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Brane-worlds in T-dual bulks

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Abstract

We consider brane-world models with a Schwarzschild–AdS black hole bulk. In the particular case of a flat black hole horizon geometry, we study the behaviour of the brane cosmological equations when T-duality transformations act on the bulk. We find that the scale factor is inverted and that either the Friedmann equation or the energy conservation equation are unchanged. However, these become both invariant if we include a tension in the brane action. In this case, the T-duality in the bulk is completely equivalent to the scale factor duality on the brane.

1. Introduction

One important lesson learned from string theory is that even though two or more theories appear to be very different, they might actually be the same theory seen from different points of view. With this spirit in mind, in this Letter we consider two aspects of string cosmology which have attracted the attention of many researchers in the last decade: pre-Big Bang scenario and brane-world models. The former is a theory essentially based on the $O(d,d)$ symmetry group of certain cosmological backgrounds which appear in low energy string theory (for a comprehensive review see [1]). One of the element of the symmetry group manifests itself through the invariance of the equations of motion under the inversion of the scale factor. This element corresponds to a T-duality transformation along the time direction, also known as scale factor duality. Together with the time-reversal symmetry of the field equations, scale factor duality smoothly connect a pre-Big Bang phase of growing curvature to the present expanding phase of decreasing curvature. The transition between the two phases is expected to occur in a non-perturbative string theory regime, in such a way that the standard cosmology is recovered after the Big Bang. This requirement is often called the graceful exit problem, since the details are still under investigation.

The second idea consist in considering our Universe as a 3-dimensional surface embedded in a 5-dimensional bulk space–time (good reviews can be found in [2,3]). Matter is confined on the brane, together with all the fundamental forces except gravity, which is a 5-dimensional field. Nevertheless, gravity on the brane is approximately Newtonian, and this property holds even when the extent of the extra-dimension is infinite [4]. Among the appealing fea-

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tures of this model there is also a possible explanation for the huge hierarchy between the electro-weak scale and the Planck scale. Moreover, if the brane is not static, an observer living on it then observes an effective 4-dimensional cosmology described by a modified Friedmann equation, which, under appropriate conditions, matches with the standard one at late times.

A connection between brane-world models and pre-Big Bang scenario is expected for at least one reason, i.e., symmetry. Indeed, on one hand we have the pre-Big Bang scenario which is based on a large $O(d,d)$ symmetry group acting on the action. On the other hand, we have branes embedded in bulks which, in most cases, have themselves a large number of isometries. For example, in the Randall–Sundrum model proposed in [4], the brane glues together two slices of anti-de Sitter space, i.e., a maximally symmetric space. Therefore, it could be interesting to see what happens to the effective cosmology on the brane under the action of the isometry group on the bulk. As a concrete example, in this Letter we examine two space–times which are related by T-duality, and which can host a brane with a flat FLRW induced metric. As we will shortly see, the duality transformations on the bulk induce the inversion of the scale factor on the brane. This simple fact leads to think of a possible connection with the scale factor duality of pre-Big Bang scenario. We will see that this connection exists, provided that some conditions are satisfied by the brane matter Lagrangian.

In Section 2, we briefly introduce two space–times related by T-duality in the context of type IIA and type IIB string theory. Their compactification to five dimensions leads to two black hole metrics, one of these being the well-known AdS–Schwarzschild black hole with toroidal horizon. In Section 3 we review the cosmological equations induced on the brane moving in the AdS–Schwarzschild black hole bulk. These are then compared to the ones obtained on the brane embedded in the dual bulk in Section 3. Finally, in Section 4 we examine a simple solution to the dual Friedmann equations, and we conclude with a summary and few remarks. We also add an appendix, where we show how the Lanczos–Israel junction conditions in string frame can be obtained from the ones written in Einstein frame by means of a conformal transformation.

2. The T-dual backgrounds

We consider the generalization of the $AdS_5 \times S^5$ compactification of type IIB string theory, obtained by replacing the AdS_5 space–time with a 5-dimensional topological black hole ([5], see also [6,7]). The 10-dimensional metric is

