Doctoral School in Mathematics

Local coherence of hearts in the derived category of a commutative ring

Lorenzo Martini

Università degli Studi di Trento
Dipartimento di Matematica

Università degli Studi di Verona
Dipartimento di Informatica

2022

Doctoral thesis in Mathematics
Joint doctoral programme in Mathematics, XXXIV cycle

Department of Mathematics,
University of Trento

Department of Computer Science,
University of Verona

Supervisor: Prof. Francesca Mantese
Coadvisor: Prof. Carlos E. Parra

The present document has been typeset in LaTeX using the font *Forgery*, created by the author using the software FontForge.

Table of Contents

Abstract

Approximation theory is a fundamental tool in order to study the representation theory of a ring R. Roughly speaking, it consists in determining suitable additive or abelian subcategories of the whole module category Mod-R with nice enough functorial properties. For instance, torsion theory is a well suited incarnation of approximation theory. Of course, such an idea has been generalised to the additive setting itself, so that both Mod-R and other interesting categories related with R may be linked functorially. By the seminal work [6] of Beĭlinson, Bernstein and Deligne (1982), the derived category $D(R)$ of the ring turns out to admit useful torsion theories, called t-structures: they are pairs of full subcategories of $D(R)$ whose intersection, called the heart, is always an abelian category. The so-called standard t-structure of $D(R)$ has as its heart the module category Mod-R itself. Since then a lot of results devoted to the module theoretic characterisation of the hearts have been achieved, providing evidence of the usefulness of the t-structures in the representation theory of R. In 2020, following a research line promoted by many other authors, Saorín and Šťovíček proved in [55] that the heart of any compactly generated t-structure is always a locally finitely presented Grothendieck categories (actually, this is true for any t-structure in a triangulated category with coproducts). Essentially, this means that the hearts of $D(R)$ come equipped with a finiteness condition miming that one valid in Mod-R. In the present thesis we tackle the problem of characterising when the hearts of certain compactly generated t-structures of a *commutative* ring are even locally coherent. In this commutative context, after the works of Neeman and Alonso, Jeremías and Saorín [1], compactly generated t-structures turned out to be very interesting over a noetherian ring, for they are in bijection with the Thomason filtrations of the prime spectrum. In other words, they are classified by geometric objects, moreover their constituent subcategories have a precise cohomological description. However, if the ascending chain condition lacks, such classification is somehow partial, though provided by Hrbek [23]. The crucial point is that the constituents of the t-structures have a different description w.r.t. that available in the noetherian setting, yet if one copies the latter for an arbitrary ring still obtains a t-structure, but it is not clear whether it must be compactly generated. Consequently, pursuing the study of the local coherence of the hearts given by a Thomason filtration, we ended by considering two t-structures. Our technique in order to face the lack of the ascending chain condition relies on a further approximation of the hearts by means of suitable torsion theories. The main results of the thesis are the following: we prove that for the so-called weakly bounded below Thomason filtrations the two t-structures have the same heart (therefore it is always locally finitely presented), and we show that they coincide if and only they are both compactly generated. Moreover, we achieve a complete characterisation of the local coherence for the hearts of the Thomason filtrations of finite length.

Introduction

When dealing with an arbitrary abelian category, it is crucial to know how far it is from being a category of modules over an associative ring. Indeed, module categories are the nicest abelian categories one can work with, since on the one hand their objects and morphisms can be treated elementwise, on the other hand they provide several fundamental universal constructions (indeed they are cocomplete and complete) and distinguished objects (they have generators, cogenerators, enough injectives and enough projectives); moreover, they come equipped with some finiteness conditions (they are locally finitely presented). The aforementioned categorical gap between a given abelian category and a category of modules is usually expressed by the existence of a functor between these latter, hence by the characterisation of the conditions under which such functor transfers specific module-theoretic properties to the abelian category. A lot of results dealing with this instance are well-known in the literature. Among them, Freyd and Mitchell (see e.g. [40, IV.5]) proved that cocomplete abelian categories $\mathcal{A}$ with a small projective generator P are precisely the categories of modules over the endomorphism ring R of the generator (essentially, using the functor $\mathrm{Hom}_{\mathcal{A}}(P,-)\colon \mathcal{A} \to \text{Mod-}R$); more recently, these kind of categorical approach has been widely used for certain abelian subcategories of triangulated categories, and in tilting and silting theories (see e.g. [12, 53, 45, 4, 55]).

Possibly, the most interesting and studied abelian categories are the Grothendieck ones, namely those cocomplete abelian categories with a family of generators and exact direct limits. They naturally appear in many algebraic and geometric settings, and turn out to be the optimal generalisation of module categories: the celebrated Gabriel–Popescu theorem classifies them as a suitable *localisation* of categories of modules. More precisely, any Grothendieck category is the quotient of a module category, modulo a *hereditary torsion class* (see Thm. 1.16). In general, in such localisation process the locally finite presentability of modules is not inherited by the corresponding Grothendieck category. Consequently, for a Grothendieck category $\mathcal{G}$ it might be very useful to know when it is locally finitely presented. This finiteness condition has also been decoded by means of the Gabriel–Popescu Theorem in [55, Proposition 2.10] (see also Prop. 1.19) and it turns out to be related with an additional property of the relevant hereditary torsion pair, namely its *finite type*.

The present thesis is focused on the study of certain locally finitely presented Grothendieck categories $\mathcal{G}$, in order to characterise when their finitely presented objects form an abelian subcategory; this instance is termed the *local coherence* (see Def. 1.3).

A very interesting family of Grothendieck categories we want to study comes from the world of triangulated categories, still by means of a specific notion of "torsion theory" therein; for this reason, it is favorable to recall the notion of torsion theory both for abelian and triangulated categories, and explain the importance and versatility of the announced Grothendieck categories. Roughly, a *torsion theory* of an additive category is a pair of mutually orthogonal full subcategories, through which any object of the given category can be functorially approximated (by a short exact sequence or an exact triangle, depending on the relevant context). In the abelian setting, we will deal with the notion of *torsion pair* (see subsec. 1.2.3); in the triangulated setting, we will deal with the notion of *t-structure*, which actually provides the family of categories we are interested in. More precisely (see subsec. 1.3.2), given a triangulated category $\mathcal{D}$ and a t-structure $(\mathcal{U}, \mathcal{V})$ on it, the *heart* $\mathcal{C} := \mathcal{U} \cap \mathcal{V}$ of the t-structure is an *abelian* subcategory of $\mathcal{D}$, whose short exact sequences are precisely the triangles of $\mathcal{D}$ with all the vertices in $\mathcal{C}$. Each t-structure on $\mathcal{D}$ with heart $\mathcal{C}$ comes equipped with a cohomological functor, $H_{\mathcal{C}} \colon \mathcal{D} \to \mathcal{C}$ say, which allows to perform a cohomological algebra on $\mathcal{D}$ with terms in the abelian category $\mathcal{C}$.

In view of what we said in the first paragraphs, the problem of characterising additional categorical and homological properties on the heart of a t-structure has been intensively investigated in the literature. The problem is widely open, though decisive results have been achieved in two fundamental frameworks, as we now resume and comment.

One family of hearts is given by the so called *Happel–Reiten–Smalø t-structures*, which are defined in the derived category of a Grothendieck category by means of a torsion pair of the latter (see Ex. 1.35(2)). HRS hearts have been studied by many authors, in order to characterise both their Grothendieck and module-theoretic properties (see e.g. [12, 37, 44, 45, 48]). The crucial result in this context is [44, (Addendum) Theorem 1.2], which establishes that the HRS heart of a torsion pair is a Grothendieck category if and only if the torsion pair is of finite type. Moreover, some results on the locally finite presentability of the HRS hearts are known, e.g. when the ambient Grothendieck category is in turn locally finitely presented (see [48, Theorem 6.1], which generalises [54, Theorem 5.2], dealing with the local coherence instead).

The second family of hearts of our interest is given by the hearts of a *compactly generated* t-structure of a triangulated category with coproducts (see Def. 1.36). Notice that, though the derived category of a Grothendieck category has coproducts, HRS t-structures are not compactly generated in general (see [8, Corollary 4.2]). The hearts of compactly generated t-structures turned out to behave surprisingly well as abelian categories: by very recent results, [4, 56] among others, and eventually [55, Theorem 8.31], they are always locally finitely presented Grothendieck categories. In fact, compactly generated t-structures, and more generally compactly generated triangulated categories (e.g. the derived category of a Grothendieck one), may be thought of the triangulated analogue of locally finitely presented Grothendieck categories, at least for they provide the right setting in which to perform a fruitful purity theory (cf. [34] and Def. 1.27). Moreover, the compactly generated t-structures generalise to another important class of t-structures, namely the *smashing* ones, and in case the ambient category is compactly generated and enhanced by a strong and stable derivator, to the *homotopically smashing* t-structures; also in these cases fundamental results concerning the hearts and their finiteness properties have been achieved (see subsections 1.3.3 and 1.3.4 for a discussion in the derived category).

Among these two families of t-structures and hearts we are interested in, those defined in the derived category of a commutative noetherian ring R have additional crucial properties. The main one of these properties, due to Alonso, Jeremías and Saorín [1], is a complete classification of the compactly generated t-structures in geometric terms, i.e. using the so-called *sp-filtrations* of the spectrum of the ring (explained below in more details). By virtue of this classification, many features of the t-structures can be interpreted geometrically, and indeed fundamental results concerning the local coherence of the hearts have been achieved in the recent literature. We will give a survey of these results in Chap. 2, providing also a new reformulation of a sufficient condition for the local coherence.

In [23], Hrbek generalised such classification for the compactly generated t-structures of an arbitrary commutative ring R (in this case, the sp-filtrations are called *Thomason filtrations*). However, a lot of the results holding true in the noetherian setting are not available for arbitrary rings: for example, the description of the constituents of a compactly generated t-structure is different to that given in [1]. Moreover, if one considers the class of complexes described as in the aisle of a t-structure given by a sp-filtration, then one obtains an aisle again; we will refer to the t-structure such an aisle gives rise as the *Alonso–Jeremías–Saorín* t-structure (*AJS* for short). It is a very challenging problem to see whether, given a Thomason filtration, its compactly generated and AJS t-structures coincide. We will analyse this instance in Chap. 3, proving some results which strengthen the possibility that the answer is affermative, and based on new methods to face the lack of a chain condition on the rings.

The present thesis is devoted to the characterisation of the local coherence of the hearts associated with certain Thomason filtrations of the Zariski spectrum of a commutative ring. At this point we are now ready to expose the structure of the thesis and explain more in detail its main results.

Chap. 1 contains the notations and all the preliminary definitions and results we need, concerning abelian and Grothendieck categories (in particular their torsion pairs and TTF triples), triangulated categories (in particular their t-structures), and a brief reminder on prederivators. We introduce the notion of quasi locally coherent additive category, which is a generalisation of the notion of local coherence for abelian categories (see Def. 1.3) and we discuss the interaction of the compactly generated t-structures (and their hearts) in the derived category (see subsec. 1.3.4).

Chap. 2 is mostly a survey on the rich literature concerning the compactly generated t-structures in the derived category of a commutative noetherian ring R, in particular on the local coherence of their hearts. As mentioned before and as we will see in sec. 2.1, in this setting the compactly generated t-structures $(\mathcal{U}, \mathcal{V})$ are classified by the sp-filtrations of $\operatorname{Spec} R$, namely by decreasing mappings $\Phi\colon (\mathbb{Z}, \leq) \to (2^{\operatorname{Spec} R}, \subseteq)$ such that each $\Phi(j)$ is a union of Zariski closed subsets $V(J)$'s, $J \leq R$. In particular, the complexes of the constituents $\mathcal{U}$ and $\mathcal{V}$ have a useful cohomological description: (cf. [1] and Thm. 2.4) the objects of the *aisle* $\mathcal{U}$ are the complexes whose jth standard cohomology is *supported* in $\Phi(j)$, for all $j \in \mathbb{Z}$, while the *coaisle* $\mathcal{V}[-1]$ is formed by the complexes whose derived local cohomology at $\Phi(j)$ belongs to $D^{>j}(R)$, for all j.

As we will see in subsec. 2.2, by the previous result several properties of the t-structures can be deduced from the corresponding sp-filtration. For what does concern the local coherence of their hearts, since the paper [54] by Saorín it is known that there are two conditions which play a prominent role, namely the *restrictability* and the *intermediacy*. In general they are not necessary conditions for the local coherence; restrictability is sufficient though, and it has been proven in the very recent literature that in case of intermediacy it is necessary assuming that the heart is *cotilting* (see Ex. 1.35(3) and [38, 49, 27], whose main results in this sense have been recollect in Thm. 2.9). The intermediacy is a very easy condition to read on a sp-filtration (Φ is intermediate if $\Phi(m) = \operatorname{Spec} R$ and $\Phi(n) = \emptyset$ for some $m < n$); on the other hand, the restrictability is more subtle. Nonetheless, in [1, Theorem 4.4] it is proven that it always implies a geometric condition on the corresponding filtration, namely the *weak Cousin condition*, and in [1, Theorem 6.9] that in case the ring R has a dualising complex (e.g. it is Gorenstein) then these two conditions are equivalent.

Sec. 2.3 is the original part of the chapter, where we will prove that for any commutative noetherian ring R, to ask for a weak Cousin sp-filtration to have a restrictable t-structure is equivalent to ask for a Faltings' annihilator theorem to hold in $D^b(\text{mod-}R)$ (see Thm. 2.14). Faltings' annihilator theorem (for modules) is a very well-known statement in commutative algebra, concerning local cohomology, which relates its finiteness to its vanishing. In principle, the announced equivalence allows to extend the class of restrictable compactly generated t-structures —whence those having a locally coherent Grothendieck heart— by proving a suitable geometric incarnation of the problem, and viceversa. As a byproduct, we deduce in Cor. 2.15 that the Faltings' annihilator theorem for complexes holds true in case R has a dualising complex. In Thm. 2.17 we prove that the weak Cousin condition implies the restrictability over local rings which are universally catenary with Cohen–Macaulay formal fibers, hence extending [1, Theorem 6.9].

Chap. 3 is the very original part of the thesis, and it is devoted to the study of the hearts of the compactly generated and the AJS t-structures of an arbitrary commutative ring R (recall that the AJS t-structures are those whose aisle is described as in [1]). The techniques used in this chapter, when dealing with the local coherence of the hearts, differ from those used in the previous one: in summary, the key point for our results is to use certain TTF triples in the hearts. Such idea is motivated by a more general result, Thm. 3.2, proved in sec. 3.1, characterising the local coherence of an arbitrary Grothendieck category equipped with a TTF triple. In view of this result, a large part of the chapter is focused in specialising it to the hearts of the t-structures of our interest, so that in detecting suitable TTF triples.

Sec. 3.2 contains the definition of Thomason subset and Thomason filtration of the $\operatorname{Spec} R$, their role in the classification of the hereditary torsion pairs of finite type of Mod-R, and of the compactly generated t-structures of $D(R)$ (due to Hrbek [23]). We start analysing some crucial relations between the compactly generated t-structures and the AJS ones. In Lem. 3.7 we show that, for any Thomason filtration, if its AJS t-structure is compactly generated, then it must coincide with the corresponding compactly generated one. In Prop. 3.9 we prove a first main result on the associated heart: in case of a weakly bounded below Thomason filtration, the AJS heart is always a locally finitely presented Grothendieck category (as it occurs in the compactly generated case).

In sec. 3.3 we continue giving properties relating the hearts (of both the t-structures mentioned above) and certain subcategories involved in the study of the finitely presented complexes of the hearts, which will be useful in the sequel. We show that both any hereditary torsion class of finite type of Mod-R and the HRS t-structure it gives rise is parametrised by a Thomason filtration of finite length (respectively 0 and 1; see subsec. 3.3.2 and Ex. 3.10). The main result of the section is Thm. 3.20, in which we completely characterise the local coherence of the former category; that is, the local coherence of the hearts of length zero Thomason filtrations.

Sections 3.4 and 3.5 are the central part of the chapter: altogether they provide the machinery to apply Thm 3.2 to the case of the hearts of filtrations of finite length.

More in detail, sec. 3.4 is devoted to detecting within the hearts of a Thomason filtration a TTF triple of finite type. In this vein, weakly bounded below Thomason filtrations are shown to possess hearts providing such feature; moreover, they turn out to have additional intriguing properties. Indeed, firstly in Thm. 3.24 we prove that, for any weakly bounded below Thomason filtration, its two relevant t-structures have always the same heart. Even if this does not confirm that the t-structures coincide, at least by Lem. 3.7 it suggests that this might be the case. This said, in Thm 3.29 we show that, given any Thomason filtration, there are naturally associated weakly bounded below Thomason filtrations whose (AJS) hearts are TTF classes in the given heart. In particular, these classes are all locally coherent Grothendieck categories in case the given heart is so.

Sec. 3.5 contains the main results concerning the Thomason filtrations of finite length, hence those whose two corresponding hearts coincide. In view of Thm. 3.29, such length is taken in the vein of providing a recursive argument for the characterisation of the local coherence of the heart, namely by taking into account the local coherence of the TTF classes of finite type detected previously.

Thm 3.41, the main result of the section, identifies the local coherence in such a recursive way, by means of five conditions. These conditions are the most eligible ones, in the sense that for the crucial cases of length $0, 1, 2$, almost all translate into module-theoretic properties, as proved in Cor. 3.43 and Cor. 3.44.

Chap. 4 contains some applications of Thm. 3.41 and Cor. 3.43: Sec. 4.1 is focused on the case of the HRS hearts, while Sec. 4.2 is devoted to specialise the previous main results over a commutative noetherian ring. In more details, Cor. 4.2 is the direct application of Cor. 3.43 when the involved Thomason filtration gives the HRS heart of a hereditary torsion pair of finite type; besides, it also allows us to exhibit an example of a quasi locally coherent additive category that is not locally coherent in the usual sense (see Ex 4.5). Other further applications of such corollary are obtained by adding conditions either on the ring, i.e. when it is coherent, or on the underlying torsion pair, i.e. when it is stable. In any case, it furtherly enlights into more handleable module-theoretic conditions. Finally, in case the ring is noetherian, we recover for Thomason filtrations of finite length the useful result [54, Theorem 6.3] by Saorín, concerning the local coherence of the hearts of the restrictable t-structures discussed in Chap. 2.

Appendix A contains another incarnation of Thm. 3.2, which might be interesting on its own in view of its generality. Appendix B concludes the thesis with some open problems related to the achieved results.

Acknowledgments

I wish to express my gratitude to my supervisor Francesca Mantese, which guided me carefully during these years, providing me many possibilities to learn and grow.

My sincere thanks to Carlos Parra, Manuel Saorín, Sergio Estrada and Michal Hrbek, with whom I had the chance to collaborate with during unforgettable visiting abroad. A very special thank goes to Sinem Odabaşı, for her precious advices, altruism and friendship, and to Claudia Menini, for her unchanging support and inspiration.

I also wish to acknowledge the staff of the Doctoral School in Mathematics of the Universities of Trento and Verona, for their expertise and patience, and the staff of the departments of Mathematics of Universidad Austral de Chile, Universidad de Murcia, and Charles University of Prague, for their warm hospitality.

CHAPTER 1

Preliminaries on abelian and triangulated categories

Otherwise stated, the term "category" will always stand for locally small category, meaning that the morphisms between any pair of objects of the category form a set, instead of a proper class. We will also identify any category with its class of objects, when this does not create confusion. The term "subcategory" will always stand for full subcategory closed under isomorphisms. Any ring will be associative with identity $1 \neq 0$. We will also assume to be well-known the basic notions and constructions in category theory (see e.g. [50]).

1.1. Some finiteness conditions

1.1 Definition. Let $\mathcal{A}$ be an additive category with coproducts or, accordingly, with direct limits. An object $M \in \mathcal{A}$ is called

- *small* or *compact* if its covariant hom functor $\operatorname{Hom}_{\mathcal{A}}(M, -)\colon \mathcal{A} \to \mathrm{Ab}$ preserves coproducts;
- *finitely generated* if $\operatorname{Hom}_{\mathcal{A}}(M, -)$ commutes with direct limits of direct systems of monomorphisms;
- *finitely presented* if $\operatorname{Hom}_{\mathcal{A}}(M, -)$ commutes with direct limits.

$\mathcal{A}^c$, $\mathrm{fg}(\mathcal{A})$ and $\mathrm{fp}(\mathcal{A})$ will denote the full subcategories of compact, resp. finitely generated, resp. finitely presented objects of the category $\mathcal{A}$.

1.2 Definition. An additive category with direct limits $\mathcal{A}$ is *locally finitely presented* if $\mathrm{fp}(\mathcal{A})$ is skeletally small and $\mathcal{A} = \varinjlim \mathrm{fp}(\mathcal{A})$; this latter condition meaning that every object of $\mathcal{A}$ is isomorphic to the direct limit of a direct system of finitely presented objects of $\mathcal{A}$.

Though the notion of local coherence for an additive category rephrases that for modules, we allow the following:

1.3 Definition. Let $\mathcal{A}$ be an additive category with kernels and direct limits. Then it will be called *quasi locally coherent* if it is locally finitely presented and the kernel (in $\mathcal{A}$) of any morphism in $\mathrm{fp}(\mathcal{A})$ is finitely presented as well.

If $\mathcal{A}$ is abelian with direct limits, then it is *locally coherent* in case it is locally finitely presented and $\mathrm{fp}(\mathcal{A})$ is an abelian category.

1.4 Remark. In Ex. 4.5 we will exhibit an example of quasi locally coherent category which is not locally coherent.

1.2. Abelian categories

1.2.1. Categories of functors. Let $\mathcal{C}$ be a category and I be a small category. The category of all functors $I \to \mathcal{C}$, with natural transformations as morphisms, will be denoted by $\mathcal{C}^I$, and called the category of *I-shaped diagrams on $\mathcal{C}$*; an object $M \in \mathcal{C}^I$ will be identified with the I-indexed family of objects of $\mathcal{C}$ it gives rise, $M := (M_i \mid i \in I)$, where $M_i := M(i)$ for all $i \in I$.

1.5 Yoneda Lemma. *Let I be a small category, and let* $\mathrm{Hom}(I^{\mathrm{op}}, \mathbf{Set})$ *be the full subcategory of* $\mathbf{Set}^I$ *formed by the contravariant functors* $I \to \mathbf{Set}$*. There is a bijective map*

$$\mathrm{Hom}_{\mathcal{C}^I}(\mathrm{Hom}_I(-,i), M) \xrightarrow{\cong} M_i$$

which is natural both in $i \in I$ and $M \in$ $\mathrm{Hom}(I^{\mathrm{op}}, \mathbf{Set})$*. In particular, there is a fully faithful functor*

$$\begin{aligned} H_{(-)} \colon I &\longrightarrow \mathrm{Hom}(I^{\mathrm{op}}, \mathbf{Set}) \\ i &\longmapsto H_i := \mathrm{Hom}_I(-,i) \\ (\lambda\colon i \to j) &\longmapsto \lambda_* := \mathrm{Hom}_I(-,\lambda)\ . \end{aligned}$$

One of the most favorable cases of functor category occurs when $I = \mathcal{A}$ is a small preadditive category and $\mathcal{C} = \mathrm{Ab}$ is the category of abelian groups; following [40], in this case $\mathcal{A}$ is regarded as a ring with several objects and the subcategory of $\mathrm{Ab}^{\mathcal{A}}$ formed by the contravariant *additive* functors $\mathcal{A} \to \mathrm{Ab}$ will be denoted with one of the following notations:

$$\mathrm{Hom}(\mathcal{A}^{\mathrm{op}}, \mathrm{Ab}) = \text{Mod-}\mathcal{A},$$

and called the category of (*generalised*) *right modules over* $\mathcal{A}$ (see also [47]). Essentially due to the properties of Ab, by Yoneda lemma this category carries all the module-theoretic properties of right modules over a ring, in particular:

- it is an abelian category with products and coproducts (hence with limits and colimits) computed objectwise;
- the contravariant hom functors $(H_a \mid a \in \mathcal{A})$ of $\mathcal{A}$ form a family of small projective generators;
- it has enough injective objects (this may be seen as a consequence of the fact Mod-$\mathcal{A}$ is a *Grothendieck category*).

We set

$$\text{mod-}\mathcal{A} := \mathrm{fp}(\text{Mod-}\mathcal{A}),$$

and this subcategory consists of the contravariant functors M which fit in an exact sequence $H_b \to H_a \to M \to 0$, for some morphism $b \to a$ in $\mathcal{A}$. It is well-known that mod-$\mathcal{A}$ is an abelian category if and only if $\mathcal{A}$ has *weak kernels*; that is, if for every morphism $a \to b$ in $\mathcal{A}$ there exists $x \to a$ such that the associated sequence $H_x \to H_a \to H_b$ is exact in Mod-$\mathcal{A}$.

Another example of abelian category (which could be regarded as a functor category indeed) is that of cochain complexes over a given abelian category $\mathcal{A}$, denoted by $\mathrm{Ch}(\mathcal{A})$. Its objects are $\mathbb{Z}$-indexed sequences of objects and morphisms of $\mathcal{A}$

$$\cdots \longrightarrow M^{n-1} \xrightarrow{d_M^{n-1}} M^n \xrightarrow{d_M^n} M^{n+1} \longrightarrow \cdots$$

satisfying $d_M^n \circ d_M^{n-1} = 0$ for all $n \in \mathbb{Z}$, usually denoted by $(M^\cdot, d_M)$ or just by M. Its morphisms $f\colon M \to N$ are the so-called *cochain maps*, i.e. $\mathbb{Z}$-indexed families of morphisms $f^n\colon M^n \to N^n$ such that $f^n \circ d_M^{n-1} = d_N^{n-1} \circ f^{n-1}$, for all $n \in \mathbb{Z}$. We refer to [29, Chapter 1] for the terminology concerning $\mathrm{Ch}(\mathcal{A})$. A fundamental category related to $\mathcal{A}$ and $\mathrm{Ch}(\mathcal{A})$ is the *homotopy category* $K(\mathcal{A})$ of $\mathcal{A}$, defined as follows. A cochain map $f\colon M \to N$ is called *null homotopic*, denoted $f \simeq 0$, if there is a family of morphisms $(s^n\colon M^n \to N^{n-1})_{n\in\mathbb{Z}}$ such that $f^n = d_N^{n-1} \circ s^n + s^{n+1} \circ d_M^n$ for all $n \in \mathbb{Z}$,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & M^{n-1} & \xrightarrow{d_M^{n-1}} & M^n & \xrightarrow{d_M^n} & M^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^{n-1}} & \swarrow{\scriptstyle s^n} & \downarrow{\scriptstyle f^n} & \swarrow{\scriptstyle s^{n+1}} & \downarrow{\scriptstyle f^{n+1}} & & \\ \cdots & \longrightarrow & N^{n-1} & \xrightarrow[d_N^{n-1}]{} & N^n & \xrightarrow[d_N^n]{} & N^{n+1} & \longrightarrow & \cdots \end{array}$$

This definition induces an equivalence relation on the group $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(M, N)$ compatible with its sum, defined by setting $f \simeq g$ iff $f - g \simeq 0$, and called *homotopy*. Then $K(\mathcal{A})$ has as objects those of $\mathrm{Ch}(\mathcal{A})$, and for any pair of objects $M, N \in \mathrm{Ch}(\mathcal{A})$ it is $\mathrm{Hom}_{K(\mathcal{A})}(M, N) := \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(M, N)/\simeq$. We recall that $K(\mathcal{A})$ is not abelian in general (it is iff $\mathcal{A}$ is semisimple) but it carries a structure of triangulated category, as we recall in sec. 1.3 (see again [29] for further notions and results concerning $K(\mathcal{A})$).

1.2.2. Yoneda ext-groups. The computation of the derived functors of an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ originating in an abelian category with enough injectives or enough projectives (e.g. module categories) is well-suited, for one can exploit injective or projective resolutions. However, there is a description of the right derived functors of $\mathrm{Hom}_{\mathcal{A}}(-, M), \mathrm{Hom}_{\mathcal{A}}(M, -)\colon \mathcal{A} \to \mathrm{Ab}$ even when $\mathcal{A}$ does not provide enough such objects. We briefly resume such description following [40, VII.3], mostly for these derived functors have a precise meaning when dealing with triangulated categories.

Given $M, N \in \mathcal{A}$ and an integer $n \geq 1$, the class of exact sequences

$$0 \longrightarrow N \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

in $\mathcal{A}$ can be partitioned by "equivalence classes", forming the quotient class $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ of the *n-extensions of M by N*. When these classes are sets for all $M, N \in \mathcal{A}$ and $n \geq 1$, then they turn out to be the abelian groups appearing in the long exact sequences one obtains by applying $\mathrm{Hom}_{\mathcal{A}}(M, -)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(-, N)$) on the short exact sequences of $\mathcal{A}$. In particular, for any $n \geq 1$ the abelian group structure of $\mathrm{Ext}^n_{\mathcal{A}}(M, N)$ can be obtained inductively from that of $\mathrm{Ext}^1_{\mathcal{A}}(M, K)$ or $\mathrm{Ext}^1_{\mathcal{A}}(K, N)$, for some K, since any n-extension can be decomposed into suitable products of extensions of lower length. In the abelian group structure of $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ we are going to define, the zero element is represented by any split short exact sequence $0 \to N \to X \to M \to 0$. In order to define the relevant sum, we introduce the following scalar multiplications. Given morphisms $f\colon M' \to M$ and $g\colon N \to N'$, consider the outer exact rows of the following commutative diagram:

$$\begin{array}{rcccccccl}
\epsilon f : 0 & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \| & & \downarrow & \text{P.B.} & \downarrow f & & \\
\epsilon : 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & g\downarrow & \text{P.O.} & \downarrow & & \| & & \\
g\epsilon : 0 & \longrightarrow & N' & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

The properties of co-cartesian squares ensure that $g(\epsilon f) = (g\epsilon)f$ for all composable ϵ, f, g.

1.6 DEFINITION. Let $\epsilon : 0 \to N \to X \to M \to 0$ and $\epsilon'\colon 0 \to N \to X' \to M \to 0$ be two short exact sequences of $\mathcal{A}$. The *Baer sum* $[\epsilon] + [\epsilon']$ is the 1-extension in $\operatorname{Ext}^1_{\mathcal{A}}(M, N)$ represented by

$$\Delta_N(\epsilon \oplus \epsilon')\nabla_M ,$$

where $\epsilon \oplus \epsilon'$ is the obvious short exact sequence obtained componentwise, $\nabla_M\colon M \to M \oplus M$ and $\Delta_N\colon N \oplus N \to N$ are the canonical morphisms given by the biproducts of M and N, respectively.

Then $\operatorname{Ext}^1_{\mathcal{A}}(M, N)$ equipped with the Baer sum turns out to be an abelian group. We conclude stating the following well-known result which guarantees that the $\operatorname{Ext}^n_{\mathcal{A}}(M, N)$'s actually underlie functors playing the role of the right derived functors of the hom functors of $\mathcal{A}$.

1.7 THEOREM. *Let $\mathcal{A}$ be an abelian category and N be an object such that $\operatorname{Ext}^n_{\mathcal{A}}(M, N)$ is a set for all $M \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the assignment $M \mapsto \operatorname{Ext}^n_{\mathcal{A}}(M, N)$ defines functors*

$$\begin{aligned}\operatorname{Ext}^n_{\mathcal{A}}(-, N)\colon \mathcal{A} &\longrightarrow \text{Mod-End}_{\mathcal{A}}(N) \\ M &\longmapsto \operatorname{Ext}^n_{\mathcal{A}}(M, N)\ .\end{aligned}$$

Moreover, if $0 \to K \xrightarrow{i} X \xrightarrow{p} M \to 0$ is a short exact sequence of $\mathcal{A}$, then the functors $\operatorname{Ext}^n_{\mathcal{A}}(-, N)$ induce a long exact sequence in $\text{Mod-End}_{\mathcal{A}}(N)$

$$\begin{aligned}&0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(M, N) \xrightarrow{p^*} \operatorname{Hom}_{\mathcal{A}}(X, N) \xrightarrow{i^*} \operatorname{Hom}_{\mathcal{A}}(K, N) \xrightarrow{\partial_1} \cdots \\ &\cdots \xrightarrow{\partial_1} \operatorname{Ext}^1_{\mathcal{A}}(M, N) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(X, N) \longrightarrow \operatorname{Ext}^1_{\mathcal{A}}(K, N) \xrightarrow{\partial_2} \cdots \\ &\cdots \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(X, N) \longrightarrow \operatorname{Ext}^n_{\mathcal{A}}(K, N)\end{aligned}$$

which is natural in the given short exact sequence.

Let $\mathcal{L}$ be a subclass of $\mathcal{A}$. For any $n \in \mathbb{N}$, set, with the clear shorthand meaning,

$$\mathcal{L}^{\perp_n} := \operatorname{Ker} \operatorname{Ext}^n_{\mathcal{A}}(\mathcal{L}, -) \qquad \text{and} \qquad {}^{\perp_n}\mathcal{L} := \operatorname{Ker} \operatorname{Ext}^n_{\mathcal{A}}(-, \mathcal{L})\ .$$

Notice that the classes $\mathcal{L}^{\perp_0}(=\operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(\mathcal{L},-))$ and ${}^{\perp_0}\mathcal{L}(=\operatorname{Ker}\operatorname{Hom}_{\mathcal{A}}(-,\mathcal{L}))$ make sense even when $\mathcal{A}$ is just additive.

1.2.3. Torsion pairs.

1.8 DEFINITION. A *torsion pair* in an abelian category $\mathcal{A}$ is a pair $(\mathcal{E},\mathcal{F})$ of subcategories such that $\mathcal{E}={}^{\perp_0}\mathcal{F}$, $\mathcal{F}=\mathcal{E}^{\perp_0}$, and for every object $M\in\mathcal{A}$ there is a (functorial) short exact sequence $0\to X\to M\to Y\to 0$ with $X\in\mathcal{E}$ and $Y\in\mathcal{F}$.

Given a torsion pair $(\mathcal{E},\mathcal{F})$ in $\mathcal{A}$, the first component $\mathcal{E}$ is called the *torsion class* while $\mathcal{F}$ is the *torsionfree class* of the pair.

In the previous definition, functoriality means that the assignment $M\mapsto X$ underlies an idempotent radical functor $x\colon\mathcal{A}\to\mathcal{A}$ or, equivalently, $M\mapsto Y$ yields an idempotent coradical functor $y\colon\mathcal{A}\to\mathcal{A}$ (see e.g. [52] for a detailed reference). In any case, (the restrictions of) x and y give rise to the following adjunctions:

$$\mathcal{E}\underset{x}{\rightleftarrows}\mathcal{A}\overset{y}{\rightleftarrows}\mathcal{F}\ .$$

Let $\mathcal{L}$ be a subclass of $\mathcal{A}$. Then

$$({}^{\perp_0}(\mathcal{L}^{\perp_0}),\mathcal{L}^{\perp_0})\qquad\text{and}\qquad({}^{\perp_0}\mathcal{L},({}^{\perp_0}\mathcal{L})^{\perp_0})$$

are called, respectively, the torsion pairs *generated* and *cogenerated* by $\mathcal{L}$. When $\mathcal{A}$ has coproducts and $\mathcal{L}$ is a subcategory of $\mathcal{A}$, then $\mathcal{L}$ is a torsion class if and only if it is closed under extensions, quotient objects and coproducts; dually, if $\mathcal{A}$ has products, $\mathcal{L}$ is a torsionfree class if and only if it is closed under taking subobjects, extensions and products.

1.9 DEFINITION. Let $\mathcal{A}$ be an abelian category and $(\mathcal{E},\mathcal{F})$ be a torsion pair in $\mathcal{A}$.

- $(\mathcal{E},\mathcal{F})$ is called *hereditary* if the torsion radical $x\colon\mathcal{A}\to\mathcal{A}$ is left exact.
- If $\mathcal{A}$ has coproducts, then $(\mathcal{E},\mathcal{F})$ is called *of finite type* if the torsion radical $x\colon\mathcal{A}\to\mathcal{A}$ commutes with direct limits.

It is easy to prove (see [33]) that a torsion pair $(\mathcal{E},\mathcal{F})$ is hereditary iff the torsion class $\mathcal{E}$ is closed under subobjects, while it is of finite type iff the torsion coradical $y\colon\mathcal{A}\to\mathcal{A}$ commutes with direct limits, iff the torsionfree class $\mathcal{F}$ is closed under direct limits.

1.2.4. Localisation. Localisation is an essential tool in the approximation theory of a given category. After recalling the general definition of localisation functor taken from the seminal work [17], we pass to adapt it to the case of abelian categories (though we will resume it even for their homotopy categories).

1.10 DEFINITION. Let $\mathcal{C}$, $\mathcal{C}'$ be two categories. A functor $F\colon\mathcal{C}\to\mathcal{C}'$ is a *localisation functor at a class of morphisms Σ* of $\mathcal{C}$ if, for any category $\mathcal{D}$, the functor

$$\begin{aligned}F^*\colon\mathcal{D}^{\mathcal{C}'}&\longrightarrow\mathcal{D}^{\mathcal{C}}\\ G&\longmapsto G\circ F\end{aligned}$$

is fully faithful and its essential image consists of those functors $\mathcal{C} \to \mathcal{D}$ that map morphisms in Σ to isomorphisms of $\mathcal{D}$.

Up to equivalence of categories, a localisation of $\mathcal{C}$ at Σ consists of a category $\Sigma^{-1}\mathcal{C}$ and a functor $q\colon \mathcal{C} \to \Sigma^{-1}\mathcal{C}$ which is universal w.r.t. the property of making invertible the elements of Σ. The localisation may produce a non-locally small category; nonetheless, either the properties of $\mathcal{C}$, Σ, and the associated functor q, affect $\Sigma^{-1}\mathcal{C}$. We are interested in the case of *exact* localisation functors:

1.11 Definition. Let $\mathcal{A}, \mathcal{B}$ be two abelian categories. An exact localisation functor $F\colon \mathcal{A} \to \mathcal{B}$ (at the class Σ of morphisms f such that $F(f)$ is invertible) is called a *Serre quotient* functor.

In this case, $f \in \Sigma$ if and only if $F(\operatorname{Ker} f) = 0 = F(\operatorname{Coker} f)$, and the subcategory $\operatorname{Ker} F$ of $\mathcal{A}$ is closed under subobjects, extensions, and quotient objects; any subcategory $\mathcal{E}$ of $\mathcal{A}$ satisfying these closure properties is called a *Serre class* of $\mathcal{A}$. There is a bijection between the Serre localisation functors originating in $\mathcal{A}$ and the Serre subcategories of $\mathcal{A}$, given by the mutually inverse assignments

$$(F\colon \mathcal{A} \to \mathcal{B}) \longmapsto \operatorname{Ker} F$$
$$\mathcal{E} \longmapsto (q\colon \mathcal{A} \to \mathcal{A}/\mathcal{E})$$

where $\mathcal{A}/\mathcal{E}$ is called the *quotient category* of $\mathcal{A}$ modulo $\mathcal{E}$, and q is canonical. It is well-known that $\mathcal{A}/\mathcal{E}$ is a locally small category and, in particular, it is abelian.

1.12 Definition. Let $\mathcal{A}$ be an abelian category and $\mathcal{E}$ be a Serre class. If the Serre quotient functor $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{E}$ admits a fully faithful right adjoint $j\colon \mathcal{A}/\mathcal{E} \to \mathcal{A}$, then:

- $\mathcal{E}$ is called *localising subcategory* and,
- $q\colon \mathcal{A} \to \mathcal{A}/\mathcal{E}$ is said to be a *Gabriel quotient functor*;
- $j\colon \mathcal{A}/\mathcal{E} \to \mathcal{A}$ is called the *section functor*.

Notice that if $\mathcal{A}$ has coproducts, then the Gabriel quotient functor q commutes with coproducts, meaning that $\mathcal{E}$ is a Serre subcategory closed under coproducts, i.e. a hereditary torsion class of $\mathcal{A}$; indeed, in this particular case the displayed above assignments induce a bijection between the localising subcategories and the hereditary torsion classes of $\mathcal{A}$.

1.2.5. Grothendieck categories. In order to introduce Grothendieck categories, let us recall the following terminology on abelian categories, which enables a categorical hierarchy among them; an abelian category $\mathcal{A}$ is said to be:

- AB-3 or *cocomplete*, if it admits coproducts;
- AB-4 if it is AB-3 and for any set I the coproduct functor $\bigoplus_{i\in I}\colon \mathcal{A}^I \to \mathcal{A}$ is exact;
- AB-5 if it is AB-3 and for any directed poset I the direct limit functor $\varinjlim_{i\in I}\colon \mathcal{A}^I \to \mathcal{A}$ is exact.

1.13 Definition. An abelian category is called *Grothendieck* in case it is AB-5 and admits a set of generators.

It is very well-known (see [58, Chap. X]) that any Grothendieck category also admits products, an injective cogenerator, and provides injective envelopes for its objects; in particular, it has enough injectives (though there are Grothendieck categories without nonzero projective objects).

For example, all module categories are Grothendieck (with enough projectives). The classification of Grothendieck categories which are categories of modules is provided by the celebrated Gabriel–Popescu theorem (we will state a generalised version in Thm. 1.16).

Essentially by [58, Chap. VI, Sec. 3], we obtain the following characterisation of the hereditary torsion pairs on a Grothendieck category.

1.14 Proposition. *Let $\mathcal{G}$ be a Grothendieck category. The following statements are equivalent for a torsion pair $(\mathcal{E}, \mathcal{F})$ on $\mathcal{G}$:*

(a) *$(\mathcal{E}, \mathcal{F})$ is hereditary;*
(b) *$\mathcal{E}$ is closed under subobjects;*
(c) *$\mathcal{F}$ is closed under injective envelopes;*
(d) *$(\mathcal{E}, \mathcal{F})$ is cogenerated by an injective object $E \in \mathcal{G}$; that is, $\mathcal{E} = {}^{\perp_0}E$.*

1.15 Example. Over any Grothendieck category $\mathcal{G}$ there is a bijection between hereditary torsion classes and idempotent radicals of $\mathcal{G}$, as we now recall in the case of module categories. When $\mathcal{G} = \text{Mod-}R$ for some ring R, by [58, Chap. VI, Theorem 5.1] there is also a bijection with *right Gabriel filters* of R, i.e. sets $\mathcal{J}$ of right ideals of R satisfying to the following conditions:

(i) for any $I, J \in \mathcal{J}$, $I \cap J \in \mathcal{J}$;
(ii) if $J \in \mathcal{J}$ and I is a right ideal such that $I \supseteq J$, then $I \in \mathcal{J}$;
(iii) if $J \in \mathcal{J}$ and $\gamma \in R$, then $(J : \gamma) := \{r \in R \mid \gamma r \in J\}$ belongs to $\mathcal{J}$;
(iv) for any right ideal I, if there exists an ideal $J \in \mathcal{J}$ such that $(I : x) \in \mathcal{J}$ for all $x \in J$, then $I \in \mathcal{J}$.

The aforementioned bijections are given by the following mutually inverse assignments:

$$\begin{aligned} \mathcal{T} \longmapsto x_{\mathcal{T}} \colon \text{Mod-}R &\to \text{Mod-}R \\ M &\mapsto \sum\{X \mid X \subseteq M,\ X \in \mathcal{T}\} \\ x \longmapsto \mathcal{T}_x := \{M \in \text{Mod-}R &\mid x(M) = M\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{T} &\longmapsto \mathcal{J}_{\mathcal{T}} := \{I \leq R_R \mid R/I \in \mathcal{T}\} \\ \mathcal{J} &\longmapsto \mathcal{T}_{\mathcal{J}} := \{M \in \text{Mod-}R \mid \text{Ann}_R(x) \in \mathcal{J}\ \forall x \in M\}\ . \end{aligned}$$

In particular, the hereditary torsion pairs of finite type of Mod-R are in bijection with the right Gabriel filters *of finite type*, i.e. those right Gabriel filters such that each ideal of the filter contains a finitely generated ideal of the filter.

