General Tensor Decomposition, Moment Matrices and Applications

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Abstract

The tensor decomposition addressed in this paper may be seen as a generalisation of Singular Value Decomposition of matrices. We consider general multilinear and multihomogeneous tensors. We show how to reduce the problem to a truncated moment matrix problem and give a new criterion for flat extension of Quasi-Hankel matrices. We connect this criterion to the commutation characterisation of border bases. A new algorithm is described. It applies for general multihomogeneous tensors, extending the approach on binary forms by J. J. Sylvester. An example illustrates the algebraic operations involved in this approach and how the decomposition can be recovered from eigenvector computation.

Key words: tensor; decomposition; multihomogeneous polynomial; rank; Hankel operator; moment matrix; flat extension.

1. Introduction

Tensors are objects that appear in various contexts and applications. Matrices are tensors of order two, and are better known than tensors. But in many problems, higher order tensors are naturally used to collect information which depend on more than two variables. Typically, these data could be observations of some experimentation or of a physical phenomenon that depends on several parameters. These observations are stored in a structure called tensor, whose dimensional parameters (or modes) depend on the problem.

The tensor decomposition problem consists of decomposing a tensor (e.g. the set of observations) into a minimal sum of so-called *decomposable* tensors (i.e. tensors of rank

1). Such a decomposition which is independent of the coordinate system allows to extract geometric or invariant properties associated with the observations. For this reason, the tensor decomposition problem has a large impact in many applications. The first well known case is encountered for matrices (i.e. tensors of order 2), and is related to Singular Value Decomposition with applications e.g. to Principal Component Analysis. Its extension to higher order tensors appears in Electrical Engineering [58], in Signal processing [26], [20], in Antenna Array Processing [30] [18] or Telecommunications [61], [17], [54], [33], [29], in Chemometrics [11] or Psychometrics [39], in Data Analysis [22], [14], [31], [38], [56], but also in more theoretical domains such as Arithmetic complexity [40], [9], [57], [42]. Further numerous applications of tensor decompositions may be found in [20], [23], [56].

From a mathematical point of view, the tensors that we will consider are elements of $\mathcal{T} := S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ where $\delta_i \in \mathbb{N}$, E_i are vector spaces of dimension $n_i + 1$ over a field $\mathbb{K}$ (which is of characteristic 0 and algebraically closed), and $S^{\delta_i}(E_i)$ is the δ_i^{th} symmetric power of E_i. The set of tensors of rank 1 form a projective variety which is called the Veronese variety when $k = 1$ or the Segre variety when $\delta_i = 1, i = 1, \ldots, k$. We will call it hereafter the Segre-Veronese variety of $\mathbb{P}(\mathcal{T})$ and denote it $\Xi(\mathcal{T})$. The set of tensors which are the linear combinations of r elements of the Segre-Veronese variety are those which admit a decomposition with at most r terms of rank 1 (i.e. in $\Xi(\mathcal{T})$). The closure of this set is called the r-secant variety and denoted $\Xi_r(\mathcal{T})$. More precise definitions of these varieties will be given in Sec. 2.3.

Decomposing a tensor T consists of finding the minimal r such that this tensor is a sum of r tensors of rank 1. This minimal r is called the rank of T. By definition, a tensor of rank r is in the secant variety $\Xi_r(\mathcal{T})$. Thus analysing the properties of these secant varieties and their characterisation helps determining tensor ranks and decompositions.

The case where $k = 2$ and $\delta_1 = \delta_2 = 1$ corresponds to the matrix case, which is well known. The rank of a matrix when viewed as a tensor of order $k = 2$ is its usual rank. The case where $k = 1$ and $\delta_1 = 2$ corresponds to the case of quadratic forms and is also well understood. The rank of a symmetric tensor is the usual rank of the associated symmetric matrix. The case where $k = 1$, $\delta_1 \in \mathbb{N}$ and $n_1 = 1$ corresponds to binary forms, which have been analyzed by J.J. Sylvester in [59], [60]. A more complete description in terms of secant varieties is given in [45].

At our knowledge if $k > 1$ and if at least one of the δ_i's is larger than 1, then there is no specific result in the literature on the defining ideal of secant varieties of Segre-Veronese varieties $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ except for [16] and [41]. In the first the authors conjecture that when $\Xi_r(S^{\delta_1}(E_1) \otimes S^{\delta_2}(E_2)))$ is a defective hypersurface, then its defining equation is a determinantal equation. In the second the authors compute the equations of $\Xi(S^2(E) \otimes S^2(E) \otimes S^2(E))$ with $\dim(E) = 2$.

In the case of the secant varieties of Veronese varieties (i.e. if $k = 1$ and $\delta_1 > 1$), the knowledge of their ideal is sparse. Beside the classical results (see one for all [37]) we quote [45] as the most up-to-date paper on that subject.

About the case of secant varieties of Segre varieties (i.e. $\delta_i = 1$ for $i = 1, \ldots, k$) the only obvious case is the 2 factors Segre. For some of the non trivial cases in which equations of secant varieties of Segre varieties are known we refer to [43], [46], [3], [16].

The first method to compute a tensor decomposition, besides the case of matrices or quadratic forms which may go back to the Babylonians, is due to Sylvester for binary forms [59], [60]. Using apolarity, kernels of catalecticant matrices are computed degree by degree until a polynomial with simple roots is found. See also [21], [37]. An extension of this approach for symmetric tensors has been analyzed in [37], and yields a decomposition method in some cases (see [37][p. 187]). Some decomposition methods are also available for specific degrees and dimensions, e.g. using invariant theory [25]. In [8], there is a simplified version of Sylvester's algorithm presented in [21], which uses the mathematical interpretation of the problem in terms of secant varieties of rational normal curves. The same approach is used in [8] to give algorithms for the decompositions of symmetric tensors belonging to $\Xi_2(S^d(E))$ and to $\Xi_3(S^d(E))$. In [5] a complete rank stratification of $\Xi_4(S^d(E))$ is given.

In [10], Sylvester's approach is revisited from an affine point of view and a general decomposition method based on a flat extension criterion is described. The main contribution of the current paper is to extend this method to more general tensor spaces including classical multilinear tensors and multihomogeneous tensors. In particular we give a new and more flexible criterion for the existence of a decomposition of a given rank, which extends non trivially the result in [48] and the characterization used in [10]. This criterion is a rank condition of an associated Hankel operator. Moreover we use that criterion to write a new algorithm which checks, degree by degree, if the roots deduced from the kernel of the Hankel operator are simple. This allows to compute the rank of any given partially symmetric tensor.

This paper is an extended version of [7], with the complete proofs and with detailed examples.

In Sec. 2, we recall the notations, the geometric point of view related to secants of Segre and Veronese varieties, and the algebraic point of view based on moment matrices. In Sec. 3, we describe the algorithm and the criterion used to solve the truncated moment problem. In Sec. 4, an example of tensor decompositions from Antenna Array Processing illustrates the approach.

2. Duality, moment matrices and tensor decomposition

2.1. Notation and preliminaries

Let $\mathbb{K}$ be an algebraically closed field (e.g. $\mathbb{K} = \mathbb{C}$ the field of complex numbers). We assume that $\mathbb{K}$ is of characteristic 0. For a vector space E, its associated projective space is denoted $\mathbb{P}(E)$. For $\mathbf{v} \in E - \{0\}$ its class in $\mathbb{P}(E)$ is denoted $\overline{\mathbf{v}}$. Let $\mathbb{P}^n$ be the projective space of $E = \mathbb{K}^{n+1}$. For a subset $F = \{f_1, \ldots, f_m\}$ of a vector-space (resp. ring) R, we denote by $\langle F \rangle$ (resp. (F)) the vector space (resp. ideal) generated by F in R.

We consider hereafter the symmetric δ-th power $S^\delta(E)$ where E is a vector space of basis $x_0, \ldots, x_n$. An element of $S^\delta(E)$ is a homogeneous polynomial of degree $\delta \in \mathbb{N}$ in the variables $\mathbf{x} = (x_0, \ldots, x_n)$. For $\mathbf{x}_1 = (x_{0,1}, \ldots, x_{n_1,1})$, ... , $\mathbf{x}_k = (x_{0,k}, \ldots, x_{n_k,k})$, $S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ (with $E_i = \langle x_{0,i}, \ldots, x_{n_i,i} \rangle$) is the vector space of polynomials multihomogeneous polynomials of degree δ_i in the variables $\mathbf{x}_i$.

Hereafter, we will consider the dehomogeneization of elements in $S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$, obtained by setting $x_{0,i} = 1$ for $i = 1, \ldots, k$. We denote by $R_{\delta_1,\ldots,\delta_k}$ this space, where $R = \mathbb{K}[\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_k]$ is the space of polynomials in the variables $\underline{\mathbf{x}}_1 = (x_{1,1}, \ldots, x_{n_1,1}), \ldots, \underline{\mathbf{x}}_k = (x_{1,k}, \ldots, x_{n_k,k})$.

For $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{n_i,i}) \in \mathbb{N}^{n_i}$ $(i = 1, \ldots, k)$, let $\underline{\mathbf{x}}_i^{\alpha_i} = \prod_{j=1}^{n_i} x_{j,i}^{\alpha_{j,i}}$, $|\alpha_{\mathbf{i}}| = \sum_{j=1}^{n_i} \alpha_{j,i}$, and $\underline{\mathbf{x}}^\alpha = \prod_{j=1}^{n_i} \mathbf{x}_i^{\alpha_i}$.

An element f of $R_\delta := R_{\delta_1,\ldots,\delta_k}$ is represented as

$$f = \sum_{\alpha=(\alpha_1,\ldots,\alpha_k);|\alpha_i|\le\delta_i} f_\alpha\, \underline{\mathbf{x}}_1^{\alpha_1}\cdots\underline{\mathbf{x}}_k^{\alpha_k}.$$

The dimension of R_δ is $n_{\delta_1,\ldots,\delta_k;n_1,\ldots,n_k} = \prod_{i=1}^k \binom{n_i+\delta_i}{\delta_i}$. For $\delta \in \mathbb{N}, \alpha \in \mathbb{N}^n$ with $|\alpha| \le \delta$, let $\binom{\delta}{\alpha} = \frac{\delta!}{\alpha_1!\cdots\alpha_n!(\delta-|\alpha|)!}$. We define the apolar inner product on $R_{\delta_1,\ldots,\delta_k}$ by $\langle f|g\rangle = \sum_{|\alpha_i|\le\delta_i} f_\alpha\, g_\alpha \binom{\delta_1}{\alpha_1}^{-1}\cdots\binom{\delta_k}{\alpha_k}^{-1}$.

The dual space of a $\mathbb{K}$-vector space E is denoted $E^* = \mathrm{Hom}_{\mathbb{K}}(E,\mathbb{K})$. It is the set of $\mathbb{K}$-linear forms from E to $\mathbb{K}$. A basis of the dual space R_δ^*, is given by the set of linear forms that compute the coefficients of a polynomial in the monomial basis $(\underline{\mathbf{x}}^\alpha)_{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k};|\alpha_i|\le\delta_i}$. We denote it by $(\mathbf{d}^\alpha)_{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k};|\alpha_i|\le\delta_i}$. We identify R^* with the (vector) space of formal power series $\mathbb{K}[[\mathbf{d}]] = \mathbb{K}[[\mathbf{d}_1,\ldots,\mathbf{d}_k]] = \mathbb{K}[[d_{1,1},\ldots,d_{n_1,1},\ldots, d_{1,k},\ldots,d_{n_k,k}]]$. Any element $\Lambda \in R^*$ can be decomposed as

$$\Lambda = \sum_{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k}} \Lambda(\mathbf{x}^\alpha)\,\mathbf{d}^\alpha.$$

Remark that the action of R^* is different from the usual action of the ring of partial differential operators, which is often used in the apolar setting.

