

A MODULE-THEORETIC APPROACH TO ABELIAN AUTOMORPHISM GROUPS

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ABSTRACT. There are several examples in the literature of finite non-abelian p -groups whose automorphism group is abelian. For some time only examples that were special p -groups were known, until Jain and Yadav [JY12] and Jain, Rai and Yadav [JRY13] constructed several non-special examples. In this paper we show how a simple module-theoretic approach allows the construction of non-special examples, starting from special ones constructed by several authors, while at the same time avoiding further direct calculations.

1. INTRODUCTION

There are several examples in the literature of finite, non-abelian groups with an abelian automorphism group, starting with the work of G.A. Miller [Mil13, Mil14] of a century ago. Clearly in such a group G the central quotient $G/Z(G)$, which is isomorphic to the group of inner automorphisms of G , is abelian, so that G is of nilpotence class 2.

Until recently, all known examples of finite, non-abelian p -groups, for p an odd prime, with an abelian automorphism group were *special* p -groups, satisfying

$$(1.1) \quad \begin{aligned} G^p \leq G' = \Phi(G) = Z(G) = \Omega_1(G), \\ |G/G'| = p^n = |G^p|, \quad |G'| = p^{\binom{n}{2}}, \end{aligned}$$

for some $n \geq 4$. (See Section 3 for a generalization.)

In such a group G , the group $\text{Aut}_c(G)$ of central automorphisms, that is, those automorphisms that induce the identity on the central quotient $G/Z(G)$, is an elementary abelian group of order $p^{\binom{n}{2}}$. Several authors have constructed examples [HL74, JK75, Ear75, Hei80, CL82, Car83, Car85, Mor94, Mor95] of groups G as in (1.1), in which the relations are suitably chosen as to force all automorphisms to be central, that is, $\text{Aut}(G) = \text{Aut}_c(G)$. It follows that $\text{Aut}(G)$ is (elementary) abelian.

In Section 3 of this paper we show that this situation allows for a simple module-theoretic translation. For previous usages of this technique, see [HL74, DH75, Car83, GPS11]. Namely, both $V = G/G'$ and G' can be regarded as vector spaces over the field $\text{GF}(p)$ with p elements. The action of $\text{GL}(V)$ on $V = G/G'$ induces a

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natural action on G' . Moreover, the map $a \wedge b \mapsto [a, b]$ extends to an isomorphism of vector spaces from the exterior square $\Lambda^2 V$ to G' , and this map is also an isomorphism of $\mathrm{GL}(V)$ -modules. Consider the linear map $f : V \rightarrow \Lambda^2 V$ given by $(aG')f = a^p$, where we are identifying G' and $\Lambda^2 V$. (And we write maps on the right.) Then the statement that all automorphisms of G are central is equivalent to the statement that

$$\{ \alpha \in \mathrm{GL}(V) : \alpha \circ f = f \circ \hat{\alpha} \} = \{ 1 \},$$

where $\hat{\alpha}$ is the automorphism of $\Lambda^2 V$ induced by α .

Mahalanobis had conjectured [Mah08] that all finite p -groups whose automorphism group is abelian are special. However, Jain and Yadav [JY12] and Jain, Rai and Yadav [JRY13] have constructed several examples to show that this is not the case. These examples require ingenious and extensive calculations to determine the automorphism groups.

In the rest of this paper we show how the module-theoretic reformulation of Section 3 allow us to construct examples similar to those of [JY12, JRY13] within a conceptual framework, which does not require further calculations. In fact, we start from known examples of the form (1.1), modify them in simple ways, and use the module-theoretic setting to show that the required properties of the automorphism group of the modified groups follow rather directly from the corresponding properties of the original groups, without resorting to extra calculations. This allows us to construct examples of finite p -groups G with the following properties:

- (1) $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ is non-abelian, and G does not have abelian, nontrivial direct factors (Section 4);
- (2) $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ is elementary abelian, and $G' = \Phi(G) < Z(G)$, so that G is not special (Section 5);
- (3) $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ is abelian, and $G' < \Phi(G) = Z(G)$, so that G is not special (Section 6).

