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A PREDICATE EXTENSION OF $\mathbb{R}$EAL VALUED LOGIC

STEFANO BARATELLA

Abstract. We study a predicate extension of an unbounded real valued propositional logic that has been recently introduced. The latter, in turn, can be regarded as an extension of both the abelian logic and of the propositional continuous logic. Among other results, we prove that our predicate extension satisfies the property of weak completeness (the equivalence between satisfiability and consistency) and, under an additional assumption on the set of premisses, the property of strong completeness (the equivalence between logical consequence and provability). Eventually we discuss some topological properties of the space of types in our logic.

1. Introduction

In [2] the authors introduced a propositional real valued logic with positive and negative unbounded truth values (hereafter to be referred to as *propositional* $\mathbb{R}$*-logic*). Propositional $\mathbb{R}$-logic can be regarded as an extension of the propositional *abelian logic* of [10], in the sense that the latter is the logic of lattice ordered abelian groups and the former is the logic of lattice ordered real vector spaces, or Riesz spaces. More precisely, in [10] the authors provide a sound and strongly complete axiomatization of abelian logic with respect to the class of lattice-ordered abelian groups (equivalently: with respect to the the lattice-ordered group of the reals). In [2] the authors provide a sound and strongly complete axiomatization of propositional $\mathbb{R}$-logic with respect to the class of Riesz spaces (equivalently: with respect to the Riesz space of the reals), under the additional assumption that the set of premisses of a derivation is *Archimedean*. See Definition 19 and [2, Theorem 6.8].

Actually, for propositional $\mathbb{R}$-logic, the equivalence of satisfiability and consistency (in the following to be referred to as *weak completeness*) does not imply the equivalence of provability and logical consequence (previously and in the following referred to as *strong completeness*). In this regard, propositional $\mathbb{R}$-logic is similar to the *continuous logic* introduced in [6]. Indeed propositional $\mathbb{R}$-logic faithfully interprets the propositional continuous

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logic, which, in turn, is an extension of the $[0,1]$-valued Łukasiewicz's propositional logic. Hence one may say that propositional $\mathbb{R}$-logic embeds both abelian and propositional continuos logic. The reader is referred to [2, §5] for the details.

In this paper we suitably extend propositional $\mathbb{R}$-logic to a predicate setting and we investigate the issues of weak and strong completeness of the proposed predicate extension, that we shall simply call $\mathbb{R}$-*logic*. We point out that a predicate version of continuous logic has been introduced in [6] and weak completeness and an *approximated* strong completeness theorem (the best possible) have been proved therein. To the best of our knowledge, no predicate extension of the propositional abelian logic has been introduced so far. In [10] the authors openly admit their conflicting ideas about the treatment of quantifiers, but do not go any further.

Here is an outline of the paper: in Section 2 we introduce the formulas and the semantics of $\mathbb{R}$-logic. In Section 3 we introduce the logical axioms and the deduction rules and we prove a number of preliminary results showing that the inf-quantifier of $\mathbb{R}$-logic has the same features as the universal quantifier in first order logic. In this section we also prove a suitable version of a Deduction Theorem. In Section 4 we use a Henkin-style construction to prove the weak completeness of $\mathbb{R}$-logic. More precisely, we show that every consistent set of formulas is satisfiable in a structure that assigns to each formula a real number as its truth value (Corollary 25). The proof extends that of [2, Theorem 6.7]. Compactness and an approximated strong completeness theorem follows easily from weak completeness. Further we recall the notion of Archimedean set of formulas and we prove that, as shown in [2] for the propositional $\mathbb{R}$-logic, strong completeness characterizes, among the consistent set of formulas, those that are Archimedean (Theorem 28). Another straightforward corollary of weak completeness is a Pavelka-style completeness theorem.

In Section 5, after some preliminary considerations we introduce the notion of type and a topology on the space of types. We prove some properties of such topology and we point out that, when the language is countable, the resulting space is metrizable. We also make a brief comparison with similar results from the existing literature.

2. The framework

In this section we introduce the formulas and the semantics of a predicate extension of the propositional $\mathbb{R}$-logic of [2]. As already mentioned in the introduction we call such extension $\mathbb{R}$-*logic*. Our presentation will be mostly self-contained, but familiarity with [2] might be helpful. We will remain fairly informal. The reader can easily provide the missing details.

We work with predicate languages whose set of logical symbols contains the connectives in $\{+,\wedge\}\cup\mathbb{Q}$. Each rational number is a unary connective; $+$ and $\wedge$ are binary connectives. Among the logical symbols there are two logical constants: $\mathbf{0}$ and $\mathbf{1}$ (not to be confused with the unary connectives 0 and 1) and a quantifier $\inf_x$, for each variable x from a fixed, countably infinite set V of variables. Notice the absence of equality among the logical symbols. We also point out that we could choose to have all the reals among the connectives. However, for sake of compatibility, we stick to [2].

As is customary, we identify a language L with the set of its extralogical symbols, which are predicate and function symbols, each in some positive, finite number of arguments, or constant symbols. If s is an L-extralogical symbol, we denote its arity by n_s. Unless otherwise specified, from this point on we assume to work with a fixed language L.

Terms and atomic formulas are defined as in first order logic (without equality). In particular the logical constants $\mathbf{0}, \mathbf{1}$ are atomic formulas. The set of formulas is the least set S that contains the atomic formulas, is closed under application of the connectives and satisfies the property that, for each $\varphi \in S$ and each variable x, $\inf_x \varphi$ is in S. Notions like free or bound occurrence of a variable in a formula are defined as in classical first order logic, with $\inf_x$ playing the role of a quantifier.

We follow [2] and we write $-\phi$ for $(-1)\phi$ and $\phi - \psi$ for $\phi + (-\psi)$. We write $\phi \vee \psi$ for $-(-\phi \wedge -\psi)$. As customary in the theory of lattice ordered structures, ϕ^+ and ϕ^- are abbreviations for $\mathbf{0} \vee \phi$ and $\mathbf{0} \vee (-\phi)$ respectively. We write $|\phi|$ for $\phi \vee (-\phi)$. We introduce $\sup_x \varphi$ as an abbreviation for $-\inf_x -\varphi$.

Next we define the class of structures for $\mathbb{R}$-logic. An $\mathbb{R}$-structure (for short: structure) is a pair $M = (M, \_^M)$ (notice the minor notational abuse), where M is a nonempty set and $\_^M$ is a function that maps

(1) each predicate symbol P to a function
$$P^M : M^{n_P} \to [-1, 1];$$
(2) each function symbol f to a function $f^M : M^{n_f} \to M$;
(3) each constant symbol c to some element c^M of M;

As usual, the cardinality of a structure M is the cardinality $|M|$ of its supporting set.

Notice that, differently from continuous logic, we do not assume that M is endowed with a metric space structure. Consequently, there is no continuity requirement on the interpretations of predicate symbols.

Let M be a structure and let $a : V \to M$ be an assignment of values to the variables. If $x \in V$ and $m \in M$ we denote by $a(x/m)$ the assignment a' such that $a'(x) = m$ and is defined as a elsewhere. We let $M_a = (M, a)$. The interpretation $t^{M_a} \in M$ of a term t in M under a is defined as in first order logic. We recursively define the truth value φ^{M_a} of a formula φ as follows:

0. $\mathbf{0}^{M_a} = 0$;
1. $\mathbf{1}^{M_a} = 1$;
2. $P(t_1, \ldots, t_{n_P})^{M_a} = P^M(t_1^{M_a}, \ldots, t_{n_P}^{M_a})$;
3. $(q\psi)^{M_a} = q(\psi^{M_a})$.