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + \frac{r^2}{l^2}\,d\Sigma_3^2 + l^2\,d\Omega_5^2, \tag{1}$$

and the function $f(r)$ is defined by

$$f(r) = k + \frac{r^2}{l^2} - \frac{\mu}{r^2}, \tag{2}$$

where l and μ are constants. The parameter k determines the geometry of the horizon and from now on we set $k = 0$, i.e., we consider only flat (or toroidal) horizons with metric $d\Sigma_3^2 = \delta_{ij}\,dx^i\,dx^j$. The metric (1) must be supplemented with an anti-self Hodge dual 5-form

$$F^{(5)} = -{}^\star F^{(5)}, \tag{3}$$

in order to satisfy the fundamental equation [5]

$$R_{MN} = \frac{1}{6\cdot 4^2} F_M{}^{A_2A_3A_4A_5} F_{NA_2A_3A_4A_5}. \tag{4}$$

The 10-dimensional space–time (1) can be easily compactified, yielding a 5-dimensional AdS–Schwarzschild topological black hole with flat or toroidal horizon [8]. The metric reads

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + \frac{r^2}{l^2}\delta_{ij}\,dx^i\,dx^j, \tag{5}$$

where now l is related to the 5-dimensional cosmological constant by $\Lambda = -(6/l^2)$, and μ is the mass of the black hole. These fields are solutions to the equations of motion derived from the effective action

$${}^{(5)}S = \frac{1}{2}\int dx^5\,\sqrt{g}\,[R - 2\Lambda]. \tag{6}$$

In [9], it was shown that the solutions (1) and (3) can be mapped by T-duality transformations into new low energy solutions of type IIA or IIB string theory. In particular, by applying three T-duality transformations along the horizon coordinates, one obtains a type IIA background with metric and dilaton given by

$$ds^2 = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + \frac{l^2}{r^2}\delta_{ij}\,dx^i\,dx^j + l^2\,d\Omega_5^2, \tag{7}$$

$$e^{-2\phi} = \frac{r^6}{l^6}, \tag{8}$$

and forms

$$F^{(2)}{}_{\mu_1\mu_2} = \frac{4r^3}{l^4}\varepsilon_{\mu_1\mu_2}, \quad \mu \neq x_i, \tag{9}$$

$$F^{(8)}{}_{A_1\ldots A_8} = -\frac{4r^3}{l^4}\varepsilon_{A_1\ldots A_8}, \quad A_i \neq t, r. \tag{10}$$

In addition to these, a B-field can be switched on by boosting the metric (1) along one of the horizon direction before applying T-duality [9]. For simplicity, here we set the B-field to zero. These fields solve the equations of motion derived from the type IIA low energy string action

$$S = \frac{1}{2}\int d^{10}x\,\sqrt{-G} \times \left\{e^{-2\phi}\left[R + 4(\nabla\phi)^2\right] - \frac{1}{4}F_{MN}F^{MN}\right\}, \tag{11}$$

which can be easily compactified to 5 dimensions. Indeed, if we assume that S^5 has unit volume, we then find

$$S = \frac{1}{2}\int d^5x\,\sqrt{-g} \times \left\{e^{-2\phi}\left[R + 4(\nabla\phi)^2 + \frac{20}{l^2}\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}, \tag{12}$$

where

$$g_{\mu\nu}\,dx^\mu\,dx^\nu = -f(r)\,dt^2 + f(r)^{-1}\,dr^2 + \frac{l^2}{r^2}\delta_{ij}\,dx^i\,dx^j, \tag{13}$$

$$F_{\mathrm{tr}} = \frac{4r^3}{l^4}, \qquad \phi = -3\log\frac{r}{l}. \tag{14}$$

3. The brane-world

We now examine the behaviour of a 4-dimensional brane radially moving in the 5-dimensional compactifications of these T-dual backgrounds. In the case of a bulk with metric (5), the 4-dimensional cosmological equations are well-known [2,3]. If we add to the bulk action (6) a boundary term of the form

$$S_{\mathrm{brane}} = \int dx^4\,\sqrt{h}(L - K), \tag{15}$$

where L and K are, respectively, the matter Lagrangian and the trace of the extrinsic curvature, and if assume a $\mathbf{Z}_2$ symmetry about the brane, then we must impose the Lanczos–Israel junction conditions

$$K_{\mu\nu} = -\frac{1}{2}\left(\tau_{\mu\nu} - \frac{1}{3}\tau h_{\mu\nu}\right). \tag{16}$$

