

Approximation of rectifiable 1-currents and weak-* relaxation of the h -mass

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Abstract

Based on Smirnov's decomposition theorem we prove that every rectifiable 1-current T with finite mass $\mathbb{M}(T)$ and finite mass $\mathbb{M}(\partial T)$ of its boundary ∂T can be approximated in mass by a sequence of rectifiable 1-currents T_n with polyhedral boundary ∂T_n and $\mathbb{M}(\partial T_n)$ no larger than $\mathbb{M}(\partial T)$. Using this result we can compute the relaxation of the h -mass for polyhedral 1-currents with respect to the joint weak-* convergence of currents and their boundaries. We obtain that this relaxation coincides with the usual h -mass for normal currents. This shows that the concepts of so-called generalized branched transport and the h -mass are equivalent.

1 Introduction

Variational models for ramified transportation networks have recently attracted lots of interest (see for instance [17, 1, 11, 12, 3, 6] and the references therein). They are closely related to the measure-geometric concept of the h -mass of normal currents (as for instance introduced in [8], where h plays the role of a group metric). The main difference is that the transportation network models are defined via relaxation with respect to weak-* convergence of currents and their boundaries, while the h -mass is defined via relaxation with respect to the weaker notion of flat convergence. In [3, Prop. 2.32] the equivalence between both models was used without proof. In this note we prove the equivalence between generalized branched transport and the h -mass in full generality in theorem 5. The main tool will be a recent relaxation result by Chambolle, Ferrari, and Merlet [4] for currents with polyhedral boundary, combined with a new strong approximation result of rectifiable 1-currents by currents with polyhedral boundary and equibounded boundary mass (lemmas 7 to 9). In the remainder of the introduction we describe the above-mentioned models and corresponding notions in more detail.

Following [17] or its generalization [3], the generalized branched transport model can be introduced as follows (where our notation is chosen slightly differently to emphasize the correspondence to the h -mass later). Throughout the article we consider $\Omega \subset \mathbb{R}^d$ to be the closure of an open bounded connected domain, and we denote by $\mathcal{M}(\Omega)$ the set of Radon measures, by $\mathcal{M}_+(\Omega) \subset \mathcal{M}(\Omega)$ the subset of nonnegative measures, and by $\mathcal{M}(\Omega; \mathbb{R}^d)$ the set of \mathbb{R}^d -valued Radon measures on Ω . The total variation measure will be indicated by $|\cdot|$, the total variation of a measure by $\|\cdot\|_{\mathcal{M}}$, and weak-* convergence by $\overset{*}{\rightharpoonup}$. The notation \mathcal{H}^m denotes the m -dimensional Hausdorff measure and \llcorner the restriction of measures to Borel sets.

- Definition 1** (Generalized branched transport). *1. A unit line flux along e is a measure $\rho \in \mathcal{M}(\Omega; \mathbb{R}^d)$ of the form $\rho = \vec{e} \mathcal{H}^1 \llcorner e$, where $e \subset \Omega$ is a straight line segment with unit tangent \vec{e} .*
- 2. A polyhedral flux in Ω is a measure $\rho \in \mathcal{M}(\Omega; \mathbb{R}^d)$ of the form $\rho = \sum_{i=1}^n a_i \rho_i$ for $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$, and ρ_1, \dots, ρ_n unit line fluxes.*
- 3. If the distributional divergence in \mathbb{R}^d of $\rho \in \mathcal{M}(\Omega; \mathbb{R}^d)$ is a Radon measure, then ρ is called a mass flux, and its negative divergence is called the boundary of ρ , denoted by $\partial \rho = -\operatorname{div} \rho$. The set of mass fluxes on Ω is denoted $\mathcal{F}(\Omega)$.*

4. Let ρ_1, ρ_2, \dots be a sequence of mass fluxes. We say ρ_n converges weakly to mass flux ρ and write $\rho_n \xrightarrow{f} \rho$ as $n \rightarrow \infty$, if $\rho_n \xrightarrow{*} \rho$ and $\partial \rho_n \xrightarrow{*} \partial \rho$. We say ρ_n converges strongly to mass flux ρ and write $\rho_n \rightarrow \rho$ as $n \rightarrow \infty$, if $\|\rho_n - \rho\|_{\mathcal{M}} \rightarrow 0$.
5. A transportation cost is a subadditive, nondecreasing, lower semi-continuous function $h : [0, \infty) \rightarrow [0, \infty)$ with $h(0) = 0$.
6. Given a transportation cost h , the corresponding generalized branched transport cost of a polyhedral flux $\rho = \sum_{i=1}^n a_i \vec{e}_i \mathcal{H}^1 \llcorner e_i$ with non-overlapping line segments e_i is

$$\mathbb{J}_h(\rho) = \sum_{i=1}^n h(|a_i|) \mathcal{H}^1(e_i).$$

The generalized branched transport cost of a mass flux ρ is

$$\mathbb{J}_h(\rho) = \inf \left\{ \liminf_{n \rightarrow \infty} \mathbb{J}_h(\rho_n) \mid \rho_n \in \mathcal{F}(\Omega) \text{ polyhedral, } \rho_n \xrightarrow{f} \rho \text{ as } n \rightarrow \infty \right\},$$

the relaxation of the generalized branched transport cost on polyhedral fluxes with respect to weak convergence of mass fluxes.

The variational problem of finding optimal mass transportation schemes between a given mass source $\mu_+ \in \mathcal{M}_+(\Omega)$ and a sink $\mu_- \in \mathcal{M}_+(\Omega)$ then is

$$\min \{ \mathbb{J}_h(\rho) \mid \rho \text{ is mass flux with } \partial \rho = \mu_- - \mu_+ \}.$$

The existence of minimizers and their properties are discussed in [3]. Note that mass fluxes are also known as *divergence measure vector fields* [14] or *vector charges* [13] or *1-dimensional normal currents* [7].

The definition of the h -mass of a flat chain follows the same strategy.

Definition 2 (*h -mass of a flat chain*). 1. An m -dimensional polyhedron in Ω is an oriented polyhedral subset of an m -dimensional plane $H \subset \Omega$ with nonempty relative interior.

2. A polyhedral m -chain in Ω is a linear combination $T = \sum_{i=1}^n a_i e_i$ with $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$, and e_1, \dots, e_n m -dimensional polyhedra in Ω . A refinement of T is a polyhedral m -chain of the form $\sum_{i=1}^n \sum_{k=1}^{K_i} a_i e_i^k$, where $e_i = e_i^1 \cup \dots \cup e_i^{K_i}$ represents a disjoint partition of e_i . Two polyhedral m -chains are equivalent and identified with each other, if they have a joint refinement. When writing a polyhedral m -chain as $T = \sum_{i=1}^n a_i e_i$ we shall always tacitly assume the e_i to be pairwise disjoint (which can always be achieved).

3. The boundary of a polyhedral m -chain $T = \sum_{i=1}^n a_i e_i$ is the polyhedral $(m-1)$ -chain $\partial T = \sum_{i=1}^n a_i \partial e_i$, where ∂e_i is the sum of the oriented faces in the relative boundary of e_i .

4. The mass of a polyhedral m -chain $T = \sum_{i=1}^n a_i e_i$ is $\mathbb{M}(T) = \sum_{i=1}^n |a_i| \mathcal{H}^m(e_i)$.

5. The flat norm of a polyhedral m -chain reads

$$\mathbb{F}(T) = \inf \{ \mathbb{M}(T - \partial D) + \mathbb{M}(D) \mid D \text{ is polyhedral } (m+1)\text{-chain} \}.$$

6. The Banach space $\mathbf{F}_m(\Omega)$ of flat m -chains is the completion of the vector space of polyhedral m -chains in Ω under the flat norm. The linear boundary operator ∂ is extended continuously with respect to the flat norm onto all of $\mathbf{F}_m(\Omega)$. The mass functional $\mathbb{M}(\cdot)$ is extended onto $\mathbf{F}_m(\Omega)$ via relaxation with respect to the flat norm. The subspace of flat m -chains of finite mass and with finite mass boundary is denoted $\bar{\mathbf{F}}_m(\Omega) \subset \mathbf{F}_m(\Omega)$.

7. Let T_1, T_2, \dots be a sequence of flat m -chains. We say T_n converges in mass to the flat m -chain T and write $T_n \rightarrow T$, if $\mathbb{M}(T_n - T) \rightarrow 0$. We say T_n converges flatly to T and write $T_n \xrightarrow{b} T$, if $\mathbb{F}(T_n - T) \rightarrow 0$.

8. Given a transportation cost h , the corresponding h -mass of a polyhedral 1-chain $T = \sum_{i=1}^n a_i e_i$ is

$$\mathbb{M}_h(T) = \sum_{i=1}^n h(|a_i|) \mathcal{H}^1(e_i).$$

The h -mass of a flat 1-chain T is

$$\mathbb{M}_h(T) = \inf \left\{ \liminf_{n \rightarrow \infty} \mathbb{M}_h(T_n) \mid T_n \in \mathbf{F}_1(\Omega) \text{ polyhedral, } T_n \xrightarrow{b} T \text{ as } n \rightarrow \infty \right\},$$

the relaxation of the h -mass on polyhedral fluxes with respect to flat convergence.