4. $(\psi + \xi)^{M_a} = \psi^{M_a} + \xi^{M_a}$;
5. $(\psi \wedge \xi)^{M_a} = \min(\psi^{M_a}, \xi^{M_a})$;
6. $(\inf_x \psi)^{M_a} = \inf\{\psi^{M_{a(x/m)}} : m \in M\}$.

In order to ensure correctness of clause 6, we must prove that the set $\{\psi^{M_{a(x/m)}} : m \in M\}$ is bounded from below. Actually, an easy induction on formulas establishes the stronger result that for every formula φ there exists $n \in \mathbb{N}$ such that, for all $a : V \to M$, $-n \leq \varphi^{M_a} \leq n$. (See Proposition 5 below.)

We say that a formula φ is *satisfiable in M under a* if $0 \leq \varphi^{M_a}$. In this case we use the standard notation $M_a \models \varphi$. The meaning of $M_a \models \Gamma$, where Γ is a set of formulas, and the logical consequence relation are defined (and denoted) accordingly. Useful notations are $M_a \models \varphi = \psi$ for $\varphi^{M_a} = \psi^{M_a}$ and $M_a \models \varphi < \psi$ for $\varphi^{M_a} < \psi^{M_a}$.

One can easily prove that the truth value of a formula in a structure only depends on the assignment of values to the free variables occurring in that formula. One can also prove most of the standard results that hold for the semantics of first order logic. For this reason we feel free to use standard model theoretic notation and terminology, without further notice.

Remark 1. The previous framework and most of the results to be proved in the following can be generalized by replacing the reals with some nontrivial (i.e. different from $\{0\}$) *Dedekind complete Riesz space with order unit* (see [1])

$$R = (R, +_R, \cdot, \wedge_R, \vee_R, <_R, 0_R, 1_R),$$

where $\cdot : \mathbb{R} \times R \to R$ is scalar multiplication; the operators $\wedge_R$ and $\vee_R$ are the binary infimum and supremum respectively; 0_R is the null vector and 1_R is a designated (strong) order unit in R. Recall that an element $0 \leq_R e$ in R is an order unit if for each $r \in R$ there exists $n \in \mathbb{N}$ such that $|r| \leq ne$, where, as usual, $|r| = r \vee_R (-r)$.

The Dedekind completeness property states the existence of the infimum of each nonempty bounded from below subsets of R. This suffices for the existence of the supremum of each nonempty bounded from above subset of R. Furthermore, denoting by $\bigwedge_R$ and by $\bigvee_R$ the infimum and the supremum operators on subsets respectively, then $\bigvee_R X = -\bigwedge_R(-X)$, whenever one of the two members is defined.

A notion related to Dedekind completeness is the *Archimedean property*. A Riesz space R is Archimedean if, for every positive element $r \in R$, the infimum of the set $\{\frac{1}{n}r : n \in \mathbb{N}^+\}$ exists and is equal to 0_R. It can be easily shown that the Archimedean property is equivalent to the following: for all $0_R \leq_R r \leq_R s$ in R, if $nr \leq_R s$ for all $n \in \mathbb{N}$, then $r = 0$. Dedekind completeness implies the Archimedean property.

We recall that every Archimedean Riesz space has a Dedekind completion that is uniquely determined up to lattice isomorphism (see [9, p. 191]).

As above, an R-structure is a pair. The function $\_^M$ is R-valued. The rest is as in the real case, with the obvious changes. We just point out that

R-valued formulas may take incomparable truth values in an arbitrary Riesz space R.

3. Logical axioms and derivations

In the following, $\mathbb{Q}^+$ denotes the set of nonnegative rationals.

A *generalization* of a formula φ is a formula of the form $\inf_{x_1} \dots \inf_{x_n} \varphi$, where $x_1, \dots, x_n$ are any variables, not necessarily distinct.

Same as with first order logic, we denote by $\varphi[t/x]$ the formula obtained by substituting a term t for a variable x in a formula φ. We say that t is free for x in φ if no variable y in t is captured by an $\inf_y$ or a $\sup_y$ quantifier in $\varphi[t/x]$. In such case we say that the substitution $\varphi[t/x]$ is correct. One can easily give a recursive definition of term free for a variable in a formula.

From now on, when we write a substitution, this is implicitly assumed to be correct.

Remark 2. As in first order logic, the following holds: for every structure M, every $a : V \to M$, every term t, variable x and formula φ with t free for x in φ, if $m = t^{M_a}$ then

$$\varphi[t/x]^{M_a} = \varphi^{M_{a(x/m)}}.$$

The proof of the equality above is by straightforward induction on φ, using a recursive definition of "t free for x in φ".

We are now ready to introduce the logical axioms. For better understanding and for sake of pointing out that $\mathbb{R}$-logic is an extension of of the propositional $\mathbb{R}$-logic, we write the axioms in form of equalities or inequalities. The inequality $\varphi \leq \psi$ stands for the formula $\psi - \varphi$ (see the remark preceding [2, Theorem 3.2]). The equality $\varphi = \psi$ stands for the formula $(\varphi - \psi) \wedge (\psi - \varphi)$. We freely use abbreviations like $\varphi \leq \psi \leq \eta$, whose meaning is self-explanatory. Equalities and inequalities will be used whenever convenient. This is certainly the case with the formulation of the axioms and the rules below.

Convention. Whenever no confusion arises, from now on we will denote the logical constants $\mathbf{0}$ and $\mathbf{1}$ by 0 and 1 respectively and we will abbreviate $r\mathbf{1}$ with r.

The $\mathbb{R}$-logic has four axiom groups. The first two groups are just the corresponding group axioms of [2].

The axioms from the first group are chosen having in mind the theory of vector spaces over $\mathbb{Q}$. They are the following:

a1. $\phi + \psi = \psi + \phi$	a5. $0\,\phi = 0$
a2. $(\phi + \psi) + \xi = \psi + (\phi + \xi)$	a6. $r\,\phi + s\,\phi = (s + r)\,\phi$
a3. $\phi + 0 = \phi$	a7. $r\,\phi + r\,\psi = r\,(\phi + \psi)$
a4. $1\,\phi = \phi$	a8. $r\,(s\,\phi) = (rs)\,\phi$

The axioms from the second group are inspired by the theory of Riesz spaces. They are the following:

a9. $\phi \wedge \phi = \phi$
a10. $\phi \wedge \psi = \psi \wedge \phi$
a11. $(\phi \wedge \psi) \wedge \xi = \phi \wedge (\psi \wedge \xi)$
a12. $(\phi + \xi) \wedge (\psi + \xi) = \phi \wedge \psi + \xi$
a13. $r(\phi \wedge \psi) = (r\phi) \wedge (r\psi)$, $r \in \mathbb{Q}^+$
a14. $\phi \wedge \psi \leq \psi$

The axioms from the third group ensure that Proposition 5 below holds. Axioms a15 and a16 below are the predicate versions of axiom a15 in [2].

For every predicate symbol P we have the following axioms:

a15. $-1 \leq P(x_1, \dots, x_{n_P})$
a16. $P(x_1, \dots, x_{n_P}) \leq 1$
a17. $0 \leq 1$

The last group of axioms deals with the quantifiers and with their behavior with respect to the connectives. The axioms from this group are the following, where x is any variable:

a18. $\inf_x \varphi \leq \varphi[t/x]$, whenever the substitution is correct.
a19. $\varphi \leq \inf_x \varphi$ if x does not occur free in φ.
a20. $\inf_x(\varphi + \psi) = (\inf_x \varphi) + \psi$ if x does not occur free in ψ.
a21. $\inf_x r\varphi = r \inf_x \varphi$, for all $r \in \mathbb{Q}^+$.

