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# FANO MANIFOLDS WITH LONG EXTREMAL RAYS*

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**Abstract.** Let $X$ be a Fano manifold of pseudoindex $i_X$ whose Picard number is at least two and let $R$ be an extremal ray of $X$ with exceptional locus $\mathrm{Exc}(R)$. We prove an inequality which bounds the length of $R$ in terms of $i_X$ and of the dimension of $\mathrm{Exc}(R)$ and we investigate the border cases.

In particular we classify Fano manifolds $X$ of pseudoindex $i_X$ obtained blowing up a smooth variety $Y$ along a smooth subvariety $T$ such that $\dim T < i_X$.

**Key words.** Fano manifolds, rational curves, extremal rays

**AMS subject classifications.** 14J45; 14J40, 14E30

**1. Introduction.** A smooth complex projective variety of dimension $n$ is called Fano if its anticanonical bundle $-K_X = \wedge^n TX$ is ample. The index of $X$, $r_X$, is the largest natural number $m$ such that $-K_X = mH$ for some (ample) divisor $H$ on $X$, while the pseudoindex $i_X$ is defined as the minimum anticanonical degree of rational curves on $X$ and it is an integral multiple of $r_X$.

The pseudoindex is related to the Picard number $\rho_X$ of $X$ by a conjecture which claims that $\rho_X(i_X - 1) \leq n$, with equality if and only if $X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$; this conjecture appeared in [7] as a generalization of a similar one (with the index in place of the pseudoindex) proposed by Mukai in 1988.

A recent very important result, which can be considered a special case of the conjecture, states that $i_X \geq n+1$ if and only if $X \simeq \mathbb{P}^n$ ([11, Corollary 0.4] or [15, Theorem 1.1]). A first step towards the proof of the conjecture was made by Wiśniewski in [24], where he proved that if $i_X > \frac{n+2}{2}$ then $\rho_X = 1$. More recently several authors ([7], [21], [1], [9]) dealt with this problem but the general case is still open.

In this paper we investigate a related problem.

Let $X$ be a smooth variety of dimension $n$ and let $R$ be an extremal ray of $X$. Let $l(R) := \min\{-K_X \cdot C \mid C \text{ a rational curve in } R\}$ be the length of $R$ and $\mathrm{Exc}(R) = \{x \in X \mid x \in C \text{ a rational curve in } R\}$ be its exceptional locus.

In general the length of an extremal ray is bounded above by $n+1$, equality holding if and only if $X \simeq \mathbb{P}^n$ (again by the results in [11] or [15]), while the length of an extremal ray whose associated contraction is birational is bounded above by $n-1$, equality holding if and only if the associated contraction is the blow-up of a point in a smooth variety (see [2, Theorem 1.1]).

We first prove that, if $X$ is a Fano manifold of pseudoindex $i_X$ and $\rho_X > 1$, the following holds:

$$i_X + l(R) \leq \dim \mathrm{Exc}(R) + 2. \qquad (*)$$

Note that this is an improved statement of the conjecture in the case $\rho_X = 2$.

Then we investigate the cases in which equality holds. Equivalently we ask if on a Fano manifold of pseudoindex $i_X$ an extremal ray $R$ of maximal length does determine the structure of the variety, proving the following

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THEOREM 1.1. *Let $X$ be a Fano manifold of dimension $n$, pseudoindex $i_X$ and Picard number $\rho_X \geq 2$, and let $R$ be a fiber type or divisorial extremal ray such that*

$$i_X + l(R) = \dim \operatorname{Exc}(R) + 2.$$

*Then $X \simeq \mathbb{P}^k \times \mathbb{P}^{n-k}$ or $X \simeq Bl_{\mathbb{P}^t}(\mathbb{P}^n)$ with $0 \leq t \leq \frac{n-3}{2}$.*

We do not know how to prove a similar theorem if $R$ is an extremal ray whose associated contraction is small (i.e. $\dim \operatorname{Exc}(R) \leq n-2$). However if we replace in the assumptions the pseudoindex $i_X$ with the index $r_X$ then we have the following

THEOREM 1.2. *Let $X$ be a Fano manifold of dimension $n$, index $r_X$, and Picard number $\rho_X \geq 2$, and let $R$ be an extremal ray such that*

$$r_X + l(R) = \dim \operatorname{Exc}(R) + 2.$$

*Then, denoted by e the dimension of* $\operatorname{Exc}(R)$, *we have $X = \mathbb{P}_{\mathbb{P}^k}(\mathcal{O}^{\oplus e-k+1} \oplus \mathcal{O}(1)^{\oplus n-e})$, with $k = n - r + 1$.*

Finally we consider the next step, namely the case

$$i_X + l(R) = \dim \operatorname{Exc}(R) + 1.$$

For a fiber type or divisorial extremal ray $R$ we prove that $\rho_X \leq 3$, describing the Kleiman-Mori cone of $X$ and classifying the varieties with $\rho_X = 3$, (Theorem 5.1).

If we assume moreover that $R$ is the ray associated to a smooth blow-up, we have a complete classification:

THEOREM 1.3. *Let $X$ be a Fano manifold and let $R$ be an extremal ray whose associate contraction $\varphi_R : X \to Y$ is the blow up of a smooth variety $Y$ along a smooth subvariety $T \subset Y$, such that*

$$i_X + l(R) \geq n \qquad \text{or equivalently} \qquad i_X \geq \dim T + 1.$$

*Then $X$ is one of the following*

- *a) $Bl_{\mathbb{P}^t}(\mathbb{P}^n)$, with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$,*
- *b) $Bl_{\mathbb{P}^t}(\mathbb{Q}^n)$, with $\mathbb{P}^t$ a linear subspace of dimension $\leq \frac{n}{2} - 1$,*
- *c) $Bl_{\mathbb{Q}^t}(\mathbb{Q}^n)$, with $\mathbb{Q}^t$ a smooth quadric of dimension $\leq \frac{n}{2} - 1$ not contained in a linear subspace of $\mathbb{Q}^n$,*
- *d) $Bl_p(V)$ where $V$ is $Bl_Y(\mathbb{P}^n)$ and $Y$ is a submanifold of dimension $n-2$ and degree $\leq n$ contained in an hyperplane $H$ such that $p \notin H$,*
- *e) $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$.*

Note that if $T$ is a point the condition $i_X \geq \dim T + 1 = 1$ is empty. In this case the theorem is actually the main theorem of [6], where Fano manifolds which are the blow-up at a point of a smooth variety are classified (those varieties correspond to cases a) and b) with $t = 0$ and d) of the above theorem). That paper has been for us a very important source of inspiration.

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**2. Background material.** In (2.1) and (2.2) we recall basic definitions and facts concerning Fano-Mori contractions and families of rational curves; our notation is consistent with the one in [17] to which we refer the reader.
Afterwards, in (2.3), for the reader's convenience we recall some results of [1] and [10] which are frequently used in the rest of the paper.

**2.1. Fano-Mori contractions.** Let $X$ be a smooth complex Fano variety of dimension $n$ and let $K_X$ be its canonical divisor. By the Cone Theorem the cone of effective 1-cycles which is contained in the $\mathbb{R}$-vector space of 1-cyles modulo numerical equivalence, $\mathrm{NE}(X) \subset N_1(X)$, is polyhedral; a face of $\mathrm{NE}(X)$ is called an extremal face and an extremal face of dimension one is called an extremal ray.

From the structure of the cone it follows that

Lemma 2.1. *[6, Lemme 2.1] Let $X$ be a Fano manifold and $D$ an effective divisor on $X$. Then there exists an extremal ray $R \subset NE(X)$ such that $D \cdot R > 0$.*

To an extremal face $\sigma$ is associated a morphism with connected fibers $\varphi_\sigma : X \to W$ onto a normal variety, which contracts the curves whose numerical class is in $\sigma$; $\varphi_\sigma$ is called an extremal contraction or a Fano-Mori contraction.
A Cartier divisor $H$ such that $H = \varphi_\sigma^* A$ for an ample divisor $A$ on $W$ is called a good supporting divisor of the map $\varphi_\sigma$ (or of the face $\sigma$).
An extremal ray $R$ (and the associated extremal contraction $\varphi_R$) is called numerically effective (nef for short) or of fiber type if $\dim W < \dim X$, otherwise the ray (and the contraction) is non nef or birational. This terminology is due to the fact that there exists an effective divisor $E$ such that $E \cdot R < 0$ if and only if the ray is not nef. If the codimension of the exceptional locus of a birational ray $R$ is equal to one the ray and the associated contraction are called divisorial, otherwise they are called small.

**2.2. Families of rational curves.** Let $X$ be a normal projective variety and let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f : \mathbb{P}^1 \to X$. We consider the open subscheme $\mathrm{Hom}_{bir}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$, corresponding to those morphisms which are birational onto their image, and its normalization $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$. The group $\mathrm{Aut}(\mathbb{P}^1)$ acts on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ and the quotient exists.

Definition 2.2. The space $\mathrm{Ratcurves}^n(X)$ is the quotient of $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$ by $\mathrm{Aut}(\mathbb{P}^1)$, and the space $\mathrm{Univ}(X)$ is the quotient of the product action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1$.

Definition 2.3. We define a family of rational curves to be an irreducible component $V \subset \mathrm{Ratcurves}^n(X)$. Given a rational curve $f : \mathbb{P}^1 \to X$ we will call a family of deformations of $f$ any irreducible component $V \subset \mathrm{Ratcurves}^n(X)$ containing the equivalence class of $f$.

Given a family $V$ of rational curves, we have the following basic diagram:

$$\begin{tikzcd}
p^{-1}(V) =: U \arrow[r, "i"] \arrow[d, "p"'] & X \\
V &
\end{tikzcd}$$

where $i$ is the map induced by the evaluation $ev : \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$ and $p$ is the $\mathbb{P}^1$-bundle induced by the projection $\mathrm{Hom}^n_{bir}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to \mathrm{Hom}^n_{bir}(\mathbb{P}^1, X)$.

