MULTIPLICITY RESULTS FOR MAGNETIC FRACTIONAL PROBLEMS

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Abstract. The paper deals with the existence of multiple solutions for a boundary value problem driven by the magnetic fractional Laplacian $(-\Delta)^s_A$, that is

$$(-\Delta)^s_A u = \lambda\, f(|u|)u \ \text{ in } \Omega, \qquad u = 0 \ \text{ in } \mathbb{R}^n \setminus \Omega,$$

where λ is a real parameter, f is a continuous function and Ω is a bounded subset of $\mathbb{R}^n$. We prove that the problem admits at least two nontrivial weak solutions under two different sets of conditions on the nonlinear term f which are dual in a suitable sense.

1. Introduction

In the last years there has been an increasing interest in the study of equations driven by non-local operators. This is motivated by the fact that non–local operators appear naturally in many important problems in pure and applied mathematics. The prototype of non–local operator is the fractional Laplacian $(-\Delta)^s$ defined, up to normalization factors, for any $u \in C^\infty_0(\mathbb{R}^n)$ and $s \in (0,1)$ as

$$(-\Delta)^s u(x) = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}dy, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

where $B(x,\varepsilon)$ denotes the ball of center x and radius ε. We refer to [7, 16] and the references therein for further details on the fractional Laplacian.

In the present paper, we will focus on the so–called magnetic fractional Laplacian. This non–local operator has been recently introduced in [6, 8] and can be considered as a fractional counterpart of the magnetic Laplacian $(\nabla - \mathrm{i}A)^2$, with $A : \mathbb{R}^n \to \mathbb{R}^n$ being a L^∞_{loc}–vector field, see [9]. We refer the interested reader to [6] for further details about the physical relevance of the magnetic fractional Laplacian. In [6], it has been proved that $(-\Delta)^s_A$ has the following representation when acting on smooth complex-valued functions $u \in C^\infty_0(\mathbb{R}^n,\mathbb{C})$

$$(-\Delta)^s_A u(x) = 2 \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^n\setminus B(x,\varepsilon)} \frac{u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)}{|x-y|^{n+2s}}dy, \qquad x \in \mathbb{R}^n,$$

therefore, the operator is consistent with (1.1) if $A = 0$. As for the classical fractional Laplacian, one can define the fractional counterpart of the magnetic Sobolev spaces, see Section 2 below for the definition. In [18, 19, 22], it has been studied the stability of these fractional Sobolev norms when either $s \nearrow 1$ or $s \searrow 0$, proving a magnetic counterpart of the Bourgain–Brezis–Mironescu formula (when $s \nearrow 1$, see [3]) and the Maz'ya–Shaposhnikova formula (when $s \searrow 0$, see [11, 12]). Finally, we refer to [2, 13, 23] for multiplicity results for different equations on $\mathbb{R}^n$ and driven by the magnetic fractional Laplacian.

Inspired by the above-mentioned works, in this paper we study the existence of multiple weak solutions of the following boundary value problem

$$\begin{cases} (-\Delta)^s_A u = \lambda\, f(|u|)u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

where $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^n$ is an open and bounded set with Lipschitz boundary $\partial\Omega$.
Concerning the nonlinearity f, we will consider two different situations which can be considered *dual* in a sense that we will specify later on. As a first scenario, we will deal with $f : [0,\infty) \to \mathbb{R}$ being a *continuous* function satisfying the following conditions:

(f_1) $f(t) = o(1)$ *as* $t \to 0$;
(f_2) $f(t) = o(1)$ *as* $t \to \infty$;
(f_3) $\sup_{t\in[0,\infty)} F(t) > 0$,

where

$$F(t) := \int_0^t f(\tau)\,\tau\,d\tau, \quad \text{for any real } t > 0. \tag{1.3}$$

There are plenty of examples of continuous functions satisfying (f_1)–(f_3), e.g. $f(t) = t\chi_{[0,1]}(t) + e^{1-t}\chi_{(1,\infty)}(t)$. We observe that the nonlinear term f can be controlled from above, thanks to (f_1)–(f_3). In particular, for a suitable $c_1 > 0$

$$|f(t)t| \leq c_1 t \qquad \text{for every } t \geq 0 \text{ sufficiently large.} \tag{1.4}$$

Our first result can be stated now as follows.

Theorem 1.1. *Let $s \in (0,1)$, $n > 2s$ and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary $\partial\Omega$. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying conditions (f_1), (f_2) and (f_3). Then, there exists $\lambda^* > 0$ such that for every $\lambda > \lambda^*$ problem (1.2) has at least two nontrivial weak solutions.*

The proof of Theorem 1.1 is mainly variational and based on the application of an abstract critical point result due to Brézis and Nirenberg in [5]. Theorem 1.1 can be considered as the fractional magnetic counterpart of [15, Theorem 1] and [10, Theorem 2.1].
The second set of conditions we consider on f is the following:

(f_4) *There exist* $a_1, a_2 > 0$ *and* $q \in (2, 2_s^*)$ *such that* $|f(t)| \leq a_1 + a_2 t^{q-2}$ for any $t \geq 0$;
(f_5) *There exist* $\mu > 2$ *and* $t_0 > 0$ *such that* $0 < \mu F(t) \leq f(t)t^2$ for any $t > t_0$,

where $2_s^* := 2n/(n-2s)$ is the fractional critical Sobolev exponent. A typical example of f verifying (f_4) and (f_5) is given by $f(t) = qt^{q-2}$, with $q \in (2, 2_s^*)$. In [10], it is proved that conditions (f_4) and (f_5) imply that there exists $c_2 > 0$ such that

$$|f(t)t| \geq c_2 t \qquad \text{for every } t \geq 0 \text{ sufficiently large,} \tag{1.5}$$

which can be considered as a counterpart of (1.4).
Our second result is the following theorem.