Note that flat m -chains with finite mass and boundary mass are also known as normal m -currents [7, 4.1.23 & 4.2.23],[3, Rem. 2.29(2)].

The following remark details how mass fluxes and flat 1-chains relate to each other. In particular, flat convergence of flat 1-chains is a strictly weaker notion than weak convergence of mass fluxes, which is why in general the h -mass must be less than or equal to the generalized branched transport cost.

Remark 3 (Flat 1-chains and mass fluxes). 1. Polyhedral fluxes $\rho = \sum_{i=1}^n a_i \vec{e}_i \mathcal{H}^1 \llcorner e_i$ and polyhedral 1-chains $T = \sum_{i=1}^n a_i e_i$ can naturally be identified with each other. Analogously, there is an obvious natural identification between polyhedral 0-chains and finite discrete measures on Ω .

2. The identification between polyhedral 0-chains and discrete measures can be extended to an isomorphism $\iota_0 : \overline{\mathbf{F}}_0(\Omega) \rightarrow \mathcal{M}(\Omega)$. Likewise, the identification between polyhedral 1-chains and polyhedral fluxes can be extended to an isomorphism $\iota_1 : \overline{\mathbf{F}}_1(\Omega) \rightarrow \mathcal{F}(\Omega)$.

The isomorphisms are consistent with the notions of boundary and convergence in the following sense.

(a) For any $T \in \overline{\mathbf{F}}_1(\Omega)$ and $\rho \in \mathcal{F}(\Omega)$ with $\iota_1(T) = \rho$ we have $\iota_0(\partial T) = \partial \rho$.

(b) Let $\mu, \mu_1, \mu_2, \dots \in \mathcal{M}(\Omega)$ and $\rho, \rho_1, \rho_2, \dots \in \mathcal{F}(\Omega)$, then

$$\begin{aligned} \mu_n \xrightarrow[n \rightarrow \infty]{} \mu \text{ strongly in } \mathcal{M}(\Omega) & \text{ is equivalent to } \iota_0^{-1}(\mu_n) \xrightarrow[n \rightarrow \infty]{} \iota_0^{-1}(\mu) \text{ in mass, and} \\ \rho_n \xrightarrow[n \rightarrow \infty]{} \rho \text{ strongly in } \mathcal{F}(\Omega) & \text{ is equivalent to } \iota_1^{-1}(\rho_n) \xrightarrow[n \rightarrow \infty]{} \iota_1^{-1}(\rho) \text{ in mass.} \end{aligned}$$

(c) Let $\rho \in \mathcal{F}(\Omega)$ and ρ_1, ρ_2, \dots be a sequence in $\mathcal{F}(\Omega)$, then

$$\rho_n \xrightarrow[n \rightarrow \infty]{f} \rho \text{ implies } \iota_1^{-1}(\rho_n) \xrightarrow[n \rightarrow \infty]{b} \iota_1^{-1}(\rho).$$

Vice versa, let $T_1, T_2, \dots \in \overline{\mathbf{F}}_1(\Omega)$ have equibounded mass and boundary mass and let $T \in \overline{\mathbf{F}}_1(\Omega)$, then

$$T_n \xrightarrow[n \rightarrow \infty]{b} T \text{ implies } \iota_1(T_n) \xrightarrow[n \rightarrow \infty]{f} \iota_1(T).$$

Analogously, $\mu_n \xrightarrow{*} \mu$ in $\mathcal{M}(\Omega)$ as $n \rightarrow \infty$ implies $\iota_0^{-1}(\mu_n) \xrightarrow{b} \iota_0^{-1}(\mu)$, while $T_n \xrightarrow{b} T$ in $\overline{\mathbf{F}}_0(\Omega)$ as $n \rightarrow \infty$ implies $\iota_0(T_n) \xrightarrow{*} \iota_0(T)$ under the condition that the flat 0-chains T_n have equibounded mass.

The proof of the above essentially relies on weak-* compactness of measures with bounded mass and classical deformation theorems such as [16]; for more details see the brief summary in [3, Rem. 2.29] and the references therein.

3. Example sequences of flat (polyhedral) 1-chains in $[-1, 1]^2$ that converge flatly to 0, while the corresponding mass fluxes do not converge weakly, are

$$\begin{aligned} T_n &= \sum_{k=-n}^{n-1} \left[\left(\frac{k}{n}, 0 \right), \left(\frac{k}{n} + \frac{1}{n^2}, 0 \right) \right] \quad \text{and} \\ S_n &= n[c_1, c_2] + n[c_2, c_3] + n[c_3, c_4] + n[c_4, c_1] \\ &\quad - n[\alpha c_1, \alpha c_2] - n[\alpha c_2, \alpha c_3] - n[\alpha c_3, \alpha c_4] - n[\alpha c_4, \alpha c_1] \quad \text{with } \alpha = 1 - \frac{1}{n^2}, \end{aligned}$$

where $[a, b]$ denotes the line segment from a to b and c_1, \dots, c_4 denote the four corners of $[-1, 1]^2$. While $\iota_1(T_n) \rightarrow 0$ strongly in $\mathcal{M}([-1, 1]^2; \mathbb{R}^2)$, $\partial(\iota_1(T_n))$ diverges in $\mathcal{M}([-1, 1]^2)$. On the other hand, $\partial(\iota_1(S_n)) = 0$ for all n , but $\iota_1(S_n)$ diverges in $\mathcal{M}([-1, 1]^2; \mathbb{R}^2)$.

Corollary 4 (Bound of h -mass by branched transport cost). *Let $T \in \overline{\mathbf{F}}_1(\Omega)$ and h be a transportation cost, then $\mathbb{M}_h(T) \leq \mathbb{J}_h(\iota_1(T))$.*

In this note we show equality.

Theorem 5 (Equivalence of h -mass and branched transport cost). *Let $T \in \overline{\mathbf{F}}_1(\Omega)$ and h be a transportation cost, then $\mathbb{M}_h(T) = \mathbb{J}_h(\iota_1(T))$.*

The proof will be provided in section 3. It will be based on the following lemmas for 1-rectifiable flat chains, whose statement requires the notion of rectifiability and acyclicity introduced below.

Definition 6 (Rectifiable and acyclic mass fluxes and flat chains). *1. Let $m \in \{0, 1\}$. Given a Borel set $A \subset \Omega$ and a flat m -chain $T \in \overline{\mathbf{F}}_m(\Omega)$ with $\iota_m(T) \llcorner A \in \mathcal{F}(\Omega)$, the restriction of T to A is defined as $T \llcorner A = \iota_m^{-1}(\iota_m(T) \llcorner A)$. The restriction to A can be extended to all flat m -chains of finite mass by continuity with respect to flat convergence.*

2. A subset $\Sigma \subset \mathbb{R}^d$ is called m -rectifiable if it is contained in the countable union of m -dimensional C^1 -submanifolds, up to an \mathcal{H}^m -negligible set.

3. A (vector-valued) Radon measure ρ or a flat m -chain T are called m -rectifiable if there exists an m -rectifiable set $\Sigma \subset \mathbb{R}^d$ with $\rho = \rho \llcorner \Sigma$ or $T = T \llcorner \Sigma$, respectively. It is known that for a rectifiable Radon measure $\rho \in \mathcal{F}(\Omega)$ supported on the 1-rectifiable set Σ the Radon-Nikodym derivative $d\rho/d|\rho|$ spans the approximate tangent line to Σ at $|\rho|$ -a.e. point. Similarly for a rectifiable flat chain $T \in \overline{\mathbf{F}}_1(\Omega)$ supported on the 1-rectifiable set Σ the Radon-Nikodym derivative $d\iota_1(T)/d|\iota_1(T)|$ spans the approximate tangent line to Σ at $|\iota_1(T)|$ -a.e. point

4. A mass flux $\rho \in \mathcal{F}(\Omega)$ is called acyclic if it cannot be decomposed into $\rho = \rho^a + \rho^b$ with $\rho^b \neq 0$, $\partial\rho^b = 0$, and $\|\rho\|_{\mathcal{M}} = \|\rho^a\|_{\mathcal{M}} + \|\rho^b\|_{\mathcal{M}}$. A flat 1-chain $T \in \overline{\mathbf{F}}_1(\Omega)$ is acyclic if $\iota_1(T)$ is.

Note that the restriction for flat chains can also be defined without reference to mass fluxes as in [8].