We define $\mathrm{Locus}(V)$ to be the image of $U$ in $X$; we say that $V$ is a covering family if $\mathrm{Locus}(V) = X$. If $L \in \mathrm{Pic}(X)$ is a line bundle, we will denote by $L \cdot V$ the intersection number of $L$ and a generar member of the family $V$; moreover we will denote by $[V]$ the numerical equivalence class in $N_1(X)$ of a general member of the family $V$.

Given a family $V \subseteq \mathrm{Ratcurves}^n(X)$, we denote by $V_x$ the subscheme of $V$ parametrizing rational curves passing through $x$.

DEFINITION 2.4. Let $V$ be a family of rational curves on $X$. Then

(a) $V$ is unsplit if it is proper;
(b) $V$ is locally unsplit if for the general $x \in \mathrm{Locus}(V)$ every component of $V_x$ is proper;
(c) $V$ is generically unsplit if there is at most a finite number of curves of $V$ passing through two general points of $\mathrm{Locus}(V)$.

PROPOSITION 2.5. *[17, IV.2.6] Let $X$ be a smooth projective variety, $V$ a family of rational curves and $x \in \mathrm{Locus}(V)$ a point such that every component of $V_x$ is proper. Then*

*(a)* $\dim X - K_X \cdot V \leq \dim \mathrm{Locus}(V) + \dim \mathrm{Locus}(V_x) + 1$;
*(b)* $-K_X \cdot V \leq \dim \mathrm{Locus}(V_x) + 1$.

REMARK 2.6. The assumptions of the proposition are clearly satisfied for every $x \in \mathrm{Locus}(V)$ if $V$ is an unsplit family; the same result holds for a general $x \in \mathrm{Locus}(V)$ assuming that $V$ is generically unsplit, though this does not imply the existence of a point such that $V_x$ is proper; for example, if $X$ is the blow up at a point $p$ of $\mathbb{P}^2$, then the family $V$ of rational curves whose general member is the strict transform of a line which does not contain $p$ is generically unsplit, but $V_x$ is not proper for any $x \in Bl_p(\mathbb{P}^2)$.

Proposition 2.5, in case $V$ is the unsplit family of deformations of a minimal extremal rational curve, i.e. a curve of minimal degree in an extremal face of $X$, gives the fiber locus inequality:

PROPOSITION 2.7. *[14], [25] Let $\varphi$ be a Fano-Mori contraction of $X$ and let $E = E(\varphi)$ be its exceptional locus; let $S$ be an irreducible component of a (non trivial) fiber of $\varphi$. Then*

$$\dim E + \dim S \geq \dim X + l - 1$$

*where*

$$l = \min\{-K_X \cdot C \mid C \textit{ is a rational curve in } S\}.$$

*If $\varphi$ is the contraction of a ray $R$, then $l(R) := l$ is called the* length of the ray.

DEFINITION 2.8. We define a Chow family of rational curves to be an irreducible component $\mathcal{V} \subset \mathrm{Chow}(X)$ parametrizing rational and connected 1-cycles. If $V$ is a family of rational curves, the closure of the image of $V$ in $\mathrm{Chow}(X)$ is called the Chow family associated to $V$.
We say that $\mathcal{V}$ is quasi-unsplit if every component of any reducible cycle in $\mathcal{V}$ is numerically proportional to $\mathcal{V}$.

Let $X$ be a smooth variety, $\mathcal{V}^1, \ldots, \mathcal{V}^k$ Chow families of rational curves on $X$ and $Y$ a subset of $X$.

Definition 2.9. We denote by $\mathrm{Locus}(\mathcal{V}^1,\dots,\mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1,\dots,C_k$ with the following properties:

- $C_i$ belongs to the family $\mathcal{V}^i$;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_k$,

i.e. $\mathrm{Locus}(\mathcal{V}^1,\dots,\mathcal{V}^k)_Y$ is the set of points that can be joined to $Y$ by a connected chain of $k$ cycles belonging respectively to the families $\mathcal{V}^1,\dots,\mathcal{V}^k$.

We denote by $\mathrm{ChLocus}_m(\mathcal{V}^1,\dots,\mathcal{V}^k)_Y$ the set of points $x \in X$ such that there exist cycles $C_1,\dots,C_m$ with the following properties:

- $C_i$ belongs to a family $\mathcal{V}^j$;
- $C_i \cap C_{i+1} \neq \emptyset$;
- $C_1 \cap Y \neq \emptyset$ and $x \in C_m$,

i.e. $\mathrm{ChLocus}_m(\mathcal{V}^1,\dots,\mathcal{V}^k)_Y$ is the set of points that can be joined to $Y$ by a connected chain of at most $m$ cycles belonging to the families $\mathcal{V}^1,\dots,\mathcal{V}^k$.

Definition 2.10. We define a relation of rational connectedness with respect to $\mathcal{V}^1,\dots,\mathcal{V}^k$ on $X$ in the following way: $x$ and $y$ are in $\mathrm{rc}(\mathcal{V}^1,\dots,\mathcal{V}^k)$ relation if there exists a chain of cycles in $\mathcal{V}^1,\dots,\mathcal{V}^k$ which joins $x$ and $y$, i.e. if $y \in \mathrm{ChLocus}_m(\mathcal{V}^1,\dots,\mathcal{V}^k)_x$ for some $m$.

To the $\mathrm{rc}(\mathcal{V}^1,\dots,\mathcal{V}^k)$ relation we can associate a fibration, at least on an open subset:

Theorem 2.11. *[8],[17, IV.4.16] There exist an open subvariety $X^0 \subset X$ and a proper morphism with connected fibers $\pi : X^0 \to Z^0$ such that*

*(a) the $rc(\mathcal{V}^1,\dots,\mathcal{V}^k)$ relation restricts to an equivalence relation on $X^0$;*
*(b) the fibers of $\pi$ are equivalence classes for the $rc(\mathcal{V}^1,\dots,\mathcal{V}^k)$ relation;*
*(c) for every $z \in Z^0$ any two points in $\pi^{-1}(z)$ can be connected by a chain of at most $2^{\dim X - \dim Z^0} - 1$ cycles in $\mathcal{V}^1,\dots,\mathcal{V}^k$.*

The geometry of Fano manifolds is strongly related to the properties of families of rational curves of low degree. The following is a fundamental theorem, due to Mori:

Theorem 2.12. *[20] Through every point of a Fano manifold $X$ there exists a rational curve of anticanonical degree $\leq \dim X + 1$.*

Remark 2.13. The families $\{V^i \subset \mathrm{Ratcurves}^n(X)\}$ containing rational curves with degree $\leq \dim X + 1$ are only a finite number, so, for at least one index $i$, we have that $\mathrm{Locus}(V^i) = X$. Among these families we choose one with minimal anticanonical degree, and we call it a minimal covering family.

Let $X$ be a Fano manifold and $\pi : X^0 \to Z^0$ a proper surjective morphism on a smooth quasiprojective variety $Z^0$ of positive dimension.
A relative version of Mori's theorem, [18, Theorem 2.1], states that, for a general point $z \in Z^0$, there exists a rational curve $C$ on $X$ of anticanonical degree $\leq \dim X + 1$ which meets $\pi^{-1}(z)$ without being contained in it (an horizontal curve, for short).
As in remark 2.13 we can find a family $V$ of horizontal curves such that $\mathrm{Locus}(V)$ dominates $Z^0$ and $-K_X \cdot V$ is minimal among the families with this property. Such a family is called a minimal horizontal dominating family for $\pi$.

Lemma 2.14. *[1, Lemma 6.5] Let $X$ be a Fano manifold, let $X \overset{\pi}{\dashrightarrow} Z$ be the fibration associated to a $rc(\mathcal{V}^1,\dots,\mathcal{V}^k)$ relation and let $V$ be a minimal horizontal dominating family for $\pi$. Then*

(a) *curves parametrized by* $V$ *are numerically independent from curves contracted by* $\pi$*;*

(b) $V$ *is locally unsplit;*

(c) *if* $x$ *is a general point in* $\mathrm{Locus}(V)$ *and* $F$ *is the fiber containing* $x$*, then*

$$\dim(F \cap \mathrm{Locus}(V_x)) = 0.$$

**2.3. Chains of rational curves, numerical equivalence and cones.** In this subsection we present some results concerning the dimension, the maximum number of numerically independent curves and the cone of curves of subsets of the form $\mathrm{Locus}(V^1, \dots, V^k)_Y$ or $\mathrm{ChLocus}(V^1, \dots, V^k)_Y$ when $V^1, \dots, V^k$ are unsplit families and $Y$ is chosen in a suitable way.

Definition 2.15. Let $V^1, \dots, V^k$ be unsplit families on $X$. We will say that $V^1, \dots, V^k$ are numerically independent if the numerical classes $[V^1], \dots, [V^k]$ are linearly independent in the vector space $N_1(X)$. If moreover $C \subset X$ is a curve we will say that $V^1, \dots, V^k$ are numerically independent from $C$ if in $N_1(X)$ the class of $C$ is not contained in the vector subspace generated by $[V^1], \dots, [V^k]$.

Notation: Let $S$ be a subset of $X$. We write $N_1(S) = \langle [V^1], \dots, [V^k] \rangle$ if the numerical class in $X$ of every curve $C \subset S$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \mathbb{Q}$ and $C_i \in V^i$. We write $\mathrm{NE}(S) = \langle [V^1], \dots, [V^k] \rangle$ (or $\mathrm{NE}(S) = \langle [R_1], \dots, [R_k] \rangle$) if the numerical class in $X$ of every curve $C \subset S$ can be written as $[C] = \sum_i a_i [C_i]$, with $a_i \in \mathbb{Q}_{\geq 0}$ and $C_i \in V^i$ (or $[C_i]$ in $R_i$).

