z = 0, \\ 5 & \text{if } y > 0 \text{ and } z = 0, \\ 2 & \text{if } z > 0 \end{cases}$$

then any general quintuple fat point scheme has good postulation in degree d.

A straightforward consequence of Theorem 1 is the following statement, whose proof is contained in Remark 3.

Corollary 35. *Fix non-negative integers w, x, y, z such that*

$$35w + 20x + 10y + 4z \geqslant \binom{14}{3}.$$

Let $Y \subset \mathbb{P}^3$ be a general union of w 5-points, x 4-points, y 3-points and z 2-points. Then Y has good postulation with respect to any degree.

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Supplementary material

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