On the cactus rank of cubics forms

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Abstract

We prove that the smallest degree of an apolar 0-dimensional scheme of a general cubic form in $n+1$ variables is at most $2n+2$, when $n \geq 8$, and therefore smaller than the rank of the form. For the general reducible cubic form the smallest degree of an apolar subscheme is $n+2$, while the rank is at least $2n$.

Key words: Cactus rank, Cubic forms, Apolarity.

Introduction

The *rank* of a homogeneous form $F \in S := \mathbb{C}[x_0, \ldots, x_n]$ of degree d is the minimal number of linear forms $L_1, \ldots, L_r$ needed to write F as a sum of pure d-powers:
$F = L_1^d + \cdots + L_r^d$.
Various other notions of rank, such as *border rank* and *cactus rank*, appear in the study of higher secant varieties and are closely related to the rank. The cactus rank is the minimal length of an apolar subscheme to F, while the border rank is the minimal r such that F is a limit of forms of rank r. For an extensive description and usage of the classical concept of *apolarity*, we refer to (Iarrobino, Kanev, 1999) and (Ranestad, Schreyer, 2000) and the references therein, which go back to the late XIX century with A. Clebsch, J.

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Lüroth, T. Reye, G. Scorza and to the beginning of the XX century with E. Lasker, F. H. S. Macaulay, J. J. Sylvester, A. Terracini and E. K. Wakeford.

The notion of cactus rank is recent and coincides with scheme length introduced by Iarrobino and Kanev in (Iarrobino, Kanev, 1999). We use the name cactus rank to make the association to *cactus varieties* introduced in (Buczynska, Buczynski, 2010) in a study of higher secant varieties.

The cactus rank and the border rank are both less than or equal to the rank as is explained in Section 1, while a natural lower bound for both of them is the differential length (also called the Hankel rank), the maximum of the dimensions of the space of k-th order partials of F as k varies between 0 and d. For a general form the rank and the border rank coincide, but little is known about the cactus rank beyond these bounds, cf. (Iarrobino, Kanev, 1999).

For specific forms, more is known: For irreducible forms that do not define a cone, the cactus rank is minimal for forms of Fermat type, e.g. $F = x_0^d + \cdots + x_n^d$. In this case the rank coincides with the Hankel rank and hence also with the cactus rank and the border rank.

The first main result of this paper, Theorem 3, is that for large n and d, the cactus rank of a general form is strictly less than the rank.

For cubic forms we give more specific results: In Section 2 we show that there are cubic forms with minimal cactus rank (equal to the Hankel rank) whose border rank is strictly higher and compute the cactus rank of a general reducible cubic form.

The rank of forms has seen growing interest in recent years. Any apolar subscheme to F of minimal length is locally Gorenstein ((Buczynska, Buczynski, 2010, proof of Lemma 2.4)), therefore this work is close in line to (Iarrobino, 1994), (Iarrobino, Kanev, 1999) and (Elias, Rossi, 2011), in their study of apolarity and the local Gorenstein algebra associated to a polynomial. Applications to higher secant varieties can be found in (Chiantini, Ciliberto, 2002), (Buczynska, Buczynski, 2010) and (Landsberg, Ottaviani, 2011), while the papers (Landsberg, Teitler, 2010), (Brachat et al, 2010), (Bernardi et al, 2011), (Carlini et al, 2011) and (Oeding , Ottaviani, 2011) concentrate on effective methods to compute the rank and to compute an explicit decomposition of a form. In a different direction, the rank of cubic forms associated to canonical curves has been computed in (De Poi, Zucconi, 2011a), (De Poi, Zucconi, 2011b) and (Ballico et al, 2011).

1. Apolar Gorenstein subschemes

We consider homogeneous polynomials $F \in S := \mathbb{C}[x_0, \dots, x_n]$, and consider the dual ring $T := \mathbb{C}[y_0, \dots, y_n]$ acting on S by differentiation:

$$y_j(x_i) = \frac{d}{dx_j}(x_i) = \delta_{ij}.$$

Let S_1 and T_1 be the degree 1 parts of S and T respectively. With respect to the action above (classically know as *apolarity*), S_1 and T_1 are natural dual spaces and $\langle x_0, \dots, x_n \rangle$ and $\langle y_0, \dots, y_n \rangle$ are dual bases. In particular T is naturally the coordinate ring of $\mathbb{P}(S_1)$, the projective space of 1-dimensional subspaces of S_1, and vice versa. The annihilator of F is an ideal in T which we denote by $F^\perp \subset T$. The quotient $T_F := T/F^\perp$ is graded Artinian and Gorenstein since $F^\perp$ is homogeneous and T_F is finitely generated as a $\mathbb{C}$-module (*Artinian*) and has a 1-dimensional socle, the annihilator of the unique maximal ideal (*Gorenstein*). The socle in T_F is the degree d part of the ring (see e.g. (Iarrobino, Kanev, 1999, Lemma 2.14)).