This equation relates the brane matter stress tensor $\tau_{\mu\nu}$ to the extrinsic curvature $K_{\mu\nu}$ defined by

$$K^{\mu\nu} = \nabla^\mu n^\mu, \tag{17}$$

where n^μ is the unit vector normal to the brane and pointing into the bulk. The stress tensor components are $\tau^\mu_\nu = \mathrm{diag}(-\rho, p, p, p)$, where ρ and p are, respectively, the energy density and pressure. The vector n^μ is normalized so that the induced metric on the brane is given by

$$h_{\mu\nu}\,dx^\mu\,x^\nu = -d\tau^2 + a(\tau)^2\delta_{ij}\,dx^i\,dx^j, \tag{18}$$

where $a(\tau) = r(\tau)/l$ is the scale factor and τ is an affine parameter, usually identified with the cosmic time measured on the brane. With these settings, we can combine the $(\tau\tau)$ and the (ij) components of Eq. (16) and obtain [2,3]

$$\frac{\dot{r}^2}{r^2} = \frac{\rho^2}{36} - \frac{1}{l^2} + \frac{\mu}{r^4}, \tag{19}$$

$$\dot{\rho} = -3\frac{\dot{r}}{r}\rho(\omega + 1), \tag{20}$$

where the dot stands for differentiation with respect τ, and where we assume that $p = \omega\rho$. These expressions can be interpreted respectively as the Friedmann and the energy conservation equations by a 4-dimensional observer living on the brane, provided that we define the Hubble function

$$H = \frac{\dot{a}}{a} = \frac{\dot{r}}{r}. \tag{21}$$

We see that the Hubble function in Eq. (19) is proportional to ρ, instead of $\sqrt{\rho}$, as in standard cosmology. However [2,3], if we add a tension V to the Lagrangian of the brane action (15), then $\rho \to$

$\rho + V$, and the Friedmann equation becomes

$$H^2 = \frac{1}{18}V\rho + \frac{\rho^2}{36} + \frac{V^2}{36} + \frac{\mu}{r^4} - \frac{1}{l^2}. \tag{22}$$

Therefore, when $V \gg \rho$, we can neglect the term proportional to ρ^2 and recover the standard cosmology up to the so-called dark radiation term μ/r^4, and the constant

$$C = \frac{V^2}{36} - \frac{1}{l^2}. \tag{23}$$

If we set $\omega = -1$ then, according to Eq. (20), ρ is constant. If we also set $C = 0$ (fine-tuning), then μ must necessarily vanish [12], and we recover the standard Randall–Sundrum scenario with one (static) brane [4].

The metrics (5) and (13) are related by the inversion of the (ij) components. It is then clear that if we embed a 4-dimensional brane in the dual bulk space–time (13), then the induced metric will be

$$ds^2 = -d\tau^2 + \tilde{a}(\tau)^2\delta_{ij}\,dx^i\,dx^j, \tag{24}$$

where now

$$\tilde{a}(\tau) = \frac{l}{r(\tau)} = \frac{1}{a(\tau)} \tag{25}$$

is the new scale factor. Therefore, the T-duality transformations on the bulk generate the inversion of the scale factor of the induced metric on the brane. The natural question is then how the effective 4-dimensional cosmology is affected. In particular, if the cosmological equations are unchanged, then the T-duality symmetry of the bulk corresponds to a scale factor duality symmetry on the brane. In the next section we will see that this can indeed be the case.

4. The dual brane-world

The dual fields (13) and (14) are solutions of the equations of motion derived from the action (12) written with respect to the string frame. Therefore, in order to study the brane cosmological equations, we need the junction conditions in string frame which read (see Appendix A)

$$K_{\alpha\beta} = -\frac{1}{2}\Omega^{(q+1)}\left(\tau_{\alpha\beta} - \frac{\tau}{3}h_{\alpha\beta}\right) - h_{\alpha\beta}\Omega^{-1}n^\mu\partial_\mu\Omega, \tag{26}$$

$$n^\mu\partial_\mu = -\frac{1}{2}\Omega^{(q+1)}\frac{\partial\xi}{\partial\phi}\tau, \tag{27}$$

where Ω^2 is the conformal factor which relates string and Einstein frame, and q determines the coupling of the brane Lagrangian to the induced metric. In our case, the action (12) can be transformed in Einstein frame through the conformal rescaling

$$g_{\mu\nu} = e^{4\phi/3}\tilde{g}_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu}, \tag{28}$$

hence, according to our notations,

$$\Omega = e^{-2\phi/3}. \tag{29}$$

To determine the unit normal vector n^μ, we first impose that the brane moves along a radial geodesics with velocity $u^\mu = (\dot{t}, \dot{r}, 0, 0, 0)$, where the dot stands for differentiation with respect to the affine parameter τ [2,3]. The bulk metric (13) can then be written as