Typical elements of R^* are the linear forms that correspond to the evaluation at a point $\zeta = (\zeta_1,\ldots,\zeta_i) \in \mathbb{K}^{n_1}\times\cdots\times\mathbb{K}^{n_k}$:

$$\begin{aligned}\mathbf{1}_\zeta \,:\, R &\to \mathbb{K}\\ p &\mapsto p(\zeta)\end{aligned}$$

The decomposition of $\mathbf{1}_\zeta$ in the basis $\{\mathbf{d}^\alpha\}_{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k}}$ is

$$\mathbf{1}_\zeta = \sum_{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k}} \zeta^\alpha\,\mathbf{d}^\alpha = \sum_{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k}} \prod_{i=1}^k \zeta_i^{\alpha_i}\,\mathbf{d}_i^{\alpha_i}.$$

Notice that in characteristic 0, $\mathbf{d}^\alpha = \frac{1}{|\alpha_1|!\cdots|\alpha_k|!}\mathbf{1_0}\circ\prod\partial_{i,j}^{\alpha_{i,j}}$ is the linear form obtained by composition of differentials $\partial_{i,j}$ with respect to the variables $x_{i,j}$ and the evaluation at the origigin $\mathbf{0}$.

We recall that the dual space R^* has a natural structure of R-module [32] which is defined as follows: for all $p \in R$, and for all $\Lambda \in R^*$ consider the linear operator

$$\begin{aligned}p\star\Lambda \,:\, R &\to \mathbb{K}\\ q &\mapsto \Lambda(pq).\end{aligned}$$

In particular, we have $x_{i,j}\star\mathbf{d}_1^{\alpha_1}\cdots\mathbf{d}_j^{\alpha_j}\cdots\mathbf{d}_k^{\alpha_k} =$
$\mathbf{d}_1^{\alpha_1}\cdots\mathbf{d}_{j-1}^{\alpha_{j-1}}\,d_{1,j}^{\alpha_{1,j}}\cdots d_{i-1,j}^{\alpha_{i-1,j}}\,d_{i,j}^{\alpha_{i,j}-1}\,d_{i+1,j}^{\alpha_{i+1,j}}\cdots d_{n_j,j}^{\alpha_{n_j,j}}\,\mathbf{d}_{j+1}^{\alpha_{j+1}}\cdots\mathbf{d}_k^{\alpha_k}$ if $\alpha_{i,j} > 0$ and 0 otherwise.

2.2. Tensor decomposition

In this section, we present different formulations of the tensor decomposition problem, that we consider in this paper.

We will consider hereafter a partially symmetric tensor T of $S^{\delta_1}(E_1)\otimes\cdots\otimes S^{\delta_k}(E_k)$ where $E_i = \langle x_{0,i},\ldots,x_{n_i,i}\rangle$. It can be represented by a partially symmetric array of coefficients

$$[T] = (T_{\alpha_1,\ldots,\alpha_k})_{\alpha_i\in\mathbb{N}^{n_i+1};|\alpha_i|=\delta_i}. \tag{1}$$

For $\alpha_i \in \mathbb{N}^{n_i}$ with $|\alpha_i| \le \delta_i$, we denote $\overline{\alpha}_i = (\delta_i - |\alpha_i|, \alpha_{1,i},\ldots, \alpha_{n_i,i})$ and, with an abuse of notation, we identify $T_{\alpha_1,\ldots,\alpha_k} := T_{\overline{\alpha_1},\ldots,\overline{\alpha_k}}$.

Such a tensor is naturally associated with a (multihomogeneous) polynomial in the variables $\mathbf{x}_1 = (x_{0,1},\ldots,x_{n_1,1})$, …, $\mathbf{x}_k = (x_{0,k},\ldots,x_{n_k,k})$

$$T(\mathbf{x}) = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_k)\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k};\\ |\alpha_i|\le\delta_i}} T_\alpha\, \mathbf{x}_1^{\overline{\alpha}_1}\cdots\mathbf{x}_k^{\overline{\alpha}_k}.$$

or to an element $\underline{T}(\underline{\mathbf{x}}) \in R_{\delta_1,\ldots,\delta_k}$ obtained by substituting $x_{0,i}$ by 1 in $T(\mathbf{x})$ (for $i = 1,\ldots,k$):

$$\underline{T}(\underline{\mathbf{x}}) = \sum_{\substack{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k};\\ |\alpha_i|\le\delta_i}} T_\alpha\, \underline{\mathbf{x}}_1^{\alpha_1}\cdots\underline{\mathbf{x}}_k^{\alpha_k}.$$

An element of $R^* = \mathbb{K}[[\mathbf{d}]]$ can also be associated naturally with T:

$$T^*(\mathbf{d}) = \sum_{\substack{\alpha\in\mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_k};\\ |\alpha_i|\le\delta_i}} \binom{\delta_1}{\alpha_1}^{-1}\cdots\binom{\delta_k}{\alpha_k}^{-1} T_\alpha\, \mathbf{d}_1^{\alpha_1}\cdots\mathbf{d}_k^{\alpha_k}.$$

so that for all $T' \in R_{\delta_1,\ldots,\delta_k}$,

$$\langle T(\underline{\mathbf{x}})|T'(\underline{\mathbf{x}})\rangle = T^*(\mathbf{d})(T'(\underline{\mathbf{x}})).$$

The decomposition of tensor T can be stated as follows:

Tensor decomposition problem. *Given $T(\mathbf{x}) \in S^{\delta_1}(E_1)\otimes\cdots\otimes S^{\delta_k}(E_k)$, find a decomposition of $T(\mathbf{x})$ as a sum of products of powers of linear forms in $\mathbf{x}_j$:*

$$T(\mathbf{x}) = \sum_{i=1}^{r} \gamma_i\, \mathbf{l}_{1,i}(\mathbf{x}_1)^{\delta_1}\cdots\mathbf{l}_{k,i}(\mathbf{x}_k)^{\delta_k} \tag{2}$$

where $\gamma_i \neq 0$, $\mathbf{l}_{j,i}(\mathbf{x}_j) = l_{0,j,i}x_{0,j} + l_{1,j,i}x_{1,j} + \cdots + l_{n_j,j,i}x_{n_j,j}$ and r is the smallest possible integer for such a decomposition.

Definition 2.1. The minimal number of terms r in a decomposition of the form (2) is called the *rank* of T.

We say that $T(\mathbf{x})$ has an *affine decomposition* if there exists a minimal decomposition of $T(\mathbf{x})$ of the form (2) where r is the rank of T and such that $l_{0,j,i} \neq 0$ for $i = 1,\ldots,r$ and for $j = 1,\ldots,k$. Notice that by a generic change of coordinates in E_i, we may assume that all $l_{0,j,i} \neq 0$ and thus that T has an affine decomposition. Suppose that $T(\mathbf{x})$ has an affine decomposition. Then by scaling $\mathbf{l}_{j,i}(\mathbf{x}_j)$ and multiplying γ_i by the inverse of the δ_j^{th} power of this scaling factor, we may assume that $l_{0,j,i} = 1$. Thus, we can write the polynomial

$$\underline{T}(\underline{\mathbf{x}}) = \sum_{i=1}^{r} \gamma_i \sum_{|\alpha_i|\le\delta_i} \binom{\delta_1}{\alpha_1}\cdots\binom{\delta_k}{\alpha_k} \zeta_{1,i}^{\alpha_1}\cdots\zeta_{k,i}^{\alpha_k}\, \underline{\mathbf{x}}_1^{\alpha_1}\cdots\underline{\mathbf{x}}_k^{\alpha_k}$$

with $T_{\alpha_1,\ldots,\alpha_k} = \sum_{i=1}^{r} \gamma_i \sum_{|\alpha_i|\leq\delta_i} \binom{\delta_1}{\alpha_1}\cdots\binom{\delta_k}{\alpha_k} \zeta_{1,i}^{\alpha_1}\cdots\zeta_{k,i}^{\alpha_k}$. Equivalently, we have

$$T^*(\mathbf{d}) = \sum_{i=1}^{r} \gamma_i \sum_{|\alpha_i|\leq\delta_i} \zeta_{1,i}^{\alpha_1}\cdots\zeta_{k,i}^{\alpha_k} \mathbf{d}_1^{\alpha_1}\cdots\mathbf{d}_k^{\alpha_k}$$

so that $T^*(\mathbf{d})$ coincides on $R_{\delta_1,\ldots,\delta_k}$ with the linear form

$$\sum_{i=1}^{r} \gamma_i \, \mathbf{1}_{\zeta_{1,i},\ldots,\zeta_{k,i}} = \sum_{i=1}^{r} \gamma_i \, \mathbf{1}_{\zeta_i}$$

with $\zeta_i := (\zeta_{1,i},\ldots,\zeta_{k,i}) \in \mathbb{K}^{n_1} \times \cdots \mathbb{K}^{n_k}$.

The decomposition problem can then be restated as follows:

Interpolation problem. *Given $T^* \in R^*_{\delta_1,\ldots,\delta_k}$ which admits an affine decomposition, find the minimal number of non-zero vectors $\zeta_1,\ldots,\zeta_r \in \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_k}$ and non-zero scalars $\gamma_1,\ldots,\gamma_r \in \mathbb{K} - \{0\}$ such that*

$$T^* = \sum_{i=1}^{r} \gamma_i \, \mathbf{1}_{\zeta_i} \qquad (3)$$

on $R_{\delta_1,\ldots,\delta_k}$.

If such a decomposition exists, we say that $\Lambda = \sum_{i=1}^{r} \gamma_i \, \mathbf{1}_{\zeta_i} \in R^*$ extends $T^* \in R^*_{\delta_1,\ldots,\delta_k}$.

Hereafter, we will say that $\Lambda \in R^*$ extends $T^* \in R^*_{\delta_1,\ldots,\delta_k}$, if $\Lambda^*_{|R_{\delta_1,\ldots,\delta_k}} = T^*$.

2.3. Decomposable tensors

In this section, we analyze the set of tensors of rank 1, also called *decomposable* tensors [1]. They naturally form projective varieties, which we are going to describe using the language of projective geometry.

We begin by defining two auxiliary but very classical varieties, namely Segre and Veronese varieties.

Definition 2.2. The image of the following map

$$\begin{aligned} s_k : \mathbb{P}(E_1) \times \cdots \times \mathbb{P}(E_k) &\to \mathbb{P}(E_1 \otimes \cdots \otimes E_k) \\ (\overline{\mathbf{v_1}},\ldots,\overline{\mathbf{v_k}}) &\mapsto \overline{\mathbf{v_1} \otimes \cdots \otimes \mathbf{v}_k} \end{aligned}$$

is the so called Segre variety of k factors. We denote it by $\Xi(E_1 \otimes \cdots \otimes E_k)$.

From Definition 2.1 of the rank of a tensor and from the Interpolation Problem point of view (3), we see that a Segre variety parametrizes projective classes of rank 1 tensors $T = \mathbf{v_1} \otimes \cdots \otimes \mathbf{v_k} \in E_1 \otimes \cdots \otimes E_k$ for certain $\mathbf{v_i} \in E_i$, $i = 1,\ldots,k$.

Definition 2.3. Let (J_1, J_2) be a partition of the set $\{1,\ldots,k\}$. If $J_1 = \{h_1,\ldots,h_s\}$ and $J_2 = \{1,\ldots,k\} \backslash J_1 = \{h'_1,\ldots,h'_{k-s}\}$, the (J_1,J_2)-Flattening of $E_1 \otimes \cdots \otimes E_k$ is the following:

$$E_{J_1} \otimes E_{J_2} = (E_{h_1} \otimes \cdots \otimes E_{h_s}) \otimes (E_{h'_1} \otimes \cdots \otimes E_{h'_{k-s}}).$$

Let $E_{J_1} \otimes E_{J_2}$ be any flattening of $E_1 \otimes \cdots \otimes E_k$ as in Definition 2.3 and let $f_{J_1,J_2} : \mathbb{P}(E_1 \otimes \cdots \otimes E_k) \to \mathbb{P}(E_{J_1} \otimes E_{J_2})$ be the obvious isomorphism. Let $[T]$ be an array associated with a tensor $T \in E_1 \otimes \cdots \otimes E_k$; let $\overline{T'} = f_{J_1,J_2}(\overline{\mathbf{T}}) \in \mathbb{P}(E_{J_1} \otimes E_{J_2})$ and let $[A_{J_1,J_2}]$ be the matrix associated with T'. Then the d-minors of the matrix $[A_{J_1,J_2}]$ are said to be d-minors of $[T]$.

An array $[A] = (x_{i_1,\dots,i_k})_{0\leq i_j \leq n_j\,,\, j=1,\dots,k}$ is said to be a generic array of indeterminates of $R = \mathbb{K}[\underline{\mathbf{x}}_1, \dots, \underline{\mathbf{x}}_k]$ if the entries of $[A]$ are the independent variables of R.