Recall that a non-abelian group is said to be *purely non-abelian* if it has only trivial abelian direct factors. Non purely non-abelian examples G with non-abelian $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ were constructed by Curran [Cur82] and Malone [Mal84]. Glasby [Gla86] constructed an infinite family of purely non-abelian, finite 2-groups G with non-abelian $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$. Jain and Yadav [JY12, Theorem B] have constructed examples as in (1), that is, purely non-abelian p -groups G with non-abelian $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, for p an odd prime.

Examples as in (2) have been constructed in [JRY13, Theorem B].

Examples as in (3) have been constructed in [JY12, Theorem A, Lemma 2.1] and [JRY13, Theorem C], the latter groups having elementary abelian automorphism group.

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2. NOTATION

Throughout the paper, p denotes an odd prime.

Our iterated commutators are left-normed, $[a, b, c] = [[a, b], c]$.

A (non-abelian) p -group G is said to be *special* if $G^p \leq G' = Z(G)$. This implies that $\Phi(G) = G^p G' = Z(G)$, and that G' is elementary abelian.

A non-abelian group is said to be *purely non-abelian* if it has only trivial abelian direct factors.

We will be writing maps on the right, and group morphisms as exponents, so if $g \in G$ and $\varphi \in \text{Aut}(G)$, then g^φ denotes the action of φ on g . We also write $g^{\varphi+\psi} = g^\varphi g^\psi$, for $\varphi, \psi \in \text{Aut}(G)$, and in other similar situations.

3. THE MODULE-THEORETIC APPROACH

Suppose we have a group G as in (1.1), for an odd prime p , and some $n \geq 4$. Then G admits a presentation of the form

$$(3.1) \quad G = \langle x_1, \dots, x_n : [x_i, x_j, x_k] = 1 \text{ for all } i, j, k, \\ [x_i, x_j]^p = 1 \text{ for all } i, j, \\ x_i^p = \prod_{j < k} [x_j, x_k]^{c_{i,j,k}} \text{ for all } i \rangle,$$

where the $c_{i,j,k}$ are such that the x_i^p are independent in G' , for $i = 1, \dots, n$. Note that we have explicitly added relations that state that G' is of exponent p ; these are redundant here, as they can be deduced from $1 = [x_i^p, x_j] = [x_i, x_j]^p$, but will play a role in the generalizations in the later sections. Note also that the relations imply that $G^p \leq G' = Z(G)$, so that $G' = \Phi(G)$, and also, as we just said, $(G')^p = 1$. Moreover $|G/G'| = |G^p| = p^n$ and $|G'| = p^{\binom{n}{2}}$, as indeed required in (1.1).

Consider the vector space $V = G/G'$ over $\text{GF}(p)$. Let $\varphi \in \text{Aut}(G)$, and let α be the automorphism of $V = G/G'$ induced by φ , so that we may regard α as an element of $\text{GL}(V)$. The action of φ on G' is completely determined by α . In fact, since $G' \leq Z(G)$ we have

$$[a, b]^\varphi = [a^\varphi, b^\varphi] = [(aG')^\alpha, (bG')^\alpha].$$

We denote by $\hat{\alpha}$ the automorphism of G' induced by α (or φ). The map $a \wedge b \mapsto [a, b]$ extends to an isomorphism of vector spaces from the exterior square $\Lambda^2 V$ to G' , and this map is also an isomorphism of $\text{GL}(V)$ -modules. Thus we can associate to such a group the linear map defined by

$$f : V \rightarrow \Lambda^2 V \\ x_i \mapsto \sum_{j,k} c_{i,j,k} \cdot x_j \wedge x_k.$$

In the papers [HL74, JK75, Ear75, Hei80, CL82, Car83, Car85, Mor94, Mor95] examples are constructed of groups of the form (3.1) in which all automorphisms are central (see below for variations).

In all of these examples, to prove that all automorphisms are central one uses the fact that a map $\{x_1, \dots, x_n\} \rightarrow G$ extends to an endomorphism of G if and only if it preserves the p -th power relations in (3.1).