$$ds^2 = -\left[f(r)\dot{t}^2 - f(r)^{-1}\dot{r}^2\right]d\tau^2 + \tilde{a}^2(\tau)\delta_{ij}\,dx^i\,dx^j, \tag{30}$$

and, if we impose the normalization

$$f(r)\dot{t}^2 - f(r)^{-1}\dot{r}^2 = 1,$$

we then obtain the induced metric (24). Finally, the condition $n^\mu u_\mu = 0$ leads to $n_\alpha = (\dot{r}, -\dot{t}, 0, 0, 0)$. With these settings, the (ij) and $(\tau\tau)$ components of Eq. (26) read, respectively,

$$\frac{1}{\dot{r}}\frac{d}{d\tau}\sqrt{f + \dot{r}^2} + \frac{4}{r}\sqrt{f + \dot{r}^2} = -\frac{1}{2}\omega\rho e^{-\frac{2}{3}(q+1)\phi}, \tag{31}$$

$$\frac{1}{r}\sqrt{f + \dot{r}^2} = \frac{\rho}{6}e^{-\frac{2}{3}(q+1)\phi}, \tag{32}$$

where we also assumed that $\tau^\mu_\nu = \rho\,\mathrm{diag}(-1, \omega, \omega, \omega)$. By squaring the $(\tau\tau)$ components and using the definition (2) with $k = 0$, we find

$$\frac{\dot{r}^2}{r^2} = \frac{\rho^2}{36}e^{-\frac{4}{3}(q+1)\phi} - \frac{1}{l^2} + \frac{\mu}{r^4}. \tag{33}$$

Then, with the explicit form of the dilaton (14) and Eq. (33), the $(\tau\tau)$ component of Eq. (26) can be written as

$$\dot{\rho} = -\rho\frac{\dot{r}}{r}(3\omega + 2q + 7). \tag{34}$$

Finally, the junction condition (27) yields

$$\frac{\partial\xi}{\partial\phi} = \frac{1}{1 - 3\omega}. \tag{35}$$

These conditions are valid if we suppose that the bulk 2-form does not couple to the matter on the brane. Consequently, there are no junction conditions to be imposed on the bulk form field [13]. According to the definitions (21) and (25), the Hubble function is

$$\tilde{H} = \frac{\dot{\tilde{a}}}{\tilde{a}} = -\frac{\dot{a}}{a} = -H = -\frac{\dot{r}}{r}. \tag{36}$$

Then, Eqs. (33) and (34) reads, respectively,

$$\tilde{H}^2 = \frac{\rho^2}{36} e^{-\frac{4}{3}(q+1)\phi} - \frac{1}{l^2} + \frac{\mu}{l^4}\tilde{a}^4, \tag{37}$$

$$\dot{\rho} = \rho\tilde{H}(3\omega + 2q + 7). \tag{38}$$

We see that, for $q = -1$, the Friedmann equation (33) is the same as (19), while, for $q = -2$, the conservation equation (34) is equal to Eq. (20). Hence, under T-duality transformation on the bulk, either the Friedmann equation or the energy conservation equation are unchanged on the brane. In particular, Eq. (34) describes the non-conservation of the energy on the brane from a point of view of an observer in the bulk. To measure the eventual energy flow from the point of view of an observer living on the brane, we must write this equation in the conformal frame, usually called the Jordan frame, related to the metric $\gamma_{\mu\nu}$ (see Appendix A and [2]). Usually, through the conformal transformation

$$\gamma_{\mu\nu} = e^{2\xi(\phi)} h_{\mu\nu}, \tag{39}$$

we "distort" the brane in the bulk in such a way that the dilaton flux is tangential to brane itself, hence its contribution to the energy density vanishes.[1] In the case of branes embedded in asymptotically AdS bulks with dilaton field, one always find that, in the Jordan frame, the energy is conserved [2,3]. However, this does not happen in our model. Indeed, if we follow [2] and we replace

$$\rho \to e^{-4\xi(\phi)}\rho, \qquad \tilde{a} \to e^{\xi(\phi)}\tilde{a},$$
$$dt \to e^{\xi(\phi)}\,dt, \tag{40}$$

in Eq. (38), we obtain

$$\dot{\rho} + \rho\tilde{H}\left[\frac{3\xi'}{1+3\xi'}(4+\beta) - \beta\right] = 0, \tag{41}$$

[1] Thanks to P. Watts for suggesting this interpretation.

where $\xi' = \partial_\phi \xi(\phi)$ and $\beta = 3\omega + 2q + 7$. By using Eq. (35), we find that

$$\frac{3\xi'}{1+3\xi'}(4+\beta) - \beta = 3(\omega+1)$$
$$\iff q = -\frac{1}{2}(7 + 6\omega). \tag{42}$$

This means that the energy on the brane, measured with respect to the Jordan frame, is *not* in general conserved. The reason for this "anomaly" becomes clear once we remember that the junction conditions depend on the induced metric $h_{\alpha\beta}$ which is related to the bulk metric. The latter is a solution to the bulk equations of motion which also include the 2-form. Hence, even if we suppose that it does not couple to the matter, the 2-form does affect the brane dynamics through the bulk equations of motion. In other words, the 2-form flux through the brane does not in general vanish, even in the Jordan frame.