Given a Grothendieck category $\mathcal{G}$ and a localising subcategory $\mathcal{E}$, i.e. a hereditary torsion class, in the associated Gabriel localisation $q : \mathcal{G} \rightleftarrows \mathcal{G}/\mathcal{E} : j$

the quotient category is Grothendieck as well; in particular, we have

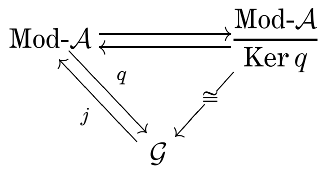
$$\mathcal{G}/\mathcal{E} \cong \operatorname{Im} j = \mathcal{E}^{\perp_0} \cap \mathcal{E}^{\perp_1} \ .$$

The aforementioned Gabriel–Popescu theorem states that any Grothendieck category is equivalent to the Gabriel quotient category of some Mod-R, say $\mathcal{G} \cong (\text{Mod-}R)/\mathcal{T}$ for some hereditary torsion class $\mathcal{T}$ of Mod-R; the following is the announced generalised version of such theorem:

1.16 Theorem ([55, Proposition 2.10]). *The following statements are equivalent for a category $\mathcal{G}$:*

(a) *$\mathcal{G}$ is a Grothendieck category;*
(b) *There is a small preadditive category $\mathcal{A}$ and a hereditary torsion class $\mathcal{T}$ in* Mod-$\mathcal{A}$ *such that $\mathcal{G}$ is equivalent to* $(\text{Mod-}\mathcal{A})/\mathcal{T}$*;*
(c) *$\mathcal{G}$ is an abelian category and there is a fully faithful functor $j\colon \mathcal{G} \to$* Mod-$\mathcal{A}$ *for some small preadditive category $\mathcal{A}$, such that j has an exact left adjoint.*

In the situation of claim (c), *we have the following commutative diagram of categories and functors*



1.2.6. Locally finitely presented Grothendieck categories. By definition, the locally finite presentability of an additive category $\mathcal{A}$ (either with direct limits or AB-3 abelian) is controlled by the full subcategory $\operatorname{fp}(\mathcal{A})$ of its finitely presented objects; indeed (see [14]), the assignment

$$\begin{aligned} \mathcal{A} &\longrightarrow \operatorname{Lex}(\operatorname{fp}(\mathcal{A})^{\mathrm{op}}, \mathrm{Ab}) \\ M &\longmapsto \operatorname{Hom}_{\mathcal{A}}(-, M)\restriction_{\operatorname{fp}(\mathcal{A})}, \end{aligned}$$

landing in the subcategory of Mod-$\mathcal{A}$ formed by the left exact contravariant functors $\operatorname{fp}(\mathcal{A}) \to \mathrm{Ab}$, is a category equivalence.

However, in general, a complete description of $\operatorname{fp}(\mathcal{A})$ is hopeless, even when $\mathcal{A} = \mathcal{G}$ is a Grothendieck category. Nonetheless, the homological algebra and the approximation theory presented so far for $\mathcal{G}$ entail crucial informations, yet necessary or sufficient conditions, concerning its locally finite presentability. Before recollecting these results, we recall that $\mathcal{G}$ is locally finitely presented if and only if $\operatorname{fp}(\mathcal{G})$ is skeletally small and $\mathcal{G}$ is generated by a set $(B_i)_{i\in I}$ of finitely presented generators; that is, any object of $\mathcal{G}$ is epimorphic image of a coproduct of some B_i's, $\mathcal{G} = \operatorname{Gen}(B_i \mid i \in I)$ say.

1.17 Proposition ([54, Proposition 3.5(1)]). *Let $\mathcal{G}$ be a locally finitely presented Grothendieck category and let $B \in \operatorname{fp}(\mathcal{G})$. For any direct system $(M_i)_{i\in I}$ in $\mathcal{G}$ the canonical homomorphism*

$$f_1\colon \varinjlim_{i\in I} \operatorname{Ext}^1_{\mathcal{G}}(B, M_i) \longrightarrow \operatorname{Ext}^1_{\mathcal{G}}(B, \varinjlim_{i\in I} M_i)$$

is injective.

1.18 PROPOSITION ([48, Proposition 1.14]). *Let $\mathcal{G}$ be a locally finitely presented Grothendieck category and $(\mathcal{E}, \mathcal{F})$ be a torsion pair on $\mathcal{G}$. Consider the following assertions:*

(a) $\mathcal{E} = \varinjlim(\mathcal{E} \cap \mathrm{fp}(\mathcal{G}))$;
(b) *there exists a set* $\mathcal{S} \subseteq \mathrm{fp}(\mathcal{G})$ *such that* $\mathcal{E} = \mathrm{Gen}\, \mathcal{S}$;
(c) $(\mathcal{E}, \mathcal{F})$ *is generated by finitely presented objects; that is there is a set* $\mathcal{S} \subseteq \mathrm{fp}(\mathcal{G})$ *such that* $\mathcal{F} = \mathcal{S}^{\perp_0}$;
(d) $(\mathcal{E}, \mathcal{F})$ *is of finite type.*

Then, implications "(a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d)" *hold true. Furthermore, when* $(\mathcal{E}, \mathcal{F})$ *is hereditary, all the assertions are equivalent.*

The following result gives sufficient conditions for a quotient of a locally finitely presented Grothendieck category to be locally finitely presented as well; therefore, by the Gabriel–Popescu theorem, it provides sufficient conditions for any Grothendieck category to be locally finitely presented.

1.19 PROPOSITION ([55, Proposition 2.11]). *Let $\mathcal{G}$ be a locally finitely presented Grothendieck category and fix a set $\mathcal{S}$ of finitely presented generators. Let $(\mathcal{E}, \mathcal{F})$ be a hereditary torsion pair in $\mathcal{G}$, and $q : \mathcal{G} \rightleftarrows \mathcal{G}/\mathcal{E} : j$ be the corresponding Gabriel localisation. The following statements are equivalent:*

(a) $\mathrm{Im}\, j$ *is closed under taking direct limits in* $\mathcal{G}$;
(b) *The section functor* j *preserves direct limits;*
(c) *The quotient functor* q *preserves finitely presented objects;*
(d) $q(\mathcal{S})$ *consists of finitely presented objects in* $\mathcal{G}/\mathcal{E}$.

When the equivalent conditions hold true, the torsion pair $(\mathcal{E}, \mathcal{F})$ is of finite type and the category $\mathcal{G}/\mathcal{E}$ is locally finitely presented, with

$$\mathrm{fp}(\mathcal{G}/\mathcal{E}) = \mathrm{add}\, q(\mathrm{fp}(\mathcal{G})) .$$

1.20 REMARK. When regarding a Grothendieck category $\mathcal{G}$ as a Gabriel quotient of some Mod-R, one can choose as R the endomorphism ring of a generator U of $\mathcal{G}$; the Gabriel–Popescu theorem establishes that the functor $j := \mathrm{Hom}_{\mathcal{G}}(U, -) \colon \mathcal{G} \to$ Mod-R, besides faithful, is full with an exact left adjoint q. Therefore, if $\mathcal{G}$ is locally finitely presented, then U can be taken finitely presented, so that $\mathrm{Hom}_{\mathcal{G}}(U, -)$ commutes with direct limits, whence $\mathrm{fp}(\mathcal{G}) = \mathrm{add}\, q(\text{mod-}R)$.

1.2.7. Locally coherent Grothendieck categories. We now give some useful results concerning locally coherent Grothendieck categories. They are particular cases of what we saw in the previous section.

1.21 PROPOSITION ([54, Proposition 3.5(2)]). *Let $\mathcal{G}$ be a locally coherent Grothendieck category and let $B \in \mathrm{fp}(\mathcal{G})$. For any direct system $(M_i)_{i \in I}$ in $\mathcal{G}$ and for all integer $j \geq 0$, the canonical homomorphism*

$$f_j \colon \varinjlim_{i \in I} \mathrm{Ext}^j_{\mathcal{G}}(B, M_i) \longrightarrow \mathrm{Ext}^j_{\mathcal{G}}(B, \varinjlim_{i \in I} M_i)$$

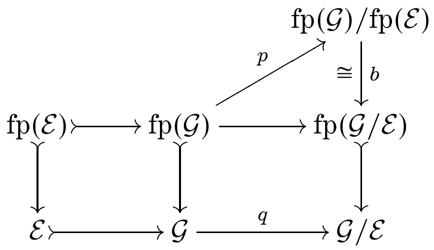
is bijective.

The following result will be vital in the body of this thesis (notice that it is a particular case of Prop. 1.19).

1.22 THEOREM ([22, Theorem 2.16], [33, Theorem 2.6]). *Let $\mathcal{G}$ be a locally coherent Grothendieck category and $q : \mathcal{G} \rightleftarrows \mathcal{G}/\mathcal{E} : j$ be a Gabriel localisation with torsion radical $x\colon \mathcal{G} \to \mathcal{E}$. The following two statements are equivalent:*

(a) *The section functor $j\colon \mathcal{G}/\mathcal{E} \to \mathcal{G}$ preserves direct limits;*
(b) *The torsion radical $x\colon \mathcal{G} \to \mathcal{E}$ preserves direct limits.*

If the previous conditions are satisfied, then $\mathcal{E}$ and $\mathcal{G}/\mathcal{E}$ are locally coherent Grothendieck categories, and the following diagram of categories and functors



in which p is a quotient functor and b an equivalence, is commutative.

1.2.8. TTF triples. Let $\mathcal{G}$ be a Grothendieck category. An interesting case of torsion theory on $\mathcal{G}$ is given by its *TTF theories*, namely triples $(\mathcal{E}, \mathcal{T}, \mathcal{F})$ of subcategories such that both $(\mathcal{E}, \mathcal{T})$ and $(\mathcal{T}, \mathcal{F})$ are torsion pairs on $\mathcal{G}$. Thus, the middle term $\mathcal{T}$ is a torsion-torsionfree class, a *TTF class* say. By the closure properties of torsion and torsionfree classes, it follows that a subcategory $\mathcal{T}$ of $\mathcal{G}$ is a TTF class if and only if it is closed under subobjects, quotients, coproducts, products and extensions. Since in a TTF triple $(\mathcal{E}, \mathcal{T}, \mathcal{F})$ the right constituent $(\mathcal{T}, \mathcal{F})$ is hereditary, as well as the left constituent $(\mathcal{E}, \mathcal{T})$ is of finite type, we say that the triple is *hereditary* (resp. *of finite type*) in case the left (resp. right) constituent is so.

1.23 EXAMPLE. TTF triples over an arbitrary ring R are well-understood (see [58, VI.6,8]): they are in bijection with idempotent two-sided ideals of R. More precisely, $\mathcal{T}$ is a TTF class in Mod-R iff there is an idempotent two-sided ideal $I \leq R$ such that $\mathcal{T}$ consists of the modules annihilated by I, i.e. $\mathcal{T} = \text{Mod-}R/I$. Consequently, the torsion modules of the left constituent $({}^{\perp_0}\mathcal{T}, \mathcal{T})$ are precisely the *I-divisible modules*, i.e. those $M \in \text{Mod-}R$ such that $MI = M$.

1.3. Triangulated categories

A *triangulated category* is an additive category $\mathcal{D}$ endowed with an additive equivalence $(-)[1] =: [1]\colon \mathcal{D} \to \mathcal{D}$ and a distinguished family of diagrams $X \to Y \to Z \to X[1]$, denoted

$$X \longrightarrow Y \longrightarrow Z \overset{+}{\longrightarrow}$$

and called (*exact*) *triangles*, satisfying certain well-known axioms (see [41]).

1.24 Example. One basic but important example of triangulated category is the homotopy category $K(\mathcal{A})$ of an abelian category $\mathcal{A}$ (see the end of subsec. 1.2.1 and [29, Chapter 1] again). In this case, the auto-equivalence of $K(\mathcal{A})$ is the usual shift functor [1], defined as in $\mathrm{Ch}(\mathcal{A})$, and the exact triangles are those isomorphic to one of the form

$$M \xrightarrow{f} N \xrightarrow{\binom{1}{0}} M(f) \xrightarrow{(0\ 1)} M[1]$$

where $M(f)$ is the mapping cone of the morphism f.

Let $\mathcal{D}$ be any triangulated category. Given a class $\mathcal{L}$ of objects of $\mathcal{D}$, and a subset $I \subseteq \mathbb{Z}$, we put

$$\mathcal{L}^{\perp_I} := \{M \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, M[j]) = 0 \text{ for all } X \in \mathcal{L} \text{ and } j \in I\}$$

and

$${}^{\perp_I}\mathcal{L} := \{M \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, X[j]) = 0 \text{ for all } X \in \mathcal{L} \text{ and } j \in I\} .$$

Usually, the set I is symbolised as "$\leq j$", "$> j$" and similar, with the obvious meaning; moreover we write just "j" in case $I = \{j\}$, accordingly with our previous notation (and what we will see in subsec. 1.3.2).

1.25 Definition ([55]). Let $(\mathcal{D}, [1])$ be a triangulated category. A subcategory $\mathcal{L}$ of the underlying additive category $\mathcal{D}$ is termed:

- *suspended* if it is closed under extensions and positive shifts; that is, in any triangle $X \to Y \to Z \xrightarrow{+}$ of $\mathcal{D}$, if $X, Z \in \mathcal{L}$ then also Y belongs to $\mathcal{L}$, and $\mathcal{L}[1] \subseteq \mathcal{L}$;
- *triangulated* if it is suspended and $\mathcal{L}[1] = \mathcal{L}$;
- *thick* if it is triangulated and closed under direct summands.

When $\mathcal{D}$ admits coproducts, then $\mathcal{L}$ is called

- *localising* if it is triangulated and closed under arbitrary coproducts.

1.26 Definition. Let $(\mathcal{D}, [1])$ be a triangulated category and $\mathcal{A}$ be an abelian category. A functor $T\colon \mathcal{D} \to \mathcal{A}$ is called *cohomological* if it maps exact triangles of $\mathcal{D}$ into exact sequences of $\mathcal{A}$.

In other words, a cohomological functor $T\colon \mathcal{D} \to \mathcal{A}$ maps any exact triangle $X \to Y \to Z \xrightarrow{+}$ of $\mathcal{D}$ into the long exact sequence in $\mathcal{A}$

$$\cdots \to T^{n-1}(Y) \longrightarrow T^{n-1}(Z) \longrightarrow T^n(X) \longrightarrow T^n(Y) \to \cdots$$

where $T^n := T \circ [n] := T \circ [1]^n$ for every integer $n \in \mathbb{Z}$.

For example, all covariant and contravariant hom functors $\mathrm{Hom}_{\mathcal{D}}(X, -)$ and $\mathrm{Hom}_{\mathcal{D}}(-, X)$ are cohomological functors $\mathcal{D} \to \mathrm{Ab}$.

Recall that, in case a triangulated category $\mathcal{D}$ has coproducts, then an object S is called *compact* if its hom functor $\mathrm{Hom}_{\mathcal{D}}(S, -)$ commutes with coproducts (see Def. 1.1). The subcategory of $\mathcal{D}$ formed by its compact objects will be denoted by $\mathcal{D}^c$.

1.27 DEFINITION. Let $\mathcal{D}$ be a triangulated category with coproducts. A triangle $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{w}{\to} X[1]$ is called *pure* if for any compact object $S \in \mathcal{D}^c$ the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{D}}(S, X) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{D}}(S, Y) \xrightarrow{v_*} \mathrm{Hom}_{\mathcal{D}}(S, Z) \longrightarrow 0$$

is short exact. In this case, u is called *pure monomorphism*, v is called *pure epimorphism*, while w is called *phantom*. Eventually:

- An object $N \in \mathcal{D}$ is said to be *pure-injective* if any pure triangle originating in N is split.
- A pure-injective object N is called *Σ-pure-injective* if $N^{(I)}$ is pure-injective for any set I.

1.28 DEFINITION. Let $(\mathcal{D}, [1]), (\mathcal{D}', [1]')$ be two triangulated categories. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is called *triangulated* if it sends exact triangles of $\mathcal{D}$ to exact triangles of $\mathcal{D}'$.

In other words, if an additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is covariant, then it is triangulated iff there exists a natural isomorphism $F \circ [1] \cong [1]' \circ F$, whereas when F is contravariant, then it is triangulated iff there is a natural isomorphism $F \circ [1] \cong [-1]' \circ F$.

1.3.1. Derived categories. As any triangulated category $\mathcal{D}$ is an additive category, one may perform on it a localisation procedure (in the sense of Def. 1.10) in order to obtain an additive category $\Sigma^{-1}\mathcal{D}$ on which transfer the triangulated structure of $\mathcal{D}$, hence obtaining a triangulated localisation functor. This is possible by requiring strong properties on Σ, more explicitly making it into a (left or right) multiplicative system compatible with the triangulation on $\mathcal{D}$; even in this case, the localisation $\Sigma^{-1}\mathcal{D}$ is not locally small in general (see [31, Chapter 10] for a detailed reference).

By localising the homotopy category $\mathcal{D} = K(\mathcal{A})$ of an abelian category $\mathcal{A}$ at the set Σ of its quasi-isomorphisms, one gets the *derived category* $D(\mathcal{A})$ of $\mathcal{A}$.

This definition of $D(\mathcal{A})$ provides a new meaning for the boundedness of a complex over $\mathcal{A}$. Indeed, since for every complex M which is acyclic at positive resp. negative degrees there are quasi-isomorphisms

$$\begin{array}{ccccccccccc}
\tau^{\leq 0}(M) & & \cdots \longrightarrow & M^{-1} & \longrightarrow & \mathrm{Ker}\, d_M^0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & & \| & & \downarrow & & \downarrow & & \\
M & & \cdots \longrightarrow & M^{-1} & \xrightarrow{d_M^{-1}} & M^0 & \xrightarrow{d_M^0} & M^1 & \longrightarrow & \cdots
\end{array}$$

and

$$\begin{array}{ccccccccccc}
M & & \cdots \longrightarrow & M^{-1} & \xrightarrow{d_M^{-1}} & M^0 & \xrightarrow{d_M^0} & M^1 & \longrightarrow & \cdots \\
\downarrow & & & \downarrow & & \downarrow & & \| & & \\
\tau^{>0}(M) & & \cdots \longrightarrow & 0 & \longrightarrow & \mathrm{Im}\, d_M^0 & \longrightarrow & M^1 & \longrightarrow & \cdots
\end{array}$$

respectively, then M is said to be (*homologically*) *bounded above* 0 (resp. *below* 0), if $H^j(M) = 0$ for all $j > 0$ (resp. $j \leq 0$). When M is an arbitrary complex, the assignments $M \mapsto \tau^{\leq 0}(M)$ and $M \mapsto \tau^{>0}(M)$ well-define the so-called (*standard*) *truncation functors* of $D(\mathcal{A})$, whose essential images are denoted by $D^{\leq 0}(\mathcal{A})$ and $D^{\geq 1}(\mathcal{A})$, respectively. Clearly, one can consider standard truncations at every degree, i.e. define $\tau^{\leq n}\colon D(\mathcal{A}) \to D^{\leq n}(\mathcal{A})$ and $\tau^{>n}\colon D(\mathcal{A}) \to D^{\geq n+1}(\mathcal{A})$, where $D^{\leq n}(\mathcal{A}) := D^{\leq 0}(\mathcal{A})[-n]$, for all integers $n \in \mathbb{Z}$. Notice there are natural isomorphisms with the standard cohomology functor:

$$H^n \cong \tau^{\leq n} \circ \tau^{>n-1} \cong \tau^{>n-1} \circ \tau^{\leq n} .$$

Let us pass to discuss the local smallness of the derived category. As usual, $q\colon K(\mathcal{A}) \to D(\mathcal{A})$ will denote the (triangulated) localisation functor.

1.29 Proposition ([61, Chapitre III]). *The following statements are equivalent for a complex E over an abelian category $\mathcal{A}$.*

(a) $\mathrm{Hom}_{K(\mathcal{A})}(W, E) = 0$ *for all acyclic complexes $W \in K(\mathcal{A})$;*
(b) *For every quasi-isomorphism $\sigma\colon M \to N$ of $K(\mathcal{A})$, the homomorphism* $\mathrm{Hom}_{K(\mathcal{A})}(\sigma, E)$ *is bijective;*
(c) *The assignment*

$$\begin{aligned} \mathrm{Hom}_{K(\mathcal{A})}(M, E) &\longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(M, E) \\ f &\longmapsto q(f) \end{aligned}$$

is a bijective correspondence for all $M \in K(\mathcal{A})$.

1.30 Definition. Let $\mathcal{A}$ be an abelian category.

- A cochain complex $E \in K(\mathcal{A})$ satisfying one of the equivalent conditions of Prop. 1.29 is called a *homotopically injective* (or *K-injective*) object.
- Given $M \in K(\mathcal{A})$, any quasi-isomorphism $M \to E$ ending in a K-injective object is called a *homotopically injective resolution* of M.

Dualising the previous result, one obtains the definition of *K-projective complex* and of *homotopically projective resolution*.

By Prop. 1.29, the derived category of an abelian category $\mathcal{A}$ is a category in the usual sense if $K(\mathcal{A})$ provides homotopically injective or projective resolutions, so that, necessarily, K-injective or K-projective complexes. However, even when $\mathcal{A}$ has enough injective or enough projective objects, these resolutions need not to exist (this occurs for the *bounded* derived category $D^b(\mathcal{A})$ of such $\mathcal{A}$). Nonetheless, we have the following crucial result, useful for our computations:

1.31 Theorem ([2, Theorem 5.4], [57]). *Let $\mathcal{G}$ be a Grothendieck category. Then each cochain complex over $\mathcal{G}$ admits a homotopically injective resolution. Moreover, the quotient functor $q\colon K(\mathcal{G}) \to D(\mathcal{G})$ admits a right adjoint functor,*

$$q : K(\mathcal{G}) \rightleftarrows D(\mathcal{G}) : \mathbf{i} .$$

If $\mathcal{G} = $ Mod-R for a ring R, then each complex also admits a homotopically projective resolution, and the quotient functor $q\colon K(R) \to D(R)$ also admits a left adjoint functor,

$$\mathbf{p} : D(R) \rightleftarrows K(R) : q .$$

Therefore, for a Grothendieck category $\mathcal{G}$ we have the following isomorphisms of abelian groups:

$$\mathrm{Hom}_{D(\mathcal{G})}(M,N) \cong \mathrm{Hom}_{D(\mathcal{G})}(M,\mathbf{i}N) \cong \mathrm{Hom}_{K(\mathcal{G})}(M,\mathbf{i}N),$$

while over the derived category $D(R)$ of a ring R, each abelian group $\mathrm{Hom}_{D(R)}(M,N)$ can be interpreted by means of the following diagram

$$\mathrm{Hom}_{K(R)}(\mathbf{p}M,N) \xleftarrow{\cong} \mathrm{Hom}_{D(R)}(M,N) \xrightarrow{\cong} \mathrm{Hom}_{K(R)}(M,\mathbf{i}N)\ .$$

Moreover, the existence of homotopically injective resolutions provides an effective description of the right derived functors: let $\mathcal{G},\mathcal{G}'$ be Grothendieck categories and $F\colon \mathrm{Ch}(\mathcal{G}) \to \mathrm{Ch}(\mathcal{G}')$ be an additive functor preserving contractible complexes and degreewise split cochain maps; then F induces a triangulated functor $K(\mathcal{G}) \to K(\mathcal{G}')$, still denoted by F. The *right derived functor* of F is the triangulated functor $\mathbf{R}F\colon D(\mathcal{G}) \to D(\mathcal{G}')$ given by the composition

$$\mathbf{R}F\colon D(\mathcal{G}) \xrightarrow{\mathbf{i}} K(\mathcal{G}) \xrightarrow{F} K(\mathcal{G}') \xrightarrow{q} D(\mathcal{G}')\ .$$

We will denote the cohomologies of an object $F(M) \in K(\mathcal{G}')$ as $H^jF(M)$, while the cohomologies of $\mathbf{R}F(M) \in D(\mathcal{G}')$ will be denoted by $R^jF(M)$, for all $j \in \mathbb{Z}$.

In case $\mathcal{G} = \text{Mod-}R$ for some ring R, we define dually the *left derived functor* of F as the triangulated functor

$$\mathbf{L}F\colon D(R) \xrightarrow{\mathbf{p}} K(R) \xrightarrow{F} K(\mathcal{G}') \xrightarrow{q} D(\mathcal{G}')\ .$$

1.32 Example. Let us recall, for a Grothendieck category $\mathcal{G}$, some basic properties of $K(\mathcal{G})$ and $D(\mathcal{G})$. There is a *hom complex bifunctor*

$$\mathcal{H}om_{\mathcal{G}}(-\,,-)\colon K(\mathcal{G})^{\mathrm{op}} \times K(\mathcal{G}) \longrightarrow K(\mathbb{Z})$$

which extends to

$$\mathbf{R}\mathcal{H}om_{\mathcal{G}}(-\,,-)\colon D(\mathcal{G})^{\mathrm{op}} \times D(\mathcal{G}) \longrightarrow D(\mathbb{Z})\ .$$

For any pair of complexes $M^\cdot, N^\cdot \in K(\mathcal{G})$ and all integers $j \in \mathbb{Z}$, one has a natural isomorphism of groups

$$H^j\mathcal{H}om_{\mathcal{G}}(M^\cdot,N^\cdot) \cong \mathrm{Hom}_{K(\mathcal{G})}(M^\cdot,N^\cdot[j])$$
$$R^j\mathcal{H}om_{\mathcal{G}}(M^\cdot,N^\cdot) \cong \mathrm{Hom}_{D(\mathcal{G})}(M^\cdot,N^\cdot[j])\ .$$

From this isomorphism, together with the definition of the right derived functors of $\mathrm{Hom}_{\mathcal{G}}(-\,,-)$ by means of the injective resolutions over $\mathcal{G}$, and Prop. 1.29, one deduces the natural isomorphism

$$\mathrm{Ext}^j_{\mathcal{G}}(M,N) \cong \mathrm{Hom}_{D(\mathcal{G})}(M[0],N[j]),$$

for any pair of objects $M,N \in \mathcal{G}$ and any integer $j \in \mathbb{Z}$.

Eventually, when $\mathcal{G} = \text{Mod-}R$ for a ring R, for any $M^\cdot \in D(R)$ and any integer $j \in \mathbb{Z}$, we have a natural isomorphism of groups

$$H^j(M^\cdot) \cong \mathrm{Hom}_{D(R)}(R[0],M^\cdot[j])\ .$$

1.33 Example. Given any ring R, there is also a *tensor complex bifunctor*

$$-\otimes_R -\colon K(R)\times K(R^{\mathrm{op}})\longrightarrow K(\mathbb{Z})$$

and its *left* derived functor

$$-\overset{\mathbf{L}}{\otimes}_R -\colon D(R)\times D(R^{\mathrm{op}})\longrightarrow D(\mathbb{Z})$$

Let now R be a commutative ring, and fix an element $\gamma\in R$. Define the *Koszul complex* of γ as the cochain complex $K(\gamma) : 0\to R\xrightarrow{\dot\gamma} R\to 0$, concentrated in degrees $\{-1,0\}$, where $\dot\gamma$ is the multiplication by γ. Given a finite sequence $(\gamma_1,\gamma_2,\dots,\gamma_n)$ of elements of R, define

$$K(\gamma_1,\gamma_2,\dots,\gamma_n)=K(\gamma_1)\otimes_R K(\gamma_2)\otimes_R\cdots\otimes_R K(\gamma_n)\ .$$

Therefore (see e.g. [43, Chapter 8]) $K(\gamma_1,\dots,\gamma_n)$ is a bounded above complex formed by (at most) $n+1$ nonzero terms, concentrated in degrees $\{-n,\dots,0\}$, and for any integer $k\geq 0$, the $-k$th term is isomorphic to a direct sum of $\binom{n}{k}$ copies of R. For example, given elements $\gamma_1,\gamma_2\in R$, the Koszul complex $K(\gamma_1,\gamma_2)$ is the complex

$$0\longrightarrow R\xrightarrow{\binom{\gamma_1}{-\gamma_2}} R^2\xrightarrow{(\gamma_1\ \gamma_2)} R\longrightarrow 0\ .$$

In the sequel we will refer to the Koszul complex of a finitely generated ideal J of rank n, denoted by $K(J)$, as the Koszul complex $K(\gamma_1,\dots,\gamma_n)$ of any set of generators of J.

1.3.2. t-structures. The corresponding notion of torsion pair for a triangulated category is the one of t-structure, introduced in the celebrated work [6], to which we will refer to. t-structures provide a useful approximation theory in their ambient triangulated category, as well as torsion pairs do in their ambient abelian category. The most powerful feature of such approximation theory is that each t-structure makes a "homological algebra" available within its triangulated category, and the relevant cohomologies belong to a suitable abelian category naturally associated with the t-structure.

Let $(\mathcal{D},[1])$ be a triangulated category. A *t-structure* in $\mathcal{D}$ is a pair $(\mathcal{U},\mathcal{V})$ of full subcategories (closed under direct summands) and satisfying the following conditions:

(i) $\mathcal{U}[1]\subseteq\mathcal{U}$;
(ii) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{U},\mathcal{V}[-1])=0$;
(iii) For any object $M\in\mathcal{D}$, there exists an exact triangle $U\to M\to V\overset{+}{\to}$ with $U\in\mathcal{U}$ and $V\in\mathcal{V}[-1]$.

Notice that $\mathcal{U}$ is a suspended subcategory of $\mathcal{D}$ (see Def. 1.25). The assignments $M\mapsto U$ and $M\mapsto V$ provided by axiom (iii) underlie the so-called *truncation functors* $\tau_{\mathcal{U}}^{\leq}$ and $\tau_{\mathcal{U}}^{>}$ of the t-structure, which are adjoint to the relevant inclusions:

$$\mathcal{U}\rightleftarrows\mathcal{D}:\tau_{\mathcal{U}}^{\leq}\qquad\text{and}\qquad\tau_{\mathcal{U}}^{>}:\mathcal{D}\rightleftarrows\mathcal{V}[-1]\ .$$

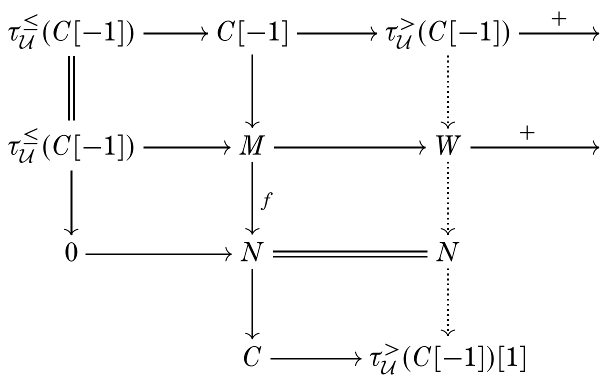
By the axioms of a triangulated category, it is readily seen that any t-structure $(\mathcal{U},\mathcal{V})$ can be expressed by means of the first component $\mathcal{U}$ via the equality

$\mathcal{V} = \mathcal{U}^{\perp_0}[1]$. $\mathcal{U}$ is called the *aisle* and its orthogonal $\mathcal{U}^{\perp_0}$ is the *coaisle* of the t-structure. We recall that $(\mathcal{U}, \mathcal{V})$ is a t-structure if, and only if, $(\mathcal{U}[n], \mathcal{V}[n])$ is a t-structure for every $n \in \mathbb{Z}$.

Let us recall the well-known and most important results from [6] on a t-structure $(\mathcal{U}, \mathcal{V})$ we are going to use in the sequel. The main one is that the intersection $\mathcal{C} := \mathcal{U} \cap \mathcal{V}$ turns out to be an abelian category, called the *heart* of the t-structure. This said, the "homological algebra" we referred to on $\mathcal{D}$ is provided by the naturally isomorphic cohomological functors $H_\mathcal{C}, \widetilde{H}_\mathcal{C} \colon \mathcal{D} \to \mathcal{C}$ defined as

$$H_\mathcal{C} := \tau^{>}_{\mathcal{U}[1]} \circ \tau^{\leq}_{\mathcal{U}} \cong \tau^{\leq}_{\mathcal{U}} \circ \tau^{>}_{\mathcal{U}[1]} =: \widetilde{H}_\mathcal{C} \ .$$

The abelian structure of $\mathcal{C}$ is described as follows. Given a morphism $f \colon M \to N$ in $\mathcal{C}$, embed it in an exact triangle of $\mathcal{D}$ by means of a cone C. Consider the approximation of $C[-1]$ within $(\mathcal{U}, \mathcal{V})$, then the following octahedron provided by a cone W of the morphism $\tau^{\leq}_{\mathcal{U}}(C[-1]) \to M$,



Define:

$$\begin{aligned} \operatorname{Ker} f &:= \tau^{\leq}_{\mathcal{U}}(C[-1]) \\ \operatorname{Im} f &:= W \\ \operatorname{Coker} f &:= \tau^{>}_{\mathcal{U}}(C[-1])[1] = H_\mathcal{C}(C) \ . \end{aligned}$$

Yet, the short exact sequences of $\mathcal{C}$ are precisely the exact triangles of $\mathcal{D}$ whose vertices belong to $\mathcal{C}$. Consequently, we have the following crucial correspondences, valid for all $M, N \in \mathcal{C}$:

$$\begin{aligned} \operatorname{Ext}^1_\mathcal{C}(M, N) &\xrightarrow{\cong} \operatorname{Hom}_\mathcal{D}(M, N[1]) \\ \operatorname{Ext}^2_\mathcal{C}(M, N) &\hookrightarrow \operatorname{Hom}_\mathcal{D}(M, N[2]) \ . \end{aligned}$$

If the ambient triangulated category $\mathcal{D}$ admits coproducts, say them denoted by the symbol $\coprod$, then the heart $\mathcal{C}$ has coproducts as well, generally distinct to those of $\mathcal{D}$; indeed, given a family $(M_i)_{i \in I}$ of objects of $\mathcal{C}$, it is not difficult to see that

$$\bigoplus_{i \in I} M_i := H_\mathcal{C}\Big(\coprod_{i \in I} M_i\Big)$$

is the coproduct of the family in $\mathcal{C}$. It is now clear how to compute direct limits. The dual notion of products and inverse limits are also available in case $\mathcal{D}$ has products.

1.34 REMARK. Given two abelian categories, it may be necessary to distinguish certain notations, e.g. those of kernels, cokernels, images, direct limits and so on. For instance, when $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is the heart of a t-structure on $D(\mathcal{A})$, we shall denote the mentioned distinguished objects of $\mathcal{C}$ by

$$\operatorname{Ker}^{(\mathcal{C})} f, \ldots, \varinjlim_{i\in I}{}^{(\mathcal{C})} M_i$$

and let unadorned those of $\mathcal{A}$.

1.35 EXAMPLE. We give three examples of t-structures discussed in the sequel.

(1) Let $\mathcal{A}$ be an abelian category. For every $n \in \mathbb{Z}$, let

$$D^{\leq n}(\mathcal{A}) := \{M \in D(\mathcal{A}) \mid H^j(M) = 0\ \forall j > n\},$$
$$D^{\geq n}(\mathcal{A}) := \{M \in D(\mathcal{A}) \mid H^j(M) = 0\ \forall j < n\}$$

be the subcategories of bounded below resp. above complexes over $\mathcal{A}$ (notice that $D^{\leq n}(\mathcal{A}) = D^{\leq 0}(\mathcal{A})[-n]$). Then $(\mathcal{D}^{\leq n}(\mathcal{A}), \mathcal{D}^{\geq n}(\mathcal{A}))$ is a t-structure of $D(\mathcal{A})$, called the (*shifted*) *standard t-structure*, and its heart is equivalent to $\mathcal{A}[-n]$.

(2) Let $\mathcal{G}$ be a Grothendieck category and $(\mathcal{E}, \mathcal{F}) =: \boldsymbol{\tau}$ be a torsion pair in $\mathcal{G}$. The *Happel-Reiten-Smalø t-structure* associated with $\boldsymbol{\tau}$ (introduced in [21]) is the t-structure of the derived category $D(\mathcal{G})$, whose members are defined respectively as

$$\mathcal{U}_{\boldsymbol{\tau}} := \{M \in D^{\leq 0}(\mathcal{G}) \mid H^0(M) \in \mathcal{E}\},$$
$$\mathcal{V}_{\boldsymbol{\tau}} := \{M \in D^{\geq -1}(\mathcal{G}) \mid H^{-1}(M) \in \mathcal{F}\}\ .$$

Therefore, the associated *HRS heart* $\mathcal{C}_{\boldsymbol{\tau}}$ consists of the cochain complexes $0 \to Y \xrightarrow{d} X \to 0$ over $\mathcal{G}$ concentrated in degrees -1 and 0 having $\operatorname{Ker} d \in \mathcal{F}$ and $\operatorname{Coker} d \in \mathcal{E}$. Yet, such heart admits $(\mathcal{F}[1], \mathcal{E}[0])$ as torsion pair. We recall that in [44] it is proved that $\mathcal{C}_{\boldsymbol{\tau}}$ is a Grothendieck category if and only if $\boldsymbol{\tau}$ is of finite type.

(3) Referring to [53, 48], let $\mathcal{D}$ be a triangulated category with products (e.g. $D(\mathcal{G})$ as usual). An object $N \in \mathcal{D}$ is called:

- *cosilting* if $({}^{\perp_{<0}}N, {}^{\perp_{>0}}N)$ is a t-structure in $\mathcal{D}$ and $N \in {}^{\perp_{>0}}N$; in this case, the t-structure is called *cosilting* induced by N.
- *cotilting* if it is cosilting and $\operatorname{Prod} M \subset {}^{\perp_{\neq 0}}M$. In this case, this latter is then termed *cotilting* t-structure induced by N.

As we will discuss in Chap. 2, cosilting and cotilting objects in the derived category of a commutative noetherian ring are strictly related to the local coherence of their hearts.

We conclude this subsection by recalling the following notions.

1.36 DEFINITION. Let $\mathcal{D}$ be a triangulated category and $(\mathcal{U}, \mathcal{V})$ be a t-structure in $\mathcal{D}$ with heart $\mathcal{C}$. The t-structure is said to be

- *left nondegenerate* (resp. *right nondegenerate*), if $\bigcap_{j\in\mathbb{Z}} \mathcal{U}[j] = 0$ (resp. $\bigcap_{j\in\mathbb{Z}} \mathcal{V}[j] = 0$). If the t-structure is left and right nondegenerate, then it is said to be *nondegenerate*;

Let $\mathcal{D}'$ be a triangulated subcategory of $\mathcal{D}$. The given t-structure is called

- *restrictable to* $\mathcal{D}'$, if the pair $(\mathcal{U} \cap \mathcal{D}', \mathcal{V} \cap \mathcal{D}')$ is a t-structure in $\mathcal{D}'$.

If $\mathcal{D}$ has coproducts, then the given t-structure is called:

- *smashing*, if the coaisle $\mathcal{V}[-1]$ is closed under coproducts or, equivalently, if the functor $H_\mathcal{C} : \mathcal{D} \to \mathcal{C}$ commutes with coproducts;
- *compactly generated*, if there exists a set $\mathcal{S}$ of compact objects in the aisle such that $\mathcal{V} = \mathcal{S}^{\perp_{<0}}$.

1.37 Remark. In a triangulated category $\mathcal{D}$ with coproducts, the compactly generated t-structures are always smashing. Moreover, by [2, Lemma 3.1], $(\mathcal{U}, \mathcal{V})$ is compactly generated if and only if there is a set $\mathcal{S}$ of compact objects of $\mathcal{D}$ such that $\mathcal{U}$ coincides with the smallest suspended subcategory of $\mathcal{D}$ containing $\mathcal{S}$; in this case we will write $\mathcal{U} = \operatorname{aisle} \mathcal{S}$. We are particularly interested in compactly generated t-structures of a triangulated category with coproducts since in [55, Theorem 8.31] it is proved that they have locally finitely presented Grothendieck hearts.

To complete the picture of the algebraic properties of the t-structures we will treat in the present thesis, we need to contextualise their ambient triangulated categories —more precisely, the derived categories— in a more abstract setting, namely to regard them as categories enhanced by derivators.

1.3.3. Reminder on derivators. We briefly recall some terminology and basic facts concerning Grothendieck prederivators, more precisely the strong and stable derivators, following [20, 59, 35, 56]. The aim is to remind that to any such derivator it is naturally associated a triangulated category, called its base, in which homotopy limits and colimits are defined; furthermore, there is a strong and stable derivator whose base is equivalent to the derived category of a fixed Grothendieck category, so that the homotopy colimits of this latter may be managed (and understood) in the base instead.

Let **Cat** be the 2-category of all categories, **cat** be the 2-category of small categories, and $\mathbf{cat}^{\mathrm{op}}$ be the 2-category obtained by **cat** reversing the arrows of the 1-cells and letting the 2-cells unchanged. A *prederivator* is a strict 2-functor $\mathbb{D} : \mathbf{cat}^{\mathrm{op}} \to \mathbf{Cat}$. Let $\mathbf{1}$ be the discrete small category consisting of one object, and let I be any small category; then each object $i \in I$ may be regarded as the functor $\mathbf{1} \to I$, $0 \mapsto i$, and similarly any morphism $\lambda : i \to j$ in I is a natural transformation of functors $i \Rightarrow j$. Then:

- $\mathbb{D}(\mathbf{1})$ is the *base* of the prederivator $\mathbb{D}$,
- $\mathbb{D}(\mathbf{1})^I$ is the category of *incoherent diagrams of shape I on $\mathbb{D}$*,
- $\mathbb{D}(I)$ is the category of *coherent diagrams of shape I on $\mathbb{D}$*.

There is a canonical *diagram functor* associated with a prederivator $\mathbb{D}$,

$$\begin{aligned} \operatorname{diag}_I : \mathbb{D}(I) &\longrightarrow \mathbb{D}(\mathbf{1})^I \\ \mathcal{X} &\longmapsto (i \mapsto \mathbb{D}(i)(\mathcal{X}) =: \mathcal{X}_i), \end{aligned}$$

which in general is not an equivalence of categories. Since $\mathbf{1}$ is a terminal object of **cat**, for every small category $I \in \mathbf{cat}$ there is a unique functor $\mathrm{pt}_I : I \to \mathbf{1}$; the *homotopy colimit* (resp. *homotopy limit*) functor is the left (resp. right) adjoint to the functor $\mathbb{D}(\mathrm{pt}_I)\colon \mathbb{D}(\mathbf{1}) \to \mathbb{D}(I)$:

$$\operatorname*{hocolim}_{i\in I} : \mathbb{D}(I) \rightleftarrows \mathbb{D}(\mathbf{1}) : \mathbb{D}(\mathrm{pt}_I)$$

and

$$\mathbb{D}(\mathrm{pt}_I) : \mathbb{D}(\mathbf{1}) \rightleftarrows \mathbb{D}(I) : \operatorname*{holim}_{i\in I} \ .$$

In general, a prederivator needs not to admit homotopy (co)limits; in fact, *derivators* are axiomatised in order to guarantee (also) their existence for all $I \in \mathbf{cat}$. Besides, the axioms of *strong and stable derivators* provide the conditions in order to equip each of their images with a triangulated structure and, moreover, their homotopy (co)limits into triangulated functors (see [20, Theorem 4.16, Corollary 4.19]). Furthermore, by [56, Theorem A], given a strong and stable derivator $\mathbb{D}$ and a t-structure $(\mathcal{U}, \mathcal{V})$ with heart $\mathcal{C}$ in the base $\mathbb{D}(\mathbf{1})$, then the classes

$$\mathcal{U}_I := \{\mathcal{X} \in \mathbb{D}(I) \mid \mathcal{X}_i \in \mathcal{U},\ \forall i \in I\}$$

and

$$\mathcal{V}_I := \{\mathcal{Y} \in \mathbb{D}(I) \mid \mathcal{Y}_i \in \mathcal{V},\ \forall i \in I\}$$

form a t-structure with heart $\mathcal{C}_I$ in the category $\mathbb{D}(I)$, and the diagram functor induces an equivalence of abelian categories $\mathcal{C}_I \cong \mathcal{C}^I$.