Lemma 7 (Approximation of rectifiable mass fluxes by mass fluxes with rectifiable boundary). *Let $\rho \in \mathcal{F}(\Omega)$ be 1-rectifiable and acyclic, then there exists a monotonically increasing sequence of ρ -measurable functions $\lambda_1, \lambda_2, \dots : \Omega \rightarrow [0, 1]$ and associated 1-rectifiable acyclic mass fluxes $\rho_1 = \lambda_1\rho, \rho_2 = \lambda_2\rho, \dots$ with $\rho_n \rightarrow \rho$ strongly as $n \rightarrow \infty$, where $\partial\rho_n$ is 0-rectifiable with $\|\partial\rho_n\|_{\mathcal{M}} \leq \|\partial\rho\|_{\mathcal{M}}$ for all $n \in \mathbb{N}$.*

Lemma 8 (Approximation of mass fluxes by fluxes with finite discrete boundary). *Let $\rho \in \mathcal{F}(\Omega)$ be acyclic with 0-rectifiable boundary $\partial\rho$, then there exists a monotonically increasing sequence of ρ -measurable functions $\kappa_1, \kappa_2, \dots : \Omega \rightarrow [0, 1]$ and associated acyclic mass fluxes $\rho_1 = \kappa_1\rho, \rho_2 = \kappa_2\rho, \dots$ with $\rho_n \rightarrow \rho$ strongly as $n \rightarrow \infty$, where $\partial\rho_n$ has finite support and $\|\partial\rho_n\|_{\mathcal{M}} \leq \|\partial\rho\|_{\mathcal{M}}$ for all $n \in \mathbb{N}$.*

The proof of both lemmas will be provided in section 2. A direct consequence is the following lemma.

Lemma 9 (Approximation of rectifiable 1-chains by 1-chains with polyhedral boundary). *Let h be a transportation cost. For any rectifiable $T \in \overline{\mathbf{F}}_1(\Omega)$ there exists a sequence $T_1, T_2, \dots \in \overline{\mathbf{F}}_1(\Omega)$ with $T_n \rightarrow T$ in mass as $n \rightarrow \infty$ such that $\mathbb{M}_h(T_n) \rightarrow \mathbb{M}_h(T)$ as $n \rightarrow \infty$ and such that ∂T_n is polyhedral with $\mathbb{M}(\partial T_n) \leq \mathbb{M}(\partial T)$ as well as $\mathbb{M}_{\tilde{h}}(T_n) \leq \mathbb{M}_{\tilde{h}}(T)$ for all $n \in \mathbb{N}$ and transportation costs \tilde{h} .*

Proof. First note that by White's structure theorem [16, Sec. 6] one can identify any 1-rectifiable flat chain $T \in \overline{\mathbf{F}}_1(\Omega)$ with a triple $[\Sigma, \theta, m]$ of a 1-rectifiable set $\Sigma \subset \Omega$ with approximate tangent $\theta : \Sigma \rightarrow \mathbb{S}^{d-1}$ and a measurable function $m : \Sigma \rightarrow \mathbb{R}$ such that

$$\iota_1(T) = m\theta\mathcal{H}^1 \llcorner \Sigma.$$

Again by [16, Sec. 6], its h -mass in this case can be expressed as

$$\mathbb{M}_h(T) = \int_{\Sigma} h(|m|) d\mathcal{H}^1.$$

Now consider $\rho = \iota_1(T) = m\theta\mathcal{H}^1\llcorner\Sigma$. By Smirnov's decomposition theorem [13, Thm. C] we can decompose $\rho = \rho^a + \rho^b$, where $\rho^a = m^a\theta\mathcal{H}^1\llcorner\Sigma \in \mathcal{F}(\Omega)$ is rectifiable and acyclic with $\partial\rho^a = \partial\rho$ and $\rho^b = m^b\theta\mathcal{H}^1\llcorner\Sigma \in \mathcal{F}(\Omega)$ (with m^b having the same sign as m^a pointwise) satisfies $\partial\rho^b = 0$. By lemma 7 there is a monotonically increasing sequence $\lambda_1, \lambda_2, \dots : \Omega \rightarrow [0, 1]$ of ρ^a -measurable functions such that $\lambda_n \rightarrow 1$ monotonically ρ^a -almost everywhere and such that $\partial(\lambda_n\rho^a)$ is 0-rectifiable with $\|\partial(\lambda_n\rho^a)\|_{\mathcal{M}} \leq \|\partial\rho^a\|_{\mathcal{M}} = \|\partial\rho\|_{\mathcal{M}}$ for all $n \in \mathbb{N}$. Denote the flat 1-chains corresponding to $\rho_n^a + \rho^b$ by $\tilde{T}_n = [\Sigma, \theta, \lambda_n m^a + m^b]$. By the Monotone Convergence Theorem

$$\mathbb{M}_h(\tilde{T}_n) = \int_{\Sigma} h(|\lambda_n m^a + m^b|) d\mathcal{H}^1 \rightarrow \int_{\Sigma} h(|m^a + m^b|) d\mathcal{H}^1 = \mathbb{M}_h(T)$$

as $n \rightarrow \infty$ so that (potentially after passing to a subsequence) we may assume

$$\mathbb{M}(\tilde{T}_n - T) = \|\rho_n^a - \rho^a\|_{\mathcal{M}} \leq \frac{1}{n} \quad \text{and} \quad \left| \mathbb{M}_h(\tilde{T}_n) - \mathbb{M}_h(T) \right| \leq \frac{1}{n}.$$

Similarly, appealing to lemma 8 instead of lemma 7, for each $n \in \mathbb{N}$ there is some sequence $T_{n,k} = [\Sigma, \theta, \kappa_k \lambda_n m^a + m^b]$, $k = 1, 2, \dots$, with $\partial T_{n,k}$ polyhedral, $\mathbb{M}(\partial T_{n,k}) \leq \mathbb{M}(\partial \tilde{T}_n) \leq \mathbb{M}(T)$, and

$$\mathbb{M}(T_{n,k} - \tilde{T}_n) = \|\kappa_k \rho_n^a - \rho_n^a\|_{\mathcal{M}} \leq \frac{1}{k} \quad \text{and} \quad \left| \mathbb{M}_h(T_{n,k}) - \mathbb{M}_h(\tilde{T}_n) \right| \leq \frac{1}{k}.$$

Thus, the sequence $T_n = T_{n,n}$ has all desired properties. \square

2 Proof of main lemmas

The proof uses Smirnov's decomposition theorem, part of which we restate for convenience.

Definition 10 (Simple oriented curve). *A simple oriented curve of finite length in Ω is a mass flux of the form*

$$\tilde{\rho} = \gamma_{\#} \dot{\gamma} \mathcal{H}^1 \llcorner [0, 1],$$

where $\gamma_{\#} \mu$ denotes the pushforward of a measure μ under a function f and $\gamma : [0, 1] \rightarrow \Omega$ is an injective Lipschitz curve. Note that, writing δ_x for the Dirac mass at x ,

$$\partial \tilde{\rho} = \delta_{\gamma(1)} - \delta_{\gamma(0)}.$$

Theorem 11 (Smirnov's decomposition theorem, [13, Thm. B-C]). *For any acyclic $\rho \in \mathcal{F}(\Omega)$ there is a set J of simple oriented curves of finite length and a nonnegative measure μ on J such that*

$$\begin{aligned} \rho &= \int_J \tilde{\rho} d\mu(\tilde{\rho}), \\ \|\rho\|_{\mathcal{M}} &= \int_J \|\tilde{\rho}\|_{\mathcal{M}} d\mu(\tilde{\rho}), \\ \|\partial\rho\|_{\mathcal{M}} &= \int_J \|\partial\tilde{\rho}\|_{\mathcal{M}} d\mu(\tilde{\rho}). \end{aligned}$$

Above, the first line means

$$\langle \rho, v \rangle = \int_J \langle \tilde{\rho}, v \rangle d\mu(\tilde{\rho})$$

for every smooth test vector field $v : \Omega \rightarrow \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between vector-valued Radon measures and continuous vector fields on Ω .

We will furthermore use the following two simple results about the generic intersection between a regular grid and a rectifiable set and a rectifiable mass flux.

Lemma 12 (Rectifiable set and grid). *Let $\Sigma \subset \Omega$ be 1-rectifiable and define the rectilinear grid*

$$\mathcal{G}_n = \left\{ y \in \mathbb{R}^d \mid y_i = \frac{m}{n} \text{ for some } m \in \mathbb{Z} \text{ and } i \in \{1, \dots, d\} \right\}$$

of grid width $\frac{1}{n}$. Then for almost every $x \in \mathbb{R}^d$ the set $\Sigma \cap (x + \mathcal{G}_n)$ is countable for all $n \in \mathbb{N}$.

Proof. Since $\mathcal{G}_n = \bigcup_{i=1}^d \mathcal{S}_n^i$ for

$$\mathcal{S}_n^i = \left\{ y \in \mathbb{R}^d \mid y_i = \frac{m}{n} \text{ for some } m \in \mathbb{N} \right\},$$

it suffices to show for fixed $1 \leq i \leq d$ that for almost every $s \in \mathbb{R}$ the set $\Sigma \cap (se_i + \mathcal{S}_n^i)$ is countable for all $n \in \mathbb{N}$ (here e_i denotes the i^{th} Cartesian unit vector). To this end it suffices to show that for almost all $r \in \mathbb{R}$ the intersection of Σ with the hyperplane

$$\mathcal{P}_r^i = \left\{ y \in \mathbb{R}^d \mid y_i = r \right\}$$

is countable. Indeed, $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n^i$ can be expressed as a countable union $\bigcup_{j=1}^{\infty} \mathcal{P}_{r_j}^i$ of such hyperplanes; thus the set of $s \in \mathbb{R}$ for which $\Sigma \cap (se_i + \mathcal{S})$ is uncountable is given by $\bigcup_{j=1}^{\infty} R_j$ with

$$R_j = \left\{ s \in \mathbb{R} \mid \Sigma \cap \mathcal{P}_{s+r_j}^i \text{ is uncountable} \right\},$$

which we show to be a nullset below. Consequently, $\Sigma \cap (se_i + \mathcal{S})$ is countable for almost all $s \in \mathbb{R}$.