This suggests that, if we chose a more appropriate brane Lagrangian, the energy might be conserved in the Jordan frame or, more importantly, even in the conformal frame defined by $h_{\mu\nu}$. In this case, both cosmological equations would be invariant under T-duality in the bulk. Therefore, we generalise the brane Lagrangian by considering the total energy density on the brane as sum the of the brane matter energy density and a *time-dependent* tension, i.e., we set $\rho = \tilde{\rho} + V(\tau)$. It is then easy to show that Eq. (34) takes the form

$$\dot{\tilde{\rho}} = -3\tilde{\rho}\frac{\dot{r}}{r}(\omega+1) = \tilde{\rho}\tilde{H}(\omega+1), \tag{43}$$

provided that we use the relation $\dot{\phi} = 3\tilde{H}$ and we assume that

$$V(\tau) = e^{a\phi}, \tag{44}$$

where a is an arbitrary constant. With this form, the time-dependent tension can also be interpreted as an effective dilatonic potential on the brane (similar solutions were found in [10], see also [11]). Also, note that the Friedmann equation (33) assume the same form of Eq. (22). Therefore we conclude that, when $q = -1$ and when there is a dilatonic potential on the brane, *both* brane cosmological equations are invariant under T-duality in the bulk.

5. Duality of brane dynamics

According to the results of the previous section, to each expanding solution to the Friedmann equation on the brane, there exists a dual contracting one, in strict analogy with the pre-Big Bang scenario. As an example, let us consider the solutions to Eqs. (19) and (20) for $\mu = 0$. In this case, Eq. (20) can be easily solved, yielding

$$\rho = \rho_0 a^{-3(\omega+1)}, \tag{45}$$

ρ_0 being an arbitrary constant. If we assume that the total energy density on the brane is given by $\rho_{\rm tot} = V + \rho$ and impose the fine-tuning $6/l = V$, then the solution to Eq. (22) reads

$$\tilde{a}^{-3(w+1)} = a^{3(\omega+1)} = At + Bt^2, \tag{46}$$

where A and B are positive constants [2]. These solutions are valid also when $\mu \neq 0$, provided that t is large. Indeed, in this case the scale factor $a = r/l$ of the type IIB background is large and the mass term in Eq. (22) is negligible. This situation corresponds to a brane moving in the asymptotic region of the black hole, where the space–time is locally anti-de Sitter [12]. In the type IIA background, for large values of t, the scale factor becomes very small, and in Eq. (37) the mass term tends again to zero. Since $\phi = 3 \log \tilde{a}$, we see that, for large t, we have large values of the effective string coupling $e^{-2\phi}$. Hence, the brane moving in the dual background of type IIA enters the strong coupling regime at late times (i.e., large r). In the type IIB background there is no dilaton, however, a breakdown of the model occurs when the brane approaches the horizon and the tension diverges [12]. Therefore, even in this simplified model we can see the duality between physics at small and large distances typical of T-duality.

6. Conclusions

We have considered two 4-dimensional branes moving respectively in two bulks related by T-duality. The branes have reciprocal scale factors and we found that either the effective Friedmann equation or the energy conservation equation are unchanged, according to which form of the coupling between brane matter and induce metric we choose. However, we showed that if we add to the brane Lagrangian a dilatonic potential, then it is possible to have both equation invariant under duality. Therefore, we have an exact equivalence between scale factor duality on the brane and T-duality in the bulk. To obtain these result we have used the junction conditions in string frame obtained, by means of a conformal transformation, from the junction conditions in Einstein frame. Finally, we analyzed a simple solution to the Friedmann equation, and we showed how the typical correspondence of T-duality between large and small distances emerges in the brane dynamics.

These results may have interesting applications in the context of the pre-Big Bang scenario, in particular in relation with the graceful exit problem. Also, the equivalence between bulk and brane duality might be extended by means of non-Abelian T-dualities [20,21] or to more general string backgrounds. We think that all these aspects deserve further investigations.