1.38 EXAMPLE. Let $\mathcal{G}$ be a Grothendieck category. For any small category $I \in \mathbf{cat}$, the assignment

$$\begin{aligned} \mathbb{D}_{\mathcal{G}}\colon \mathbf{cat}^{\mathrm{op}} &\longrightarrow \mathbf{Cat} \\ I &\longmapsto D(\mathcal{G}^I) \\ (u\colon J \to I) &\longmapsto \big(D(\mathcal{G}^I) \xrightarrow{u^*} D(\mathcal{G}^J)\big), \end{aligned}$$

where $u^* := \mathbb{D}_{\mathcal{G}}(u)$ is induced by the exact functor $\mathcal{G}^I \to \mathcal{G}^J$ given by the precomposition by u, well-defines a strong and stable derivator, called the *standard derivator of* $\mathcal{G}$. The base $\mathbb{D}_{\mathcal{G}}(\mathbf{1})$ is then equivalent to the derived category of $\mathcal{G}$. On the other hand, since $\mathrm{Ch}(\mathcal{G}^I) \cong \mathrm{Ch}(\mathcal{G})^I$ canonically, the objects of $\mathbb{D}_{\mathcal{G}}(I) = D(\mathcal{G}^I)$ can be regarded as I-shaped diagrams with values in $\mathrm{Ch}(\mathcal{G})$. In these derived categories, the homotopy (co)limits are naturally isomorphic to the total right (resp. left) derived functors of the ordinary (co)limits of complexes, i.e. for every $I \in \mathbf{cat}$ and $\mathcal{X}$ in $D(\mathcal{G}^I)$, represented by $(X_i)_{i\in I}$ in $\mathrm{Ch}(\mathcal{G})^I$, we have

$$\operatorname*{holim}_{i\in I} \mathcal{X} = \mathbf{R}\!\lim_{i\in I} X_i \quad \text{and} \quad \operatorname*{hocolim}_{i\in I} \mathcal{X} = \mathbf{L}\!\operatorname*{colim}_{i\in I} X_i \ .$$

We are mostly interested in the case of filtered homotopy colimits, namely when I is a directed poset, and that we will denote by $\underrightarrow{\mathrm{holim}}$. In this case, the ordinary colimit functor $\mathrm{Ch}(\mathcal{G})^I \to \mathrm{Ch}(\mathcal{G})$ is exact, hence for any $\mathcal{X}$ as above we have a natural isomorphism

$$\underset{i\in I}{\underrightarrow{\mathrm{holim}}}\, \mathcal{X} \cong \underset{i\in I}{\varinjlim}\, X_i \ .$$

1.3.4. Some crucial results on t-structures. We conclude this preliminary chapter recollecting further results known in the literature we will exploit in the body of the thesis, concerning the t-structures in the derived category of a Grothendieck category $\mathcal{G}$. We will implicitly regard $D(\mathcal{G})$ as the base of the standard derivator described in Ex. 1.38. In particular, following [56], we say that a t-structure $(\mathcal{U}, \mathcal{V})$ is *homotopically smashing* if $\mathcal{V}$ is closed under taking filtered homotopy colimits. By [56, Theorem B], homotopically smashing t-structures are smashing, and compactly generated t-structures are homotopically smashing. Moreover, thanks to the following result, they actually encompass a very large class of t-structures, in particular those we will deal with in the sequel.

1.39 THEOREM ([35, Theorem 4.6]). *Let $\mathcal{G}$ be a Grothendieck category and $(\mathcal{U}, \mathcal{V})$ be a nondegenerate t-structure in $D(\mathcal{G})$. The following statements are equivalent:*

(a) *$(\mathcal{U}, \mathcal{V})$ is homotopically smashing;*
(b) *$(\mathcal{U}, \mathcal{V})$ is smashing with a Grothendieck heart;*
(c) *$(\mathcal{U}, \mathcal{V})$ is cosilting, associated to a pure-injective*[1] *cosilting object of $D(\mathcal{G})$.*

The local coherence of the hearts of (compactly generated) t-structures is far from being characterised, even in derived categories of modules or locally coherent Grothendieck categories. The following result by Saorín provides some sufficient conditions for such condition to occur:

1.40 THEOREM ([54, Proposition 4.5]). *Let $\mathcal{G}$ be a locally coherent Grothendieck category and $\mathcal{S}$ be a set of finitely presented generators of $\mathcal{G}$. Let $(\mathcal{U}, \mathcal{V})$ be any t-structure in $D(\mathcal{G})$ with heart $\mathcal{C}$, and assume that the following conditions hold:*

(i) *$(\mathcal{U}, \mathcal{V})$ restricts*[2] *to $D^b(\mathrm{fp}(\mathcal{G}))$;*
(ii) *there are integers $m \leq n$ such that $D^{\leq m}(\mathcal{G}) \subseteq \mathcal{U} \subseteq D^{\leq n}(\mathcal{G})$;*
(iii) *$\mathcal{C} \cap D^b(\mathrm{fp}(\mathcal{G}))$ is a (skeletally small) class of generators of $\mathcal{C}$;*
(iv) *For any direct system $(M_i)_{i\in I}$ in $\mathcal{C}$, any $B \in \mathcal{S}$ and any integer $j \in \mathbb{Z}$, the canonical morphisms*

$$\mu_{B[j]} \colon \varinjlim_{i\in I} \mathrm{Hom}_{D(\mathcal{G})}(B[j], M_i) \longrightarrow \mathrm{Hom}_{D(\mathcal{G})}(B[j], \varinjlim_{i\in I}{}^{(\mathcal{C})} M_i)$$

are isomorphisms.

Then $\mathcal{C}$ is a locally coherent Grothendieck category, with $\mathrm{fp}(\mathcal{G}) = \mathcal{C} \cap D^b(\mathrm{fp}(\mathcal{G}))$.

Notice that conditions (iii) and (iv) are quite natural in order to ask for the heart to be locally coherent; in this sense, restrictability is not necessary in general (see [54, Remark 4.6] and Ex. 2.8) but together with (ii), known as *intermediacy* of the given t-structure, it seems to play a crucial role at least when $\mathcal{G}$ is the category of modules over a commutative noetherian ring. Indeed, as we will review and discuss in Chap. 2, there are decisive results on the local coherence in this setting, related either to algebraic aspects, from tilting theory, and to geometric ones, due to the fundamental classification of compactly generated t-structures. Such classification translates the intermediacy into a

[1] See Def. 1.27
[2] See Def. 1.36

very handleable condition; on the other hand, we will prove that for certain noetherian rings the restrictability can be rephrased precisely as well-known statements in local cohomology.

However, these results cannot be replicated for arbitrary commutative rings, as we will see in Chap. 3, at least for their proofs heavily exploit properties holding true in the noetherian setting. In fact, even if compactly generated t-structures of the derived category of a commutative ring are classified (in a geometric sense), the results we will obtain in that setting will be argued somehow “ad hoc”.

CHAPTER 2

Compactly generated t-structures over a commutative noetherian ring

2.1. Classification results

Throughout this chapter, R will denote a commutative noetherian ring. We recall the very well-known results concerning the classification of localising subcategories of Mod-R and $D(R)$, the former termed as hereditary torsion classes in the previous chapter. These classifications make use of certain subsets of the Zariski spectrum of R, meaning that many features of the corresponding localising subcategory can be decoded in geometric terms.

Even if the content of the forthcoming subsections have been generalised to arbitrary commutative rings (see subsec. 3.2 in the next chapter), the remaining results of the present chapter are not available in that setting.

2.1.1. Specialisation closed subsets. Let R be a commutative noetherian ring. The prime spectrum $\operatorname{Spec} R$, endowed with the topology whose closed are the sets of the form $V(I) := \{\boldsymbol{p} \in \operatorname{Spec} R \mid \boldsymbol{p} \supseteq I\}$, the I's being ideals of R, is called *Zariski spectrum* of R. For any prime ideal $\boldsymbol{p}$, the canonical ring homomorphism $\phi\colon R \to R_{\boldsymbol{p}}$, landing in the localisation of R at $R \setminus \boldsymbol{p}$, yields adjoint functors

$$\text{Mod-}R_{\boldsymbol{p}} \;\underset{\operatorname{Hom}_R(R_{\boldsymbol{p}},-)}{\overset{-\otimes_R R_{\boldsymbol{p}}}{\leftleftarrows}}\!\!\xrightarrow{\;\phi_*\;}\; \text{Mod-}R \qquad \text{and} \qquad D(R_{\boldsymbol{p}}) \;\underset{\mathbf{R}\operatorname{Hom}_R(R_{\boldsymbol{p}},-)}{\overset{-\otimes^{\mathbf{L}}_R R_{\boldsymbol{p}}}{\leftleftarrows}}\!\!\xrightarrow{\;\phi_*\;}\; D(R)$$

where ϕ_* is the scalar restriction functor. We can identify $\operatorname{Spec} R_{\boldsymbol{p}}$ as the subset of $\operatorname{Spec} R$ formed by those prime ideals contained in $\boldsymbol{p}$. For any module $M \in$ Mod-R, define its *support* by setting

$$\operatorname{Supp} M := \{\boldsymbol{p} \in \operatorname{Spec} R \mid M \otimes_R R_{\boldsymbol{p}} \neq 0\};$$

in particular, we have $\operatorname{Supp} R/J = V(J)$ for all ideals $J \leq R$. Recall that any hereditary torsion pair $\mathcal{T}$ in Mod-R is generated by the torsion cyclic modules R/J's (which actually are finitely presented since R is noetherian). This is to highlight that the support of a torsion module or of a finitely generated module is an important example of the following notion, which will play a vital role in the rest of the thesis.

2.1 DEFINITION. A subset Z of $\operatorname{Spec} R$ is said to be *closed under specialisation*, or *sp-closed*, if for any inclusion $\boldsymbol{q} \subseteq \boldsymbol{p}$ with $\boldsymbol{q} \in Z$, we have $\boldsymbol{p} \in Z$.

A set Z of prime ideals of R is sp-closed if and only if it is the union of Zariski closed, say $Z = \bigcup_{\lambda \in \Lambda} V(J_\lambda)$ for some set Λ. Therefore, by the above considerations, the support of a module which is torsion w.r.t. some hereditary torsion pair is a sp-closed subset of $\operatorname{Spec} R$. The interplay of sp-closed subsets with the hereditary torsion pairs is even stronger: the bijection between the hereditary torsion classes $\mathcal{T}$ of Mod-R, the Gabriel filters $\mathcal{J}$ of R, and the idempotent radical functors of Mod-R, extends to the sp-closed subsets of $\operatorname{Spec} R$ (see [16]) by means of the following mutually inverse assignments, provided on the hereditary torsion classes and the Gabriel filters:

$$\begin{aligned} \mathcal{T} &\longmapsto Z_{\mathcal{T}} := \bigcup \{\operatorname{Supp} M \mid M \in \mathcal{T}\} \\ Z &\longmapsto \mathcal{T}_Z := \{M \in \text{Mod-}R \mid \operatorname{Supp} M \subseteq Z\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{J} &\longmapsto Z_{\mathcal{J}} := \bigcup \{V(J) \mid J \in \mathcal{J}\} \\ Z &\longmapsto \mathcal{J}_Z := \{J \leq R \mid V(J) \subseteq Z\} \ . \end{aligned}$$

The associated torsion radical is usually denoted by Γ_Z, and it coincides with the 0*th local cohomology* functor:

$$\Gamma_Z := \varinjlim_{J \in \mathcal{J}_Z} \operatorname{Hom}_R(R/J, -) \ .$$

2.2 REMARK.

(1) We will refer to the *local cohomology* of a sp-closed subset Z either as the right derived functors of $\Gamma_Z \colon \text{Mod-}R \to \text{Mod-}R$, denoted by $H^j\Gamma_Z$ for all $j \geq 0$, or as the standard cohomologies of the right derived functor $\mathbf{R}\Gamma_Z \colon D(R) \to D(R)$, denoted by $R^j\Gamma_Z$, for all $j \in \mathbb{Z}$.

(2) Throughout this chapter (mostly in sec. 2.3) we will frequently drop the rounded parentheses used for the arguments of a functor or of a function, essentially to make the formulae lighter. Accordingly, this implies that no associativity is allowed in these formulae; for example, $\mathbf{R}\Gamma_Z M[-1]$ will stand for $\mathbf{R}\Gamma_Z(M)[-1]$, and so on. . .

2.1.2. Filtrations by supports. We have seen that sp-closed subsets of $\operatorname{Spec} R$ parametrise the hereditary torsion pairs of Mod-R. There is a derived category analogue of such result, due to Hopkins and Neeman, concerning the classification of localising subcategories[1] of $D(R)$, which has been furtherly specialised to the compactly generated t-structures of $D(R)$ by Alonso, Jeremías and Saorín in [1]: such t-structures are parametrised by the so-called *sp-filtration* of $\operatorname{Spec} R$.

[1] see Def. 1.25

2.3 DEFINITION. Let R be a commutative noetherian ring. A *filtration by support* of Spec R, or a *sp-filtration*, is a decreasing map $\Phi\colon (\mathbb{Z}, \leq) \to (2^{\operatorname{Spec} R}, \subseteq)$ such that any image $\Phi(n)$ is a sp-closed subset of Spec R.

2.4 THEOREM ([1, Theorem 3.11]). *Let R be a commutative noetherian ring. There is a bijection between the compactly generated t-structures $(\mathcal{U}, \mathcal{V})$ of $D(R)$ and the sp-filtrations of* Spec R, *given by the following mutually inverse assignments:*

$$(\mathcal{U}, \mathcal{V}) \longmapsto \Phi_{\mathcal{U}}\colon \mathbb{Z} \to 2^{\operatorname{Spec} R}$$

$$j \mapsto \{\boldsymbol{p} \in \operatorname{Spec} R \mid R/\boldsymbol{p}[-j] \in \mathcal{U}\}$$

$$\Phi \longmapsto (\mathcal{U}_{\Phi}, \mathcal{V}_{\Phi})$$

$$\mathcal{U}_{\Phi} := \operatorname{aisle}(R/\boldsymbol{p}[-j] \mid \boldsymbol{p} \in \Phi(j),\ j \in \mathbb{Z})$$

$$= \{M \in D(R) \mid \operatorname{Supp} H^j M \in \Phi(j),\ \forall j \in \mathbb{Z}\},$$

$$\mathcal{V}_{\Phi}[-1] := \{M \in D(R) \mid \mathbf{R}\Gamma_{\Phi(j)} M \in D^{>j}(R),\ \forall j \in \mathbb{Z}\}\ .$$

2.5 REMARK.

(1) The previous classification yields a plethora of fruitful informations concerning the compactly generated t-structures of $D(R)$, which are visible directly on the relevant sp-filtration. For instance, a t-structure $(\mathcal{U}, \mathcal{V})$ associated with a sp-filtration Φ is intermediate if and only there are integers $m \leq n$ such that $\Phi(m) = \operatorname{Spec} R$ and $\Phi(n) = \emptyset$. Notice that any intermediate (compactly generated) t-structure of $D(R)$ is nondegenerate.

(2) Recall that in $D(R)$ the *Telescope Conjecture* holds true: any smashing localising subcategory of $D(R)$ is generated by compact objects. Very recently, Hrbek and Nakamura proved in [25] a non-triangulated version of the conjecture, namely that any homotopically smashing t-structure is compactly generated. Therefore, in $D(R)$, Thm. 1.39 by Laking identifies the nondegenerate cosilting t-structures induced by a pure-injective cosilting object with the compactly generated ones.

(3) As we will show in in Ex. 3.10, the HRS t-structure induced by the torsion pair associated with a sp-closed subset of Spec R is a particular case of intermediate sp-filtration, hence it is compactly generated. (In general, HRS t-structures are not compactly generated; see [8].)

2.2. On the local coherence of the hearts

The very general Thm 1.40 by Saorín specialises to the following[2]:

2.6 THEOREM ([54, Theorem 6.3]). *Let R be a commutative noetherian ring and $(\mathcal{U}, \mathcal{V})$ be a compactly generated t-structure of $D(R)$. If the t-structure restricts to $D^b(\text{mod-}R)$, then its heart $\mathcal{H}$ is a locally coherent Grothendieck category, with* $\operatorname{fp}(\mathcal{H}) = \mathcal{H} \cap D^b(\text{mod-}R)$.

[2] In the sequel we will always deal with t-structures given by filtrations, and we will denote their hearts by $\mathcal{H}$.

In other words, the local coherence of a heart always occurs in case of a restrictable compactly generated t-structure. The converse is not true in general, see [54, Remark 4.6]. We will exhibit another interesting counterexample, after recalling the following notion.

2.7 DEFINITION. We say that a sp-filtration Φ satisfies the *weak Cousin condition* in case for all $j \in \mathbb{Z}$ and any maximal inclusion $\boldsymbol{q} \subset \boldsymbol{p}$ of prime ideals, if $\boldsymbol{q} \in \Phi(j)$ then $\boldsymbol{p} \in \Phi(j-1)$.

In [1, Theorem 4.4] it is proved that the weak Cousin condition on Φ is necessary for the restrictability of the associated t-structure, moreover by [1, Theorem 6.9] the two conditions are equivalent if the ring has a dualising complex.

2.8 EXAMPLE. Let $R = \mathbb{Z}$ and consider the following intermediate sp-filtration:

$$\Phi : \cdots = \operatorname{Spec} \mathbb{Z} \supset \operatorname{Max} \mathbb{Z} \supseteq \operatorname{Max} \mathbb{Z} \supset \emptyset,$$

concentrated in degrees $\{-2, -1, 0\}$, where $\operatorname{Max} \mathbb{Z} = \operatorname{Spec} \mathbb{Z} \setminus \{0\}$. The t-structure associated with Φ is not restrictable to $D^b(\text{mod-}\mathbb{Z})$ for it does not satisfies the weak Cousin condition. Being $\mathbb{Z}$ a hereditary ring, the complexes of $D(\mathbb{Z})$ are described cohomologically, in particular it is readily seen that the complexes M of the aisle $\mathcal{U}$ (resp. of the coaisle $\mathcal{U}^{\perp_0}$) are precisely those such that

$$H^j(M) \in \begin{cases} \text{Mod-}\mathbb{Z} & \text{if } j < -1 \\ \mathcal{T}_{(0)} & \text{if } j = -1, 0 \\ \{0\} & \text{if } j > 0, \end{cases} \qquad \text{resp. } H^j(M) \in \begin{cases} \{0\} & \text{if } j < -1 \\ \text{Mod-}\mathbb{Q} & \text{if } j = -1 \\ \mathcal{F}_{(0)} & \text{if } j = 0 \\ \text{Mod-}\mathbb{Z} & \text{if } j > 0, \end{cases}$$

where $(\mathcal{T}_{(0)}, \mathcal{F}_{(0)})$ is the canonical torsion pair of Mod-$\mathbb{Z}$ formed by the torsion and torsionfree abelian groups. Consequently, the heart is $\mathcal{H} \cong \text{Mod-}\mathbb{Q}[0] \times \mathcal{T}_{(0)}[-2]$, so it is locally coherent (actually, it is even *locally noetherian*, see [35, 5.2]). It is proven in [5, 26] that the pure-injective cosilting object inducing the t-structure is

$$N := \mathbb{Q}[2] \oplus \prod_{p \text{ prime}} J_p[1],$$

where J_p is the ring of p-adic integers. We point out the following two facts:

(1) N is neither an *elementary cogenerator*[3] nor a Σ-pure-injective object of $D(\mathbb{Z})$ (see Def. 1.27). These claims follow essentially since $\prod_p J_p$ is not an elementary cogenerator of Mod-$\mathbb{Z}$ (the proof of [51, Example 5.3.51], showing that the J_p's are not elementary cogenerators of Mod-$\mathbb{Z}$, applies on their product, since this latter is a reduced torsionfree abelian group) whence in particular it is not a Σ-pure-injective abelian group by [51, Example 5.3.54]. More in details, if N is an elementary cogenerator, then $H^j(N)$ would be so as abelian group, for every $j \in \mathbb{Z}$, contradiction

[3] For any ring R, an object N in $D(R)$ is an *elementary cogenerator* if the class of objects which purely embed in a product of copies of N coincides with the smallest class containing N closed under pure subobjects, products and direct homotopy colimits (cf. [35], and [51, sec. 5.3.5] for the theory of elementary cogenerators in Mod-R).

by $j = -1$; on the other hand, if N is Σ-pure-injective, then any direct summand would be so, including $H^{-1}(N)[1]$, whence $H^{-1}(N)$ would be Σ-pure-injective as abelian group, contradiction. This fact provides an example of a locally noetherian heart $\mathcal{H}$ induced by a pure-injective cosilting object which is not Σ-pure-injective, unfortunately invalidating [35, Proposition 5.6].

(2) Referring to the forthcoming Thm. 2.9, $\mathcal{H}$ is induced by a pure-injective cosilting object which is not cotilting.

A decisive result concerning the interplay between the restrictability of the compactly generated t-structures and the local coherence of their hearts is provided in case of intermediate t-structures (see [38, Corollary 4.2], [49, Theorem 6.16] and [27, Theorem 3.5]).

2.9 Theorem. *Let R be a commutative noetherian ring and $(\mathcal{U}, \mathcal{V})$ be an intermediate compactly generated t-structure of $D(R)$. Then $(\mathcal{U}, \mathcal{V})$ restricts to $D^b(\text{mod-}R)$ if and only if it is cotilting with a locally coherent Grothendieck heart.*

2.3. Restrictability and Faltings' annihilator theorem

In the previous section we saw that, in spite of intermediacy, in general the restrictability seems to elude a characterisation in geometric terms, albeit there are classes of noetherian rings in which such description is possible.

In this section we will deepen such possibility by proving that, indipendently of the (commutative noetherian) ring, the situation in which the weak Cousin condition implies the restrictability is equivalent to a Faltings' annihilator theorem *for complexes* to hold.

We recall the celebrated Faltings' annihilator theorem *for modules* in the following result, in which the *depth* of a module is denoted as "dp" and the *height* as "ht" (see [11] for a reference in commutative algebra).

2.10 Theorem ([32]). *Let R be a commutative noetherian ring, let Z be a sp-closed subset of* $\operatorname{Spec} R$, *M a finitely generated R-module and n any integer. Let us state the following:*

(a) *For every $\boldsymbol{q}$ with $V(\boldsymbol{q}) \nsubseteq Z$ and every $\boldsymbol{p}$ with $V(\boldsymbol{p}) \subseteq Z \cap V(\boldsymbol{q})$, it is* $\operatorname{dp}_{R_{\boldsymbol{q}}} M_{\boldsymbol{q}} + \operatorname{ht} \boldsymbol{p}/\boldsymbol{q} > n$;

(b) *There exists an ideal J of R such that $V(J) \subseteq Z$ and $JH^j\Gamma_Z M = 0$ for all $j = 0, \ldots, n$;*

(c) *$H^j\Gamma_Z M$ is a finitely generated R-module for all $j = 0, \ldots, n$.*

Then one always has "(b) $\Leftrightarrow$ (c) $\Rightarrow$ (a)" *and the statements are equivalent whenever R has a dualising complex or it is the homomorphic image of a Cohen–Macaulay ring.*

In case the three statements are equivalent, then the result is also known as Faltings' annihilator theorem (for modules).

This said, we formulate a possible incarnation of Faltings' annihilator theorem for complexes in the following conjecture, in which for every ideal J of R and every complex $X \in D(R)$ we define

$$\mathrm{dp}_R(J, X) := \inf\{j \in \mathbb{Z} \mid R^j\Gamma_{V(J)}X \neq 0\} =: \inf \mathbf{R}\Gamma_{V(J)}X,$$

and put just $\mathrm{dp}_R X$ in case R is local and J is its maximal ideal. It is known that for any sp-closed subset Z of $\operatorname{Spec} R$, we have $\inf \mathbf{R}\Gamma_Z X = \inf\{\mathrm{dp}_R(J, X) \mid V(J) \subseteq Z\}$.

2.11 Finite Conjecture. *Let R be a commutative noetherian ring. Let $X \in D^b(\text{mod-}R)$, let Z be a sp-closed subset of $\operatorname{Spec} R$ and n any integer. The following statements are equivalent:*

(a) *For every $\boldsymbol{q}$ with $V(\boldsymbol{q}) \not\subseteq Z$ and every $\boldsymbol{p}$ with $V(\boldsymbol{p}) \subseteq Z \cap V(\boldsymbol{q})$, it is*

$$\mathrm{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} + \mathrm{ht}\, \boldsymbol{p}/\boldsymbol{q} > n;$$

(b) *$R^j\Gamma_Z X_{\boldsymbol{q}}$ is a finitely generated R-module for all $j \leq n$.*

Consider now the following statement:

2.12 Restrictability Conjecture. *Let R be a commutative noetherian ring. Let Φ be an intermediate sp-filtration of $\operatorname{Spec} R$ with associated t-structure $(\mathcal{U}, \mathcal{V})$ in $D(R)$. The following statements are equivalent:*

(1) *$(\mathcal{U}, \mathcal{V})$ restricts to $D^b(\text{mod-}R)$;*
(2) *Φ satisfies the weak Cousin condition.*

Recall that a criterion for the restrictability of intermediate compactly generated t-structures of $D(R)$ is given by [1, Lemma 5.7].

We want to prove that conjectures 2.12 and 2.11 are in fact equivalent. This fact, in principle, allows to extend the class of restrictable compactly generated t-structures (hence those with locally coherent hearts) looking at an important result in commutative algebra, and viceversa.

2.13 Remark. During the writing of the present thesis, Ryo Takahashi proved in [60] that our finite conjecture 2.11 is true for a certain class of commutative noetherian rings, called *CM-excellent rings*, also achieving the subsequent results of the section, though with different techniques.

2.14 Theorem. *Let R be a commutative noetherian ring of finite Krull dimension. Then the finite conjecture holds true if and only if the restrictability conjecture holds true.*

Proof. Let us assume that the finite conjecture holds true, and let Φ be an intermediate sp-filtration of $\operatorname{Spec} R$.
"(1) $\Rightarrow$ (2)" This implication is true for any commutative noetherian ring by [1, Theorem 4.4].
"(2) $\Rightarrow$ (1)" W.l.o.g. we let Φ be the sp-filtration[4]

$$\operatorname{Spec} R = \Phi(0) \supsetneq \Phi(1) \supseteq \cdots \supseteq \Phi(n-1) \supseteq \Phi(n) \supsetneq \emptyset\,.$$

[4]In the sequel we will always concentrate the nonempty sp-subsets of a sp-filtration in negative degrees; however, in this case it is more convenient to put $\Phi(0) = \operatorname{Spec} R$ and $\Phi(n) \neq \emptyset$ for $n > 0$ in order to make such n agree with that in finite conjecture 2.11(a).

Let us consider the sp-filtration Φ', associated to Φ, introduced in [1, 5.3]: in our hypothesis also Φ' fulfils the weak Cousin condition; in particular, arguing by induction on n, we shall assume that the t-structure $(\mathcal{U}', \mathcal{V}')$ of Φ' restricts to $D^b(\text{mod-}R)$. In view of [1, Lemma 5.7], we shall conclude our proof by showing that $R^n\Gamma_{\Phi(n)}X$ is a finitely generated R-module for every complex $X \in \mathcal{V}' \cap D^b(\text{mod-}R)$. To this end, we exploit the finite conjecture 2.11; that is, we show that for any inclusion of prime ideals $\boldsymbol{q} \subseteq \boldsymbol{p}$ with $\boldsymbol{q} \notin \Phi(n) \ni \boldsymbol{p}$, we have $\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} + \operatorname{ht} \boldsymbol{p}/\boldsymbol{q} > n$. By [26], the map

$$\begin{aligned} f_\Phi =: f\colon \operatorname{Spec} R &\longrightarrow \mathbb{Z} \\ \boldsymbol{r} &\longmapsto \sup\{j \mid \boldsymbol{r} \in \Phi(j)\} \end{aligned}$$

is order preserving, and by the weak Cousin hypothesis is such that $f(\boldsymbol{p}) - f(\boldsymbol{q}) \leq \operatorname{ht} \boldsymbol{p}/\boldsymbol{q}$, whence $\operatorname{ht} \boldsymbol{p}/\boldsymbol{q} + f(\boldsymbol{q}) \geq n$. By Thm. 2.4, we have $\mathbf{R}\Gamma_{V(\boldsymbol{q})}X \in D^{>f(\boldsymbol{q})}(R)$, whence $\mathbf{R}\Gamma_{V(\boldsymbol{q})}X_{\boldsymbol{q}} \in D^{>f(\boldsymbol{q})}(R_{\boldsymbol{q}})$ by exactness of $(-)_{\boldsymbol{p}}$, and consequently $\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} > f(\boldsymbol{q})$, thus we are done.

In order to prove the converse, we shall suppose that the finite conjecture is false and exhibit an intermediate weak Cousin sp-filtration whose t-structure does not restrict to $D^b(\text{mod-}R)$. Since one always has that "(b) ⇒ (a)" ([15, Theorem 4.5]), then we shall assume that for all integers n there exist $X \in D^b(\text{mod-}R)$ and a sp-filtration Z such that $\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} + \operatorname{ht} \boldsymbol{p}/\boldsymbol{q} > n$ for all $\boldsymbol{q} \notin Z$ and $\boldsymbol{p} \in Z \cap V(\boldsymbol{q})$, and that $R^j\Gamma_Z X$ is a finitely generated R-module for all $j < n$ but $R^n\Gamma_Z X$ is *not*. Notice that we can assume in fact that $R^j\Gamma_Z X = 0$ for all $j < n$. This said, by the approximation $\mathbf{R}\Gamma_Z X \to X \to \mathbf{R}Q_Z X \overset{+}{\to}$ of X in the t-structure associated to Z (see [1, 1.6] for the notation) we also get a triangle $\tau^{\leq n}\mathbf{R}\Gamma_Z X \to X \to X' \overset{+}{\to}$ for some $X' \in D(R)$. We claim that X' satisfies the same stated properties of X. Indeed, applying the triangulated functor $(-)_{\boldsymbol{q}}$ we get $X'_{\boldsymbol{q}} \cong X_{\boldsymbol{q}}$, while the local cohomology sequence yields $R^j\Gamma_Z X' = 0$ for all $j < n$ and $R^n\Gamma_Z X' \cong R^n\Gamma_Z X$. Consider the following sp-filtration Φ of $\operatorname{Spec} R$:

$$\Phi(j) := \begin{cases} \emptyset & j > n \\ Z & j = n \\ \Phi(j+1) \cup W & j < n, \end{cases}$$

where W is the set of those primes $\boldsymbol{p}$ which are maximal under some other prime $\boldsymbol{q} \in \Phi(j)$. Observe that in general Φ might be not intermediate, though it is so by our hypothesis on the dimension of R. Moreover, it fulfils the weak Cousin condition. Let us show that the t-structure $(\mathcal{U}, \mathcal{V})$ of Φ is not restrictable. Our claim will follow once we prove that $X \in \mathcal{V}'$ i.e. that $\mathbf{R}\Gamma_{\Phi(j)}X \in D^{>j}(R)$ for all $j < n$ (see again Thm. 2.4). By [26, (2.16)] this occurs if and only if $\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} > j$ for all $\boldsymbol{q} \in \Phi(j)$. For any such j, we have two cases: $\boldsymbol{q} \in \Phi(n)$ and $\boldsymbol{q} \in \Phi(j) \setminus \Phi(j+1)$. In the first one, i.e. when $\boldsymbol{q} \in Z$, we have $\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} = n$ $(> j)$ because $\inf \mathbf{R}\Gamma_Z X = n$, and we are done; in the second case, we have $\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} > n - \operatorname{ht} \boldsymbol{p}/\boldsymbol{q} \geq n - (n-j) = j$ for some $\boldsymbol{p} \in Z$. Consequently, the t-structure $(\mathcal{U}', \mathcal{V}')$ restricts to $D^b(\text{mod-}R)$, but $R^n\Gamma_Z X$ is not finitely generated, and we conclude by [1, Lemma 5.7] again. □

2.15 COROLLARY. *Faltings' annihilator theorem for complexes of $D(R)$ holds true for rings R having a dualising complex.*

Proof. Rings with a dualising complex fulfil the restrictability conjecture by [1, Theorem 6.9]. □

We are now concerned in proving that the equivalent conjectures of Thm. 2.14 are verified by local rings which are universally catenary and whose formal fibers are Cohen–Macaulay (see [11]). As usual, for any local ring R with maximal ideal $\boldsymbol{m}$, we will denote by $\widehat{R}$ the $\boldsymbol{m}$-adic completion of R, and by $\lambda\colon R \to \widehat{R}$ the canonical ring homomorphism, which is injective and faithfully flat. For any sp-closed subset Z of $\operatorname{Spec} R$ we have a sp-closed subset $\widehat{Z}$ in $\operatorname{Spec}\widehat{R}$ defined as $\widehat{Z} := \bigcup_{\boldsymbol{p}\in Z} V(\boldsymbol{p}\widehat{R}) = \{\boldsymbol{P} \in \operatorname{Spec}\widehat{R} \mid \boldsymbol{P}\cap R \in Z\}$. Let us prove formerly that certain quantities appearing in the finite conjecture 2.11, namely

$$s_Z(X) := \min\{\operatorname{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} + \operatorname{ht} \boldsymbol{p}/\boldsymbol{q} \mid \boldsymbol{q}\notin Z,\ \boldsymbol{p}\in Z\cap V(\boldsymbol{q})\},$$

are invariant when passing to the completion.

2.16 LEMMA. *Let R be a universally catenary noetherian local ring with Cohen–Macaulay formal fibers. For any sp-closed subset Z of* $\operatorname{Spec} R$ *and any complex $X \in D^b(\text{mod-}R)$, we have*

$$s_Z(X) = s_{\widehat{Z}}(\widehat{X}),$$

where $\widehat{X} = X \otimes^{\mathbf{L}}_R \widehat{R}$.

Proof. Recall that for any prime ideal $\boldsymbol{P}$ of $\operatorname{Spec}\widehat{R}$, if $\boldsymbol{p} := \boldsymbol{P}\cap R$ we have a flat local ring homomorphism $R_{\boldsymbol{p}} \to \widehat{R}_{\boldsymbol{P}}$ and, by [30, Corollary 2.6],

$$\operatorname{dp}_{\widehat{R}_{\boldsymbol{P}}} X_{\boldsymbol{p}} \otimes_{R_{\boldsymbol{p}}} \widehat{R}_{\boldsymbol{P}} = \operatorname{dp}_{R_{\boldsymbol{p}}} X_{\boldsymbol{p}} + \operatorname{dp} \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}}\ .$$

In this case, the complex in the left hand term is

$$X_{\boldsymbol{p}} \otimes_{R_{\boldsymbol{p}}} \widehat{R}_{\boldsymbol{P}} = (X \otimes_{R_{\boldsymbol{p}}} R_{\boldsymbol{p}}) \otimes_{R_{\boldsymbol{p}}} (\widehat{R} \otimes_{\widehat{R}} \widehat{R}_{\boldsymbol{P}}) \cong \widehat{X}_{\boldsymbol{P}}$$

while the ring in the right hand term is

$$\widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}} = \widehat{R}_{\boldsymbol{P}} \otimes_{R_{\boldsymbol{p}}} \kappa(\boldsymbol{p}) = (\widehat{R} \otimes_{\widehat{R}} \widehat{R}_{\boldsymbol{P}}) \otimes_{R_{\boldsymbol{p}}} \kappa(\boldsymbol{p}) \cong (\widehat{R} \otimes_R \kappa(\boldsymbol{p})) \otimes_{\widehat{R}} \widehat{R}_{\boldsymbol{P}},$$

and since by hypothesis $\widehat{R} \otimes_R \kappa(\boldsymbol{p})$ is Cohen–Macaulay, then also $\widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}}$ is a local Cohen–Macaulay ring. Altogether, the first display of the proof actually is

$$\operatorname{dp}_{\widehat{R}_{\boldsymbol{P}}} \widehat{X}_{\boldsymbol{P}} = \operatorname{dp}_{R_{\boldsymbol{p}}} X_{\boldsymbol{p}} + \dim \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}},$$

for any prime ideal $\boldsymbol{P}$ of $\operatorname{Spec}\widehat{R}$. This said, let $\boldsymbol{Q}, \boldsymbol{P}$ be prime ideals of $\widehat{R}$ such that $\boldsymbol{Q} \notin \widehat{Z}$ and $\boldsymbol{P} \in \widehat{Z}\cap V(\boldsymbol{Q})$; by setting $\boldsymbol{q} := \boldsymbol{Q}\cap R$ and $\boldsymbol{p} := \boldsymbol{P}\cap R$, it is clear that $\boldsymbol{q}\notin Z$ and $\boldsymbol{p}\in Z\cap V(\boldsymbol{q})$. We have

$$\begin{aligned}\operatorname{ht} \boldsymbol{P}/\boldsymbol{Q} - \operatorname{ht}\boldsymbol{p}/\boldsymbol{q} &= \dim \widehat{R}/\boldsymbol{Q} - \dim\widehat{R}/\boldsymbol{P} - (\dim R/\boldsymbol{q} - \dim R/\boldsymbol{p})\\ &= (\dim \widehat{R}/\boldsymbol{Q} - \dim R/\boldsymbol{q}) - (\dim \widehat{R}/\boldsymbol{P} - \dim R/\boldsymbol{p})\ . \qquad (1)\end{aligned}$$

Since R is universally catenary by hypothesis, then $\widehat{R}/\boldsymbol{p}\widehat{R}$ (and $\widehat{R}/\boldsymbol{q}\widehat{R}$) is a equidimensional ring, meaning that for any minimal prime ideal $\boldsymbol{P'} \in V(\boldsymbol{p}\widehat{R})$ we have

$$(\dim R/\boldsymbol{p} =)\ \dim \widehat{R}/\boldsymbol{p}\widehat{R} = \dim \widehat{R}/\boldsymbol{P'}$$

(same argument for $\widehat{R}/\boldsymbol{q}\widehat{R}$). We then have

$$\dim(\widehat{R}/\boldsymbol{p}\widehat{R})_{\boldsymbol{P}} = \dim \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}} = \dim \widehat{R}_{\boldsymbol{P}}/\boldsymbol{P}'_{\boldsymbol{P}}\ .$$

W.l.o.g. we can assume $\boldsymbol{P'} \subseteq \boldsymbol{P}$, so that the previous display yields

$$\mathrm{ht}\,\boldsymbol{P}/\boldsymbol{P'} = \dim \widehat{R}/\boldsymbol{P'} - \dim \widehat{R}/\boldsymbol{P} = \dim R/\boldsymbol{p} - \dim \widehat{R}/\boldsymbol{P}\ .$$

Altogether, by repeating the argument taking $\boldsymbol{Q}$ and $\boldsymbol{q}$, (1) is

$$\mathrm{ht}\,\boldsymbol{P}/\boldsymbol{Q} - \mathrm{ht}\,\boldsymbol{p}/\boldsymbol{q} = \dim \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}} - \dim \widehat{R}_{\boldsymbol{Q}}/\boldsymbol{q}\widehat{R}_{\boldsymbol{Q}}\ .$$

Eventually, we compute:

$$\begin{aligned}
&\mathrm{dp}_{\widehat{R}_{\boldsymbol{Q}}} \widehat{X}_{\boldsymbol{Q}} + \mathrm{ht}\,\boldsymbol{P}/\boldsymbol{Q} - \mathrm{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}} - \mathrm{ht}\,\boldsymbol{p}/\boldsymbol{q} \\
&\qquad = (\mathrm{dp}_{\widehat{R}_{\boldsymbol{Q}}} \widehat{X}_{\boldsymbol{Q}} - \mathrm{dp}_{R_{\boldsymbol{q}}} X_{\boldsymbol{q}}) + (\mathrm{ht}\,\boldsymbol{P}/\boldsymbol{Q} - \mathrm{ht}\,\boldsymbol{p}/\boldsymbol{q}) \\
&\qquad = \dim \widehat{R}_{\boldsymbol{Q}}/\boldsymbol{q}\widehat{R}_{\boldsymbol{Q}} + \dim \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}} - \dim \widehat{R}_{\boldsymbol{Q}}/\boldsymbol{q}\widehat{R}_{\boldsymbol{Q}} \geq 0
\end{aligned}$$

which proves that $s_{\widehat{Z}}(\widehat{X}) \geq s_Z(X)$.

Conversely, let $\boldsymbol{p}$ and $\boldsymbol{q}$ be prime ideals of R such that $\boldsymbol{q} \notin Z$ and $\boldsymbol{p} \in Z \cap V(\boldsymbol{q})$. By going up property we can choose prime ideal $\boldsymbol{P}$ of the formal fiber $\widehat{R} \otimes_R \kappa(\boldsymbol{p})$ which is minimal in $V(\boldsymbol{p}\widehat{R})$. Since $\operatorname{Spec} \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}}$ consists of one element, we have $\dim \widehat{R}_{\boldsymbol{P}}/\boldsymbol{p}\widehat{R}_{\boldsymbol{P}} = 0$. Using the going down property, we can find a prime $\boldsymbol{Q} \subseteq \boldsymbol{P}$ such that $\boldsymbol{Q} \cap R = \boldsymbol{q}$, and these $\boldsymbol{P}, \boldsymbol{Q}$ yield $s_{\widehat{Z}}(\widehat{X}) \leq s_Z(X)$. □

2.17 THEOREM. *Faltings' annihilator theorem for complexes of $D(R)$ holds true for universally catenary local rings with Cohen–Macaulay formal fibers.*

Proof. We will prove that for any sp-closed subset Z and $X \in D^b(\text{mod-}R)$, the local cohomology $R^j\Gamma_Z X$ is finitely generated for all $j < s_Z(X)$. By the previous lemma we have $s_Z(X) = s_{\widehat{Z}}(\widehat{X}) =: s$, moreover the complete local ring $\widehat{R}$ has a dualising complex (by Cohen Structure Theorem, see [62, Sec. 10.160, Lemma 47.15.8]), so that the finite conjecture 2.11 is valid, and since $\mathrm{dp}_{\widehat{R}_{\boldsymbol{Q}}} \widehat{X}_{\boldsymbol{Q}} + \mathrm{ht}\,\boldsymbol{P}/\boldsymbol{Q} \geq s$ (for all $\boldsymbol{Q} \notin Z$ and $\boldsymbol{P} \in Z \cap V(\boldsymbol{Q})$ as usual) we have that $R^j\Gamma_{\widehat{Z}}\widehat{X} \cong R^j\Gamma_Z X \otimes_R \widehat{R}$ are finitely generated $\widehat{R}$-modules for all $j < s$ (the previous isomorphism follows by generalising [62, Lemma 47.9.3] to sp-closed subsets). We conclude our proof by faithful flatness of the homomorphism $\lambda\colon R \to \widehat{R}$, since it reflects finiteness (that is, finite generation descents along λ, see e.g. [62, 35.3]). □

CHAPTER 3

Compactly generated t-structures over a commutative ring

As we announced in the introduction and in the previous chapter, several results on compactly generated t-structures of a commutative noetherian ring are available for arbitrary commutative rings, though the relevant proofs rely on very different arguments. In the present chapter we will provide instances of these arguments, working on a commutative ring and pursuing the characterisation of the local coherence of the hearts of its compactly generated t-structures. In more details, we achieve a characterisation (Thm. 3.41) by means of a general criterion (Thm. 3.2), valid for any Grothendieck category equipped with a TTF triple. Large part of the chapter is focused in detecting within the hearts suitable TTF triples, in order to apply Thm. 3.2. For this reason, we will start the chapter by proving such criterion, which is our key tool to get around the lack of the ascending chain condition on the rings.

3.1. A criterion for the local coherence

As usual, given a torsion pair $(\mathcal{E}, \mathcal{F})$ in a Grothendieck category $\mathcal{G}$, the adjunctions it gives rise with the inclusion functors originating in the subcategories of the pair will be denoted by

$$\mathcal{E} \underset{x}{\rightleftarrows} \mathcal{G} \overset{y}{\rightleftarrows} \mathcal{F} .$$

There is a notion of *restrictability* for torsion pairs in a locally finitely presented Grothendieck category:

3.1 Definition. Let $\mathcal{G}$ be a locally finitely presented Grothendieck category. A torsion pair $(\mathcal{E}, \mathcal{F})$ of $\mathcal{G}$ is said to *restrict* to $\mathrm{fp}(\mathcal{G})$ if for any $B \in \mathrm{fp}(\mathcal{G})$, also $x(B)$ and $y(B)$ are finitely presented.

In fact one just checks that $x(B)$ is finitely generated, for then $y(B)$ is a cokernel in $\mathcal{G}$ of a morphism in $\mathrm{fp}(\mathcal{G})$, whence finitely presented as well.