This is clear in one direction. If $\varphi \in \text{End}(G)$, and $y_i = x_i^\varphi$, then the y_i satisfy

$$(3.2) \quad y_i^p = \prod_{j < k} [y_j, y_k]^{c_{i,j,k}}, \quad \text{for } i = 1, \dots, n.$$

Conversely, suppose we have a map $\{x_1, \dots, x_n\} \rightarrow G$ such that $x_i \mapsto y_i$, and assume that the y_i satisfy (3.2). Consider the free group F in the variables z_1, \dots, z_n in the variety defined by the laws $[a, b, c], [a, b]^p$, that is, the variety of groups of nilpotence class at most 2, with derived subgroup of exponent dividing p . Clearly G belongs to this variety, and the epimorphism $F \rightarrow G$ determined by $z_i \mapsto x_i$ has kernel N generated by the

$$z_i^{-p} \prod_{j < k} [z_j, z_k]^{c_{i,j,k}}, \quad \text{for } i = 1, \dots, n.$$

Identify G with F/N , by taking $x_i = z_i N$. Since the y_i satisfy (3.2), the morphism $F \rightarrow G$ which sends $z_i \mapsto y_i$ has N in its kernel, and thus induces a morphism $G = F/N \rightarrow G$ which sends $x_i = z_i N \mapsto y_i$. This is the required endomorphism of G .

Thus one starts with a map $\{x_1, \dots, x_n\} \rightarrow G$, and assuming it extends to $\varphi \in \text{Aut}(G)$, one computes the value of $x^p \varphi$, as x ranges over a basis of $V = G/G'$, in two ways. First, since $p > 2$, the p -th power map induces a morphism $G/G' \rightarrow G'$. Thus, in the notation just introduced, $(x^p)^\varphi = (x^\varphi)^p = ((xG')^\alpha)^p$, and this yields one value of $(x^p)^\varphi$. Another value comes from $(x^p)^\varphi = (x^p)^{\hat{\alpha}}$, as $x^p \in G'$. Equating these two values, one obtains (quadratic) equations for the entries of $\alpha \in \text{GL}(V)$ as a matrix with respect to a suitable basis of $V = G/G'$, and these equations are exploited to show that $\alpha = 1$, that is $\varphi \in \text{Aut}_c(G)$.

As we said in the Introduction, in module-theoretic terms, the fact that $\text{Aut}(G) = \text{Aut}_c(G)$ is thus seen to be equivalent to saying that the only element of $\text{GL}(V)$ that commutes with f is the identity,

$$\{\alpha \in \text{GL}(V) : \alpha \circ f = f \circ \hat{\alpha}\} = \{1\}.$$

It might be remarked here that in [HL74, DH75, Car85, GPS11] a more general fact has been used. This states that there is an automorphism φ of G which induces the automorphism α of $V = G/G'$ if and only if

$$(3.3) \quad \alpha \circ f = f \circ \hat{\alpha}.$$

The direct implication is clear. Conversely, if (3.3) holds, choose elements $y_i \in G$ such that $(x_i G')\alpha = y_i G'$. Because of (3.3), the y_i satisfy (3.2), and thus the map $\{x_1, \dots, x_n\} \rightarrow G$ that sends $x_i \mapsto y_i$ extends to an automorphism φ of G , which induces α .

The above setup admits a slight generalization. In some examples of groups with $\text{Aut}(G) = \text{Aut}_c(G)$, the presentation (3.1) includes some relations within G' , so that some products of commutators vanish. However, we require that the condition $Z(G) = \Phi(G)$ still holds, that is, no generator is made central by adding these commutator relations, and also that the relation $|G/G'| = |G^p|$ is preserved. Some examples, for instance in [Mor94], make do without the last requirement, but we will need it for the examples in the following sections.

So we have presentations of the form

$$(3.4) \quad \begin{aligned} G = \langle x_1, \dots, x_n : & [x_i, x_j, x_k] = 1 \text{ for all } i, j, k, \\ & [x_i, x_j]^p = 1 \text{ for all } i, j, \\ & \prod_{j < k} [x_j, x_k]^{d_{l,j,k}} = 1 \text{ for } l = 1, \dots, t, \\ & x_i^p = \prod_{j < k} [x_j, x_k]^{c_{i,j,k}} \text{ for all } i \rangle, \end{aligned}$$

for some t , with $p^n = |G/G'| = |G^p|$, and $Z(G) = \Phi(G)$.

When determining the automorphisms of G , starting with $\alpha \in \text{GL}(V)$ (where $V = G/G'$), we have to make sure that the automorphism $\hat{\alpha}$ leaves invariant the subspace

$$(3.5) \quad K = \left\langle \sum_{j < k} d_{l,j,k} \cdot x_j \wedge x_k : l = 1, \dots, t \right\rangle$$

of $\Lambda^2 V$, which corresponds to the extra relations within G' .