To show that $\Sigma \cap \mathcal{P}_r^i$ is countable for almost all $r \in \mathbb{R}$ it suffices to cite the coarea formula for rectifiable sets [7, 3.2.22(2)] with $W = \Sigma$, $f(x) = x_i$ which states that $\Sigma \cap \mathcal{P}_r^i$ is \mathcal{H}^0 -measurable and \mathcal{H}^0 -rectifiable for almost all $r \in \mathbb{R}$. \square

Lemma 13 (Smirnov curves and grid). *Let $\rho \in \mathcal{F}(\Omega)$ be acyclic and 1-rectifiable so that $\rho = \rho \llcorner \Sigma$ for a 1-rectifiable set $\Sigma \subset \Omega$ and there exists a decomposition*

$$\rho = \int_J \tilde{\rho} d\mu(\tilde{\rho})$$

into simple oriented curves by Smirnov's decomposition theorem. Then for almost all $x \in \mathbb{R}^d$, $\mu(J_x^n) = 0$ for all $n \in \mathbb{N}$ with

$$J_x^n = \left\{ \tilde{\rho} = \gamma_{\#} \hat{\gamma} \mathcal{H}^1 \llcorner [0, 1] \in J \mid \gamma([0, 1]) \cap (x + \mathcal{G}_n) \not\subset \Sigma \right\},$$

that is, for any $n \in \mathbb{N}$ the intersection of μ -almost every Smirnov curve with $(x + \mathcal{G}_n)$ lies in Σ .

Proof. Obviously it suffices to prove the statement for fixed $n \in \mathbb{N}$, which we shall assume in the following. Below, we will denote the Lipschitz curve associated with a simple oriented curve $\tilde{\rho} \in J$ by $\gamma_{\tilde{\rho}}$ and the complement of Σ by Σ^c .

Step 1. We first show for any Lipschitz curve $\gamma : [0, 1] \rightarrow \Omega$ that

$$\mathcal{H}^d(A^\gamma) > 0 \text{ implies } \mathcal{H}^1(\gamma([0, 1]) \setminus \Sigma) > 0,$$

where

$$A^\gamma = \left\{ x \in \mathbb{R}^d \mid \gamma([0, 1]) \cap (x + \mathcal{G}_n) \not\subset \Sigma \right\}.$$

To prove it, assume by contradiction that $\mathcal{H}^1(\gamma([0, 1]) \setminus \Sigma) = 0$. For $i = 1, \dots, d$, denote

$$\mathcal{G}_n^i = \left\{ y \in \mathbb{R}^d \mid y_i = \frac{m}{n} \text{ for some } m \in \mathbb{Z} \right\}$$

and let

$$A^{\gamma, i} = \left\{ x \in \mathbb{R}^d \mid \gamma([0, 1]) \cap (x + \mathcal{G}_n^i) \not\subset \Sigma \right\}.$$

Since $\mathcal{G}_n = \cup_i \mathcal{G}_n^i$, then $A^\gamma \subset \cup_i A^{\gamma,i}$. Hence the assumption $\mathcal{H}^d(A^\gamma) > 0$ implies the existence of an index i such that $\mathcal{H}^d(A^{\gamma,i}) > 0$. Applying for every $m \in \mathbb{Z}$ the coarea formula [7, 3.2.22(3)] with $W = \gamma([0, 1])$, $Z = \mathbb{R}$, $f(x) = x_i + \frac{m}{n}$, $g = \mathbf{1}_{\Sigma^c}$, we deduce from the equality $\mathcal{H}^1(\gamma([0, 1]) \setminus \Sigma) = 0$ that for \mathcal{H}^1 -a.e $z \in \mathbb{R}$ the set $A^{\gamma,i} \cap \{x_i = z\}$ is empty, which contradicts $\mathcal{H}^d(A^{\gamma,i}) > 0$.

Step 2. We show $\mathcal{H}^d(A^{\gamma\tilde{\rho}}) = 0$ for μ -almost every $\tilde{\rho} \in J$. Indeed, we have

$$0 = \|\rho\|_{\mathcal{M}} - \|\rho \llcorner \Sigma\|_{\mathcal{M}} \geq \int_J \|\tilde{\rho}\|_{\mathcal{M}} d\mu(\tilde{\rho}) - \int_J \|\tilde{\rho} \llcorner \Sigma\|_{\mathcal{M}} d\mu(\tilde{\rho}) = \int_J \|\tilde{\rho} \llcorner \Sigma^c\|_{\mathcal{M}} d\mu(\tilde{\rho}),$$

thus μ -almost every $\tilde{\rho} \in J$ satisfies

$$\mathcal{H}^1(\gamma_{\tilde{\rho}}([0, 1]) \setminus \Sigma) = \int_0^1 \mathbf{1}_{\Sigma^c}(\gamma_{\tilde{\rho}})|\dot{\gamma}_{\tilde{\rho}}| d\mathcal{H}^1 = \int_{\Sigma^c} d|\tilde{\rho}| = \|\tilde{\rho} \llcorner \Sigma^c\|_{\mathcal{M}} = 0.$$

By the previous step this implies the desired result.

Step 3. Finally we show $\mathcal{H}^d(\{x \in \mathbb{R}^d \mid \mu(J_x^n) > 0\}) = 0$, which concludes the proof. Indeed, let us introduce the function

$$h : \mathbb{R}^d \times J \rightarrow \{0, 1\}, \quad h(x, \tilde{\rho}) = \begin{cases} 1 & \text{if } \tilde{\rho} \in J_x^n, \\ 0 & \text{otherwise,} \end{cases}$$

then by Fubini's theorem we have

$$\int_{\mathbb{R}^d} \mu(J_x^n) dx = \int_J \int_{\mathbb{R}^d} h(x, \tilde{\rho}) dx d\mu(\tilde{\rho}) = \int_J \mathcal{H}^d(A^{\gamma\tilde{\rho}}) d\mu(\tilde{\rho}),$$

which is zero by the previous step. Thus, $\mu(J_x^n) = 0$ for almost all $x \in \mathbb{R}^d$, as desired. \square

Proof of lemma 7. Since ρ is rectifiable, there is a 1-rectifiable set $\Sigma \subset \Omega$ with $\rho = \rho \llcorner \Sigma$. Using Smirnov's decomposition theorem we decompose ρ into simple oriented curves,

$$\rho = \int_J \tilde{\rho} d\mu(\tilde{\rho}).$$

Now, for $n \in \mathbb{N}$ consider the rectilinear grids \mathcal{G}_{2^n} from lemma 12 with grid size 2^{-n} and note $\mathcal{G}_{2^n} \subset \mathcal{G}_{2^m}$ for $m \geq n$. Since Σ is 1-rectifiable, by lemmas 12 and 13 there exists $x \in \Omega$ such that for all $n \in \mathbb{N}$ the intersection $(x + \mathcal{G}_{2^n}) \cap \Sigma$ is countable and μ -almost all $\tilde{\rho}$ intersect $(x + \mathcal{G}_{2^n})$ in points which belong to Σ . Now define for each simple oriented curve $\tilde{\rho} = \gamma_{\#} \dot{\gamma} \mathcal{H}^1 \llcorner [0, 1]$ the pruned curve

$$\tilde{\rho}_n = \gamma_{\#} \dot{\gamma} \mathcal{H}^1 \llcorner [t_n^{\gamma,l}, t_n^{\gamma,r}] \quad \text{with} \quad t_n^{\gamma,l} = \min\{t \in [0, 1] \mid \gamma(t) \in x + \mathcal{G}_{2^n}\}, \quad t_n^{\gamma,r} = \max\{t \in [0, 1] \mid \gamma(t) \in x + \mathcal{G}_{2^n}\}$$

(if γ does not intersect $x + \mathcal{G}_{2^n}$ we shall define $\tilde{\rho}_n = 0$ by convention). Next set

$$\rho_n = \int_J \tilde{\rho}_n d\mu(\tilde{\rho}).$$

Using the properties of the Smirnov decomposition we obtain

$$\|\rho\|_{\mathcal{M}} = \int_J \|\tilde{\rho}\|_{\mathcal{M}} d\mu(\tilde{\rho}) = \int_J \|\tilde{\rho}_n\|_{\mathcal{M}} + \|\tilde{\rho} - \tilde{\rho}_n\|_{\mathcal{M}} d\mu(\tilde{\rho}) \geq \|\rho_n\|_{\mathcal{M}} + \|\rho - \rho_n\|_{\mathcal{M}}.$$