It is a classical result due to R. Grone (see [35]) that a set of equations for a Segre variety is given by all the 2-minors of a generic array. In [36] it is proved that, if $[A]$ is a generic array in R of size $(n_1 + 1) \times \cdots \times (n_k + 1)$ and $I_d([A])$ is the ideal generated by the d-minors of $[A]$, then $I_2([A])$ is a prime ideal, therefore:

$$I(\Xi(E_1 \otimes \cdots \otimes E_k)) = I_2([A]).$$

We introduce now the Veronese variety. Classically it is defined to be the d-tuple embedding of $\mathbb{P}^n$ into $\mathbb{P}^{\binom{n+d}{d}-1}$ via the linear system associated with the sheaf $\mathcal{O}(d)$ with $d > 0$. We give here an equivalent definition.

Let E be an $n + 1$ dimensional vector space. With the notation $S^d(E)$ we mean the vector subspace of $E^{\otimes d}$ of symmetric tensors.

Definition 2.4. The image of the following map

$$\begin{aligned} \nu_d : \mathbb{P}(E) &\to \mathbb{P}(S^d(E)) \\ \overline{\mathbf{v}} &\mapsto \overline{\mathbf{v}^{\otimes \mathbf{d}}} \end{aligned}$$

is the so called Veronese variety. We indicate it with $\Xi(S^d(E))$.

With this definition it is easy to see that the Veronese variety parametrizes symmetric rank 1 tensors.

Observe that if we take the vector space E to be a vector space of linear forms $\langle x_0, \dots, x_n\rangle$ then the image of the map ν_d above parametrizes homogeneous polynomials that can be written as d-th powers of linear forms.

The Veronese variety $\Xi(S^d(E)) \subset \mathbb{P}(S^d(E))$ can be also viewed as $\Xi(S^d(E)) = \Xi(E^{\otimes d}) \cap \mathbb{P}(S^d(E))$.

Let $[A] = (x_{i_1,\dots,i_d})_{0\leq i_j\leq n,\, j=1,\dots,d}$ be a generic symmetric array. It is a known result that:

$$I(\Xi(S^d(E))) = I_2([A]). \tag{4}$$

See [62] for the set theoretical point of view. In [53] the author proved that $I(\Xi(S^d(E)))$ is generated by the 2-minors of a particular catalecticant matrix (for a definition of "Catalecticant matrices" see e.g. either [53] or [34]). A. Parolin, in his PhD thesis ([52]), proved that the ideal generated by the 2-minors of that catalecticant matrix is actually $I_2([A])$.

We are now ready to describe the geometric object that parametrizes partially symmetric tensors $T \in S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$. Let us start with the rank 1 partially symmetric tensors.

Definition 2.5. Let $E_1, \dots, E_k$ be vector spaces of dimensions $n_1 + 1, \dots, n_k + 1$ respectively. The Segre-Veronese variety $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ is the embedding of $\mathbb{P}(E_1) \otimes \cdots \otimes \mathbb{P}(E_k)$ into $\mathbb{P}^{N-1} \simeq \mathbb{P}(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$, where $N = \left(\Pi_{i=1}^{k}\binom{n_i+\delta_i}{d_i}\right)$,

given by sections of the sheaf $\mathcal{O}(\delta_1, \ldots, \delta_k)$.
I.e. $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ is the image of the composition of the following two maps:

$$\mathbb{P}(E_1) \times \cdots \times \mathbb{P}(E_k) \stackrel{\nu_{\delta_1} \times \cdots \times \nu_{\delta_k}}{\longrightarrow} \mathbb{P}^{\binom{n_1+\delta_1}{\delta_1}-1} \times \cdots \times \mathbb{P}^{\binom{n_k+\delta_k}{\delta_k}-1}$$

and $\mathbb{P}^{\binom{n_1+\delta_1}{\delta_1}-1} \times \cdots \times \mathbb{P}^{\binom{n_k+\delta_k}{\delta_t}-1} \stackrel{s}{\longrightarrow} \mathbb{P}^{N-1}$, where each ν_{δ_i} is a Veronese embedding of $\mathbb{P}(E_i)$ as in Definition 2.4, then $\mathrm{Im}(\nu_{\delta_1} \times \cdots \times \nu_{\delta_k}) = \Xi(S^{\delta_1}(E_1)) \times \cdots \times \Xi(S^{\delta_k}(E_k))$ and $\mathrm{Im}(s)$ is the Segre variety of k factors. Therefore the Segre-Veronese variety is the Segre re-embedding of the product of k Veronese varieties.

If $(\delta_1, \ldots, \delta_k) = (1, \ldots, 1)$ then the corresponding Segre-Veronese variety is nothing else than the classical Segre variety of $\mathbb{P}(E_1 \otimes \cdots \otimes E_k)$.

If $k = 1$ then the corresponding Segre-Veronese variety is nothing else than the classical Veronese variety of $\mathbb{P}(S^{\delta_1}(E_1))$.

Observe that $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ can be viewed as the intersection with the Segre variety $\Xi(E_1^{\otimes\delta_1} \otimes \cdots \otimes E_k^{\otimes\delta_k})$ that parametrizes rank one tensors and the projective subspace $\mathbb{P}(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)) \subset \mathbb{P}(E_1^{\otimes\delta_1} \otimes \cdots \otimes E_k^{\otimes\delta_k})$ that parametrizes partially symmetric tensors: $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)) = \Xi(E_1^{\otimes\delta_1} \otimes \cdots \otimes E_k^{\otimes\delta_k}) \cap \mathbb{P}(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$.

In [6] it is proved that if $[A]$ is a generic array of indeterminates associated with the multihomogeneous polynomial ring $S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ (i.e. it is a generic partially symmetric array), the ideal of the Segre-Veronese variety $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ is

$$I(\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))) = I_2([A])$$

with $\delta_i > 0$ for $i = 1, \ldots, k$.

Now if we consider the vector spaces of linear forms $E_i \simeq S^1(E_i)$ for $i = 1, \ldots, k$, we get that the Segre-Veronese variety $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ parametrizes multihomogenoeus polynomials $F \in S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ of the type $F = \mathbf{l}_1^{\delta_1} \cdots \mathbf{l}_k^{\delta_k}$ where $\mathbf{l}_i$ are linear forms in $S^1(E_i)$ for $i = 1, \ldots, k$.

From this observation we understand that the tensor decomposition problem of finding a minimal decomposition of type (2) for an element $T \in S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ is equivalent to finding the minimum number of elements belonging to the Segre-Veronese variety $\Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ whose span contains $\overline{T} \in \mathbb{P}(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$.

The natural geometric objects that are associated with this kind of problems are the higher secant varieties of the Segre-Veronese varieties that we are going to define.

Definition 2.6. Let $X \subset \mathbb{P}^N$ be any projective variety and define

$$X_s^0 := \bigcup_{\overline{\mathbf{P}}_1, \ldots, \overline{\mathbf{P}}_s \in X} \langle \overline{\mathbf{P}}_1, \ldots, \overline{\mathbf{P}}_s \rangle.$$

The s-th secant variety $X_s \subset \mathbb{P}^N$ of X is the Zariski closure of X_s^0.

Observe that the generic element of X_s is a point $\overline{\mathbf{P}} \in \mathbb{P}^N$ that can be written as a linear combination of s points of X. In fact a generic element of X_s is an element of X_s^0. Therefore if X is the Segre-Veronese variety, then the generic element of $\Xi_s(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ is the projective class of a partially symmetric tensor $T \in S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ that can be written as a linear combination of s linearly independent partially symmetric tensors of rank 1. Unfortunately not all the elements of $\Xi_s(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ are of this form. In fact if $\overline{T} \in \Xi_s(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)) \setminus \Xi_s^0(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ then the rank of T is strictly bigger than s.

Definition 2.7. The minimum integer s such that $\overline{T} \in \mathbb{P}(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ belongs to $\Xi_s(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ is called the border rank of T.

In order to find the border rank of a tensor $T \in S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$ we should need a set of equations for $\Xi_s(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ for $s > 1$. The knowledge of the generators of the ideals of secant varieties of homogeneous varieties is a very deep problem that is solved only in very particular cases (see eg. [51], [46], [44], [45], [12]).

From a computational point of view, there is a very direct and well known way of getting the equations for the secant variety, which consists in introducing parameters or unknowns for the coefficients of $\mathbf{l}_{i,j}$ and γ_i in (2), to expand the polynomial and identify its coefficients with the coefficients of T. Eliminating the coefficients of $\mathbf{l}_{i,j}$ and γ_i yields the equations of the secant variety.

Unfortunately this procedure is far from being computationally practical, because we have to deal with high degree polynomials in many variables, with a lot of symmetries. This is why we need to introduce moment matrices and to use a different kind of elimination.

2.4. Moment matrices

In this section, we recall the algebraic tools and the properties we need to describe and analyze our algorithm. Refer e.g. to [10], [32], [50] for more details.

As above, we denote by $R = \mathbb{K}[\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_k]$ the space of polynomials in the variables $\underline{\mathbf{x}}_1 = (x_{1,1}, \ldots, x_{n_1,1}), \ldots, \underline{\mathbf{x}}_k = (x_{1,k}, \ldots, x_{n_k,k})$. For any $\Lambda \in R^*$, define the bilinear form Q_Λ, such that $\forall a, b \in R$, $Q_\Lambda(a, b) = \Lambda(ab)$. The matrix of Q_Λ in the monomial basis, of R is $\mathbb{Q}_\Lambda = (\Lambda(\mathbf{x}^{\alpha+\beta}))_{\alpha,\beta}$, where $\alpha, \beta \in \mathbb{N}^N$ with $N = n_1 + \cdots + n_k$. Similarly, for any $\Lambda \in R^*$, we define the Hankel operator H_Λ from R to R^* as

$$\begin{aligned} H_\Lambda \,:\, R &\to R^* \\ p &\mapsto p \star \Lambda. \end{aligned}$$

The matrix of the linear operator H_Λ in the monomial basis, and in the dual basis, $\{\mathbf{d}^\alpha\}$, is $\mathbb{H}_\Lambda = (\Lambda(\mathbf{x}^{\alpha+\beta}))_{\alpha,\beta}$, where $\alpha, \beta \in \mathbb{N}^N$. The following relates the Hankel operators with the bilinear forms. For all $a, b \in R$, thanks to the R-module structure, the following holds:

$$Q_\Lambda(a, b) = \Lambda(ab) = a \star \Lambda(b) = H_\Lambda(a)(b).$$

In what follows, we will identify H_Λ and Q_Λ.

Definition 2.8. Given $B = \{b_1, \ldots, b_r\}, B' = \{b'_1, \ldots, b'_{r'}\} \subset R$, we define

$$H_\Lambda^{B,B'} : \langle B \rangle \to \langle B' \rangle^*,$$

as the restriction of H_Λ to the vector space $\langle B \rangle$ and the projection of R^* in $\langle B' \rangle^*$. Let $\mathbb{H}_\Lambda^{B,B'} = (\Lambda(b_i\, b'_j))_{1 \le i \le r, 1 \le j \le r'}$. If $B' = B$, we also use the notation H_Λ^B and $\mathbb{H}_\Lambda^B$.

If B, B' are linearly independent, then $\mathbb{H}_\Lambda^{B,B'}$ is the matrix of $H_\Lambda^{B,B'}$ in this basis $\{b_1, \ldots, b_r\}$ of $\langle B \rangle$ and the dual basis of B' in $\langle B' \rangle^*$.

From the definition of the Hankel operators, we can deduce that a polynomial $p \in R$ belongs to the kernel of $\mathbb{H}_\Lambda$ if and only if $p \star \Lambda = 0$, which in turn holds if and only if for all $q \in R$, $\Lambda(pq) = 0$.

Proposition 2.9. Let I_Λ be the kernel of H_Λ. Then, I_Λ is an ideal of R.

Proof. Let $p_1, p_2 \in I_\Lambda$. Then for all $q \in R$, $\Lambda((p_1 + p_2)q) = \Lambda(p_1 q) + \Lambda(p_2 q) = 0$. Thus, $p_1 + p_2 \in I_\Lambda$.