Theorem 1.2. *Let $s \in (0,1)$, $n > 2s$ and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary $\partial\Omega$. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function satisfying conditions (f_4) and (f_5). Then for every $\rho > 0$ and any $\lambda \in (0, \Lambda(\rho))$, with*

$$\Lambda(\rho) := \frac{2q}{a_1 c_2^2 q + 2a_2 c_q^q \rho^{\frac{q-2}{2}}}, \qquad \text{where } c_2 \text{ and } c_q \text{ are given in (2.12)},$$

problem (1.2) has at least two nontrivial weak solutions, one of which has norm strictly less than ρ.

The approach in Theorem 1.2 is still variational but based on the application of another abstract result due to Ricceri in [20]. Theorem 1.2 is the fractional magnetic version of [20, Theorem 4] which has been subsequently refined and extended in [1, 14, 17].

The paper is organized as follows. In Section 2, we introduce the necessary functional and variational setup to study the boundary value problem (1.2). In Section 3, we prove Theorem 1.1. Finally, in Section 4, we prove Theorem 1.2.

2. Functional and Variational Setup

Throughout the paper, we indicate with $|A|$ the n-dimensional Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. Moreover, for every $z \in \mathbb{C}$ we will denote by $\Re z$ its real part, and by $\overline{z}$ its complex conjugate. Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by $L^2(\Omega, \mathbb{C})$ the space of measurable functions $u : \Omega \to \mathbb{C}$ such that

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 \, dx \right)^{1/2} < \infty,$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{C}$. For every $A \in L^\infty_{loc}(\mathbb{R}^n)$, we consider the semi–norm

$$[u]_{H^1_A(\Omega)} := \left(\int_\Omega |\nabla u(x) - \mathrm{i} A(x) u(x)|^2 \, dx \right)^{1/2},$$

and following [9], we define $H^1_A(\Omega)$ as the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H^1_A(\Omega)} < \infty$, endowed with the norm

$$\|u\|_{H^1_A(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + [u]^2_{H^1_A(\Omega)} \right)^{1/2}.$$

We also indicate with $H^1_{0,A}(\Omega)$ the closure of $C^\infty_0(\Omega, \mathbb{C})$ in $H^1_A(\Omega)$.

For any $s \in (0,1)$, the magnetic Gagliardo semi-norm is set as

$$[u]_{H^s_A(\Omega)} := \left(\iint_{\Omega \times \Omega} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{1/2}.$$

We denote by $H^s_A(\Omega)$ the space of functions $u \in L^2(\Omega, \mathbb{C})$ such that $[u]_{H^s_A(\Omega)} < \infty$, normed with

$$\|u\|_{H^s_A(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + [u]^2_{H^s_A(\Omega)} \right)^{1/2}. \tag{2.1}$$

For $A = 0$, this definition is consistent with the usual fractional space $H^s(\Omega)$. We stress out that $C^\infty_0(\mathbb{R}^n, \mathbb{C}) \subseteq H^s_A(\mathbb{R}^n)$, see [6, Proposition 2.2].

In order to define weak solutions of problem (1.2), we introduce the functional space

$$X_{0,A} := \{ u \in H^s_A(\mathbb{R}^n) : \; u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \},$$

which generalizes to the magnetic framework the space introduced in [21]. As in [6], we define the following real scalar product on $X_{0,A}$

$$\langle u, v \rangle_{X_{0,A}} := \Re \iint_{\mathbb{R}^{2n}} \frac{\left(u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y) \right) \overline{\left(v(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} v(y) \right)}}{|x-y|^{n+2s}} \, dxdy, \tag{2.2}$$

which induces the following norm

$$\|u\|_{X_{0,A}} := \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right)^{1/2}. \tag{2.3}$$

We now state and prove some properties of space $X_{0,A}$ which will be useful in the sequel.

Lemma 2.1. *There exists a constant $C > 1$, depending only on n, s and Ω, such that*

$$\|u\|^2_{X_{0,A}} \le \|u\|^2_{H^s_A(\mathbb{R}^n)} \le C \|u\|^2_{X_{0,A}}, \tag{2.4}$$

for any $u \in X_{0,A}$. Thus, (2.3) is a norm on $X_{0,A}$ equivalent to (2.1).