3.2 Theorem. *Let $\mathcal{G}$ be a Grothendieck category equipped with a TTF triple of finite type $(\mathcal{E}, \mathcal{T}, \mathcal{F})$. Consider the following three statements:*

(a) *$\mathcal{G}$ is locally coherent;*

(b) *The following conditions are satisfied:*
  (i) $\mathcal{E}$ *and* $\mathcal{T}$ *are quasi locally coherent;*
  (ii) *For every* $P \in \mathrm{fp}(\mathcal{E})$, *the functor* $\mathrm{Ext}^1_{\mathcal{G}}(P, -)$ *commutes with direct limits of direct systems of* $\mathcal{T}$*;*
  (iii) *For every* $Q \in \mathrm{fp}(\mathcal{T})$, *the functor* $\mathrm{Ext}^1_{\mathcal{G}}(Q, -)$ *commutes with direct limits of direct systems of* $\mathcal{E}$*.*

(c) *Conditions* (i), (ii) *of part* (b) *hold true, and moreover*
  (iii)' *The torsion pair* $(\mathcal{E}, \mathcal{T})$ *restricts to* $\mathrm{fp}(\mathcal{G})$.

Then "(a) $\Leftrightarrow$ (b) $\Rightarrow$ (c)", *and the statements are all equivalent in case* $\mathcal{G}$ *is locally finitely presented.*

Proof. We will denote by

$$\mathcal{E} \underset{x}{\rightleftarrows} \mathcal{G} \overset{y}{\rightleftarrows} \mathcal{T}$$

the adjunction provided by the left constituent of the TTF triple. Notice that the finite type the TTF triple implies, by [48, Lemma 2.4], that $\mathrm{fp}(\mathcal{E}), \mathrm{fp}(\mathcal{T}) \subseteq \mathrm{fp}(\mathcal{G})$. In turn, the local coherence of $\mathcal{G}$ always implies conditions (ii), (iii) and (iii)': the first two conditions are guaranteed by [54, Proposition 3.5(2)]; to see the third, for every $B \in \mathrm{fp}(\mathcal{G})$ we have $x(B) \in \mathrm{fp}(\mathcal{G})$, since the latter occurs as the kernel of the epimorphism $B \to y(B)$ in $\mathrm{fp}(\mathcal{G})$.
"(a) $\Rightarrow$ (b)" By what we just observed, we only have to check condition (i). $\mathcal{T}$ is a locally coherent Grothendieck category thanks to Thm. 1.22, thus it is quasi locally coherent. Now, let us show that $\mathcal{E}$ is locally finitely presented. Let $X \in \mathcal{E}$ and $(B_i)_{i\in I}$ be a direct system in $\mathrm{fp}(\mathcal{G})$ such that $X = \varinjlim_{i\in I} B_i$. Since $(\mathcal{E}, \mathcal{T})$ is of finite type, we have $X = x(X) = \varinjlim_{i\in I} x(B_i)$, thus $\mathcal{E}$ is locally finitely presented since each $x(B_i)$ belongs to $\mathrm{fp}(\mathcal{E})$. It remains to show that $\mathcal{E}$ is quasi locally coherent. By the previous part, it suffices to check that the kernel in $\mathcal{E}$ of an epimorphism $f\colon P \to P'$ in $\mathrm{fp}(\mathcal{E})$ is finitely presented as well. Notice that f is an epimorphism also in $\mathcal{G}$, therefore $\mathrm{Ker}\, f \in \mathrm{fp}(\mathcal{G})$ by the local coherence hypothesis. Our claim then follows since $\mathrm{Ker}^{(\mathcal{E})}(f) = x(\mathrm{Ker}\, f)$ and $(\mathcal{E}, \mathcal{T})$ restricts to $\mathrm{fp}(\mathcal{G})$.

Let us now show that if $\mathcal{G}$ is locally finitely presented, then "(c) $\Rightarrow$ (a)". We have to prove that the kernel of any epimorphism $f\colon B \to B'$ in $\mathrm{fp}(\mathcal{G})$ is finitely presented as well. Since the torsion pair $(\mathcal{E}, \mathcal{T})$ restricts to $\mathrm{fp}(\mathcal{G})$ by (iii)', the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & x(B) & \longrightarrow & B & \longrightarrow & y(B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & x(B') & \longrightarrow & B' & \longrightarrow & y(B') & \longrightarrow & 0 \end{array}$$

lives in $\mathrm{fp}(\mathcal{G})$. Besides q, also p is an epimorphism, being $\mathrm{Coker}\, p \in \mathcal{E} \cap \mathcal{T} = 0$. Therefore, p and q are epimorphisms in $\mathrm{fp}(\mathcal{E})$ and $\mathrm{fp}(\mathcal{T})$ respectively, hence by hypothesis (i) we obtain that $\mathrm{Ker}^{(\mathcal{E})}(p)$ and $\mathrm{Ker}^{(\mathcal{T})}(q) = \mathrm{Ker}\, q$ are finitely presented objects of $\mathcal{G}$. Thus, once we prove that $\mathrm{Ker}\, p \in \mathrm{fp}(\mathcal{G})$, we infer that $\mathrm{Ker}\, f$ is finitely presented by extension-closure (see [48, Corollary 1.8]), applied on the short exact sequence $0 \to \operatorname{Ker} p \to \operatorname{Ker} f \to \operatorname{Ker} q \to 0$ provided by the snake lemma. Consider the approximation

$$0 \longrightarrow \operatorname{Ker}^{(\mathcal{E})}(p) \longrightarrow \operatorname{Ker} p \longrightarrow y(\operatorname{Ker} p) \longrightarrow 0$$

of the relevant kernel within $(\mathcal{E}, \mathcal{T})$, and let us prove that the third term is finitely presented in $\mathcal{G}$. We have the following pushout diagram:

$$\begin{array}{ccccc}
\operatorname{Ker}^{(\mathcal{E})}(p) & = & \operatorname{Ker}^{(\mathcal{E})}(p) & & \\
\downarrow & & \downarrow & & \\
\operatorname{Ker} p & \rightarrowtail & x(B) & \overset{p}{\twoheadrightarrow} & x(B') \\
\downarrow & \text{P.O.} & \downarrow & & \| \\
y(\operatorname{Ker} p) & \rightarrowtail & C & \twoheadrightarrow & x(B')
\end{array}$$

whose second column tells us that the pushout C is finitely presented as well. Eventually, given a direct system $(M_i)_{i \in I}$ of objects of $\mathcal{T}$, applying the functors

$$\varinjlim_{i \in I} \operatorname{Ext}^k_{\mathcal{G}}(-, M_i) \quad \text{and} \quad \operatorname{Ext}^k_{\mathcal{G}}(-, \varinjlim_{i \in I} M_i) \qquad (k \in \mathbb{N} \cup \{0\})$$

on the second exact row, thanks to hypothesis (ii), by the five lemma we get that $\operatorname{Hom}_{\mathcal{G}}(y(\operatorname{Ker} p), -)$ preserves direct limits of $\mathcal{T}$; that is, $y(\operatorname{Ker} p)$ is a finitely presented object of $\mathcal{T}$, hence of $\mathcal{G}$, as desired.

In order to conclude the proof, we now show that condition (b) implies that $\mathcal{G}$ is locally finitely presented and (c). For the first claim we will follow the pattern of the proof of [48, Lemma 2.5]. Let M be an arbitrary object of $\mathcal{G}$ and consider its approximation $0 \to x(M) \to M \to y(M) \to 0$ within $(\mathcal{E}, \mathcal{T})$. Since $\mathcal{T}$ is locally finitely presented by (i), there exists a direct system $(Q_i)_{i \in I}$ in $\operatorname{fp}(\mathcal{T})$ such that $y(M) = \varinjlim_{i \in I} Q_i$. We have the pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & x(M) & \longrightarrow & M_i & \longrightarrow & Q_i & \longrightarrow & 0 \\
 & & \| & & \downarrow & \text{P.B.} & \downarrow & & \\
0 & \longrightarrow & x(M) & \longrightarrow & M & \longrightarrow & y(M) & \longrightarrow & 0
\end{array}$$

and the M_i's form a direct system in $\mathcal{G}$ whose direct limit is M. Once we show that $M_i \in \operatorname{Gen}(\operatorname{fp}(\mathcal{G}))$ for all $i \in I$, then we conclude our first claim (see the proof of [48, Lemma 2.5]). Consider the extension $\xi_i : 0 \to x(M) \to M_i \to Q_i \to 0$ provided by the previous diagram. Since $\mathcal{E}$ is locally finitely presented, there exists a direct system $(P_\lambda)_{\lambda \in \Lambda} \subseteq \operatorname{fp}(\mathcal{E})$ such that $x(M) = \varinjlim_{\lambda \in \Lambda} P_\lambda$. By hypothesis (iii), we obtain

$$\xi_i \in \operatorname{Ext}^1_{\mathcal{G}}(Q_i, \varinjlim_{\lambda \in \Lambda} P_\lambda) \cong \varinjlim_{\lambda \in \Lambda} \operatorname{Ext}^1_{\mathcal{G}}(Q_i, P_\lambda),$$

i.e., by definition of Yoneda ext-group, there is an index $\gamma \in \Lambda$ such that ξ_i factors as the pushout diagram (see again [48] for details)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_\gamma & \longrightarrow & N_\gamma & \longrightarrow & Q_i & \longrightarrow & 0 \\ & & \downarrow & \text{P.O.} & \downarrow & & \| & & \\ 0 & \longrightarrow & x(M) & \longrightarrow & M_i & \longrightarrow & Q_i & \longrightarrow & 0 \end{array}$$

in which N_γ is a finitely presented object of $\mathcal{G}$ by [48, Corollary 1.8]. Moreover, it is

$$M_i = \varinjlim_{\lambda \geq \gamma} N_\lambda$$

so that our first claim is proved. Let us check that condition (iii)' holds true. Let $B \in \mathrm{fp}(\mathcal{G})$ and let us consider its approximation $0 \to x(B) \to B \to y(B) \to 0$ within $(\mathcal{E}, \mathcal{T})$. We only have to show that $x(B) \in \mathrm{fp}(\mathcal{E}) \subseteq \mathrm{fp}(\mathcal{G})$, since $y(B) \in \mathrm{fp}(\mathcal{T}) \subseteq \mathrm{fp}(\mathcal{G})$. The approximation yields the following long exact sequence of covariant functors:

$$0 \to \mathrm{Hom}_{\mathcal{G}}(y(B), -) \to \mathrm{Hom}_{\mathcal{G}}(B, -) \to \mathrm{Hom}_{\mathcal{G}}(x(B), -) - \cdots$$
$$\cdots \to \mathrm{Ext}^1_{\mathcal{G}}(y(B), -) \to \mathrm{Ext}^1_{\mathcal{G}}(B, -)$$

which, when restricted to $\mathcal{E}$, by hypothesis (iii), [48, Proposition 1.6] and the five lemma, gives that $x(B) \in \mathrm{fp}(\mathcal{E})$. □

3.2. Classification results

The classification of the localising subcategories of the module and the derived category of a commutative noetherian ring have been suitably generalised to arbitrary commutative ring, as we now recall.

3.2.1. Thomason subsets.

3.3 Definition. Let R be any commutative ring. A subset Z of $\mathrm{Spec}\, R$ is said to be *Thomason* if there exists a family $\mathcal{B}_Z$ of finitely generated ideals of R such that $Z = \bigcup_{J \in \mathcal{B}_Z} V(J)$.

In other words, Thomason subsets generalise specialisation closed subsets. There is a bijection, described precisely as in the previous chapter, between hereditary torsion pairs *of finite type* $\mathcal{T}$ of Mod-R, whence with Gabriel filters of finite type $\mathcal{J}$ of R, and Thomason subsets Z of $\mathrm{Spec}\, R$. More precisely, given a Thomason subset Z of $\mathrm{Spec}\, R$, we have associated a hereditary torsion class of finite type $\mathcal{T}_Z := \{M \in \text{Mod-}R \mid \mathrm{Supp}\, M \subseteq Z\}$ and a Gabriel filter of finite type $\mathcal{J}_Z := \{J \leq R \mid V(J) \subseteq Z\}$. By the following results, we can regard any Thomason subset Z as the union of Zariski closed indexed over the set $\mathcal{I}_Z$ of all finitely generated ideals in $\mathcal{J}_Z$.

3.4 Proposition. *Let Z, $\mathcal{T}_Z$, $\mathcal{J}_Z$ and $\mathcal{I}_Z$ be as above.*

(i) *$\mathcal{T}_Z$ is a Grothendieck category, and $\mathrm{fp}(\mathcal{T}_Z) = \mathcal{T}_Z \cap \text{mod-}R$;*
(ii) *$\mathcal{T}_Z = \mathrm{Gen}(R/J \mid J \in \mathcal{I}_Z)$.*

Proof.
(i) It is well-known that $\mathcal{T}_Z$ is a Grothendieck category (we deduce it in Prop. 3.19). Let us show the equality in the second part of the statement. The inclusion "$\supseteq$" is clear, while "$\subseteq$" follows by [48, Lemma 1.11] since $(\mathcal{T}_Z, \mathcal{F}_Z)$ is a torsion pair of finite type.
(ii) The inclusion "$\supseteq$" is clear from the properties of a torsion class. Conversely, since $\mathcal{T}_Z$ is a hereditary torsion class of Mod-R, every torsion object is the direct limit of a direct system of finitely generated objects of $\mathcal{T}_Z$, hence it suffices to show that each module M in such direct system is the homomorphic image of the direct sum of some R/J's, where each J is a finitely generated ideal in $\mathcal{J}_Z$. Since M is a finitely generated module, then $\operatorname{Supp} M = V(\operatorname{Ann}_R(M))$ (see e.g. [36, Exercise 23, p. 58]). Therefore, $V(\operatorname{Ann}_R(M)) \subseteq Z$, and since $\mathcal{J}_Z$ is a Gabriel filter of finite type, $\operatorname{Ann}_R(M)$ contains a finitely generated ideal J of the filter. This means $MJ = 0$ i.e. M is a R/J-module, in fact finitely generated over R/J as well, so that there exists an epimorphism $(R/J)^n \to M$ for some positive integer n. □

3.5 Corollary. *Let $Z = \bigcup_{J \in \mathcal{B}_Z} V(J)$ be any Thomason set and $\mathcal{J}_Z$ the associated Gabriel filter. Then $Z = \bigcup_{J \in \mathcal{I}_Z} V(J)$.*

Proof. The right-ward inclusion $Z \subseteq \bigcup_{J \in \mathcal{I}_Z} V(J)$ is clear since $\mathcal{B}_Z \subseteq \mathcal{I}_Z$. Conversely, let $\boldsymbol{p}$ be a prime ideal containing some finitely generated ideal J in $\mathcal{J}_Z$, and let us prove that $\boldsymbol{p}$ contains an ideal in $\mathcal{B}_Z$. The module $R/\boldsymbol{p}$ is torsion by Prop. 3.4, whence $\operatorname{Supp} R/\boldsymbol{p} = V(\boldsymbol{p}) \subseteq Z$, so we are done for $\boldsymbol{p} \in V(\boldsymbol{p})$. □

3.2.2. Thomason filtrations. Filtrations by supports are then naturally replaced by the so-called *Thomason filtrations*. We actually allow the following:

3.6 Definition. A *Thomason filtration* of Spec R is a decreasing map $\Phi\colon (\mathbb{Z}, \leq) \to (2^{\operatorname{Spec} R}, \subseteq)$ such that $\Phi(n)$ is a Thomason subset of Spec R for all $n \in \mathbb{Z}$.

A Thomason filtration Φ will be called:

- *weakly bounded below* if there is $l \in \mathbb{Z}$ such that $\Phi(l) = \Phi(l-i)$ for all $i \geq 0$; if $\Phi(l) = \operatorname{Spec} R$, then Φ is called *bounded below*;
- *bounded above* if there exists $r \in \mathbb{Z}$ such that $\Phi(r+1) = \emptyset$.

In these cases, we say that Φ is (*weakly*) *bounded below* l and *bounded above* r, respectively.

- A Thomason filtration Φ weakly bounded below and bounded above will be called *of finite length*.

Without loss of generality, we will always assume a Thomason filtration of finite length to be bounded above 0, so that if it is weakly bounded below $-l$, with $\Phi(-l+1) \neq \Phi(-l)$, we will say that it has *length* l.

Of course the bounded above and bounded below Thomason filtrations are intermediate. Contrarily to the previous one, in this chapter we will prominently deal with Thomason filtrations of finite lenght, i.e. with filtrations with a weaker condition than the intermediacy.

Thomason filtrations of Spec R classify the compactly generated t-structures of $D(R)$, but *not* precisely as described in the previous chapter. In [23, Lemma 3.7], Hrbek generalises Thm. 2.4 for *bounded below* compactly generated t-structures in $D(R)$: they are in bijection with the bounded below Thomason filtrations of Spec R, via the mutually inverse assignments

$$\begin{aligned}(\mathcal{U},\mathcal{V}) &\longmapsto \Phi_{\mathcal{U}}\colon \mathbb{Z} \to 2^{\operatorname{Spec} R}\\ &\qquad j \mapsto \bigcup_{\substack{M\in \text{Mod-}R\\ M[-j]\in\mathcal{U}}} \operatorname{Supp} M,\\ \Phi &\longmapsto (\mathcal{U}_\Phi,\mathcal{V}_\Phi)\\ &\qquad \mathcal{U}_\Phi := \{M\in D(R) \mid \operatorname{Supp} H^n(M)\subseteq \Phi(n),\ \forall n\in\mathbb{Z}\}\\ &\qquad\quad = \{M\in D(R) \mid H^n(M)\in \mathcal{T}_{\Phi(n)},\ \forall n\in\mathbb{Z}\}\ .\end{aligned}$$

Notice that the aisle of the t-structure associated with Φ has the same description as in Thm. 2.4, but this is not true for the coaisle; actually, so far for the latter no characterisation is available.

Another bijective correspondence, involving arbitrary compactly generated t-structures and Thomason filtrations, is shown in [23, Theorem 5.1]. However, in this case even the relevant aisle, still denoted by $\mathcal{U}_\Phi$, has a different description w.r.t. the previous ones, indeed we have

$$\mathcal{U}_\Phi = \operatorname{aisle}(K(J)[-n] \mid V(J)\subseteq \Phi(n),\ \forall n\in\mathbb{Z}),$$

where "K" denotes the Koszul complex (see Ex. 1.33 and Rem. 1.37). The crucial fact, proved by Hrbek, is that if we denote by $\mathcal{U}_\Phi^\#$ the class of complexes whose cohomologies are supported along Φ (i.e. copying the definition of the noetherian or bounded below cases) then still it an aisle of $D(R)$. However, it is not known whether this latter aisle gives rise to a compactly generated t-structure, nor if the two aisles $\mathcal{U}_\Phi$ and $\mathcal{U}_\Phi^\#$ coincide. Nevertheless, by [24, Lemma 3.6] they coincide in the bounded below derived category of the ring:

$$\mathcal{U}_\Phi \cap D^+(R) = \mathcal{U}_\Phi^\# \cap D^+(R)\ .$$

Consequently, the cohomologies of the complexes of $\mathcal{U}_\Phi$ are always supported on the Thomason subsets; moreover, for any module $X\in\mathcal{T}_{\Phi(j)}$, $j\in\mathbb{Z}$, by the previous equality we see that its stalk $X[-j]$ belongs to $\mathcal{U}_\Phi$.

In order to make clearer the distinction between the above two t-structures associated to a Thomason filtration Φ, we will call the t-structure $(\mathcal{U}_\Phi^\#, (\mathcal{U}_\Phi^\#)^{\perp_0}[1]) =: (\mathcal{U}_\Phi^\#, \mathcal{V}_\Phi^\#)$ and its heart $\mathcal{H}^\#$ the *Alonso-Jeremías–Saorín t-structure*, resp. *AJS heart*, induced by Φ.

3.7 LEMMA. *Let Φ be any Thomason filtration of* Spec R. *Then $\mathcal{U}_\Phi = \mathcal{U}_\Phi^\#$ if and only if the AJS t-structure $(\mathcal{U}_\Phi^\#, \mathcal{V}_\Phi^\#)$ is compactly generated in $D(R)$.*

Proof.
"$\Rightarrow$" There is nothing to prove since $\mathcal{U}_\Phi$ is the aisle of a compactly generated t-structure.

"$\Leftarrow$" By hypothesis, there exists a family $\mathcal{S} \subseteq D(R)^c$ of compact complexes such that $\mathcal{U}_\Phi^\# = \operatorname{aisle}\mathcal{S}$. By a well-known characterisation of $D(R)^c$ due to Rickard, the objects of $\mathcal{S}$ belong in particular to $D^+(R)$. In other words, we have $\mathcal{S} \subseteq \mathcal{U}_\Phi^\# \cap D^+(R)$, and this latter coincides with $\mathcal{U}_\Phi \cap D^+(R)$, thus we infer $\mathcal{U}_\Phi^\# \subseteq \mathcal{U}_\Phi$ by minimality in containing $\mathcal{S}$, and we are done. □

3.8 Remark.

(1) Henceforth, accordingly with our previous notation, when a Thomason filtration Φ is fixed we will denote the two t-structures it gives rise by $(\mathcal{U}, \mathcal{V})$ and $(\mathcal{U}^\#, \mathcal{V}^\#)$, i.e. omitting any subscript referring to Φ. The corresponding hearts will be denoted by $\mathcal{H}$ and $\mathcal{H}^\#$, respectively. Moreover, the Gabriel filter and the torsion pair associated with each Thomason subset $\Phi(n)$ will be denoted just $\mathcal{J}_n$ and by $(\mathcal{T}_n, \mathcal{F}_n)$, respectively; in turn, the adjunction of the relevant torsion radical and coradical to the inclusions in Mod-R will be denoted by

$$\mathcal{T}_n \underset{x_n}{\rightleftarrows} \text{Mod-}R \overset{y_n}{\rightleftarrows} \mathcal{F}_n \ .$$

(2) Albeit so far in the literature there is no evidence of that the aisles $\mathcal{U}$ and $\mathcal{U}^\#$ should coincide, in this thesis we will prove results suggesting somehow that this might be the case. The main one in this sense is Thm. 3.24, in which we will prove that when Φ is a weakly bounded below Thomason filtration, then $\mathcal{H} = \mathcal{H}^\#$.

(3) In particualar, the previous hearts are locally finitely generated Grothendieck categories. As mentioned at the beginning of the chapter, one first aim to pursue is to detect in the heart of a (weakly bounded below) Thomason filtration a TTF triple (of finite type); we will solve this problem in Thm. 3.29.

As claimed in the previous remark, our interest will be focused mostly on the weakly bounded below filtrations. Let us conclude this section with the following result, which has a vital role throughout the chapter.

3.9 Proposition. *Let Φ be a weakly bounded below k Thomason filtration of* Spec R.

(i) *The AJS heart $\mathcal{H}^\#$ is contained in $D^{\geq k}(R)$; in particular, when Φ has length l (≥ 0) we have $\mathcal{H} \subseteq D^{[-l,0]}(R)$.*

(ii) *$\mathcal{H}^\#$ is an exact abelian subcategory of $\mathcal{H}$ and an* AB-3 *abelian category. Moreover, the coproducts hence the direct limits of $\mathcal{H}^\#$ are computed as in $\mathcal{H}$.*

(iii) *$\mathcal{H}^\#$ is a locally finitely presented Grothendieck category.*

Proof.

(i) Let us prove that for every $M \in \mathcal{H}^\#$ we have $\tau^{\leq k-1}(M) = 0$. Notice that, by definition, $\mathcal{T}_n = \mathcal{T}_k$ for every $n \leq k$. Thus, $\tau^{\leq k-1}(M) \in \mathcal{U}^\#[1]$ since

$$H^j(\tau^{\leq k-1}(M)[-1]) = \begin{cases} H^{j-1}(M) \in \mathcal{T}_{j-1} = \mathcal{T}_k = \mathcal{T}_j & \text{if } j \leq k \\ 0 & \text{if } j > k. \end{cases}$$

Therefore, in the exact triangle $\tau^{\leq k-1}(M) \to M \to \tau^{>k-1}(M) \overset{+}{\to}$ the first edge is the zero morphism. By [41, Corollary 1.2.7] we obtain the decomposition $\tau^{>k-1}(M) \cong M \oplus \tau^{\leq k-1}(M)[1]$, thus our claim follows at once by additivity of the standard cohomology. Thus we have $\mathcal{H}^\# = (\mathcal{U}^\# \cap \mathcal{V}^\#) \cap D^{\geq k}(R)$ and this latter coincides with $\mathcal{H} \cap D^{\geq k}(R)$ by [24, Lemma 3.6]. The second part of the statement is a consequence of the previous part.

(ii) We recall that, by [44, Proposition 3.2], the heart of any t-structure of $D(R)$ is an AB-3 abelian category. The fact that the kernels and the cokernels of $\mathcal{H}^\#$ are the same as in $\mathcal{H}$ is clear by the construction of these objects (see subsec. 1.3.2). Let us prove that for any family $(M_i)_{i\in I}$ of objects of $\mathcal{H}^\#$ the coproduct $\bigoplus_{i\in I}^{(\mathcal{H})} M_i$ belongs to $\mathcal{H}^\#$, whence the thesis. By [24, Lemma 3.6] the coproduct belongs to $\mathcal{U}^\#$. On the other hand, since $(\mathcal{U}, \mathcal{V})$ is a compactly generated hence smashing t-structure of $D(R)$, we have $\bigoplus_{i\in I}^{(\mathcal{H})} M_i = \coprod_{i\in I} M_i$, and this latter object belongs to $\mathcal{V} \cap D^{\geq k}(R)$, hence to $\mathcal{V}^\#$ by [ibid.].

(iii) Since $\mathcal{H}^\#$ is an AB-3 category with direct limits computed as in $\mathcal{H}$, which is an AB-5 category, then in turn $\mathcal{H}^\#$ is an AB-5 abelian category. Recall now that, by [44, Lemma 3.1], for any t-structure $(\mathcal{X}, \mathcal{Y})$ with heart $\mathcal{C}$ the restriction $H_{\mathcal{C}}\restriction_{\mathcal{X}} : \mathcal{X} \to \mathcal{C}$ is left adjoint to the inclusion $\mathcal{C} \hookrightarrow \mathcal{X}$. We claim that the restriction $L := H_{\mathcal{H}^\#}\restriction_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^\#$ is left adjoint to the inclusion $\mathcal{H}^\# \hookrightarrow \mathcal{H}$. Indeed, for all $M \in \mathcal{H}$ and $M' \in \mathcal{H}^\#$ we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}^\#}(L(M), M') &= \mathrm{Hom}_{D(R)}(H_{\mathcal{H}^\#}\restriction_{\mathcal{U}^\#}(M), M') && (M \in \mathcal{H} \subseteq \mathcal{U} \subseteq \mathcal{U}^\#) \\ &\cong \mathrm{Hom}_{\mathcal{U}^\#}(M, M') && \text{(by adjunction)} \\ &= \mathrm{Hom}_{\mathcal{H}}(M, M') && (M, M' \in \mathcal{H}). \end{aligned}$$

This said, let us prove that for any generator U of $\mathcal{H}$, the object $L(U)$ is a generator of $\mathcal{H}^\#$. Let $M \in \mathcal{H}^\#$. Then there is a set α and an epimorphism $U^{(\alpha)} \to M$ in $\mathcal{H}$, whence applying the left adjoint functor L we obtain an epimorphism $L(U)^{(\alpha)} \to L(M)$. Now, by definition of L, we have $L(M) = \tau^{>}_{\mathcal{U}^\#}(M[-1])[1]$, and being $M[-1] \in (\mathcal{U}^\#)^{\perp_0}$ since $M \in \mathcal{H}^\#$, then actually $\tau^{>}_{\mathcal{U}^\#}(M[-1]) = M[-1]$, i.e. $L(M) = M$, and we are done.

The previous paragraph also shows that the right adjoint to L is a direct limit preserving functor, hence by [33, Lemma 2.5] $L(\mathcal{S})$ is a set of finitely presented generators for $\mathcal{H}^\#$ for any set $\mathcal{S}$ of finitely presented generators of $\mathcal{H}$. □

3.10 Example. We show that the HRS heart of a hereditary torsion pair of finite type of Mod-R can be realised as the heart of an intermediate Thomason filtration (i.e. bounded both below and above).

Let Z be a proper Thomason subset, and let $\mathcal{H}_{\boldsymbol{\tau}}$ the HRS heart of the torsion pair $\boldsymbol{\tau} := (\mathcal{T}_Z, \mathcal{F}_Z)$ corresponding to Z (see Ex. 1.35(2)). Consider now the Thomason filtration

$$\Phi : \mathrm{Spec}\, R \supset Z \supset \emptyset$$

(concentrated in degrees $\{-1, 0\}$) and let us prove that $\mathcal{H} = \mathcal{H}^\# = \mathcal{H}_{\boldsymbol{\tau}}$. The first equality is guaranteed by [23, Lemma 3.7].

Let us prove that $\mathcal{H} \subseteq \mathcal{H}_{\boldsymbol{\tau}}$. For every $M \in \mathcal{H}$ we have $H^0(M) \in \mathcal{T}_Z$, so it remains to verify that $H^{-1}(M) \in \mathcal{F}_Z$. This follows by Lem. 3.11 and Prop. 3.9.

Conversely, let us prove the inclusion $\mathcal{H}_{\boldsymbol{\tau}} \subseteq \mathcal{H}$ by showing that both the torsion and torsionfree classes $\mathcal{F}_Z[1]$ and $\mathcal{T}_Z[0]$ approximating $\mathcal{H}_{\boldsymbol{\tau}}$ are contained in $\mathcal{H}$, whence the conclusion by the extension-closure of the heart. The fact that $\mathcal{T}_Z[0] \subseteq \mathcal{H}$ is clear by definition of the t-structure $(\mathcal{U}, \mathcal{V})$.

On the other hand, let $Y \in \mathcal{F}_Z$. Since $\operatorname{Supp} H^{-1}(Y[1]) = \operatorname{Supp} Y$ is contained in the spectrum i.e. in $\Phi(-1)$, whereas $\operatorname{Supp} H^j(Y[1]) = \emptyset$ for all $j \neq -1$, we have $\mathcal{F}_Z[1] \subseteq \mathcal{U}$. Let now $M \in \mathcal{U} \subseteq D^{\leq 0}(R)$; out of the exact triangle $\tau^{\leq -1}(M) \to M \to H^0(M)[0] \xrightarrow{+}$ provided by the standard t-structure of $D(R)$, applying the cohomological functor $\operatorname{Hom}_{D(R)}(-, Y[0])$ we obtain the exact sequence

$$\operatorname{Hom}_{D(R)}(H^0(M)[0], Y[0]) \to \operatorname{Hom}_{D(R)}(M, Y[0]) \to \operatorname{Hom}_{D(R)}(\tau^{\leq -1}(M), Y[0])$$

whence we obtain the remaining inclusion $\mathcal{F}_Z[1] \subseteq \mathcal{U}^{\perp_0}[1]$, for the left hand term is zero by [61] and since $(\mathcal{T}_Z, \mathcal{F}_Z)$ is a torsion pair in Mod-R, and for the right hand term being clearly zero as well.

3.3. Some useful results

Let us introduce some further notions and useful results that will be crucial throughout the rest of the chapter.

3.11 Lemma [46, Lemma 4.2(3)]. *Let Φ be a Thomason filtration of* $\operatorname{Spec} R$ *and let M be a complex in the associated heart $\mathcal{H}$. If m is the least integer such that $H^m(M) \neq 0$, then*

$$H^m(M) \in \mathcal{T}_m \cap \mathcal{F}_{m+1} \cap \operatorname{Ker} \operatorname{Ext}^1_R(\mathcal{T}_{m+2}, -) .$$

Proof. By what we recalled before of Lem. 3.7, we only need to check that $H^m(M)$ belongs to the last two classes of the displayed intersection. By hypothesis, $H^m(M)[-m] \cong \tau^{\leq m}(M)$, hence for every $X \in \mathcal{T}_{m+1}$ we obtain

$$\begin{aligned} \operatorname{Hom}_R(X, H^m(M)) &\cong \operatorname{Hom}_{D(R)}(X[-m], H^m(M)[-m]) \\ &\cong \operatorname{Hom}_{D(R)}(X[-m], \tau^{\leq m}(M)) . \end{aligned}$$

The latter group is zero, indeed we have $X[-m] \in \mathcal{U}^{\#}[1] \cap D^+(R) = \mathcal{U}[1] \cap D^+(R) \subseteq \mathcal{U}[1]$, so that the covariant hom functor of the stalk complex applied on exact triangle $\tau^{>m}(M)[-1] \to \tau^{\leq m}(M) \to M \xrightarrow{+}$ yields the claimed vanishing by the axioms of t-structure. Therefore, the least nonzero cohomology of M is an object of $\mathcal{F}_{m+1}$.

On the other hand, by Verdier's thesis [61], for every $X \in \mathcal{T}_{m+2}$ we have

$$\operatorname{Ext}^1_R(X, H^m(M)) \cong \operatorname{Hom}_{D(R)}(X[-m], H^m(M)[-m+1]),$$

and the right-hand group is zero by the previous argument, i.e. by applying the hom functor of $X[-m] \in \mathcal{U}[2]$ on the rotation of the above triangle. □

Notice that, again by [24, Lemma 3.6], that the previous lemma also holds true for any objects M of $\mathcal{H}^{\#}$ because, within the notation used in the statement, we have $M \in \mathcal{H}^{\#} \cap D^{\geq m}(R)$.

Let us recall some basic facts concerning Koszul complexes and their cohomology (see e.g. [43, Chap. 8] and Ex. 1.33). For any finitely generated ideal J one has:

- $K(J) \in D^{[-n,0]}(R) \cap D(R)^c$, where $n = \operatorname{rank} J$;
- $H^0(K(J)) \cong R/J$;
- $JH^{-j}(K(J)) = 0$ or, equivalently, $\operatorname{Supp} H^{-j}(K(J)) \subseteq V(J)$, for all $j = 0, \dots, n$.

The last two items tell us that the Koszul cohomologies are torsion modules w.r.t. the torsion pair associated with $V(J)$.

The following result will be resumed in sec. 3.4 and stengthened in Thm 3.24.

3.12 Proposition. *Let Φ be a Thomason filtration of* Spec R. *For any $n \in \mathbb{Z}$ and for any ideal $J \in \mathcal{I}_n$, we have $H_{\mathcal{H}}(K(J)[-n]) = H_{\mathcal{H}^\#}(K(J)[-n])$.*

Proof. For all n and J as stated, by [24, Lemma 3.6] we have $K(J)[-n] \in \mathcal{U} \cap D^+(R) = \mathcal{U}^\# \cap D^+(R)$. On the other hand, we have $\tau^{\leq n-1}(K(J)[-n]) \in \mathcal{U}^\#[1] \cap D^+(R) = \mathcal{U}[1] \cap D^+(R) \subseteq \mathcal{U}[1]$, so out of the triangle $\tau^{\leq n-1}(K(J)[-n]) \to K(J)[-n] \to R/J[-n] \overset{+}{\to}$, provided by the basic properties of the Koszul complex, we infer that

$$H_{\mathcal{H}}(K(J)[-n]) \cong H_{\mathcal{H}}(R/J[-n])$$

and

$$H_{\mathcal{H}^\#}(K(J)[-n]) \cong H_{\mathcal{H}^\#}(R/J[-n]) \ .$$

We now anticipate an argument which will be formalised in sec. 3.4. There exists a (weakly bounded below) Thomason filtration Φ_n associated to Φ whose t-structure has its aisle $\mathcal{U}_n^\#$ and the heart $\mathcal{H}_n^\#$ contained, respectively, in $\mathcal{U}^\#$ and $\mathcal{H}^\#$ (see Lem. 3.23); moreover, $R/J[-n] \in \mathcal{U}_n^\#$, thus out of the triangle

$$\tau^{\leq}_{\mathcal{U}_n^\#}(R/J[-n-1])[1] \longrightarrow R/J[-n] \longrightarrow H_{\mathcal{H}_n^\#}(R/J[-n]) \overset{+}{\longrightarrow}$$

we infer that $H_{\mathcal{H}^\#}(R/J[-n]) = H_{\mathcal{H}_n^\#}(R/J[-n])$. On the other hand, the first vertex of the triangle belongs to $D^+(R)$ for the other two vertices do, therefore it also belongs to $\mathcal{U}[1]$ by the usual argument of [24]. Thus, the displayed triangle also shows that $H_{\mathcal{H}}(R/J[-n]) = H_{\mathcal{H}_n^\#}(R/J[-n])$, and we are done. □

3.13 Lemma. *Let Φ be any Thomason filtration of* Spec R, *let $\mathcal{C}$ be either $\mathcal{H}$ or the AJS heart $\mathcal{H}^\#$, and let $(M_i)_{i\in I}$ be a family of objects of the heart $\mathcal{C}$. Then, for all $n \in \mathbb{Z}$,*

$$\bigoplus_{i\in I} H^n(M_i) \cong H^n\Big(\bigoplus_{i\in I}^{(\mathcal{C})} M_i\Big) \ .$$

Moreover, if I is directed, then

$$\varinjlim_{i\in I} H^n(M_i) \cong H^n(\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i) \ .$$

Proof. Since each stalk complex of the ring R is a compact object of $D(R)$, we have

$$\begin{aligned} H^n\Big(\bigoplus_{i\in I}^{(\mathcal{C})} M_i\Big) &= \operatorname{Hom}_{D(R)}(R[-n], \bigoplus_{i\in I}^{(\mathcal{C})} M_i) \\ &\cong \bigoplus_{i\in I} \operatorname{Hom}_{D(R)}(R[-n], M_i) = \bigoplus_{i\in I} H^n(M_i) \ . \end{aligned}$$

Notice that this result in fact holds true for the heart of any smashing t-structure of $D(R)$.

Let now I be a directed set. By [56, Proposition 5.4], $R[-n]$ is a *homotopically finitely presented* object of $D(R)$, meaning that its covariant hom functor commutes with direct homotopy colimits; by [56, Corollary 5.8], the direct homotopy colimits of the heart $\mathcal{H}$ are canonically isomorphic to the underlying direct limits; consequently, we can repeat the proof above by replacing the coproducts of $(M_i)_{i\in I}$ with its direct limit, and we are done. □

Slightly diverting from [46], we fix the following notation: given any Thomason filtration Φ, for any $n \in \mathbb{Z}$ we set

$$\mathcal{TF}_n := \mathcal{T}_n \cap \mathcal{F}_{n+1}$$
$$\mathcal{TFT}_n := \mathcal{T}_n \cap \mathcal{F}_{n+1} \cap \operatorname{Ker}\operatorname{Ext}^1_R(\mathcal{T}_{n+2}, -) .$$

It is readily seen that $\mathcal{TF}_n$ is closed under subobjects and that $\mathcal{TFT}_n$ is closed under kernels; moreover, we will show in Rem. 3.35(1) that the latter category has direct limits, so it will make sense to consider the subcategory of its finitely presented objects (mostly in order to study its quasi local coherence), which will play a crucial role in the subsequent sections.

3.14 Corollary. *Let Φ be a Thomason filtration of* Spec R. *Then the class*

$$\mathcal{HTF}_n := \{M \in \text{Mod-}R \mid M[-n] \in \mathcal{H}\}$$

coincides with

$$\mathcal{HTF}^{\#}_n := \{M \in \text{Mod-}R \mid M[-n] \in \mathcal{H}^{\#}\} .$$

Moreover, $\mathcal{HTF}_n$ is a subcategory of Mod-R *closed under direct limits, for every $n \in \mathbb{Z}$.*

Proof. The equality between the two classes of modules follows since the corresponding stalk complexes belong to the relevant hearts and to $D^+(R)$. Let now $(M_i)_{i\in I} \in \mathcal{HTF}_n$ be a direct system, so that $(M_i[-n])_{i\in I}$ is a direct system of $\mathcal{H}$. By Lem. 3.13 we obtain

$$H^n(\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i[-n]) \cong \varinjlim_{i\in I} M_i,$$

while in any degree different from n the direct limit has no cohomology. Therefore,

$$\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i[-n] \cong \Big(\varinjlim_{i\in I} M_i\Big)[-n]$$

i.e. direct limits of $\mathcal{HTF}_n$ are computed precisely as in Mod-R. □

3.3.1. Bounded above Thomason filtrations. We continue giving useful results concerning the bounded above Thomason filtrations.

3.15 Lemma. *Let Φ be a Thomason filtration bounded above r. Then*

$$\mathcal{HTF}_{r-1} = \mathcal{TF}_{r-1} .$$

Proof. Notice that, by boundedness of Φ, we have $\mathcal{U} \subseteq \mathcal{U}^{\#} \subseteq D^{\leq r}(R)$.

This said, let $M \in \mathcal{HTF}_{r-1}$. Then $M = H^{r-1}(M[-r+1])$, hence by Lem. 3.11 we obtain $M \in \mathcal{T}_{r-1} \cap \mathcal{F}_r$.

Conversely, let us prove that the stalk concentrated in degree $r-1$ of a module $M \in \mathcal{T}_{r-1} \cap \mathcal{F}_r$ belongs to the heart associated with Φ. $M[-r+1]$ belongs to $\mathcal{U}^{\#} \cap D^+(R)$, hence it lands in $\mathcal{U}$. On the other hand, we see that $M[-r]$ belongs to the coaisle $(\mathcal{U}^{\#})^{\perp_0}$, since for every $U \in \mathcal{U}^{\#}$ the standard approximation $\tau^{\leq r-1}(U) \to U \to H^r(U)[-r] \xrightarrow{+}$ (provided by the boundedness of Φ) yields, by [61], the desired vanishing $\mathrm{Hom}_{D(R)}(U, M[-r]) = 0$. By the usual argument of [24], we infer that $M[-r+1] \in \mathcal{V}$. □

3.16 Remark. As we shall deduce by Prop. 3.19 (which does not depend on the forthcoming results; see however the comments after its proof), the torsion class corresponding to any nonempty Thomason subset is a locally finitely presented Grothendieck category. In particular, for a Thomason filtration as above, by Lem. 3.15 and [48, Corollary 5.3] we have

$$\mathrm{fp}(\mathcal{TF}_{r-1}) = \mathrm{add}\, y_r(\mathrm{fp}(\mathcal{T}_{r-1})) = \mathrm{add}\, y_r(\mathcal{T}_{r-1} \cap \text{mod-}R)\ .$$

3.17 Lemma. *Let Φ be a Thomason filtration bounded above r. Then:*

(i) *For every $J \in \mathcal{I}_r$, it is $H_{\mathcal{H}^{\#}}(K(J)[-r]) \cong R/J[-r]$;*
(ii) *For every $J \in \mathcal{I}_{r-1}$, it is $H_{\mathcal{H}^{\#}}(K(J)[-r+1]) \cong y_r(R/J)[-r+1]$.*

Proof. Notice that, by definition of the aisles $\mathcal{U}$ and $\mathcal{U}^{\#}$, and by Prop. 3.12, it makes no difference in working within the compactly generated t-structure or in the AJS one. For the sake of clearness, we will continue in working in the AJS t-structure.

(i) Let $J \in \mathcal{I}_r$. The complexes $K := K(J)[-r]$ and $M := H_{\mathcal{H}^{\#}}(K)$ fit as the vertexes of the triangle

$$U[1] \longrightarrow K \longrightarrow M \xrightarrow{+}$$

provided by the object $U := \tau^{\leq}_{\mathcal{U}^{\#}}(K[-1])$. We will prove that $H^r(M) \cong R/J$ and that $\tau^{\leq r-1}(M) = 0$. Fix $j \leq r-1$ and consider the exact sequence $H^j(K) \to H^j(M) \to H^{j+2}(U)$ in Mod-R. The Koszul cohomology $H^j(K)$ is an object of $\mathcal{T}_r$, hence of $\mathcal{T}_{j+1}$, so that $H^j(M) \in \mathcal{T}_{j+1}$ since in turn $\mathcal{T}_{j+1} \supseteq \mathcal{T}_{j+2}$. It follows $\tau^{\leq r-1}(M) \in \mathcal{U}^{\#}[1]$, and from the triangle

$$\tau^{\leq r-1}(M) \longrightarrow M \longrightarrow \tau^{>r-1}(M) \xrightarrow{+}$$

we deduce $\tau^{>r-1}(M) \cong M \oplus \tau^{\leq r-1}(M)[1]$ by [41, Corollary 1.2.7], whence $\tau^{\leq r-1}(M)[1] \in D^{\leq r-1}(R) \cap D^{\geq r}(R) = 0$. Now, the first displayed triangle yields the following exact sequence in Mod-R:

$$H^{r+1}(U) \longrightarrow H^r(K)(\cong R/J) \longrightarrow H^r(M) \longrightarrow H^{r+2}(U),$$

whence we obtain $H^r(M) \cong R/J$ since $\Phi(r+1) = \Phi(r+2) = \emptyset$ and $U \in \mathcal{U}^{\#}$.