Together with the triangle inequality this implies $\|\rho\|_{\mathcal{M}} = \|\rho_n\|_{\mathcal{M}} + \|\rho - \rho_n\|_{\mathcal{M}}$, which in turn implies equality of the total variation measures, $|\rho| = |\rho_n| + |\rho - \rho_n|$, as well as parallelism of the Radon–Nikodym derivatives $\frac{d\rho}{d|\rho|}$ and $\frac{d\rho_n}{d|\rho_n|}$, ρ_n -a.e. Consequently,

$$\rho_n = \lambda_n \rho$$

for some ρ -measurable $\lambda_n : \Omega \rightarrow [0, 1]$. Replacing ρ with ρ_m for $m > n$ in the above argument implies $\rho_n = \lambda_{n,m} \rho_m$ for some ρ -measurable $\lambda_{n,m} : \Omega \rightarrow [0, 1]$ so that $\lambda_n = \lambda_{n,m} \lambda_m \leq \lambda_m$. Furthermore,

$$\|\rho - \rho_n\|_{\mathcal{M}} = \left\| \int_J \tilde{\rho} - \tilde{\rho}_n \, d\mu(\tilde{\rho}) \right\|_{\mathcal{M}} \leq \int_J \|\tilde{\rho} - \tilde{\rho}_n\|_{\mathcal{M}} \, d\mu(\tilde{\rho}) \rightarrow 0$$

by the Monotone Convergence Theorem. Finally, using that $\partial\tilde{\rho}_n = (\partial\tilde{\rho}_n)_{\mathcal{L}}(\Sigma \cap (x + \mathcal{G}_{2^n}))$ for μ almost all $\tilde{\rho}$ due to our choice of x , we can compute

$$\partial\rho_n \llcorner (\Sigma \cap (x + \mathcal{G}_{2^n})) = \left(\int_J \partial\tilde{\rho}_n \, d\mu(\tilde{\rho}) \right) \llcorner (\Sigma \cap (x + \mathcal{G}_{2^n})) = \int_J (\partial\tilde{\rho}_n)_{\mathcal{L}}(\Sigma \cap (x + \mathcal{G}_{2^n})) \, d\mu(\tilde{\rho}) = \int_J \partial\tilde{\rho}_n \, d\mu(\tilde{\rho}) = \partial\rho_n$$

so that $\partial\rho_n$ is 0-rectifiable, and

$$\|\partial\rho\|_{\mathcal{M}} = \int_J \|\partial\tilde{\rho}\|_{\mathcal{M}} \, d\mu(\tilde{\rho}) \geq \int_J \|\partial\tilde{\rho}_n\|_{\mathcal{M}} \, d\mu(\tilde{\rho}) \geq \|\partial\rho_n\|_{\mathcal{M}}. \quad \square$$

Proof of lemma 8. Again use Smirnov's decomposition theorem to decompose ρ into simple oriented curves,

$$\rho = \int_J \tilde{\rho} \, d\mu(\tilde{\rho}) \quad \text{with} \quad \|\partial\rho\|_{\mathcal{M}} = \int_J \|\partial\tilde{\rho}\|_{\mathcal{M}} \, d\mu(\tilde{\rho}).$$

Denote the parameterization associated with a simple oriented curve $\tilde{\rho}$ by $\gamma_{\tilde{\rho}} : [0, 1] \rightarrow \Omega$. Since $\partial\rho$ is rectifiable by assumption, there is a countable set S of points with $\partial\rho = \partial\rho \llcorner S$. This implies $\gamma_{\tilde{\rho}}(0), \gamma_{\tilde{\rho}}(1) \in S$ for μ -almost all $\tilde{\rho}$. For each $x \in S$ and $t = 0, 1$ we now introduce the set

$$J_x^t = \{\tilde{\rho} \in J \mid x = \gamma_{\tilde{\rho}}(t)\},$$

which due to $t \in \{0, 1\}$ does not depend on the particular choice of parameterizations $\gamma_{\tilde{\rho}}$ for curves $\tilde{\rho}$. Note that each J_x^t is μ -measurable (indeed, it is the preimage of x under the mapping $\tilde{\rho} \mapsto \gamma_{\tilde{\rho}}(t)$ with $t = 0, 1$, which is continuous with respect to the underlying topology on the space of simple oriented curves, the weak-* topology, [for instance because the map \$\rho \mapsto \text{div} \rho\$ is continuous](#)). Also note that $J_x^0 \cap J_y^1$ is disjoint from $J_z^0 \cap J_w^1$ whenever $(x, y) \neq (z, w)$ so that

$$\sum_{(x,y) \in S \times S} \mu(J_x^0 \cap J_y^1) = \int_{\bigcup_{(x,y) \in S \times S} J_x^0 \cap J_y^1} d\mu(\tilde{\rho}) = \int_J d\mu(\tilde{\rho}) = \frac{1}{2} \int_J \|\partial\tilde{\rho}\|_{\mathcal{M}} \, d\mu(\tilde{\rho}) = \frac{1}{2} \|\partial\rho\|_{\mathcal{M}} < \infty.$$

Thus, since $S \times S$ is countable, it is straightforward to see that we can arrange all its elements $(x, y) \in S \times S$ in decreasing order with respect to $\mu(J_x^0 \cap J_y^1)$. Denote by (x_i, y_i) the i^{th} element of $S \times S$ and define

$$\rho_n = \sum_{i=1}^n \int_{J_{x_i}^0 \cap J_{y_i}^1} \tilde{\rho} \, d\mu(\tilde{\rho})$$

for $n \in \mathbb{N}$. In the same manner as in the previous proof we obtain $\rho_n = \kappa_n \rho$ for a monotonically increasing sequence of ρ -measurable functions $\kappa_n : \Omega \rightarrow [0, 1]$ as well as $\|\rho - \rho_n\|_{\mathcal{M}} \rightarrow 0$. Furthermore, $\partial\rho_n = \partial\rho_n \llcorner \{x_1, \dots, x_n, y_1, \dots, y_n\}$ and

$$\|\partial\rho_n\|_{\mathcal{M}} \leq \sum_{i=1}^n \int_{J_{x_i}^0 \cap J_{y_i}^1} \|\partial\tilde{\rho}\|_{\mathcal{M}} \, d\mu(\tilde{\rho}) \leq \|\partial\rho\|_{\mathcal{M}}. \quad \square$$

3 Weak-* relaxation of the polyhedral h -mass

The strategy to prove theorem 5 is to first restrict to transportation costs h with $h(m) \geq \alpha m$ for some $\alpha > 0$ and all $m > 0$ and to separately consider two cases: If the right derivative $h'(0)$ of the transportation

cost h in 0 is finite, then one can prove equivalence of both relaxations directly by construction. Otherwise [5, Prop. 2.8] implies that $\mathbb{M}_h(\cdot)$ is only finite on 1-rectifiable flat chains. In that case we employ our new approximation lemma for 1-rectifiable flat chains to reduce theorem 5 to the case of chains with polyhedral boundary. This case in turn has already been solved by Chambolle, Ferrari, and Merlet [4] (under the above condition on h). The proof for general transportation cost h can then be reduced to costs with $h(m) \geq \alpha m$ using a representation theorem for $\mathbb{M}_h(T)$.

Proof of theorem 5 for $h(m) \geq \alpha m$. First consider the case $h'(0) < \infty$. Let $\tilde{T}_n \xrightarrow{b} T$ be a sequence of polyhedral 1-chains with $\lim_{n \rightarrow \infty} \mathbb{M}_h(\tilde{T}_n) = \mathbb{M}_h(T) < \infty$ (if $\mathbb{M}_h(T) = \infty$ there is nothing to prove). Due to our growth condition on h we have $\alpha \mathbb{M}(\tilde{T}_n) \leq \mathbb{M}_h(\tilde{T}_n) \rightarrow \mathbb{M}_h(T)$ so that the \tilde{T}_n have equibounded mass. If the boundaries $\partial \tilde{T}_n$ also have equibounded mass, then $\iota_1(\tilde{T}_n) \xrightarrow{f} \iota_1(T)$ and thus

$$\mathbb{J}_h(\iota_1(T)) \leq \liminf_{n \rightarrow \infty} \mathbb{J}_h(\iota_1(\tilde{T}_n)) = \liminf_{n \rightarrow \infty} \mathbb{M}_h(\tilde{T}_n) = \mathbb{M}_h(T)$$

as desired. Otherwise, let $\mu_{\pm}^1, \mu_{\pm}^2, \dots \in \mathcal{M}_+(\Omega)$ be finite linear combinations of Dirac masses such that $\mu_{-}^n - \mu_{+}^n \xrightarrow{*} \iota_0(\partial T)$. Since \tilde{T}_n converges flatly, also $\partial \tilde{T}_n \xrightarrow{b} \partial T$ and thus

$$\iota_0^{-1}(\mu_{-}^n - \mu_{+}^n) - \partial \tilde{T}_n \xrightarrow{b} 0 \quad \text{as } n \rightarrow \infty.$$

Consequently, and as this null sequence in $\mathbf{F}_0(\Omega)$ is polyhedral, there exist polyhedral 1-chains D_n with $D_n \rightarrow 0$ in mass such that

$$\mathbb{M}(\iota_0^{-1}(\mu_{-}^n - \mu_{+}^n) - \partial \tilde{T}_n - \partial D_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Now define $T_n = \tilde{T}_n + D_n$, then both $\mathbb{M}(T_n)$ and $\mathbb{M}(\partial T_n)$ are equibounded, and $T_n \xrightarrow{b} T$. Therefore, we have $\iota_1(T_n) \xrightarrow{f} \iota_1(T)$ and

$$\begin{aligned} \mathbb{J}_h(\iota_1(T)) &\leq \liminf_{n \rightarrow \infty} \mathbb{J}_h(\iota_1(T_n)) = \liminf_{n \rightarrow \infty} \mathbb{M}_h(\tilde{T}_n + D_n) \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{M}_h(\tilde{T}_n) + \mathbb{M}_h(D_n) \leq \liminf_{n \rightarrow \infty} \mathbb{M}_h(\tilde{T}_n) + h'(0)\mathbb{M}(D_n) = \lim_{n \rightarrow \infty} \mathbb{M}_h(\tilde{T}_n) = \mathbb{M}_h(T), \end{aligned}$$

where we have used that the transportation cost h is subadditive (so that the h -mass is subadditive).