If $p \in I_\Lambda$ and $p' \in R$, then for all $q \in R$, $\Lambda(pp'q) = 0$ holds. Thus $pp' \in I_\Lambda$ and I_Λ is an ideal. □

Let $\mathcal{A}_\Lambda = R/I_\Lambda$ be the quotient algebra of polynomials modulo the ideal I_Λ, which, as Proposition 2.9 states, is the kernel of H_Λ. The rank of H_Λ is the dimension of $\mathcal{A}_\Lambda$ as a $\mathbb{K}$-vector space.

Definition 2.10. For any $B \subset R$, let $B^+ = B \cup \underline{\mathbf{x}}_1 B \cup \cdots \cup \underline{\mathbf{x}}_k B$ and $\partial B = B^+ \setminus B$.

For the proof of the following proposition we refer to [10].

Proposition 2.11. Assume that $\operatorname{rank}(H_\Lambda) = r < \infty$ and let $B = \{b_1, \ldots, b_r\} \subset R$ such that $\mathbb{H}_\Lambda^B$ is invertible. Then $b_1, \ldots, b_r$ is a basis of $\mathcal{A}_\Lambda$. If $1 \in \langle B \rangle$ the ideal I_Λ is generated by $\ker H_\Lambda^{B^+}$.

Proposition 2.12. Assume that $\mathbb{K}$ is algebraically closed and of characteristic 0. If $\operatorname{rank}(H_\Lambda) = r < \infty$, then $\mathcal{A}_\Lambda$ is of dimension r over $\mathbb{K}$ and there exist $\zeta_1, \ldots, \zeta_d \in \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_k}$, where $d \leq r$, and $p_i \in \mathbb{K}[\partial] = \mathbb{K}[\partial_1, \ldots, \partial_n]$, with $\partial_1 = (\partial_{1,1}, \ldots, \partial_{n_1,1}), \ldots, \partial_k = (\partial_{1,k}, \ldots, \partial_{n_k,k})$, such that

$$\Lambda = \sum_{i=1}^{d} \mathbf{1}_{\zeta_i} \circ p_i(\partial) \tag{5}$$

Moreover the multiplicity of ζ_i is the dimension of the vector space spanned the inverse system generated by $\mathbf{1}_{\zeta_i} \circ p_i(\partial)$.

Proof. Since $\operatorname{rank}(\mathbb{H}_\Lambda) = r$, the dimension of the vector space $\mathcal{A}_\Lambda$ is also r. Thus the number of zeros of the ideal I_Λ, say $\{\zeta_1, \ldots, \zeta_d\}$ is at most r, viz. $d \leq r$. We can apply the structure Theorem [32, Th. 7.34, p. 185] in order to get the decomposition. □

In characteristic 0, the inverse system generated by $\mathbf{1}_{\zeta_i} \circ p_i(\partial)$ is isomorphic to the vector space generated by p_i and its derivatives of any order with respect to the variables ∂_i. In general characteristic, a similar result holds but we replace the differentials by the dual variables $d_{i,j}$ and the derivations by the product by the "inverse" of the variables [50], [32].

Definition 2.13. For $T^* \in R^*_{\delta_1,\ldots,\delta_k}$, we call *generalized decomposition* of T^* a decomposition such that $T^* = \sum_{i=1}^{d} \mathbf{1}_{\zeta_i} \circ p_i(\partial)$ where the sum for $i = 1, \ldots, d$ of the dimensions of the vector spaces spanned by the inverse system generated by $\mathbf{1}_{\zeta_i} \circ p_i(\partial)$ is minimal. This minimal sum of dimensions is called the length of f.

This definition extends the definition introduced in [37] for binary forms. The length of T^* is the rank of the corresponding Hankel operator H_Λ.

Theorem 2.14. Let $\Lambda \in R^*$, then $\Lambda = \sum_{i=1}^{r} \gamma_i \mathbf{1}_{\zeta_i}$ with $\gamma_i \neq 0$ and ζ_i distinct points of $\mathbb{K}^n$, iff rank $H_\Lambda = r$ and I_Λ is a radical ideal.

Proof. Let $\Lambda = \sum_{i=1}^{r} \gamma_i \mathbf{1}_{\zeta_i}$, with $\gamma_i \neq 0$ and ζ_i distinct points of $\mathbb{K}^n$. Let $\{e_1, \ldots, e_r\}$ be a family of interpolation polynomials at these points: $e_i(\zeta_j) = 1$ if $i = j$ and 0 otherwise. Let I_ζ be the ideal of polynomials which vanish at $\zeta_1, \ldots, \zeta_r$. It is a radical ideal. We have clearly $I_\zeta \subset I_\Lambda$. For any $p \in I_\Lambda$, and $i = 1, \ldots, r$, we have $p \star \Lambda(e_i) = \Lambda(p\, e_i) = p(\zeta_i) = 0$, which proves that $I_\Lambda = I_\zeta$ is a radical ideal. As the quotient $\mathcal{A}_\Lambda$ is generated by the interpolation polynomials $e_1, \ldots, e_r$, H_Λ is of rank r.

Conversely, if rank $H_\Lambda = r$, by Proposition 2.12, where $\Lambda = \sum_{i=1}^{r} \mathbf{1}_{\zeta_i} \circ p_i(\partial)$ is a polynomial of degree 0, since the multiplicity of ζ_i is 1. This concludes the proof of the equivalence. □

In the binary case this also corresponds to the border rank of T^*, therefore the r-th minors of the Hankel operator give equations for the r-th secant variety to the rational normal curves [37].

In order to compute the zeroes of an ideal I_Λ when we know a basis of $\mathcal{A}_\Lambda$, we exploit the properties of the operators of multiplication in $\mathcal{A}_\Lambda$: $M_a : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$, such that $\forall b \in \mathcal{A}_\Lambda$, consider $M_a(b) = a\, b$ and its transposed operator $M_a^t : \mathcal{A}_\Lambda^* \to \mathcal{A}_\Lambda^*$, such that for $\forall \gamma \in \mathcal{A}_\Lambda^*, M_a^\top(\gamma) = a \star \gamma$.

The following proposition expresses a similar result, based on the properties of the duality.

Proposition 2.15. For any linear form $\Lambda \in R^*$ such that rank $H_\Lambda < \infty$ and any $a \in \mathcal{A}_\Lambda$, we have

$$H_{a\star\Lambda} = M_a^t \circ H_\Lambda \tag{6}$$

Proof. By definition, $\forall p \in R, H_{a\star\Lambda}(p) = a\, p \star \Lambda = a \star (p \star \Lambda) = M_a^\top \circ H_\Lambda(p)$. □

We have the following well-known theorem:

Theorem 2.16 ([28,27,32])**.** Assume that $\mathbb{K}$ is algebraically closed and of characteristic 0. If $\mathcal{A}_\Lambda$ is a finite dimensional vector space, then $\Lambda = \sum_{i=1}^{d} \mathbf{1}_{\zeta_i} \circ p_i(\partial)$ for $\zeta_i \in \mathbb{K}^n$ and $p_i(\partial) \in \mathbb{K}[\partial] = \mathbb{K}[\partial_1, \ldots, \partial_n]$ and

- the eigenvalues of the operators M_a and M_a^t, are given by $\{a(\zeta_1), \ldots, a(\zeta_r)\}$.
- the common eigenvectors of the operators $(M_{x_i}^t)_{1\leq i\leq n}$ are (up to scalar) $\mathbf{1}_{\zeta_i}$.

Using the previous proposition, one can recover the points $\zeta_i \in \mathbb{K}^n$ by eigenvector computation as follows. Assume that $B \subset R$ with $|B| = \text{rank}(H_\Lambda)$, then equation (6) and its transposition yield

$$\mathbb{H}_{a\star\Lambda}^B = \mathbb{M}_a^t \mathbb{H}_\Lambda^B = \mathbb{H}_\Lambda^B\, \mathbb{M}_a,$$

where $\mathbb{M}_a$ is the matrix of multiplication by a in the basis B of $\mathcal{A}_\Lambda$. By Theorem 2.16, the common solutions of the generalized eigenvalue problem

$$(\mathbb{H}_{a\star\Lambda} - \lambda\, \mathbb{H}_\Lambda)\mathbf{v} = \mathbb{O} \tag{7}$$

for all $a \in R$, yield the common eigenvectors $\mathbb{H}_\Lambda^B \mathbf{v}$ of $\mathbb{M}_a^t$, that is the evaluation $\mathbf{1}_{\zeta_i}$ at the roots. Therefore, these common eigenvectors $\mathbb{H}_\Lambda^B \mathbf{v}$ are up to a scalar, the vectors $[b_1(\zeta_i), \ldots, b_r(\zeta_i)]$ $(i = 1, \ldots, r)$. Notice that it is sufficient to compute the common eigenvectors of $(\mathbb{H}_{x_i\star\Lambda}, \mathbb{H}_\Lambda)$ for $i = 1, \ldots, n$

If $\Lambda = \sum_{i=1}^{d} \gamma_i \mathbf{1}_{\zeta_i}$ ($\gamma_i \neq 0$), then the roots are simple, and one eigenvector computation is enough: for any $a \in R$, $\mathbb{M}_a$ is diagonalizable and the generalized eigenvectors $\mathbb{H}_\Lambda^B \mathbf{v}$ are, up to a scalar, the evaluation $\mathbf{1}_{\zeta_i}$ at the roots.

Coming back to our problem of partially symmetric tensor decomposition, $T^* \in R^*_{\delta_1,\ldots,\delta_k}$ admits an affine decomposition of rank r iff T^* coincide on $R_{\delta_1,\ldots,\delta_k}$ with

$$\Lambda = \sum_{i=1}^{r} \gamma_i \, \mathbf{1}_{\zeta_i},$$

for some distinct $\zeta_1, \ldots, \zeta_r \in \mathbb{K}^{n_1} \times \cdots \times \mathbb{K}^{n_k}$ and some $\gamma_i \in \mathbb{K} - \{0\}$. Then, by theorem 2.14, H_Λ is of rank r and I_Λ is radical.

Conversely, given H_Λ of rank r with I_Λ radical which coincides on $R_{\delta_1,\ldots,\delta_k}$ with T^*, by proposition 2.12, $\Lambda = \sum_{i=1}^{r} \gamma_i \, \mathbf{1}_{\zeta_i}$ and extends T^*, which thus admits an affine decomposition.

The problem of decomposition of T^* can then be reformulated as follows:

Truncated moment problem. *Given $T^* \in R^*_{\delta_1,\ldots,\delta_k}$, find the smallest r such that there exists $\Lambda \in R^*$ which extends T^* with H_Λ of rank r and I_Λ a radical ideal.*

In the next section, we will describe an algorithm to solve the truncated moment problem.

3. Algorithm

In this section, we first describe the algorithm from a geometric point of view and consider the algebraic computation it induces. Then we characterize under which conditions, the element T^* can be extended to $\Lambda \in R^*$ with H_Λ is of rank r. The algorithm is described in 3.1. It extends the one in [10] which applies only to symmetric tensors. The approach used in [8] for the rank of tensors in $\Xi_2(S^d(E))$ and in $\Xi_3(S^d(E))$ allows to avoid to loop again at step 4: if one doesn't get simple roots, then it is possible to use other techniques to compute the rank. Unfortunately the mathematical knowledge on the stratification by rank of secant varieties is nowadays not complete, hence the techniques developped in [8] cannot be used to improve algorithms for higher border ranks yet.

Algorithm 3.1: DECOMPOSITION ALGORITHM

Input: a tensor $T \in S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)$.
Output: a minimal decomposition of T.
Set $r = 0$;

(1) Determine if T^* can be extended to $\Lambda \in R^*$ with $\operatorname{rank} H_\Lambda = r$;
(2) Find if there exists r distinct points $P_1, \ldots, P_r \in \Xi(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ such that $T \in \langle P_1, \ldots, P_r \rangle \simeq \mathbb{P}^{r-1}$; Equivalently compute the roots of $\ker H_\Lambda$ by generalized eigenvector computation (7) and check that the eigenspaces are simple;
(3) If the answer to 2 is YES, then it means that $T \in \Xi_r^o(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k)) \setminus \Xi_{r-1}(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$; therefore the rank of T is actually r and we are done;
(4) If the answer to 2 is NO, then it means that $T \notin \Xi_r^o(S^{\delta_1}(E_1) \otimes \cdots \otimes S^{\delta_k}(E_k))$ hence its rank is bigger than r; Repeat this procedure from step 2 with $r + 1$.