The following lemma is a generalization of proposition 2.5 and of [7, Theoreme 5.2]

Lemma 2.16. *[1, Cfr. Lemma 5.4] Let* $Y \subset X$ *be a closed subset and* $V$ *an unsplit family. Assume that* $[V] \notin NE(Y)$ *and that* $Y \cap \mathrm{Locus}(V) \neq \emptyset$*. Then for a general* $y \in Y \cap \mathrm{Locus}(V)$

(a) $\dim \mathrm{Locus}(V)_Y \geq \dim(Y \cap \mathrm{Locus}(V)) + \dim \mathrm{Locus}(V_y)$;

(b) $\dim \mathrm{Locus}(V)_Y \geq \dim Y - K_X \cdot V - 1$.

*Moreover, if* $V^1, \dots, V^k$ *are numerically independent unsplit families such that curves contained in* $Y$ *are numerically independent from curves in* $V^1, \dots, V^k$ *then either* $\mathrm{Locus}(V^1, \dots, V^k)_Y = \emptyset$ *or*

(c) $\dim \mathrm{Locus}(V^1, \dots, V^k)_Y \geq \dim Y + \sum -K_X \cdot V^i - k$.

Remark 2.17. In [1, Lemma 5.4] parts (a) and (b) were proved in the stronger assumption that curves contained in $Y$ are numerically independent from curves in $V$. The proof is the same.

Lemma 2.18. *[21, Lemma 1] Let* $Y \subset X$ *be a closed subset and* $V$ *an unsplit family of rational curves. Then every curve contained in* $\mathrm{Locus}(V)_Y$ *is numerically equivalent to a linear combination with rational coefficients*

$$\lambda C_Y + \mu C_V,$$

*where* $C_Y$ *is a curve in* $Y$*,* $C_V$ *belongs to the family* $V$ *and* $\lambda \geq 0$*.*

Corollary 2.19. *[1, Corollary 4.4] If* $X$ *is rationally connected with respect to some (quasi) unsplit families* $V^1, \dots, V^k$ *then* $N_1(X) = \langle [V^1], \dots, [V^k] \rangle$*.*

PROPOSITION 2.20. *[1, Corollary 4.2], [10, Corollary 2.23]*

*(a) Let $V$ be a quasi-unsplit family of rational curves and $x$ a point in* $\operatorname{Locus}(V)$. *Then $NE(\operatorname{ChLocus}_m(V)_x) = \langle [V] \rangle$ for every $m \geq 1$.*

*(b) Let $V$ be a family of rational curves and $x$ a point in $X$ such that $V_x$ is proper. Then $NE(\operatorname{Locus}(V_x)) = \langle [V] \rangle$.*

*(c) Let $\sigma$ be an extremal face of $NE(X)$, $F$ a fiber of the associated contraction and $V$ an unsplit family independent from $\sigma$. Then $NE(\operatorname{ChLocus}_m(V)_F) = \langle \sigma, [V] \rangle$ for every $m \geq 1$.*

COROLLARY 2.21. *Let $D \subset X$ be an effective divisor and $V$ an unsplit family such that $[V] \notin NE(D)$ and that $D \cdot V > 0$; then, for every $x \in \operatorname{Locus}(V)$ we have $\dim \operatorname{Locus}(V_x) = 1$; in particular, if $V$ is the family of deformations of a minimal extremal rational curve in a ray $R$ then every non trivial fiber of $\varphi_R$ is one dimensional.*

*Proof.* Since $D \cdot V > 0$, for every $x \in \operatorname{Locus}(V)$ we have $D \cap \operatorname{Locus}(V_x) \neq \emptyset$, and so $\dim(D \cap \operatorname{Locus}(V_x)) \geq \dim \operatorname{Locus}(V_x) - 1$. It follows that $\dim \operatorname{Locus}(V_x) = 1$, since a curve in the intersection would be a curve in $D$ whose numerical class is proportional to $[V]$. □

**3. Some technical results.** In order to make the exposition clearer, we collect in this section two technical lemmata we will use in the proofs of the main theorems.

LEMMA 3.1. *Let $X$ be a Fano manifold whose cone of curves is generated by a divisorial extremal ray $R_1$ with exceptional locus $E$ and a fiber type extremal ray $R_2$, and let $V$ be a quasi unsplit covering family of rational curves.*
*Then $[V] \in R_2$; in particular $E \cdot V > 0$.*

*Proof.* Consider the rc$\mathcal{V}$ fibration $X \overset{\pi}{\dashrightarrow} Z$; by proposition 2.20 we have $\dim Z > 0$ since $V$ is quasi unsplit and $\rho_X = 2$ and $V$ is extremal by [10, Lemma 2.28] since $X$ has not small contractions.
The last assertion follows from lemma 2.1, since $E \cdot R_1 < 0$. □

LEMMA 3.2. *Let $X$ be a Fano manifold of dimension $n$, pseudoindex $i_X \geq 2$ and Picard number $\rho_X = 2$ whose extremal contractions are:*

1. *the blow up $\varphi : X \to Y$ of a smooth variety along a smooth subvariety $T \subset Y$, associated to an extremal ray $R_1$, with exceptional locus $E = \operatorname{Exc}(R_1)$;*
2. *a fiber type contraction associated to a ray $R_2$.*

*Suppose that $i_X + l(R_1) \geq n$ and that there exists a covering family $V$ of rational curves of degree $\leq n+1$ such that $E \cdot V = 0$. Then $V$ is not quasi unsplit and all the reducible cycles in the associated Chow family $\mathcal{V}$ have two irreducible components, $C_1$ and $C_2$, where $C_1$ and $C_2$ are curves in the rays $R_1$ and $R_2$ respectively.*

*Proof.* First of all we note that, since $E \cdot V = 0$, by lemma 3.1, $V$ is not a quasi unsplit family.
Let $C = \sum C_i$ be a reducible cycle in $\mathcal{V}$. At least one of the components of $C$, let it be $C_1$, has negative intersection with $E$; in fact, if $E \cdot C_i = 0$ for every $i$ the effective divisor $E$ would be numerically trivial on the whole NE$(X)$ since $\rho_X = 2$.
Denote by $V^1$ a family of deformations of $C_1$; if $V^1$ is not unsplit then there exists a reducible cycle $\sum C_{1j}$ in $\mathcal{V}^1$, and for at least one of the components, call it $C_{11}$, we have $E \cdot C_{11} < 0$.
Denote by $V^{11}$ a family of deformations of $C_{11}$. If $V^{11}$ is not unsplit, we repeat the argument, and the procedure terminates because $-K_X \cdot V > -K_X \cdot V^1 > -K_X \cdot V^{11} >$

$\cdots > 0$.
Therefore every reducible cycle $\sum C_i$ in $\mathcal{V}$ has an irreducible component on which $E$ is negative and such that its family of deformations is unsplit.
Let $\Gamma$ be one of these components and $W$ a family of deformations of $\Gamma$; since $E \cdot \Gamma < 0$ we have $\mathrm{Locus}(W) \subset E$. We claim that $[W] \in R_1$. Assume by contradiction that $W$ is independent from $R_1$.
Denote by $F$ a non trivial fiber of $\varphi$ meeting $\mathrm{Locus}(W)$; by proposition 2.7 we have $\dim F \geq \dim X - \dim E + l(R_1) - 1 \geq l(R_1)$, and by lemma 2.16 b) we have $\dim \mathrm{Locus}(W)_F \geq -K_X \cdot W + \dim F - 1$. Combining the inequalities we have

$$n - 1 \geq \dim \mathrm{Locus}(W)_F \geq -K_X \cdot W + \dim F - 1 \geq i_X + l(R_1) - 1 \geq n - 1.$$

This forces $\mathrm{Locus}(W) = E$, so $F \subset \mathrm{Locus}(W)$ and we can apply part a) of lemma 2.16 and get

$$i_X - 1 + \dim F = \dim \mathrm{Locus}(W)_F \geq \dim F + \dim \mathrm{Locus}(W_y) \geq \dim F + i_X - 1,$$

the last equality following from proposition 2.5. Therefore $\dim \mathrm{Locus}(W_y) = i_X - 1$ so $W$ is covering, by proposition 2.5, a contradiction.
It follows that $[W] \in R_1$ and, for every reducible cycle $\sum_{i=1}^{k} C_i$ in $\mathcal{V}$, we have

$$n + 1 \geq -K_X \cdot V = -K_X \cdot \sum_{i=1}^{k} C_i \geq l(R_1) + (k-1) i_X \geq n. \tag{1}$$

Hence $k = 2$ and every reducible cycle has two components, $C_1$, which belongs to $R_1$ and $C_2$.

Let us assume, and prove it later, that $V$ is not locally unsplit. Since $V$ is also covering and $\mathrm{Locus}(V_1) = E$, the family $V_2$ of deformations of $C_2$ must be a covering family. We have $-K_X \cdot V_2 \leq i_X + 1 < 2 i_X$, hence $V_2$ is an unsplit family; therefore, by lemma 3.1, its numerical class belongs to the ray $R_2$.

To conclude we have therefore to show that $V$ is not locally unsplit. Assume that the contrary holds.
From (1) it follows that $-K_X \cdot V \geq n$. On the other hand we can not have $-K_X \cdot V = n + 1$, otherwise $\rho_X = 1$ by proposition 2.20 b). So we have $-K_X \cdot V = n$ and, again by inequality (1), that $l(R_1) + i_X = n$, so $l(R_1) \leq n - 2$ and therefore the center of the blow up, $T$, has dimension $\dim T = n - 1 - l(R_1) \geq 1$.
By proposition 2.5, for a general $x \in X$, $D_x = \mathrm{Locus}(V_x)$ is a divisor; we claim that this divisor is zero on $R_1$. To prove it recall that $R_1$ is a divisorial ray, so, by proposition 2.7, for every nontrivial fiber $F$ of $\varphi_{R_1}$ we have $\dim F \geq \dim X - \dim E + l(R) - 1 \geq l(R) \geq i_X \geq 2$. Moreover, by proposition 2.20 b), $\mathrm{NE}(D_x) = \langle [V] \rangle$, so we can apply corollary 2.21 to the divisor $D_x$ and to the unsplit family associated to $R_1$ to get $D_x \cdot R_1 = 0$.
On the other hand $\varphi(D_x)$ is an effective, hence ample divisor on $Y$, so it meets $T$. It follows that $D_x \cap E \neq \emptyset$; this, together with $D_x \cdot R_1 = 0$ implies that $D_x$ contains fibers of $\varphi_{R_1}$, a contradiction with $\mathrm{NE}(D_x) = \langle [V] \rangle$. □

## 4. A bound on the length.

LEMMA 4.1. *Let $X$ be a Fano manifold with $\rho_X \geq 2$, let $R$ be an extremal ray of $X$ and denote by* Exc($R$) *its exceptional locus. Then there exists a family of rational*

*curves $V$ whose general member does not belong to $R$ such that* $\mathrm{Exc}(R) \cap \mathrm{Locus}(V) \neq \emptyset$ *and, for some* $x \in \mathrm{Exc}(R) \cap \mathrm{Locus}(V)$*,* $V_x$ *is proper.*
*Moreover, if $R$ is not nef and $W$ is a minimal covering family, then, among the families of deformations of irreducible components of cycles in the associated Chow family $\mathcal{W}$, there is a family $V$ as above and one of the following happens*

*a)* $\mathrm{Exc}(R) \subset \overline{\mathrm{Locus}(V)}$.

