STRUCTURE OF PRESCRIBED GRADIENT DOMAINS FOR NON-INTEGRABLE VECTOR FIELDS

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Abstract. Let $F \in C^1(\Omega, \mathbb{R}^n)$ and $f \in C^2(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$ with n even. We describe the structure of the set of points in Ω at which the equality $Df = F$ and a certain non-integrability condition on F hold. This result generalizes the second statement of [1, Theorem 3.1].

1. Introduction

Let us consider an open subset Ω of $\mathbb{R}^n$ and a vector field $F \in C^1(\Omega, \mathbb{R}^n)$. Then, whatever the choice of $f \in C^2(\Omega)$, the set

$$A_{f,F} := \{x \in \Omega \mid Df(x) = F(x)\}$$

has to be somehow "scarcely dense" at each point $x \in \Omega$ satisfying the condition

$$(DF(x))^t \neq DF(x). \tag{1.1}$$

Actually, if such a condition occurs then the point x cannot be in the interior of $A_{f,F}$, by the Schwarz theorem on mixed derivatives. More interestingly, by [3, Theorem 2.1], the point x cannot be a $(n+1)$-density point of the set $A_{f,F}$, i.e., it must be

$$\limsup_{r \to 0+} \frac{\mathcal{L}^n(B(x,r) \setminus A_{f,F})}{r^{n+1}} > 0.$$

In the special case when $n = 2m$ and

$$F_0(x_1, \dots, x_{2m}) = (2x_{m+1}, \dots, 2x_{2m}, -2x_1, \dots, -2x_m)$$

the condition (1.1) is trivially satisfied everywhere by F_0 and, according to the Balogh's result [1, Theorem 3.1], the set A_{f,F_0} is covered by countably many m-dimensional Lipschitz graphs. In particular the Hausdorff dimension of A_{f,F_0} cannot exceed m. This theorem implies immediately that, given a C^2 hypersurface S in $\mathbb{R}^{2m+1}$, the set of points at which the tangent space of S coincides with the space spanned by the left-invariant horizontal vector fields of the Heisenberg group $\mathbb{H}^m$ over $\mathbb{R}^{2m+1}$ (namely the characteristic set of S) has Hausdorff dimension less or equal to m, compare [1, Theorem 1.2].

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The main goal of this short paper is a theorem generalizing Balogh's result above and describing the structure of the set of points in $A_{f,F}$ satisfying a certain condition which is stronger than (1.1). In order to formulate such a condition and to state our result, let us consider the canonical basis $\{e_1, \dots , e_n\}$ of $\mathbb{R}^n$ and a real $n \times n$ matrix M. Then define

$$\Gamma(M) := \sum_{i=1}^{n} Me_i \wedge e_i$$

and (if h is a positive integer)

$$\Gamma(M)^h := \underbrace{\Gamma(M) \wedge \cdots \wedge \Gamma(M)}_{h \text{ times}}.$$

Now we can finally state our structure theorem. It establishes that *if $n = 2m$ then the set $A^*_{f,F}$ of all points $x \in A_{f,F}$ such that*

$$\Gamma(DF(x))^m \neq 0 \tag{1.2}$$

is covered by a finite family of m-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$ (Theorem 4.1 below).

In the special case considered by Balogh, i.e., when $F = F_0$, the condition (1.2) is verified at every $x \in \mathbb{R}^n$. Indeed a standard computation yields

$$\Gamma(DF_0(x))^m = C(m)\, e_1 \wedge \cdots \wedge e_{2m}$$

for all $x \in \mathbb{R}^n$, where $C(m)$ is a positive constant depending only on m. Hence $A_{f,F_0} = A^*_{f,F_0}$, thus Balogh's result follows trivially from our theorem (Corollary 4.1 below).

The proof of our structure theorem is provided in Section 4 and is an elementary argument combining the classical implicit function theorem with some basic results from multilinear algebra which are developed in Section 3.