We are going to characterize now under which conditions T^* can be extended to $\Lambda \in R^*$ with H_Λ of rank r (step 1).

We need the following technical property on the bases of $\mathcal{A}_\Lambda$, that we will consider:

Definition 3.1. Let B be a subset of monomials in $R = \mathbb{K}[\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_k]$. We say that B is connected to 1 if $\forall m \in B$ either $m = 1$ or there exists $i \in [1, n]$ and $m' \in B$ such that $m = \underline{\mathbf{x}}_i \, m'$.

Let $B, B' \subset R_{\delta_1,\ldots,\delta_k}$ be two sets of monomials connected to 1. We consider the formal Hankel matrix

$$\mathcal{H}_\Lambda^{B,B'} = (h_{\alpha+\beta})_{\alpha\in B',\beta\in B},$$

with $h_\alpha = T^*(\mathbf{x}^\alpha) = c_\alpha$ if $\mathbf{x}^\alpha \in R_{\delta_1,\ldots,\delta_k}$ and otherwise h_α is a variable. The set of these new variables is denoted $\mathbf{h}$.

Suppose that $\mathcal{H}_\Lambda^{B,B'}$ is invertible in $\mathbb{K}(\mathbf{h})$, then we define the formal multiplication operators

$$\mathcal{M}_{i,l}^{B,B'}(\mathbf{h}) := (\mathcal{H}_\Lambda^{B,B'})^{-1}\mathcal{H}_{x_{i,l}\star\Lambda}^{B,B'}$$

for every variable $x_{i,l} \in R$.

We use the following theorems which extend the results of [48] to the cases of distinct sets of monomials indexing the rows and columns of the Hankel operators. They characterize the cases where $\mathbb{K}[\mathbf{x}] = B \oplus I_\Lambda$:

Theorem 3.2. Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ and $B' = \{\mathbf{x}^{\beta'_1}, \ldots, \mathbf{x}^{\beta'_r}\}$ be two sets of monomials of in $R_{\delta_1,\ldots,\delta_k}$, connected to 1 and let Λ be a linear form that belongs to $(\langle B' \cdot B^+\rangle_{\delta_1,\ldots,\delta_k})^*$. Let $\Lambda(\mathbf{h})$ be the linear form of $\langle B' \cdot B^+\rangle^*$ defined by $\Lambda(\mathbf{h})(\mathbf{x}^\alpha) = \Lambda(\mathbf{x}^\alpha)$ if $\mathbf{x}^\alpha \in R_{\delta_1,\ldots,\delta_k}$ and $h_\alpha \in \mathbb{K}$ otherwise. Then, $\Lambda(\mathbf{h})$ admits an extension $\tilde{\Lambda} \in R^*$ such that $H_{\tilde{\Lambda}}$ is of rank r with B and B' basis of $A_{\tilde{\Lambda}}$ iff

$$\mathcal{M}_{i,l}^{B}(\mathbf{h}) \circ \mathcal{M}_{j,q}^{B}(\mathbf{h}) - \mathcal{M}_{j,q}^{B}(\mathbf{h}) \circ \mathcal{M}_{i,l}^{B}(\mathbf{h}) = 0 \qquad (8)$$

$(0 \le l, q \le k, 1 \le i \le n_l, 1 \le j \le n_q)$ and $\det(\mathcal{H}_{\Lambda(\mathbf{h})}^{B',B}) \neq 0$. Moreover, such a $\tilde{\Lambda}$ is unique.

Proof. If there exists $\tilde{\Lambda} \in R^*$ which extends $\Lambda(\mathbf{h})$, with $H_{\tilde{\Lambda}}$ of rank r and B and B' basis of $A_{\tilde{\Lambda}}$ then $\mathcal{H}_{\Lambda(\mathbf{h})}^{B',B}$ is invertible and the tables of multiplications by the variables $x_{i,l}$:

$$\mathcal{M}_{i,l} := (\mathcal{H}_{\Lambda(\mathbf{h})}^{B',B})^{-1}\mathcal{H}_{x_{i,l}\star\Lambda(\mathbf{h})}^{B',B}$$

(Proposition 2.15) commute.

Conversely suppose that these matrices commute and consider them as linear operators on $\langle B\rangle$. Then by [49], we have such a decomposition $R = \langle B\rangle \oplus I$ where I is an ideal of R. As a matter of fact, using commutation relation and the fact that B is connected to 1, one can easily prove that the following morphism:

$$\pi : R \longrightarrow \langle B\rangle$$

$$p \to p(\mathcal{M})(1)$$

is a projection on $\langle B\rangle$ whose kernel is the ideal I of R (note that for any $p \in \langle B\rangle$, $p(M)(1) = p$).

We define $\tilde{\Lambda} \in R^*$ as follows: $\forall p \in \mathbb{R}, \tilde{\Lambda}(p) = \Lambda(p(\mathcal{M})(1))$ where $p(\mathcal{M})$ is the operator obtained by substitution of the variables $x_{i,l}$ by the commuting operators $\mathcal{M}_{i,l}$. Notice that $p(\mathcal{M})$ is also the operator of multiplication by p modulo I.

Let us prove by induction on the degree of $b' \in B'$ that for all $b \in B$:

$$\Lambda(\mathbf{h})(b'\, b) = \Lambda(\mathbf{h})(b'(\mathcal{M})(b)) \tag{9}$$

and thus by linearity that

$$\Lambda(\mathbf{h})(b'\, p) = \Lambda(\mathbf{h})(b'(\mathcal{M})(p)) \tag{10}$$

for all $p \in \langle B \rangle$.

The property is obviously true for $b' = 1$. Suppose now that $b' \in B'$ is a monomial of degree strictly greater than zero. As B' is connected to 1, one has $b' = x_{j,q}\, b''$ for some variable $x_{j,q} \in R$ and some element $b'' \in B'$ of degree smaller than the degree of b'. By construction of the operators of multiplication $(\mathcal{M}_{i,l})$, we have

$$\Lambda(\mathbf{h})(b'\, b) = \Lambda(\mathbf{h})(b''\, x_{j,q}\, b) = \Lambda(\mathbf{h})(b''\, \mathcal{M}_{j,q}(b)).$$

Finally we have that $\Lambda(\mathbf{h})(b'\, b) = \Lambda(\mathbf{h})(b'(\mathcal{M})(b))$ and (9) is proved.
Let us deduce now that $\tilde{\Lambda}$ extends $\Lambda(\mathbf{h})$ i.e that for all $b^+ \in B^+$ ans $b' \in B'$ we have:

$$\tilde{\Lambda}(b'\, b^+) := \Lambda(\mathbf{h})((b'\, b^+)(\mathcal{M})(1)) = \Lambda(\mathbf{h})(b'\, b^+).$$

Indeed, from (10) we have:

$$\Lambda(\mathbf{h})((b'\, b^+)(\mathcal{M})(1)) = \Lambda(\mathbf{h})((b'(\mathcal{M})\, b^+(\mathcal{M})(1)) = \Lambda(\mathbf{h})(b'\, [b^+(\mathcal{M})(1)])$$

as $b^+(\mathcal{M})(1)$ belongs to $\langle B \rangle$. Then, by definition of the multiplication operators $(\mathcal{M}_{i,l})$ we have

$$\Lambda(\mathbf{h})(b'\, [b^+(\mathcal{M})(1)]) = \Lambda(\mathbf{h})(b'\, b^+).$$

Thus, we have

$$\tilde{\Lambda}(b'\, b^+) = \Lambda(\mathbf{h})(b'\, b^+) \tag{11}$$

for all $b^+ \in B^+$ ans $b' \in B'$ (i.e $\tilde{\Lambda}$ extends $\Lambda(\mathbf{h})$).

We eventually need to prove that $I_{\tilde{\Lambda}} = I := \mathrm{Ker}(\pi)$. By the definition of $\tilde{\Lambda}$ we obviously have that $I \subset I_{\tilde{\Lambda}}$. Let us prove that $I_{\tilde{\Lambda}} \subset I$: assume p belongs to $I_{\tilde{\Lambda}}$, then from (11)

$$\tilde{\Lambda}(b'\, p(\mathcal{M})(1)) = \Lambda(\mathbf{h})(b'\, p(\mathcal{M})(1)) = 0$$

for all $b' \in B'$. As $p(\mathcal{M})(1) \in \langle B \rangle$ and $\det(\mathcal{H}_{\Lambda}^{B',B})(\mathbf{h}) \neq 0$, we deduce that $p(\mathcal{M})(1) = 0$ and that p belongs to I. Thus we have $I_{\tilde{\Lambda}} \subset I$.

Eventually, $\tilde{\Lambda}$ extends $\Lambda(\mathbf{h})$ with $I_{\tilde{\Lambda}} = I := \mathrm{Ker}(\pi)$ and $\mathcal{A}_{\tilde{\Lambda}}$ equal to $R/I \simeq \langle B \rangle$ which is a zero dimensional algebra of multiplicity r with basis B.

If there exists another $\Lambda' \in R^*$ which extends $\Lambda(\mathbf{h}) \in \langle B' \cdot B^+ \rangle^*$ with $\operatorname{rank} H_{\Lambda'} = r$, by proposition 2.11, $\ker H_{\Lambda'}$ is generated by $\ker H_{\Lambda'}^{B',B^+}$ and thus coincides with $\ker H_{\tilde{\Lambda}}$. As Λ' coincides with $\tilde{\Lambda}$ on B, the two elements of R^* must be equal. This ends the proof of the theorem. □

The degree of these commutation relations is at most 2 in the coefficients of the multiplications matrices $\mathcal{M}_{i,l}$. A direct computation yields the following, for $m \in B$:

- If $x_{i,l}, m \in B$, and $x_{j,q}\, m \in B$ then $(\mathcal{M}_{i,l}^B \circ \mathcal{M}_{j,q}^B - \mathcal{M}_{j,q}^B \circ \mathcal{M}_{i,l}^B)(m) \equiv 0$ in $\mathbb{K}(\mathbf{h})$.
- If $x_{i,l} m \in B$, $x_{j,q}\, m \notin B$ then $(\mathcal{M}_{i,l}^B \circ \mathcal{M}_{j,q}^B - \mathcal{M}_{j,q}^B \circ \mathcal{M}_{i,l}^B)(m)$ is of degree 1 in the coefficients of $\mathcal{M}_{i,l}, \mathcal{M}_{j,q}$.

- If $x_{i,l}m \notin B$, $x_{j,q}\, m \notin B$ then $(\mathcal{M}^B_{i,l} \circ \mathcal{M}^B_{j,q} - \mathcal{M}^B_{j,q} \circ \mathcal{M}^B_{i,l})(m)$ is of degree 2 in the coefficients of $\mathcal{M}_{i,l}, \mathcal{M}_{j,q}$.

We are going to give an equivalent characterization of the extension property, based on rank conditions:

Theorem 3.3. Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ and $B' = \{\mathbf{x}^{\beta'_1}, \ldots, \mathbf{x}^{\beta'_r}\}$ be two sets of monomials in $R_{\delta_1,\ldots,\delta_k}$, connected to 1. Let Λ be a linear form in $(\langle B'^+ * B^+\rangle_{\delta_1,\ldots,\delta_k})^*$ and $\Lambda(\mathbf{h})$ be the linear form of $\langle B'^+ \cdot B^+\rangle^*$ defined by $\Lambda(\mathbf{h})(\mathbf{x}^\alpha) = \Lambda(\mathbf{x}^\alpha)$ if $\mathbf{x}^\alpha \in R_{\delta_1,\ldots,\delta_k}$ and $h_\alpha \in \mathbb{K}$ otherwise. Then, $\Lambda(\mathbf{h})$ admits an extension $\tilde{\Lambda} \in R^*$ such that $H_{\tilde{\Lambda}}$ is of rank r with B and B' basis of $A_{\tilde{\Lambda}}$ iff all $(r+1)\times(r+1)$-minors of $\mathcal{H}^{B'^+,B^+}_{\Lambda(\mathbf{h})}$ vanish and $\det(\mathcal{H}^{B',B}_{\Lambda(\mathbf{h})}) \neq 0$.