*b) There exists a reducible cycle* $C_R + C_V + \sum_{i=1}^{k} C_i$ *in* $\mathcal{W}$ *with* $[C_R] \in R$ *and* $C_V$ *a curve in* $V$*.*

*Proof.* If $R$ is a nef ray it's enough to choose $V$ as the family of deformation of a minimal extremal rational curve in any ray $R_1 \neq R$, so we can assume that $R$ is not nef.
Let $W$ be a minimal covering family for $X$. Note that, since $W$ is covering, it is certainly independent from $R$: indeed, since $R$ is birational, curves whose numerical classes are in $R$ are contained in $\mathrm{Exc}(R)$.
If there exists $x \in \mathrm{Exc}(R)$ such that $W_x$ is unsplit then we are done, otherwise for every $x \in \mathrm{Exc}(R)$ there exists in $\mathcal{W}$ a reducible cycle $\sum_{i_x=1}^{m_x} C_{i_x}$, with rational components, passing through $x$.
As the point $x$ varies in $\mathrm{Exc}(R)$ the families of deformations of the curves $C_{i_x}$ are a finite number, since their anticanonical degree is bounded by $-K_X \cdot W \leq \dim X + 1$ so, calling these families $T^1, \dots, T^l$ for at least one index $j$ we have $\mathrm{Exc}(R) \subset \overline{\mathrm{Locus}(T^j)}$.
If $T^j$ is independent from $R$ then let $W^1 = T^j$, otherwise take a reducible cycle in $\mathcal{W}$ of the form $C_j + \sum_{i \neq j} C_i$, with $C_j$ in the family $T^j$, passing through a point $x \in \mathrm{Exc}(R)$. Since $[W] = [C_j + \sum_{i \neq j} C_i]$ is independent from $R$ and every component which is proportional to $R$ is contained in $\mathrm{Exc}(R)$ there exists an irreducible component $C_k$ independent from $R$ which meets $\mathrm{Exc}(R)$. In this case denote by $W^1$ the family of deformations of $C_k$.
We have thus found a family $W^1$ which is independent from $R$ such that $\overline{\mathrm{Locus}(W^1)} \cap \mathrm{Exc}(R) \neq \emptyset$. Moreover either we can choose $W^1$ such that $\mathrm{Exc}(R) \subset \overline{\mathrm{Locus}(W^1)}$ or there exists a reducible cycle in $\mathcal{W}$ with one component belonging to $R$. Let $x_1 \in \mathrm{Locus}(W^1) \cap \mathrm{Exc}(R)$. If $W^1_{x_1}$ is unsplit we set $V = W^1$ and we are are done, otherwise we repeat the argument, replacing $W$ with $W^1$ and $\mathrm{Exc}(R)$ with $\mathrm{Locus}(W^1) \cap \mathrm{Exc}(R)$. Since $n + 1 > \deg W > \deg W^1 > \cdots > 0$ the procedure terminates. □

REMARK 4.2. Both cases a) and b) of the lemma are possible; we give an example of case b), which is less intuitive. Let $X$ be the blow up of $\mathbb{P}^5$ along a Veronese surface $S$; $X$ is a Fano manifold with $\rho_X = 2$ and pseudoindex $i_X = 2$ whose other contraction is the blow up of the dual $\mathbb{P}^5$ with center a Veronese surface. Let $W$ be the family of rational curves in $X$ whose general member is the strict transform of a line meeting $S$ in one point; it is not difficult to prove that this family is a minimal covering family. If $x$ is a point outside the two exceptional divisors $E_1$ and $E_2$ then $W_x$ is unsplit, while, if $x$ is contained in $E_1 \cup E_2$ curves in $W_x$ degenerate to reducible cycles $C_1 + C_2$, with $C_i$ a line in a fiber of the contraction of $E_i$, so we are in case b). Note that the family $V$ given by the lemma is the family of curves of minimal degree in $E_2$, so $\mathrm{Exc}(R_1) \not\subset \overline{\mathrm{Locus}(V)}$ and we are not in case a).

*Proof of inequality (*).* Let $V$ and $x \in \mathrm{Exc}(R) \cap \mathrm{Locus}(V)$ be as in lemma 4.1. Let $\varphi_R : X \to Y$ be the extremal contraction associated to $R$ and let $F_x$ be the fiber of $\varphi_R$ which contains $x$. The numerical class of every curve in $F_x$ is in $R$ and, by proposition 2.20, b) the numerical class of every curve in $\mathrm{Locus}(V_x)$ is proportional to $[V]$ so, since $V$ is independent from $R$, we have $\dim \operatorname{Locus}(V_x) \cap F_x = 0$ . Moreover, by inequalities 2.5 and 2.7, we have $\dim \operatorname{Locus}(V_x) \geq i_X - 1$ and $\dim F_x \geq \dim X - \dim \operatorname{Exc}(R) + l(R) - 1$. Combining these inequalities we get

$$\begin{aligned} \dim X &\geq \dim \operatorname{Locus}(V_x) + \dim F_x \\ &\geq i_X + \dim X - \dim \operatorname{Exc}(R) + l(R) - 2 \end{aligned} \tag{2}$$

which gives

$$i_X + l(R) \leq \dim \operatorname{Exc}(R) + 2,$$

and the proposition is proved. □

## 5. The border cases.

*Proof of 1.1.* First of all note that, since the length of a fiber type extremal ray is $\leq n+1$, equality holding if and only if $X \simeq \mathbb{P}^n$, and the length of a birational extremal ray is $\leq n-1$, the assumptions of the theorem imply $i_X \geq 2$.
Let $V$ be the family given by lemma 4.1, let $x \in \operatorname{Exc}(R)$ be a point such that $V_x$ is proper and let $F_x$ be the fiber of $\varphi_R$ containing $x$. If equality holds in (*), then equality holds everywhere in (2); in particular we have

$$\dim F_x = l(R) + \dim X - \dim \operatorname{Exc}(R) - 1 \tag{3}$$

$$\dim \operatorname{Locus}(V_x) = i_X - 1. \tag{4}$$

Let $V_R$ be a family of curves in $R$ such that $-K_X \cdot V_R = l(R)$; in particular note that this family is unsplit.
We clearly have $\operatorname{Locus}(V_R) \subset \operatorname{Exc}(R)$ and, for every $x \in \operatorname{Locus}(V_R)$, denoting by $F_x$ the fiber of $\varphi_R$ containing $x$, $\operatorname{Locus}((V_R)_x) \subset F_x$; combining equation 3 with inequality 2.5 for the family $V_R$ we get

$$\dim X + l(R) - 1 = \dim \operatorname{Exc}(R) + \dim F_x =$$

$$\geq \dim \operatorname{Locus}(V_R) + \dim \operatorname{Locus}((V_R)_x) \geq \dim X + l(R) - 1,$$

hence $\operatorname{Locus}(V_R) = \operatorname{Exc}(R)$ (and $\operatorname{Locus}((V_R)_x) = F_x$).

Equality 4, together with inequality 2.5 yields that $\dim \operatorname{Locus}(V) = n$, so $V$ is a covering family, and that $-K_X \cdot V = i_X$, so $V$ is unsplit.

If $\varphi_R : X \to Y$ is of fiber type then we can apply [21, Theorem 1] to $V$ and $V_R$ to get that $X \simeq \mathbb{P}^{i_X - 1} \times \mathbb{P}^{l(R)-1}$.

Suppose now that $\varphi_R : X \to Y$ is divisorial and call $E$ the divisor $\operatorname{Exc}(R)$.
By (3) we have $\dim F_x = l(R)$; note that, since $V$ is covering and unsplit, this equality holds for every non trivial fiber $F$, hence we can apply [2, Theorem 5.1] and we obtain that $Y$ is smooth and $\varphi_R$ is the blow up of $Y$ along a smooth subvariety $T$.
Let $F$ be any non trivial fiber of $\varphi_R$; by lemma 2.16 b) we have

$$\dim \operatorname{Locus}(V)_F \geq \dim F + i_X - 1 \geq n,$$

so, by proposition 2.20 c) we have $NE(X) = \langle [R], [R_V] \rangle$, where $R_V$ is the ray spanned by the numerical class of $V$.

The target $Y$ of $\varphi_R$ is a smooth variety with $\rho_Y = 1$ covered by rational curves, hence a Fano manifold; let $V_Y$ be a minimal dominating family of rational curves for $Y$ and let $V^*$ be the family of deformations of the strict transform of a general curve in $V_Y$. By [17, Proposition 3.7] a general member of $V_Y$ does not meet $T$, which has codimension at least two in $Y$, hence $E \cdot V^* = 0$. Therefore, by lemma 3.2, the family $V^*$ is not quasi unsplit and all the reducible cycles in the associated Chow family $\mathcal{V}^*$ have two irreducible components, $C_R$ and $C_V$, where $C_R$ and $C_V$ are curves in the rays $R$ and $R_V$ respectively. In particular

$$(5) \qquad n+1 \geq -K_Y \cdot V_Y = -K_X \cdot V^* = -K_X \cdot (C_R + C_V) \geq l(R) + i_X = n+1,$$

and $Y \simeq \mathbb{P}^n$ by the proof of [15, Theorem 1.1]. (Note that the assumptions of the quoted result are different, but the proof actually works in our case, since for a very general $y$ the pointed family $V_{Y_y}$ has the properties 1-3 in [15, Theorem 2.1]).