Proof. Let $u \in X_{0,A}$. In order to show (2.4), it is enough to see that there exists a constant $\widetilde{C} = \widetilde{C}(n,s,\Omega) > 0$ such that

$$\|u\|^2_{L^2(\Omega)} \leq \widetilde{C} \iint_{\mathbb{R}^{2n}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{n+2s}}\, dxdy. \tag{2.5}$$

By [6, Lemma 3.1] we have the pointwise diamagnetic inequality

$$|u(x) - e^{\mathrm{i}(x-y)\cdot A(\frac{x+y}{2})}u(y)| \geq ||u(x)| - |u(y)||, \quad \text{for a.e. } x, y \in \mathbb{R}^n,$$

from which we immediately have

$$\begin{aligned}\iint_{\mathbb{R}^{2n}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{n+2s}}\, dxdy &\geq \iint_{\mathbb{R}^{2n}} \frac{||u(x)| - |u(y)||^2}{|x-y|^{n+2s}}\, dxdy \\ \geq \int_{\mathcal{C}\Omega} \left(\int_\Omega \frac{||u(x)| - |u(y)||^2}{|x-y|^{n+2s}}\, dy \right) dx &= \int_{\mathcal{C}\Omega} \left(\int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s}} dy \right) dx,\end{aligned}$$

where the last equality follows from the fact that $u = 0$ a.e. in $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Since Ω is bounded, there exists $R > 0$ such that $\Omega \subseteq B_R$ and $|B_R \setminus \Omega| > 0$. For this, it follows that

$$\int_{\mathcal{C}\Omega} \left(\int_\Omega \frac{|u(y)|^2}{|x-y|^{n+2s}} dy \right) dx \geq \int_{B_R \setminus \Omega} \left(\int_\Omega \frac{|u(y)|^2}{|2R|^{n+2s}} dy \right) dx = \frac{|B_R \setminus \Omega|}{(2R)^{n+2s}}\, \|u\|^2_{L^2(\Omega)}\,,$$

which yields (2.5). Now, we observe that

$$\|u\|^2_{H^s_A(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} + [u]^2_{H^s_A(\mathbb{R}^n)} \leq (\widetilde{C} + 1)\|u\|^2_{X_{0,A}},$$

therefore setting $C := \widetilde{C} + 1$, we get the first part of lemma.
By (2.5) it follows that if $\|u\|_{X_{0,A}} = 0$ then $u = 0$ a.e. in $\mathbb{R}^n$, which implies that $\|\cdot\|_{X_{0,A}}$ is a norm. This is enough to conclude the proof. □

Making use of Lemma 2.1 and proceeding exactly as in [21, Lemma 7], we immediately get that $\left(X_{0,A}, \langle\cdot,\cdot\rangle_{X_{0,A}}\right)$ is a real separable Hilbert space.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Then*

$$X_{0,A} \hookrightarrow H^s(\Omega, \mathbb{C}). \tag{2.6}$$

Furthermore, if the boundary of Ω is Lipschitz the injection

$$X_{0,A} \hookrightarrow L^p(\Omega, \mathbb{C}) \tag{2.7}$$

is compact for any $p \in [1, 2^*_s)$.

Proof. Let $u \in X_{0,A}$. We have

$$\begin{aligned}\|u\|^2_{H^s(\Omega)} &= \int_\Omega |u(x)|^2 dx + \iint_{\Omega\times\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \\ &\leq \int_\Omega |u(x)|^2 dx + \iint_{\Omega\times\Omega} \frac{|e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \\ &\quad + \iint_{\Omega\times\Omega} \frac{|u(x)|^2 |e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)} - 1|^2}{|x-y|^{n+2s}} dxdy \\ &\leq \|u\|^2_{H^s_A(\Omega)} + D\end{aligned} \tag{2.8}$$

where we denote

$$\begin{aligned} D := & \iint_{\Omega\times\Omega} \frac{|u(x)|^2|e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}-1|^2}{|x-y|^{n+2s}}dxdy \\ = & \int_\Omega |u(x)|^2 \left(\int_{\Omega\cap\{y\in\mathbb{R}^n:\ |x-y|>1\}} \frac{|e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}-1|^2}{|x-y|^{n+2s}}dy\right)dx \\ & + \int_\Omega |u(x)|^2 \left(\int_{\Omega\cap\{y\in\mathbb{R}^n:\ |x-y|\le1\}} \frac{|e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}-1|^2}{|x-y|^{n+2s}}dy\right)dx := D_1+D_2. \end{aligned} \tag{2.9}$$

Since $|e^{\mathrm{i}t}-1|\le 2$, we get

$$D_1 \le 4\int_\Omega |u(x)|^2\left(\int_{\Omega\cap\{y\in\mathbb{R}^n:\ |x-y|>1\}}\frac{1}{|x-y|^{n+2s}}dy\right)dx \le C\|u\|^2_{L^2(\Omega)}. \tag{2.10}$$

Considering that Ω is bounded, there exists a compact set $K\subset\mathbb{R}^n$ such that $K\supset\Omega$. Thus, it follows that

$$D_2 \le \int_K |u(x)|^2\left(\int_{K\cap\{y\in\mathbb{R}^n:\ |x-y|\le1\}}\frac{|e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}-1|^2}{|x-y|^{n+2s}}dy\right)dx.$$

Since A is locally bounded and $K\subset\mathbb{R}^n$ is compact, we have

$$|e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}-1|^2 \le C|x-y|^2, \qquad \text{for } |x-y|\le 1,\ x,y\in K,$$

from which

$$D_2 \le \int_K |u(x)|^2\left(\int_{K\cap\{y\in\mathbb{R}^n:\ |x-y|\le1\}}\frac{1}{|x-y|^{n+2s-2}}dy\right)dx \le C\|u\|^2_{L^2(\Omega)}. \tag{2.11}$$