Definition 1. A subscheme $X \subset \mathbb{P}(S_1)$ is apolar to $F \in S$ if its homogeneous ideal $I_X \subset T$ is contained in $F^\perp$.

Any apolar subscheme to F of minimal length is locally Gorenstein ((Buczynska, Buczynski, 2010, proof of Lemma 2.4)), therefore we concentrate on finite local Gorenstein schemes. More precisely, we consider finite subschemes $\Gamma \subset \mathbb{P}(S_1)$ isomorphic to Spec$R$, where R is a local Artinian Gorenstein $\mathbb{C}$-algebra. The ring R does not have to be graded. On the other hand, if $R = \mathbb{C}[y_1, \ldots, y_n]/I$ is a local Artinian Gorenstein algebra, then I is the annihilator $f^\perp$, via apolarity, of some polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ (cf. (Iarrobino, 1994, Lemma 1.2)). If the polynomial f is homogeneous, then R is graded. This is the case of the form F above. Now, we consider the affine scheme Spec$R$ for possibly inhomogeneous polynomials.

In fact, the homogeneous polynomial $F \in S$ admits some natural finite local apolar Gorenstein subschemes. Let $S_{x_0} := \mathbb{C}[x_1, \ldots, x_n]$ and $T_{y_0} := \mathbb{C}[y_1, \ldots, y_n]$. Consider the polynomial $f = F(1, x_1, \ldots, x_n) \in S_{x_0}$ and the Artinian Gorenstein quotient $T_f := T_{y_0}/f^\perp$. We show that the image of the natural embedding Spec$(T_f) \subset \mathbb{P}(S_1)$ is apolar to $F \in S$.
What we have just described for the special case of $f \in S_{x_0}$, the dehomogenisation of $F \in S$ by x_0, can be repeated with any other linear form $l \in S_1$. In fact, F admits a natural apolar Gorenstein subscheme for any linear form in S.

Any nonzero linear form $l \in S$ belongs to a basis $(l, l_1, \ldots, l_n)$ of S_1, with dual basis $(l', l'_1, \ldots, l'_n)$ of T_1. In particular the homogeneous ideal in T of the point $[l] \in \mathbb{P}(S_1)$ is generated by $\{l'_1, \ldots, l'_n\}$, while $\{l_1, \ldots, l_n\}$ generates the ideal of the point $\phi([l]) \in \mathbb{P}(T_1)$, where $\phi : \mathbb{P}(T_1) \to \mathbb{P}(S_1),\ y_i \mapsto x_i, i = 0, \ldots, n$.

The form $F \in S$ defines a hypersurface $\{F = 0\} \subset \mathbb{P}(T_1)$. The Taylor expansion of F with respect to the point $\phi([l])$ may naturally be expressed in the coordinates functions $(l, l_1, \ldots, l_n)$. Thus there exist $a_0, \ldots, a_d \in \mathbb{C}$ such that

$$F = a_0 l^d + a_1 l^{d-1} f_1(l_1, \ldots, l_n) + \cdots + a_d f_d(l_1, \ldots, l_n).$$

We denote the corresponding dehomogenisation of $F \in S$ with respect to $l \in S_1$ by $F_l \in S_l$, i.e.

$$F_l = a_0 + a_1 f_1(l_1, \ldots, l_n) + \cdots + a_d f_d(l_1, \ldots, l_n).$$

Notice that the subscript number of f_i refers to the degree i of the form, distinct from the subscript form of F_l that indicate dehomogenisation with respect to l.

Also, we denote the subring of T generated by $\{l'_1, \ldots, l'_n\}$ by $T_{l'}$. It is the natural coordinate ring of the affine subspace $\{l' \neq 0\} \subset \mathbb{P}(S_1)$.