Thus in the module-theoretic setting we have

- (1) a vector space V of dimension at least 4 over $\text{GF}(p)$,
- (2) a subspace K of $\Lambda^2 V$ such that $v \wedge V \not\subseteq K$ for $0 \neq v \in V$ (this is a translation of the requirement $Z(G) = \Phi(G)$), and
- (3) an *injective* linear map

$$(3.6) \quad \begin{aligned} f : V &\rightarrow \Lambda^2 V / K \\ v_i &\mapsto \sum_{j,k=1}^n c_{i,j,k} \cdot v_j \wedge v_k, \end{aligned}$$

where v_1, \dots, v_n is a basis of V .

We say that such a triple (V, K, f) is a *Trivial Automorphism Triple*, or *TAT*, if in addition to (1)-(3) above, we have

- (4) $\{ \alpha \in \text{GL}(V) : K^{\hat{\alpha}} = K, \alpha \circ f = f \circ \bar{\alpha} \} = \{ 1 \}$.

Here $\hat{\alpha}$ and $\bar{\alpha}$ denote the automorphisms induced by α respectively on $\Lambda^2 V$ and, since K is left invariant by $\hat{\alpha}$, on $\Lambda^2 V / K$. Thus every group G of the form (3.4) for which $\text{Aut}(G) = \text{Aut}_c(G)$ yields a TAT.

Conversely, given a TAT (V, K, f) , we write succinctly

$$(3.7) \quad G = \langle x : x^p = xf, K \rangle$$

for the presentation (3.4), where K is as in (3.5) and f as in (3.6). Our discussion shows that for such a group G we have $\text{Aut}(G) = \text{Aut}_c(G)$.

4. A GROUP G WITH ALL AUTOMORPHISMS CENTRAL, BUT $\text{Aut}(G)$ NON-ABELIAN

In this section we apply the techniques of Section 3 to construct an example of a finite, purely non-abelian p -group G such that $\text{Aut}(G) = \text{Aut}_c(G)$ is non-abelian.

Let us first review why $\text{Aut}(G) = \text{Aut}_c(G)$ is abelian in a group (3.7), where (V, K, f) is a TAT. (This would also follow from [AY65, Theorem 4], reported as [JRY13, Theorem 2.7].)

An observation of Guerboussa and Daoud [GD13], which is a specialization of results of H. Laue [Lau85], comes handy. This states that if G is purely non-abelian (a condition that is satisfied here, as we require $Z(G) \leq \Phi(G)$), then $\text{Aut}_c(G)$ under composition is isomorphic to $\text{Hom}(G, Z(G))$ under the circle operation $\gamma \circ \delta = \gamma + \delta + \gamma\delta$, where $\gamma\delta$ denotes composition. (Recall that $g^{\gamma+\delta} = g^\gamma g^\delta$, and note that $\gamma + \delta = \delta + \gamma$, as $g^\gamma, g^\delta \in Z(G)$.) This is readily seen, as if φ is a central automorphism, then the map γ given by $x^\gamma = x^{-1}x^\varphi = [x, \varphi]$ is in $\text{Hom}(G, Z(G))$.

In the case of (3.7), we have $G' = Z(G)$, so if $\gamma, \delta \in \text{Hom}(G, Z(G))$, then $\text{im}(\gamma), \text{im}(\delta) \leq Z(G) = G' \leq \ker(\gamma), \ker(\delta)$, so that $\gamma\delta = 0 = \delta\gamma$, and $\gamma \circ \delta = \gamma + \delta + \gamma\delta = \gamma + \delta = \delta + \gamma = \delta \circ \gamma$. This shows that $\text{Aut}(G)$ is abelian.

We now exploit an idea of Zurek [Zur82]. We start with a TAT (V, K, f) and construct a group as the following variation on (3.7)

$$G = \langle x : x^{p^2} = xf, K \rangle.$$

By this we mean a presentation like (3.4), where we have replaced the p -th powers in the last line of relations with p^2 -th powers. Note that G is still in the variety of groups of class 2, and derived subgroup of exponent p . Also $G' < \Phi(G) = Z(G)$, and the p^2 -th power map is a morphism $G/\Phi(G) \rightarrow G'$ which, with the proper identifications, is the same as $f : V \rightarrow \Lambda^2/K$. The same arguments of the previous section apply to show that the fact that (V, K, f) is a TAT implies that all automorphisms of G are central.