(ii) Let $J \in \mathcal{I}_{r-1}$, $K := K(J)[-r+1]$ and $M := H_{\mathcal{H}^{\#}}(K[1])$. The thesis follows as in the previous part, namely by proving that $H^j(M) = 0$ for every $j \neq r-1$ and that $H^{r-1}(M) \cong y_r(R/J)$. To this aim, look at the long exact cohomology sequence arising from $U[1] \to K \to M \overset{+}{\to}$, in which $U := \tau^{\leq}_{\mathcal{U}^{\#}}(K[-1])$, and use Lem. 3.11 again. □

3.18 Proposition. *Let Φ be a Thomason filtration bounded above r. The following hold true for a module $X \in \text{Mod-}R$:*

(i) $X \in \text{fp}(\mathcal{T}_r)$ *if and only if* $X[-r] \in \text{fp}(\mathcal{H})$. *In particular,* $H^r(\text{fp}(\mathcal{H}))[-r] \subseteq \text{fp}(\mathcal{H})$*;*

(ii) $X \in \text{fp}(\mathcal{HTF}_{r-1})$ *if and only if* $X[-r+1] \in \text{fp}(\mathcal{H})$.

Proof. Notice that $T[-r] \in \mathcal{H} \cap \mathcal{H}^{\#}$ for all $T \in \mathcal{T}_r$ (the proof is similar to that in Ex. 3.10).

(i) Let X be a finitely presented object of $\mathcal{T}_r$ i.e. an object of $\text{fp}(\mathcal{T}_r) = \text{mod-}R \cap \mathcal{T}_r$. By (the proof of) Prop. 3.4 there exists in $\mathcal{T}_r \cap \text{mod-}R$ an exact row $(R/J')^n \overset{\alpha}{\to} (R/J)^m \to X \to 0$, which can be embedded in the following diagram in $D(R)$ by taking the stalk complexes:

$$\begin{array}{ccccccc}
(\operatorname{Ker}\alpha)[-r] & \longrightarrow & (R/J')^n[-r] & \longrightarrow & (\operatorname{Im}\alpha)[-r] & \overset{+}{\longrightarrow} & \\
 & & & & \downarrow & & \\
 & & & & (R/J)^m[-r] & & \\
 & & & & \downarrow & & \\
 & & & & X[-r] & & \\
 & & & & \downarrow{\scriptstyle +} & &
\end{array}$$

By Lem. 3.17(i) and [56, Lemma 6.3], for every $I \in \mathcal{I}_r$ the stalk $R/I[-r]$ is a finitely presented object of $\mathcal{H}$. Moreover, since the triangles of the diagram are in $\mathcal{H}$ (by what we noted at the beginning of the proof), then they actually are short exact sequences of $\mathcal{H}$, hence $X[-r] \cong \operatorname{Coker}^{(\mathcal{H})}(\alpha[-r])$ and it is finitely presented being the cokernel of a map between finitely presented complexes.

Conversely, let X be a module whose stalk $X[-r]$ is a finitely presented complex of $\mathcal{H}$. Then clearly $X \in \mathcal{T}_r$; moreover, for all direct systems of modules $(X_i)_{i\in I}$ in $\mathcal{T}_r$, so that $X_i[-r] \in \mathcal{H}$ for all $i \in I$, by [61] we deduce the natural isomorphism

$$\varinjlim_{i\in I} \operatorname{Hom}_R(X, X_i) \cong \operatorname{Hom}_R(X, \varinjlim_{i\in I} X_i),$$

whence $X \in \text{mod-}R$ since $(\mathcal{T}_r, \mathcal{F}_r)$ is a torsion pair of finite type (see [48, Lemma 2.4]).

The second part of the statement readily follows by the previous one, since out of the exact triangle $\tau^{\leq r-1}(B) \to B \to H^r(B)[-r] \overset{+}{\to}$ approximating a finitely presented complex B of the heart $\mathcal{H}$, by [61] we infer that $H^r(B)$ is a finitely presented object of $\mathcal{T}_r$.

(ii) If X is a module whose stalk $X[-r+1]$ is a finitely presented complex of the heart $\mathcal{H}$, then by definition of $\mathcal{HTF}_{r-1}$ and by Corollary 3.14, Lem. 3.15 and [61], for every direct system of modules $(M_i)_{i\in I}$ in $\mathcal{HTF}_{r-1}$ we obtain the following commutative diagram

$$\begin{array}{ccc} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{H}}(X[-r+1], M_i[-r+1]) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{H}}(X[-r+1], \varinjlim_{i\in I}^{(\mathcal{H})} M_i[-r+1]) \\ \cong\downarrow & & \downarrow\cong \\ \varinjlim_{i\in I} \mathrm{Hom}_R(X, M_i) & \longrightarrow & \mathrm{Hom}_R(X, \varinjlim_{i\in I} M_i) \end{array}$$

showing that X is a finitely presented object of $\mathcal{HTF}_{r-1}$.

Conversely, let X be a module in $\mathrm{fp}(\mathcal{TF}_{r-1}) = \mathrm{add}\, y_r(\mathrm{fp}(\mathcal{T}_{r-1}))$ (see Rem. 3.16), so that there exists a finitely presented object B of $\mathcal{T}_{r-1}$ such that $X \leq_\oplus y_r(B)^n$ for some $n \in \mathbb{N}$, hence we shall prove the statement on $y_r(B)^n$, in particular by showing that $y_r(B)[-r+1] \in \mathrm{fp}(\mathcal{H})$. By Prop. 3.4 there is an exact sequence $(R/J')^n \xrightarrow{a} (R/J)^m \to B \to 0$ in Mod-R for some positive integers m, n and ideals J', J in $\mathcal{I}_{r-1}$. By Lem. 3.17(ii) and Prop. 3.12, we have the exact row

$$H_{\mathcal{H}^\#}(K(J')[-r+1])^n \xrightarrow{y_r(a)[-r+1]} H_{\mathcal{H}^\#}(K(J)[-r+1])^m \longrightarrow \cdots$$
$$\cdots \to \mathrm{Coker}^{(\mathcal{H}^\#)}(y_r(a)[-r+1]) \to 0$$

in the AJS heart $\mathcal{H}^\#$, whose first two terms are finitely presented objects of $\mathcal{H}$, so $\mathrm{Coker}^{(\mathcal{H}^\#)}(y_r(a)[-r+1])$ turns out to be finitely presented of $\mathcal{H}$ once we prove that it belongs to $D^+(R)$. Let us prove that in fact such object is a stalk complex. To prove this, consider the canonical short exact sequences of $\mathcal{H}^\#$

$$0 \to \mathrm{Ker}^{(\mathcal{H}^\#)}(y_r(a)[-r+1]) \to H_{\mathcal{H}^\#}(K(J')[-r+1])^n \longrightarrow \cdots$$
$$\cdots \to \mathrm{Im}^{(\mathcal{H}^\#)}(y_r(a)[-r+1]) \to 0$$

and

$$0 \to \mathrm{Im}^{(\mathcal{H}^\#)}(y_r(a)[-r+1]) \to H_{\mathcal{H}^\#}(K(J)[-r+1])^m \longrightarrow \cdots$$
$$\cdots \to \mathrm{Coker}^{(\mathcal{H}^\#)}(y_r(a)[-r+1]) \to 0,$$

say them $0 \to K \to M' \to L \to 0$ and $0 \to L \to M \to N \to 0$ respectively. Since their middle terms are stalk complexes concentrated in degree $r-1$, they yield $H^r(L) = 0$ and $H^r(N) = 0$, respectively. On the other hand, from the second exact row, we have $H^{j-1}(N) \cong H^j(L) \in \mathcal{T}_j$ for all $j \leq r-2$, and $H^{r-2}(N)$ is a submodule of $H^{r-1}(L) \in \mathcal{T}_{r-1}$. Hence $\tau^{\leq r-2}(N) \in \mathcal{U}^\#[1]$, so that $N \cong \tau^{>r-2}(N) = \tau^{\geq r-1}(N) = H^{r-1}(N)[-r+1]$. Therefore, the very first displayed exact row $M' \to M \to N \to 0$ gives, by exactness,

$$N = \mathrm{Coker}^{(\mathcal{H}^\#)}(y_r(a)[-r+1]) \cong D[-r+1],$$

for some $D \in \mathcal{TF}_{r-1}$; whence $D[-r+1] \in \mathrm{fp}(\mathcal{H})$. Once we prove that $y_r(B) \cong D$, then we get the thesis. By the long exact sequence in cohomology of the previous two short exact sequences, we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & y_r(R/J')^n & \longrightarrow & y_r(R/J)^m & \longrightarrow & \operatorname{Coker} y_r(\alpha) & \longrightarrow & 0 \\
 & & {\scriptstyle\delta}\downarrow & \searrow\!\!\!\!\rightarrow \operatorname{Im}\delta \rightarrowtail \nearrow & \| & & \downarrow{\scriptstyle q} & & \\
0 & \longrightarrow & H^{r-1}(L) & \longrightarrow & y_r(R/J)^m & \xrightarrow{p} & D & \longrightarrow & 0
\end{array}$$

where $\operatorname{Coker}\delta = H^r(K) \in \mathcal{T}_r$ and p is an epimorphism since $H^r(L) = 0$. We deduce that $D \cong y_r(\operatorname{Coker} y_r(\alpha))$; set now $C := \operatorname{Coker} y_r(\alpha)$. On the other hand, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & x_r(R/J')^n & \longrightarrow & (R/J')^n & \longrightarrow & y_r(R/J')^n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle x_r(\alpha)} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle y_r(\alpha)} & & \\
0 & \longrightarrow & x_r(R/J)^m & \longrightarrow & (R/J)^m & \longrightarrow & y_r(R/J)^m & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{Coker} x_r(\alpha) & \longrightarrow & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

The short exact sequence $0 \to A \to B \xrightarrow{g} C \to 0$ provided by the factorisation of the morphism $\operatorname{Coker} x_r(\alpha) \to B$ through its image A yields that this latter is an object of $\mathcal{T}_r$. Consequently, we deduce $D \cong y_r(B)$ by the snake lemma applied on the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & x_r(B) & \longrightarrow & B & \longrightarrow & y_r(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle g} & & \downarrow & & \\
0 & \longrightarrow & x_r(C) & \longrightarrow & C & \longrightarrow & y_r(C) & \longrightarrow & 0
\end{array}$$

and this concludes the proof. □

3.3.2. Lenght zero Thomason filtrations. We conclude the section by fixing a proper Thomason subset Z of $\operatorname{Spec} R$ and studying the heart $\mathcal{H}_Z^\#$ given by the weakly bounded Thomason filtration

$$\Phi : \cdots = Z = Z = \cdots = Z \supset \emptyset\,.$$

3.19 PROPOSITION. *Let Φ be as above. Then the heart $\mathcal{H}_Z^\#$ is equivalent to $\mathcal{T}_Z$.*

Proof. We have $\mathcal{H}_Z^\# \subseteq \mathcal{T}_Z[0]$ by Prop. 3.9; on the other hand, by a similar argument to that of Ex. 3.10, we obtain $\mathcal{T}_Z[0] \subseteq \mathcal{H}_Z^\#$. □

Consequently, we get that for any Thomason subset $Z \neq \emptyset$, its torsion class $\mathcal{T}_Z$ is a locally finitely presented Grothendieck category by Prop. 3.9 (in fact, by Thm 3.24, $\mathcal{H}_Z^\#$ coincides with the heart $\mathcal{H}_Z$ of the compactly generated t-structure associated with Z). Now, the following result completely characterises the local coherence of $\mathcal{H}_Z^\#$; that is, by [18, Theorem 2.2], that of any hereditary torsion class of finite type in Mod-R.

3.20 Theorem. *Let Z be a nonempty Thomason subset. The following statements are equivalent:*

(a) *The torsion class $\mathcal{T}_Z$ is a locally coherent Grothendieck category; that is, $\mathcal{T}_Z \cap \text{mod-}R$ is an exact abelian subcategory of $\mathcal{T}_Z$.*
(b) *$(J:\gamma)$ is a finitely generated ideal for every $J \in \mathcal{I}_Z$ and for all $\gamma \in R$;*
(c) *R/J is a coherent commutative ring for every $J \in \mathcal{I}_Z$.*

Proof. Let us recall that $\mathcal{I}_Z$ is the family of finitely generated ideals in the Gabriel filter associated with the Thomason subset Z.
"(a) $\Rightarrow$ (b)" For every $J \in \mathcal{I}_Z$ and for all $\gamma \in R$, the ideal $J+\gamma R$ is in $\mathcal{I}_Z$ hence $R/(J+\gamma R)$ is a finitely presented (torsion) module (see Prop. 3.4). In turn, $(J+\gamma R)/J \cong \gamma R/(J \cap \gamma R)$ is so, being the kernel of the epimorphism $R/J \to R/(J+\gamma R)$ in $\mathcal{T}_Z \cap \text{mod-}R$. The conclusion follows from the short exact sequence $0 \to (J:\gamma) \to R \to \gamma R/(J\cap \gamma R) \to 0$.
"(b) $\Rightarrow$ (a)" Let $f\colon M \to M'$ be a R-linear map in $\mathcal{T}_Z \cap \text{mod-}R$. By the well-known closure properties of this latter class of modules, we only need to verify that $\operatorname{Ker} f$ is a finitely presented module, and clearly it suffices to consider f as an epimorphism. Furthermore, from the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & (R/J)^n & \longrightarrow & M' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & \operatorname{Ker} f & \longrightarrow & M & \xrightarrow{f} & M' & \longrightarrow & 0
\end{array}$$

in which the epimorphism α is provided by (the proof of) Prop. 3.4(ii), we argue that a "backward" argument on the extension-closure of the finitely presented modules shows that our claim is equivalent to requiring that $\operatorname{Ker}\alpha$ is finitely presented. Indeed, we have the following exact diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker}(\alpha\circ\mu) & \longrightarrow & (R/J)^{n-1} & \longrightarrow & \operatorname{Im}(\alpha\circ\mu) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}\alpha & \longrightarrow & (R/J)^n & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \longrightarrow & R/J & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

where μ is the canonical split monomorphism and the third exact row is given by the snake lemma, so that $\operatorname{Ker}\alpha$ is finitely presented if C and $\operatorname{Ker}(\alpha\circ\mu)$ are so. Now, once we prove that C is finitely presented, we can repeat the previous argument for each $n \geq k \geq 2$, achieving the validity at the base $k=2$. In other words, $\operatorname{Ker}\alpha$ is finitely presented iff C is finitely presented. Let us prove that C is a finitely presented module. It is finitely generated for $\operatorname{Ker}\alpha$ being so.

Consider now the pullback diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & J' & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & \text{P.B.} & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R/J & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & R/J' & = & R/J' & & \end{array}$$

in which J' is a finitely generated ideal by extension closure, so that an element of $\mathcal{I}_Z$. Let us prove the claim by induction on the rank of J'. If $J' = \gamma_1 R$, then it is $J' = J + \gamma_1 R$, whence $C \cong J'/J \cong \gamma_1 R/(J \cap \gamma_1 R)$. We conclude by hypothesis (b) applied on the short exact sequence $0 \to (J : \gamma_1) \to R \to C \to 0$. Notice that we just proved that for any ideal $J \in \mathcal{I}_Z$ and any $\gamma \in R$, then $(J + \gamma R)/J$ is finitely presented. This said, assume $J' = \gamma_1 R + \gamma_2 R$. Then again $J' = J + \gamma_1 R + \gamma_2 R$, with $J + \gamma_1 R$ being in $\mathcal{I}_Z$, and from the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & J + \gamma_1 R & \longrightarrow & (J + \gamma_1 R)/J & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J & \longrightarrow & J' & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & J'/(J + \gamma_1 R) & = & J'/(J + \gamma_1 R) & & \end{array}$$

we obtain, by the inductive base, that $(J + \gamma_1 R)/J$ and $J'/(J + \gamma_1 R)$ are finitely presented, hence C is so by extension-closure. This argument clearly applies at every finite rank of J', so C is finitely presented.
"(b) $\Rightarrow$ (c)" Let J'/J be a finitely generated ideal of R/J (so that J'/J is a finitely generated module over R) and let us prove that it is finitely presented. J' is in $\mathcal{G}_Z$, and by the short exact sequence $0 \to J'/J \to R/J \to R/J' \to 0$ in Mod-R we deduce that R/J' is a finitely presented R-module. By the hypothesis "(b) $\Leftrightarrow$ (a)" we get that J'/J is finitely presented over R, hence over R/J.
"(c) $\Rightarrow$ (b)" Assume that R/J is a coherent ring for each $J \in \mathcal{I}_Z$, and let $\gamma \in R$. By the short exact sequence $0 \to (J : \gamma) \to R \to \gamma R/(J \cap \gamma R) \to 0$ we shall prove that $\gamma R/(J \cap \gamma R) \cong (J + \gamma R)/J$ is a finitely presented R-module. $(J + \gamma R)/J$ is a finitely generated hence a finitely presented ideal of R/J, so there is a presentation $0 \to K \to (R/J)^n \to (J + \gamma R)/J \to 0$ with $n \in \mathbb{N}$ and K a finitely generated R/J-module. Since the scalar restriction functor R/J-Mod $\to$ Mod-R is exact, and since K is also a finitely generated R-module, such presentation lifts to Mod-R so that $(J + \gamma R)/J$ is finitely presented, as desired. □

3.21 Corollary. *Let R be a coherent commutative ring and Z be a Thomason subset. Then $\mathcal{T}_Z$ is a locally coherent Grothendieck category.*

Proof. It follows by the previous theorem, since any factor ring R/J is coherent for every finitely generated ideal J (see [36, (c) p. 143]). □

For the sake of the reader, let us exhibit a Thomason subset for which one of the equivalent conditions of Thm. 3.20 is not satisfied. This example will be resumed in Chap. 4.

3.22 EXAMPLE. In [9, Appendix A] Bravo and Parra consider the ring $R := \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$, whose sum is componentwise and its multiplication is defined by

$$(m, a) \cdot (n, b) := (mn, mb + na + ab),$$

where $ma := (ma_1, ma_2, \ldots)$ and $ab := (a_1b_1, a_2b_2, \ldots)$. In [9, Lemma A.1] it is proved that R is a commutative non-coherent ring, namely for the ideal generated by any $(2m, a)$ is finitely generated and not finitely presented. This fact entails at once a (somehow trivial) example of a Thomason subset whose torsion class is not a locally coherent Grothendieck category, namely $\operatorname{Spec} R$ itself, since the resulting torsion class is Mod-R.

Nonetheless, let us show that, over R as above, there are proper Thomason subsets of $\operatorname{Spec} R$ and finitely generated ideals which do not satisfy Thm 3.20(b). For instance, consider

$$J := (0, e_1)R, \quad Z := V(J), \quad \text{and} \quad \gamma := (2, e_2) \in R,$$

where e_n is the standard basis vector of $(\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$, so that $J \in \mathcal{I}_Z$. We compute:

$$\begin{aligned} J &= \{(0, e_1) \cdot (m, a) \mid (m, a) \in R\} \\ &= \{(0, e_1(m + a_1)) \mid (m, a) \in R\} \end{aligned}$$

and

$$\begin{aligned} (J : \gamma) &= \{(m, a) \in R \mid (2, e_2)(m, a) \in J\} \\ &= \{(m, a) \in R \mid (2m, e_2m + e_2a) \in J\} \\ &= \{(m, a) \in R \mid (2m, e_2(m + a_2)) \in J\} \\ &= \operatorname{Ann}_R(\gamma)\ . \end{aligned}$$

Now, out of the presentation $0 \to \operatorname{Ann}_R(\gamma) \to R \to \gamma R \to 0$, since γR is not finitely presented ([9, Lemma A.1]), then $(J : \gamma)$ is not finitely generated, as claimed.

3.4. Arbitrary Thomason filtrations

Let Φ be any Thomason filtration of $\operatorname{Spec} R$, let $(\mathcal{U}, \mathcal{V})$ be the corresponding compactly generated t-structure with heart $\mathcal{H}$, and let $(\mathcal{U}^{\#}, \mathcal{V}^{\#})$ be the induced AJS t-structure, with heart $\mathcal{H}^{\#}$. For any $k \in \mathbb{Z}$ we define

$$\Phi_k(n) := \begin{cases} \Phi(k) & \text{if } n < k \\ \Phi(n) & \text{if } n \geq k. \end{cases}$$

Thus, Φ_k is a weakly bounded below k Thomason filtration, naturally associated with Φ. We will denote by $(\mathcal{U}_k, \mathcal{V}_k)$ and $\mathcal{H}_k$, respectively, the corresponding compactly generated t-structure and the heart, and by $(\mathcal{U}_k^{\#}, \mathcal{V}_k^{\#})$ and $\mathcal{H}_k^{\#}$ the induced AJS t-structure and heart. It is clear that at each degree in which Φ_k and

Φ have the same Thomason subsets, namely for all $n \geq k$, their corresponding torsion pairs coincide as well; in this case we will denote these latter just as $(\mathcal{T}_n, \mathcal{F}_n)$, i.e. as those associated with $\Phi(n)$. Throughout this section, k will denote an arbitrary, but fixed, integer.

3.23 LEMMA. *Let Φ be a Thomason filtration of* Spec R. *Then $\mathcal{H}_k^{\#} \subseteq \mathcal{H}^{\#}$.*

Proof. Given $M \in \mathcal{H}_k^{\#}$, then clearly $M \in \mathcal{U}^{\#}$ so it remains to prove that $M[-1] \in (\mathcal{U}^{\#})^{\perp_0}$. This follows immediately by applying the functor $\mathrm{Hom}_{D(R)}(-, M[-1])$ on the approximation $\tau^{\leq k-1}(U) \to U \to \tau^{>k-1}(U) \xrightarrow{+}$ of an arbitrary object $U \in \mathcal{U}^{\#}$ within the shifted standard t-structure of $D(R)$, bearing in mind that $\mathcal{H}_k^{\#} \subseteq D^{\geq k}(R)$, by Prop. 3.9, and that $\tau^{>k-1}(U) \in \mathcal{U}_k^{\#}$. □

The following is the second main result of the chapter.

3.24 THEOREM. *Let Φ be a Thomason filtration of* Spec R. *The following hold:*

(i) *For any compact object S either in $\mathcal{U}$ or $\mathcal{U}^{\#}$, we have $H_{\mathcal{H}}(S) = H_{\mathcal{H}^{\#}}(S)$;*
(ii) *If Φ is weakly bounded below, then the heart $\mathcal{H}$ of the compactly generated t-structure associated with Φ coincides with the corresponding AJS heart $\mathcal{H}^{\#}$.*

Proof. Recall that we have $D(R)^c \cap \mathcal{U} = D(R)^c \cap \mathcal{U}^{\#}$.
(i) Let $S \in D(R)^c \cap \mathcal{U}$. Then there is an integer k such that $S \in \mathcal{U}_k^{\#}$. Consequently, by the triangle

$$\tau^{\leq}_{\mathcal{U}_k^{\#}}(S[-1])[1] \longrightarrow S \longrightarrow H_{\mathcal{H}_k^{\#}}(S) \xrightarrow{+}$$

we see that the first vertex belongs to $D^+(R)$ for the other two do so (the third vertex belongs to $D^+(R)$ by Prop. 3.9), moreover we deduce

$$H_{\mathcal{H}}(S) = H_{\mathcal{H}}(H_{\mathcal{H}_k^{\#}}(S))$$

and

$$H_{\mathcal{H}^{\#}}(S) = H_{\mathcal{H}^{\#}}(H_{\mathcal{H}_k^{\#}}(S)),$$

since $\mathcal{U}_k^{\#} \cap D^+(R) \subseteq \mathcal{U}^{\#} \cap D^+(R) = \mathcal{U} \cap D^+(R)$. By Lem. 3.23, the object of the previous display coincides with $H_{\mathcal{H}_k^{\#}}(S)$, which is a complex of $\mathcal{H}^{\#} \cap D^+(R)$, i.e. of $\mathcal{H} \cap D^+(R)$. Consequently, the objects in the previous displays coincide, as desired.
(ii) Recall that, by [55, Theorem 8.31], $\mathcal{S} := \mathrm{add}\, H_{\mathcal{H}}(D(R)^c \cap \mathcal{U})$ is a set of (finitely presented) generators for $\mathcal{H}$, and by the previous part we obtain the equality

$$\mathrm{add}\, H_{\mathcal{H}}(D(R)^c \cap \mathcal{U}) = \mathrm{add}\, H_{\mathcal{H}^{\#}}(D(R)^c \cap \mathcal{U}^{\#})\ .$$

Thanks to the hypothesis on Φ we have, by Prop. 3.9, that $\mathcal{H}^{\#} \subseteq \mathcal{H}$, whereas by the proof of the proposition we see that $H_{\mathcal{H}^{\#}}(\mathcal{S})$ is a set of generators for $\mathcal{H}^{\#}$. Therefore, we infer $\mathcal{H} = \mathcal{H}^{\#}$ since $H_{\mathcal{H}^{\#}}(\mathcal{S}) = \mathcal{S}$, $\mathcal{H}^{\#}$ is an exact abelian subcategory of $\mathcal{H}$, and the coproducts in $\mathcal{H}^{\#}$ are computed as in $\mathcal{H}$. □

3.25 COROLLARY. *For any Thomason filtration Φ of* Spec R, *we have $\mathcal{H}_k^{\#} = \mathcal{H}_k$ and $\mathcal{H}_k \subseteq \mathcal{H}$.*

Proof. The stated identity among the two hearts is ensured by the previous theorem. In order to prove the stated inclusion, we will imitate the proof of Lem. 3.23. Let $M \in \mathcal{H}_k$, then $M \in \mathcal{U}_k \subseteq \mathcal{U}$, since the compact objects generating $\mathcal{U}_k$ belong to $\mathcal{U}$. Let us prove that $M[-1] \in \mathcal{U}^{\perp_0}$. This follows immediately by applying the functor $\mathrm{Hom}_{D(R)}(-, M[-1])$ on the approximation $\tau^{\leq k-1}(U) \to U \to \tau^{>k-1}(U) \xrightarrow{+}$ of an arbitrary object $U \in \mathcal{U}$ within the shifted standard t-structure of $D(R)$, bearing in mind that $\mathcal{H}_k = \mathcal{H}_k^{\#} \subseteq D^{\geq k}(R)$, by Prop. 3.9, and that $\tau^{>k-1}(U) \in \mathcal{U}_k^{\#} \cap D^{+}(R) \subseteq \mathcal{U}_k$. □

3.26 Corollary. *If Φ is a Thomason filtration of finite length, then the statements* (i) *and* (ii) *of Proposition 3.18 hold true even replacing $\mathcal{H}$ by $\mathcal{H}^{\#}$.*

3.27 Corollary. *Let Φ be a Thomason filtration. For every $M \in \mathcal{H}^{\#}$ the following assertions hold:*

(i) *there exists in $\mathcal{H}^{\#}$ a short exact sequence $0 \to A \to M \to B \to 0$ with $A \in {}^{\perp_0}(\mathcal{H}_k^{\#})$ and $B \in \mathcal{H}_k^{\#}$ (the orthogonal being computed w.r.t. $\mathcal{H}^{\#}$);*

(ii) *there exists in $\mathcal{H}^{\#}$ a short exact sequence $0 \to A \to M \to B \to 0$ with $A \in \mathcal{H}_k^{\#}$ and $B \in (\mathcal{H}_k^{\#})^{\perp_0}$ (the orthogonal being computed w.r.t. $\mathcal{H}^{\#}$).*

Proof.

(i) Let $M \in \mathcal{H}^{\#}$, and consider the octahedron:

$$\begin{array}{ccccccc}
\tau^{\leq k-1}(M) & \longrightarrow & A & \longrightarrow & U[1] & \xrightarrow{+} & \\
\| & & \downarrow & & \downarrow & & \\
\tau^{\leq k-1}(M) & \longrightarrow & M & \longrightarrow & \tau^{>k-1}(M) & \xrightarrow{+} & \\
 & & \downarrow & & \downarrow & & \\
 & & H_{\mathcal{H}_k^{\#}}(\tau^{>k-1}(M)) & = & H_{\mathcal{H}_k^{\#}}(\tau^{>k-1}(M)) & & \\
 & & {\scriptstyle +}\downarrow & & \downarrow{\scriptstyle +} & &
\end{array}$$

provided by $U := \tau^{\leq}_{\mathcal{U}_k^{\#}}(\tau^{>k-1}(M)[-1])$ and a cone A (notice that $\tau^{>k-1}(M) \in \mathcal{U}_k^{\#}$). Since $B := H_{\mathcal{H}_k^{\#}}(\tau^{>k-1}(M))$ actually is in $\mathcal{H}_k^{\#}$, hence in $\mathcal{H}^{\#}$ by Lem. 3.23, we only have to check that A belongs to $\mathcal{H}^{\#}$ and that it is left orthogonal to $\mathcal{H}_k^{\#}$ in $\mathcal{H}^{\#}$. From the first vertical triangle we see that $A \in \mathcal{V}^{\#}$, whereas by the first horizontal one we deduce that $A \in \mathcal{U}^{\#}$. Moreover, using once again the first horizontal triangle, we infer that $A \in {}^{\perp_0}(\mathcal{H}_k^{\#})$ since $\mathcal{H}_k^{\#} \subseteq D^{\geq k}(R)$, as desired. Thus, the first vertical triangle yields the stated short exact sequence of $\mathcal{H}^{\#}$.

(ii) Consider the approximation $A \to M \to B \xrightarrow{+}$ of M within the t-structure $(\mathcal{U}_k^{\#}, \mathcal{V}_k^{\#})$, thus surely B is right orthogonal to $\mathcal{H}_k^{\#}$ in $\mathcal{H}^{\#}$. It remains to check that $A \in \mathcal{V}_k^{\#}$ and that $B \in \mathcal{H}^{\#}$. The first claim holds true by extension-closure of the coaisle applied on the rotated triangle $B[-2] \to A[-1] \to M[-1] \xrightarrow{+}$, and since $\mathcal{U}_k^{\#} \subseteq \mathcal{U}^{\#}$. On the other hand, B belongs to the aisle $\mathcal{U}^{\#}$ in view of the rotated triangle $M \to B \to A[1] \xrightarrow{+}$, while for every $U \in \mathcal{U}^{\#}$, by the approximation

$$\tau^{\leq k-1}(U) \longrightarrow U \longrightarrow \tau^{>k-1}(U) \xrightarrow{+},$$

we have $\tau^{>k-1}(U) \in \mathcal{U}_k^{\#}$, whence $\mathrm{Hom}_{D(R)}(\tau^{>k-1}(U), B[-1]) = 0$. Therefore, once we show that $\mathrm{Hom}_{D(R)}(\tau^{\leq k-1}(U), B[-1]) = 0$, we conclude the proof. Our claim follows at once by applying the covariant hom functor of $\tau^{\leq k-1}(U)$ on the triangle $M[-1] \to B[-1] \to A \xrightarrow{+}$, bearing in mind that $A \in \mathcal{H}_k^{\#} \subseteq D^{\geq k}(R)$. □

3.28 Corollary. *Let Φ be a Thomason filtration. Then the heart $\mathcal{H}_k^{\#}$ is closed in $\mathcal{H}^{\#}$ under taking products and coproducts.*

Proof. Let $(M_i)_{i\in I}$ be a family of objects of $\mathcal{H}_k^{\#}$ with product $(\prod_{i\in I} M_i, (\pi_i)_{i\in I})$ in $\mathcal{H}_k^{\#}$. We have to prove that such pair satisfies the universal property of the product in $\mathcal{H}^{\#}$. So, let $M \in \mathcal{H}^{\#}$ and $(f_i)_{i\in I}$ be a family of morphisms $f_i\colon M \to M_i$ in $\mathcal{H}^{\#}$. By Cor. 3.27(i) we obtain the following commutative diagram,

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & B & \xrightarrow{+} & \\
\downarrow & & \downarrow{\scriptstyle f_i} & & \vdots{\scriptstyle g_i} & & \\
0 & \longrightarrow & M_i & = & M_i & \xrightarrow{+} &
\end{array}$$

hence a family of morphisms $g_i\colon B \to M_i$ in $\mathcal{H}_k^{\#}$ inducing a unique morphism $g\colon B \to \prod_{i\in I} M_i$ such that $\pi_i \circ g = g_i$ for all $i \in I$. The composition $g \circ \beta$ yields the existence of a morphism $M \to \prod_{i\in I} M_i$ in $\mathcal{H}^{\#}$ such that $\pi_i \circ (g \circ \beta) = f_i$ for all $i \in I$. Uniqueness of $g \circ \beta$ w.r.t. the latter property is a byproduct of the construction of the triangle made in Cor. 3.27, namely for both A and B are uniquely determined up to isomorphism, together with the fact that β is an epimorphism in $\mathcal{H}^{\#}$.

The proof concerning the coproduct is dual. □

3.29 Theorem. *For any Thomason filtration Φ of* Spec R *and for any integer k, the following assertions hold true:*

- (i) *the heart $\mathcal{H}_k^{\#}$ ($= \mathcal{H}_k$, see Thm. 3.24) is a TTF class in $\mathcal{H}^{\#}$;*
- (ii) *If $\mathcal{H} = \mathcal{H}^{\#}$ (e.g. when R is noetherian or Φ is weakly bounded below, see Thm. 3.24), then $\mathcal{H}_k^{\#}$ is a TTF class of finite type in $\mathcal{H}^{\#}$.*

Proof.

(i) In order to prove that $\mathcal{H}_k^{\#}$ is a TTF class in $\mathcal{H}^{\#}$, by Cor. 3.28 we only have to show that the former heart is closed under subobjects, quotient objects and extensions. The closure under extensions is obvious since the relevant aisle and coaisle fulfil it. So, let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathcal{H}^{\#}$ with $M \in \mathcal{H}_k^{\#}$. Clearly, L and N belong to $\mathcal{V}_k^{\#}$. By Prop. 3.9, applied on the Thomason filtration Φ_k, we deduce $H^j(M) = 0$ for all $j < k-1$. Thus we have $H^j(N) \cong H^{j+1}(L) \in \mathcal{T}_{j+1}$ $(*)$ for all $j < k-2$, and that $H^{k-1}(N)$ is a submodule of $H^k(L)$, i.e. it belongs to $\mathcal{T}_k$. Moreover, it follows $\tau^{\leq k-1}(N) \in \mathcal{U}^{\#}[1]$ and consequently, by the usual argument of the proof of

Prop. 3.9, that $\tau^{\leq k-1}(N)=0$. By $(*)$, we infer $\tau^{\leq k-1}(L)=0$ as well. Therefore, $N, L\in\mathcal{H}^{\#}\cap D^{\geq k}(R)\subseteq\mathcal{U}^{\#}\cap D^{\geq k}(R)\subseteq\mathcal{U}_k^{\#}$, and this concludes the proof (as stated, the equality $\mathcal{H}_k^{\#}=\mathcal{H}_k$ follows by Thm 3.24).

(ii) Suppose that $\mathcal{H}^{\#}=\mathcal{H}$. We have to prove that the class $(\mathcal{H}_k^{\#})^{\perp_0}$, right orthogonal of $\mathcal{H}_k^{\#}$ in $\mathcal{H}^{\#}$ $(=\mathcal{H})$, is closed under taking direct limits of the latter heart. We claim that

$$\mathcal{H}_k^{\perp_0}=\mathcal{U}_k^{\perp_0}\cap\mathcal{H},$$

where the orthogonal of $\mathcal{U}_k$ is computed in $D(R)$, so that the inclusion "$\supseteq$" is clear. Let $M\in\mathcal{H}_k^{\perp_0}$ and apply the functor $\mathrm{Hom}_{D(R)}(-,M)$ on the exact triangle $\tau_{\mathcal{U}_k}^{\leq}(U[-1])[1]\to U\to H_{\mathcal{H}_k}(U)\overset{+}{\to}$ associated with an arbitrary object $U\in\mathcal{U}_k$, to get $\mathrm{Hom}_{D(R)}(U,M)=0$ since $\mathcal{U}_k\subseteq\mathcal{U}$ and $\mathcal{H}_k^{\perp_0}\subseteq\mathcal{H}$. This proves our claim. Now, for any direct system $(M_i)_{i\in I}$ in $\mathcal{H}_k^{\perp_0}$, by [56, Corollary 5.8] we have the natural isomorphism

$$\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i\cong\underset{i\in I}{\underrightarrow{\mathrm{holim}}}\,M_i;$$

moreover, since $(\mathcal{U}_k,\mathcal{V}_k)$ is a compactly generated hence homotopically smashing t-structure of $D(R)$, by [56, Corollary 5.6] we infer that $\varinjlim_{i\in I}^{(\mathcal{H})} M_i\in\mathcal{H}\cap\mathcal{U}_k^{\perp_0}=\mathcal{H}_k^{\perp_0}$. Therefore, the class $\mathcal{H}_k^{\perp_0}$ is closed under direct limits taken in $\mathcal{H}$. Eventually, we conclude by our assumption together with the identities $\mathcal{H}_k^{\#}=\mathcal{H}_k$ and

$$\begin{aligned}(\mathcal{H}_k^{\#})^{\perp_0}&=\{M\in\mathcal{H}^{\#}\mid\mathrm{Hom}_{D(R)}(\mathcal{H}_k^{\#},M)=0\}\\&=\{M\in\mathcal{H}\mid\mathrm{Hom}_{D(R)}(\mathcal{H}_k,M)=0\}=\mathcal{H}_k^{\perp_0}\,.\end{aligned}$$ □

3.30 Corollary. *Let Φ be a Thomason filtration bounded above r and weakly bounded below. Then $\mathcal{T}_r[-r]$ is a TTF class of finite type in $\mathcal{H}^{\#}$.*

Proof. Thanks to the boundedness of Φ, we have $\mathcal{H}_r^{\#}=\mathcal{T}_r[-r]$, so the conclusion follows by the previous theorem. Notice that in this case the left constituent of the TTF triple is $(\tau^{\leq r-1}(\mathcal{H}^{\#}),\mathcal{T}_r[-r])$, for there are no nonzero morphisms between the members of the pair and, by Cor. 3.27(i), for every $M\in\mathcal{H}^{\#}$ its standard approximation $\tau^{\leq r-1}(M)\to M\to H^r(M)[-r]\overset{+}{\to}$ yields a functorial short exact sequence in $\mathcal{H}^{\#}$. □

3.31 Remark. The existence in $\mathcal{H}^{\#}$ of TTF triples carries useful information, both on the members of the triples and on the local coherence of $\mathcal{H}^{\#}$ itself. More precisely:

(1) Within the notation of Cor. 3.27(i), the torsion class ${}^{\perp_0}(\mathcal{H}_k^{\#})$ consists of those complexes M of $\mathcal{H}^{\#}$ which fit in an exact triangle $\tau^{\leq k-1}(M)\to M\to U[1]\overset{+}{\to}$ for some object $U\in\mathcal{U}_k^{\#}$.

(2) The torsion class $\mathcal{H}_k^{\#}$ is a locally finitely presented category by Prop. 3.9. Moreover, if Φ is weakly bounded below, then we have

$$\mathrm{fp}({}^{\perp_0}(\mathcal{H}_k^{\#})),\mathrm{fp}(\mathcal{H}_k^{\#})\subseteq\mathrm{fp}(\mathcal{H}^{\#})$$

by [48, Lemma 2.4]. Furthermore, by Thm 3.2, both ${}^{\perp_0}(\mathcal{H}^\#_k)$ and $\mathcal{H}^\#_k$ are quasi locally coherent categories[1] in case $\mathcal{H}^\#$ is locally coherent.

(3) If Φ is weakly bounded below, then thanks to Thm 3.24 and [24, Lemma 3.6], for any module $X \in \mathcal{T}_j$ the triangle

$$U[1] \longrightarrow X[-j] \longrightarrow H_{\mathcal{H}}(X[-j]) \stackrel{+}{\longrightarrow}$$

can be taken either with $U \in \mathcal{U}$ or $U \in \mathcal{U}^\#$, since both $X[-j]$ and $H_{\mathcal{H}}(X[-j])$ belong to $D^+(R)$.

(4) In order to distinguish the torsion radicals and coradicals of each torsion pair $({}^{\perp_0}(\mathcal{H}^\#_k), \mathcal{H}^\#_k)$ of $\mathcal{H}^\#$ to those of each torsion pair $(\mathcal{T}_k, \mathcal{F}_k)$ of Mod-R we dealt with so far, we will use the following notation

$$ {}^{\perp_0}(\mathcal{H}^\#_k) \underset{\boldsymbol{x}_k}{\leftrightarrows} \mathcal{H}^\# \overset{\boldsymbol{y}_k}{\rightleftarrows} \mathcal{H}^\#_k;$$

furthermore, we will drop the index in case the value of the integer is clear from the context.

3.32 Remark. Let Φ be any Thomason filtration, and $k \in \mathbb{Z}$. Then the composition $H^{-k} \circ H_{\mathcal{H}} \circ [k]$ defines a functor

$$\begin{aligned} \Sigma^{-k} \colon \mathcal{TF}_{-k} &\longrightarrow \mathcal{TFT}_{-k} \\ X &\longmapsto H^{-k}(H_{\mathcal{H}}(X[k])) \end{aligned}$$

equipped with a functorial monomorphism $\sigma \colon \mathrm{id} \Rightarrow \Sigma^{-k}$ such that $\operatorname{Coker} \sigma_X \in \mathcal{T}_{-k+2}$. Indeed, for every $X \in \mathcal{TF}_{-k}$, i.e. $X \in \mathcal{T}_{-k} \cap \mathcal{F}_{-k+1}$, its stalk $X[k]$ is an object of $\mathcal{U} \cap \mathcal{U}_{-k}$ (see Rem. 3.31(3)), hence $H_{\mathcal{H}}(X[k]) \cong H_{\mathcal{H}_{-k}}(X[k])$, so that the least nonzero cohomology of the latter complex is at degree $-k$ by Prop. 3.9. Therefore, by Lem. 3.11, Σ^{-k} is well-defined on objects. Let now $f \colon X \to X'$ be a morphism in $\mathcal{TF}_{-k}$. Then we have a diagram

$$\begin{array}{ccccccc} U[1] & \longrightarrow & X[k] & \longrightarrow & H_{\mathcal{H}}(X[k]) & \stackrel{+}{\longrightarrow} \\ \vdots & & \downarrow{\scriptstyle f[k]} & & \vdots{\scriptstyle h} & \\ U'[1] & \longrightarrow & X'[k] & \longrightarrow & H_{\mathcal{H}}(X'[k]) & \stackrel{+}{\longrightarrow} \end{array}$$

for some $U, U' \in \mathcal{U}$, which can be completed to a morphism of triangles since the composition $U[1] \to X[k] \stackrel{f[k]}{\to} X'[k] \to H_{\mathcal{H}}(X'[k])$ is the zero map. We have $h = H_{\mathcal{H}}(f[k])$, hence Σ^{-k} actually is a functor. This said, apply the standard cohomology H^{-k} on the first triangle of the previous commutative diagram, to obtain the exact sequence

$$0 \longrightarrow H^{-k}(U[1]) \longrightarrow X \stackrel{\sigma_X}{\longrightarrow} \Sigma^{-k}(X) \longrightarrow H^{-k+1}(U[1]) \longrightarrow 0$$

in which $H^{-k}(U[1]) \in \mathcal{T}_{-k+1} \cap \mathcal{F}_{-k+1} = 0$ by assumption on X. Therefore, the σ_X's are monomorphisms, moreover they form a natural transformation in view of the construction of the functor Σ^{-k}. Finally, $\operatorname{Coker} \sigma_X \in \mathcal{T}_{-k+2}$ being isomorphic to $H^{-k+1}(U[1])$.