Lemma 2. *The Artinian Gorenstein scheme $\Gamma(F_l)$ defined by $F_l^\perp \subset T_{l'}$ is apolar to F, i.e. the homogenisation $(F_l^\perp)^h \subset F^\perp \subset T$.*

Proof. If $g \in F_l^\perp \subset \mathbb{C}[l'_1, \ldots, l'_n]$, then $g = g_1 + \cdots + g_r$ where g_i is homogeneous in degree i. Similarly $F_l = f = f_0 + \cdots + f_d$. The annihilation $g(f) = 0$ means that for each $e \geq 0$, $\sum_j g_j f_{e+j} = 0$. Homogenizing we get

$$g^h = G = (l')^{r-1} g_1 + \cdots + g_r, \quad f^h = F = l^d f_0 + \cdots + f_d$$

and

$$G(F) = \sum_e \sum_j l^{d-r-e} g_j f_{e+j} = \sum_e l^{d-r-e} \sum_j g_j f_{e+j} = 0.$$

□

Remark 1. (Suggested by Mats Boij) The ideal $(F_l^\perp)^h$ may be obtained without dehomogenising F. Write $F = l^e F_{d-e}$, such that l does not divide F_{d-e}. Consider the form $l^{d-e}F_{d-e}$ of degree $2(d-e)$. Unless $d-e=0$, i.e. $F = l^d$, the degree $d-e$ part of the annihilator $(l^{d-e}F_{d-e})^\perp_{d-e}$ generates an ideal in $(l)^\perp$ and the saturation of $(l^{d-e}F_{d-e})^\perp_{d-e}$ coincides with $(F_l^\perp)^h$. In fact if $G \in T_{d-e}$ then

$$G(l^{d-e}F_{d-e}) = G(l^{d-e})F_{d-e} + lG(l^{d-e-1})F_{d-e}$$

so $G(l^{d-e}F_{d-e}) = 0$ only if $G(l^{d-e}) = 0$.

Apolarity was used classically to characterizes powersum decompositions of F, cf. (Iarrobino, Kanev, 1999), (Ranestad, Schreyer, 2000) and the references therein. In fact, the annihilator of a power of a linear form $l^d \in S$ is the ideal of the corresponding point $p_l \in \mathbb{P}_T$ in degrees at most d. Therefore $F = \sum_{i=1}^r l_i^d$ only if $I_\Gamma \subset F^\perp$ where $\Gamma = \{p_{l_1}, \ldots, p_{l_r}\} \subset \mathbb{P}_T$. On the other hand, if $I_{\Gamma,d} \subset F_d^\perp \subset T_d$, then any differential form that annihilates each l_i^d also annihilates F, so, by duality, $[F]$ must lie in the linear span of the $[l_i^d]$ in $\mathbb{P}(S_d)$. Thus $F = \sum_{i=1}^r l_i^d$ if and only if $I_\Gamma \subset F^\perp$.

The various notions of rank for F listed in the introduction are therefore naturally defined by apolarity : The *cactus rank* $cr(F)$ is defined as

$$cr(F) := \min\{\text{length of a scheme } \Gamma \mid \Gamma \subset \mathbb{P}(T_1), \dim\Gamma = 0, I_\Gamma \subset F^\perp\},$$

the *smoothable rank* $sr(F)$ is defined as

$$sr(F) := \min\{\text{length of a scheme } \Gamma \mid \Gamma \subset \mathbb{P}(T_1) \text{ smoothable}, \dim\Gamma = 0, I_\Gamma \subset F^\perp\}$$

and the *rank* $r(F)$ is defined as

$$r(F) := \min\{\text{length of a scheme } \Gamma \mid \Gamma \subset \mathbb{P}(T_1) \text{ smooth}, \dim\Gamma = 0, I_\Gamma \subset F^\perp\}.$$

A smoothable scheme of length r in $\mathbb{P}(T_1)$ is an element in the irreducible component of the Hilbert scheme containing the smooth schemes of $\mathbb{P}(T_1)$ of length r.

The separate notion of *border rank*, $br(F)$, often considered, is not defined by apolarity. It is the minimal r, such that F is the limit of polynomials of rank r. These notions of rank coincide with the notions of length of annihilating schemes in Iarrobino and Kanev's book (Iarrobino, Kanev, 1999, Definition 5.66). Thus cactus rank coincides with the scheme length, $cr(F) = l\text{sch}(F)$, and smoothable rank coincides with the smoothable scheme length, $sr(F) = l\text{schsm}(F)$, while border rank coincides with length $br(F) = l(F)$. In addition they consider the differential length $l\text{diff}(F)$, the maximum of the dimensions of the space of k-th order partials of F as k varies between 0 and $\deg F$. This length is the maximal rank of a catalecticant or Hankel matrix at F.