The inference rules are listed below. The non-replaceability of rules r2, r3 below with logical axioms is discussed in [2].

r1. (Modus Ponens)

$$\frac{\varphi \qquad \varphi \leq \psi}{\psi}$$

r2. (Positive Homogeneity)

$$\frac{\varphi}{r\varphi} \qquad \text{for } r \in \mathbb{Q}^+$$

r3. (Restriction)

$$\frac{\varphi \leq \psi}{\varphi \wedge 0 \leq \psi \wedge 0}$$

r4. (Generalization)

$$\frac{\varphi}{\inf_x \varphi}$$

if x does not occur free in any of the premises, different from axioms, on which the derivation of φ depends

The notion of derivation is the standard one in natural deduction systems with axioms. Equivalently, $\mathbb{R}$-logic can be formulated in Hilbert style. Our choice is just a matter of slight convenience.

As usual, we write $\Gamma \vdash \varphi$ if there exists a derivation of φ whose set of assumptions is included in Γ. When speaking of the assumptions in a derivation, we always mean the extralogical ones.

Let φ be a formula. A *generalization* of φ is a formula of the form $\inf_{x_1} \dots \inf_{x_n} \varphi$, for some variables $x_1, \dots, x_n$.

Remark 3. Rule r4 implies that if ψ is a generalization of some axiom then ψ is a logical theorem, namely $\vdash \psi$.

Notice that the deduction rules in [2] can be easily shown to be equivalent to r1 − r3 above. In particular, concerning the rule of Positive Linearity from [2], for each positive rational r we can produce the following sketch of a derivation (as is customary, a double line stands for a bit of derivation that is not fully detailed):

$$\dfrac{\dfrac{\varphi \leq \psi}{\varphi + r^{-1}\eta \leq \psi + r^{-1}\eta}}{r\varphi + \eta \leq r\psi + \eta}$$

The proof sketch above can be easily expanded to a fully detailed derivation from the vector space axioms, by using rules r1 and r2. It follows that, for all $r \in \mathbb{Q}^+$,

$$\frac{\varphi \leq \psi}{r\varphi + \eta \leq r\psi + \eta}$$

is a derived rule. Conversely, one can easily derive r2 from the latter.

Since $\mathbb{R}$-logic extends the propositional $\mathbb{R}$-logic of [2], we will freely use [2, Theorem 5.3] to establish whether two formulas are provably equivalent.

Remark 4. Rule r4 is replaceable by axioms. Actually, let $\mathbb{R}'$-logic be $\mathbb{R}$-logic without rule r4 and with the additional axioms

a22. $\varphi \wedge 0 + \psi \wedge 0 \leq (\varphi + \psi) \wedge 0$
a23. $\inf_x \varphi + \inf_x \psi \leq \inf_x(\varphi + \psi)$
a24. $\inf_x(\varphi \wedge 0) = (\inf_x \varphi) \wedge 0$, where x is any variable.
a25. all generalizations of the axioms a1 − a24.

Let $\vdash'$ the provability relation in $\mathbb{R}'$-logic. We claim that, for every set $\Gamma \cup \{\varphi\}$ of formulas,

$$\Gamma \vdash \varphi \iff \Gamma \vdash' \varphi.$$

To get the right-to-left implication, we prove that each instance of a22 − a25 is a theorem of $\mathbb{R}$-logic. As for a22, this follows from an application of [2, Proposition 5.2]. Provability of a23 − a25 is a matter of routine, with the help of [2, Proposition 3.1].

Concerning the converse implication, it suffices to show that, for every set $\Gamma \cup \{\varphi\}$ of formulas and every variable x not occurring free in Γ, if $\Gamma \vdash' \varphi$ then $\Gamma \vdash' \inf_x \varphi$. The proof is by induction on a derivation of φ from Γ in $\mathbb{R}'$-logic. The only nontrivial case is when the last rule applied is r3. It is convenient to deal with formulas. By inductive assumption we get $\Gamma \vdash' \inf_x(\psi - \varphi)$. It follows by a24 that $\Gamma \vdash' \inf_x((\psi - \varphi) \wedge 0)$. By a22, a25 and a23 we get $\Gamma \vdash' \inf_x((\psi - \varphi) \wedge 0) \leq \inf_x(\psi \wedge 0 - \varphi \wedge 0)$ and the conclusion follows by applying Modus Ponens.

Proposition 5. *For every formula φ there exists some natural number n such that* $\vdash -n \leq \varphi \leq n$.

Proof. By induction on formulas. The atomic cases are taken care by the axioms from the third group. Here we only show that

$$\vdash -1 \leq P(t_1, \ldots, t_{n_P}) \leq 1,$$

where P is a predicate symbol and $t_1, \ldots, t_{n_P}$ are terms. By a15, recalling Remark 3, we have $\vdash \inf_{x_1} \ldots \inf_{x_{n_P}} (P(x_1, x_2, \ldots, x_{n_P}) + 1)$. From a18 and the vector space axioms, by applying r1 we get:

$$\begin{array}{c} \inf\limits_{x_1} \ldots \inf\limits_{x_{n_P}} (P(x_1, x_2, \ldots, x_{n_P}) + 1) \\ \hline\hline \inf\limits_{x_2} \ldots \inf\limits_{x_{n_P}} (P(t_1, x_2, \ldots, x_{n_P}) + 1) \\ \hline\hline -1 \leq P(t_1, \ldots, t_{n_P}) \end{array}$$

Applying the same argument, we get the other inequality. Finally, by [2, Proposition 3.1] we get the conclusion.

All nonatomic cases are straightforward. In particular, if φ is of the form $\inf_x \psi$ and, for some $n \in \mathbb{N}$, $\vdash -n \leq \psi \leq n$, then we get $\vdash -n \leq \inf_x \psi$ from $\vdash -n \leq \psi$ by r4 and a20. Moreover, from $\psi \leq n$ we get $\vdash \inf_x \psi \leq n$ by a18, the Riesz space axioms and r1. □

In the following we will provide only sketches of the formal derivations that are required in some of the proofs. The reader can easily figure out the missing details.

Corollary 6. *For every formula* φ, $\; -1 \vdash \varphi$.

Proof. By Proposition 5, let $n \in \mathbb{N}$ be such that $\vdash -n \leq \varphi$. Then

$$\dfrac{\begin{array}{c} -1 \\ \hline\hline -n \end{array} \quad -n \leq \varphi}{\varphi}$$

□

The content of the previous corollary is that -1 plays the role of a contradiction.

From this point on, Γ denotes some set of formulas and φ, ψ, η denote formulas.

Theorem 7. *(Soundness Theorem) If* $\Gamma \vdash \varphi$ *then* $\Gamma \models \varphi$.

Proof. First of all notice that all the axioms are valid and that all the rules do preserve validity. Then argue by induction on a derivation of φ from Γ. □

Our aim is to prove, under suitable assumptions, the converse of the Soundness Theorem. In order to do that we need to establish a series of preliminary results. So doing, we will freely refer to the results in [2], whenever applicable.

Theorem 8. *(Deduction Theorem.) The following are equivalent:*

(1) $\Gamma, \varphi \vdash \psi$;
(2) *there exists* $r \in \mathbb{Q}^+$ *such that* $\Gamma \vdash \psi + r\varphi^-$.

Proof. (2) $\Rightarrow$ (1) follows easily from $\varphi \vdash \varphi^- = 0$ (see [2, Proposition 3.5]).

To show (1) $\Rightarrow$ (2) we argue by induction on a derivation $\mathcal{D}$ of ψ whose set of assumptions is included in $\Gamma \cup \{\varphi\}$. If $\mathcal{D}$ is atomic or the last rule applied in $\mathcal{D}$ is one of r1–r3, basically the proof is that of [2, Theorem 3.6]. The only remaining case is when the last rule applied in $\mathcal{D}$ is r4. Hence ψ is of the form $\inf_x \eta$, for some formula η, and $\mathcal{D}$ is as follows:

$$\cfrac{\begin{array}{c}\Gamma, \varphi \\ \vdots\, \mathcal{D}_1 \\ \eta\end{array}}{\inf_x \eta}$$

By the inductive assumption applied to the subderivation $\mathcal{D}_1$, there exists some $r \in \mathbb{Q}^+$ such that $\Gamma \vdash \eta + r\varphi^-$.