We have eventually to mention a few main results strictly related to this subject that have been published after [1]. In paper [4], B. Franchi, R. Serapioni and F. Serra Cassano extended the Balogh's covering type argument to all stratified groups of step two. The generalization to arbitrary stratified groups has been proved by V. Magnani through a different approach based on a coarea inequality [7]. More recently, an estimate for the size of tangencies of submanifolds with respect to a non-involutive distribution, which generalizes the Balogh's theorem above, has been provided in [2].

2. Notation

If m, n are positive integers with $m \le n$ then $I(n, m)$ is the set of integer multi-indices $(\alpha_1, \dots , \alpha_m)$ such that $1 \le \alpha_1 < \ldots < \alpha_m \le n$. If $\alpha \in I(n, m)$ then $\bar{\alpha}$ is the complement in $I(n, n-m)$ of α. Moreover $\sigma(\alpha, \bar{\alpha})$ is the sign of the permutation of $(1, \dots , n)$ into $(\alpha, \bar{\alpha})$. The space of m-vectors in $\mathbb{R}^n$ is denoted by $\Lambda^m\mathbb{R}^n$. If $\{e_1, \dots , e_n\}$ is a basis of $\mathbb{R}^n$ and $\alpha = (\alpha_1, \dots , \alpha_m) \in I(n, m)$, then we put $e_\alpha := e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_m}$. Recall that

$\{e_\alpha\}_{\alpha\in I(n,m)}$ is a basis of $\Lambda^m\mathbb{R}^n$. Let $M_n(\mathbb{R})$ denote the space of all $n\times n$ matrices with real entries and consider $M\in M_n(\mathbb{R})$. Then M^t is the transpose of M. Moreover, if $1\le m\le n$, define the linear operator $\Lambda^m M:\Lambda^m\mathbb{R}^n\to\Lambda^m\mathbb{R}^n$ as the one such that $\Lambda^m M(e_\alpha)=Me_{\alpha_1}\wedge\cdots\wedge Me_{\alpha_m}$, for all $\alpha=(\alpha_1,\dots,\alpha_m)\in I(n,m)$. Observe that, for all $\beta\in I(n,m)$, one has

$$\Lambda^m M(e_\beta)=\sum_{\alpha\in I(n,m)}(\det M_{\alpha,\beta})e_\alpha$$

where $M_{\alpha,\beta}$ is the $m\times m$ submatrix of M with rows $\alpha_1,\dots,\alpha_m$ and columns $\beta_1,\dots,\beta_m$. The symmetric group of degree m is denoted by S_m. Finally, if Ω is an open subset of $\mathbb{R}^n$, $x\in\Omega$ and $\psi\in C^1(\Omega,\mathbb{R}^m)$, then $D\psi(x)$ denotes the matrix of the differential of ψ at x (with respect to the canonical basis).

3. Some basic preliminaries from multilinear algebra.

Proposition 3.1. *Let $\{e_1,\dots,e_n\}$ be the canonical basis of $\mathbb{R}^n$. Consider the operator $\Gamma:M_n(\mathbb{R})\to\Lambda^2\mathbb{R}^n$ defined as*

$$\Gamma(M):=\sum_{i=1}^n Me_i\wedge e_i,\quad M\in M_n(\mathbb{R}).$$

Then Γ has the following properties:

(1) *It is linear, namely if $r,s\in\mathbb{R}$ and $M,N\in M_n(\mathbb{R})$ then*

$$\Gamma(rM+sN)=r\,\Gamma(M)+s\,\Gamma(N).$$

(2) *For all $M\in M_n(\mathbb{R})$ one has*

$$\Gamma(M)=\sum_{\substack{i,j=1\\ i<j}}^n(M_{ij}-M_{ji})\,e_i\wedge e_j$$

where $M_{ij}:=(Me_j)\cdot e_i$. Hence:

  (i) *$\Gamma(M^t)=-\Gamma(M)$;*
  (ii) *$\Gamma(M)=0$ if and only if M is symmetric.*