To show that $\text{Aut}(G)$ is non-abelian, we use again the observation of Guerboussa and Daoud: let $n = \dim(V)$, and let x_1, \dots, x_n be a basis for V . Define $\gamma, \delta \in \text{Hom}(G, Z(G))$ by

$$\begin{aligned} x_1^\gamma &= x_1^p, & x_i^\gamma &= 1, \text{ for } i > 1 \\ x_1^\delta &= x_2^p, & x_i^\delta &= 1, \text{ for } i > 1. \end{aligned}$$

Then $x_1^{\gamma\delta} = (x_1^p)^\delta = x_2^{p^2} \neq 1 = (x_2^p)^\gamma = x_1^{\delta\gamma}$, so that $\text{Aut}(G) = \text{Aut}_c(G)$ is non-abelian.

5. GROUPS WITH $\text{Aut}(G) = \text{Aut}_c(G)$ ABELIAN, AND $G' = \Phi(G) < Z(G)$

In this section we construct an example of a finite p -group G with $\text{Aut}(G) = \text{Aut}_c(G)$ elementary abelian, and $G' = \Phi(G) < Z(G)$, so that G is not special.

We start with a TAT (V, K, f) such that $K + Vf$ is a proper subspace of $\Lambda^2 V$ (for instance $K = \{0\}$ will do), and construct first a group

$$(5.1) \quad H = \langle x : x^p = xf, K \rangle$$

as in (3.4).

Then we construct a group G as a central product $H\langle z \rangle$, where z has order p^2 , and we amalgamate $z^p \in H' \setminus H^p$. (The extra condition on the TAT in fact implies that $H^p < H'$.)

We have $W = G/G' \cong V \oplus \langle w \rangle$, where $V = H/H'$ and w is the image of z . Consider $\varphi \in \text{Aut}(G)$. Clearly $\langle w \rangle = Z(G)G'/G'$ is left invariant by φ . Write the action of φ on W in matrix form

$$(5.2) \quad \begin{bmatrix} \alpha & \lambda \\ 0 & \mu \end{bmatrix},$$

with respect to a basis of V , extended with w . Here $\alpha \in \text{GL}(V)$, $\mu \in \text{GF}(p)$, and λ is a column vector of length n . (Since our automorphisms act on the right, elements of V and W are row vectors.)

Consider the embedding $\iota : V \rightarrow V \oplus \langle w \rangle = W$. The composition

$$V \xrightarrow{\iota} V \oplus \langle w \rangle \xrightarrow{f} \Lambda^2 V$$

equals f . Moreover, the action of φ on $G' \cong \Lambda^2 V$ is completely determined by α , as $z \in Z(G)$. Therefore we have

$$v^\alpha f = (vf)^{\hat{\alpha}},$$

so that $\alpha = 1$, as (V, K, f) is a TAT. This implies that $\hat{\alpha} = 1$ on $\Lambda^2 V$, and since the p -th power map $V \oplus \langle w \rangle \rightarrow \Lambda^2 V$ is injective, also $\lambda = 0$ and $\mu = 1$. It follows that the matrix in (5.1) is the identity, and thus φ induces the identity modulo G' . This shows that $\text{Aut}(G) = \text{Aut}_c(G)$.

We have shown that all automorphisms of G are trivial modulo $G' < Z(G)$. Since G' is elementary abelian, $\text{Aut}(G)$ is also elementary abelian, as it is isomorphic to $(\text{Hom}(G/G', G'), \circ)$.

6. GROUPS WITH $\text{Aut}(G) = \text{Aut}_c(G)$ ABELIAN, AND $G' < \Phi(G) = Z(G)$

In this section we construct a finite p -group G with $\text{Aut}(G) = \text{Aut}_c(G)$ abelian and $G' < \Phi(G) = Z(G)$; in particular G is not special.

Let us start from a TAT $(V, \{0\}, f)$, with $|V| = p^n$. Define a special p -group

$$H = \langle x : x^p = xf \rangle$$

as in (3.7), with $K = \{0\}$.