If the assumption φ is not used in in $\mathcal{D}_1$, then $\Gamma \vdash \inf_x \eta$. Since $\vdash \varphi^-$, it follows that $\Gamma \vdash \inf_x \eta + r\varphi^-$.

If φ is used in $\mathcal{D}_1$ then, by the side condition of r4, the variable x does not occur free neither in φ^- nor in any of the formulas from Γ that are effectively used in $\mathcal{D}_1$. Therefore from $\Gamma \vdash \eta + r\varphi^-$ we get that $\Gamma \vdash \inf_x(\eta + r\varphi^-)$. Then, by a20, $\Gamma \vdash (\inf_x \eta) + r\varphi^-$.

□

We can now state the following:

Proposition 9. *The following are equivalent:*

(1) $\Gamma \vdash \psi$;
(2) $\Gamma, \varphi \vdash \psi$ *and* $\Gamma, -\varphi \vdash \psi$.

Proof. Same as the proof of [2, Proposition 3.7].

□

The following is a series of syntactic results that will be useful later on.

Lemma 10. *For all* $x, y \in V$

$$\vdash \inf_x \inf_y \varphi = \inf_y \inf_x \varphi.$$

Proof. $\vdash \inf_x \inf_y \varphi \leq \inf_y \varphi$ and $\vdash \inf_y \varphi \leq \varphi$ are instances of a18. Therefore $\vdash \inf_x \inf_y \varphi \leq \varphi$, by r1. A double application of r4 and a20 yields $\vdash \inf_x \inf_y \varphi \leq \inf_y \inf_x \varphi$. By swapping x and y we are done. □

Remark 11. If $\vdash \psi = \eta$ then $\vdash \inf_x \psi = \inf_x \eta$, for each variable x. We just show one inequality, the other can be symmetrically proved:

$$\vdash \eta \leq \psi \;\Rightarrow \vdash \inf_x \eta \leq \psi \Rightarrow \vdash \inf_x \eta \leq \inf_x \psi$$

When saying that some symbol S does not occur in a formula φ, we mean that there is no occurrence at all of S in φ. In such case we often write $S \notin \varphi$. If there is some free occurrence of the variable x in φ we write $x \in \mathrm{FV}(\varphi)$. We extend the notation above to a set of formulas, with the obvious meaning.

We write $\varphi \equiv \psi$ to say that φ and ψ are syntactically the same formula.

We write $\varphi\{x/c\}$ for the formula obtained by replacing each occurrence of the constant symbol c with the variable x in the formula φ.

Lemma 12. *Let $x, y \notin \varphi$ and let c be a constant symbol. Then*

$$\vdash \inf_x \varphi\{x/c\} = \inf_y \varphi\{y/c\}.$$

Proof. To get $\vdash \inf_x \varphi\{x/c\} \leq \inf_y \varphi\{y/c\}$, notice that y is free for x in $\varphi\{x/c\}$ and that $\varphi\{x/c\}[y/x] \equiv \varphi\{y/c\}$. Then use a18, r4 and a20. □

Lemma 13. *Let $y \notin \varphi$. Then*

$$\vdash \inf_x \varphi = \inf_y \varphi[y/x].$$

Proof. We only prove $\vdash \inf_y \varphi[y/x] \leq \inf_x \varphi$.

Since x is free for y in $\varphi[y/x]$ then $\vdash \inf_y \varphi[y/x] \leq \varphi[y/x][x/y]$. Notice that $\varphi[y/x][x/y] \equiv \varphi$, by the assumption that y does not occur in φ. Therefore $\vdash \inf_y \varphi[y/x] \leq \varphi$. The conclusion follows from application of r4 and from a20. □

Remark 14. An immediate consequence of Lemma 13 is that for any formula φ and any finite set X of variables there exists a formula φ' such that:

(1) φ and φ' differ only for the names of some bound variables;
(2) $\vdash \varphi = \varphi'$;
(3) the set of bound variables of φ' is disjoint from X.

Any φ' satisfying (1) and (2) above will be called an *alphabetic variant* of φ.

Theorem 15. *If $\Gamma \vdash \varphi$, $x \notin \mathrm{FV}(\Gamma)$, $x \notin \varphi$ and $c \notin \Gamma$ then*

$$\Gamma \vdash \inf_x \varphi\{x/c\}.$$

Proof. By induction on a derivation $\mathcal{D}$ of φ from Γ.

(1) $\mathcal{D}$ is atomic. We examine two subcases. Subcase 1: $\varphi \in \Gamma$. Then $c \notin \varphi$ and so $\varphi\{x/c\} \equiv \varphi$. Since $x \notin \mathrm{FV}(\Gamma)$ then $\Gamma \vdash \inf_x \varphi$. Subcase 2: φ is an axiom. Then $\varphi\{x/c\}$ is an axiom, since $x \notin \varphi$. Therefore $\vdash \inf_x \varphi\{x/c\}$.
(2) $\mathcal{D}$ ends with an application of $r1$. Then $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\, \mathcal{D}_1 \\ \psi\end{array} \qquad \begin{array}{c}\Gamma \\ \vdots\, \mathcal{D}_2 \\ \psi \leq \varphi\end{array}}{\varphi}$$

for some formula ψ. Let y be a variable such that $y \notin \mathrm{FV}(\Gamma)$ and $y \notin (\psi \leq \varphi)$. By inductive assumption applied to $\mathcal{D}_1$ and $\mathcal{D}_2$ we get $\Gamma \vdash \inf_y \psi\{y/c\}$ and $\Gamma \vdash \inf_y(\varphi\{y/c\} - \psi\{y/c\})$ respectively. Hence $\Gamma \vdash \psi\{y/c\}$ and $\Gamma \vdash \psi\{y/c\} \leq \varphi\{y/c\}$. By r1, $\Gamma \vdash \varphi\{y/c\}$. Since $y \notin \mathrm{FV}(\Gamma)$ then $\Gamma \vdash \inf_y \varphi\{y/c\}$. By Lemma 12, $\Gamma \vdash \inf_x \varphi\{x/c\}$.

(3) $\mathcal{D}$ ends with an application of $r2$. Then $\varphi \equiv r\psi$, for some $r \in \mathbb{Q}^+$ and $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\,\mathcal{D}_1\\ \psi\end{array}}{r\psi}$$

By inductive assumption applied to $\mathcal{D}_1$, we get $\Gamma \vdash \inf_x \psi\{x/c\}$. Rule r2, axiom a21 and $(r\psi)\{x/c\} \equiv r(\psi\{x/c\})$ yield the conclusion.

(4) $\mathcal{D}$ ends with an application of $r3$. Then $\varphi \equiv \eta \wedge 0 \leq \psi \wedge 0$, for some formulas ψ, η and $x \notin (\eta \leq \psi)$ and $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\,\mathcal{D}_1\\ \eta \leq \psi\end{array}}{\eta \wedge 0 \leq \psi \wedge 0}$$

By inductive assumption applied to $\mathcal{D}_1$, we get $\Gamma \vdash \inf_x (\psi\{x/c\} - \eta\{x/c\})$. Therefore $\Gamma \vdash \psi\{x/c\} - \eta\{x/c\}$. By r3 and r4 we get $\Gamma \vdash \inf_x (\psi\{x/c\} \wedge 0 - \eta\{x/c\} \wedge 0)$. Moreover $(\psi \wedge 0 - \eta \wedge 0)\{x/c\} \equiv \psi\{x/c\} \wedge 0 - \eta\{x/c\} \wedge 0$, hence the conclusion.