Inequalities between these ranks valid for any form F are summarized in (Iarrobino, Kanev, 1999, Lemma 5.17). Clearly, by the definitions above,

$$cr(F) \leq sr(F) \leq r(F).$$

Furthermore,

$$br(F) \leq sr(F), \quad \text{while} \quad l\text{diff}(F) \leq br(F) \quad \text{and} \quad l\text{diff}(F) \leq cr(F).$$

For a general form F in S of degree d the rank, the smoothable rank and the border rank coincide and equals, by the Alexander-Hischowitz theorem (see (Alexander, Hirschowitz, 1995)),

$$br(F) = sr(F) = r(F) = \left\lceil \frac{1}{n+1}\binom{n+d}{d} \right\rceil,$$

when $d > 2$, $(n,d) \neq (2,4), (3,4), (4,3), (4,4)$. The local Gorenstein subschemes considered above show that the cactus rank for a general polynomial may be smaller. Let

$$N_d = \begin{cases} 2\binom{n+k}{k} & \text{when } d = 2k+1 \\ \binom{n+k}{k} + \binom{n+k+1}{k+1} & \text{when } d = 2k+2 \end{cases} \tag{1}$$

and denote by Diff(F) the subspace of S generated by the partials of F of all orders, i.e. of order $0, \dots, d = \deg F$.

Theorem 3. *Let $F \in S = \mathbb{C}[x_0, \dots, x_n]$ be a homogeneous form of degree d, and let $l \in S_1 = \langle x_0, \dots, x_n \rangle$ be any linear form. Let F_l be a dehomogenisation of F with respect to l. Then*

$$cr(F) \leq \dim_K \mathrm{Diff}(F_l).$$

In particular,

$$cr(F) \leq N_d.$$

Proof. According to Lemma 2 the subscheme $\Gamma(F_l) \subset \mathbb{P}(T_1)$ is apolar to F. The subscheme $\Gamma(F_l)$ is affine and has length equal to

$$\dim_k T_{l'}/F_l^{\perp} = \dim_K \mathrm{Diff}(F_l).$$

If all the partial derivatives of F_l of order at most $\lfloor \frac{d}{2} \rfloor$ are linearly independent, and the partial derivatives of higher order span the space of polynomials of degree at most $\lfloor \frac{d}{2} \rfloor$, then

$$\dim_K \mathrm{Diff}(\mathrm{F_l}) = 1 + n + \binom{n+1}{n-1} + \cdots + \binom{n + \lfloor \frac{d}{2} \rfloor}{n-1} + \cdots + n + 1 = N_d.$$

Clearly this is an upper bound so the theorem follows. □

Local apolar subschemes of minimal length may not be of the kind $\Gamma(F_l)$, described above. In fact, even quadratic forms have local apolar subschemes of length equal to its rank that are not of the kind $\Gamma(F_l)$ (cf. (Ranestad, Schreyer, 2011, Corollary 2.7)).

Question 1. What is the cactus rank $cr(n, d)$ for a general form $F \in \mathbb{C}[x_0, \dots, x_n]_d$?

2. Cubic forms

If $F \in S$ is a general cubic form, then the cactus rank according to Theorem 3 is at most $2n + 2$.

If F is a general reducible cubic form in S and l is a linear factor, then $f = F_l$ is a quadratic polynomial and $\Gamma(f)$ is smoothable of length at most $n + 2$: The partials of a nonsingular quadratic polynomial in n variables form a vector space of dimension $n + 2$, so this is the length of $\Gamma(f)$. On the other hand let E be an elliptic normal curve of degree $n+2$ in $\mathbb{P}^{n+1}$. Let $T(E)$ be the homogeneous coordinate ring of E. A quotient of $T(E)$ by two general linear forms is Artinian Gorenstein with Hilbert function $(1, n, 1)$ isomorphic to T_q for a quadric q of rank n. A quotient of $T(E)$ by two general inhomogeneous linear polynomials is the coordinate ring of $n + 2$ distinct points. Thus T_f is isomorphic to T_q and $\Gamma(f)$ is smoothable.