[1] See Def. 1.3

3.33 LEMMA. *Let Φ be a Thomason filtration, $k \in \mathbb{Z}$ and $X \in \mathcal{T}_{-k}$. Consider the following assertions:*

(a) $H_{\mathcal{H}}(X[k]) \in \mathrm{fp}(\mathcal{H})$;
(b) $H_{\mathcal{H}}(y_{-k+1}(X)[k]) \in \mathrm{fp}(\mathcal{H})$;
(c) $H_{\mathcal{H}}(\Sigma^{-k}(y_{-k+1}(X))[k]) \in \mathrm{fp}(\mathcal{H})$;
(d) $\Sigma^{-k}(y_{-k+1}(X)) \in \mathrm{fp}(\mathcal{TFT}_{-k})$.

Then "(a) ⇔ (b) ⇔ (c) ⇒ (d)"; moreover, if Φ is weakly bounded below, then all the assertions are equivalent. In this case, the subclass of $\mathcal{T}_{-k}$ of modules satisfying the previous equivalent conditions will be denoted by $\Sigma\mathcal{T}_{-k}$.

Proof.
"(a) ⇔ (b)" Consider the approximation $0 \to x_{-k+1}(X) \to X \to y_{-k+1}(X) \to 0$ of X within the torsion pair $(\mathcal{T}_{-k+1}, \mathcal{F}_{-k+1})$ of Mod-R. Then $x_{-k+1}(X)[k] \in \mathcal{U}[1]$ hence applying the functor $H_{\mathcal{H}}$ on the triangle involving the stalk complexes of the sequence, we obtain $H_{\mathcal{H}}(X[k]) \cong H_{\mathcal{H}}(y_{-k+1}(X)[k])$, and we are done.
"(b) ⇔ (c)" Since $y_{-k+1}(X) \in \mathcal{TF}_{-k}$, in view of Rem. 3.32 we have a short exact sequence

$$0 \longrightarrow y_{-k+1}(X) \longrightarrow \Sigma^{-k}(y_{-k+1}(X)) \longrightarrow \operatorname{Coker} \sigma_{y_{-k+1}(X)} \longrightarrow 0$$

say it $0 \to Y \to S \to C \to 0$ for short, in which $C \in \mathcal{T}_{-k+2}$. By applying the functor $H_{\mathcal{H}}$ on the triangle involving the stalk complexes of such sequence, we obtain $H_{\mathcal{H}}(Y[k]) \cong H_{\mathcal{H}}(S[k])$, whence the thesis.
"(c) ⇒ (d)" The proof is similar to that of Thm. 3.24. In view of the previous notation, we have $S[k] \in \mathcal{U}^{\#}_{-k}$, hence $H_{\mathcal{H}}(S[k]) = H_{\mathcal{H}^{\#}_{-k}}(S[k])$. We now apply Lem. 3.36 on the heart $\mathcal{H}^{\#}_{-k}$, obtaining that $H^{-k}(H_{\mathcal{H}^{\#}_{-k}}(S[k])) \in \mathrm{fp}(\mathcal{TFT}_{-k})$, but this latter object coincides with S by Lem. 3.34(ii). Notice that, even if our filtration Φ_{-k} is not of finite length, the arguments of the cited forthcoming results still hold true for the cohomologies involved in the present proof.

If Φ is weakly bounded below, we have $\mathcal{H}^{\#}_{-k} = \mathcal{H}_{-k}$ and $\mathcal{H}^{\#} = \mathcal{H}$, by Thm. 3.24. By Rem. 3.31(2) we have that $\mathrm{fp}(\mathcal{H}_{-k}) \subseteq \mathrm{fp}(\mathcal{H})$, eventually by Lem. 3.36 applid on $\mathcal{H}_{-k}$ we infer that "(d) ⇒ (c)". □

3.5. Thomason filtrations of finite length

The present section is devoted to deepen the approximation theory of the AJS heart $\mathcal{H}^{\#}$ associated with a Thomason filtration Φ of finite length, in order to characterise its local coherence. In this vein, the main tools we have at our disposal are:

- the TTF classes of finite type $\mathcal{H}^{\#}_k$ detected in Thm 3.29, for they allow to specialise Thm 3.2;
- Thm 3.24, which ensures that all these hearts coincide with those of the compactly generated t-structures corresponding to Φ.

In view of this latter result, we will make the identifications

$$\mathcal{H} = \mathcal{H}^{\#} \qquad \text{and} \qquad \mathcal{H}_k = \mathcal{H}^{\#}_k \quad \text{for all } k \in \mathbb{Z},$$

so that, within $\mathcal{H} = \mathcal{H}^{\#}$, we also have ${}^{\perp_0}(\mathcal{H}_k^{\#}) = {}^{\perp_0}\mathcal{H}_k$ and $(\mathcal{H}_k^{\#})^{\perp_0} = \mathcal{H}_k^{\perp_0}$. Bearing in mind Rem. 3.31, it is then natural to seek for a recursive argument, namely a result which takes in account the local coherence of each heart $\mathcal{H}_k$. Therefore, we set $l+1$ to be the length of Φ.

3.34 Lemma. *Let Φ be a Thomason filtration of length $l+1$. Then:*

(i) *For every $X \in \mathcal{T}_{-l-1}$, we have $H_{\mathcal{H}}(X[l+1]) \in {}^{\perp_0}\mathcal{H}_{-l}$;*
(ii) *For every $X \in \mathcal{TFT}_{-l-1}$ there exist $U \in \mathcal{U}_{-l+2}$ and a triangle $U[1] \to X[l+1] \to H_{\mathcal{H}}(X[l+1]) \xrightarrow{+}$. In particular, $H^{-l-1}(H_{\mathcal{H}}(X[l+1])) = X$.*
(iii) *for all $M \in {}^{\perp_0}\mathcal{H}_{-l}$, there exists in $\mathcal{H}$ a functorial short exact sequence $0 \to L \to W \to M \to 0$, in which $L \in \mathcal{H}_{-l+1}$ and $W \cong H_{\mathcal{H}}(X[l+1])$, where $X = H^{-l-1}(M)$;*
(iv) ${}^{\perp_0}\mathcal{H}_{-l} = \mathrm{Gen}(H_{\mathcal{H}}(K(J)[l+1]) \mid J \in \mathcal{I}_{-l-1})$.

Proof. We will often exploit the characterisation of the torsion class ${}^{\perp_0}\mathcal{H}_{-l}$ deduced from Cor. 3.27 (see Rem. 3.31(1)).
(i) Given $X \in \mathcal{T}_{-l-1}$, let $M := H_{\mathcal{H}}(X[l+1])$ and consider the exact triangle $U[1] \to X[l+1] \to M \xrightarrow{+}$ given by some object $U \in \mathcal{U}$ (see Rem. 3.31(3)). Let us show that M satisfies the aforementioned characterisation of the torsion class ${}^{\perp_0}\mathcal{H}_{-l}$. Applying the standard cohomology on the above triangle we obtain $H^j(M) \cong H^{j+2}(U)$ for all $j \geq -l+1$ and that $H^{-l}(M) \leq H^{-l+2}(U)$, where the former two are modules in the torsion class $\mathcal{T}_{j+2}$. We claim that $\tau^{\geq -l}(M)[-1] \in \mathcal{U}_{-l}$, whence the conclusion thanks to the triangle

$$H^{-l-1}(M)[l+1] \longrightarrow M \longrightarrow \tau^{\geq -l}(M) \xrightarrow{+} .$$

Indeed, we have

$$H^j(\tau^{\geq -l}(M)[-1]) = H^{j-1}(\tau^{\geq -l}(M)) = \begin{cases} 0 & \text{if } j-1 < -l, \\ H^{j-1}(M) & \text{if } j-1 \geq -l, \end{cases}$$

hence, when $j-1 \geq -l$, we have $H^{j-1}(M) \cong H^{j+1}(U) \in \mathcal{T}_{j+1} \subseteq \mathcal{T}_j$, as desired.
(ii) Let $X \in \mathcal{TFT}_{-l-1}$ and $U[1] \to X[l+1] \to M \xrightarrow{+}$ as in part (i). The long exact sequence in standard cohomology yields

$$0 \longrightarrow H^{-l}(U) \longrightarrow X \longrightarrow H^{-l-1}(M) \longrightarrow H^{-l+1}(U) \longrightarrow 0$$

in which in fact $H^{-l}(U) = 0$ for it belongs simultaneously to $\mathcal{T}_{-l}$ and $\mathcal{F}_{-l}$ by assumption on X. Moreover, the resulting extension of $H^{-l-1}(M)$ is split by assumption on X again, meaning that $H^{-l+1}(U) = 0$ as well. Consequently, $U \in D^{\geq -l+2}(R) \cap \mathcal{U}$, as desired.
(iii) Let $M \in {}^{\perp_0}\mathcal{H}_{-l}$, so that by Prop. 3.9 and Rem. 3.31(1) there exists $U \in \mathcal{U}_{-l}$ and an exact triangle $H^{-l-1}(M)[l+1] \to M \to U[1] \xrightarrow{+}$, in which we set $X := H^{-l-1}(M)$. The long exact sequence in standard cohomology yields in particular $U \in \mathcal{U}_{-l} \cap D^{[-l+1,0]}(R) \subseteq \mathcal{U}_{-l+1}$. The usual triangle of U w.r.t. the

heart of $(\mathcal{U}_{-l+1}, \mathcal{V}_{-l+1})$ gives the following octahedron

$$\begin{array}{ccccccc}
U'[1] & = & U'[1] & & & & \\
\downarrow & & \downarrow & & & & \\
U & \longrightarrow & X[l+1] & \longrightarrow & M & \overset{+}{\longrightarrow} & \\
\downarrow & & \downarrow & & \| & & \\
L & \longrightarrow & W & \longrightarrow & M & \overset{+}{\longrightarrow} & \\
+\downarrow & & \downarrow + & & & &
\end{array}$$

for some $U' \in \mathcal{U}_{-l+1}$, so that $L \cong H_{\mathcal{H}_{-l+1}}(U)$, and a cone W, which actually belongs to $\mathcal{H}$ by extension-closure applied on the second horizontal triangle. Applying the t-cohomological functor $H_{\mathcal{H}}$ on the second vertical triangle we obtain $W \cong H_{\mathcal{H}}(X[l+1])$, hence the former triangle is a functorial short exact sequence of $\mathcal{H}$; indeed, it is the image under $H_{\mathcal{H}}$ of the first horizontal triangle, which is in turn functorial.

(iv) Let M and X be as in part (iii). By Prop. 3.4 we know that there exist a family $(J_i)_{i\in I}$ of finitely generated ideals in the Gabriel filter associated with the torsion class $\mathcal{T}_{-l-1}$, and an epimorphism $\varphi\colon \bigoplus_{i\in I}(R/J_i)^{(\alpha_i)} \to X$. Applying $H_{\mathcal{H}}$ on the associated triangle of the stalk complexes concentrated in degrees $-l-1$, bearing in mind that it commutes with coproducts of $D(R)$, we obtain the exact sequence of $\mathcal{H}$

$$H_{\mathcal{H}}(\operatorname{Ker}(\varphi)[l+1]) \longrightarrow \bigoplus_{i\in I} H_{\mathcal{H}}(R/J_i[l+1])^{(\alpha_i)} \longrightarrow \overbrace{H_{\mathcal{H}}(X[l+1])}^{\cong W} \longrightarrow 0\,.$$

Thus, our claim follows once we prove that $H_{\mathcal{H}}(K(J)[l+1]) \cong H_{\mathcal{H}}(R/J[l+1])$ for all $J \in \mathcal{I}_{-l-1}$, since M is an epimorphic image of W in $\mathcal{H}$. Shifting by $l+1$ the standard approximation $\tau^{\leq -1}(K(J)) \to K(J) \to R/J[0] \overset{+}{\to}$ of the Koszul complex $K(J)$, we see that $\tau^{\leq -1}(K(J))[l+1] = (\tau^{\leq -1}(K(J)[l]))[1]$ is an object of the aisle $\mathcal{U}[1]$. Therefore, applying the functor $H_{\mathcal{H}}$ on the result in triangle, we conclude. □

3.35 Remark.

(1) For all Thomason filtrations of finite length and $k \in \mathbb{Z}$, the class $\mathcal{TFT}_k$ is closed under direct limits (of Mod-R). Indeed, let $(X_i)_{i\in I}$ be a direct system in $\mathcal{TFT}_k$, and for all $i \in I$ consider $H_{\mathcal{H}_k}(X_i[-k]) \cong H_{\mathcal{H}}(X_i[-k])$; since Φ_k is weakly bounded below, by using the proof of Lem. 3.34(ii) we get that $H^k(H_{\mathcal{H}}(X_i[-k])) \cong X_i$. On the other hand, $\varinjlim^{(\mathcal{H})}_{i\in I} H_{\mathcal{H}}(X_i[-k])$ belongs to $\mathcal{H}_k$, which in turn is contained in $D^{\geq k}(R)$, and consequently $H^k(\varinjlim^{(\mathcal{H})}_{i\in I} H_{\mathcal{H}}(X_i[-k])) \in \mathcal{TFT}_k$ by Lem. 3.11. But this latter module is isomorphic to $\varinjlim_{i\in I} H^k(H_{\mathcal{H}}(X_i[-k])) \cong \varinjlim_{i\in I} X_i$, as desired (see also the proof of Cor. 3.14).

(2) For every $X \in \mathcal{TFT}_{-l-1}$ and $M \in \mathcal{H}_{-l+1}$ we have $\operatorname{Ext}^1_{\mathcal{H}}(H_{\mathcal{H}}(X[l+1]), M) = 0$. Indeed, by Lem. 3.34(ii) there are $U \in \mathcal{U}_{-l+2}$ and a triangle

$U[1] \to X[l+1] \to H_{\mathcal{H}}(X[l+1]) \xrightarrow{+}$, hence applying $\mathrm{Hom}_{D(R)}(-, M[1])$ on the triangle we obtain, by [61], the desired vanishing of the ext-group since in the exact sequence

$$\mathrm{Hom}_{D(R)}(U[2], M[1]) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(H_{\mathcal{H}}(X[l+1]), M) \longrightarrow \mathrm{Hom}_{D(R)}(X[l+1], M[1])$$

the first term is zero by axioms of t-structure, as well as the third since $M[1] \in D^{\geq -l}(R)$.

3.36 Lemma. *Let Φ be a Thomason filtration of length $l+1$, and $X \in \mathcal{T}_{-l-1}$. Then $H_{\mathcal{H}}(X[l+1]) \in \mathrm{fp}(\mathcal{H})$ if and only if the functor $\mathrm{Hom}_R(X, -)$ commutes with direct limits of direct systems in $\mathcal{TFT}_{-l-1}$.*

In particular, for all $B \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$ we have $H_{\mathcal{H}}(B[l+1]) \in \mathrm{fp}(\mathcal{H})$.

Proof.
"⇒" Let $X \in \mathcal{T}_{-l-1}$ and suppose that $H_{\mathcal{H}}(X[l+1])$ is a finitely presented object of $\mathcal{H}$. Let $(X_i)_{i \in I}$ be a direct system in $\mathcal{TFT}_{-l-1}$. For each complex $X_i[l+1] \in \mathcal{U}$ we have a triangle $U_i[1] \to X_i[l+1] \to H_{\mathcal{H}}(X_i[l+1]) \xrightarrow{+}$, say M_i its last vertex, for some $U_i \in \mathcal{U}$, in which $X_i \cong H^{-l-1}(M_i)$ for all $i \in I$, by Lem. 3.34(ii). On the other hand, out of the triangle $U[1] \to X[l+1] \to M \xrightarrow{+}$ corresponding to $X[l+1]$, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim_{i \in I} \mathrm{Hom}_{D(R)}(M, M_i) & \longrightarrow & \varinjlim_{i \in I} \mathrm{Hom}_{D(R)}(X[l+1], M_i) & \longrightarrow 0 \\ & \downarrow \cong & & \downarrow & \\ 0 \longrightarrow & \mathrm{Hom}_{D(R)}(M, \varinjlim^{(\mathcal{H})}_{i \in I} M_i) & \longrightarrow & \mathrm{Hom}_{D(R)}(X[l+1], \varinjlim^{(\mathcal{H})}_{i \in I} M_i) & \longrightarrow 0 \end{array}$$

in which the left hand vertical homomorphism is bijective by hypothesis, thus the right hand one is so. Eventually, by Prop. 3.9 we have a triangle

$$H^{-l-1}(\varinjlim^{(\mathcal{H})}_{i \in I} M_i)[l+1] \longrightarrow \varinjlim^{(\mathcal{H})}_{i \in I} M_i \longrightarrow \tau^{>-l-1}(\varinjlim^{(\mathcal{H})}_{i \in I} M_i) \xrightarrow{+}$$

whose first vertex is $(\varinjlim_{i \in I} H^{-l-1}(M_i))[l+1]$ since the standard cohomologies commute with direct limits, hence applying $\mathrm{Hom}_{D(R)}(X[l+1], -)$ on such triangle we see, by [61, Proposition 3.1.3, page 191], that the right hand isomorphism of the previous diagram actually is

$$\varinjlim_{i \in I} \mathrm{Hom}_R(X, H^{-l-1}(M_i)) \longrightarrow \mathrm{Hom}_R(X, \varinjlim_{i \in I} H^{-l-1}(M_i))$$

i.e. the desired one showing that $\mathrm{Hom}_R(X, -)$ commutes with direct limits of direct systems in $\mathcal{TFT}_{-l-1}$.
"⇐" Let $X \in \mathcal{T}_{-l-1}$ be a module whose functor $\mathrm{Hom}_R(X, -)$ commutes with direct limits of direct systems in $\mathcal{TFT}_{-l-1}$. Let $(M_i)_{i \in I}$ be a direct system in $\mathcal{H}$, and consider the direct system of approximating triangles $(H^{-l-1}(M_i)[l+1] \to M_i \to \tau^{>-l-1}(M_i) \xrightarrow{+})_{i \in I}$ in $D(R)$. Applying $\mathrm{Hom}_{D(R)}(X[l+1], -)$ we obtain,

as in the previous part of the proof, the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim_{i\in I} \mathrm{Hom}_R(X, H^{-l-1}(M_i)) & \longrightarrow & \varinjlim_{i\in I} \mathrm{Hom}_{D(R)}(X[l+1], M_i) & \longrightarrow 0 \\ & \cong \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathrm{Hom}_R(X, \varinjlim_{i\in I} H^{-l-1}(M_i)) & \longrightarrow & \mathrm{Hom}_{D(R)}(X[l+1], \varinjlim_{i\in I}^{(\mathcal{H})} M_i) & \longrightarrow 0 \end{array}$$

in which the left hand vertical homomorphism is bijective by hypothesis. Eventually, applying the functors $\mathrm{Hom}_{D(R)}(-, M_i)$'s on the usual triangle $U[1] \to X[l+1] \to M \xrightarrow{+}$ (see the previous part of the proof), we obtain again that the right hand isomorphism of the previous diagram is the desired one. □

3.37 Corollary. *Let Φ be a Thomason filtration of length $l+1$. For every $B \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$, there exist $n \in \mathbb{N}$, ideals $J_1, \dots, J_n \in \mathcal{I}_{-l-1}$, and*

(i) *an epimorphism in $\mathcal{H}$*

$$\bigoplus_{k=1}^{n} H_{\mathcal{H}}(K(J_k)[l+1]) \twoheadrightarrow H_{\mathcal{H}}(B[l+1]);$$

(ii) *integers $k_1, \dots, k_n$, and a homomorphism in* Mod-R

$$f\colon \bigoplus_{i=1}^{n} \Sigma^{-l-1}(y_{-l}(R/J_i)^{k_i}) \longrightarrow B$$

with $\mathrm{Coker}\, f \in \mathcal{T}_{-l}$.

Proof.
(i) By Lem. 3.36 we know that $H_{\mathcal{H}}(B[l+1])$ is a finitely presented object of the heart. On the other hand, by Lem. 3.34(iv) there are families $(J_i)_{i\in I}$ of ideals in $\mathcal{I}_{-l-1}$, a set Λ and an epimorphism

$$p\colon \Big(\bigoplus_{i\in I} H_{\mathcal{H}}(K(J_i)[l+1])\Big)^{(\Lambda)} \twoheadrightarrow H_{\mathcal{H}}(B[l+1])\ .$$

For every finite subset $\overline{I} \subset I$, every $i \in \overline{I}$, and every finite subset $A \subset \Lambda$, consider the composition

$$H_{\mathcal{H}}(K(J_i)[l+1])^{(A)} \xrightarrow{\varepsilon_i^A} \Big(\bigoplus_{i\in I} H_{\mathcal{H}}(K(J_i)[l+1])\Big)^{(\Lambda)} \xrightarrow{p} H_{\mathcal{H}}(B[l+1])$$

where ε_i^A is the split monomorphism. Then

$$H_{\mathcal{H}}(B[l+1]) = \mathrm{Im}\, p = \sum_{\substack{i\in\overline{I} \\ \overline{I}\subset I \\ A\subset\Lambda}} \mathrm{Im}(p\circ\varepsilon_i^A),$$

hence being the former a finitely presented complex, there exist finite subsets $\overline{I} \subset I$ and $A \subset \Lambda$ such that $H_{\mathcal{H}}(B[l+1]) = \sum_{i\in\overline{I}} \mathrm{Im}(p\circ\varepsilon_i^A)$, as desired.

(ii) Let p be as in part (i) and define $f := H^{-l-1}(p)$. In view of the proof of Lem. 3.34(iv), in the heart $\mathcal{H}$ we have exact rows

$$H_{\mathcal{H}}(\mathrm{Ker}(f)[l+1]) \longrightarrow \bigoplus_{i=1}^{n} H_{\mathcal{H}}(R/J_i[l+1]) \xrightarrow{\beta} H_{\mathcal{H}}(\mathrm{Im}(f)[l+1]) \longrightarrow 0$$

and

$$H_{\mathcal{H}}(\mathrm{Im}(f)[l+1]) \xrightarrow{\alpha} H_{\mathcal{H}}(B[l+1]) \longrightarrow H_{\mathcal{H}}(\mathrm{Coker}(f)[l+1]) \longrightarrow 0$$

in which $\alpha \circ \beta = p$, whence α is an epimorphism, so that $H_{\mathcal{H}}(\mathrm{Coker}(f)[l+1]) = 0$. Consequently, the usual triangle of $D(R)$ ending in this latter complex of $\mathcal{H}$ shows that $\mathrm{Coker}(f)[l+1]$ is isomorphic to the object $U[1]$ for some $U \in \mathcal{U}$, meaning that $\mathrm{Coker}\, f \cong H^{-l-1}(U[1]) \in \mathcal{T}_{-l}$. □

We now pass to consider some necessary conditions to the local coherence of the heart of a Thomason filtration of finite length.

3.38 Proposition. *Let Φ be a Thomason filtration of length $l+1$. If $\mathcal{H}$ is a locally coherent Grothendieck category and $P \in {}^{\perp_0}\mathcal{H}_{-l}$, then $P \in \mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$ if and only if the following conditions hold true:*

(i) $H^{-l-1}(P) \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$;
(ii) $\mathrm{Hom}_{D(R)}(\tau^{\geq -l}(P)[-1], -)$ *commutes with direct limits of direct systems in* $\mathcal{H}_{-l+1}$.

Proof.
"⇒" Let $P \in \mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$. By Lem. 3.34(iii) there exists $L \in \mathcal{H}_{-l+1}$ and a short exact sequence $0 \to L \to H_{\mathcal{H}}(X[l+1]) \to P \to 0$ in $\mathcal{H}$, in which $X := H^{-l-1}(P)$. Set $W := H_{\mathcal{H}}(X[l+1])$, and consider the exact sequence of covariant functors

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{H}}(P,-) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(W,-) \longrightarrow \mathrm{Hom}_{\mathcal{H}}(L,-) \longrightarrow \cdots$$
$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(P,-) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}}(W,-)\ .$$

When we restrict these functors to $\mathcal{H}_{-l+2}$, we obtain $\mathrm{Hom}_{\mathcal{H}}(W,-)\restriction = 0$ by Lem. 3.34(i), hence $\mathrm{Hom}_{\mathcal{H}}(P,-)\restriction = 0$, moreover $\mathrm{Ext}^1_{\mathcal{H}}(W,-)\restriction = 0$ by Rem. 3.35(2). Therefore, there is a natural isomorphism $\mathrm{Hom}_{\mathcal{H}}(L,-)\restriction \cong \mathrm{Ext}^1_{\mathcal{H}}(P,-)\restriction$, and by local coherence of $\mathcal{H}$ together with [54, Proposition 3.5(2)] we get that $L \in \mathrm{fp}(\mathcal{H}_{-l+2}) \subseteq \mathrm{fp}(\mathcal{H})$. By extension-closure of $\mathrm{fp}(\mathcal{H})$ (see [48, Corollary 1.8]), we have that W is a finitely presented object of $\mathcal{H}$, whence $X \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$ by Lem. 3.36. This proves part (i), so let us show part (ii). By Prop. 3.9 we have an exact triangle $H^{-l-1}(P)[l+1] \to P \to \tau^{\geq -l}(P) \overset{+}{\to}$, say $X[l+1]$ the first vertex, as in part (i). By Rem. 3.31(1), we know that $\tau^{\geq -l}(P) \in \mathcal{U}[1]$. Thus, by applying the functor $H_{\mathcal{H}}$ on such triangle we obtain the exact row

$$0 \longrightarrow H_{\mathcal{H}}(\tau^{\geq -l}(P)[-1]) \longrightarrow H_{\mathcal{H}}(X[l+1]) \longrightarrow P \longrightarrow 0,$$

which actually coincides with the short exact sequence $0 \to L \to W \to P \to 0$ provided by Lem. 3.34(iii). Therefore, by local coherence of $\mathcal{H}$ together with

Lem. 3.36, $H_{\mathcal{H}}(\tau^{\geq -l}(P)[-1])$ is a finitely presented complex of the heart. Moreover, we have the triangle $U[1] \to \tau^{\geq -l}(P)[-1] \to H_{\mathcal{H}}(\tau^{\geq -l}(P)[-1]) \overset{+}{\to}$ provided by $U := \tau_{\mathcal{U}}^{\leq}(\tau^{\geq -l}(P)[-2])$, so given a direct system $(M_i)_{i\in I}$ of complexes in $\mathcal{H}_{-l+1}$ and applying the functors

$$F := \varinjlim_{i\in I} \mathrm{Hom}_{D(R)}(-, M_i) \qquad \text{and} \qquad G := \mathrm{Hom}_{D(R)}(-, \varinjlim_{i\in I}{}^{(\mathcal{H})} M_i)$$

on the previous triangle, we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & F(H_{\mathcal{H}}(\tau^{\geq -l}(P)[-1])) & \longrightarrow & F(\tau^{\geq -l}(P)[-1]) & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow & & \\ 0 & \longrightarrow & G(H_{\mathcal{H}}(\tau^{\geq -l}(P)[-1])) & \longrightarrow & G(\tau^{\geq -l}(P)[-1]) & \longrightarrow & 0 \end{array}$$

yielding the thesis. Notice that in this way we proved that our condition (ii) is equivalent to $L \in \mathrm{fp}(\mathcal{H})$.
"$\Leftarrow$" Let $P \in {}^{\perp_0}\mathcal{H}_{-l}$ and consider the short exact sequence $0 \to L \to W \to P \to 0$ of $\mathcal{H}$ provided by Lem. 3.34(iii). Then $L \in \mathrm{fp}(\mathcal{H})$ by what we said at the end of the proof of the previous part (ii), whereas $W \in \mathrm{fp}(\mathcal{H})$ by Lem. 3.36. Therefore, P is finitely presented as well, being a cokernel of a morphism in $\mathrm{fp}(\mathcal{H})$. □

3.39 Corollary. *Let Φ be a weakly bounded below Thomason filtration such that its heart $\mathcal{H}$ is a locally coherent Grothendieck category. If $B \in \mathrm{fp}(\mathcal{H})$ and m is the least nonzero cohomology degree of B, then we have $H^m(B) \in \mathrm{fp}(\mathcal{TFT}_m)$.*

Proof. By definition of m and by Prop. 3.9, we have $B \in \mathcal{H}_m$. Moreover, since $\mathcal{H}$ is locally coherent, so is $\mathcal{H}_m$ being a TTF class of finite type. In particular, the torsion pair $({}^{\perp_0}\mathcal{H}_{m+1}, \mathcal{H}_{m+1})$ of $\mathcal{H}_m$ *restricts* to $\mathrm{fp}(\mathcal{H}_m)$, meaning that (see Thm. 3.2) the approximation[2] $0 \to \boldsymbol{x}(B) \to B \to \boldsymbol{y}(B) \to 0$ of B within the torsion pair actually is in $\mathrm{fp}(\mathcal{H}_m)$. By the proof of Prop. 3.38, we get $H^m(\boldsymbol{x}(B)) \in \mathrm{fp}(\mathcal{TFT}_m)$, and being $\boldsymbol{y}(B) \in \mathcal{H}_{m+1} \subseteq D^{\geq m+1}(R)$, it follows $H^m(B) \cong H^m(\boldsymbol{x}(B))$, and we are done. □

3.40 Proposition. *Let Φ be a Thomason filtration of length $l+1$. If the heart $\mathcal{H}$ is locally coherent, then*

(i) $\mathrm{fp}(\mathcal{TFT}_{-l-1})$ *is closed under kernels (in* Mod-R*);*
(ii) *For all $B \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$, there exists a R-linear map*

$$f\colon \bigoplus_{i=1}^{n} \Sigma^{-l-1}(y_{-l}(R/J_i)^{k_i}) \longrightarrow B$$

with $\mathrm{Coker}\, f \in \Sigma\mathcal{T}_{-l}$*;*
(iii) *For all morphisms f in* $\mathrm{fp}(\mathcal{TFT}_{-l-1})$ *with* $\mathrm{Coker}\, f \in \mathcal{T}_{-l}$*, then* $\mathrm{Coker}\, f \in \Sigma\mathcal{T}_{-l}$*.*

Proof.
(i) Given $f\colon B \to B'$ a homomorphism in $\mathrm{fp}(\mathcal{TFT}_{-l-1})$, we have to show that $\mathrm{Ker}\, f \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$. Consider the following diagram in $D(R)$ obtained by

[2] See Rem. 3.31(4)

approximating the stalk complexes of the modules within the t-structure $(\mathcal{U}, \mathcal{V})$:

$$\begin{array}{ccccccc} U[1] & \xrightarrow{a} & B[l+1] & \longrightarrow & H_{\mathcal{H}}(B[l+1]) & \xrightarrow{+} & \\ \downarrow & & \downarrow{\scriptstyle f[l+1]} & & \downarrow{\scriptstyle q} & & \\ U'[1] & \longrightarrow & B'[l+1] & \xrightarrow{b'} & H_{\mathcal{H}}(B'[l+1]) & \xrightarrow{+} & \end{array}$$

Since $b' \circ f[l+1] \circ a = 0$, the dotted vertical maps actually exist and they complete the diagram to a morphism of triangles. Now, by Lem. 3.34(i) we have that $H_{\mathcal{H}}(B[l+1]) =: M$ and $H_{\mathcal{H}}(B'[l+1]) =: M'$ are complexes of ${}^{\perp_0}\mathcal{H}_{-l}$, whereas by Lem. 3.36 we have that q is a morphism in $\mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$. This said, by the hypothesis of quasi local coherence of ${}^{\perp_0}\mathcal{H}_{-l}$ we infer that $\boldsymbol{x}(\mathrm{Ker}^{(\mathcal{H})}(q))$ is a finitely presented object of ${}^{\perp_0}\mathcal{H}_{-l}$, so that of $\mathcal{H}$. Moreover, notice that $H^{-l-1}(q) = f$, and that the standard cohomology sequences associated with the following two sequences of $\mathcal{H}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{x}(\mathrm{Ker}^{(\mathcal{H})}(q)) & \longrightarrow & \mathrm{Ker}^{(\mathcal{H})}(q) & \longrightarrow & \boldsymbol{y}(\mathrm{Ker}^{(\mathcal{H})}(q)) & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & M & & & & \\ & & & & \downarrow{\scriptstyle q} & & & & \\ & & & & M' & & & & \end{array}$$

yield

$$\begin{aligned} H^{-l-1}(\boldsymbol{x}(\mathrm{Ker}^{(\mathcal{H})}(q)) &= H^{-l-1}(\mathrm{Ker}^{(\mathcal{H})}(q)) \\ &= \mathrm{Ker}\, H^{-l-1}(q) = \mathrm{Ker}\, f, \end{aligned}$$

where the second equality follows by applying the functor H^{-l-1} to commutative diagram of $\mathcal{H}$ obtained by the factorisation of q through its kernel and image. Now, since $\mathrm{Ker}\, f = H^{-l-1}(\boldsymbol{x}(\mathrm{Ker}^{(\mathcal{H})}(q)))$, by Prop. 3.38 we infer that $\mathrm{Ker}\, f \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$, as desired.

(ii) We know that there is a homomorphism $f\colon \bigoplus_{i=1}^{n} \Sigma^{-l-1}(y_{-l}(R/J_i)^{k_i}) \to B$ having cokernel in $\mathcal{T}_{-l}$ (see Cor. 3.37(ii)). Let us rename the corresponding canonical exact sequence by $0 \to K \to N \xrightarrow{f} B \to C \to 0$, and let $L := \mathrm{Im}\, f$. Since $N, B \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$, by part (i) we know that $K \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$ as well. In turn, $L \in \mathcal{TF}_{-l-1}$ and $H_{\mathcal{H}}(L[l+1])$ is finitely presented being a cokernel in $\mathrm{fp}(\mathcal{H})$, by Lem. 3.36. On the other hand, since $C \in \mathcal{T}_{-l}$, we have $C[l+1] \in \mathcal{U}[1]$ whence $H_{\mathcal{H}}(C[l+1]) = 0$. This said, in the heart we have the commutative diagram with exact row

$$\begin{array}{ccccccccc} H_{\mathcal{H}}(B[l]) & \longrightarrow & H_{\mathcal{H}}(C[l]) & \longrightarrow & H_{\mathcal{H}}(L[l+1]) & \longrightarrow & H_{\mathcal{H}}(B[l+1]) & \longrightarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & & \\ & & H & & H' & & & & \end{array}$$

in which $H_{\mathcal{H}}(B[l]) \in \mathcal{H}_{-l+2}$ (as we will show at the end of the proof), so that also H belongs to such TTF class of $\mathcal{H}$; then $H_{\mathcal{H}}(C[l]) \in {}^{\perp_0}\mathcal{H}_{-l+1}$ by an adaptation of Lem. 3.34(i), so that also H' belongs to such torsion class of $\mathcal{H}$; eventually, the remaining terms of the diagram belong to ${}^{\perp_0}\mathcal{H}_{-l} \cap \mathrm{fp}(\mathcal{H})$ by lemmata 3.34(i) and 3.36. Since $\mathcal{H}$ is locally coherent by assumption, we infer that $H' \in \mathrm{fp}(\mathcal{H})$. Now we take the standard cohomologies of the extension of $H_{\mathcal{H}}(C[l])$ to see that $H^{-l}(H_{\mathcal{H}}(C[l])) \cong H^{-l}(H') \in \mathrm{fp}(\mathcal{TFT}_{-l})$ by Cor. 3.39. On the other hand, the standard cohomology sequence of the triangle $U[1] \to C[l] \to H_{\mathcal{H}}(C[l]) \xrightarrow{+}$ provided by some object $U \in \mathcal{U}$ yields

$$H^{-l+1}(U) \longrightarrow C \longrightarrow H^{-l}(H_{\mathcal{H}}(C[l])) \longrightarrow H^{-l+2}(U) \longrightarrow 0$$

and the canonical factorisations of the first two homomorphisms give indeed the approximation of C within the torsion pair $(\mathcal{T}_{-l+1}, \mathcal{F}_{-l+1})$. Therefore, $H^{-l}(H_{\mathcal{H}}(C[l])) = \Sigma^{-l}(y_{-l+1}(C))$ by means of the monomorphism induced by the natural transformation σ (see Rem. 3.32), and we are done.

As announced above, let us now show that $H_{\mathcal{H}}(B[l]) \in \mathcal{H}_{-l+2}$. By Lem. 3.34(ii) there exist $U \in \mathcal{U}_{-l+2}$ and a triangle $U[1] \to B[l+1] \to H_{\mathcal{H}}(B[l+1]) \xrightarrow{+}$, whence $H_{\mathcal{H}}(H_{\mathcal{H}}(B[l+1])[j]) = 0$ for $j = -1, -2$, meaning that $H_{\mathcal{H}}(B[l]) = H_{\mathcal{H}}(U) = H_{\mathcal{H}_{-l+2}}(U)$, as claimed.

(iii) This is a consequence of the proof of part (ii). □

3.5.1. A characterisation of the local coherence. We are now ready to state and prove the characterisation of the local coherence of the heart associated with a Thomason filtration of finite length of the prime spectrum of a commutative ring. We recall once again that such result is a specialisation of Thm. 3.2.

3.41 THEOREM. *Let Φ be a Thomason filtration of length $l+1$. Then $\mathcal{H}$ is a locally coherent Grothendieck category if and only if the following conditions hold true:*

1. *$\mathcal{H}_{-l}$ is a locally coherent Grothendieck category;*
2. *For every $B \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$, the functor $\mathrm{Ext}^1_{\mathcal{H}}(H_{\mathcal{H}}(B[l+1]), -)$ commutes with direct limits of direct systems in $\mathcal{H}_{-l}$;*
3. *For every $B \in \mathrm{fp}(\mathcal{TFT}_{-l-1})$, the functor $\mathrm{Ext}^1_{\mathcal{H}}(H_{\mathcal{H}}(B[l+1]), -)$ commutes with direct limits of direct systems in ${}^{\perp_0}\mathcal{H}_{-l}$;*
4. *For all $P \in \mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$, in the functorial short exact sequence $0 \to L \to W \to P \to 0$ of Lem. 3.34, we have $L \in \mathrm{fp}(\mathcal{H})$;*
5. *The torsion pair $({}^{\perp_0}\mathcal{H}_{-l}, \mathcal{H}_{-l})$ restricts to $\mathrm{fp}(\mathcal{H})$.*

Proof. Let us assume that the heart $\mathcal{H}$ associated with Φ is a locally coherent Grothendieck category, and let us show that the five stated conditions hold true. (1) $\mathcal{H}_{-l}$ is a locally coherent Grothendieck category since it is a TTF class of finite type in $\mathcal{H}$. (2) and (3) follow by Lem. 3.36 and [54, Proposition 3.5(2)]. (4) follows by the proof of Prop. 3.38. (5) holds true by hypothesis on $\mathcal{H}$ and since the torsion pair $(\mathcal{H}_{-l}, \mathcal{H}_{-l}^{\perp_0})$ is of finite type.

Conversely, let us show that the five stated conditions imply the local coherence of the heart $\mathcal{H}$. More in details, we want to exploit Thm. 3.2, which characterises the local coherence of an arbitrary Grothendieck category equipped with a TTF triple of finite type. Notice that hypothesis (iii)' of Thm. 3.2 coincides with our hypothesis (5)

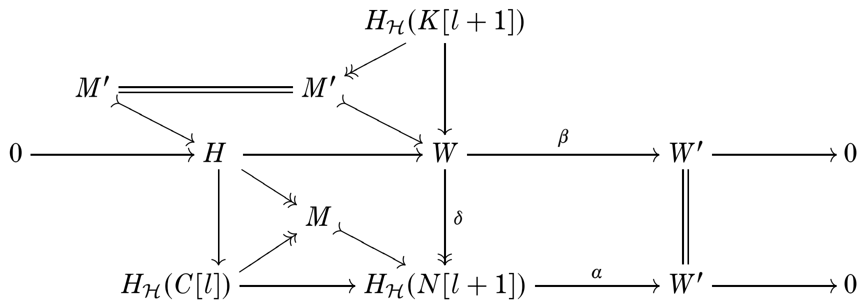
Concerning condition (i) of Thm. 3.2, thanks to our hypothesis (1) we need to check that the torsion class ${}^{\perp_0}\mathcal{H}_{-l}$ is quasi locally coherent. We know that $\mathcal{H}$ is a locally finitely presented Grothendieck category by Prop. 3.9 and Thm. 3.24, and by imitating the proof of "(a) ⇒ (b)" in Thm. 3.2 we deduce that ${}^{\perp_0}\mathcal{H}_{-l}$ is locally finitely presented as well, thus it remains to prove that $\mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$ is closed under taking kernels; in particular, it suffices to check that for every epimorphism $p\colon P \to P'$ in $\mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$, we have $\boldsymbol{x}(\mathrm{Ker}^{(\mathcal{H})}(f)) \in \mathrm{fp}({}^{\perp_0}\mathcal{H}) \subseteq \mathrm{fp}(\mathcal{H})$. The following diagram provided by Lem. 3.34(iii)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & W & \longrightarrow & P & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle\alpha} & & \Big\downarrow{\scriptstyle\beta} & & \Big\downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & L' & \longrightarrow & W' & \longrightarrow & P' & \longrightarrow & 0 \end{array}$$

can be completed to a commutative diagram, since in $D(R)$ the composition $W \to P \overset{p}{\to} P' \to L[1]$ yields an element of $\mathrm{Ext}^1_{\mathcal{H}}(W, L)$, which is zero by Rem. 3.35(2); consequently α is defined by the universal property of the kernel. By condition (4), the objects L, L' are finitely presented complexes of $\mathcal{H}$, while W, W' are so by extension-closure. Moreover, since $W' \in {}^{\perp_0}\mathcal{H}_{-l}$ by Lem. 3.34(i), then β is an epimorphism since its cokernel in $\mathcal{H}$ is a quotient in both the torsion classes ${}^{\perp_0}\mathcal{H}_{-l}$ and $\mathcal{H}_{-l}$. The snake lemma applied on the previous commutative diagram gives us the exact row

$$0 \longrightarrow \mathrm{Ker}^{(\mathcal{H})}(\alpha) \longrightarrow \mathrm{Ker}^{(\mathcal{H})}(\beta) \longrightarrow \mathrm{Ker}^{(\mathcal{H})}(p) \longrightarrow \mathrm{Coker}^{(\mathcal{H})}(\alpha) \longrightarrow 0$$

in which the outer terms are finitely presented by hypotheses (1). This said, by [48, Corollary 1.8] and hypothesis (5), our claim will follow once we check that $H := \mathrm{Ker}^{(\mathcal{H})}(\beta)$ is a finitely presented object. Let $X = H^{-l-1}(P)$ so that $W = H_{\mathcal{H}}(X[l+1])$ (similarly for W'), and consider $f := H^{-l-1}(\beta)$, with $K := \mathrm{Ker}\, f$, $N := \mathrm{Im}\, f$ and $C := \mathrm{Coker}\, f$. By applying the functor $H_{\mathcal{H}}$ to the triangles $K[l+1] \to X[l+1] \to N[l+1] \overset{+}{\to}$ and $C[l] \to N[l+1] \to X'[l+1] \overset{+}{\to}$ obtained out of the canonical short exact sequences in Mod-R associated to f, we get the commutative diagram of $\mathcal{H}$ with exact rows



in which $\beta = \alpha \circ \delta$ (this also implies $H_{\mathcal{H}}(C[l+1]) = 0$ i.e. that $C \in \mathcal{T}_{-l}$, whence $H_{\mathcal{H}}(C[l]) \in \mathcal{H}_{-l}$) and the epimorphism $H \to M$ is provided by the universal property of the kernel. Moreover, by the snake lemma, the image M' of the morphism $H_{\mathcal{H}}(K[l+1]) \to W$ induces the short exact sequence $0 \to M' \to H \to M \to 0$, which actually is the approximation of H within the torsion pair $({}^{\perp_0}\mathcal{H}_{-l}, \mathcal{H}_{-l})$ (see Lem. 3.34(i)). Thus, we reduced our claim to check that $M', M \in \mathrm{fp}(\mathcal{H})$. We have $M' \in \mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l}) \subseteq \mathrm{fp}(\mathcal{H})$ by hypothesis (3) applied on the short exact sequence $0 \to M' \to W \to H_{\mathcal{H}}(N[l+1]) \to 0$, whereas $M \in \mathrm{fp}(\mathcal{H}_{-l}) \subseteq \mathrm{fp}(\mathcal{H})$ thanks to hypothesis (2) applied on the short exact sequence $0 \to M \to H_{\mathcal{H}}(N[l+1]) \to W' \to 0$.