(5) $\mathcal{D}$ ends with an application of $r4$. Then $\varphi \equiv \inf_y \psi$, for some variable y and some formula ψ, and $\mathcal{D}$ is of the form

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\,\mathcal{D}_1\\ \psi\end{array}}{\inf_y \psi}$$

Notice that $x \not\equiv y$. Since $x \notin \psi$ we can apply the inductive assumption to Γ, ψ and $\mathcal{D}_1$ to get $\Gamma \vdash \inf_x \psi\{x/c\}$. Since y does not occur free in Γ, then $\Gamma \vdash \inf_y \inf_x \psi\{x/c\}$. By Lemma 10 and by $\inf_y \psi\{x/c\} \equiv (\inf_y \psi)\{x/c\}$ we are done.

□

4. Completeness

From now on, Γ denotes a set of formulas. We recall some notions and related results from [2].

As usual, we say that Γ is *satisfiable* if there exist a structure M and an assignment $a : V \to M$ such that $M_a \models \Gamma$.

We say that Γ is *consistent* if $\Gamma \nvdash \varphi$, for some formula φ. By Corollary 6, Γ is consistent if and only if $\Gamma \nvdash -1$.

We say that Γ is *total* if it is consistent and, for every formula φ, $\Gamma \vdash \varphi$ or $\Gamma \vdash -\varphi$, possibly both. Notice that the deductive closure of a total set Γ induces a preorder on the set of formulas.

Proposition 16. *Every consistent set Γ has a maximal consistent extension. Moreover every maximal consistent extension of Γ is closed under provability and total.*

Proof. The existence of some maximal consistent extension Γ' of Γ follows from a routine application of Zorn's Lemma. Clearly, if $\Gamma' \vdash \varphi$ then $\varphi \in \Gamma'$.

If, for some formula φ, none of $\varphi, -\varphi$ is in Γ' then, by maximal consistency, $\Gamma', \varphi \vdash -1$ and $\Gamma', -\varphi \vdash -1$. Hence, by Proposition 9, $\Gamma' \vdash -1$. It follows that at least one of $\varphi, -\varphi$ is in Γ'. Hence Γ' is total.

□

Remark 17. If $\Gamma \nvdash \varphi$ then, by Proposition 9, $\Gamma, -\varphi$ is consistent. In the following we shall repeatedly use this this fact, often without further mention.

We write $\mathbb{Q}^+\varphi$ for the set $\{r\varphi : r \in \mathbb{Q}^+\}$.

Proposition 18. *If Γ is consistent and $\Gamma \vdash \mathbb{Q}^+\varphi \leq \psi$ then $\Gamma, -\varphi$ is consistent.*

Proof. Same as the proof of [2, Proposition 6.4]. □

A formula ξ is a *unit* for Γ if, for every φ, there is some $r \in \mathbb{Q}^+$ such that $\Gamma \vdash \varphi \leq r\xi$. Notice that the logical constant 1 is a unit for every set of formulas, as a consequence of Proposition 5.

Definition 19. *A set Γ of formulas is Archimedean if for every φ, ψ such that $\Gamma \vdash \mathbb{Q}^+\varphi \leq \psi$ then $\Gamma \vdash -\varphi$.*

Clearly, for every R-structure M and every assignment $a : V \to M$, the set $\{\varphi : M_a \models \varphi\}$ is Archimedean.

We let $\Gamma^+ = \{\varphi : \Gamma \vdash \varphi\}$ and $\Gamma^- = \{\varphi : \Gamma \vdash -\varphi\}$.

For sake of completeness, we provide a detailed proof of the following result from [2].

Proposition 20. *The following are equivalent:*

(1) Γ *is maximal consistent;*
(2) Γ *is closed under deduction, total and Archimedean;*
(3) Γ *is closed under deduction, total and every $\xi \in \Gamma^+ \setminus \Gamma^-$ is a unit for T.*

Proof. (1) $\Rightarrow$ (2). By Proposition 16, it suffices to show that Γ is Archimedean. This follows at once from Proposition 18 by maximal consistency.
(2) $\Rightarrow$ (3). Assume (2) and, for sake of contradiction, let $\xi \in \Gamma^+ \setminus \Gamma^-$ which is not a unit. Then there is some formula φ such that, for all $r \in \mathbb{Q}^+$,

$\Gamma \not\vdash \varphi \leq r\xi$. Being total, $\Gamma \vdash \mathbb{Q}^+\xi \leq \varphi$. Since Γ is Archimedean, $\Gamma \vdash -\xi$: a contradiction.
(3) $\Rightarrow$ (1). Assume (3). Let $\xi \notin \Gamma$. Totality and closure under deduction imply that $-\xi \in \Gamma$. Hence, by assumption, $-\xi$ is a unit for Γ. Therefore for every φ there exists $r \in \mathbb{Q}^+$ such that $\Gamma \vdash r\xi \leq -\varphi$. From $\Gamma, \xi \vdash r\xi$ we get $\Gamma, \xi \vdash \varphi$ for every φ. Hence $\Gamma \cup \{\xi\}$ is inconsistent. □

Our next aim is to prove that every consistent set of formulas is satisfiable. We use a Henkin style construction.

Definition 21. *Let Γ be a set of L-formulas. We say that Γ has the Henkin property in L if, for every formula φ and every variable x, there exists some L-constant symbol c such that $(\varphi[c/x] \leq \inf_x \varphi) \in \Gamma$.*

As is customary, the cardinality of a language L (denoted by $|L|$) is the maximum between the cardinality of the L-extralogical symbols and $\aleph_0$.

Proposition 22. *Let Γ be a set of L-formulas. Then there exist a language L' extending L and a set Γ' of L'-formulas extending Γ with the properties that:*

(1) $|L| = |L'|$;
(2) Γ' *has the Henkin property in* L';
(3) *if* Γ *is consistent then* Γ' *is consistent.*

Proof. We recursively define an increasing sequence $(L_n)_{n\in\omega}$ of languages and an increasing sequence $(\Gamma_n)_{n\in\omega}$, where each Γ_n is a set of L_n-formulas, as follows:

$$\begin{aligned} L_0 &= L \\ L_{n+1} &= L_n \cup \{c_{\varphi,x} : \varphi \text{ is an } L_n\text{-formula and } x \in V\}, \end{aligned}$$

where each $c_{\varphi,x}$ is a constant symbol not occurring in L_n.

$$\begin{aligned} \Gamma_0 &= \Gamma \\ \Gamma_{n+1} &= \Gamma_n \cup \{\varphi[c_{\varphi,x}/x] \leq \inf_x \varphi : \varphi \text{ is an } L_n\text{-formula and } x \in V\}. \end{aligned}$$

Finally, we let $L' = \bigcup_{n\in\omega} L_n$ and $\Gamma' = \bigcup_{n\in\omega} \Gamma_n$.

Properties (1) and (2) are clearly satisfied, so we deal with (3). It suffices to show that if Γ_n is consistent, then Γ_{n+1} is consistent, for all $n \in \omega$. Without loss of generality, we assume $n = 0$. Let m be the smallest cardinality of a set $\Delta \subset \Gamma_1 \setminus \Gamma$ such that $\Gamma \cup \Delta$ is inconsistent. Let Δ of cardinality m be as above, say

$$\Delta = \{\varphi[c_{\varphi_i,x_i}/x_i] \leq \inf_{x_i} \varphi_i : 1 \leq i \leq m\}$$

Let ψ_i be $\varphi[c_{\varphi_i,x_i}/x_i] \leq \inf_{x_i} \varphi_i$, for $2 \leq i \leq m$. For notational simplicity let us write $\varphi[c/x] \leq \inf_x \varphi$ for ψ_1 and let us call it ψ.