Combining (2.8)–(2.11), we have

$$\|u\|^2_{H^s(\Omega)} \le C\|u\|^2_{H^s_A(\Omega)} \le C\|u\|^2_{H^s_A(\mathbb{R}^n)} \le \widetilde{C}\|u\|^2_{X_{0,A}}$$

where last inequality is given by (2.4). This concludes the proof of (2.6).
By (2.6) and the hypothesis on the boundary of Ω, (2.7) directly follows from [7, Corollary 7.2]. □

By (2.7), for every $p\in[1,2^*_s)$ the number

$$c_p = \sup_{u\in X_{0,A}\setminus\{0\}}\frac{\|u\|_{L^p(\Omega)}}{\|u\|_{X_{0,A}}}, \tag{2.12}$$

is well–defined and strictly positive. We conclude this section providing a variational formulation of the problem (1.2). We will say that a function $u\in X_{0,A}$ is a weak solution of (1.2) if

$$\begin{aligned} & \Re\iint_{\mathbb{R}^{2n}} \frac{\left(u(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)\right)\overline{\left(v(x)-e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}v(y)\right)}}{|x-y|^{n+2s}}dxdy \\ & \qquad = \lambda\Re\int_\Omega f(|u(x)|)u(x)\overline{v(x)}\,dx, \quad \text{for every } v\in X_{0,A}. \end{aligned} \tag{2.13}$$

Clearly, the weak solutions of (1.2) are the critical points of the Euler–Lagrange functional associated with (1.2), that is

$$\mathcal{J}_\lambda(u) := \Phi(u)-\lambda\Psi(u), \quad u\in X_{0,A} \tag{2.14}$$

where

$$\Phi(u) := \frac{1}{2}\iint_{\mathbb{R}^{2n}} \frac{|u(x) - e^{\mathrm{i}(x-y)\cdot A\left(\frac{x+y}{2}\right)}u(y)|^2}{|x-y|^{n+2s}}\,dxdy, \qquad \Psi(u) := \int_\Omega F(|u(x)|)dx \tag{2.15}$$

and F is defined as in (1.3). It is easy to see that $\mathcal{J}_\lambda$ is well–defined and of class $C^1(X_{0,A}, \mathbb{R})$.

3. Proof of Theorem 1.1

Throughout this section, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying conditions (f_1), (f_2) and (f_3), without further mentioning.
The proof of Theorem 1.1 is based on the application of the following abstract theorem in critical point theory. For the sake of completeness, let us recall that a functional $J : E \to \mathbb{R}$ of class $C^1(E)$, on a Banach space E and dual space E^*, is said to satisfy the *Palais-Smale condition* (PS) if any *Palais–Smale sequence* associated with J has a strongly convergent subsequence in E. A sequence $\{u_j\}_{j\in\mathbb{N}}$ in E is called a *Palais–Smale sequence* if $\{J(u_j)\}_{j\in\mathbb{N}}$ is bounded and $\|J'(u_j)\|_{E^*} \to 0$ as $j \to \infty$.

Theorem 3.1 (Theorem 4 of [5]). *Let $(E, \|\cdot\|)$ be a Banach space which admits a decomposition*

$$E = E_1 \oplus E_2,$$

with $\dim(E_2) < \infty$. Let $J : E \to \mathbb{R}$ be a C^1 functional such that:
- *(a) $J(0) = 0$;*
- *(b) J satisfies the Palais-Smale condition (PS);*
- *(c) J is bounded from below;*
- *(d) $\inf_{u\in E} J(u) < 0$.*

Let us also suppose that there exists a positive constant $R > 0$ such that

$$\begin{cases} J(u) \geq 0, & u \in E_1,\ \|u\| \leq R; \\ J(u) \leq 0, & u \in E_2,\ \|u\| \leq R. \end{cases} \tag{3.1}$$

Then J admits at least two nonzero critical points.

Before proving Theorem 1.1, we introduce three technical lemmas necessary to verify that the functional $\mathcal{J}_\lambda$ satisfies the assumptions required to apply Theorem 3.1.

Lemma 3.2. *For every $\lambda \in \mathbb{R}$, the functional $\mathcal{J}_\lambda$ is bounded from below, coercive and satisfies the (PS) condition.*

Proof. If $\lambda = 0$ the results follows by [4, Proposition 3.32]. Let $\lambda \neq 0$. By (f_2), for any $\varepsilon > 0$ there exists $r_\varepsilon = r(\varepsilon) > 0$ such that

$$|f(t)t| \leq \varepsilon t, \quad \text{for any } t > r_\varepsilon.$$

Let $\delta_\varepsilon := \max_{t\leq r_\varepsilon} |f(t)t| > 0$. We get

$$|f(t)t| \leq \varepsilon t + \delta_\varepsilon, \quad \text{for any } t \geq 0, \tag{3.2}$$

from which

$$|F(t)| \leq \frac{\varepsilon}{2}t^2 + \delta_\varepsilon t, \quad \text{for any } t \geq 0. \tag{3.3}$$

Then, by (3.3), for every $\varepsilon > 0$ and for any $u \in X_{0,A}$ we have that

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geq \frac{1}{2}\|u\|^2_{X_{0,A}} - |\lambda|\left|\int_\Omega F(|u(x)|)dx\right| \geq \frac{1}{2}\|u\|^2_{X_{0,A}} - |\lambda|\left(\frac{\varepsilon}{2}\|u\|^2_{L^2(\Omega)} + \delta_\varepsilon\|u\|_{L^1(\Omega)}\right) \\ &\geq \frac{1}{2}(1 - \varepsilon|\lambda|c_2)\|u\|^2_{X_{0,A}} - \delta_\varepsilon|\lambda|c_1\|u\|_{X_{0,A}},\end{aligned}$$

where last inequality is a consequence of injection (2.7), with c_1 and c_2 as in (2.12). By fixing $\varepsilon < 1/|\lambda|c_2$, it follows that $\mathcal{J}_\lambda$ is bounded from below and coercive.