By equation 5 we also have $-K_X \cdot C_R = l(R)$ and $-K_X \cdot C_V = i_X$, so $C_R$ and $C_V$ are minimal extremal rational curves; in particular $E \cdot C_R = -1$ and therefore, since $E \cdot V^* = 0$ we have $E \cdot C_V = 1$.
Let $\psi : X \to Z$ be the contraction of the ray $R_V$; we know that $E \cdot C_V > 0$, so every fiber of $\psi$ meets a non trivial fiber $F$ of $\varphi_R$ and therefore its dimension is $n - \dim F = i_X - 1$, since fibers of different extremal ray contractions can meet only in points.
Let now $G$ be a general fiber of $\psi$; $G$ is smooth, and, by adjunction

$$K_G + (\dim G + 1)E_{|G} = \mathcal{O}_G,$$

so $G$ is a projective space and $E \cap G$ is an hyperplane which dominates $T$. Therefore $T$ is a projective space by [19, Theorem 4.1].

The bound on the dimension of $T$ follows from the fact that $\dim T = n - l(R) - 1 = i_X - 2$ and $2i_X \leq l(R) + i_X = n+1$. □

THEOREM 5.1. *Let $X$ be a Fano manifold of Picard number $\rho_X \geq 2$, and let $R$ be a fiber type or divisorial extremal ray such that*

$$i_X + l(R) = \dim \mathrm{Exc}(R) + 1.$$

*Then $\rho_X \leq 3$ and $\rho_X = 3$ if and only if $X$ is*
- *a) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2}$,*
- *b) $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$,*
- *c) $Bl_p(V)$ where $V$ is $Bl_Y(\mathbb{P}^n)$ and $Y$ is a submanifold of dimension $n-2$ and degree $\leq n$ contained in an hyperplane $H$ such that $p \notin H$.*

*If $\rho_X = 2$, except for the cases*
- *d) $Bl_p(\mathbb{Q}^n)$,*
- *e) $Bl_{\mathbb{P}^{l(R)-2}}(\mathbb{P}^n)$.*

*the cone of curves $NE(X)$ is generated by $R$ and by a fiber type extremal ray and moreover $i_X \geq 2$.*

*Proof.* If $i_X = 1$ and $R$ is divisorial we have $l(R) \geq n-1$ so, by [2, Theorem 1.1], $X$ is the blow up at a point of a smooth variety $X'$; by [6, Theorem 1.1] we are in case c) or in case d).
If $i_X = 1$ and $R$ is of fiber type then $l(R) = n$; in particular $\varphi_R : X \to B$ is equidimensional with $(n-1)$-dimensional fibers over a smooth curve $B$. The general

fiber of $\varphi_R$ is a projective space by [11, Corollary 0.4] or [15, Theorem 1.1].
Over an open Zariski subset $U$ of $B$ the morphism $p$ is a projective bundle. By taking the closure in $X$ of a hyperplane section of $p$ defined over the open set $U$ we get a global relative hyperplane section divisor (we use $\rho(X/B) = 1$) hence $p$ is a projective bundle globally by [13, Lemma 2.12].
Since $X$ is a Fano manifold $B \simeq \mathbb{P}^1$. Write $X = \mathbb{P}_{\mathbb{P}^1}(\oplus_{i=0}^{n-1}\mathcal{O}(a_i))$ with $0 \le a_0 \le \cdots \le a_{n-1}$. A straightforward computation shows that $X$ is Fano if and only if either all the $a_i$ are zero or all the $a_i$ but $a_{n-1}$ are zero and $a_{n-1} = 1$. In the first case $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$ and $i_X = 2$, in the second case $X = Bl_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$ and we are in case e).

From now on we can assume $i_X \ge 2$.
Let $V$ be the family given by lemma 4.1, let $x \in \operatorname{Exc}(R)$ be a point such that $V_x$ is proper and let $F_x$ be the fiber of $\varphi_R$ containing $x$. First of all we prove that $V$ is an unsplit family. In fact, if $V$ were not unsplit then $-K_X \cdot V \ge 2i_X$ and $\dim \operatorname{Locus}(V_x) \ge 2i_X - 1$.
In this case we would have

$$\dim \operatorname{Locus}(V_x) + \dim F_x \ge 2i_X - 1 + n + l(R) - \dim \operatorname{Exc}(R) - 1 =$$

$$\ge n + i_X - 1 > n$$

and so $\dim \operatorname{Locus}(V_x) \cap F_x \ge 1$, a contradiction, since $V$ is independent from $R$.

Now we divide the proof in two cases, according to the type of $R$.

**Case 1:** $R$ is nef.

Let $\varphi_R : X \to Z$ be the contraction associated to $R$, and let $V_R$ be a minimal dominating family of curves in $R$; we claim that $V_R$ is unsplit. If this were not the case then $-K_X \cdot V_R \ge 2l(R)$, so, for a general $x \in X$ we would have $\dim \operatorname{Locus}(V_R)_x \ge 2l(R) - 1 \ge i_X + l(R) - 1 = n$ and $\rho_X$ would be one by proposition 2.20 b).

Recall that, according to the proof of lemma 4.1, in this case $V$ is the family of deformations of a minimal extremal rational curve in a ray $R_1$ different from $R$.

Suppose that $R_1$ is not nef; by inequality 2.7, if $G$ is a nontrivial fiber of the associated contraction we have $\dim G \ge i_X$ and, by lemma 2.16

$$\dim \operatorname{Locus}(R)_G \ge \dim G + l(R) - 1 \ge i_X + l(R) - 1 = n.$$

It follows that $\dim G = i_X = l(R_1)$ and $X = \operatorname{Locus}(R)_G$, so $\operatorname{NE}(X) = \langle [R], [R_1] \rangle$ by proposition 2.20 c).
Since $\dim G = i_X = l(R_1)$ for every nontrivial fiber of the contraction associated to $R_1$, this contraction is a smooth blow up by [2, Theorem 5.1].
We can repeat the second part of the proof of theorem 1.1, replacing $R$ with $R_1$ and $R_V$ with $R$ and obtain that $X = Bl_{\mathbb{P}^{l(R)-2}}(\mathbb{P}^n)$, so we are in case e).

Suppose now that $R_1$ is nef and consider the $\operatorname{rc}(R, R_1)$ fibration $\pi_{R,R_1} : X -> Z$. Let $F$ be a general fiber of $\pi_{R,R_1}$ and $x \in F$ a point; $F$ contains $\operatorname{Locus}(R, R_1)_x$ which has dimension $\ge i_X + l(R) - 2 = n - 1$ by lemma 2.16, so $\dim Z \le 1$.

Suppose that $\dim Z = 1$ and let $V'$ be a minimal horizontal dominating family for $\pi_{R,R_1}$; by lemma 2.14 c), if $x'$ is a general point in $\operatorname{Locus}(V')$, we have

$\dim \mathrm{Locus}(V')_{x'} = 1$ and so $-K_X \cdot V' = 2 = i_X$.
In particular $V'$ is unsplit and, by 2.5, covering. We can apply [21, Theorem 1] to $V$, $V_R$ and $V'$ to conclude that $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^{n-2}$ and we are in case a).

If $\dim Z = 0$ then $X$ is rc$(R, R_1)$-connected and $\rho_X = 2$ by corollary 2.19; in this case we clearly have $\mathrm{NE}(X) = \langle [R], [R_1] \rangle$.

**Case 2:** $R$ is not nef.

Let $W$ be a minimal covering family for $X$ and let $V$ be a family as in lemma 4.1, chosen among the families of deformations of irreducible components of cycles in $\mathcal{W}$.

**Step 1** V is an unsplit covering family.

Suppose that $V$ is not a covering family. Then, by inequality 2.5, $\dim \mathrm{Locus}(V_x) \geq i_X$ and therefore $E := \mathrm{Exc}(R)$ is not contained in $\mathrm{Locus}(V)$. Indeed, in this case, by lemma 2.16 a), denoted by $F$ a nontrivial fiber of $\varphi_R$, we would have $\dim \mathrm{Locus}(V)_F \geq \dim F + \dim \mathrm{Locus}(V_x) = n$, a contradiction.
So we are in case b) of lemma 4.1 and there exists a reducible cycle $C_R + C_V + \sum_{i=i}^{k} C_i$ in $\mathcal{W}$ with $[C_R] \in R$ and $C_V$ in $V$. Hence we have

$$n \geq -K_X \cdot W \geq -K_X \cdot (C_R + C_V + \sum_{i=1}^{k} C_i) \geq l(R) + i_X + k i_X = n + k i_X$$

forcing $-K_X \cdot W = n$ and $k = 0$.
We have thus proved that in $\mathcal{W}$ there exists a reducible cycle $C_R + C_V$, with $[C_R]$ in $R$ and $C_V$ in $V$.
Let $D = \mathrm{Locus}(W_x)$ for a general $x \in X$; by proposition 2.20 b) $\mathrm{NE}(D) = \langle [W] \rangle$.
By corollary 2.21, since the fibers of $\varphi_R$ are at least two dimensional we have $D \cdot R = 0$; by the same corollary, since $\dim \mathrm{Locus}(V_x) \geq 2$ we have $D \cdot V = 0$. This implies also that $D \cdot W = D \cdot (C_R + C_V) = 0$.
By lemma 2.1 there exists an extremal ray $R_1$ such that $D \cdot R_1 > 0$; let $V^1$ be a family of deformations of a minimal curve in $R_1$. By lemma 2.16 b) we have $\dim \mathrm{Locus}(V^1)_D \geq \dim D + i_X - 1 \geq n$, hence $X = \mathrm{Locus}(V^1)_D$ and $\rho_X = 2$.
This is a contradiction, since $D$ is zero on $R$ and $V$ and so, if $\rho_X = 2$ it would be zero on the entire cone. Therefore $V$ is a covering family as claimed.