Therefore $\Gamma, \psi, \psi_2, \ldots, \psi_m, \vdash -1$. By the Deduction Theorem, there exists $r \in \mathbb{Q}^+$ such that $\Gamma, \psi_2, \ldots, \psi_m \vdash 1 \leq r\psi^-$, where ψ^- is $(\varphi[c/x] - \inf_x \varphi) \vee 0$.

By axiom a18 and [2, Proposition 3.1] we get

$$\vdash (\varphi[c/x] - \inf_x \varphi) \vee 0 = \varphi[c/x] - \inf_x \varphi.$$

Therefore $\Gamma, \psi_2, \ldots, \psi_m \vdash 1 \leq r(\varphi[c/x] - \inf_x \varphi)$. Let y be a variable such that $y \notin \mathrm{FV}(\Gamma \cup \{\psi_2, \ldots, \psi_m\})$ and $y \notin (\varphi[c/x] - \inf_x \varphi)$. Since c does not occur in $\Gamma, \psi_2, \ldots, \psi_m, \varphi$, by Theorem 15 and by axioms, we get $\Gamma, \psi_2, \ldots, \psi_m \vdash 1 \leq r(\inf_y \varphi[c/x]\{y/c\} - \inf_x \varphi)$. Moreover $\varphi[c/x]\{y/c\} \equiv \varphi[y/x]$ (recall that $c \notin \varphi$ and notice that y is free for x in φ). By Lemma 13 and by axioms we get $\Gamma, \psi_2, \ldots, \psi_m \vdash -1$, contradicting the minimality of m.

□

Corollary 23. *Let Γ be a consistent set of L-formulas. Then there exist a language L' extending L, of the same cardinality as L, and a set Γ' of L'-formulas with the property that Γ' is maximal consistent and has the Henkin property in L'.*

Proof. Apply Proposition 22 and notice that the Henkin property is preserved by extension to a maximal consistent superset. □

The next result is crucial to get completeness results.

Theorem 24. *Let Γ be maximal consistent and with the Henkin property in L. Then there exist an $\mathbb{R}$-structure M and an assignment $a : V \to M$ such that $M_a \models \Gamma$. Furthermore M can be chosen such that $|M| \leq |L|$.*

Proof. We proceed as in the proof of [2, Theorem 6.7]. Notice that, by Proposition 20, Γ is total and Archimedean.

If φ is a formula we let

$$\Gamma_\varphi = \{r \in \mathbb{Q} : \Gamma \vdash \varphi \leq r\} \quad \text{and} \quad \Gamma^\varphi = \{r \in \mathbb{Q} : \Gamma \vdash r \leq \varphi\}.$$

Totality and the Archimedean property of Γ imply that, for every formula φ,

(1) $\emptyset \neq \Gamma^\varphi \leq \Gamma_\varphi \neq \emptyset$;
(2) Γ_φ is bounded from below and Γ^φ is bounded from above;
(3) $\inf \Gamma_\varphi = \sup \Gamma^\varphi$.

Let M be the set of L-terms and let $a : V \to M$ be the inclusion map. For every function symbol f we define $f^M : M^{n_f} \to M$ as follows: $f^M(t_1, \ldots, t_{n_f}) = f(t_1, \ldots, t_{n_f})$ and, for every constant symbol c, we let $c^M = c$. An easy induction on terms shows that, for every term t, $t^{M_a} = t$.

For every atomic formula φ, we let

$$\varphi^M = \inf \Gamma_\varphi = \sup \Gamma^\varphi \tag{4}$$

So doing, we actually define the interpretation P^M, for each predicate symbol P. Moreover $\inf \Gamma_0 = 0$ and $\inf \Gamma_1 = 1$, in accordance with the semantics of the logical constants $0, 1$ respectively.

We prove by induction that (4) extends to all formulas. The propositional cases are as in [2, Theorem 6.7].

We are left with the case when φ is of the form $\inf_x \psi$.

For each term t, let ψ_t be an alphabetic variant of ψ such that t is free for x in ψ_t. By Remark 14, Remark 2 and by the Soundness Theorem, we get $\psi^{M_{a(x/t)}} = \psi_t^{M_{a(x/t)}} = (\psi_t[t/x])^{M_a}$, for all $t \in M$. Therefore

$$\begin{aligned}(\inf_x \psi)^{M_a} &= \inf\{\psi^{M_{a(x/t)}} :\ t \in M\}\\ &= \inf\{(\psi_t[t/x])^{M_a} :\ t \in M\}\\ &= \inf\{\inf\{r \in \mathbb{Q} :\ \Gamma \vdash \psi_t[t/x] \leq r\} : t \in M\}\\ &= \inf(\textstyle\bigcup_{t\in M}\{r \in \mathbb{Q} :\ \Gamma \vdash \psi_t[t/x] \leq r\})\\ &\geq \inf\{r \in \mathbb{Q} :\ \Gamma \vdash \inf_x \psi \leq r)\end{aligned}$$

where the third equality from top follows from the inductive assumption and the inequality follows from a18 and from $\vdash \inf_x \psi = \inf_x \psi_t$ (recall that $\vdash \psi = \psi_t$ and Remark 11), for all $t \in M$.

Next we prove the reverse inequality. By the Henkin property there exists some constant symbol c such that $(\psi[c/x] \leq \inf_x \psi) \in \Gamma$. It follows by a18 that $\Gamma \vdash \psi[c/x] = \inf_x \psi$. Therefore, by inductive assumption,

$$\psi[c/x]^{M_a} = \inf\{r \in \mathbb{Q} :\ \Gamma \vdash \inf_x \psi \leq r\}.$$

Since $(\inf_x \psi)^{M_a} \leq \psi^{M_{a(x/c)}} = \psi[c/x]^{M_a}$, we are done.

Finally, it is easy to check that $M_a \models \Gamma$. □

In the literature, the equivalence of consistency and satisfiability is often referred to as *weak completeness*.

Corollary 25. *(Weak Completeness Theorem) The following are equivalent for a set* Γ *of* L*-formulas:*

(1) Γ *is consistent;*
(2) *there exist an* $\mathbb{R}$*-structure* M *with the property that* $|M| \leq |L|$ *and an assignment* $a : V \to M$ *such that* $M_a \models \Gamma$.

A straightforward consequence of Corollary 25 is an approximated strong completeness theorem, in the vein of that proved in [6] for continuous logic.

Theorem 26. *(Approximated Strong Completeness Theorem) The following are equivalent for every set* $\Gamma \cup \{\varphi\}$ *of formulas:*

1. $\Gamma \vdash r \leq \varphi$ *for all* $0 > r \in \mathbb{Q}$*;*
2. $\Gamma \models \varphi$.

Proof. (1) ⇒ (2) follows from soundness and from the Archimedean property. As for (2) ⇒ (1), assume that $\Gamma \nvdash r \leq \varphi$, for some negative rational r. Hence $\Gamma \cup \{\varphi \leq r\}$ is consistent. Let $M \models \Gamma \cup \{\varphi \leq r\}$. Then $\Gamma \not\models \varphi$. □

Of course, another straightforward consequence of the Weak Completeness Theorem is the following:

Corollary 27. *(Compactness Theorem) A set of formulas is satisfiable if and only if it is finitely satisfiable.*

As in [2], Archimedean set of formulas are those for which a *strong completeness* theorem holds.