Now, it remains to check the validity of the *Palais–Smale condition*. For this, let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $X_{0,A}$ such that

$$\{\mathcal{J}_\lambda(u_j)\}_{j\in\mathbb{N}} \text{ is bounded and } \mathcal{J}'_\lambda(u_j) \to 0 \text{ as } j \to \infty. \tag{3.4}$$

By the coercivity of $\mathcal{J}_\lambda$ and (3.4), the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $X_{0,A}$. Thus, by the reflexivity of the space $X_{0,A}$ and Lemma 2.2, there exists $u \in X_{0,A}$ such that, up to a subsequence, still relabeled $\{u_j\}_{j\in\mathbb{N}}$, we have

$$u_j \rightharpoonup u \text{ in } X_{0,A} \text{ and } u_j \to u \text{ in } L^p(\Omega,\mathbb{C}) \text{ for any } p \in [1, 2^*_s), \tag{3.5}$$

as $j \to \infty$.

By (3.2) with $\varepsilon = 1$ and Hölder inequality, we get

$$\begin{aligned}\int_\Omega |f(|u_j(x)|)u_j(x)(\overline{u_j(x)-u(x)})|dx &\le \int_\Omega |u_j(x)||u_j(x)-u(x)|dx + \delta_1 \int_\Omega |u_j(x)-u(x)|dx \\ &\le \|u_j\|_{L^2(\Omega)}\|u_j-u\|_{L^2(\Omega)} + \delta_1\|u_j-u\|_{L^1(\Omega)},\end{aligned}$$

and by (3.5)

$$\int_\Omega |f(|u_j(x)|)u_j(x)(\overline{u_j(x)-u(x)})|dx \to 0 \quad \text{as } j \to \infty. \tag{3.6}$$

By differentiating $\mathcal{J}_\lambda$ we immediately have

$$\mathcal{J}'_\lambda(u_j)(u_j-u) = \langle u_j, u_j-u\rangle_{X_{0,A}} - \lambda \int_\Omega f(|u_j(x)|)u_j(x)(\overline{u_j(x)-u(x)})dx$$

from which, by (3.4), (3.6) and since $|\mathcal{J}'_\lambda(u_j)(u_j-u)| \le \|\mathcal{J}'_\lambda(u_j)\|_{(X_{0,A})^*}\|u_j-u\|_{X_{0,A}}$, it follows that

$$\langle u_j, u_j-u\rangle_{X_{0,A}} = \|u_j\|^2_{X_{0,A}} - \langle u_j, u\rangle_{X_{0,A}} \to 0 \quad \text{as } j \to \infty.$$

Thus, using (3.5) we get $\|u_j\|_{X_{0,A}} \to \|u\|_{X_{0,A}}$ as $j \to \infty$, and so by [4, Proposition 3.32] we conclude $u_j \to u$ in $X_{0,A}$ as $j \to \infty$. □

Lemma 3.3. *There exists $\lambda^* > 0$ such that $\inf\limits_{u\in X_{0,A}} \mathcal{J}_\lambda(u) < 0$ for any $\lambda > \lambda^*$.*

Proof. Let $\lambda > 0$. Since Ω is bounded, we can pick a point $x_0 \in \Omega$ and $\tau > 0$ such that $\overline{B(x_0,\tau)} \subset \Omega$. By condition (f_3), we can find a $\bar{t} > 0$ such that $F(\bar{t}) > 0$. Therefore, we can also fix $\sigma_0 \in (0,1)$ such that

$$F(\bar{t})\sigma_0^n - (1-\sigma_0^n) \max_{t\le\bar{t}} |F(t)| > 0. \tag{3.7}$$

Let $\widetilde{u} : \mathbb{R}^n \to \mathbb{R}$ be such that:

- (u_1) $\widetilde{u} \in C_0^\infty(\mathbb{R}^n,\mathbb{R})$, *with* $\operatorname{supp}(\widetilde{u}) \subset B(x_0,\tau)$;
- (u_2) $|\widetilde{u}(x)| \le |\bar{t}|$, *if* $x \in B(x_0,\tau) \setminus B(x_0,\sigma_0\tau)$;
- (u_3) $\widetilde{u}(x) := \begin{cases} 0, & x \in \mathbb{R}^n \setminus B(x_0,\tau) \\ \bar{t}, & x \in B(x_0,\sigma_0\tau). \end{cases}$

By [6, Proposition 2.2] we have $\widetilde{u} \in H^s_A(\mathbb{R}^n)$, and since $\widetilde{u} = 0$ in $\mathbb{R}^n \setminus \Omega$ we conclude that $\widetilde{u} \in X_{0,A}$. We claim that

$$\Psi(\widetilde{u}) \ge \left[F(\bar{t})\sigma_0^n - (1-\sigma_0^n) \max_{t\le\bar{t}} |F(t)|\right] \omega_n \tau^n > 0, \tag{3.8}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Indeed, by (u_2) we have that

$$\begin{aligned}\int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(|\widetilde{u}(x)|)dx &\geq -\int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} |F(|\widetilde{u}(x)|)|dx \\ &\geq -\max_{t\leq \bar{t}} |F(t)| \int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} dx \\ &= -\max_{t\leq \bar{t}} |F(t)|(1-\sigma_0^n)\omega_n\tau^n.\end{aligned} \tag{3.9}$$