Proof. First, if such a $\tilde{\Lambda}$ exists then $\mathcal{H}^{B',B}_{\Lambda(\mathbf{h})}$ is invertible and $\mathcal{H}_{\tilde{\Lambda}}$ is of rank r. Thus all the $(r+1)\times(r+1)$-minors of $\mathcal{H}^{B'^+,B^+}_{\Lambda(\mathbf{h})}$ are equal to zero.

Reciprocally, assume all $(r+1)\times(r+1)$-minors of $\mathcal{H}^{B'^+,B^+}_{\Lambda(\mathbf{h})}$ vanish and $\det(\mathcal{H}^{B',B}_{\Lambda(\mathbf{h})}) \neq 0$ then one can consider the same operators:

$$\mathcal{M}^{B,B'}_{i,l}(\mathbf{h}) := (\mathcal{H}^{B,B'}_{\Lambda(\mathbf{h})})^{-1}\mathcal{H}^{B,B'}_{x_{i,l}\star\Lambda(\mathbf{h})}.$$

By definition of these operators one has that

$$\Lambda(\mathbf{h})(x_{i,l}b\, b') = \Lambda(\mathbf{h})(\mathcal{M}_{i,l}(b)\, b') \tag{12}$$

for all $b \in B$ and $b' \in B'$. As the rank of $\mathcal{H}^{B'^+,B^+}_{\Lambda(\mathbf{h})}$ is equal to the rank of $\mathcal{H}^{B',B}_{\Lambda(\mathbf{h})}$ we easily deduce that (12) is also true for $b' \in B'^+$. Thus we have

$$\Lambda(\mathbf{h})(x_{j,q}x_{i,l}b\, b') = \Lambda(\mathbf{h})(x_{i,l}b\, b' x_{j,q}) = \Lambda(\mathbf{h})(\mathcal{M}_{i,l}(b)\, b' x_{j,q}) =$$
$$= \Lambda(\mathbf{h})(x_{j,q}\, \mathcal{M}_{i,l}(b)\, b') = \Lambda(\mathbf{h})([\mathcal{M}_{j,q}\, \mathcal{M}_{i,l}](b)\, b')$$

for all $b \in B$, b' in B' and $x_{i,l}, x_{j,q} \in R$. Then,

$$\Lambda(\mathbf{h})([\mathcal{M}_{j,q}\, \mathcal{M}_{i,l}](b)\, b') = \Lambda(\mathbf{h})([\mathcal{M}_{i,l}\, \mathcal{M}_{j,q}](b)\, b') = \Lambda(\mathbf{h})(x_{j,q}x_{i,l}b\, b')$$

for all $b \in B$ and $b' \in B'$. As $\mathcal{H}^{B',B}_{\Lambda(\mathbf{h})}$ is invertible we deduce

$$[\mathcal{M}_{j,q}\, \mathcal{M}_{i,l}](b) = [\mathcal{M}_{i,l}\, \mathcal{M}_{j,q}](b)$$

for all $b \in B$. Thus

$$\mathcal{M}_{j,q}\, \mathcal{M}_{i,l} = \mathcal{M}_{i,l}\, \mathcal{M}_{j,q}.$$

Finally, we conclude the proof by using Theorem 3.2 □

Proposition 3.4. Let $B = \{\mathbf{x}^{\beta_1}, \ldots, \mathbf{x}^{\beta_r}\}$ and $B' = \{\mathbf{x}^{\beta'_1}, \ldots, \mathbf{x}^{\beta'_r}\}$ be two sets of monomials in $R_{\delta_1,\ldots,\delta_k}$, connected to 1 and Λ be a linear form on $\langle B'^+ \cdot B^+\rangle$, and let $\partial B = B^+ \setminus B$ be as in Definition 2.10. Then, Λ admits an extension $\tilde{\Lambda}$ in R^* such that $H_{\tilde{\Lambda}}$ is of rank r with B and B' two basis of $\mathcal{A}_{\tilde{\Lambda}}$ iff

$$\mathbb{H}^+ = \begin{pmatrix} \mathbb{H} & \mathbb{G}' \\ \mathbb{G}^t & \mathbb{J} \end{pmatrix}, \tag{13}$$

with $\mathbb{H}^+ = \mathbb{H}_\Lambda^{B'^+,B^+}$ $\mathbb{H} = \mathbb{H}_\Lambda^{B',B}$ and

$$\mathbb{G} = \mathbb{H}^t\,\mathbb{W}, \mathbb{G}' = \mathbb{H}\,\mathbb{W}', \mathbb{J} = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}'. \tag{14}$$

for some matrices $\mathbb{W} \in \mathbb{K}^{B\times\partial B'}$, $\mathbb{W}' \in \mathbb{K}^{B'\times\partial B}$.

Proof. According to Theorem 3.3, $\Lambda \in \langle B'^+ * B^+\rangle^*$ admits a (unique) extension $\tilde{\Lambda} \in R^*$ such that $H_{\tilde{\Lambda}}$ is of rank r with B and B' two basis of $\mathcal{A}_{\tilde{\Lambda}}$, iff $H_{\tilde{\Lambda}}^{B'^+,B^+} = H_\Lambda^{B'^+,B^+}$ is of rank r. Let us decompose H_Λ^+ as (13) with $\mathbb{H}^+ = \mathbb{H}_\Lambda^{B'^+,B^+}$, $\mathbb{H} = \mathbb{H}_\Lambda^{B',B}$.

If we have $\mathbb{G} = \mathbb{H}^t\,\mathbb{W}, \mathbb{G}' = \mathbb{H}\,\mathbb{W}', \mathbb{J} = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}'$, then

$$\mathbb{H}^+ = \begin{pmatrix} \mathbb{H} & \mathbb{H}\,\mathbb{W}' \\ \mathbb{W}^t\mathbb{H} & \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}' \end{pmatrix}$$

is clearly of rank $\leq \operatorname{rank}\mathbb{H}$.

Conversely, suppose that $\operatorname{rank}\mathbb{H}^+ = \operatorname{rank}\mathbb{H}$. This implies that the image of $\mathbb{G}'$ is in the image of $\mathbb{H}$. Thus, there exists $\mathbb{W}' \in \mathbb{K}^{B'\times\partial B}$ such that $\mathbb{G}' = \mathbb{H}\,\mathbb{W}'$. Similarly, there exists $\mathbb{W} \in \mathbb{K}^{B\times\partial B'}$ such that $\mathbb{G} = \mathbb{H}^t\,\mathbb{W}$. Thus, the kernel of $(\mathbb{H}\ \mathbb{G}')$ (resp. $(\mathbb{H}^t\ \mathbb{G})$) contains $\begin{pmatrix} \mathbb{W}' \\ -\mathbb{I} \end{pmatrix}$ (resp. $\begin{pmatrix} \mathbb{W} \\ -\mathbb{I} \end{pmatrix}$). As $\operatorname{rank}\mathbb{H} = \operatorname{rank}\mathbb{H}^+ = r$, the kernel of $(\mathbb{H}\ \mathbb{G}')$ (resp. $(\mathbb{H}^t\ \mathbb{G})$) is the kernel of $\mathbb{H}^+$ (resp $\mathbb{H}^{+t}$). Thus we have $\mathbb{J} = \mathbb{G}^t\,\mathbb{W}' = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}'$. □

Notice that if $\mathbb{H}$ is invertible, $\mathbb{W}'$, $\mathbb{W}$ are uniquely determined.

Introducing new variables $\mathbf{w}$, $\mathbf{w}'$ for the coefficients of the matrices $\mathbb{W}, \mathbb{W}$, solving the linear system $\mathbb{G} = \mathbb{H}^t\,\mathbb{W}, \mathbb{G}' = \mathbb{H}\,\mathbb{W}'$, and reporting the solutions in the equation $\mathbb{J} = \mathbb{W}^t\,\mathbb{H}\,\mathbb{W}'$, we obtain a new set of equations, bilinear in $\mathbf{w}$, $\mathbf{w}'$, which characterize the existence of an extension Λ on R^*.

This leads to the following system in the variables $\mathbf{h}$ and the coefficients $\mathbf{w}$ of matrix $\mathbb{W}$. It characterizes the linear forms $\Lambda \in R^*_{\delta_1,\ldots,\delta_k}$ that admit an extension $\tilde{\Lambda} \in R^*$ such that $H_{\tilde{\Lambda}}$ is of rank r with B a basis of $\mathcal{A}_{\tilde{\Lambda}}$.

$$\mathcal{H}_\Lambda^{B,\partial B}(\mathbf{h}) - \mathcal{H}_\Lambda^{B}(\mathbf{h})\,\mathbb{W}(\mathbf{w}) = 0, \quad \mathcal{H}_\Lambda^{\partial B,\partial B}(\mathbf{h}) - \mathbb{W}^t(\mathbf{w})\,\mathcal{H}_\Lambda^{B}(\mathbf{h})\,\mathbb{W}(\mathbf{w}) = 0 \tag{15}$$

with $\det(\mathcal{H}_\Lambda^B(\mathbf{h})) \neq 0$.

The matrix $\mathcal{H}_\Lambda^{B^+}$ is a quasi-Hankel matrix [50], whose structure is imposed by equality (linear) constraints on its entries. If $\mathbb{H}$ is known (i.e. $B \times B \subset R_{\delta_1,\ldots,\delta_k}$), the number of independent parameters in $\mathcal{H}_\Lambda^{B,B^+}(\mathbf{h})$ or in $\mathbb{W}$ is the number of monomials in $B \times \partial B - R_{\delta_1,\ldots,\delta_k}$. By Proposition 3.4, the rank condition is equivalent to the quadratic relations $\mathbb{J} - \mathbb{W}^t\mathbb{H}^t\,\mathbb{W} = 0$ in these unknowns.

If $\mathbb{H}$ is not completely known, the number of parameters in $\mathbb{H}$ is the number of monomials in $B \times B - R_{\delta_1,\ldots,\delta_k}$. The number of independent parameters in $\mathcal{H}_\Lambda^{B,\partial B}(\mathbf{h})$ or in $\mathbb{W}$ is then $B \times \partial B - R_{\delta_1,\ldots,\delta_k}$.

The system (15) is composed of linear equations deduced from quasi-Hankel structure, quadratic relations for the entries in $B \times \partial B$ and cubic relations for the entries in $B \times \partial B$ in the unknown parameters $\mathbf{h}$ and $\mathbf{w}$.

We are going to use explicitly these characterizations in the new algorithm we propose in Algorithm 3.1, for minimal tensor decomposition.

4. Examples and applications

There exist numerous areas in which decomposing a tensor into a sum of rank-one terms is useful. These fields range from arithmetic complexity [13] to chemistry [56]. One nice application is worth to be emphasized, namely wireless transmissions [55]: one or several signals are wished to be extracted from noisy measurements, received on an array of sensors and disturbed by interferences. The approach is deterministic, which makes the difference compared to approaches based on data statistics [23]. The array of sensors is composed of J subarrays, each containing I sensors. Subarrays do not need to be disjoint, but must be deduced from each other by a translation in space. If the transmission is narrow band and in the far field, then the measurements at time sample t recorded on sensor i of subarray j take the form:

$$T(i,j,t) = \sum_{p=1}^{r} A_{ip} B_{jp} C_{tp}$$

if r waves impinge on the array. Matrices A and B characterize the geometry of the array (subarray and translations), whereas matrix C contains the signals received on the array. An example with $(I, J) = (4, 4)$ is given in Figure 1. Computing the decomposition of tensor T allows to extract signals of interest as well as interferences, all included in matrix C. Radiating sources can also be localized with the help of matrix A if the exact location of sensors of a subarray are known. Note that this framework applies in radar, sonar or telecommunications.

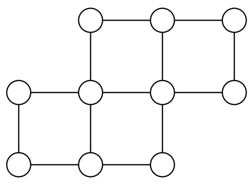

Fig. 1. Array of 10 sensors decomposed into 4 subarrays of 4 sensors each.

4.1. Best approximation of lower multilinear rank

By considering a kth order tensor as a linear map from one linear space onto the tensor product of the others, one can define the ith mode rank, which is nothing else but the rank of that linear operator. Since there are k distinct possibilities to build such a linear operator, one defines a k-tuple of ranks $(r_1, \ldots r_k)$, called the *multilinear rank* of the kth order tensor. It is known that tensor rank is bounded below by all mode ranks r_i:

$$r \geq r_i, \forall 1 \leq i \leq k \tag{16}$$

This inequality gives us an easily accessible lower bound. Let's turn now to an upper bound.