Theorem 28. *(Completeness Theorem for Archimedean sets.) For each* Γ *the following are equivalent:*

1. Γ *is Archimedean;*
2. *for every formula* φ*, if* $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$*.*

Proof. (1) $\Rightarrow$ (2). Suppose Γ is Archimedean. Let φ be such that $\Gamma \nvdash -\varphi$. Let $0 < r \in \mathbb{Q}^+$ be such that $\Gamma \nvdash r\varphi \leq 1$. Then $\Gamma \cup \{1 \leq r\varphi\}$ is consistent (see Remark 17). By Corollary 25, let M and $a : V \to M$ be such that $M_a \models \Gamma \cup \{1 \leq r\varphi\}$. It follows that $\Gamma \not\models -\varphi$.
(2) $\Rightarrow$ (1). Suppose that $\Gamma \nvdash -\varphi$, for some formula φ and let ψ be any formula. By (2) there exist a structure M and an assignment a such that $M_a \models \Gamma$ and $\varphi^{M_a} > 0$. By soundness, we get immediately $\Gamma \nvdash \mathbb{Q}^+\varphi \leq \psi$. □

The content of Theorem 28 is that, for Archimedean sets of formulas, $\mathbb{R}$-logic is strongly complete with respect to the class of structures.

5. The space of types

In this section we define the set of types and we endow it with a compact Hausdorff topology. Such topology has similarities with the *logic topology* defined in [5, §8] and shares many of the properties of the latter. Moreover, when the language is countable, it is metrizable. Hence a metric can be defined directly from $\mathbb{R}$-logic. At this point, it is worth recalling the framework of *compact abstract theories* introduced in [3] and the result by the same author that every countable Hausdorff compact abstract theory admits a metric [4]. A consequence of the latter is that, under reasonable assumptions, a classical structure admits a metric which is type-definable. In the following, we get an analogous result with respect to $\mathbb{R}$-logic. Ours is not a by-product of a general theory, but is a specific result to $\mathbb{R}$-logic, in the vein of those in [5, §8] for continuous logic. Similarities with the results in [5, §8] will be pointed out in the following.

We begin with a definition that is inspired by [11]. Some language L is fixed throughout.

Definition 29. *Let* Γ *be a consistent set of* L*-formulas and let* φ *be an* L*-formula.*

1. *The degree of truth of* φ *with respect to* Γ *(notation:* φ^T_Γ*) is defined as follows:*
$$\varphi^T_\Gamma = \inf\{\varphi^{M_a} : M \text{ is some structure, } a : V \to M \text{ and } M_a \models \Gamma\}.$$
2. *The degree of provability of* φ *with respect to* Γ *(notation:* φ^P_Γ*) is defined as follows:*
$$\varphi^P_\Gamma = \sup\{r \in \mathbb{Q} : \Gamma \vdash r \leq \varphi\}.$$

When Γ *is the empty set we omit the subscript.*

Regarding the previous definition, note that the consistency of Γ, Proposition 5 and the Soundness Theorem imply that the set

$$\{\varphi^{M_a} : M \text{ is a structure, } a : V \to M \text{ and } M_a \models \Gamma\}$$

is nonempty and bounded from below, hence φ^T_Γ exists and is finite. Similarly, the set $\{r \in \mathbb{Q} : \Gamma \vdash r \leq \varphi\}$ is nonempty and bounded from above.

The following instance of a so-called Pavelka-style completeness theorem (see [11]) is actually a corollary of the Weak Completeness Theorem.

Corollary 30. *Let Γ be a consistent set of L-formulas and φ be an L-formula. Then $\varphi^T_\Gamma = \varphi^P_\Gamma$.*

Proof. In order to prove $\varphi^T_\Gamma \leq \varphi^P_\Gamma$ it suffices to show that, for all $r \in \mathbb{Q}$, if $r > \varphi^P_\Gamma$ then $r \geq \varphi^T_\Gamma$. Let $r \in \mathbb{Q}$ be such that $r > \varphi^P_\Gamma$. Then $\Gamma \not\vdash r \leq \varphi$. By Remark 17, $\Gamma, \varphi \leq r$ is consistent. By the Weak Completeness Theorem, there exist a structure M and an assignment $a : V \to M$ such that $r \geq \varphi^{M_a}$. Therefore $r \geq \varphi^T_\Gamma$.

The inequality $\varphi^P_\Gamma \leq \varphi^T_\Gamma$ follows from the Soundness Theorem: let $r \in \mathbb{Q}$ be such that $\Gamma \vdash r \leq \varphi$. From $\Gamma \models r \leq \varphi$ we get at once $r \leq \varphi^T_\Gamma$.

□

Let L_n be the set of L-formulas whose free variables are among $x_1, \dots, x_n$.

In the current setting we can formulate the notion of *logical distance* introduced in [5]: if $\varphi, \psi \in L_n$ we let

$$d(\varphi, \psi) = \sup\{|\varphi - \psi|^{M_a} : M \text{ is some structure and } a : V \to M\}.$$

The logical distance is a pseudo metric. It is related to the notion of logical equivalence as follows: first of all let us say that φ, ψ are *logically equivalent* if, for all M and a as above, $\varphi^{M_a} = \psi^{M_a}$. Then φ, ψ are logically equivalent if and only if $d(\varphi, \psi) = 0$.

Clearly, the logical distance can be defined in terms of the degree of truth as follows:

$$d(\varphi, \psi) = -(-|\varphi - \psi|)^T$$

and, from Corollary 30, we get

$$d(\varphi, \psi) = \inf\{r \in \mathbb{Q} : \vdash |\varphi - \psi| \leq r\}.$$

Finally we define the notion of *n-type* and a topology on the set of types.

For simplicity we deal with parameter-free types only. In a slightly different form, the following arguments carry through even in presence of parameters.

From now on Γ will denote some maximal consistent set of L-sentences (equivalently: Γ is the set of L-sentences which are true in some L-structure).

Let $n \in \mathbb{N}$. An *n-type* is a set p of L_n-formulas for which there exist a structure M and an assignment $a : V \to M$ such that $M \models \Gamma$ and $p = \{\varphi : M_a \models \varphi\}$. We denote the set of n-types by $S_n(\Gamma)$.

For notational simplicity, we fix Γ as above and we write S_n for $S_n(\Gamma)$.

We endow the reals with the standard topology. To each L_n-formula φ we associate the mapping

$$\begin{array}{rccc} \bar{\varphi}: & S_n & \to & \mathbb{R} \\ & p & \mapsto & \varphi_p^P, \end{array}$$

where φ_p^P is the previously defined degree of provability of φ with respect to the type p.

Remark 31. Let $p \in S_n$. By definition of type, $\varphi_p^P = \varphi^{M_a}$ for some (any) $\mathbb{R}$-structure M and some (any) assignment $a : V \to M$ such that $M \models \Gamma$ and $p = \{\psi \in L_n : M_a \models \psi\}$. Notice also that, if we had the reals as unary connectives, for each $p \in S_n$ there would be a unique $r \in \mathbb{R}$ such that $(\varphi = r) \in p$ and we could simply define $\bar{\varphi}(p) = r$.

Let τ_n be the initial topology on S_n with respect to the family of mappings

$$F_n = \{\bar{\varphi} : \ \varphi \text{ is an } L_n\text{-formula}\}.$$

Since F_n separates points, τ_n is Hausdorff. It is easy to see that the basic open sets in the topology τ_n are of the form

$$[\varphi] := \{p \in S_n : (r \leq \varphi) \in p \text{ for some } 0 < r \in \mathbb{Q}\},$$

for some L_n-formula φ. Notice the similarities with the *logic topology* defined in [5].

Let $\varphi \in L_n$. We claim that

$$S_n \setminus [\varphi] = \{p \in S_n : -\varphi \in p\}. \tag{5}$$

The left-to-right inclusion follows by noticing that

$$\begin{array}{rcl} S_n \setminus [\varphi] & = & \{p \in S_n : \text{for all } \ 0 < r \in \mathbb{Q} \ (r \leq \varphi) \notin p\} \\ & \subseteq & \{p \in S_n : \text{for all } \ 0 < r \in \mathbb{Q} \ (\varphi \leq r) \in p\} \\ & \subseteq & \{p \in S_n : -\varphi \in p\}, \end{array}$$

by definition of type.