**Step 2** $\rho_X \leq 3$.

Let $F$ be a nontrivial fiber of $\varphi_R$; by lemma 2.16 b) we have

$$\dim \mathrm{Locus}(V)_F \geq \dim F + i_X - 1 \geq n - 1$$

If $X = \mathrm{Locus}(V)_F$ then, by proposition 2.20 c) $\mathrm{NE}(X) = \langle [R], [V] \rangle$ and we are done. Note that this is always the case if $\dim F > l(R)$, so we assume from now on that $\varphi_R$ is equidimensional with fibers of dimension $l(R)$, hence it is a smooth blow up by [2, Theorem 5.1].
An irreducible component of $\mathrm{Locus}(V)_F$ is thus a divisor $D \subset X$ such that $\mathrm{NE}(D) = \langle [R], [V] \rangle$. If $D \cdot V > 0$ then $X = \mathrm{ChLocus}_2(V)_F$ and $\mathrm{NE}(X) = \langle [R], [V] \rangle$ again by proposition 2.20 c), so we can assume $D \cdot V = 0$.
By lemma 2.1 there exists an extremal ray $R_1$ such that $D \cdot R_1 > 0$.

If $R_1 \not\subset \mathrm{NE}(D)$ then, by lemma 2.16 b), denoted by $V^1$ a family of deformations of a minimal extremal rational curve in $R_1$, we have $\dim \mathrm{Locus}(V^1)_D = n$. By lemma

2.18 $N_1(X) = \langle [R], [V], [V^1] \rangle$, so $\rho_X \leq 3$, equality holding if and only if $R_1$ is not contained in the vector subspace of $N_1(X)$ spanned by $R$ and $[V]$.

If $R_1 \subset \mathrm{NE}(D)$ then $R_1 = R$ because $D \cdot V = 0$. It follows that $\mathrm{Locus}(R)_D = E$, so $N_1(E) = \langle [R], [V] \rangle$.
If $E \cdot V > 0$ then $\mathrm{Locus}(V)_E = X$ and $N_1(X) = \langle [R], [V] \rangle$ by lemma 2.18, so $\rho_X = 2$. We claim that we cannot have $E \cdot V = 0$; in fact, in this case every curve of $V$ which meets $E$ is entirely contained in $E$, so $E = \mathrm{Locus}(V)_F = D$ and we have $D \cdot R < 0$, a contradiction.

**Step 3** $\rho_X = 2$, description of the cone.

We have to prove that $\mathrm{NE}(X) = \langle [R], [R_1] \rangle$ where $R_1$ is a fiber type extremal ray. By step two this is the case if for a nontrivial fiber $F$ of $\varphi_R$ either we have $X = \mathrm{Locus}(V)_F$ or an irreducible component of $\mathrm{Locus}(V)_F$ is a divisor $D$ such that $D \cdot V > 0$. We can therefore assume that an irreducible component of $\mathrm{Locus}(V)_F$ is a divisor $D$ such that $D \cdot V = 0$; moreover we know that there exists an extremal ray $R_1$ of $X$ on which $D$ is positive.

If $R_1 \not\subset \mathrm{NE}(D)$ then $\mathrm{NE}(X) = \langle [R], [R_1] \rangle$ and moreover, by corollary 2.21 the contraction associated to $R_1$ has nontrivial one dimensional fibers, and so it is of fiber type, since $i_X \geq 2$ by proposition 2.7.

If $R_1 \subset \mathrm{NE}(D)$ then $R_1 = R$ thus, if $V$ is not extremal, $D$ is negative on an extremal ray $R_2$, and so $\mathrm{Exc}(R_2) \subset D$, against $\mathrm{NE}(D) = \langle [R], [V] \rangle$. Therefore $V$ is extremal and $\mathrm{NE}(X) = \langle [R], [V] \rangle$.

**Step 4** $\rho_X = 3$, description of the cone.

By step two, if $\rho_X = 3$, then $\mathrm{Locus}(V)_F$ has dimension $n-1$; moreover, denoted by $D$ one irreducible component of $\mathrm{Locus}(V)_F$ we have $D \cdot V = 0$ and $D \cdot R_1 > 0$ for a ray $R_1$ not contained in the vector subspace of $N_1(X)$ spanned by $R$ and $[V]$.
Since $\mathrm{NE}(D) = \langle [R], [V] \rangle$, by corollary 2.21, every nontrivial fiber of the contraction associated to $R_1$ is one dimensional. Combining this with $i_X \geq 2$, by inequality 2.7, we have that $V^1$ is a covering unsplit family.
By lemma 2.16, denoting again by $F$ a nontrivial fiber of $\varphi_R$, we have $\dim \mathrm{Locus}(V, V^1)_F = \dim \mathrm{Locus}(V^1, V)_F = n$ (and $-K_X \cdot V = -K_X \cdot V^1 = i_X = 2$), so $X = \mathrm{Locus}(V, V^1)_F = \mathrm{Locus}(V^1, V)_F$.
We can write $X = \mathrm{Locus}(V, V^1)_F = \mathrm{Locus}(V)_{\mathrm{Locus}(V^1)_F}$ and therefore, by lemma 2.18 and proposition 2.20, the numerical class of every curve in $X$ can be written as a linear combination $a[V] + b[V^1] + c[R]$ with $b, c \geq 0$.
On the other hand $X = \mathrm{Locus}(V^1, V)_F = \mathrm{Locus}(V^1)_{\mathrm{Locus}(V)_F}$, so the numerical class of every curve in $X$ can be written as a linear combination $a[V] + b[V^1] + c[R]$ with $a, c \geq 0$. By the uniqueness of the decomposition it follows that $\mathrm{NE}(X) = \langle [V], [V^1], [R] \rangle$.

**Step 5** If $\rho_X = 3, i_X \geq 2$ and $R$ is not nef then $X \simeq Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$.

We have thus proved that the cone of curves of $X$ is generated by $R$, which is the ray associated to a smooth blow up $\varphi_R : X \to Y$, and by other two fiber type extremal rays, call them $R_1$ and $R_2$, which both have length two. In particular we have $i_X = 2$, so $l(R) = n-2$ and $\dim F_R = n-2$ for every nontrivial fiber of $\varphi_R$.
Moreover, since $E = \mathrm{Exc}(R)$ is non negative on $R_1$ and $R_2$, by [25, Proposition 3.4] $Y$ is a Fano manifold.

The effective divisor $E$ is positive on at least one of the rays $R_i$ by lemma 2.1; let us assume that $E \cdot R_1 > 0$. Let $\sigma$ be the extremal face spanned by $R$ and $R_1$ and consider the associated contraction $\varphi_\sigma$.
Let $x \in X$ be a point, let $\Gamma_1$ be a curve in $R_1$ through $x$ and let $F$ be a nontrivial fiber of $\varphi_R$ meeting $\Gamma_1$. The fiber of $\varphi_\sigma$ through $x$ contains $\mathrm{Locus}(R_1)_F$, which has dimension $n-1$ by lemma 2.16, so the target of $\varphi_\sigma$ is a smooth curve, which has to be rational since $X$ is Fano. We have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi_R"] \arrow[dr, "\varphi_\sigma"'] & Y \arrow[d, "\psi_\sigma"] \\
& \mathbb{P}^1
\end{tikzcd}$$

The general fiber $F_\sigma$ of $\varphi_\sigma$ is, by adjunction, a Fano manifold of index $\geq 2$ which has a divisorial extremal ray of length $\dim F_\sigma - 1$, so, by theorem 1.1, $F_\sigma \simeq Bl_p\mathbb{P}^{n-1}$. It follows that the general fiber of $\psi_\sigma$ is $\mathbb{P}^{n-1}$. The Fano manifold $Y$ has a fiber type extremal ray $\psi_\sigma$ of length $\dim Y$ while the other ray is of fiber type, since the associated contraction contracts the images of curves in $R_2$. Therefore $i_Y \geq 2$.
We can thus apply theorem 1.1 to conclude that $Y \simeq \mathbb{P}^1 \times \mathbb{P}^{n-1}$. Let $T \simeq \mathbb{P}^1$ be the center of the blow up; we claim that $T$ is a fiber of the projection $Y \to \mathbb{P}^{n-1}$. By contradiction, assume that this is not the case. Let $C \simeq \mathbb{P}^1$ be a fiber of the projection $Y \to \mathbb{P}^{n-1}$ meeting $T$ and let $\widetilde{C}$ be the strict transform of $C$.
By the canonical bundle formula we have

$$-K_X \cdot \widetilde{C} = -K_Y \cdot C - l(R)E \cdot \widetilde{C} \leq 2 - l(R) \leq 0,$$

and so $X$ is not a Fano manifold, a contradiction. □

## 6. Blow ups.

*Proof of 1.3.* If $i_X + l(R) = n+1$, by theorem 1.1 we have that $X = Bl_{\mathbb{P}^t}(\mathbb{P}^n)$, with $t \leq \frac{n-3}{2}$.

We can thus assume that $i_X + l(R) = n$. By theorem 5.1, if $\rho_X \geq 3$, then $X$ is either $Bl_{\mathbb{P}^1 \times \{p\}}(\mathbb{P}^1 \times \mathbb{P}^{n-1})$ or $Bl_p(V_d)$ where $V_d$ is $Bl_Y(\mathbb{P}^n)$ and $Y$ is a submanifold of dimension $n-2$ and degree $\leq n$ contained in an hyperplane which does not contain $p$. Note that case a) of theorem 5.1 has been excluded since it is not a blow up.