Proposition 4.1. [4] The rank of a tensor of order 3 and dimensions $n_1 \times n_2 \times n_3$, with $n_1 \leq n_2$, is bounded by

$$r \leq n_1 + n_2 \lfloor n_3/2 \rfloor \tag{17}$$

This bound on maximal rank has not been proved to be always reached, and it is likely to be quite loose for large values of n_i. Nevertheless, it is sufficient for our reasoning.

There are two issues to address. First, the algorithm we have proposed is not usable in large dimensions (e.g. significantly larger than 10). The idea is then to reduce dimensions n_i down to r_i before executing the algorithm, if necessary. Second, another problem in practice is the presence of measurement errors or modeling inaccuracies, which increase the tensor rank to its generic value. We do not know how to reduce tensor rank back to its exact value. The practical solution is then to compute the best approximate of lower multilinear rank $(r_1, \dots r_k)$, as explained in [24]. This best approximate always exists, and inequality (17) shows that reducing dimensions will indirectly reduce tensor rank. To compute it, it suffices to minimize $||T - (U^{(1)}, U^{(2)}, U^{(3)}) \cdot C||$ with respect to the three matrices $U^{(i)}$, each of size $n_i \times r_i$, under the constraint $U^{(1)\mathsf{H}} U^{(1)} = I$. If properly initialized by a truncated HOSVD, a few iterations of any iterative algorithm will do it [47]. The tensor of reduced dimensions is then given by $C = (U^{(1)\mathsf{H}}, U^{(2)\mathsf{H}}, U^{(3)\mathsf{H}}) \cdot T$.

4.2. Number of solutions

In the above mentioned applications, it is necessary to have either a unique solution, or a finite set of solutions from which the most realistic one can be picked up. For this reason, it is convenient to make sure that the tensor rank is not too large, as pointed out by the following propositions.

Proposition 4.2. [2] A generic symmetric tensor of order $d \geq 3$ and rank r admits a finite number of decompositions into a sum of rank one terms if $r < r_E(d, n)$, where:

$$r_E(k, n) = \left\lceil \frac{\binom{n+d-1}{d}}{n} \right\rceil \tag{18}$$

Rank r_E is usually referred to as the expected rank of order d and dimension n.

Note that this result is true for generic tensors of rank r, which means that there exists a set of exceptions, of null measure.

This proposition has not yet been entirely extended to unconstrained tensors, which we are interested in. However, some partial results are available in the literature: in [1, Thm. 4.4], [15, Prop. 3.3] and [16, Thm. 2.4] it is for example shown the following:

Theorem 4.3. A generic tensor of order $k \geq 3$, rank r and dimensions n_i such that $n_1 \geq \cdots \geq n_k \geq r_E(k, \mathbf{n}) + 1$, admits a finite number of decompositions into a sum of rank one terms if and only if $r \leq r_E(k, \mathbf{n})$, where:

$$r_E(k, \mathbf{n}) = \prod_{i=1}^{k-1} (n_i + 1) - \sum_{i=1}^{k-1} n_i. \tag{19}$$

The hypothesis $n_1 \geq \cdots \geq n_k \geq r_E(k, \mathbf{n}) + 1$ of Theorem 4.3 is known as “unbalanced case”.

On the other hand, a sufficient condition for uniqueness has been proposed by Kruskal [40], but the bound is more restrictive:

Proposition 4.4. [40] A tensor of order $k \geq 3$ and rank r admits a finite number of decompositions into a sum of rank one terms if:

$$r \leq \frac{\sum_{i=1}^{k} \kappa_i}{2} - 1 \tag{20}$$

where κ_i denote the so-called Kruskal's ranks of loading matrices, which generically equal the dimensions n_i if the rank r is larger than the latter.

4.3. Computer results

If we consider a $4 \times 4 \times 7$ unconstrained tensor, it has an expected rank equal to 9, whereas Kruskal's bound generically equals 6. So it is interesting to consider a tensor with such dimensions but with rank $6 < r < 9$. In such conditions, we expect that there are almost surely a finite number of solutions. This tensor would correspond to measurements received on the array depicted in Figure 1, if 7 time samples are recorded. In [19] L. Chiantini and G. Ottaviani claim that a computer check shows that for a generic $4 \times 4 \times 7$ tensor of rank 7, uniqueness of the decomposition holds.

• As an illustration of our algorithm, we consider a $4 \times 4 \times 4$ tensor whose affine representation is given by:

$T := 4 + 7\,a_1 + 8\,a_2 + 9\,a_3 + 5\,b_1 - 2\,b_2 + 11\,b_3 + 6\,c_1 + 8\,c_2 + 6\,c_3 + 21\,a_1\,b_1 + 28\,a_2\,b_1 + 11\,a_3\,b_1 - 14\,a_1\,b_2 - 21\,a_2\,b_2 - 10\,a_3\,b_2 + 48\,a_1\,b_3 + 65\,a_2\,b_3 + 28\,a_3\,b_3 + 26\,a_1\,c_1 + 35\,a_2\,c_1 + 14\,a_3\,c_1 + 18\,b_1\,c_1 - 10\,b_2\,c_1 + 40\,b_3\,c_1 + 36\,a_1\,c_2 + 48\,a_2\,c_2 + 18\,a_3\,c_2 + 26\,b_1\,c_2 - 9\,b_2\,c_2 + 55\,b_3\,c_2 + 38\,a_1\,c_3 + 53\,a_2\,c_3 + 14\,a_3\,c_3 + 26\,b_1\,c_3 - 16\,b_2\,c_3 + 58\,b_3\,c_3 + 68\,a_1\,b_1\,c_1 + 91\,a_2\,b_1\,c_1 + 48\,a_3\,b_1\,c_1 - 72\,a_1\,b_2\,c_1 - 105\,a_2\,b_2\,c_1 - 36\,a_3\,b_2\,c_1 + 172\,a_1\,b_3\,c_1 + 235\,a_2\,b_3\,c_1 + 112\,a_3\,b_3\,c_1 + 90\,a_1\,b_1\,c_2 + 118\,a_2\,b_1\,c_2 + 68\,a_3\,b_1\,c_2 - 85\,a_1\,b_2\,c_2 - 127\,a_2\,b_2\,c_2 - 37\,a_3\,b_2\,c_2 + 223\,a_1\,b_3\,c_2 + 301\,a_2\,b_3\,c_2 + 151\,a_3\,b_3\,c_2 + 96\,a_1\,b_1\,c_3 + 129\,a_2\,b_1\,c_3 + 72\,a_3\,b_1\,c_3 - 114\,a_1\,b_2\,c_3 - 165\,a_2\,b_2\,c_3 - 54\,a_3\,b_2\,c_3 + 250\,a_1\,b_3\,c_3 + 343\,a_2\,b_3\,c_3 + 166\,a_3\,b_3\,c_3.$

If we consider $B' := (1, b_1, b_2, b_3)$ and $B := (1, a_1, a_2, a_3)$, the corresponding matrix $\mathbb{H}_\Lambda^{B',B}$ is equal to

$$\mathbb{H}_\Lambda^{B',B} = \begin{pmatrix} 4 & 7 & 8 & 9 \\ 5 & 21 & 28 & 11 \\ -2 & -14 & -21 & -10 \\ 11 & 48 & 65 & 28 \end{pmatrix}$$

and is invertible. Moreover, the transposed operators of multiplication by the variables c_1, c_2, c_3 are known:

$$^t\mathbb{M}_{c_1}^B = \begin{pmatrix} 0 & 11/6 & -2/3 & -1/6 \\ -2 & -41/6 & 20/3 & 19/6 \\ -2 & -85/6 & 37/3 & 29/6 \\ -2 & 5/2 & 0 & 1/2 \end{pmatrix}$$

$$^t\mathbb{M}_{c_2}^B = \begin{pmatrix} -2 & 23/3 & -13/3 & -1/3 \\ -6 & 1/3 & 7/3 & 13/3 \\ -6 & -28/3 & 29/3 & 20/3 \\ -6 & 14 & -7 & 0 \end{pmatrix}$$

$$
{}^t\mathbb{M}_{c_3}^B = \begin{pmatrix} 0 & 3/2 & 0 & -1/2 \\ -2 & -33/2 & 14 & 11/2 \\ -2 & -57/2 & 23 & 17/2 \\ -2 & 3/2 & 2 & -1/2 \end{pmatrix}
$$

whose eigenvalues are respectively $(-1, 2, 4, 1)$, $(-2, 4, 5, 1)$ and $(-3, 2, 6, 1)$. The corresponding common eigenvectors are:

$$
v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 5 \\ 7 \\ 3 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
$$

We deduce that the coordinates $(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3)$ of the 4 points of evaluation are:

$$
\zeta_1 := \begin{pmatrix} -1 \\ -2 \\ 3 \\ ? \\ ? \\ ? \\ -1 \\ -2 \\ -3 \end{pmatrix}, \zeta_2 := \begin{pmatrix} 2 \\ 2 \\ 2 \\ ? \\ ? \\ ? \\ 2 \\ 4 \\ 2 \end{pmatrix}, \zeta_3 := \begin{pmatrix} 5 \\ 7 \\ 3 \\ ? \\ ? \\ ? \\ 4 \\ 5 \\ 6 \end{pmatrix}, \zeta_4 := \begin{pmatrix} 1 \\ 1 \\ 1 \\ ? \\ ? \\ ? \\ 1 \\ 1 \\ 1 \end{pmatrix},
$$

Then, computing the same way the operators of multiplication ${}^t\mathbb{M}_{c_1}^{B'}, {}^t\mathbb{M}_{c_2}^{B'}, {}^t\mathbb{M}_{c_3}^{B'}$ and their common eigenvectors, we deduce:

$$
\zeta_1 = \begin{pmatrix} -1 \\ -2 \\ 3 \\ -1 \\ -1 \\ -1 \\ -1 \\ -2 \\ -3 \end{pmatrix}, \zeta_2 := \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 3 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \zeta_3 = \begin{pmatrix} 5 \\ 7 \\ 3 \\ 3 \\ -4 \\ 8 \\ 4 \\ 5 \\ 6 \end{pmatrix}, \zeta_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
$$

Finally, we have to solve the following linear system in $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$:

$T = \gamma_1 (1 + a_1 + a_2 + a_3) (1 + b_1 + b_2 + b_3) (1 + c_1 + c_2 + c_3)$
$+ \gamma_2 (1 - a_1 - 2 a_2 + 3 a_3) (1 - b_1 - b_2 - b_3) (1 - c_1 - 2 c_2 - 3 c_3)$
$+ \gamma_3 (1 + 2 a_1 + 2 a_2 + 2 a_3) (1 + 2 b_1 + 2 b_2 + 3 b_3) (1 + 2 c_1 + 4 c_2 + 2 c_3)$
$+ \gamma_4 (1 + 5 a_1 + 7 a_2 + 3 a_3) (1 + 3 b_1 - 4 b_2 + 8 b_3) (1 + 4 c_1 + 5 c_2 + 6 c_3),$
We get $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 1$.