(3) *If $\{u_1,\dots,u_n\}$ is any arbitrary orthonormal basis in $\mathbb{R}^n$ and $M\in M_n(\mathbb{R})$, then*

$$\sum_{i=1}^n Mu_i\wedge u_i=\Gamma(M).$$

Moreover, in the special case when n is even, i.e. $n = 2m$, the following identities hold for all $M \in M_{2m}(\mathbb{R})$:

$$\Gamma(M)^m := \underbrace{\Gamma(M) \wedge \cdots \wedge \Gamma(M)}_{m \text{ times}} = C(m) \sum_{\alpha \in I(2m,m)} [(\Lambda^m M)\, e_\alpha] \wedge e_\alpha$$
$$= C(m) \left(\sum_{\alpha \in I(2m,m)} \sigma(\alpha, \bar{\alpha})\, det\, M_{\alpha,\bar{\alpha}} \right) e_1 \wedge \cdots \wedge e_{2m}$$

with $C(m) := m!\,(-1)^{\frac{m(m-1)}{2}}$.

Proof. Assertion (1) follows immediately from the definition of Γ, while (2) is a standard computation:

$$\begin{aligned}
\Gamma(M) &= \sum_{i,j=1}^{n} M_{ji}\, e_j \wedge e_i \\
&= \sum_{\substack{i,j=1 \\ i<j}}^{n} M_{ji}\, e_j \wedge e_i \,+\, \sum_{\substack{i,j=1 \\ i>j}}^{n} M_{ji}\, e_j \wedge e_i \\
&= \sum_{\substack{i,j=1 \\ i<j}}^{n} M_{ji}\, e_j \wedge e_i \,+\, \sum_{\substack{i,j=1 \\ i<j}}^{n} M_{ij}\, e_i \wedge e_j \\
&= \sum_{\substack{i,j=1 \\ i<j}}^{n} (M_{ij} - M_{ji})\, e_i \wedge e_j.
\end{aligned}$$

Also (3) is very easy:

$$\begin{aligned}
\sum_{i=1}^{n} Me_i \wedge e_i &= \sum_{i,h,k=1}^{n} (e_i \cdot u_h)(e_i \cdot u_k)\, Mu_h \wedge u_k \\
&= \sum_{h,k=1}^{n} \left[\sum_{i=1}^{n} (e_i \cdot u_h)(e_i \cdot u_k) \right] Mu_h \wedge u_k \\
&= \sum_{h,k=1}^{n} (u_h \cdot u_k)\, Mu_h \wedge u_k \\
&= \sum_{h=1}^{n} Mu_h \wedge u_h.
\end{aligned}$$

Finally, one has

$$
\begin{aligned}
\Gamma(M)^m &= \left(\sum_{i_1=1}^{2m} Me_{i_1} \wedge e_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_m=1}^{2m} Me_{i_m} \wedge e_{i_m}\right) \\
&= (-1)^{1+2+\cdots+m-1} \sum_{i_1,\dots,i_m=1}^{2m} Me_{i_1} \wedge \cdots \wedge Me_{i_m} \wedge e_{i_1} \wedge \cdots \wedge e_{i_m} \\
&= (-1)^{\frac{m(m-1)}{2}} \sum_{\alpha\in I(2m,m)} \sum_{\lambda\in S_m} Me_{\alpha_{\lambda(1)}} \wedge \cdots \wedge Me_{\alpha_{\lambda(m)}} \wedge e_{\alpha_{\lambda(1)}} \wedge \cdots \wedge e_{\alpha_{\lambda(m)}} \\
&= (-1)^{\frac{m(m-1)}{2}} \sum_{\alpha\in I(2m,m)} \#(S_m)\ [(\Lambda^m M)\, e_\alpha] \wedge e_\alpha \\
&= (-1)^{\frac{m(m-1)}{2}} m! \sum_{\alpha\in I(2m,m)} [(\Lambda^m M)\, e_\alpha] \wedge e_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
[(\Lambda^m M)\, e_\alpha] \wedge e_\alpha &= \sum_{\beta\in I(2m,m)} (\det M_{\beta,\alpha})\, e_\beta \wedge e_\alpha = (\det M_{\bar\alpha,\alpha})\, e_{\bar\alpha} \wedge e_\alpha \\
&= \sigma(\bar\alpha,\alpha)\, (\det M_{\bar\alpha,\alpha})\, e_1 \wedge \cdots \wedge e_{2m}.
\end{aligned}
$$