Eventually, let us prove that Thm. 3.2(ii) holds true. Given $P \in \mathrm{fp}({}^{\perp_0}\mathcal{H}_{-l})$ we have a short exact sequence $0 \to L \to W \to P \to 0$ by Lem. 3.34(iii), in which $L \in \mathrm{fp}(\mathcal{H})$ by hypothesis (4). The sequence yields an exact row

$$0 \to \mathrm{Hom}_{\mathcal{H}}(P,-) \to \mathrm{Hom}_{\mathcal{H}}(W,-) \to \mathrm{Hom}_{\mathcal{H}}(L,-) - \cdots$$
$$\cdots \to \mathrm{Ext}^1_{\mathcal{H}}(P,-) \to \mathrm{Ext}^1_{\mathcal{H}}(W,-) \to \mathrm{Ext}^1_{\mathcal{H}}(L,-)$$

of covariant functors $\mathcal{H} \to \mathrm{Ab}$. When restricted to $\mathcal{H}_{-l}$, the first three functors commute with direct limits, whereas the last two do so respectively by hypotheses (2) and (1), and by [54, Proposition 3.5(2)], so that $\mathrm{Ext}^1_{\mathcal{H}}(P,-)\restriction_{\mathcal{H}_{-l}}$ commutes with the desired direct limits. □

3.42 Remark. The previous theorem provides a recursive argument for the construction of a Thomason filtration of finite length whose heart is a locally coherent Grothendieck category. However, one practical issue is to check conditions (2) and (3) when the length of the filtration, i.e. l, is greater than 2. Nonetheless, for $0 \leq l \leq 2$ (that are values involving interesting classes of abelian categories, e.g. torsion classes of Mod-R and certain HRS hearts, as we have already seen), the conditions of the theorem simplify so that most of them can be rephrased in module-theoretic ones, as we will show in the following results.

The length zero case has been treated in subsec. 3.3.2, and it consists in a characterisation of the local coherence of an arbitrary hereditary torsion class of finite type in Mod-R.

3.43 Corollary. *Let Φ be a Thomason filtration of length 1. Then $\mathcal{H}$ is a locally coherent Grothendieck category if and only if the following conditions are satisfied:*

(1) *$\mathcal{T}_0$ is locally coherent;*
(2) *For all $P \in \mathrm{fp}(\mathcal{TFT}_{-1})$, the functor $\mathrm{Hom}_R(P,-)$ commutes with direct limits of direct systems in $\mathcal{T}_0$;*
(3) *For all $P \in \mathrm{fp}(\mathcal{TFT}_{-1})$, the functor $\mathrm{Ext}^1_R(P,-)$ commutes with direct limits of direct systems in $\mathcal{TFT}_{-1}$;*
(4) *For all $Q \in \mathrm{fp}(\mathcal{T}_0)$, the functor $\mathrm{Ext}^2_R(Q,-)$ commutes with direct limits of direct systems in $\mathcal{TFT}_{-1}$.*

Proof. First, notice that $H_{\mathcal{H}}(P[1]) = P[1]$ for all $P \in \mathcal{TFT}_{-1}$, that $\mathcal{H}_0 = \mathcal{T}_0[0]$ and that ${}^{\perp_0}\mathcal{H}_0 = \mathcal{TFT}_{-1}[1]$. Thus, the stated conditions (1), (2) and (3) are exactly the corresponding ones of Thm. 3.41, since $l = 0$. In turn, condition (4) of the theorem is clearly satisfied since $L \in \mathcal{H}_1 = 0$ (see Lem. 3.34). Let us check condition (5) of the theorem. We claim that it is implied by our condition (4). Let

$B \in \mathrm{fp}(\mathcal{H})$ and consider its approximation $0 \to H^{-1}(B)[1] \to B \to H^0(B)[0] \to 0$ within the torsion pair $({}^{\perp_0}\mathcal{H}_0, \mathcal{H}_0) = (\mathcal{TFT}_{-1}[1], \mathcal{T}_0[0])$; we have to prove that the outer terms are finitely presented objects of $\mathcal{H}$. We recall that $H^0(B)[0] \in \mathrm{fp}(\mathcal{H})$ by Cor. 3.26(i); in particular, we have $H^0(B) \in \mathrm{fp}(\mathcal{T}_0)$. Let $(X_i)_{i\in I}$ be a direct system of modules in $\mathcal{TFT}_{-1}$. Applying the functors

$$F^k := \varinjlim_{i\in I} \mathrm{Ext}^k_{\mathcal{H}}(-, X_i[1]) \quad \text{and} \quad G^k := \mathrm{Ext}^k_{\mathcal{H}}(-, \varinjlim_{i\in I} X_i[1]) \qquad (k \in \mathbb{N} \cup \{0\})$$

on the previous approximation, say it $0 \to Y[1] \to B \to X[0] \to 0$ for short, we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \to & F^0(X[0]) & \to & F^0(B) & \to & F^0(Y[1]) & \to & F^1(X[0]) & \to & F^1(B) \\ & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ 0 & \to & G^0(X[0]) & \to & G^0(B) & \to & G^0(Y[1]) & \to & G^1(X[0]) & \to & G^1(B) \end{array}$$

in which, using [61, 6], f_1 is an isomorphism by Cor. 3.26(i), f_2 is iso and f_5 is monic, and f_4 is an isomorphism by hypothesis (4), so we are done by the five lemma.

In order to conclude, it remains to prove that if $\mathcal{H}$ is locally coherent, then our hypothesis (4) is satisfied. Let $Q \in \mathrm{fp}(\mathcal{T}_0)$. By Cor. 3.26(i) again, we have $Q[0] \in \mathrm{fp}(\mathcal{H})$, hence $\mathrm{Ext}^1_{\mathcal{H}}(Q[0], -)$ preserves direct limits by [54, Proposition 3.5(2)]; in particular, it commutes with direct limits of $\mathcal{TFT}_{-1}[1]$, which is our thesis by [61, 6]. □

3.44 Corollary. *Let Φ be a Thomason filtration of length 2. Then $\mathcal{H}$ is a locally coherent Grothendieck category if and only if the following conditions are satisfied:*

1. *$\mathcal{H}_{-1}$ is locally coherent (cf. Cor. 3.43);*
2. *For all $P \in \mathrm{fp}(\mathcal{TFT}_{-2})$, the functor $\mathrm{Hom}_R(P, -)$ preserves direct limits of direct systems in $\mathcal{TFT}_{-1}$;*
3. *The following conditions hold true:*
   - (3.i) *For all $J \in \mathcal{I}_{-2}$, the functor $\mathrm{Ext}^1_{\mathcal{H}}(\Sigma^{-2}(y_{-1}(R/J))[2], -)$ preserves direct limits of direct systems in ${}^{\perp_0}\mathcal{H}_{-1}$;*
   - (3.ii) *$\mathrm{fp}(\mathcal{TFT}_{-2})$ is closed under kernels in Mod-R.*
   - (3.iii) *For all morphisms f in $\mathrm{fp}(\mathcal{TFT}_{-2})$, we have $\Sigma^{-2}(\mathrm{Im}\, f)/\mathrm{Im}\, f \in$ mod-R;*
4. *For all exact sequences of Mod-R of the form $0 \to Y \to M \xrightarrow{f} N \to X \to 0$ such that $Y \in \mathrm{fp}(\mathcal{TFT}_{-2})$, $X \in \mathrm{fg}(\mathcal{T}_0)$ and $\mathrm{Cone}(f[1]) \in \mathcal{H}$, we have $X \in \mathrm{fp}(\mathcal{T}_0)$.*
5. *For all $P \in \mathrm{fp}(\mathcal{H})$, the following conditions hold true:*
   - (5.i) *$H^{-2}(P) \in \mathrm{fp}(\mathcal{TFT}_{-2})$;*
   - (5.ii) *$x_0(H^{-1}(P)) \in \mathrm{fp}(\mathcal{T}_0)$.*

Proof. It is clear that our hypothesis (1) corresponds exactly to condition (1) of Thm. 3.41.

Let us prove that our hypothesis (2) is equivalent to Thm. 3.41(2). Notice again that for all $P \in \mathcal{TFT}_{-2}$ we have $H_{\mathcal{H}}(P[2]) = P[2]$. This said, any direct system $(M_i)_{i\in I}$ of $\mathcal{H}_{-1}$ is approximated by $(0 \to H^{-1}(M_i)[1] \to M_i \to H^0(M_i)[0] \to 0)_{i\in I}$ within the left constituent of the TTF triple given by the TTF class $\mathcal{H}_0$ (see the proof of Cor. 3.43). Thus, by applying the cohomological functor $\mathrm{Hom}_{D(R)}(P[2], -)$ on the direct limit of the previous approximation and using [61, 6], we obtain the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim_{i\in I} \mathrm{Hom}_R(P, H^{-1}(M_i)) & \longrightarrow & \varinjlim_{i\in I} \mathrm{Ext}^1_{\mathcal{H}}(P[2], M_i) & \longrightarrow 0 \\ & \downarrow & & \downarrow & \\ 0 \longrightarrow & \mathrm{Hom}_R(P, \varinjlim_{i\in I} H^{-1}(M_i)) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{H}}(P[2], \varinjlim_{i\in I} M_i) & \longrightarrow 0 \end{array}$$

which shows the desired equivalence, since for all $M \in \mathcal{H}_{-1}$ and $Y \in \mathcal{TFT}_{-1}$, we have $H^{-1}(M) \in \mathcal{TFT}_{-1}$ and $Y[1] \in \mathcal{H}_{-1}$.

Let us show that Thm. 3.41 implies our condition (3).

(3.i) Let $J \in \mathcal{I}_{-2}$. By the approximating triangle $\tau^{\leq -1}(K(J)[2]) \to K(J)[2] \to R/J[2] \xrightarrow{+}$ of the Koszul complex $K(J)[2]$ within the standard t-structure of $D(R)$, since the first vertex belongs to $\mathcal{U}[3]$ by the proof of Lem. 3.34(iv), we obtain $H_{\mathcal{H}}(K(J)[2]) \cong H_{\mathcal{H}}(R/J[2])$, and these are finitely presented objects of $\mathcal{H}$ by [56, Lemma 6.3]. Let us call M such complex; it fits in an exact triangle $U[1] \to K(J)[2] \to M \xrightarrow{+}$ provided by some $U \in \mathcal{U}$, whose standard cohomology exact sequence yields

$$0 \longrightarrow H^{-1}(U) \longrightarrow R/J \xrightarrow{d} H^{-2}(M) \longrightarrow H^0(U) \longrightarrow 0 .$$

On the one hand we infer that M is a stalk, i.e. $M \cong H^{-2}(M)[2]$, whence in turn $H^{-2}(M) \in \mathrm{fp}(\mathcal{TFT}_{-2})$ by Lem. 3.36; on the other hand, we have $\mathrm{Im}\, d \in \mathcal{TF}_{-2}$ and $H^0(U) \in \mathcal{T}_0$, thus

$$H^{-2}(M) \cong \Sigma^{-2}(\mathrm{Im}\, d) \cong \Sigma^{-2}(y_{-1}(R/J))$$

and we conclude by Lem. 3.33 and [54, Proposition 3.5(2)].

(3.ii) It follows by Prop. 3.40(i).

(3.iii) Let $f\colon B \to B'$ be a morphism in $\mathrm{fp}(\mathcal{TFT}_{-2})$. In view of Rem. 3.32, we have to prove that $\mathrm{Coker}\, \sigma_{\mathrm{Im}\, f}$ is a finitely presented R-module. We have $\mathrm{Ker}\, f \in \mathrm{fp}(\mathcal{TFT}_{-2})$ by part (3.ii), so by the exact sequence

$$0 \longrightarrow H_{\mathcal{H}}(\mathrm{Im}(f)[1]) \longrightarrow \mathrm{Ker}(f)[2] \longrightarrow B[2] \longrightarrow H_{\mathcal{H}}(\mathrm{Im}(f)[2]) \longrightarrow 0$$

of the heart $\mathcal{H}$ we obtain that the outer terms are finitely presented, in particular we infer $\Sigma^{-2}(\mathrm{Im}\, f) \in \mathrm{fp}(\mathcal{TFT}_{-2})$ by Lem. 3.33. On the other hand, from the short exact sequence $0 \to \mathrm{Im}\, f \to \Sigma^{-2}(\mathrm{Im}\, f) \to \mathrm{Coker}\, \sigma_{\mathrm{Im}\, f} \to 0$ we obtain the triangle

$$\Sigma^{-2}(\mathrm{Im}\, f)[0] \longrightarrow \mathrm{Coker}(\sigma_{\mathrm{Im}\, f})[0] \longrightarrow \mathrm{Im}(f)[1] \longrightarrow \Sigma^{-2}(\mathrm{Im}\, f)[1]$$

whence

$$H_{\mathcal{H}}(\mathrm{Im}(f)[1]) \cong H_{\mathcal{H}}(\mathrm{Coker}(\sigma_{\mathrm{Im}\, f})[0]) = \mathrm{Coker}(\sigma_{\mathrm{Im}\, f})[0]$$

and the latter term belongs to $\mathrm{fp}(\mathcal{H})$. Then, by Cor. 3.26(i) we obtain that $\mathrm{Coker}\, \sigma_{\mathrm{Im}\, f} \in \text{mod-}R$, as desired.

Conversely, let us prove that our hypotheses (2) and (3) imply Thm. 3.41(3). Let $B \in \mathrm{fp}(\mathcal{TFT}_{-2})$. By Cor. 3.37(ii) there exists an R-linear map

$$f\colon \bigoplus_{i=1}^{n} \Sigma^{-2}(y_{-1}(R/J_i)^{k_i}) \longrightarrow B,$$

which we rename $f\colon N \to B$, with cokernel $C \in \mathcal{T}_{-1}$ and kernel $K \in \mathrm{fp}(\mathcal{TFT}_{-2})$ by hypothesis (3.ii). Let $f = \mu \circ \beta$ be the canonical factorisation of f through its image L. Consider the following commutative diagram with exact rows in $\mathcal{H}$

$$\begin{array}{ccccccccc}
 & & & & & & H_{\mathcal{H}}(C[1]) & & \\
 & & & & & & \big\downarrow \gamma & & \\
0 \longrightarrow & H_{\mathcal{H}}(L[1]) & \xrightarrow{\lambda} & K[2] & \xrightarrow{v[2]} & N[2] & \xrightarrow{H_{\mathcal{H}}(\beta[2])} & H_{\mathcal{H}}(L[2]) & \longrightarrow 0 \\
 & & & \big\downarrow & & \big\| & & \big\downarrow H_{\mathcal{H}}(\mu[2]) & \\
 & 0 & \longrightarrow & H & \longrightarrow & N[2] & \xrightarrow{f[2]} & B[2] & \longrightarrow 0
\end{array}$$

in which $f[2]$ is an epimorphism since its cone in $D(R)$ belongs to $\mathcal{U}[1]$, whereas λ and γ are monomorphisms since $H_{\mathcal{H}}(N[1]) = 0$ and $H_{\mathcal{H}}(B[1]) = 0$, respectively. Moreover, notice that $H_{\mathcal{H}}(C[1]) \cong y_{-1}(C)[1]$, in particular it belongs to $\mathcal{H}_{-1}$. The snake lemma yields a short exact sequence $0 \to \mathrm{Im}^{(\mathcal{H})}(v[2]) \to H \to y_{-1}(C)[1] \to 0$ in which the outer terms are finitely presented objects, as we now show. On the one hand, $\mathrm{Im}^{(\mathcal{H})}(v[2])$ is finitely presented for being a cokernel in $\mathrm{fp}(\mathcal{H})$; indeed, $H_{\mathcal{H}}(L[1]) \cong \mathrm{Coker}(\sigma_L)[0]$ is finitely presented by hypothesis (3.iii) and Cor. 3.26(i). On the other hand, we have

$$H_{\mathcal{H}}(L[2]) = H_{\mathcal{H}}(\Sigma^{-2}(L)[2]) = \Sigma^{-2}(L)[2]$$

and $\Sigma^{-2}(L) \in \mathrm{fp}(\mathcal{TFT}_{-2})$ by Lem. 3.33; moreover, by our condition (2) (i.e. Thm. 3.41(2)) in view of the exact column of the previous diagram, we infer that $y_{-1}(C)[1] \in \mathrm{fp}(\mathcal{H}_{-1}) \subseteq \mathrm{fp}(\mathcal{H})$. By extension-closure, we have $H \in \mathrm{fp}(\mathcal{H})$ as well. Thus, the second exact row of the previous diagram induces the exact sequence of covariant functors

$$\begin{aligned}
0 \to \mathrm{Hom}_{\mathcal{H}}(B[2], -) \to \mathrm{Hom}_{\mathcal{H}}(N[2], -) \to \mathrm{Hom}_{\mathcal{H}}(H, -) - \cdots \\
\cdots \to \mathrm{Ext}^1_{\mathcal{H}}(B[2], -) \to \mathrm{Ext}^1_{\mathcal{H}}(N[2], -) \to \mathrm{Ext}^1_{\mathcal{H}}(H, -)
\end{aligned}$$

in which, since $\mathrm{Ext}^1_{\mathcal{H}}(N[2], -)$ restricted to ${}^{\perp_0}\mathcal{H}_{-1}$ preserves direct limits by (3.i), then also $\mathrm{Ext}^1_{\mathcal{H}}(B[2], -)\restriction$ does so, as desired.

Let us prove that Thm. 3.41 implies our condition (4). First notice that if $X \in \mathrm{fg}(\mathcal{T}_0)$, then there exists $B \in \mathrm{fp}(\mathcal{T}_0)$ and an epimorphism $p\colon B \to X$, whence a short exact sequence $0 \to \mathrm{Ker}(p)[0] \to B[0] \to X[0] \to 0$ in $\mathcal{H}$, which shows that $X[0] \in \mathrm{fg}(\mathcal{H})$. Let now $0 \to Y \to M \xrightarrow{f} N \to X \to 0$ be as in the statement.

Then we obtain the following diagram of $D(R)$

$$\begin{array}{ccccccc} & & & & Y[2] & & \\ & & & & \downarrow & & \\ M[1] & \xrightarrow{f[1]} & N[1] & \longrightarrow & \mathrm{Cone}(f[1]) & \xrightarrow{+} & \\ & & & & \downarrow & & \\ & & & & X[1] & & \\ & & & & \downarrow + & & \end{array}$$

and the rotation of the vertical triangle is a short exact sequence of $\mathcal{H}$ by hypothesis on the cone. In particular, by $0 \to X[0] \to Y[2] \to \mathrm{Cone}(f[1]) \to 0$, being $X[0] \in \mathrm{fg}(\mathcal{H})$ and $Y[2] \in \mathrm{fp}(\mathcal{H})$ (see Lem. 3.33(d)), we infer that $\mathrm{Cone}(f[1])$ is a finitely presented object of $\mathcal{H}$. By [54, Proposition 3.5(2)], the functor $\mathrm{Ext}^1_{\mathcal{H}}(\mathrm{Cone}(f[1]), -)$ commutes with direct limits, in particular those of $\mathcal{T}_0[0]$, but the relevant restriction of the functor is naturally isomorphic to $\mathrm{Hom}_R(X, -)\restriction_{\mathcal{T}_0}$, and we are done.

Let us prove that our conditions (4) and (5.i) imply Thm. 3.41(4). Let $P \in \mathrm{fp}({}^{\perp_0}\mathcal{H}_{-1})$, and consider the associated short exact sequence $0 \to L \xrightarrow{\varepsilon} W \to P \to 0$ as in Lem. 3.34(iii), so that with $L \in \mathcal{H}_0$ and $W = H^{-2}(P)[2]$. By hypothesis (5.i) and Lem. 3.33 we know that $W \in \mathrm{fp}(\mathcal{H})$, thus $L \in \mathrm{fg}(\mathcal{H})$. Therefore, there exists an epimorphism $Q \to L$ originating in a finitely presented complex Q of $\mathcal{H}$, whence we have the epimorphism $H^0(Q) \to H^0(L)$ originating in $H^0(Q) \in \mathrm{fp}(\mathcal{T}_0)$, whence $H^0(L) \in \mathrm{fg}(\mathcal{T}_0)$. Now, since ε is a morphism in

$$\mathrm{Hom}_{D(R)}(L, W) \cong \mathrm{Hom}_{D(R)}(H^0(L)[0], H^{-2}(W)[2]) \cong \mathrm{Ext}^2_R(H^0(L), H^{-2}(W)),$$

then it is represented by an exact sequence

$$0 \longrightarrow H^{-2}(W) \longrightarrow X_2 \xrightarrow{f} X_1 \longrightarrow H^0(L) \longrightarrow 0$$

of Mod-R, in which $\mathrm{Cone}(f[1]) \cong B$. By (4.ii), we deduce that $H^0(L) \in \mathrm{fp}(\mathcal{T}_0)$, i.e. $L \cong H^0(L)[0] \in \mathrm{fp}(\mathcal{H})$ by Cor. 3.26(i).

It remains to treat condition (5). Part (5.i) has been proved in Cor. 3.39. On the other hand, for any $P \in \mathrm{fp}(\mathcal{H})$ consider the approximation $0 \to \boldsymbol{x}(P) \to P \to \boldsymbol{y}(P) \to 0$ within the torsion pair $({}^{\perp_0}\mathcal{H}_{-1}, \mathcal{H}_{-1})$. Its cohomology long exact sequence breaks up in the following exact rows of Mod-R:

$$0 \longrightarrow H^{-2}(\boldsymbol{x}(P)) \longrightarrow H^{-2}(P) \longrightarrow 0$$
$$0 \longrightarrow H^{-1}(\boldsymbol{x}(P)) \longrightarrow H^{-1}(P) \longrightarrow H^{-1}(\boldsymbol{y}(P)) \longrightarrow 0$$
$$0 \longrightarrow H^{0}(P) \longrightarrow H^{0}(\boldsymbol{y}(P)) \longrightarrow 0$$

where the only non-trivial fact is that $H^0(\boldsymbol{x}(P)) = 0$, but this follows since the epimorphism $\boldsymbol{x}(P) \to H^0(\boldsymbol{x}(P))[0]$ is zero by axiom of torsion pair. This said, we have $H^{-1}(\boldsymbol{x}(P)) \in \mathcal{T}_0$ since $\boldsymbol{x}(P) \in {}^{\perp_0}\mathcal{H}_{-1}$, and $H^{-1}(\boldsymbol{y}(P)) \in \mathcal{T}\mathcal{F}\mathcal{T}_{-1} \subseteq \mathcal{F}_0$. Therefore, by the second displayed exact row we deduce $H^{-1}(\boldsymbol{x}(P)) \cong$

$x_0(H^{-1}(P))$. Moreover, by rotating the approximation of $\boldsymbol{x}(P)$ within the standard t-structure of $D(R)$ we obtain the short exact sequence

$$0 \longrightarrow x_0(H^{-1}(P))[0] \longrightarrow H^{-2}(P)[2] \longrightarrow \boldsymbol{x}(P) \longrightarrow 0$$

of $\mathcal{H}$. Now, bearing in mind part (5.i), if $\mathcal{H}$ is locally coherent, then our condition (5.ii) holds true by Cor. 3.26(i); conversely, if $x_0(H^{-1}(P)) \in \mathrm{fp}(\mathcal{T}_0)$, then $\boldsymbol{x}(P) \in \mathrm{fp}(\mathcal{H})$ for being a cokernel of a morphism in $\mathrm{fp}(\mathcal{H})$. □

CHAPTER 4

Applications

4.1. On the HRS hearts

We apply Cor. 3.43 in the case of the HRS heart $\mathcal{H}_{\boldsymbol{\tau}}$ associated with a hereditary torsion pair of finite $\boldsymbol{\tau} := (\mathcal{T}, \mathcal{F})$ in Mod-R; indeed, in Ex. 3.10 we have seen that $\mathcal{H}_{\boldsymbol{\tau}}$ can be realised as the AJS heart associated with the Thomason filtration length 1

$$\Phi : \operatorname{Spec} R \supset Z \supset \emptyset,$$

where Z is the Thomason subset that corresponds to the torsion class $\mathcal{T}$.

Firstly, we see that for an arbitrary torsion pair $\boldsymbol{\tau}$ of Mod-R, one necessary condition for the local coherence of $\mathcal{H}_{\boldsymbol{\tau}}$ is that $\boldsymbol{\tau}$ must be hereditary of finite type. Notice that this follows by [28, Proposition 2.6] since the locally finite presentability of the heart is equivalent to $\mathcal{T} = \varinjlim \operatorname{fp}(\mathcal{T})$ by [48]; however, we now achieve such result with a different argument.

4.1 Proposition. *Let $\boldsymbol{\tau} := (\mathcal{T}, \mathcal{F})$ be any torsion pair in* Mod-R. *If the associated HRS heart $\mathcal{H}_{\boldsymbol{\tau}}$ is a locally finitely presented Grothendieck category, then $\boldsymbol{\tau}$ is hereditary* (*of finite type*).

Proof. By [44] the torsion pair is necessarily of finite type; moreover, since $\mathcal{H}_{\boldsymbol{\tau}}$ is locally finitely presented, by [48, Theorem 6.1] we have in particular $\mathcal{T} = \varinjlim(\mathcal{T} \cap \text{mod-}R)$. Therefore, $\mathcal{T} \cap \text{mod-}R$ is a set (up to isomorphism), whose right orthogonal in Mod-R coincides with $\mathcal{F}$, hence by [8, Theorem 3.3] $\boldsymbol{\tau}$ is a *tCG torsion pair*; that is, its HRS t-structure $(\mathcal{U}_{\boldsymbol{\tau}}, \mathcal{V}_{\boldsymbol{\tau}})$ in $D(R)$ is compactly generated. Consequently, by [23, Lemma 3.7] there exists a Thomason filtration Φ such that $(\mathcal{U}_{\boldsymbol{\tau}}, \mathcal{V}_{\boldsymbol{\tau}}) = (\mathcal{U}_{\Phi}, \mathcal{V}_{\Phi})$. We claim that $\mathcal{T} = \mathcal{T}_{\Phi(0)}$, whence $\mathcal{T}$ turns out to be a hereditary torsion class. This readily follows thanks to the equality $\mathcal{U}_{\boldsymbol{\tau}} = \mathcal{U}_{\Phi}$, namely by taking the 0th cohomology of the stalk $X[0]$ for a module X either in $\mathcal{T}$ or in $\mathcal{T}_{\Phi(0)}$ (see Rem. 3.31(3)). ☐

We recall that the converse of the previous result is known in the literature (see [18, Theorem 2.2] and [23, 48, 55]).

4.2 Corollary. *Let* $\boldsymbol{\tau} := (\mathcal{T}, \mathcal{F})$ *be a torsion pair in* Mod-R, *say with adjunctions*

$$\mathcal{T} \underset{x}{\rightleftarrows} \text{Mod-}R \overset{y}{\rightleftarrows} \mathcal{F}\ .$$

The associated HRS heart $\mathcal{H}_{\boldsymbol{\tau}}$ *is a locally coherent Grothendieck category if and only if* $\boldsymbol{\tau}$ *is hereditary of finite type and the following four conditions hold:*

(i) *The torsion class* $\mathcal{T}$ *is locally coherent;*
(ii) *For every* $B \in$ mod-R, *the functor* $\mathrm{Hom}_R(y(B), -)$ *commutes with direct limits of direct systems in* $\mathcal{T}$*;*
(iii) *For all* $B \in$ mod-R, *the functor* $\mathrm{Ext}^1_R(y(B), -)$ *commutes with direct limits of direct systems of* $\mathcal{F}$*;*
(iv) *For every finitely generated ideal* J *in the Gabriel filter associated with* $\mathcal{T}$, *the functor* $\mathrm{Ext}^2_R(R/J, -)$ *commutes with direct limits of* $\mathcal{F}$.

Proof. The necessity of the torsion pair being hereditary and of finite type has been proved in Prop. 4.1; this said, we shall prove the present Corollary by showing that the listed four conditions are equivalent to the corresponding ones of Cor. 3.43.

It is clear that our hypothesis (i) is precisely Cor. 3.43(1). On the other hand, we have $\mathcal{T}\mathcal{F}\mathcal{T}_{-1} = \mathcal{T}\mathcal{F}_{-1} = \mathcal{F}_0$, thus $\mathrm{fp}(\mathcal{T}\mathcal{F}\mathcal{T}_{-1}) = \mathrm{add}\, y(\text{mod-}R)$ (see Rem. 3.16). The previous equality together with the additivity of the bifunctors $\mathrm{Hom}_R(-,-)$ and $\mathrm{Ext}^1_R(-,-)$ shows that also our hypotheses (ii) and (iii) are equivalent to the corresponding conditions of Cor. 3.43. Moreover, it is clear that Cor. 3.43(4) implies our condition (iv). Let us prove that our hypotheses (i) and (iv) imply Cor. 3.43(4). Let $Q \in \mathrm{fp}(\mathcal{T})$ and let $(Y_i)_{i\in I}$ be a direct systems of modules in $\mathcal{F}$. By Prop. 3.4 there exist a finitely generated ideal J in the Gabriel filter of the torsion pair $(\mathcal{T}, \mathcal{F})$ and a short exact sequence $0 \to X \to (R/J)^n \to Q \to 0$, for some $n \in \mathbb{N}$ and X a torsion module (we have $X \in \mathrm{fp}(\mathcal{T})$ by (i)). By applying the functors

$$L^k := \varinjlim_{i\in I} \mathrm{Ext}^k_R(-, Y_i) \quad \text{and} \quad \Gamma^k := \mathrm{Ext}^k_R(-, \varinjlim_{i\in I} Y_i) \qquad (k \geq 1)$$

on the above short exact sequence, we get the commutative diagram with exact rows

$$\begin{array}{ccccccccccccc}
0 \longrightarrow & L^1(Q) & \longrightarrow & L^1((R/J)^n) & \longrightarrow & L^1(X) & \longrightarrow & L^2(Q) & \longrightarrow & L^2((R/J)^n) & \longrightarrow & L^2(X) \\
& \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_4} & & \downarrow{\scriptstyle g_5} & & \downarrow{\scriptstyle g_6} \\
0 \longrightarrow & \Gamma^1(Q) & \longrightarrow & \Gamma^1((R/J)^n) & \longrightarrow & \Gamma^1(X) & \longrightarrow & \Gamma^2(Q) & \longrightarrow & \Gamma^2((R/J)^n) & \longrightarrow & \Gamma^2(X)
\end{array}$$

in which g_1, g_2, g_3 are isomorphisms since $Q[0], (R/J)^n[0], X[0] \in \mathrm{fp}(\mathcal{H})$ by Cor. 3.26(i), g_5 is iso by condition (iv), while g_6 is a monomorphism by [48, Proposition 1.6], so that g_4 is iso as well. This concludes the proof. □

4.3 Remark. A more general characterisation of the local coherence of the HRS hearts has been achieved in [48, Sec. 7] in the context of locally finitely presented Grothendieck categories.

4.4 Example. Let us exhibit an example of hereditary torsion pair of finite type whose HRS heart is not locally coherent. Consider the non-coherent commutative ring $R := \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$ introduced in Ex. 3.22. For any nonzero tuple $a \in (\mathbb{Z}/2\mathbb{Z})^{(\mathbb{N})}$, the non unitary element $e := (1, a)$ is idempotent, and the ideal $J := eR$ is idempotent as well. Therefore, J gives rise to a TTF triple $(\mathcal{E}, \mathcal{T}, \mathcal{F})$ in Mod-R which is *split*; that is (see [58, Proposition VI.8.5]), in which $\mathcal{E} = \mathcal{F}$ and both the torsion pairs $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{T})$ are hereditary. In particular, $(\mathcal{T}, \mathcal{F})$ is of finite type for being $\mathcal{F} = \operatorname{Ker}\operatorname{Hom}_R(R/J, -)$; moreover it restricts to mod-R being split. We claim that it does not have a locally coherent HRS heart. Assume, by contradiction, that such $\mathcal{H}$ is locally coherent. Since the ring R is non-coherent, there exists an R-linear epimorphism $f\colon M \to N$ in mod-R such that $\operatorname{Ker} f$ is not finitely presented. In the exact row

$$0 \longrightarrow \operatorname{Ker} x(f) \longrightarrow \operatorname{Ker} f \xrightarrow{\ a\ } \operatorname{Ker} y(f) \longrightarrow \operatorname{Coker} x(f) \longrightarrow 0$$

provided by the snake lemma, we have $\operatorname{Ker} y(f) \in \operatorname{fp}(\mathcal{F})$ by conditions (ii) and (iii) of the previous corollary. On the other hand, $\operatorname{Ker} x(f), \operatorname{Coker} x(f) \in \operatorname{fp}(\mathcal{T})$ since the torsion pair is split, so that $x(M)$ and $x(N)$ are finitely presented objects of $\mathcal{T}$, which is locally coherent (as proved in the previous corollary). By hypotheses (i) and (ii) we infer that $\operatorname{Im} a \in$ mod-R, thus we get the contradiction $\operatorname{Ker} f \in$ mod-R by the extension-closure of the finitely presented modules.

4.5 Example. Let us exhibit an example of a quasi locally coherent category in which the finitely presented objects do not form an abelian category. Consider the ordinary torsion pair $(\mathcal{T}, \mathcal{F})$ formed by the torsion and the torsionfree abelian groups. Since $\mathbb{Z}$ is noetherian, the torsion pair is of finite type and restricts to the finitely presented $\mathbb{Z}$-modules, hence by [54, Theorem 5.2] its HRS heart $\mathcal{H}$ is locally coherent. By Ex. 3.10, we know that there exists a proper Thomason subset Z of $\operatorname{Spec}\mathbb{Z}$ such that $\mathcal{H}$ is the heart of the Thomason filtration $\operatorname{Spec}\mathbb{Z} \supset Z \supset \emptyset$. We then obtain a TTF triple of finite type

$$({}^{\perp_0}\mathcal{H}_Z, \mathcal{H}_Z, \mathcal{H}_Z^{\perp_0})$$

in which ${}^{\perp_0}\mathcal{H}_Z$ ($\cong \mathcal{F}[1]$) is a quasi locally coherent category by Thm. 3.2. We claim that this torsion class is not locally coherent, i.e. that the subcategory $\operatorname{fp}({}^{\perp_0}\mathcal{H}_Z)$ is not abelian. By contradiction, assume that this is the case and let $M \in \mathcal{T}$ be a finitely presented nonzero $\mathbb{Z}$-module, say it presented by the exact sequence $\mathbb{Z}^n \xrightarrow{f} \mathbb{Z}^m \to M \to 0$. Now, $\mathbb{Z}^n[1]$ and $\mathbb{Z}^m[1]$ are finitely presented objects of $\mathcal{H}$ ([56, Lemma 6.3]), therefore from the exact row

$$0 \longrightarrow \operatorname{Ker}^{({}^{\perp_0}\mathcal{H}_Z)}(f[1]) \longrightarrow \mathbb{Z}^n[1] \xrightarrow{f[1]} \mathbb{Z}^m[1] \longrightarrow \operatorname{Coker}^{(\mathcal{H})}(f[1]) \longrightarrow 0$$

which lives in $\operatorname{fp}({}^{\perp_0}\mathcal{H}_Z)$ by quasi local coherence, we would have

$$\operatorname{Coker}^{(\mathcal{H})}\operatorname{Ker}^{({}^{\perp_0}\mathcal{H}_Z)}(f[1]) \cong \operatorname{Ker}^{({}^{\perp_0}\mathcal{H}_Z)}\operatorname{Coker}^{(\mathcal{H})}(f[1])\ .$$

On the other hand, the canonical short exact sequences in Ab given by the factorisation of f through its image yield, once one takes the shifted stalk complexes, the following commutative diagram with exact rows in $\mathcal{H}$,

$$\begin{array}{ccccccccc}
 & & & & & & M[0] & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & (\operatorname{Ker} f)[1] & \longrightarrow & \mathbb{Z}^n[1] & \longrightarrow & (\operatorname{Im} f)[1] & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ker}^{(\mathcal{H})}(f[1]) & \longrightarrow & \mathbb{Z}^n[1] & \xrightarrow{f[1]} & \mathbb{Z}^m[1] & \dashrightarrow & 0
\end{array}$$

whence we see that $f[1]$ is an epimorphism, so that

$$\operatorname{Ker}^{({}^{\perp_0}\mathcal{H}_Z)} \operatorname{Coker}^{(\mathcal{H})}(f[1]) \cong \mathbb{Z}^m[1],$$

and being

$$\operatorname{Coker}^{(\mathcal{H})} \operatorname{Ker}^{({}^{\perp_0}\mathcal{H}_Z)}(f[1]) \cong (\operatorname{Im} f)[1],$$

we conclude $M[0] = 0$, contradiction.

4.1.1. When the ring is coherent. When the ring is coherent, our previous characterisations furtherly lighten, as we shall prove in Cor. 4.7. Let us start with an interesting example, which can be deduced formerly by [54, Theorem 5.2].

4.6 Example. Let R be a commutative coherent ring, and let $(\mathcal{X}, \mathcal{Y})$ be a torsion pair in the abelian category mod-R; by [14, p. 1666] $(\varinjlim \mathcal{X}, \varinjlim \mathcal{Y}) =: (\mathcal{T}, \mathcal{F})$ is a torsion pair (of finite type) in Mod-R. We claim that the associated HRS heart in $D(R)$ is a locally coherent Grothendieck category, namely by showing that the torsion pair is hereditary and satisfies the four conditions of the previous Corollary.

The torsion pair $(\mathcal{T}, \mathcal{F})$ is hereditary by the same argument of the proof of Prop. 4.1, namely for it is a tCG torsion pair.

(i) Since Mod-R is a locally coherent Grothendieck category, then $\mathcal{T}$ is so.

On the other hand, the torsion pair $(\mathcal{T}, \mathcal{F})$ restricts to mod-R, so $y(B)$ is a finitely presented module for all $B \in$ mod-R, whence it is clear that conditions (ii), (iii) and (iv) of Cor. 4.2 hold true, since R is coherent.

4.7 Corollary. *Let R be a commutative coherent ring and $\boldsymbol{\tau} := (\mathcal{T}, \mathcal{F})$ be a torsion pair in* Mod-R. *Then the HRS heart $\mathcal{H}_{\boldsymbol{\tau}}$ is a locally coherent Grothendieck category if and only if*

- (i) *The torsion pair $\boldsymbol{\tau}$ is hereditary of finite type;*
- (ii) *For all $B \in$ mod-R, the functor $\operatorname{Hom}_R(y(B), -)$ commutes with direct limits of direct systems in $\mathcal{T}$;*
- (iii) *For all $B \in$ mod-R, the functor $\operatorname{Ext}^1_R(y(B), -)$ commutes with direct limits of direct systems of $\mathcal{F}$.*

4.1.2. When the torsion pair is stable. We equip the torsion pairs of Mod-R with a homological condition, namely we consider the stable torsion pairs, i.e. those whose torsion class is closed under taking injective envelopes. As we shall see, such a homological condition translates into a finiteness one and, in particular, the necessary and sufficient conditions for the local coherence of the involved HRS hearts furtherly simplify. In fact, our assumption is consistent and independent from the previous subsection, thanks to Ex. 4.4, which exhibits a non-trivial TTF triple $(\mathcal{F}, \mathcal{T}, \mathcal{F})$ over a non-coherent commutative ring, in which $(\mathcal{T}, \mathcal{F})$ is stable for $(\mathcal{F}, \mathcal{T})$ being hereditary.

We need some auxiliary preliminary results, which in fact specialise the conditions of Cor. 4.2 within the stability assumption.

4.8 Lemma. *If $(\mathcal{T}, \mathcal{F})$ is a stable hereditary torsion pair of* Mod-R, *then for every $X, Y \in \mathcal{T}$ we have $\operatorname{Ext}^k_R(X, Y) \cong \operatorname{Ext}^k_{\mathcal{T}}(X, Y)$, for all $k \in \mathbb{N} \cup \{0\}$.*

Proof. By the adjuction $\jmath : \mathcal{T} \rightleftarrows \text{Mod-}R : x$ and by [42, Proposition 2.28], we have the adjoint pair

$$\mathbf{L}\jmath : D(\mathcal{T}) \rightleftarrows D(R) : \mathbf{R}x$$

of derived functors. In particular, for all $X, Y \in \mathcal{T}$, regarding the stalk of X as an object of $D(\mathcal{T})$ and the stalk of Y as a complex of $D(R)$, being x an exact functor by hereditariness, we have the natural isomorphism

$$\begin{aligned}
\operatorname{Hom}_{D(R)}(\mathbf{L}\jmath(X[0]), Y[n]) &\cong \operatorname{Hom}_{D(\mathcal{T})}(X[0], \mathbf{R}x(Y[n])) \\
&= \operatorname{Hom}_{D(\mathcal{T})}(X[0], x(\mathbf{i}Y[n])) \\
&\cong \operatorname{Hom}_{D(\mathcal{T})}(X[0], \mathbf{i}Y[n]) \\
&\cong \operatorname{Hom}_{D(\mathcal{T})}(X[0], Y[n]),
\end{aligned}$$

where $\mathbf{i}$ is the homotopically injective coresolution functor, computed equivalently either on $D(R)$ or in $D(\mathcal{T})$, for $\mathcal{T}$ being a stable torsion class and an exact subcategory of Mod-R. By [61, Proposition 3.1.3, page 191], the latter group of the display is isomorphic to $\operatorname{Ext}^n_{\mathcal{T}}(X, Y)$, so we claim that the first displayed group is isomorphic to $\operatorname{Ext}^n_R(X, Y)$. Indeed, we have

$$\begin{aligned}
\operatorname{Hom}_{D(R)}(\mathbf{L}\jmath(X[0]), Y[n]) &= \operatorname{Hom}_{D(R)}(\jmath(\mathbf{p}X[0]), Y[n]) \\
&\cong \operatorname{Hom}_{D(R)}(\mathbf{p}X[0], Y[n]) \\
&\cong \operatorname{Hom}_{D(R)}(X[0], Y[n]),
\end{aligned}$$

where $\mathbf{p} \colon D(R) \to K(R)$ is the homotopically projective resolution functor. □

4.9 Lemma. *Let $(\mathcal{T}, \mathcal{F})$ be a stable torsion pair of* Mod-R. *Assume that conditions* (i) *and* (iv) *of Cor. 4.2 hold true. Then, for every finitely generated ideal J in the Gabriel filter associated with $\mathcal{T}$, it is $R/J \in \operatorname{fp}_3(R)$, i.e. the functors $\operatorname{Ext}^k_R(R/J, -)$ commute with direct limits for $k = 0, 1, 2$.*

Proof. Let $(M_i)_{i\in I}$ be a direct system in Mod-R and consider the direct system $(0 \to X_i \to M_i \to Y_i \to 0)_{i\in I}$ formed by the approximations of its members within $(\mathcal{T}, \mathcal{F})$. Since R/J is a finitely presented torsion module (so that $R/J[0]$ is a finitely presented object of $\mathcal{H}$), by applying the functors

$$L^k := \varinjlim_{i\in I} \operatorname{Ext}^k_R(R/J, -) \quad \text{and} \quad \Gamma^k := \operatorname{Ext}^k_R(R/J, \varinjlim_{i\in I}(-)) \qquad (k \in \mathbb{N} \cup \{0\})$$

on the latter direct system, we find at once that $L^0(M_i) \cong \Gamma^0(M_i)$; moreover, in the following commutative diagram with exact rows,

$$\begin{array}{ccccccccccccc} L^1(X_i) & \rightarrowtail & L^1(M_i) & \longrightarrow & L^1(Y_i) & \longrightarrow & L^2(X_i) & \longrightarrow & L^2(M_i) & \longrightarrow & L^2(Y_i) & \longrightarrow & L^3(X_i) \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} & & \downarrow{\scriptstyle f_6} & & \downarrow{\scriptstyle f_7} \\ \Gamma^1(X_i) & \rightarrowtail & \Gamma^1(M_i) & \longrightarrow & \Gamma^1(Y_i) & \longrightarrow & \Gamma^2(X_i) & \longrightarrow & \Gamma^2(M_i) & \longrightarrow & \Gamma^2(Y_i) & \longrightarrow & \Gamma^3(X_i) \end{array}$$

the canonical maps f_1, f_4, f_7 are isomorphisms by hypothesis (i) and Lem. 4.8, f_6 is an isomorphism by hypothesis (iv), while f_3 is iso as well by [61, 6] and since $R/J[0] \in \mathrm{fp}(\mathcal{H})$. Therefore, by the five lemma we deduce that f_2 and f_5 are isomorphisms, as desired. □

4.10 REMARK. By the proof of [19, Lemma 2.14] every indexing set I is the union of a well-ordered chain of directed subposets $(I_\alpha \mid \alpha < \lambda)$, where each I_α has cardinality less than I. Moreover, for every direct system $(M_i)_{i\in I}$ of R-modules, $(\varinjlim_{i\in I_\alpha} M_i \mid \alpha < \lambda)$ is a well-ordered direct system satisfying

$$\varinjlim_{i\in I} M_i = \varinjlim_{\alpha<\lambda} \varinjlim_{i\in I_\alpha} M_i \ .$$

4.11 LEMMA. *Let* $(\mathcal{T}, \mathcal{F})$ *be a stable torsion pair in* Mod-R. *Assume that condition* (ii) *of Cor. 4.2 holds true. Then, for every* $B \in$ mod-R *and every direct system* $(M_i)_{i\in I}$ *in* $\mathcal{T}$, *the canonical homomorphism*

$$\varinjlim_{i\in I} \mathrm{Ext}^1_R(y(B), M_i) \longrightarrow \mathrm{Ext}^1_R(y(B), \varinjlim_{i\in I} M_i)$$

is injective.