We can thus assume, from now on, that $\rho_X = 2$; again by theorem 5.1 either $X \simeq Bl_p(\mathbb{Q}^n)$ or $\rho_X = 2$, $i_X \geq 2$ and the cone of curves of $X$ is generated by $R$ and by a fiber type extremal ray $R_V$. (Case e) of theorem 5.1 has been excluded since in that case $R$ is a fiber type ray).

The target $Y$ of $\varphi_R$ is a smooth variety with $\rho_Y = 1$ covered by rational curves, hence a Fano manifold; let $V_Y$ be a minimal dominating family of rational curves for $Y$ and let $V^*$ be the family of deformations of the strict transform of a general curve in $V_Y$. By [17, Proposition 3.7] a general member of $V_Y$ does not meet $T$, which has codimension at least two in $Y$, hence $E \cdot V^* = 0$.

Since $E \cdot V^* = 0$, by lemma 3.2, the family $V^*$ is not quasi unsplit and all the reducible cycles in the associated Chow family $\mathcal{V}^*$ have two irreducible components, $C_1$ and $C_2$, where $C_1$ and $C_2$ are curves in the rays $R$ and $R_V$ respectively.
Let $\Gamma_R$ and $\Gamma_V$ be curves in $R$ and $R_V$ respectively with minimal anticanonical degree.

Since $\varphi_R$ is a smooth blow up $E \cdot \Gamma_R = -1$, hence the numerical class of every curve in $R$ is an integral multiple of $[\Gamma_R]$; in particular we can write $[C_1] = m_1[\Gamma_R]$ with $m_1$ a positive integer. By the canonical bundle formula

$$(6) \qquad \begin{aligned} n+1 \geq -K_Y \cdot V_Y = -K_X \cdot V^* = -K_X \cdot (C_1 + C_2) \geq \\ \geq m_1 l(R) + i_X \geq (m_1 - 1) l(R) + n. \end{aligned}$$

Recalling that $l(R) \geq i_X \geq 2$ we have $m_1 = 1$, i.e. $[C_1] = [\Gamma_R]$. It follows that $E \cdot C_2 = 1$, so $[C_2] = [\Gamma_V]$ and $[V^*] = [\Gamma_R + \Gamma_V]$.

Consider now the contraction of $R_V$, $\psi : X \to Z$ and let $F$ be any fiber of $\psi$. Since $E \cdot \Gamma_V > 0$ the fiber $F$ meets a nontrivial fiber $F_R$ of $\varphi_R$ and therefore $\dim F \leq n - \dim F_R = i_X$.
On the other hand, by inequality 2.7 $\dim F \geq l(R_V) - 1 \geq i_X - 1$, so the length of $R_V$ is either $i_X$ or $i_X + 1$. In the first case, by equation 6 we have $-K_Y \cdot V_Y = n$, while in the second we have $-K_Y \cdot V_Y = n + 1$.
The contraction $\psi$ is supported by $K_X + i_X E$ in the first case and by $K_X + (i_X + 1)E$ in the second; in both cases, since for every fiber of $\psi$ we have $i_X - 1 \leq \dim F \leq i_X$, the target variety $Z$ is smooth by [3, Theorem 5.1].

The general fiber of $\psi$ has dimension either $i_X - 1$ or $i_X$, so the dimension of $Z$ is either $l(R) + 1$ or $l(R)$. We divide the proof in two cases, accordingly.

**Case 1** $\dim Z = l(R) + 1$.

In this case $\psi$ is supported by $K_X + i_X E$, its general fiber has dimension $i_X - 1$ and it is a projective space $\mathbb{P}^{i_X - 1}$ by [3, Theorem 5.1], while jumping fibers, if they exist, have dimension $i_X$ and are projective spaces $\mathbb{P}^{i_X}$, again by [3, Theorem 5.1].

We claim that, for at least one fiber $F$ of $\psi$, we have $E \cap F = \mathbb{P}^{i_X - 1}$. The claim is clearly true if either $E$ contains a fiber of dimension $i_X - 1$ or, being $E \cdot \Gamma_V = 1$, if $\psi$ has a jumping fiber ($E$ cannot contain a jumping fiber $F$, otherwise, by lemma 2.16 a) we will have $\dim E \geq \dim \mathrm{Locus}(R)_F \geq i_X + l(R) \geq n$).
Suppose by contradiction that neither of these two possibilities happens. The restriction of $\psi$ to $E$ is thus an equidimensional morphism with general fiber a projective space, such that $E$ restricted to the general fiber is $\mathcal{O}_{\mathbb{P}}(1)$, so $\psi$ makes $E$ a projective bundle over $Z$.
Therefore $E$, which is also a projective bundle over $T$, has two projective bundle structures and $\rho_E = 2$ so, by [22, Theorem 2], $E$ is the projectivization of the tangent bundle of a projective space, but this is impossible since the two fibrations of $E$ have fibers of dimension $i_X - 2$ and $l(R)$ and these two dimensions are different, being $l(R) \geq i_X$, so the claim is proved.

It follows that either $\psi$ has a jumping fiber or $E$ contains a fiber of $\psi$; in both cases $T$, the center of the blow up, is dominated by the intersection of $E$ with this fiber, and so it is a projective space of dimension $i_X - 1$ by [19, Theorem 4.1].

To finish the proof, we have to show that $Y \simeq \mathbb{Q}^n$, and we will do this proving the existence of a line bundle $L_Y \in \mathrm{Pic}(Y)$ such that $-K_Y = nL_Y$ and applying the Kobayashi-Ochiai theorem [16].
Take a line $l$ in $T$ and denote by $Y_l$ the inverse image $\varphi_R^{-1}(l)$; $Y_l$ is a projective bundle over a smooth rational curve, so a toric variety. The restriction $\psi_{|Y_l} : Y_l \to Z$ is thus a surjective morphism from a toric variety to a smooth variety with Picard number

one, so $Z$ is a projective space by [22, Theorem 1].
Let $L$ be the line bundle $\psi^*\mathcal{O}_{\mathbb{P}}(1)+E$; we have $L\cdot R=0$ and therefore there exists $L_Y\in \operatorname{Pic}(Y)$ such that $\varphi_R^*L_Y=L$.
Moreover, since $L\cdot V^*=1$ we have $L_Y\cdot V_Y=1$, so, recalling that $-K_Y\cdot V_Y=\dim Y$ we get $-K_Y=nL_Y$ and we conclude that $Y\simeq \mathbb{Q}^n$ by the Kobayashi-Ochiai theorem [16].

**Case 2** $\dim Z=l(R)$.

In this case, as noted above, every fiber of $\psi$ has dimension $i_X$. The contraction $\psi$ is supported either by $K_X+(i_X+1)E$ and it is a projective bundle or by $K_X+i_XE$ and it is a quadric bundle, by [3, Theorem 5.1].
Every nontrivial fiber of $\varphi_R$ dominates $Z$ so, by [19, Theorem 4.1] $Z$ is a projective space.

Let $L$ be the line bundle $\psi^*\mathcal{O}_{\mathbb{P}}(1)+E$; we have $L\cdot R=0$ so there exists $L_Y\in \operatorname{Pic}(Y)$ such that $\varphi_R^*L_Y=L$.
Moreover, since $L\cdot V^*=1$ we have $L_Y\cdot V_Y=1$.

**Case 2a** $\psi:X\to Z$ is a projective bundle.

In this case $-K_Y\cdot V_Y=n+1$, so $-K_Y=(n+1)L_Y$ and $Y$ is a projective space by the Kobayashi-Ochiai theorem [16]. The intersection of $E$ with the general fiber of $\psi$ is thus a projective space and therefore the center $T$ of the blow up is a linear space by [19, Theorem 4.1].

**Case 2b** $\psi:X\to Z$ is a quadric bundle.

In this case $-K_Y\cdot V_Y=n$, so $-K_Y=nL_Y$ and $Y$ is a smooth quadric by the Kobayashi-Ochiai theorem.
The intersection of $E$ with the general fiber of $\psi$ is thus a smooth quadric, so the center $T$ of the blow up is either a linear space or a smooth quadric by [23, Proposition 8].

Actually the first case can be excluded by direct computation, since the blow up of a quadric along a linear subspace is not a quadric bundle over $\mathbb{P}^{l(R)}$.
In the second case let $\Pi\simeq\mathbb{P}^{i_X}$ be the linear subspace of dimension $i_X$ of $\mathbb{P}^{n+1}$ which contains $T\simeq\mathbb{Q}^{i_X-1}$.
Two cases are possible: either $Y\supseteq\Pi$ or $Y\cap\Pi=T$. The first case has to be excluded because, if $Y\supseteq\Pi$ the blow up of $\mathbb{Q}^n$ along $T$ does not give rise to a Fano manifold.
To see this, take a line $l\subset\Pi$ not contained in $T$; by the canonical bundle formula, if $X=Bl_T\mathbb{Q}^n$ we have

$$-K_X\cdot\tilde{l}=-K_Y\cdot l-l(R)E\cdot\tilde{l}\le n-2l(R)\le 0.$$

Finally note that in both cases the bound on the dimension of the center follows from the fact that $i_X\le l(R)$ and so $2i_X\le l(R)+i_X\le n$. □

## 7. Varieties with a polarization.

*Proof of 1.2.* Let $V$ be the family given by lemma 4.1, let $x\in\operatorname{Exc}(R)$ be a point such that $V_x$ is unsplit and let $F_x$ be the fiber of $\varphi_R$ containing $x$.
First of all we prove that $\rho_X=2$ and that the cone of curves of $X$ is generated by $R$ and by the ray spanned by $[V]$.
We are assuming that equality holds in (*), so equality holds everywhere in (2); in particular we have

$$\dim F_x=l(R)+\dim X-\dim\operatorname{Exc}(R)-1=\dim X-r_X+1 \tag{7}$$

$$\dim\operatorname{Locus}(V_x)=r_X-1. \tag{8}$$

This forces $-K_X \cdot V = r_X$, so the family $V$ is unsplit. Moreover, by inequality 2.5 $V$ is a covering family.
Therefore, by lemma 2.16 we have $\dim \mathrm{Locus}(V)_{F_x} \geq \dim F_x + r_X - 1 = \dim X$, so, by proposition 2.20 c), we have $NE(X) = \langle [V], [R] \rangle$.