Now assume $h'(0) = \infty$. By [5, Prop. 2.8], $\mathbb{M}_h(\cdot)$ is only finite on 1-rectifiable flat 1-chains so that it suffices to show $\mathbb{J}_h(\iota_1(T)) \leq \mathbb{M}_h(T)$ for a 1-rectifiable $T \in \overline{\mathbf{F}}_1(\Omega)$. By lemma 9 there is a sequence T_n converging in mass to T such that $\mathbb{M}_h(T_n) \rightarrow \mathbb{M}_h(T)$ and ∂T_n is polyhedral with equibounded mass. Due to the equibounded mass and boundary mass we have $\iota_1(T_n) \xrightarrow{f} \iota_1(T)$ and thus, by definition of the relaxation,

$$\mathbb{J}_h(\iota_1(T)) \leq \liminf_{n \rightarrow \infty} \mathbb{J}_h(\iota_1(T_n)).$$

Since T_n has polyhedral boundary, by Chambolle, Ferrari, and Merlet [4, Thm. 1.2] we know that

$$\mathbb{M}_h(T_n) = \lim_{j \rightarrow \infty} \mathbb{M}_h(T_n^j)$$

for a sequence T_n^1, T_n^2, \dots of polyhedral flat 1-chains **converging to T_n in flat norm**, with equibounded mass and $\partial T_n^j = \partial T_n$ for all $j \in \mathbb{N}$. Thus we have $\iota_1(T_n^j) \xrightarrow{f} \iota_1(T_n)$ so that

$$\mathbb{J}_h(\iota_1(T)) \leq \liminf_{n \rightarrow \infty} \mathbb{J}_h(\iota_1(T_n)) \leq \liminf_{n \rightarrow \infty} \liminf_{j \rightarrow \infty} \mathbb{J}_h(\iota_1(T_n^j)) = \liminf_{n \rightarrow \infty} \lim_{j \rightarrow \infty} \mathbb{M}_h(T_n^j) = \liminf_{n \rightarrow \infty} \mathbb{M}_h(T_n) = \mathbb{M}_h(T)$$

as desired. □

To also cover the case of general transportation costs h , let us first note that h can be approximated by a sequence of superlinear transportation costs. To this end, define the indicator function of a set A as $\chi_A(m) = 0$ if $m \in A$ and $\chi_A(m) = \infty$ else and recall that the lower semi-continuous subadditive envelope of a function $\psi : [0, \infty) \rightarrow [0, \infty]$ is the lower semi-continuous subadditive function [2, Def. 5.16 and Prop. 5.17] defined as

$$m \mapsto \sup\{\phi(m) \mid \phi : [0, \infty) \rightarrow [0, \infty) \text{ is lower semi-continuous subadditive with } \phi \leq \psi\}.$$

Lemma 14 (Superlinear approximation of transportation costs). *Let h be a transportation cost. For $M > 0$ define the transportation cost $h_M : [0, \infty) \rightarrow [0, \infty)$ to be the lower semi-continuous subadditive envelope of the function $m \mapsto h(m) + \chi_{[0, M]}(m)$. Then there exists some $\alpha > 0$ such that*

$$\begin{aligned} \alpha m &\leq h_M(m) && \text{for all } m \geq 0, \\ h(m) &\leq h_M(m) && \text{for all } m \geq 0, \\ h_M(m) &\leq \frac{2h(M)}{M}m && \text{for all } m \geq M, \\ h_M(m) &= h(M) && \text{for all } m \leq M. \end{aligned}$$

Proof. That h_M is a transportation cost with $h_M \geq h$ as well as $h_M(m) = h(m)$ for $m \leq M$ follows directly from the properties of the lower semi-continuous subadditive envelope (see [2, Prop. 5.17]). Furthermore, for $m \geq M$ let $k \in \mathbb{N}$ and $r \in [0, M)$ such that $m = kM + r$. Then by the subadditivity of h_M we obtain

$$h_M(m) \leq kh_M(M) + h_M(r) = kh(M) + h(r) \leq (k+1)h(M) \leq 2kh(M) \leq \frac{2h(M)}{M}(kM + r) = \frac{2h(M)}{M}m.$$

Finally, by [10, Thm. 5 and its proof] we have $h(m) \geq \alpha m$ for all $m \in [0, M]$ with

$$\alpha = \inf \left\{ \frac{h(m)}{m} \mid m \in \left(\frac{M}{2}, M \right] \right\} > 0$$

so that $h_M(m) \geq \alpha m$ for all $m \geq 0$. □

We further require the representation theorem for $\mathbb{M}_h(T)$ from [3, Prop. 2.32, last three bullet points of the proof]. To state it, we use that by [15, Thm. 4.2] any $T \in \overline{\mathbf{F}}_1(\Omega)$ can be uniquely decomposed into

$$T = T^{\text{rec}} + T^{\text{diff}}$$

with $T^{\text{rec}}, T^{\text{diff}} \in \overline{\mathbf{F}}_1(\Omega)$ a rectifiable and a diffuse flat chain, that is, there exists a triple $[\Sigma, \theta, m]$ of a 1-rectifiable set $\Sigma \subset \Omega$ with approximate tangent $\theta : \Sigma \rightarrow \mathbb{S}^{d-1}$ and a measurable function $m : \Sigma \rightarrow \mathbb{R}$ such that

$$\iota_1(T^{\text{rec}}) = m\theta\mathcal{H}^1 \llcorner \Sigma,$$

while $|\iota_1(T^{\text{diff}})|(S) = 0$ for any 1-rectifiable set $S \subset \Omega$.

Theorem 15 (Representation of $\mathbb{M}_h(T)$ [3]). *Let $T \in \overline{\mathbf{F}}_1(\Omega)$ have the decomposition $T = T^{\text{rec}} + T^{\text{diff}}$, where T^{rec} is associated with the triple $[\Sigma, \theta, m]$. Then*

$$\mathbb{M}_h(T) = \int_{\Sigma} h(|m|) d\mathcal{H}^1 + h'(0)\mathbb{M}(T^{\text{diff}}),$$

where $h'(0) \in [0, \infty]$ denotes the right derivative of h in 0.

Now we are prepared to finish the proof of theorem 5.

Proof of theorem 5. The case of a transportation cost h with $h(m) \geq \alpha m$ for some $\alpha > 0$ and all $m > 0$ has already been treated before. Thus it remains to show the result for transportation costs h with $h(m)/m \rightarrow 0$ as $m \rightarrow \infty$. Indeed if there is no $\alpha > 0$ such that $h(m) \geq \alpha m$, then the only possibility is that $h(m)/m \rightarrow 0$ as $m \rightarrow \infty$, because we showed that $h(m) \leq h_M(m) \leq \frac{2h(M)}{M}m$ for all $m \geq M$. Let $T \in \overline{\mathbf{F}}_1(\Omega)$, and let h_M

denote the transportation cost from lemma 14 for arbitrary $M > 0$. By $h_M \geq h$ and the definition of the branched transport cost we have $\mathbb{J}_h(\iota_1(T)) \leq \mathbb{J}_{h_M}(\iota_1(T))$. On the other hand, by theorem 15 we have

$$\begin{aligned} \mathbb{M}_{h_M}(T) &= h'_M(0)\mathbb{M}(T^{\text{diff}}) + \int_{\Sigma} h_M(|m|) d\mathcal{H}^1 \\ &= h'(0)\mathbb{M}(T^{\text{diff}}) + \int_{\Sigma} h(|m|) d\mathcal{H}^1 + \int_{\{x \in \Sigma \mid |m(x)| > M\}} h_M(|m|) - h(|m|) d\mathcal{H}^1 \\ &= \mathbb{M}_h(T) + \int_{\{x \in \Sigma \mid |m(x)| > M\}} h_M(|m|) - h(|m|) d\mathcal{H}^1 \\ &\leq \mathbb{M}_h(T) + \frac{2h(M)}{M} \int_{\{x \in \Sigma \mid |m(x)| > M\}} |m| d\mathcal{H}^1 \leq \mathbb{M}_h(T) + \frac{2h(M)}{M} \mathbb{M}(T), \end{aligned}$$

where the first inequality follows from the fact that $2h(M)|m|/M \geq h_M(|m|) \geq h(|m|) \geq 0$ on the set $\{|m| \geq M\}$. Furthermore, h_M satisfies the growth condition for which we have already proved equality between the h -mass and the branched transport cost. Thus we can summarize

$$\mathbb{J}_h(\iota_1(T)) \leq \mathbb{J}_{h_M}(\iota_1(T)) = \mathbb{M}_{h_M}(T) \leq \mathbb{M}_h(T) + \frac{2h(M)}{M} \mathbb{M}(T),$$

and the result follows from letting $M \rightarrow \infty$. \square

4 Consequences

Here we briefly mention a few implications of the previous results on generalized branched transport models. We concentrate on models (which we call *admissible* below) in which the generalized branched transport cost metrizes weak-* convergence.