As for the converse inclusion, let p be such that $-\varphi \in p$. Therefore $(\varphi \leq r) \in p$ for all $0 < r \in \mathbb{Q}$. If it were that for some $0 < s \in \mathbb{Q}$, $(2s \leq \varphi) \in p$, then from $(\varphi \leq s) \in p$ we would get $-s \in p$: a contradiction. Hence $p \in S_n \setminus [\varphi]$.

We characterize the τ_n-closed sets as follows:

Proposition 32. *Let $C \subseteq S_n$. The following are equivalent:*

(1) *C is closed in the topology τ_n;*
(2) *there exists some set Δ of L_n-formulas such that*

$$C = \{p \in S_n : \Delta \subseteq p\}.$$

Proof. (1) $\Rightarrow$ (2). Let C be closed and let $\Psi \subseteq L_n$ be such that $C = \bigcap_{\psi \in \Psi}(S_n \setminus [\psi])$. By (5)

$$C = \bigcap_{\psi \in \Psi} \{p \in S_n : -\psi \in p\} = \{p \in S_n : \{-\psi : \psi \in \Psi\} \subseteq p\}.$$

(2) $\Rightarrow$ (1). Let C be as in (2). Again by (5), we get that for all $\delta \in \Delta$ and all $p \in S_n$, $\delta \in p$ if and only if $p \in S_n \setminus [-\delta]$. Therefore, for all $p \in S_n$, $\Delta \subseteq p$ if and only if $p \in \bigcap_{\delta\in\Delta}(S_n \setminus [-\delta])$. Hence C is closed. □

Theorem 33. *The space (S_n, τ_n) is compact.*

Proof. Let $\mathcal{F}$ be a family of closed subsets of S_n with the finite intersection property. For each $C \in \mathcal{F}$ let Δ_C be a set of L_n-formulas as in (2) of Proposition 32. By the fip the set $\Gamma \cup \bigcup_{C\in\mathcal{F}} \Delta_C$ is finitely satisfiable, hence it is satisfiable by the Compactness Theorem. Let $p \in S_n$ be such that $\bigcup_{C\in\mathcal{F}} \Delta_C \subseteq p$. Then $p \in \bigcap \mathcal{F}$. □

We consider the space $C(S_n, \mathbb{R})$ of continuous functions from S_n to $\mathbb{R}$ endowed with the topology of uniform convergence, i.e. the topology associated to the norm $\|f\| = \sup\{|f(p)| : p \in S_n\}$.

By definition of τ_n, $F_n \subseteq C(S_n, \mathbb{R})$. The following is an analogue of [5, Proposition 8.10]. We prove it in detail for sake of completeness.

Proposition 34. *The set F_n is dense in $C(S_n, \mathbb{R})$.*

Proof. We prove that:

(1) The set F_n separates points. Let $p \neq q$ be n-types and, without loss of generality, let $\varphi \in p \setminus q$. Let $M_a, N_b \models \Gamma$ be such that $p = \{\psi \in L_n : M_a \models \psi\}$ and $q = \{\psi \in L_n : N_b \models \psi\}$ respectively. Then $0 \leq \varphi^{M_a}$; $\varphi^{N_b} < 0$ and, by Remark 31, $\bar{\varphi}(p) \neq \bar{\varphi}(q)$.
(2) The constant function 1 belongs to F_n. Actually, $\bar{1}(p) = 1$ for all $p \in S_n$.
(3) For all $\bar{\varphi} \in F_n$ and all $t \in \mathbb{R}$ the function $t\bar{\varphi}$ belongs to the closure of F_n. To prove this, let $(r_k)_{k\in\mathbb{N}}$ be a sequence of rationals converging to t. The equality $\overline{r_k\varphi} = r_k\overline{\varphi}$ holds for all $k \in \mathbb{N}$ and $(r_k\bar{\varphi}_k)_{k\in\mathbb{N}}$ converges uniformly to $s\bar{\varphi}$.
(4) For all $\bar{\varphi}, \bar{\eta} \in F_n$, the function $\bar{\varphi} + \bar{\eta}$ belongs to F_n. We claim that $\bar{\varphi} + \bar{\eta} = \overline{\varphi + \eta}$. This follows from Remark 31 and from $(\varphi + \eta)^{M_a} = \varphi^{M_a} + \eta^{M_a}$.
(5) For all $\bar{\varphi}, \bar{\eta} \in F_n$, the function $\max(\bar{\varphi}, \bar{\eta})$ belongs to F_n. We claim that $\max(\bar{\varphi}, \bar{\eta}) = \overline{(\varphi - \eta) \vee 0 + \eta}$. This follows from Remark 31 and from $\max(\varphi^{M_a}, \eta^{M_a}) = ((\varphi - \eta) \vee 0 + \eta)^{M_a}$.

By applying a suitable version of the Stone-Weierstrass Theorem (see [7, Theorem 7.29]) we finally get that F_n is dense in $C(S_n, \mathbb{R})$.

□

Next we notice that the space (S_n, τ_n) is regular and T1. Therefore, under the assumption that the language L is countable, (S_n, τ_n) is metrizable, by the Urysohn's Metrization Theorem (see [8]).

Actually, even when the language is uncountable, it is possible to define a nontrivial extended metric $\delta : S_n \times S_n \to [0, +\infty]$ as follows:

$$\delta(p, q) = \sup\{|\bar{\varphi}(p) - \bar{\varphi}(q)| : \ \varphi \in L_n\}.$$

The properties of a metric can be easily verified as well as the fact that, for every $\varphi \in L_n$, the function $\bar{\varphi}$ is uniformly continuous with the respect to the metric δ. Furthermore, the topology induced by δ is finer than the topology τ_n previously defined: let $[\varphi] \in \tau_n$ and, for each $p \in [\varphi]$, let r_p be a positive rational such that $(r_p \leq \varphi) \in p$ and $r_p < \bar{\varphi}(p)$. It is straightforward to check that the open δ-ball $B(p, r_p/2)$ is contained in $[\varphi]$. Therefore

$$[\varphi] = \bigcup_{p \in \varphi} B(p, r_p/2).$$

Proposition 35. *The space (S_n, δ) is complete.*

Proof. Let $(p_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in S_n. Then, for each formula $\varphi \in L_n$, the sequence $(\bar{\varphi}(p_k))_{k\in\mathbb{N}}$ is Cauchy. Let r_φ be its limit and let $(s_l^\varphi)_{l\in\mathbb{N}}$, $(t_l^\varphi)_{l\in\mathbb{N}}$ be rational sequences converging to r_φ from below and from above respectively. Let

$$\Delta = \Gamma \cup \{s_k^\varphi \leq \varphi,\, \varphi \leq t_k^\varphi : \ \varphi \in L_n \text{ and } k \in \mathbb{N}\}.$$

For every finite subset Σ of Δ there exists a sufficiently large k such that $\Sigma \subseteq p_k$. Therefore Δ is finitely satisfiable, hence satisfiable. Let p be a type such that $\Delta \subseteq p$. Then $(p_k)_{k\in\mathbb{N}}$ converges to p. □

It is straightforward to check that, for each $\varphi \in L_n$, the function $\bar{\varphi}$ is uniformly continuous with respect to the metric δ.

Eventually we notice that, if a structure M is equipped with a distance d_M (which might not be one of the relations in M: this is certainly the case when d_M is unbounded) and the interpretations in M of predicate and function symbols are continuous (uniformly continuous) functions, then it follows easily that, for all $n \in \mathbb{N}$, the interpretation $\varphi^M : M^n \to \mathbb{R}$ of each formula $\varphi \in L_n$ is a continuous (uniformly continuous) function. This suggests further investigations in the line of continuous logic.

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