Let $\psi : X \to Z$ be the contraction of the ray $R_V$ spanned by $[V]$, which is of fiber type since $V$ is a covering family; curves parametrized by $V$ have anticanonical degree $r_X$, so they are minimal extremal curves in $R_V$ which has length $r_X$.
By inequality 2.7, every fiber of $\psi$ has dimension $\geq l(R_V) - 1 = r_X - 1$, so $\dim Z \leq n - r_X + 1$. Again by inequality 2.7 the fibers of $\varphi_R$ have dimension $\geq n - e + l - 1 = n - r_X + 1$, so they dominate $Z$. In particular every fiber of $\psi$ meets a fiber $F_R$ of $\varphi_R$ and so its dimension is $\leq \dim X - \dim F_R = r_X - 1$; therefore the contraction $\psi : X \to Z$ is equidimensional.
Moreover we also have that the dimension of every fiber of $\varphi_R$ is $\leq \dim Z \leq n - r_X + 1$, so $\varphi_R$ is equidimensional with fibers of dimension $n - r_X + 1$ and $\dim Z = n - r_X + 1$.

Denote by $H$ the divisor such that $-K_X = r_X H$. The general fiber $G$ of $\psi$ is, by generic smoothness and adjunction, a projective space $\mathbb{P}^{r_X - 1}$ and $H_G \simeq \mathcal{O}(1)$, so, by [13, Lemma 2.12], $\psi$ is a projective bundle over $Z$, $X = \mathbb{P}_Z(\mathcal{E})$, with $\mathcal{E} = \varphi_R^* H$. In particular $Z$ is a smooth Fano variety of Picard number one.
The canonical bundle formula yields

$$\psi^*(K_Z + \det \mathcal{E}) = K_X + r_X H = \mathcal{O}_X,$$

and so $-K_Z = \det \mathcal{E}$. Note also that, if $C_R$ is a curve in $R$ then

$$H \cdot C_R = \frac{-K_X \cdot C_R}{r_X} \geq \frac{l(R)}{r_X}. \tag{9}$$

Let $V_Z$ be a minimal covering family for $Z$ and $C$ a curve in $V_Z$; Let $\nu : \mathbb{P}^1 \to C \subset Z$ be the normalization of $C$ and let $Z_C$ be the fiber product $Z_C = \mathbb{P}^1 \times_C X$.

$$\begin{tikzcd}
Z_C \arrow[r, "\bar{\nu}"] \arrow[d, "p"'] & X \arrow[d, "\psi"] \\
\mathbb{P}^1 \arrow[r, "\nu"'] & Z
\end{tikzcd}$$

The variety $Z_C$ is a projective bundle over $\mathbb{P}^1$, $Z_C = \mathbb{P}^1_{\mathbb{P}}(\nu^*\mathcal{E})$; the vector bundle $\nu^*\mathcal{E}$ is ample, so we can write $\nu^*\mathcal{E} \simeq \oplus_{i=0}^{r_X - 1} \mathcal{O}(a_i)$ with $a_i > 0$ and $a_i \leq a_{i+1} \quad \forall i$.
Denote by $m$ the maximum index $i$ such that $a_i = a_0$ and rewrite $\nu^*\mathcal{E}$ in the following way

$$\nu^*\mathcal{E} \simeq \oplus^{m+1} \mathcal{O}(a_0) \oplus_{i=m+1}^{r_X - 1} \mathcal{O}(a_i).$$

The cone of curves $\mathrm{NE}(Z_C)$ is generated by the class of a line in a fiber of $p$ and by the class of a section $C_0$ corresponding to a surjection $\nu^*\mathcal{E} \to \mathcal{O}(a_0)$.
The cone of curves $\mathrm{NE}(X)$ is generated by the class $[V]$ of a line in a fiber of $\varphi_V$ and by the class of $\Gamma_R$, a minimal extremal curve in $R$.
The morphism $\bar{\nu}$ induces a map of spaces of cycles $N_1(Z_C) \to N_1(X)$ which allows us to identify $\mathrm{NE}(Z_C)$ with a subcone of $\mathrm{NE}(X)$.
Since $\bar{\nu}(Z_C)$ contains lines in the fibers of $\psi$ and contains curves in the fibers of

$\varphi_R$ (since for dimensional reasons $\dim(\bar{\nu}(Z_C) \cap F_R) \geq 1$), we have an identification $\mathrm{NE}(Z_C) \simeq \mathrm{NE}(X)$.
In particular $F_R \cap \bar{\nu}(Z_C)$, which is a curve whose numerical class in $X$ is a multiple of $[\Gamma_R]$, is the image of a curve $\Gamma$ whose numerical class in $Z_C$ is a multiple of $[C_0]$.
By [21, Lemma 3] the curve $\Gamma$ is the union of disjoint minimal sections, so $\bar{\nu}(Z_C) \cap F_R$ consists of the images via $\bar{\nu}$ of disjoint minimal sections.
On the other hand, if $C_0$ is a minimal section, then $\bar{\nu}(C_0)$ is a curve whose numerical class is in $R$, so it is contained in a fiber of $\varphi_R$.
It follows that the dimension of $\varphi_R(\mathrm{Exc}(R))$ is the dimension of the space parametrizing minimal sections, which is $m$. Therefore

$$m = \dim \mathrm{Exc}(R) - \dim F_R = l(R) + r_X - 2 - \dim F_R.$$

Moreover, since $[C_0] \in R$ we have, by equation 9

$$a_0 = H \cdot C_0 \geq \frac{l(R)}{r_X},$$

hence $a_i \geq a_0 + 1 \geq \dfrac{l(R)}{r_X} + 1$ for $i = m+1, \ldots, r_X - 1$.
It follows that

$$\dim Z + 1 \geq -K_Z \cdot C = \det \mathcal{E} \cdot C =$$

$$= (m+1)a_0 + \sum_{m+1}^{r_X - 1} a_i \geq (m+1)\frac{l(R)}{r_X} + (r_X - m - 1)\left(\frac{l(R)}{r_X} + 1\right) =$$

$$= l(R) + r_X - m - 1 = \dim F_R + 1 = \dim Z + 1.$$

Therefore $Z$ admits a minimal dominating family of degree $\dim Z + 1$, hence $Z$ is a projective space of dimension $n - r_X + 1$ by the proof of [15, Theorem 1.1].
Since equality holds everywhere we also have $a_0 = 1$, $a_i = 2 \quad i = m+1, \ldots, r_X - 1$, so the splitting type of $\mathcal{E}$ on lines of $Z$ is uniform.
If $\dim \mathrm{Exc}(R) \leq \dim X - 2$ then $\mathrm{rk}\, \mathcal{E} = r_X \leq l(R) < n - r + 1 = \dim Z$, therefore $\mathcal{E}$ is decomposable by [12] and $\mathcal{E} \simeq \oplus^{m+1}\mathcal{O}(1) \oplus^{r_X - 1 - m} \mathcal{O}(2)$.
If $\dim \mathrm{Exc}(R) = \dim X - 1$ then $\mathrm{rk}\, \mathcal{E} = \dim Z$ and the splitting type of $\mathcal{E}$ is $(1, \ldots, 1, 2)$, so, by [12], either $\mathcal{E}$ is decomposable or $\mathcal{E}$ is the tangent bundle of $Z = \mathbb{P}^{\dim Z}$, but the second case has to be excluded since $X$ has a divisorial contraction.
Finally, if $\mathrm{Exc}(R) = X$ then the splitting type of $\mathcal{E}$ is $(1, \ldots, 1)$, so $\mathcal{E}$ is decomposable by [4, Proposition 1.2] and $X$ is a product of projective spaces. □

PROPOSITION 7.1. *Let $X$ be a Fano manifold of Picard number $\rho_X = 2$, index $r_X \geq 2$, and let $R$ be a fiber type or divisorial extremal ray such that $r_X + l(R) = \dim \mathrm{Exc}(R) + 1$. Then, if $R$ is divisorial either $X$ is as in theorem 1.3 or $X$ has the structure of a projective bundle over a smooth variety.*
*If $R$ is of fiber type then $X$ is a projective bundle or a quadric bundle or the projectivization of a Bănică sheaf over a smooth variety $Y$.*

*Proof.* By theorem 5.1 either $X \simeq Bl_{\mathbb{P}^{l(R)-2}}(\mathbb{P}^n)$ or the cone of curves $\mathrm{NE}(X)$ is generated by $R$ and by a fiber type extremal ray; let $\psi : X \to Z$ be the contraction

of this ray.
Let $H$ be the line bundle such that $-K_X = r_X H$, let $A \in \mathrm{Pic}(Z)$ be an ample divisor and let $H' = H + \psi^* A$. The contraction $\psi$ is supported by $K_X + r_X H'$.

If $R$ is divisorial then every fiber of $\varphi_R$ has dimension $\geq l(R)$. If equality holds for every fiber, $\varphi_R$ is a smooth blow up by [2, Theorem 5.1], so $X$ is as in theorem 1.3.
We can therefore assume that there exists a fiber $F$ of $\varphi_R$ of dimension $\geq l(R)+1$. The contraction $\psi : X \to Z$ has fibers of dimension $\geq r_X - 1 \geq n - l(R) - 1$, so $\dim Z \leq l(R) + 1$. It follows that $F$ dominates $Z$ and meets every fiber of $\psi$, forcing the equidimensionality of $\psi$.
We can now conclude that $X$ is a projective bundle over $Z$ by [13, Lemma 2.12] since $H \cdot V = 1$.

If $R$ is of fiber type then every fiber of $\varphi_R$ has dimension $\geq l(R) - 1$ and so the contraction $\psi : X \to Z$ has fibers of dimension $\leq n - l(R) + 1 \leq r_X$, so we can conclude by [3, Theorem 5.1] and [5, Proposition 2.5]. □

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