Definition 16 (Admissible transportation cost). *A transportation cost h is admissible if there exists a concave function $\beta : [0, \infty) \rightarrow [0, \infty)$ with $h(m) \leq \beta(m)$ for all m and $\int_0^1 \frac{\beta(m)}{m^{2-1/a}} dm < \infty$.*

Remark 17 (Metrization property). *By [3, Cor. 2.24] the admissibility condition on h implies that the generalized branched transport cost*

$$d_{\mathbb{J}_h}(\mu_+, \mu_-) = \min \{ \mathbb{J}_h(\rho) \mid \rho \text{ is mass flux with } \partial\rho = \mu_- - \mu_+ \}$$

between two measures $\mu_+, \mu_- \in \mathcal{M}_+(\Omega)$ metrizes weak- convergence on the set of probability measures.*

Similarly to [4] we now show that one may also prescribe the boundary during the relaxation.

Theorem 18 (Relaxation under prescribed boundary). *Let $\mu_+, \mu_- \in \mathcal{M}_+(\Omega)$ with equal mass and fix arbitrary sequences $\mu_{\pm}^1, \mu_{\pm}^2, \dots \in \mathcal{M}_+(\Omega)$ of finite linear combinations of Dirac masses with $\mu_+^n(\Omega) = \mu_-^n(\Omega)$ and $\mu_{\pm}^n \xrightarrow{*} \mu_{\pm}$ as $n \rightarrow \infty$. For any flat 1-chain $T \in \overline{\mathbf{F}}_1(\Omega)$ with $\iota_0(\partial T) = \mu_- - \mu_+$ and an admissible transportation cost h we have $\mathbb{M}_h(T) = \mathbb{J}_h(\iota_1(T)) = \mathbb{M}_h^{(\mu_{\pm}^n)}(T) = \mathbb{J}_h^{(\mu_{\pm}^n)}(\iota_1(T))$ for*

$$\begin{aligned} \mathbb{J}_h^{(\mu_{\pm}^n)}(\rho) &= \inf \left\{ \liminf_{n \rightarrow \infty} \mathbb{J}_h(\rho_n) \mid \rho_n \in \mathcal{F}(\Omega) \text{ polyhedral, } \rho_n \xrightarrow{*} \rho \text{ as } n \rightarrow \infty, \partial\rho_n = \mu_-^n - \mu_+^n \right\}, \\ \mathbb{M}_h^{(\mu_{\pm}^n)}(T) &= \inf \left\{ \liminf_{n \rightarrow \infty} \mathbb{M}_h(T_n) \mid T_n \in \mathbf{F}_1(\Omega) \text{ polyhedral, } T_n \xrightarrow{b} T \text{ as } n \rightarrow \infty, \partial T_n = \iota_0^{-1}(\mu_-^n - \mu_+^n) \right\}. \end{aligned}$$

Proof. By definition we have

$$\mathbb{M}_h(T) = \mathbb{J}_h(\iota_1(T)) \leq \mathbb{M}_h^{(\mu_{\pm}^n)}(T) \leq \mathbb{J}_h^{(\mu_{\pm}^n)}(\iota_1(T)),$$

so it suffices to show $\mathbb{J}_h^{(\mu_{\pm}^n)}(\iota_1(T)) \leq \mathbb{J}_h(\iota_1(T))$. Abbreviate $\rho = \iota_1(T)$ and consider a sequence $\bar{\rho}_1, \bar{\rho}_2, \dots$ of polyhedral fluxes with $\bar{\rho}_n \xrightarrow{f} \rho$ and $\mathbb{J}_h(\bar{\rho}_n) \rightarrow \mathbb{J}_h(\rho)$ as $n \rightarrow \infty$. Next, by the Jordan Decomposition Theorem

we can decompose $\partial\bar{\rho}_n - \mu_+^n + \mu_+^n = \nu_+^n - \nu_-^n$ with $\nu_\pm^n \in \mathcal{M}_+(\Omega)$. Obviously, $\nu_-^n - \nu_+^n \xrightarrow{*} 0$ as $n \rightarrow \infty$ so that, [since \$h\$ is admissible](#), by [3, Cor.2.24] there exists a sequence $\hat{\rho}_1, \hat{\rho}_2, \dots$ of equibounded mass fluxes with $\partial\hat{\rho}_n = \nu_-^n - \nu_+^n$ and $\mathbb{J}_h(\hat{\rho}_n) \rightarrow 0$. Since ν_+^n and ν_-^n are finite linear combinations of Dirac masses, the $\hat{\rho}_n$ can be chosen as polyhedral fluxes, [by \[4, Theorem 1.2\]](#). Finally define the sequence $\rho_n = \bar{\rho}_n + \hat{\rho}_n$, $n \in \mathbb{N}$, of polyhedral fluxes, then $\partial\rho_n = \mu_+^n - \mu_-^n$ as well as $\rho_n \xrightarrow{f} \rho$ (because $\bar{\rho}_n \xrightarrow{f} \rho$ while $\hat{\rho}_n \xrightarrow{f} 0$ by [lower semi-continuity of \$\mathbb{J}_h\(\cdot\)\$](#)) and $\mathbb{J}_h(\rho_n) \leq \mathbb{J}_h(\bar{\rho}_n) + \mathbb{J}_h(\hat{\rho}_n) \rightarrow \mathbb{J}_h(\rho)$ as $n \rightarrow \infty$, as desired. \square

As also emphasized in [4], the latter result is particularly useful for the development of phasefield approximations of generalized branched transport or minimal h -mass problems. Indeed, when proving Γ -convergence of a phasefield functional to the minimal h -mass problem with prescribed boundary, a recovery sequence can typically only be constructed for polyhedral fluxes, in particular with polyhedral boundary. The above result implies that this is indeed sufficient.

Finally we state that the generalized branched transport problem and the problem of minimizing the h -mass are equivalent.

Theorem 19 (Branched transport problem and minimal h -mass). *Let h be an admissible transportation cost and $\mu_+, \mu_- \in \mathcal{M}_+(\Omega)$ with equal mass, then*

$$\min \{ \mathbb{J}_h(\rho) \mid \rho \in \mathcal{F}(\Omega), \partial\rho = \mu_- - \mu_+ \} = \min \{ \mathbb{M}_h(T) \mid T \in \mathbf{F}_1(\Omega), \partial T = \iota_0^{-1}(\mu_- - \mu_+) \} ,$$

and the minimizers of both problems are related by ι_1 .

Proof. Let us abbreviate

$$\begin{aligned} d_{\mathbb{J}_h}(\mu_+, \mu_-) &= \inf \{ \mathbb{J}_h(\rho) \mid \rho \in \mathcal{F}(\Omega), \partial\rho = \mu_- - \mu_+ \} , \\ d_{\mathbb{M}_h}(\mu_+, \mu_-) &= \inf \{ \mathbb{M}_h(T) \mid T \in \mathbf{F}_1(\Omega), \partial T = \iota_0^{-1}(\mu_- - \mu_+) \} . \end{aligned}$$

The existence of minimizers for $d_{\mathbb{J}_h}(\mu_+, \mu_-)$ is shown in [3, Cor.2.20], and since each minimizer ρ for $d_{\mathbb{J}_h}(\mu_+, \mu_-)$ induces a competitor $\iota_1^{-1}(\rho)$ for $d_{\mathbb{M}_h}(\mu_+, \mu_-)$ with $\mathbb{M}_h(\iota_1^{-1}(\rho)) = \mathbb{J}_h(\rho)$, we have $d_{\mathbb{J}_h}(\mu_+, \mu_-) \geq d_{\mathbb{M}_h}(\mu_+, \mu_-)$ and only need to show the opposite inequality. To this end consider two sequences $\mu_\pm^1, \mu_\pm^2, \dots$ of nonnegative finite linear combinations of Dirac masses such that $\mu_+^n(\Omega) = \mu_-^n(\Omega) = \mu_+(\Omega)$ and $\mu_\pm^n \xrightarrow{*} \mu_\pm$ as $n \rightarrow \infty$, and abbreviate $M = \mu_+(\Omega)$.