□

Remark 3.1. From (1) and (2) of Proposition 3.1, it follows at once the following identity which will be useful below:

$$\Gamma(M - M^t) = 2\Gamma(M) = -2\Gamma(M^t)$$

for all $M \in M_n(\mathbb{R})$.

Remark 3.2. Let $h \geq 2$ and $n \geq 2h$. Then the identity $\Gamma(M)^h = 0$ holds whenever M is symmetric (by (2) of Proposition 3.1) but it can occur even when M is nonsymmetric. For example, for

$$M_{ij} = \begin{cases} 1 & \text{if } i = 1 \text{ and } j = 2 \\ 0 & \text{otherwise} \end{cases}$$

one has $\Gamma(M) = e_1 \wedge e_2$, hence $\Gamma(M)^h = 0$.

4. The structure theorem. Statement and proof.

Definition 4.1. *Given $f \in C^2(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^{2m})$, where Ω is an open subset of $\mathbb{R}^{2m}$, define the sets*

$$A_{f,F} := \{x \in \Omega \,|\, Df(x) = F(x)\}$$

and

$$A^*_{f,F} := \{x \in A_{f,F} \,|\, \Gamma(DF(x))^m \neq 0\}.$$

Remark 4.1. As we have already pointed out in the Introduction, under the assumptions of Definition 4.1, a point $x_0 \in \Omega$ cannot be a $(2m+1)$-density point of the set $A_{f,F}$ if $\Gamma(DF(x_0)) \neq 0$ (by [3, Theorem 2.1]). In particular one has the following property: There is no point $x_0 \in A^*_{f,F}$ which is a $(2m+1)$-density point of the set$A^*_{f,F}$. In our main result, namely Theorem 4.1 below, we will prove that something very much stronger than this property actually holds for $A^*_{f,F}$. In particular, the set $A^*_{f,F}$ must be so thin that its Hausdorff dimension cannot exceed m.

We need the following lemma which will be proved by a simple argument combining Proposition 3.1 and the implicit function theorem.

Lemma 4.1. *Let $\Phi \in C^1(\Omega, \mathbb{R}^{2m})$, where Ω be an open subset of $\mathbb{R}^{2m}$. Then*

$$\{x \in \Omega \,|\, \Phi(x) = 0,\, \Gamma(D\Phi(x))^m \neq 0\} \tag{4.1}$$

is a relatively closed subset of Ω which is covered by a finite family $\{\Sigma_\alpha \,|\, \alpha \in I(2m,m)\}$ of m-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$.

Proof. For $\alpha \in I(2m,m)$, define

$$\Phi_\alpha := (\Phi_{\alpha_1}, \dots, \Phi_{\alpha_m}), \quad \Sigma_\alpha := \{x \in \Omega \,|\, \Phi_\alpha(x) = 0,\, \mathrm{rank}\, D\Phi_\alpha(x) = m\}\,.$$

Observe that each Σ_α has to be a m-dimensional regularly imbedded C^1 submanifold of $\mathbb{R}^{2m}$ by a standard application of the implicit function theorem, e.g. compare [6, Theorem 4.3.1] or [5, Ch. 1, Theorem 3.2]. The conclusion follows from the last identity in Proposition 3.1, with $M = D\Phi(x)$. □

Now we are ready to prove our main result.