Proof. We formerly prove the statement in case I is a well ordered directed poset.

If I is a finite set, there exist indices $\bar{\imath}, \bar{\jmath} \in I$ such that $\varinjlim_{i\in I} M_i = M_{\bar{\imath}}$ and $\varinjlim_{i\in I} \mathrm{Ext}^1_R(y(B), M_i) = \mathrm{Ext}^1_R(y(B), M_{\bar{\jmath}})$; moreover, there exists $k \geq \bar{\imath}, \bar{\jmath}$ making the displayed canonical map an isomorphism indeed.

If I is infinite, by [23, Lemma 3.5] there exists a direct system $(0 \to M_i \to E_i \to C_i \to 0)_{i\in I}$ in which E_i is the injective envelope of M_i, so that the direct system is in $\mathcal{T}$ by the stability hypothesis. Therefore, the canonical homomorphism displayed in the statement factors through the kernel of the map

$$\mathrm{Ext}^1_R(y(B), \varinjlim_{i\in I} M_i) \longrightarrow \mathrm{Ext}^1_R(y(B), \varinjlim_{i\in I} E_i)$$

by means of an isomorphism, thanks to the snake lemma and the assumption on $y(B)$ (similarly to the proof [48, Proposition 1.6]). In other words, our statement is true for well ordered directed posets.

This said, the general case follows as soon as we write $I = \bigcup_{\alpha<\lambda} I_\alpha$ as in Rem. 4.10; indeed, by the argument of the previous part (applied twice) and by AB-5 condition of abelian groups, we obtain the following composition of monomorphisms

$$\varinjlim_{\alpha<\lambda} \varinjlim_{i\in I_\alpha} \mathrm{Ext}^1_R(y(B), M_i) \hookrightarrow \varinjlim_{\alpha<\lambda} \mathrm{Ext}^1_R(y(B), \varinjlim_{i\in I_\alpha} M_i) \hookrightarrow \mathrm{Ext}^1_R(y(B), \varinjlim_{\alpha<\lambda} \varinjlim_{i\in I_\alpha} M_i),$$

which coincides with the natural map of the statement. □

4.12 COROLLARY. *Let* $(\mathcal{T}, \mathcal{F})$ *be a stable torsion pair in* Mod-R. *Then its HRS heart* $\mathcal{H}$ *is a locally coherent Grothendieck category if and only if the torsion pair is of finite type and the following three conditions hold:*

(i) $\mathrm{fp}(\mathcal{T}) \subseteq \mathrm{fp}_3(R)$*;*
(ii) $\mathrm{fp}(\mathcal{F}) \subseteq$ mod-R*;*
(iii) *For all* $B \in$ mod-R, *the functor* $\mathrm{Ext}^1_R(y(B), -)$ *commutes with direct limits of direct systems of* $\mathcal{F}$.

Proof. We shall prove that the stated conditions are equivalent to the ones of Cor. 4.2. Let us start by proving that our three hypotheses imply the conditions of the corollary.
(i) By Prop. 3.19, Prop. 4.1, and [55], $\mathcal{T}$ is a locally finitely presented Grothendieck category. It remains to show that $\mathrm{fp}(\mathcal{T})$ is an exact abelian subcategory of $\mathcal{T}$, and this follows by condition (i), namely for the kernel of any epimorphism in $\mathrm{fp}(\mathcal{T})$ is finitely presented as well.
(ii) It follows immediately by our hypothesis (ii).
(iv) It follows immediately by our hypothesis (i).

Let us prove that the four conditions of Cor. 4.2 imply our hypotheses (i) and (ii).
(i) Let $B \in \mathrm{fp}(\mathcal{T})$; by Cor. 3.26 there exist finitely generated ideals J', J in the Gabriel filter associated with $\mathcal{T}$ and an exact row $(R/J')^n \overset{\alpha}{\to} (R/J)^m \to B \to 0$ for some $n, m \in \mathbb{N}$. By Lem. 4.9, R/J and R/J' are objects of $\mathrm{fp}_3(R)$, thus, being $\mathrm{Ker}\,\alpha$ a finitely presented torsion module by Cor. 4.2(i), in view e.g. of [10] we infer that $\mathrm{Im}\,\alpha \in \mathrm{fp}_2(R)$ and consequently that $B \in \mathrm{fp}_3(R)$.
(ii) Since $\mathrm{fp}(\mathcal{F}) = \mathrm{add}\, y(\text{mod-}R)$, we shall prove our assertion (ii) on torsionfree modules of the form $y(B)$, where $B \in$ mod-R. Let $(M_i)_{i\in I}$ be a direct system in Mod-R and consider its approximation $(0 \to X_i \to M_i \to Y_i \to 0)_{i\in I}$ within $(\mathcal{T}, \mathcal{F})$. By applying the functors

$$L^k := \varinjlim_{i\in I} \mathrm{Ext}^1_R(y(B), -) \quad \text{and} \quad \Gamma^k := \mathrm{Ext}^k_R(y(B), \varinjlim_{i\in I}(-)) \qquad (k \in \mathbb{N} \cup \{0\})$$

on the latter direct system, we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^0(X_i) & \longrightarrow & L^0(M_i) & \longrightarrow & L^0(Y_i) & \longrightarrow & L^1(X_i) \\ & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_4} \\ 0 & \longrightarrow & \Gamma^0(X_i) & \longrightarrow & \Gamma^0(M_i) & \longrightarrow & \Gamma^0(Y_i) & \longrightarrow & \Gamma^1(X_i) \end{array}$$

in which g_1 is an isomorphism by Cor. 4.2(ii), g_3 is isomorphism since $y(B) \in \mathrm{fp}(\mathcal{F})$, and g_4 is monic by Lem. 4.11. By the five lemma, we conclude that $y(B)$ is a finitely presented module. □

4.2. On the restrictability over a commutative noetherian ring

We want to recover Thm. 2.6 for Thomason filtrations of finite length; that is, we want to check that in case the commutative ring R is noetherian, given a Thomason filtration Φ of length $l+1$ whose (compactly generated) t-structure $(\mathcal{U}, \mathcal{V})$ restricts to $D^b(\text{mod-}R)$, then its heart is locally coherent and $\text{fp}(\mathcal{H}) = \mathcal{H} \cap D^b(\text{mod-}R)$.

4.13 Lemma. *Let R be a commutative noetherian ring. For every Thomason filtration Φ of* Spec R, *if $X \in \text{fp}(\mathcal{T}_k)$, then $H_{\mathcal{H}}(X[-k]) \in \text{fp}(\mathcal{H})$.*

Proof. Notice that it suffices to prove the statement for the heart $\mathcal{H}_k$, instead of $\mathcal{H}$, thanks to Thm. 3.29. This said, we conclude by Lem. 3.36 using the hypotheses on R and X. □

4.14 Lemma. *Let R be a commutative noetherian ring, and let Φ be a Thomason filtration whose t-structure $(\mathcal{U}, \mathcal{V})$ restricts to $D^b(\text{mod-}R)$. For all integers $k \in \mathbb{Z}$, if $X \in \mathcal{T}_k$ is such that $H_{\mathcal{H}}(X[-k]) \in \text{fp}(\mathcal{H})$, then $H_{\mathcal{H}}(X[-k]) \in D^b(\text{mod-}R)$.*

Proof. Let $(X_i)_{i\in I}$ be the canonical direct system of finitely generated submodules of X in $\mathcal{T}_k$, such that $X = \varinjlim_{i\in I} X_i$. Since each torsion pair associated with Φ is of finite type, we have

$$y_{k+1}(X) \cong \varinjlim_{i\in I} y_{k+1}(X_i);$$

and the direct system $(y_{k+1}(X_i))_{i\in I}$ lives in $\mathcal{TF}_k$, and $y_{k+1}(X)$ as well. Therefore, in view of Rem. 3.32, we get a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & y_{k+1}(X_i) & \xrightarrow{\sigma_i} & \Sigma^k(y_{k+1}(X_i)) & \longrightarrow & C_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & y_{k+1}(X) & \xrightarrow{\sigma_X} & \Sigma^k(y_{k+1}(X)) & \longrightarrow & C_X & \longrightarrow & 0
\end{array}$$

where $C_X, C_i \in \mathcal{T}_{k+2}$, for all $i \in I$, and the vertical R-linear maps are the canonical ones. Moreover, we have

$$\Sigma^k(y_{k+1}(X)) \cong H^k(H_{\mathcal{H}}(X[-k])) \ .$$

We claim that the vertical R-linear map of the previous diagram is an isomorphism. First, observe that the functor $H_{\mathcal{H}} \colon D(R) \to \mathcal{H}$ commutes with coproduct (see [4, Lemma 3.3]). This said, consider the following diagram of triangles of $D(R)$,

$$\begin{array}{ccccccc}
 & & & & (\text{Ker} f)[-k+1] & & \\
 & & & & \downarrow & & \\
\coprod_{i\leq j} X_{ji}[-k] & \xrightarrow{f[-k]} & \coprod_{i\in I} X_i[-k] & \longrightarrow & Z & \xrightarrow{+} & \\
 & & & & \downarrow & & \\
 & & & & X[-k] & & \\
 & & & & \downarrow{+} & &
\end{array}$$

where $f\colon \bigoplus_{i\leq j} X_{ji} \to \bigoplus_{i\in I} X_i$ is the colimit-defining homomorphism of X in Mod-R. Applying $H_{\mathcal{H}}$ on the diagram, we obtain the following one with exact rows

$$\begin{array}{ccccccc}
\bigoplus\limits_{i\leq j} H_{\mathcal{H}}(X_{ji}[-k]) & \longrightarrow & \bigoplus\limits_{i\in I} H_{\mathcal{H}}(X_i[-k]) & \longrightarrow & \varinjlim\limits_{i\in I}{}^{(\mathcal{H})} H_{\mathcal{H}}(X_i[-k]) & \longrightarrow & 0 \\
\cong\downarrow & & \downarrow\cong & & \vdots\cong & & \\
H_{\mathcal{H}}(\coprod\limits_{i\leq j} X_{ji}[-k]) & \longrightarrow & H_{\mathcal{H}}(\coprod\limits_{i\in I} X_i[-k]) & \longrightarrow & H_{\mathcal{H}}(Z) & \longrightarrow & 0 \\
 & & & & \downarrow\cong & & \\
 & & & & H_{\mathcal{H}}(X[-k]) & &
\end{array}$$

where the dotted morphism is given by the universal property of the cokernel, and the last vertical morphism is iso since $(\operatorname{Ker} f)[-k+1] \in \mathcal{U}[1]$. The diagram show that $H_{\mathcal{H}}(X[-k]) \cong \varinjlim_{i\in I}^{(\mathcal{H})} H_{\mathcal{H}}(X_i[-k])$, and being the former a finitely presented object of $\mathcal{H}$, its identity morphism factors through $H_{\mathcal{H}}(X_{\bar{\imath}}[-k])$, for some index $\bar{\imath} \in I$. Consequently, the resulting R-linear map

$$H^k(H_{\mathcal{H}}(X_{\bar{\imath}}[-k])) \longrightarrow H^k(H_{\mathcal{H}}(X[-k]))$$

is a split epimorphism, by additivity of the standard cohomology functors. Such homomorphism coincides with the relevant vertical R-linear map in the middle of the first commutative diagram of the proof. Eventually, from the following commutative diagram with canonical morphisms

$$\begin{array}{ccccccccc}
0 & \longrightarrow & x_{k+1}(X_{\bar{\imath}}) & \longrightarrow & X_{\bar{\imath}} & \longrightarrow & y_{k+1}(X_{\bar{\imath}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & x_{k+1}(X) & \longrightarrow & X & \longrightarrow & y_{k+1}(X) & \longrightarrow & 0
\end{array}$$

by the snake lemma we deduce that $y_{k+1}(X_{\bar{\imath}}) \to y_{k+1}(X)$ is monic as well, so that, in the first commutative diagram, the left vertical map is injective too. Altogether, in such diagram, by the snake lemma again, we infer that the middle vertical map is an isomorphism, since its kernel is a submodule of the kernel of $C_{\bar{\imath}} \to C_X$, which belongs to $\mathcal{T}_{k+2} \subseteq \mathcal{T}_{k+1}$. On the other hand, $y_{k+1}(X_{\bar{\imath}})[-k-1]$ belongs to $D^b(\text{mod-}R)$, and since the t-structure restricts by hypothesis, the associated triangle, call it

$$U \longrightarrow y_{k+1}(X_{\bar{\imath}})[-k-1] \longrightarrow V \overset{+}{\longrightarrow},$$

lives in $D^b(\text{mod-}R)$. Now, in view of the proof of Lem. 3.33, we compute:

$$\begin{aligned}
H_{\mathcal{H}}(X[-k]) &\cong H_{\mathcal{H}}(y_{k+1}(X)[-k]) \cong H_{\mathcal{H}}(\Sigma^k(y_{k+1}(X))[-k]) \\
&\cong H_{\mathcal{H}}(\Sigma^k(y_{k+1}(X_{\bar{\imath}}))[-k]) \cong H_{\mathcal{H}}(y_{k+1}(X_{\bar{\imath}})[-k]) = V[1],
\end{aligned}$$

where the equality follows from the previous triangle. Since $V[1]$ belongs to $D^b(\text{mod-}R)$, we conclude. □

4.15 Proposition. *For every complex $M \in D^b(\text{mod-}R)$, every direct system $(X_i)_{i\in I}$ in* Mod-R, *and integer $k \in \mathbb{Z}$, the canonical morphism*

$$\varinjlim_{i\in I} \operatorname{Hom}_{D(R)}(M, X_i[k]) \longrightarrow \operatorname{Hom}_{D(R)}(M, (\varinjlim_{i\in I} X_i)[k])$$

is an isomorphism.

Proof. Let us start by showing that the displayed homomorphism is epic. Consider the standard triangle $\tau^{\leq -k-1}(M) \to M \to \tau^{>-k-1}(M) \overset{+}{\to}$ and apply the cohomological contravariant hom functors of the stalks $X_i[k]$ and $(\varinjlim_{i\in I} X_i)[k]$, to get the commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \varinjlim_{i\in I} \operatorname{Hom}_{D(R)}(\tau^{>-k-1}(M), X_i[k]) & \longrightarrow & \varinjlim_{i\in I} \operatorname{Hom}_{D(R)}(M, X_i[k]) & \longrightarrow 0 \\
& \downarrow & & \downarrow & \\
0 \longrightarrow & \operatorname{Hom}_{D(R)}(\tau^{>-k-1}(M), (\varinjlim_{i\in I} X_i)[k])) & \longrightarrow & \operatorname{Hom}_{D(R)}(M, (\varinjlim_{i\in I} X_i)[k]) & \longrightarrow 0
\end{array}$$

In order to conclude our claim, it suffices to check that the left vertical homomorphism is epic. Consider the triangle $H^{-k}(M)[k] \to \tau^{>-k-1}(M) \to \tau^{>-k-2}(M) \overset{+}{\to}$, which we rename $H \to V_1 \to V_2 \overset{+}{\to}$ for brevity. Applying the same functors of before, call them respectively Γ_i^k and Γ^k, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\varinjlim_{i\in I} \Gamma_i^k(V_2[-1]) & \to & \varinjlim_{i\in I} \Gamma_i^k(H) & \to & \varinjlim_{i\in I} \Gamma_i^k(V_1) & \to & \varinjlim_{i\in I} \Gamma_i^k(V_2) & \to & \varinjlim_{i\in I} \Gamma_i^k(H[1]) \\
\downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} & & \downarrow{\scriptstyle \mu_4} & & \downarrow{\scriptstyle \mu_5} \\
\Gamma^k(V_2[-1]) & \longrightarrow & \Gamma^k(H) & \longrightarrow & \Gamma^k(V_1) & \longrightarrow & \Gamma^k(V_2) & \longrightarrow & \Gamma^k(H[1])
\end{array}$$

where μ_2 and μ_5 are isomorphisms since $H^{-k}(M)$ is finitely presented. By the snake lemma, we reduced to prove that μ_4 is epic. Moreover, arguing inductively, bearing in mind that M belongs to $D^b(\text{mod-}R)$, it suffices to check our claim for a truncation $\tau^{>j}(M)$, for some $j \leq -k$, having just two nonzero cohomologies. In such case, we have the following standard triangle:

$$H^{j-1}(M)[-j+1] \longrightarrow \tau^{>j}(M) \longrightarrow H^{j-2}(M)[-j+2] \overset{+}{\longrightarrow},$$

say it $\widetilde{H} \to \widetilde{V}_1 \to \widetilde{V}_2 \overset{+}{\to}$. Applying the functors Γ_i^k and Γ^k as above, we obtain

$$\begin{array}{ccccccccc}
\varinjlim_{i\in I} \Gamma_i^k(\widetilde{V}_2[-1]) & \to & \varinjlim_{i\in I} \Gamma_i^k(\widetilde{H}) & \to & \varinjlim_{i\in I} \Gamma_i^k(\widetilde{V}_1) & \to & \varinjlim_{i\in I} \Gamma_i^k(\widetilde{V}_2) & \to & \varinjlim_{i\in I} \Gamma_i^k(\widetilde{H}[1]) \\
\downarrow{\scriptstyle \widetilde{\mu}_1} & & \downarrow{\scriptstyle \widetilde{\mu}_2} & & \downarrow{\scriptstyle \widetilde{\mu}_3} & & \downarrow{\scriptstyle \widetilde{\mu}_4} & & \downarrow{\scriptstyle \widetilde{\mu}_5} \\
\Gamma^k(\widetilde{V}_2[-1]) & \longrightarrow & \Gamma^k(\widetilde{H}) & \longrightarrow & \Gamma^k(\widetilde{V}_1) & \longrightarrow & \Gamma^k(\widetilde{V}_2) & \longrightarrow & \Gamma^k(\widetilde{H}[1])
\end{array}$$

whose vertical homomorphisms but $\widetilde{\mu}_3$ are bijective, since $H^{j-1}(M)$ and $H^{j-2}(M)$ are finitely presented. By the five lemma, $\widetilde{\mu}_3$ is bijective as well.

The injectivity of the stated homomorphism follows by the recursive argument of the previous part. Indeed, since $V_2[-1]$ belongs to $D^b(\text{mod-}R)$, the homomorphism μ_1 is surjective, so that μ_3 is an isomorphism if and only if μ_4 is so; this said, since in the last step of the previous argument we proved that $\widetilde{\mu}_3$ is bijective, it follows that μ_4 will be bijective as well. □

4.16 PROPOSITION. *Let Φ be a Thomason filtration of finite length. Then for every complex $M \in \mathcal{H} \cap D^b(\text{mod-}R)$ we have that $M \in \text{fp}_2(\mathcal{H})$; that is, the functors $\text{Hom}_{\mathcal{H}}(M, -)$ and $\text{Ext}^1_{\mathcal{H}}(M, -)$ preserve direct limits.*

Proof. Let $l + 1$ be the length of Φ. Let $(M_i)_{i\in I}$ be a direct system of $\mathcal{H}$. Recall that the standard cohomology functors preserve direct limits of $\mathcal{H}$, so that

$$H^k(\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i) \cong \varinjlim_{i\in I} H^k(M_i)$$

for all $k \in \mathbb{Z}$. Let us prove that, for all integers $k \in \mathbb{Z}$, the canonical morphism

$$\varinjlim_{i\in I} \text{Hom}_{D(R)}(M, M_i[k]) \longrightarrow \text{Hom}_{D(R)}(M, (\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i)[k])$$

is an isomorphism. Consider the following family of triangles:

$$\big(H^{-l-1}(M_i)[l+1+k] \longrightarrow M_i[k] \longrightarrow \tau^{>-l-1}(M_i)[k] \overset{+}{\longrightarrow}\big)_{i\in I}$$

and

$$H^{-l-1}(\varinjlim_{i\in I} M_i)[l+1+k] \longrightarrow (\varinjlim_{i\in I} M_i)[k] \longrightarrow \tau^{>-l-1}(\varinjlim_{i\in I} M_i)[k] \overset{+}{\longrightarrow}$$

let us rename them by

$$\big(H_i[k] \longrightarrow M_i[k] \longrightarrow V_i[k] \overset{+}{\longrightarrow}\big)_{i\in I}$$

and

$$H[k] \longrightarrow L[k] \longrightarrow V[k] \overset{+}{\longrightarrow}$$

for short. Apply the functor $\Delta := \text{Hom}_{D(R)}(M, -)$ and the direct limit functor on these triangles to get

$$\begin{array}{ccccccccc}
\cdots \to & \varinjlim_{i\in I} \Delta(H_i[k]) & \to & \varinjlim_{i\in I} \Delta(M_i[k]) & \to & \varinjlim_{i\in I} \Delta(V_i[k]) & \to & \varinjlim_{i\in I} \Delta(H_i[k+1]) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \Delta(H[k]) & \longrightarrow & \Delta(L[k]) & \longrightarrow & \Delta(V[k]) & \longrightarrow & \Delta(H[k+1]) & \longrightarrow \cdots
\end{array}$$

By the previous proposition, all the vertical homomorphisms in which the H's and their shifting do appear are bijective. Our claim then reduces to prove that all the vertical morphisms involving the V's are bijective. Arguing recursively, we have to show that the canonical homomorphism

$$\varinjlim_{i\in I} \text{Hom}_{D(R)}(M, \tau^{>-1}(M_i)[k]) \longrightarrow \text{Hom}_{D(R)}(M, \tau^{>-1}(\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i)[k])$$

is bijective, but this occurs, by the previous proposition, since we have

$$\tau^{>-1}(M_i) = H^0(M_i)[0] \qquad \text{and} \qquad \tau^{>-1}(\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i) \cong H^0(\varinjlim_{i\in I}{}^{(\mathcal{H})} M_i)[0]$$
$$\cong (\varinjlim_{i\in I} H^0(M_i))[0] \,.$$

Thus, we conclude by [6] letting $k = 0, 1$ in the second display of the present proof, bearing in mind that $M \in \mathcal{H}$. □

We are now ready to recover the aforementioned result of Saorín for Thomason filtrations of finite length:

4.17 COROLLARY ([54, Theorem 6.3]). *Let R be a commutative noetherian ring, and Φ be a Thomason filtration of finite length. If the associated t-structure restricts to $D^b(\text{mod-}R)$, then the heart $\mathcal{H}$ is locally coherent and* $\text{fp}(\mathcal{H}) = \mathcal{H} \cap D^b(\text{mod-}R)$.

Proof. By [55, Theorem 8.31], we know that $\mathcal{H}$ is a locally finitely presented Grothendieck category, with $\text{fp}(\mathcal{H}) = \text{add}\, H_{\mathcal{H}}(\mathcal{U} \cap D(R)^c)$. By the hypothesis on R, it is clear that $D(R)^c \subseteq D^b(\text{mod-}R)$, and since the t-structure restricts to $D^b(\text{mod-}R)$, we infer that $\text{fp}(\mathcal{H}) = \mathcal{H} \cap D^b(\text{mod-}R)$. Now, by [48, Proposition 1.13(2)] and in view of Prop. 4.16, we obtain that $\mathcal{H}$ is locally coherent. □

Let us conclude this section with the following result, which shows that over a commutative noetherian ring any Thomason filtration of length 1 has locally coherent heart (recall that [54, Remark 4.6] and Ex. 2.8 exhibit an example of a Thomason filtration of length 1 with locally coherent heart whose t-structure does not restrict to $D^b(\text{mod-}R)$).

4.18 COROLLARY. *Let R be a commutative noetherian ring. If Φ is a Thomason filtration of length* 0 *or* 1, *then its heart $\mathcal{H}$ is locally coherent.*

Proof. The length 0 case has been proved in Cor. 3.21. For the length 1 case we will show that the four conditions of Cor. 3.43 hold true. In our hypotheses, only conditions (2) and (3) of such Corollary are not trivially satisfied. However, these conditions follow by the following inclusion

$$\text{fp}(\mathcal{T}\mathcal{F}\mathcal{T}_{-1}) \subseteq \text{fp}(\mathcal{T}_{-1}),$$

which we now prove. Indeed, let $B \in \text{fp}(\mathcal{T}\mathcal{F}\mathcal{T}_{-1})$ and $(B_i)_{i \in I}$ be the canonical direct system in mod-R such that $B = \varinjlim_{i \in I} B_i$. Notice that the direct system lives in $\mathcal{T}\mathcal{F}\mathcal{T}_{-1} = \mathcal{T}_{-1} \cap \mathcal{F}_0$, hence by hypothesis the identity of B factors through $B_{\bar{\imath}}$, for some index $\bar{\imath} \in I$, whence B is finitely presented in $\mathcal{T}_{-1}$. □

APPENDIX A

On the criterion for the local coherence

In view of the generality of Thm. 3.2, we give a similar reformulation of such result, which might be interesting on its own. We recall these two easy lemmata.

A.1 Lemma. *Let $\mathcal{G}$ be a locally coherent Grothendieck category and $(\mathcal{E}, \mathcal{F})$ be a torsion pair of $\mathcal{G}$. If there is a set $\mathcal{S}$ of finitely presented generators of $\mathcal{G}$ such that $x(S)$ is finitely presented for all $S \in \mathcal{S}$, then the torsion pair $(\mathcal{E}, \mathcal{F})$ restricts to* $\mathrm{fp}(\mathcal{G})$.

Proof. Let $B \in \mathrm{fp}(\mathcal{G})$ and let us prove that its torsion part $x(B)$ is finitely presented as well. By hypothesis, there exists an exact sequence $\bigoplus_{j=1}^{m} S_j \xrightarrow{f} \bigoplus_{i=1}^{n} S_i \to B \to 0$, and applying y we get $y(B) = \operatorname{Coker} y(f)$, which is finitely presented for each $y(S_k)$ in the sequence being so. By local coherence of $\mathcal{G}$, we conclude $x(B) \in \mathrm{fp}(\mathcal{G})$. □

A.2 Lemma. *Let $\mathcal{G}$ be a locally finitely presented Grothendieck category endowed with a TTF triple $(\mathcal{E}, \mathcal{T}, \mathcal{F})$. If the left constituent $(\mathcal{E}, \mathcal{T})$ resticts to* $\mathrm{fp}(\mathcal{G})$, *then the right constituent $(\mathcal{T}, \mathcal{F})$ is of finite type (i.e. the TTF triple is of finite type).*

Proof. Let $y\colon \mathcal{G} \to \mathcal{T}$ be the torsion coradical in the left constituent of the TTF triple. Then $\mathcal{T} = \operatorname{Gen} y(\mathrm{fp}(\mathcal{G}))$, whence the conclusion by Prop. 1.18. □

A.3 Theorem. *Let $\mathcal{G}$ be a Grothendieck category equipped with a TTF triple $(\mathcal{E}, \mathcal{T}, \mathcal{F})$. The following statements are equivalent:*

- (a) *$\mathcal{G}$ is locally coherent and the TTF triple is of finite type;*
- (b) *The following conditions are satisfied:*
  - (i) *$\mathcal{G}$ is locally finitely presented;*
  - (ii) *The left constituent $(\mathcal{E}, \mathcal{T})$ restricts to* $\mathrm{fp}(\mathcal{G})$*;*
  - (iii) *Any morphism either in $\mathcal{E} \cap \mathrm{fp}(\mathcal{G})$ or in $\mathcal{T} \cap \mathrm{fp}(\mathcal{G})$ has a finitely presented kernel.*
- (c) *Conditions* (i), (ii) *above are satisfied; moreover:*
  - (iii)' *Any (epi)morphism in $\mathcal{E} \cap \mathrm{fp}(\mathcal{G})$ with kernel in $\mathcal{T}$ has finitely presented kernel;*
  - (iv) *$\mathcal{E}$ and $\mathcal{T}$ are quasi locally coherent.*

Proof.
"(a) ⇒ (b)" Conditions (i) and (iii) are trivial. Condition (ii) has been proved in Thm. 3.2.
"(b) ⇒ (a)" The fact that the TTF triple is of finite type is a consequence of Lem. A.2. In order to prove that $\mathcal{G}$ is locally coherent we exploit [48, Proposition 2.10], whose condition (1) is clearly satisfied, as well as half of condition (2), namely in case of a morphism $f\colon B \to C$ in $\mathrm{fp}(\mathcal{G}) \cap \mathcal{E}$. It remains to prove that if $B \in \mathrm{fp}(\mathcal{G}) \cap \mathcal{T} = \mathrm{fp}(\mathcal{T})$ and $C \in \mathrm{fp}(\mathcal{G}) \cap \mathcal{E} = \mathrm{fp}(\mathcal{E})$, then $\operatorname{Ker} f$ is finitely presented. We have $\operatorname{Coker} f \in \mathrm{fp}(\mathcal{G}) \cap \mathcal{E}$, whence $\operatorname{Im} f \in \mathrm{fp}(\mathcal{G}) \cap \mathcal{T}$ by hypothesis (iii) and since $\mathcal{T}$ is a TTF class, and the same hypothesis entails the conclusion.
"(b) ⇒ (c)" Condition (iii) implies at once (iii)'. Conditions (i, ii) imply, by Lem. A.2, that besides the left hand one, also the right constituent of the TTF triple is of finite type, hence the quasi local coherence of $\mathcal{E}$ and $\mathcal{T}$ follows by hypothesis (iii).
"(c) ⇒ (b)" The part of the statement concerning $\mathcal{T} \cap \mathrm{fp}(\mathcal{G})$ is clear, since this latter coincides with $\mathrm{fp}(\mathcal{T})$ by Lem. A.2 and since $\mathcal{T}$ is a locally coherent category closed in $\mathcal{G}$ under taking subobjects. So, let $\phi\colon B \to B'$ be a morphism in $\mathrm{fp}(\mathcal{E})$ with kernel K (in $\mathcal{G}$). Without loss of generality we shall assume that ϕ is an epimorphism, since $\operatorname{Coker} \phi \in \mathrm{fp}(\mathcal{E})$ and hypothesis (iv) yields $\operatorname{Im} \phi \in \mathrm{fp}(\mathcal{E})$ as well. We then get the following pushout diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & B & \xrightarrow{\ \phi\ } & B' & \longrightarrow & 0 \\
 & & \big\downarrow & \text{P.O.} & \big\downarrow f & & \big\| & & \\
0 & \longrightarrow & y(K) & \longrightarrow & M & \xrightarrow{\ g\ } & B' & \longrightarrow & 0
\end{array}$$

whence $x(K) \cong \operatorname{Ker} f$, which are finitely presented objects by hypothesis (ii). Consequently, $M \in \mathcal{E} \cap \mathrm{fp}(\mathcal{G})$, and therefore $y(K)$ is finitely presented being an object of $\mathcal{T}$ that occurs as a kernel of an epimorphism in $\mathcal{E} \cap \mathrm{fp}(\mathcal{G})$. By extension closure of $\mathrm{fp}(\mathcal{G})$ we are done. □

APPENDIX B

Open problems

We conclude the present thesis with some open problems and possible future directions related to the discussed or proved results. Essentially, those concerning the local coherence of the hearts associated to a sp-filtration are motivated by Saorín's Thm. 2.6, which establishes that restrictability is a sufficient and not necessary condition for such finiteness condition.

Thm 2.14 relates the local coherence of the hearts of compactly generated t-structures over a noetherian ring to an important result in commutative algebra; we detected in Thm. 2.17 a class of rings for which the weak Cousin sp-filtrations are restrictable, using arguments mutuated by commutative algebra, but we are convinced that other classes of rings might be obtained in this way (see also [60]). In particular, a derived version of the so-called Faltings' *Local–Global principle* for sp-closed subsets would give a decisive result in this sense[1]; with such result at disposal, we think we should extend Thm. 2.17 and the results in [60] to larger classes of (finite dimensional) noetherian rings.

We exhibited in Ex. 2.8 a pure-injective cosilting object inducing a locally noetherian heart which is not cotilting. Following [35], this naturally leads to the problem of identify the (pure-injective) cosilting objects whose induced heart is locally noetherian.

In Cor. 4.18 we proved that over a noetherian ring, the sp-filtrations of length zero and one have always a locally coherent heart. A natural question would be whether such fact occurs in general for sp-filtrations of finite length, or at least give a bound on the length for this to happen. With this result, one then has a rich source of locally coherent Grothendieck categories at hand.

Eventually, albeit the local coherence is far from being characterised in general, i.e. for an arbitrary locally finitely presented Grothendieck category $\mathcal{G}$, of course it is not the only aspect one may investigate furtherly. For instance, on any such $\mathcal{G}$ a notion of *flatness* can be introduced, as we now recall, and we think that the hearts studied in this thesis might provide interesting example in

[1] One possible formulation would be precisely [15, Theorem 3.8], though by a private communication with the authors and Michal Hrbek it seems that there is a crucial error in the proof invalidating the result.

this context. Flatness is formulated by means of the *purity*[2] of $\mathcal{G}$: a short exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{G}$ is said to be *pure* if it yields short exact sequences under every functor $\mathrm{Hom}_{\mathcal{G}}(B, -)$, with $B \in \mathrm{fp}(\mathcal{G})$; in such case, $M' \to M$ is called a *pure monomorphism* and $M \to M''$ a *pure epimorphism*; an object $L \in \mathcal{G}$ is said to be *flat* in case any epimorphism of $\mathcal{G}$ landing in L is pure. Notice that this notion of flatness is indipendent both of the existence of a monoidal structure on $\mathcal{G}$ (hence of a tensor product) and of the existence of enough projective objects, though evidently any projective object would be flat[3]. A conjecture by Cuadra and Simson[4] asks whether a locally finitely presented Grothendieck category with enough flat objects has enough projectives indeed. The conjecture is open, and so far no evidence of its truthfulness arose in the literature, however, it is natural to expect that a counterexample might be provided by a locally coherent Grothendieck category, as soon as a generating set of flat objects is detected.

[2] We gave in Def. 1.27 the corresponding notion for triangulated categories with coproducts.

[3] There are Grothendieck categories without nonzero flat objects though, see S. Estrada, M. Saorín, *Locally finitely presented categories with no flat objects*, Forum Math. **27** (2015), 269–301

[4] See *Open Problems* 2.9 in J. Cuadra, D. Simson, *Flat Comodules and Perfect Coalgebras*, Comm. Alg. **35**:10, 3164–3194 (2007)

Bibliography

[1] L. Alonso Tarrío, A. López Jeremías, M. Saorín, *Compactly generated t-structures on the derived category of a noetherian ring*, Journal of Algebra, **324**(3):313–346, 2010

[2] L. Alonso Tarrío, A. López Jeremías, M. Souto, *Localization in categories of complexes and bounded resolutions*, Canad. J. Math. **52** (2000), no. 2, 225–247

[3] L. Angeleri Hügel, M. Hrbek, *Silting Modules over Commutative Rings*, International Mathematics Research Notices, Vol. 2017, No. 13, pp. 4131–4151

[4] L. Angeleri Hügel, F. Marks, J. Vitória, *Torsion pairs in silting theory*, Pacific Journal of Mathematics, Vol. **291** (2017), No. 2, 257–278

[5] L. Angeleri Hügel, D. Pospíšil, J. Šťovíček, J. Trilifaj, *Tilting, cotilting, and spectra of commutative noetherian rings*, Transactions Amer. Math. Soc. **366** (2014), 3487–3517

[6] A. Beĭlinson, J. Bernstein, P. Deligne, *Faisceaux Pervers*, in "*Analysis and topology on singular spaces*", I, Luminy 1981, Astèrisque **100**, Soc. Math. France, Paris (1982), 5–171

[7] M. Bondarko, *On perfectly generated weight structures and adjacent t-structures*, Math. Z. **300**, 1421–1454 (2022). https://doi.org/10.1007/s00209-021-02815-6

[8] D. Bravo, C. E. Parra, *tCG torsion pairs*, J. Algebra Appl. Vol. **18**, no. 17 (2019)

[9] D. Bravo, C.E. Parra, *Torsion pairs over n-hereditary rings*, Communications in Algebra, 47:5, 1892–1907 (2019)

[10] D. Bravo, M. Pérez, *Finiteness conditions and cotorsion pairs*, Journal of Pure and Applied Algebra, **221** (2017), 1249–1267

[11] W. Bruns, H. J. Herzog, *Cohen–Macaulay rings*, Second edition, Cambridge University Press (1998)

[12] R. Colpi, E. Gregorio, F. Mantese, *On the Heart of a faithful torsion theory*, Journal of Algebra, **307** (2007), 841–863

[13] R. Colpi, F. Mantese, A. Tonolo, *When the heart of a faithful torsion pair is a module category*, Journal of Pure and Applied Algebra, **215** (2011), 2923–2936

[14] W. Crawley-Boevey, *Locally finitely presented additive categories*, Comm. Algebra, **22**(5):1641–1674, 1994

[15] K. Divaani-Aazar, M. R. Zargar, *The derived category analogues of Faltings Local–global Principle and Annihilator Theorems*, J. Algebra Appl. (2018)

[16] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France, **90** (1962), 323–448

[17] P. Gabriel, M. Zisman, *Calculus of Fractions and Homotopy Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band **35**. Springer–Verlag New York, 1967

[18] G. Garkusha, M. Prest, *Torsion classes of finite type and spectra*, K-theory and Noncommutative Geometry, pages 393–412, 2008

[19] R. Göbel, J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, Two volumes, De Gruyter Expositions in Mathematics 41, 2012

[20] M. Groth, *Derivators, pointed derivators and stable derivators*, Algebraic & Geometric Topology **13** (2013), 313–374

[21] D. Happel, I. Reiten, S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. **120** (1996)

[22] I. Herzog, *The Ziegler spectrum of a locally coherent Grothendieck category*, Proc. Lond. Math. Soc. **74**(3) 1997, 503–558

[23] M. Hrbek, *Compactly generated t-structures in the derived category of a commutative ring*, Mathematische Zeitschrift, month 6, 2019

[24] M. Hrbek, J. Hu, R. Zhu, *Gluing compactly generated t-structure over stalks of affine schemes*, preprint available at ArXiv:2101.09966

[25] M. Hrbek, T. Nakamura, *Telescope conjecture for homotopically smashing t-structures over commutative noetherian rings*, J. Pure Appl. Algebra **225** (2021)

[26] M. Hrbek, T. Nakamura, J. Šťovíček, *Tilting complexes and codimension functions over commutative noetherian rings*, preprint available at arXiv:2207.01309

[27] M. Hrbek, S. Pavon, *Singular equivalences to locally coherent hearts of commutative noetherian rings*, preprint available at arXiv:2109.13853

[28] M. Hrbek, J. Šťovíček, *Tilting classes over commutative rings*, Forum Math. 2019

[29] T. Holm, P. Jørgensen, R. Rouquier (editors), Triangulated Categories, London Mathematical Society, LNS 375, Cambridge University Press, 2011

[30] S. Iyengar, *Depth for complexes, and intersection theorems*, Math. Z **230**, 545–567 (1999).

[31] M. Kashiwara, P. Schapira, *Categories and Sheaves*, Grundleheren der Math. 332, Springer-Verlag, 2006

[32] T. Kawasaki, *On Faltings' annihilator theorem*, Proc. Amer. Math. Soc. **136**, no. 4, 1205–1211 (2008)

[33] H. Krause, *The spectrum of a locally coherent category*, J. Pure Appl. Algebra **114**(3), 1997, 259–271

[34] H. Krause, *Smashing subcategories and the telescope conjecture: an algebraic approach*, Invent. Math. **139**:1 (2000), 99–133, https://doi.org/10.1007/s002229900022

[35] R. Laking, *Purity in compactly generated derivators and t-structures with Grothendieck hearts*, Mathematische Zeitschrift (2019), https://doi.org/10.1007/s00209-019-02411-9

[36] T. Y. Lam, *Lectures on Modules and Rings*, GTM 189, Springer-Verlag New York, 1999

[37] F. Mantese, A. Tonolo *On the heart associated with a torsion pair*, Topology and its Applications, **159** (2012), 2483–2489

[38] F. Marks, A. Zvonareva, *Lifting and restricting t-structures*, preprint available at arXiv:2108.00471

[39] B. Mitchell, *Theory of Categories*, Academic Press, 1965

[40] B. Mitchell, *Rings with several objects*, Advances in Mathematics 8(1) (1972), 1–161

[41] A. Neeman, *Triangulated Categories*, Ann. Math. Stud., vol. 148, Princeton University Press, 2001

[42] P. Nicolás, M. Saorín, *Classical derived functors as fully faithful embeddings*, Proc. 46th Japan Symposium on Ring Theory and Representation Theory (Tokyo 2013), editor Kikumasa, I. Yamaguchi University (2014), 137–187

[43] D. G. Northcott, *Lessons on Rings, Modules and Multiplicities*, Cambridge University Press, 1968

[44] C. E. Parra, M. Saorín, *Direct limits in the heart of a t-structure: the case of a torsion pair*, J. Pure Appl. Algebra **219**, 4117–4143 (2015). *Addendum* in J. Pure Appl. Algebra **220**, 2567–2469 (2016)

[45] C. E. Parra, M. Saorín, *On hearts which are module categories*, J. Math. Soc. Japan, Volume **68**, Number 4 (2016), 1421-1460.

[46] C. E. Parra, M. Saorín, *Hearts of t-structures in the derived category of a commutative noetherian ring*, Trans. Amer. Math. Soc. **369** (2017), 7789–7827

[47] C. E. Parra, M. Saorín, S. Virili, *Torsion Pairs in Categories of Modules over a Preadditive Category*, Bull. Iran. Math. Soc. (2020), https://doi.org/10.1007/s41980-020-00433-2

[48] C. E. Parra, M. Saorín, S. Virili, *Locally finitely presented and coherent hearts*, preprint available at arXiv:1908.00649v1

[49] S. Pavon, J. Vitória, *Hearts for commutative Noetherian rings: torsion pairs and derived equivalences*, Doc. Math. **26**, 829–871 (2021)

[50] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, Academic Press, London, 1973

[51] M. Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications **121**, Cambridge University Press, 2009

[52] C. Psaroudakis, J. Vitória, *Recollements of Module Categories*, Appl. Categor. Struct. **22**, 579—593 (2014)

[53] C. Psaroudakis, J. Vitória, *Realisation functor in tilting theory*, J. Math. Z. (2018) **288**:965–1028

[54] M. Saorín, *On locally coherent hearts*, Pacific Journal of Mathematics, Vol. **287** (2017), No. 1, 199–221

[55] M. Saorín, J. Šťovíček, *t-Structures with Grothendieck hearts via functor categories*, preprint available at arXiv:2003.01401v3

[56] M. Saorín, J. Šťovíček, S. Virili, *t-Structures on stable derivators and Grothendieck hearts*, preprint available at arXiv:1708.07540v2

[57] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988) 121–154

[58] B. B. Stenström, *Rings of Quotients*, Grundleheren der Math. 217, Springer-Verlag, 1975

[59] J. Šťovíček, *Derived equivalences induced by big cotilting modules*, Adv. Math. **263** (2014):45–87

[60] R. Takahashi, *Faltings' annihilator theorem and t-structures of derived categories*, preprint available at arXiv:2208.04455

[61] J. L. Verdier, *Des Catégories Dérivés des Catégories Abéliennes*, Astèrisque **239**, Soc. Math. France, Marseilles (1996)

[62] The Stacks project, https://stacks.math.columbia.edu