Let us first restrict ourselves to the case where $h'(0) < \infty$. By the triangle inequality (which follows from the subadditivity of $\mathbb{M}_h(\cdot)$) we have

$$d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq d_{\mathbb{M}_h}(\mu_+^n, \mu_-^n) - d_{\mathbb{M}_h}(\mu_+^n, \mu_+) - d_{\mathbb{M}_h}(\mu_-, \mu_-^n) ,$$

where without loss of generality we may assume $\mathbb{F}(\iota_0^{-1}(\mu_+^n - \mu_+)) + \mathbb{F}(\iota_0^{-1}(\mu_- - \mu_-^n)) \leq \frac{1}{n}$ as well as $d_{\mathbb{M}_h}(\mu_+^n, \mu_+) + d_{\mathbb{M}_h}(\mu_-, \mu_-^n) < \frac{1}{n}$ due to the admissibility of h . Now let $T_n \in \mathbf{F}_1(\Omega)$ with $\partial T_n = \iota_0^{-1}(\mu_-^n - \mu_+^n)$ and $\mathbb{M}_h(T_n) \leq d_{\mathbb{M}_h}(\mu_+^n, \mu_-^n) + \frac{1}{n}$. By definition of $\mathbb{M}_h(\cdot)$ there exists a polyhedral 1-chain \bar{T}_n with $\mathbb{M}_h(\bar{T}_n) \leq \mathbb{M}_h(T_n) + \frac{1}{n}$ and $\mathbb{F}(\bar{T}_n - T_n) \leq \frac{1}{n}$ as well as $\mathbb{F}(\partial\bar{T}_n - \partial T_n) \leq \frac{1}{n}$. The latter implies $\mathbb{F}(\partial\bar{T}_n - \iota_0^{-1}(\mu_- - \mu_+)) \leq \frac{2}{n}$ and thus the existence of a flat 1-chain $S_n \in \bar{\mathbf{F}}_1(\Omega)$ with $\mathbb{M}(\partial(\bar{T}_n + S_n) - \iota_0^{-1}(\mu_- - \mu_+)) + \mathbb{M}(S_n) \leq \frac{2}{n}$. Letting $\bar{\mu}_+^n, \bar{\mu}_-^n \in \mathcal{M}_+(\Omega)$ be the positive and the negative part of $\iota_0(\partial(\bar{T}_n + S_n))$, we can summarize

$$\begin{aligned} d_{\mathbb{M}_h}(\mu_+, \mu_-) &\geq d_{\mathbb{M}_h}(\mu_+^n, \mu_-^n) - \frac{1}{n} \geq \mathbb{M}_h(T_n) - \frac{2}{n} \geq \mathbb{M}_h(\bar{T}_n) - \frac{3}{n} \geq \mathbb{M}_h(\bar{T}_n) + h'(0)\mathbb{M}(S_n) - \frac{3+2h'(0)}{n} \\ &\geq \mathbb{M}_h(\bar{T}_n + S_n) - \frac{3+2h'(0)}{n} = \mathbb{J}_h(\iota_1^{-1}(\bar{T}_n + S_n)) - \frac{3+2h'(0)}{n} \geq d_{\mathbb{J}_h}(\bar{\mu}_+^n, \bar{\mu}_-^n) - \frac{3+2h'(0)}{n} . \end{aligned}$$

where in the fifth inequality we used that $\mathbb{M}_h \leq h'(0)\mathbb{M}$. This follows directly from the fact that $h(m) \leq mh'(0)$, which can be found in [9, Theorem 16.3.3].

For $n \rightarrow \infty$ we obtain the desired inequality if we can show $\lim_{n \rightarrow \infty} d_{\mathbb{J}_h}(\bar{\mu}_+^n, \bar{\mu}_-^n) \geq d_{\mathbb{J}_h}(\mu_+, \mu_-)$. Note that $\bar{\mu}_\pm^n \rightarrow \mu_\pm$ strongly and let $\hat{\mu}_+^n, \hat{\mu}_-^n \in \mathcal{M}_+(\Omega)$ be the positive and the negative part of $\mu_+ - \bar{\mu}_+^n - \mu_- + \bar{\mu}_-^n$. Then, [using again \$\mathbb{M}_h \leq h'\(0\)\mathbb{M}\$](#) , we get $d_{\mathbb{J}_h}(\hat{\mu}_+^n, \hat{\mu}_-^n) \leq h'(0)W_1(\hat{\mu}_+^n, \hat{\mu}_-^n) \leq h'(0)\text{diam}(\Omega)\|\hat{\mu}_+^n\|_{\mathcal{M}} \rightarrow 0$ as

$n \rightarrow \infty$, where W_1 denotes the Wasserstein-1 distance and $\text{diam}(\Omega)$ denotes the intrinsic diameter of Ω (the largest geodesic distance between two points in Ω). Thus, taking the limit $n \rightarrow \infty$ in

$$d_{\mathbb{J}_h}(\mu_+, \mu_-) \leq d_{\mathbb{J}_h}(\bar{\mu}_+^n, \bar{\mu}_-^n) + d_{\mathbb{J}_h}(\hat{\mu}_+^n, \hat{\mu}_-^n)$$

yields the desired result.

Now consider the case $h'(0) = \infty$. For any $N \in \mathbb{N}$ define h^N to be the lower semi-continuous subadditive envelope of $m \mapsto \min\{h(m), Nm\}$. Using $(h^N)'(0) < \infty$ we thus obtain

$$d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq d_{\mathbb{M}_{h^N}}(\mu_+, \mu_-) = d_{\mathbb{J}_{h^N}}(\mu_+, \mu_-).$$

Let further $\rho_N \in \mathcal{F}(\Omega)$ denote a minimizer for the right-hand side (which exists by [3, Cor. 2.20]) should we maybe explain why h^N is admissible? so that via theorem 18 we have

$$d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq d_{\mathbb{J}_{h^N}}(\mu_+, \mu_-) = \mathbb{J}_{h^N}(\rho_N) = \mathbb{J}_{h^N}^{\left(\frac{\mu_+^n}{\mu_-^n}\right)}(\rho_N) = \lim_{n \rightarrow \infty} \mathbb{J}_{h^N}(\rho_N^n),$$

where ρ_N^n is a polyhedral flux with $\partial \rho_N^n = \mu_+^n - \mu_-^n$. By [3, Lem. 2.5] we can reduce the right-hand side even further by replacing ρ_N^n with an acyclic flux $\tilde{\rho}_N^n$ of the same boundary, which by [3, Lem. 2.9] has multiplicity bounded by M . Since by [10, Thm. 5 and its proof] we have $h^N(m) \geq \alpha_N m \geq \alpha_1 m$ for all $m \in [0, M]$ with

$$\alpha_N = \inf \left\{ \frac{h^N(m)}{m} \mid m \in \left(\frac{M}{2}, M\right] \right\} > 0,$$

we see $d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq \lim_{n \rightarrow \infty} \mathbb{J}_{h^N}(\tilde{\rho}_N^n) \geq \lim_{n \rightarrow \infty} \alpha_1 \|\tilde{\rho}_N^n\|_{\mathcal{M}}$. Hence, the $\tilde{\rho}_N^n$ have equibounded mass and converge weakly as mass fluxes (up to a subsequence) to some $\tilde{\rho}_N$ with $\|\tilde{\rho}_N\|_{\mathcal{M}} \leq d_{\mathbb{M}_h}(\mu_+, \mu_-)/\alpha_1$. Summarizing, we obtain

$$d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq \lim_{n \rightarrow \infty} \mathbb{J}_{h^N}(\tilde{\rho}_N^n) \geq \mathbb{J}_{h^N}(\tilde{\rho}_N)$$

for all $N \in \mathbb{N}$, where $\tilde{\rho}_N$ with $\partial \tilde{\rho}_N = \mu_+ - \mu_-$ has mass bounded by $d_{\mathbb{M}_h}(\mu_+, \mu_-)/\alpha_1$. Again restricting to a subsequence (still indexed by N) we have $\tilde{\rho}_N \xrightarrow{f} \rho$ for some $\rho \in \mathcal{F}(\Omega)$ and thus

$$d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq \lim_{N \rightarrow \infty} \mathbb{J}_{h^N}(\tilde{\rho}_N) \geq \liminf_{N \rightarrow \infty} \mathbb{J}_{h^L}(\tilde{\rho}_N) \geq \mathbb{J}_{h^L}(\rho) = \mathbb{M}_{h^L}(l_1^{-1}(\rho))$$

for any $L \in \mathbb{N}$. Using the representation theorem 15 and the Monotone Convergence Theorem as $L \rightarrow \infty$ we arrive at

$$d_{\mathbb{M}_h}(\mu_+, \mu_-) \geq \mathbb{M}_h(l_1^{-1}(\rho)) = \mathbb{J}_h(\rho) \geq d_{\mathbb{J}_h}(\mu_+, \mu_-)$$

as desired. □

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