• We consider now an example with 6 time samples, that is an element of $\mathbb{R}^4 \otimes \mathbb{R}^4 \otimes \mathbb{R}^6$: $T := 1046\, a_1 b_1 c_1 + 959\, a_1 b_1 c_2 + 660\, a_1 b_1 c_3 + 866\, a_1 b_1 c_4 + 952\, a_1 b_1 c_5 - 1318\, a_1 b_2 c_1 - 1222\, a_1 b_2 c_2 - 906\, a_1 b_2 c_3 - 1165\, a_1 b_2 c_4 - 1184\, a_1 b_2 c_5 - 153\, a_1 b_3 c_1 + 52\, a_1 b_3 c_2 + 353\, a_1 b_3 c_3 + 354\, a_1 b_3 c_4 + 585\, a_1 b_3 c_5 + 852\, a_2 b_1 c_1 + 833\, a_2 b_1 c_2 + 718\, a_2 b_1 c_3 + 903\, a_2 b_1 c_4 + 828\, a_2 b_1 c_5 - 1068\, a_2 b_2 c_1 - 1060\, a_2 b_2 c_2 - 992\, a_2 b_2 c_3 - 1224\, a_2 b_2 c_4 - 1026\, a_2 b_2 c_5 + 256\, a_2 b_3 c_1 + 468\, a_2 b_3 c_2 + 668\, a_2 b_3 c_3 + 748\, a_2 b_3 c_4 + 1198\, a_2 b_3 c_5 - 614\, a_3 b_1 c_1 - 495\, a_3 b_1 c_2 - 276\, a_3 b_1 c_3 - 392\, a_3 b_1 c_4 - 168\, a_3 b_1 c_5 + 664\, a_3 b_2 c_1 + 525\, a_3 b_2 c_2 + 336\, a_3 b_2 c_3 + 472\, a_3 b_2 c_4 + 63\, a_3 b_2 c_5 + 713\, a_3 b_3 c_1 + 737\, a_3 b_3 c_2 + 791\, a_3 b_3 c_3 + 965\, a_3 b_3 c_4 + 674\, a_3 b_3 c_5 - 95\, a_1 b_1 + 88\, a_1 b_2 + 193\, a_1 b_3 + 320\, a_1 c_1 + 285\, a_1 c_2 + 134\, a_1 c_3 + 188\, a_1 c_4 + 382\, a_1 c_5 - 29\, a_2 b_1 - 2\, a_2 b_2 + 198\, a_2 b_3 + 292\, a_2 c_1 + 269\, a_2 c_2 + 138\, a_2 c_3 + 187\, a_2 c_4 + 406\, a_2 c_5 + 119\, a_3 b_1 - 139\, a_3 b_2 + 20\, a_3 b_3 - 222\, a_3 c_1 - 160\, a_3 c_2 + 32\, a_3 c_3 + 9\, a_3 c_4 - 229\, a_3 c_5 + 122\, b_1 c_1 + 119\, b_1 c_2 + 112\, b_1 c_3 + 140\, b_1 c_4 + 108\, b_1 c_5 - 160\, b_2 c_1 - 163\, b_2 c_2 - 176\, b_2 c_3 - 214\, b_2 c_4 - 117\, b_2 c_5 + 31\, b_3 c_1 + 57\, b_3 c_2 + 65\, b_3 c_3 + 73\, b_3 c_4 + 196\, b_3 c_5 - 35\, a_1 - 21\, a_2 + 54\, a_3 - 3\, b_1 - 3\, b_2 + 24\, b_3 + 50\, c_1 + 46\, c_2 + 20\, c_3 + 29\, c_4 + 63\, c_5 - 6.$

If we take $B = \{1, a_1, a_2, a_3, b_1, b_2\}$ and $B' = \{1, c_1, c_2, c_3, c_4, c_5\}$ we obtain the following known submatrix of H_Λ:

$$\mathbb{H}_\Lambda^{B',B} = \begin{bmatrix} -6 & -35 & -21 & 54 & -3 & -3 \\ 50 & 320 & 292 & -222 & 122 & -160 \\ 46 & 285 & 269 & -160 & 119 & -163 \\ 20 & 134 & 138 & 32 & 112 & -176 \\ 29 & 188 & 187 & 9 & 140 & -214 \\ 63 & 382 & 406 & -229 & 108 & -117 \end{bmatrix}$$

which is invertible. Thus, the rank is at least 6. Let us find if $H_{\tilde{\Lambda}}$ can be extended to a rank 6 Hankel matrix H_Λ. If we look at $H_\Lambda^{B'^+,B^+}$, several coefficients are unknown. Yet, as will see, they can be determined by exploiting the commutation relations, as follows.

The columns $\mathbb{H}^{B',\{m\}}$ are also known for $m \in \{b_3, a_1 b_1, a_2 b_1, a_3 b_1, a_1 b_2, a_2 b_2, a_3 b_2\}$. Thus we deduce the relations between these monomials and B by solving the system

$$\mathbb{H}_\Lambda^{B',B} X = \mathbb{H}_\Lambda^{B',\{m\}}.$$

This yields the following relations in $\mathcal{A}_\Lambda$:
$b_3 \equiv -1. - 0.02486\, a_1 + 1.412\, a_2 + 0.8530\, a_3 - 0.6116\, b_1 + 0.3713\, b_2$, $a_1 b_1 \equiv -2. + 6.122\, a_1 -$

$3.304\,a_2 + .6740\,a_3 + .7901\,b_1 - 1.282\,b_2, a_2\,b_1 \equiv -2. + 4.298\,a_1 - 1.546\,a_2 + 1.364\,a_3 + .5392\,b_1 - 1.655\,b_2, a_3\,b_1 \equiv -2. - 3.337\,a_1 + 5.143\,a_2 + 1.786\,a_3 - 2.291\,b_1 + 1.699\,b_2, a_1\,b_2 \equiv -2. + 0.03867\,a_1 - 0.1967\,a_2 + 1.451\,a_3 - 2.049\,b_1 + 3.756\,b_2, a_2\,b_2 \equiv -2. + 3.652\,a_1 - 3.230\,a_2 + .9425\,a_3 - 2.562\,b_1 + 4.198\,b_2, a_3\,b_2 \equiv -2. + 6.243\,a_1 - 7.808\,a_2 - 1.452\,a_3 + 5.980\,b_1 + 0.03646\,b_2$

Using the first relation on b_3, we can reduce $a_1\,b_3, a_2\,b_3, a_3\,b_3$ and obtain 3 linear dependency relations between the monomials in $B \cup \{a_1^2, a_1a_2, a_1a_3, a_2^2, a_2a_3, a_3^2\}$. Using the commutation relations $\frac{\mathrm{lcm}(m_1,m_2)}{m_1}N(m_1) - \frac{\mathrm{lcm}(m_1,m_2)}{m_2}N(m_2)$, for $(m_1, m_2) \in \{(a_1\,b_1, a_2\,b_1), (a_1\,b_2, a_2\,b_2), (a_2\,b_2, a_3\,b_2)\}$ where $N(m_i)$ is the reduction of m_i with respect to the prevision relations, we obtain 3 new linear dependency relations between the monomials in $B \cup \{a_1^2, a_1a_2, a_1a_3, a_2^2, a_2a_3, a_3^2\}$. From these 6 relations, we deduce the expression of the monomials in $\{a_1^2, a_1a_2, a_1a_3,\ a_2^2, a_2a_3, a_3^2\}$ as linear combinations of monomials in B:

$a_1^2 \equiv 12.08\,a_1 - 5.107\,a_2 + .2232\,a_3 - 2.161\,b_1 - 2.038\,b_2 - 2., a_1\,a_2 \equiv 8.972\,a_1 - 1.431\,a_2 + 1.392\,a_3 - 3.680\,b_1 - 2.254\,b_2 - 2., a_1\,a_3 \equiv -11.56\,a_1 + 9.209\,a_2 + 2.802\,a_3 + 1.737\,b_1 + .8155\,b_2 - 2., a_2^2 \equiv -2. + 6.691\,a_1 + 2.173\,a_2 + 2.793\,a_3 - 5.811\,b_1 - 2.846\,b_2, a_2\,a_3 \equiv -2. - 11.87\,a_1 + 9.468\,a_2 + 2.117\,a_3 + 3.262\,b_1 + 0.01989\,b_2, a_3^2 \equiv -2. + 16.96\,a_1 - 8.603\,a_2 + 1.349\,a_3 - 6.351\,b_1 - .3558\,b_2.$

Now, we are able to compute the matrix of multiplication by a_1 in B, which is obtained by reducing the monomials $B \cdot a_1 = \{a_1, a_1^2, a_1\,a_2, a_1\,a_3, a_1\,b_1, a_1\,b_2\}$ by the computed relations:

$$M_{a_1} := \begin{bmatrix} 0.0 & -2.0 & -2.0 & -2.0 & -2.0 & -2.0 \\ 1.0 & 12.08 & 8.972 & -11.56 & 6.122 & 0.03867 \\ 0.0 & -5.107 & -1.431 & 9.209 & -3.304 & -0.1967 \\ 0.0 & 0.2232 & 1.392 & 2.802 & 0.6740 & 1.451 \\ 0.0 & -2.161 & -3.680 & 1.737 & 0.7901 & -2.049 \\ 0.0 & -2.038 & -2.254 & 0.8155 & -1.282 & 3.756 \end{bmatrix}.$$

The eigenvectors of the transposed operator normalized so that the first coordinate is 1 are:

$$\begin{bmatrix} 1.0 \\ 5.0 \\ 7.003 \\ 3.0 \\ 3.0 \\ -4.0 \end{bmatrix}, \begin{bmatrix} 1.0 \\ 2.999 \\ 4.0 \\ -4.999 \\ -2.999 \\ 4.999 \end{bmatrix}, \begin{bmatrix} 1.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \\ 2.0 \end{bmatrix}, \begin{bmatrix} 1.0 \\ 8.001 \\ 6.002 \\ -7.002 \\ 4.001 \\ -5.001 \end{bmatrix}, \begin{bmatrix} 1.0 \\ -1.0 \\ -2.0 \\ 3.0 \\ -1.0 \\ -1.0 \end{bmatrix}, \begin{bmatrix} 1.0 \\ 0.9999 \\ 0.9999 \\ 0.9999 \\ 0.9999 \\ 0.9999 \end{bmatrix}$$

They correspond to the vectors of evaluation of the monomial vector B at the roots of I_Λ. Thus we know the coordinates a_1, a_2, a_3, b_1, b_2 of these roots. By expanding the polynomial

$\gamma_1 (1 + a_1 + a_2 + a_3)) (1 + b_1 + b_2 + b_3) (1 + \cdots) + \gamma_2 (1 - a_1 - 2\, a_2 + 3\, a_3) (1 - b_1 - b_2 - b_3) (1 + \cdots) + \gamma_3 (1 + 2\, a_1 + 2\, a_2 + 2\, a_3) (1 + 2\, b_1 + 2\, b_2 + 3\, b_3) (1 + \cdots) + \gamma_4 (1 + 5\, a_1 + 7\, a_2 + 3\, a_3) (1 + 3\, b_1 - 4\, b_2 + 8\, b_3) (1 + \cdots) + \gamma_5 (1 + 8\, a_1 + 6\, a_2 - 7\, a_3) (1 + 4\, b_1 - 5\, b_2 - 3\, b_3) (1 + \cdots) + \gamma_6 (1 + 3\, a_1 + 4\, a_2 - 5\, a_3) (1 - 3\, b_1 + 5\, b_2 + 4\, b_3) (1 + \cdots)$

(where the $\cdots$ are linear terms in the c_i's) and identifying the coefficients of T which do not depend on $c_1, \ldots, c_5$, we obtain a linear system in γ_i, whose unique solution is $(2, -1, -2, 3, -5, -3)$. This allows us to compute the value Λ for any monomials in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. In particular, we can compute the entries of $\mathbb{H}_\Lambda^{B,B}$. Solving the system $\mathbb{H}_\Lambda^{B,B} X = \mathbb{H}_\Lambda^{B,B'}$, we deduce the relations between the monomials in B' and B in $\mathcal{A}_\Lambda$ and in particular $c_1, \ldots, c_5$ as linear combinations of monomials in B. This allows us to recover the missing coordinates and yields the following decomposition:

$T := 2\,(1+a_1+a_2+a_3)\,(1+b_1+b_2+b_3)\,(1+c_1+c_2+c_3+c_4+c_5) - (1-a_1-2\,a_2+3\,a_3)\,(1-b_1-b_2-b_3)\,(1-c_1-2\,c_2-3\,c_3-4\,c_4+5\,c_5) - 2\,(1+2\,a_1+2\,a_2+2\,a_3)\,(1+2\,b_1+2\,b_2+3\,b_3)\,(1+2\,c_1+2\,c_2+2\,c_3+2\,c_4+2\,c_5) + 3\,(1+5\,a_1+7\,a_2+3\,a_3)\,(1+3\,b_1-4\,b_2+8\,b_3)\,(1+4\,c_1+5\,c_2+6\,c_3+7\,c_4+8\,c_5) - 5\,(1+8\,a_1+6\,a_2-7\,a_3)\,(1+4\,b_1-5\,b_2-3\,b_3)\,(1-6\,c_1-5\,c_2-2\,c_3-3\,c_4-5\,c_5) - 3\,(1+3\,a_1+4\,a_2-5\,a_3)\,(1-3\,b_1+5\,b_2+4\,b_3)\,(1-3\,c_1-2\,c_2+3\,c_3+3\,c_4-7\,c_5)$.

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