As seen in Section 4, if $V = H/H'$, then the group of central automorphisms of H is isomorphic to $\text{Hom}(H, Z(H)) \cong \text{Hom}(V, \Lambda^2 V)$ under the circle operation. The latter contains the subspace

$$\{x \mapsto x \wedge v : v \in V\}$$

of the *inner* maps, which correspond to the inner automorphisms of H . Since $v \wedge v = 0$ for all $v \in V$, none of these inner maps is injective. Therefore if we choose an *injective* $\gamma \in \text{Hom}(H, Z(H))$ (for instance, $x \mapsto x^p$ will do), this will not be inner. Let $m > 1$, and consider the extension G of H by an element y such that $[x, y] = x\gamma$, for $x \in H$, and $y^{p^m} \in H' \setminus H^p$. The choice of γ ensures that $Z(G) \leq \Phi(G)$, and actually $Z(G) = \Phi(G)$.

We have $G/\Phi(G) \cong \langle u \rangle \oplus V$, where u is the image of y . Let $\varphi \in \text{Aut}(H)$. Since V is the image in $\langle u \rangle \oplus V$ of $\Omega_2(G)$, the automorphism φ induces on $\langle u \rangle \oplus V$ a

linear map which can be described by the matrix

$$(6.1) \quad \begin{bmatrix} \tau & \sigma \\ 0 & \alpha \end{bmatrix},$$

with respect to a basis which begins with u , and continues with a basis of V . Here $\alpha \in \text{GL}(V)$, $\tau \in \text{GF}(p)$, and σ is a row vector of length n . Since $G' = H'$, we have that α alone determines the action $\hat{\alpha}$ of φ on G' .

As in the previous section, the composition of the embedding $V \xrightarrow{f} \langle u \rangle \oplus V$ with the p -th power map equals f . Since $(V, \{0\}, f)$ is a TAT, we obtain $\alpha = \hat{\alpha} = 1$. In particular φ acts trivially on $G' \cong \Lambda^2 V$. This also implies $\tau = 1$.

Let x_1, \dots, x_n be elements of H such that their images v_1, \dots, v_n are a basis of $V = H/H'$. Now if in $\langle u \rangle \oplus V$ we have

$$u\varphi = u + \sigma_1 v_1 + \dots + \sigma_n v_n,$$

where $\sigma = [\sigma_1, \dots, \sigma_n]$ with respect to the basis of the v_i , then

$$\begin{aligned} [u, v_1] &= [u, v_1]\varphi = [u\varphi, v_1] \\ &= [u + \sigma_1 v_1 + \dots + \sigma_n v_n, v_1] \\ &= [u, v_1] + \sigma_2 [v_2, v_1] + \dots + \sigma_n [v_n, v_1], \end{aligned}$$

so that

$$\sigma_2 [v_2, v_1] + \dots + \sigma_n [v_n, v_1] = 0.$$

Now the elements $v_2 \wedge v_1, \dots, v_n \wedge v_1$ are independent in $\Lambda^2 V$. Since $K = \{0\}$, we have $G' \cong \Lambda^2 V$, so that $\sigma_2 = \dots = \sigma_n = 0$. Considering $[u, v_2]\varphi = [u, v_2]$, we see that also $\sigma_1 = 0$, so that $\sigma = 0$, the matrix in (6.1) is the identity, and thus φ induces the identity modulo $Z(G) = \Phi(G)$.

Using once more the approach of Section 4 (or appealing again to [AY65, Theorem 4]), we have that $\text{Aut}(G) = \text{Aut}_c(G)$ is isomorphic to $(\text{Hom}(G, Z(G)))$ under the circle operation. Take two elements $\gamma_1, \gamma_2 \in \text{Hom}(G, Z(G))$, which are easily seen to be of the form

$$\gamma_k : y \mapsto y^{p^{s_k}} \pmod{G'}, \quad x_i \mapsto 1 \pmod{G'}, \text{ for all } i,$$

for some s_1, s_2 . Since $G' \leq \ker(\gamma_k)$, and $(G')^p = 1$, we have

$$y\gamma_1\gamma_2 = y^{p^2 s_1 s_2}, \quad x_i\gamma_1\gamma_2 = 1, \text{ for all } i,$$

from which it follows that $\text{Aut}(G) = \text{Aut}_c(G)$ is abelian. Moreover $G' < \langle G', y^p \rangle = \Phi(G) = Z(G)$.

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