Theorem 4. *For a general cubic form $F \in \mathbb{C}[x_0, \dots, x_n]$, the cactus rank is*

$$cr(F) \leq 2n + 2.$$

For a general reducible cubic form $F \in \mathbb{C}[x_0, \dots, x_n]$ with $n > 1$, the cactus rank and the smoothable rank are

$$cr(F) = sr(F) = n + 2.$$

Proof. It remains to show that for a general reducible cubic form $cr(F) \geq n+2$. On the one hand, if $\Gamma \subset \mathbb{P}_T$ has length less than $n+1$ it is contained in a hyperplane, so $I_\Gamma \subset F^\perp$ only if the latter contains a linear form. If $\{F=0\}$ is not a cone, this is not the case. On the other hand, if $\Gamma \subset \mathbb{P}_T$ has length $n+1$, then, for the same reason, this subscheme must span $\mathbb{P}_T$. Its ideal in that case is generated by $\binom{n+1}{2}$ quadratic forms. If F is general, $F_2^\perp$ is also generated by $\binom{n+1}{2}$ quadrics, so they would have to coincide. For $F_2^\perp$ to generate the ideal of a scheme of length $n+1$ is a closed condition on cubic forms F. If $F = x_0(x_0^2 + \cdots + x_n^2)$, then

$$F_2^\perp = \langle y_1y_2, \ldots, y_{n-1}y_n, y_0^2 - y_1^2, \ldots, y_0^2 - y_n^2 \rangle.$$

In particular $\dim F_2^\perp = \binom{n+1}{2}$, but the quadrics $F_2^\perp$ do not have any common zeros, so $cr(F) \geq n+2$. The general reducible cubic must therefore also have cactus rank at least $n+2$ and the theorem follows. □

Remark 2. By (Landsberg, Teitler, 2010, Theorem 1.3) the lower bound for the rank of a reducible cubic form that depends on $n+1$ variables and not less, is $2n$.

If $F = x_0F_1(x_1, \ldots, x_n)$ where F_1 is a quadratic form of rank n, then

$$cr(F) = sr(F) = n+1,$$

the same as for a Fermat cubic, while the rank is at least $2n$.

We give an example with $cr(F) = l\mathrm{diff}(F) = n+1 < sr(F)$.

Example 1. Let $G \in \mathbb{C}[x_1, \ldots, x_m]$ be a cubic form such that the scheme $\Gamma(G) = \mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_m]/G^\perp)$ has length $2m+2$ and is not smoothable. By (Iarrobino, 1984, Section 4A) examples occur for $m \geq 6$. Denote by $G_1 = y_1(G), \ldots, G_m = y_m(G)$ the first order partials of G. Let

$$F = G + x_0x_1x_{m+1} + x_0x_2x_{m+2} + \cdots + x_0x_mx_{2m} + x_0^2x_{2m+1} \in \mathbb{C}[x_0, \ldots, x_{2m+1}].$$

Then

$$F_{x_0} = G + x_1x_{m+1} + \cdots + x_mx_{2m} + x_{2m+1}$$

and

$$\mathrm{Diff}(F_{x_0}) = \langle F_{x_0}, G_1 + x_{m+1}, \ldots, G_m + x_{2m}, x_1, \ldots, x_m, 1 \rangle$$

so $\dim\mathrm{Diff}(F_{x_0}) = 2m+2$. Therefore $\Gamma(F_{x_0})$ is apolar to F and computes the cactus rank of F. Since $\{F=0\}$ is not a cone, $\Gamma(F_{x_0})$ is nondegenerate, so its homogeneous ideal is generated by the quadrics in the ideal of $F^\perp$. In particular $\Gamma(F_{x_0})$ is the unique apolar subscheme of length $2m+2$. Since this is not smoothable, the smoothable rank is strictly bigger.

By Theorem 4 the cactus rank of a generic cubic form $F \in \mathbb{C}[x_0, \ldots, x_n]$ is at most $2n+2$. The first n for which $2n+2$ is smaller than the rank $r(F) = \lceil \frac{1}{n+1}\binom{n+3}{3} \rceil$ of the generic cubic form in $n+1$ variables is $n=8$, where $r(F) = 19$ and $cr(F) \leq 18$.

Conjecture 1. The cactus rank $cr(F)$ of a general homogeneous cubic $F \in k[x_0, \ldots, x_n]$ equals the rank when $n \leq 7$ and equals $2n+2$ when $n \geq 8$.

For a general cubic form, the rank is ≤ 10 when $n \leq 5$, while it is 12 when $n=6$. Now, any local Artinian Gorenstein scheme of length at most 10 is smoothable (cf. (Casnati, Notari, 2011)), so the conjecture holds for $n \leq 5$. Casnati and Notari has recently extended their result to length at most 11, (cf. (Casnati, Notari, 2012)), which means that the conjecture holds also when $n = 6$. There are nonsmoothable local Gorenstein algebras of length 14 (cf. (Iarrobino, 1984)), so for $n \geq 7$ a different argument is needed to confirm or disprove the conjecture.

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