On the other hand, since $F(0)=0$, by (u_3) we have that

$$\int_{\mathbb{R}^n\setminus B(x_0,\tau)} F(|\widetilde{u}(x)|)dx = 0. \tag{3.10}$$

Therefore, combining (3.7), (3.9), (3.10) and (u_3), we get

$$\begin{aligned}\int_\Omega F(|\widetilde{u}(x)|)dx &= \int_{B(x_0,\sigma_0\tau)} F(|\widetilde{u}(x)|)dx + \int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(|\widetilde{u}(x)|)dx \\ &= \int_{B(x_0,\sigma_0\tau)} F(\bar{t})dx + \int_{B(x_0,\tau)\setminus B(x_0,\sigma_0\tau)} F(|\widetilde{u}(x)|)dx \\ &\geq \left[F(\bar{t})\sigma_0^n - (1-\sigma_0^n)\max_{t\leq\bar{t}}|F(t)|\right]\omega_n\tau^n > 0,\end{aligned}$$

which yields the claim (3.8).

Thus, denoting with $\lambda^* := \Phi(\widetilde{u})/\Psi(\widetilde{u})$, by (2.14) and (3.8) we have

$$\inf_{u\in X_{0,A}} \mathcal{J}_\lambda(u) \leq \mathcal{J}_\lambda(\widetilde{u}) = \Phi(\widetilde{u}) - \lambda\Psi(\widetilde{u}) < 0,$$

for any $\lambda > \lambda^*$. This concludes the proof. □

Lemma 3.4. *For every $\lambda\in\mathbb{R}$, there exists $R>0$ such that $\mathcal{J}_\lambda(u)\geq 0$, for any $u\in X_{0,A}$ with $\|u\|_{X_{0,A}}\leq R$.*

Proof. Fix $\nu\in(2,2^*_s)$. By (f_1), for any $\sigma>0$ there exists $r_\sigma = r(\sigma)>0$ such that

$$|f(t)t| \leq \sigma t, \quad \text{for any } t< r_\sigma. \tag{3.11}$$

Let $\delta_1>0$ be as in (3.3) with $\varepsilon=1$ and define $\kappa_\sigma = \left(\frac{1}{2r_\sigma^{\nu-2}} + \frac{\delta_1}{r_\sigma^{\nu-1}}\right) > 0$. If $t\geq r_\sigma$, a simple calculation gives

$$|F(t)| \leq \kappa_\sigma t^\nu, \quad \text{for any } t\geq r_\sigma$$

and using (3.11) we conclude

$$|F(t)| \leq \frac{\sigma}{2}t^2 + \kappa_\sigma t^\nu, \quad \text{for any } t\geq 0. \tag{3.12}$$

By (2.12) and (3.12), we get

$$\begin{aligned}\mathcal{J}_\lambda(u) &\geq \frac{1}{2}\|u\|^2_{X_{0,A}} - |\lambda|\left|\int_\Omega F(|u(x)|)dx\right| \\ &\geq \frac{1}{2}\|u\|^2_{X_{0,A}} - \frac{|\lambda|\sigma}{2}\|u\|^2_{L^2(\Omega)} - |\lambda|k_\sigma\|u\|^\nu_{L^\nu(\Omega)} \\ &\geq \frac{1}{2}\|u\|^2_{X_{0,A}} - \frac{|\lambda|\sigma c_2^2}{2}\|u\|^2_{X_{0,A}} - |\lambda|k_\sigma c_\nu^\nu\|u\|^\nu_{X_{0,A}} \\ &= \frac{1}{2}(1-|\lambda|\sigma c_2^2)\|u\|^2_{X_{0,A}} - |\lambda|k_\sigma c_\nu^\nu\|u\|^\nu_{X_{0,A}}.\end{aligned} \tag{3.13}$$

Let us fix $\sigma \in (0, 1/|\lambda| c_2^2)$. Since $\nu \in (2, 2_s^*)$, by (3.13) we can find $R > 0$ sufficiently small such that

$$\mathcal{J}_\lambda(u) \geq 0, \quad \text{for } \|u\|_{X_{0,A}} \leq R.$$

This concludes the proof. □

Proof of Theorem 1.1. We want to apply Theorem 3.1 to the functional $\mathcal{J}_\lambda : X_{0,A} \to \mathbb{R}$. First of all, let us consider the following decomposition of the Hilbert space $X_{0,A}$,

$$X_{0,A} = X_{0,A} \oplus \{0\},$$

where the direct sum has to be intended with respect to the scalar product set in (2.2). By Lemmas 3.2 and 3.3, the functional $\mathcal{J}_\lambda$ satisfies conditions (b), (c) and (d) of Theorem 3.1, for any $\lambda > \lambda^*$ with λ^* given in Lemma 3.3. Furthermore, it is immediate to see that $\mathcal{J}_\lambda(0) = 0$. Hence, by Lemma 3.4 we have (3.1), which concludes the proof. □