Theorem 4.1 (Structure theorem). *Let $f \in C^2(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^{2m})$, where Ω is an open subset of $\mathbb{R}^{2m}$. Then $A^*_{f,F}$ is a relatively closed subset of Ω which is covered by a finite family $\{\Sigma_\alpha \,|\, \alpha \in I(2m,m)\}$ of m-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$. In particular, the Hausdorff dimension of $A^*_{f,F}$ has to be less or equal to m.*

Proof. Consider

$$\Phi := Df - F \in C^1(\Omega, \mathbb{R}^{2m}).$$

Since

$$D\Phi = D^2 f - DF$$

one has

$$\Gamma(D\Phi(x) - (D\Phi(x))^t) = \Gamma((DF(x))^t - DF(x))$$

for all $x \in \Omega$. By recalling Remark 3.1, we get

$$\Gamma(D\Phi(x)) = -\Gamma(DF(x))$$

for all $x \in \Omega$, hence

$$A^*_{f,F} = \{x \in \Omega \,|\, \Phi(x) = 0,\, \Gamma(D\Phi(x))^m \neq 0\}.$$

The conclusion follows from Lemma 4.1. □

From Theorem 4.1 we obtain the following corollary.

Corollary 4.1. *Consider* $F_0 : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ *defined as*

$$F_0(x_1, \dots, x_{2m}) := (2x_{m+1}, \dots, 2x_{2m}, -2x_1, \dots, -2x_m)$$

and let Ω *be an open subset of* $\mathbb{R}^{2m}$*. Then, for all* $f \in C^2(\Omega)$*, the set* A_{f,F_0} *is a relatively closed subset of* Ω *which is covered by a finite family* $\{\Sigma_\alpha \,|\, \alpha \in I(2m, m)\}$ *of* m*-dimensional regularly imbedded* C^1 *submanifolds of* $\mathbb{R}^{2m}$*. In particular, the Hausdorff dimension of* A_{f,F_0} *has to be less or equal to* m*.*

Proof. Observe that

$$DF_0 = \begin{pmatrix} 0 & 2I_m \\ -2I_m & 0 \end{pmatrix}$$

where I_m denotes the identity $m \times m$ matrix. Thus, for all $x \in \mathbb{R}^{2m}$, we obtain

$$\begin{aligned} \Gamma(DF_0(x)) &= \sum_{i=1}^{2m} DF_0(x)e_i \wedge e_i = \sum_{i=1}^{m} (-2e_{m+i}) \wedge e_i + \sum_{i=m+1}^{2m} 2e_{i-m} \wedge e_i \\ &= 4 \sum_{i=1}^{m} e_i \wedge e_{m+i} \end{aligned}$$

hence

$$\begin{aligned} \Gamma(DF_0(x))^m &= 4^m \sum_{i_1, \dots, i_m = 1}^{m} (e_{i_1} \wedge e_{m+i_1}) \wedge \cdots \wedge (e_{i_m} \wedge e_{m+i_m}) \\ &= 4^m \sum_{\lambda \in S_m} (e_{\lambda(1)} \wedge e_{m+\lambda(1)}) \wedge \cdots \wedge (e_{\lambda(m)} \wedge e_{m+\lambda(m)}) \\ &= 4^m \sum_{\lambda \in S_m} (e_1 \wedge e_{m+1}) \wedge \cdots \wedge (e_m \wedge e_{2m}) \\ &= 4^m m! (-1)^{\frac{m(m-1)}{2}} e_1 \wedge \cdots \wedge e_{2m}. \end{aligned}$$

In particular one has $\Gamma(DF_0(x))^m \neq 0$ for all $x \in \mathbb{R}^{2m}$, so that $A_{f,F_0} = A^*_{f,F_0}$. The conclusion follows from Theorem 4.1. □

Remark 4.2. Corollary 4.1 improves slightly the second statement in [1, Theorem 3.1], which yields immediately the following interesting fact: the characteristic set of a codimension 1 submanifold of class C^2 in the Heisenberg group $\mathbb{H}^m$ has Hausdorff dimension less or equal to m, compare [1, Theorem 1.2].

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