Remark 3.5. *We point out that in Theorem 1.1 the lower threshold λ^* for parameter λ is not optimal, since $\lambda^* := \Phi(\widetilde{u})/\Psi(\widetilde{u})$. However, we show that problem (1.2) admits only the trivial solution when $\lambda \in (0, 1/c_2^2 \max_{t\geq 0} |f(t)|)$, where c_2 is given in (2.12) and $\max_{t\geq 0} |f(t)| < \infty$, by (f_1), (f_2) and the continuity of f.*

Let us consider a nontrivial weak solution u_0 of problem (1.2). If $\lambda \in (0, 1/c_2^2 \max_{t\geq 0} |f(t)|)$, by (2.12) and (2.13) we have

$$\begin{aligned}
\|u_0\|_{X_{0,A}}^2 &= \lambda \int_\Omega f(|u_0(x)|)|u_0(x)|^2 dx \\
&\leq \lambda \max_{t\geq 0} |f(t)| \, \|u_0\|_{L^2(\Omega)}^2 \\
&\leq \lambda \max_{t\geq 0} |f(t)| c_2^2 \, \|u_0\|_{X_{0,A}}^2 < \|u_0\|_{X_{0,A}}^2,
\end{aligned}$$

which yields a contradiction.

4. Proof of Theorem 1.2

Throughout this section, we assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying conditions (f_4) and (f_5), without further mentioning.

The proof of Theorem 1.2 is mainly based on the application of the following result.

Theorem 4.1 (Theorem 6 of [20])**.** *Let $(E, \|\cdot\|)$ be a reflexive real Banach space. Let Φ, Ψ : $E \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous and coercive. Further, assume that Ψ is sequentially weakly continuous. In addition, assume that, for each $\gamma > 0$, the functional $I_\gamma : E \to \mathbb{R}$,*

$$I_\gamma(z) := \gamma\Phi(z) - \Psi(z), \quad z \in E,$$

satisfies (PS).

Then, for every $\rho > \inf_E \Phi$ and every

$$\gamma > \inf_{u\in\Phi^{-1}(-\infty,\rho)} \frac{\sup_{v\in\Phi^{-1}(-\infty,\rho)} \Psi(v) - \Psi(u)}{\rho - \Phi(u)},$$

the following alternative holds:

either the functional I_γ has a strict global minimum in $\Phi^{-1}(-\infty, \rho)$, or I_γ has at least two critical points one of which lies in $\Phi^{-1}(-\infty, \rho)$.

Here, we consider the functional $\mathcal{I}_\lambda : X_{0,A} \to \mathbb{R}$, given by

$$\mathcal{I}_\lambda(u) := \frac{1}{\lambda}\Phi(u) - \Psi(u), \quad u \in X_{0,A}$$

with Φ and Ψ defined as in (2.15). To apply Theorem 4.1, we first prove that $\mathcal{I}_\lambda$ satisfies the Palais–Smale condition.

Lemma 4.2. *For every $\lambda > 0$, the functional $\mathcal{I}_\lambda$ satisfies the (PS) condition.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $X_{0,A}$ verifying (3.4).

We first show that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $X_{0,A}$. By (f_4) we have

$$|F(t)| \leq \frac{a_1}{2}t^2 + \frac{a_2}{q}t^q, \quad \text{for any } t \geq 0, \tag{4.1}$$

and so, using again (f_4), we have that for any $j \in \mathbb{N}$

$$\begin{aligned} &\left|\int_{\Omega\cap\{x\in\mathbb{R}^n:\ |u_j(x)|\leq t_0\}} \left[F(|u_j(x)|) - \frac{1}{\mu}f(|u_j(x)|)|u_j(x)|^2\right] dx\right| \\ &\qquad \leq \left[\frac{a_1(\mu+2)}{2\mu}t_0^2 + \frac{a_2(\mu+q)}{q\mu}t_0^q\right]|\Omega| =: C, \end{aligned} \tag{4.2}$$

with t_0 and μ defined in (f_5). Thus, by (f_5) and (4.2) we have for any $j \in \mathbb{N}$

$$\begin{aligned} \mathcal{I}_\lambda(u_j) - \frac{1}{\mu}\mathcal{I}'_\lambda(u_j)(u_j) &\geq \left(\frac{1}{2\lambda} - \frac{1}{\mu\lambda}\right)\|u_j\|^2_{X_{0,A}} \\ &\quad - \int_{\Omega\cap\{x\in\mathbb{R}^n:\ |u_j(x)|\leq t_0\}} \left[F(|u_j(x)|) - \frac{1}{\mu}f(|u_j(x)|)|u_j(x)|^2\right] dx \\ &\geq \left(\frac{1}{2\lambda} - \frac{1}{\mu\lambda}\right)\|u_j\|^2_{X_{0,A}} - C. \end{aligned} \tag{4.3}$$

Since $\{u_j\}_{j\in\mathbb{N}}$ satisfies (3.4) with $\mathcal{I}_\lambda$, we know there exist a $\widetilde{C} > 0$ such that for any $j \in \mathbb{N}$

$$|\mathcal{I}_\lambda(u_j)| \leq \widetilde{C}, \quad \left|\mathcal{I}'_\lambda(u_j)\left(\frac{u_j}{\|u_j\|_{X_{0,A}}}\right)\right| \leq \widetilde{C}. \tag{4.4}$$

Combining (4.3) and (4.4) we prove the boundedness of $\{u_j\}_{j\in\mathbb{N}}$, since $\mu > 2$ in (f_5). Using (f_4) we conclude as in Lemma 3.2. □

We now study functional Ψ, introduced in (2.15).

Lemma 4.3. *The functional Ψ is sequentially weakly continuous on $X_{0,A}$.*

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $X_{0,A}$ such that $u_j \rightharpoonup u$ in $X_{0,A}$. By Lemma 2.2 and [4, Theorem 4.9], up to a subsequence, still relabeled $\{u_j\}_{j\in\mathbb{N}}$, we have

$$\begin{aligned} &u_j \to u \text{ in } L^p(\Omega,\mathbb{C}) \text{ and } u_j \to u \text{ a.e. in } \Omega \text{ as } j \to \infty, \\ &|u_j(x)| \leq h_p(x) \text{ for a.e. } x \in \Omega \text{ and for any } j \in \mathbb{N}, \end{aligned} \tag{4.5}$$

for any $p \in [1, 2^*_s)$, with $h_p \in L^p(\Omega)$. Hence, by (4.1) and (4.5) we get

$$|F(|u_j(x)|)| \leq \frac{a_1}{2}|u_j(x)|^2 + \frac{a_2}{q}|u_j(x)|^q \leq \left(\frac{a_1}{2}(h_2(x))^2 + \frac{a_2}{q}(h_q(x))^q\right) \in L^1(\Omega) \tag{4.6}$$

for a.e. $x \in \Omega$ and for every $j \in \mathbb{N}$. By the continuity of F and (4.5), we also have

$$F(|u_j(x)|) \to F(|u(x)|) \quad \text{a.e. in } \Omega \quad \text{as } j \to \infty. \tag{4.7}$$

Using (4.6) and (4.7) and the Lebesgue dominated convergence theorem we conclude

$$\int_\Omega F(|u_j(x)|)dx \to \int_\Omega F(|u(x)|)dx \quad \text{as } j \to \infty.$$

It follows that the map

$$u \to \Psi(u)$$

is continuous from $X_{0,A}$ endowed with the weak topology to $\mathbb{R}$. □

Proof of Theorem 1.2. By Lemma 4.2, $\mathcal{I}_\lambda$ satisfies the *Palais-Smale condition*. By (2.15) we immediately see that Φ is coercive and sequentially weakly lower semicontinuous, while by Lemma 4.3 the functional Ψ is sequentially weakly continuous. Let $q \in (2, 2_s^*)$ be as in (f_4). For every $\rho > 0$ let

$$0 < \lambda < \frac{2q}{a_1 c_2^2 q + 2a_2 c_q^q \rho^{\frac{q-2}{2}}},$$

where c_2, c_q are as in (2.12). We claim that

$$\frac{1}{\lambda} > \Theta(\rho) := \inf_{u\in\Phi^{-1}(-\infty,\rho)} \frac{\sup_{v\in\Phi^{-1}(-\infty,\rho)} \Psi(v) - \Psi(u)}{\rho - \Phi(u)}. \tag{4.8}$$

Since $\Phi(0) = 0$ and $\Psi(0) = 0$, then

$$\Theta(\rho) \le \frac{\sup_{v\in\Phi^{-1}(-\infty,\rho)} \Psi(v)}{\rho} = \frac{\sup_{\{v\in X_{0,A}\colon \|v\|_{X_{0,A}} < \rho^{1/2}\}} \Psi(v)}{\rho}. \tag{4.9}$$

On the other hand, it holds true that

$$\frac{\sup_{\{v\in X_{0,A}\colon \|v\|_{X_{0,A}} < \rho^{1/2}\}} \Psi(v)}{\rho} \le \frac{a_1 c_2^2}{2} + \frac{a_2 c_q^q}{q}\rho^{\frac{q-2}{2}}, \tag{4.10}$$

indeed, by (4.1) we have

$$\Psi(v) \le \frac{a_1}{2}\|v\|^2_{L^2(\Omega)} + \frac{a_2}{q}\|v\|^q_{L^q(\Omega)}$$

and so (4.10) follows by Lemma 2.2. By (4.9) and (4.10) we infer

$$\Theta(\rho) \le \frac{a_1 c_2^2}{2} + \frac{a_2 c_q^q}{q}\rho^{\frac{q-2}{2}}$$

which yields the claim (4.8). Now we prove that $\mathcal{I}_\lambda$ cannot have a strict global minimum in $\Phi^{-1}((-\infty,\rho))$. By (f_5) and arguing as in [14, Remark 3.2], we have $F(tv) \ge t^\mu F(v)$ for all $t \ge 1$ and $v \ge t_0$. Hence, it follows that

$$\mathcal{I}_\lambda(tu_0) = \frac{1}{\lambda}\Phi(tu_0) - \Psi(tu_0) \le \frac{t}{\lambda}\Phi(u_0) - t^\mu \int_{\{x\in\Omega\colon |u_0(x)|\ge t_0\}} F(|u_0(x)|)dx + c_F|\Omega|,$$

for every $u_0 \in X_{0,A}$, where $c_F = \max_{t\le t_0} |F(t)|$. Choosing u_0 such that

$$|\{x \in \Omega : |u_0(x)| \ge t_0\}| > 0,$$

recalling that $\mu > 2$ and $F(t) > 0$ for $t \ge t_0$, we get

$$\lim_{t\to\infty} \mathcal{I}_\lambda(tu_0) = -\infty.$$

Applying Theorem 4.1 we conclude the proof. □

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