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Appendix A. Junction conditions in string frame

The Lanczos–Israel matching conditions in string frame have been found in various works (see, for example, [14,15]), by extremizing the bulk action implemented by the appropriate boundary terms. Here, we propose a much simpler approach based on the conformal mapping between the action written in Einstein frame and in string frame. Consider the generic action in Einstein frame with a boundary term

$$\tilde{S} = \frac{1}{2}\int dx^5 \sqrt{\tilde{g}} \left\{ \tilde{R} - \frac{4}{3}(\partial\phi)^2 + \tilde{L}_{\rm bulk} \right\} + \int dx^4 \sqrt{\tilde{h}}\, \tilde{L}_{\rm brane}, \tag{A.1}$$

where $\tilde{L}_{\rm bulk}$ and $\tilde{L}_{\rm brane}$ denote generic matter Lagrangians in the bulk and in the brane, respectively, ϕ is the dilaton field, $\tilde{h}_{\mu\nu} = \tilde{g}_{\mu\nu} - \tilde{n}_\mu \tilde{n}_\nu$ is the induced metric on the brane, and $\tilde{n}^\mu$ is the unit vector normal to the brane. By assuming $\mathbf{Z}_2$ symmetry about the brane, the junction conditions to be satisfied by the metric read [17–19]

$$\tilde{K}_{\mu\nu} = -\frac{1}{2}\left(\tilde{\tau}_{\mu\nu} - \frac{1}{3}\tilde{\tau}\tilde{h}_{\mu\nu}\right), \tag{A.2}$$

where $\tilde{\tau}_{\mu\nu}$ is the stress tensor of the brane matter, and $\tilde{K}^{\mu\nu} = \tilde{\nabla}^{\mu}\tilde{n}^{\nu}$ is the extrinsic curvature of the brane. We also assume that the matter fields confined on the brane are coupled to a metric $\tilde{\gamma}_{\mu\nu}$, conformally related to the induced metric through

$$\tilde{\gamma}_{\alpha\beta} = e^{2\tilde{\xi}(\phi)}\tilde{h}_{\alpha\beta}. \tag{A.3}$$

Then, the dilaton field must satisfy the junction condition [2]

$$\tilde{n}^{\mu}\tilde{\nabla}_{\mu}\phi = -\frac{1}{2}\frac{\partial\tilde{\xi}}{\partial\phi}\tilde{\tau}. \tag{A.4}$$

The action can be written in string frame by means of a conformal rescaling of the metric

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{A.5}$$

where Ω is in general a function of the dilaton field. If we rescale the induced metric and the unit normal vector according to

$$\tilde{h}_{\mu\nu} = \Omega^2 h_{\mu\nu}, \qquad \tilde{n}_{\alpha} = \Omega n_{\alpha}, \tag{A.6}$$

then we ensure that $n_{\mu}n^{\mu} = 1$ and that $h_{\alpha}{}^{\beta}$ is a projection operator [16]. Note also that $\tilde{h}_{\alpha}{}^{\beta} = h_{\alpha}{}^{\beta}$, and $h_{\mu\nu} = g_{\mu\nu} - n_{\mu}n_{\nu}$. By using the transformation law for the connection coefficients [16]

$$\tilde{\Gamma}^{\alpha}{}_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} + \Omega^{-1} \times \left(\delta^{\alpha}{}_{\mu}\partial_{\nu}\Omega + \delta^{\alpha}{}_{\nu}\partial_{\mu}\Omega - g_{\mu\nu}g^{\alpha\beta}\partial_{\beta}\Omega\right), \tag{A.7}$$

we find that the extrinsic curvature in string frame reads

$$\tilde{K}_{\alpha\beta} = \Omega K_{\alpha\beta} + h_{\alpha\beta}n^{\mu}\partial_{\mu}\Omega. \tag{A.8}$$

In order to write the right-hand side of Eq. (A.2) in string frame, we need to know how the Lagrangian $\tilde{L}_{\mathrm{brane}}$ transforms under the conformal rescaling (A.5). The transformation will in general depend on the field content of the brane Lagrangian, and here we simply suppose that, under the transformation (A.5),

$$\tilde{L}_{\mathrm{brane}} = \Omega^{q} L_{\mathrm{brane}}, \tag{A.9}$$

for some real number q. Consequently, if we define in Einstein frame

$$\tilde{\tau}^{\mu\nu} = \frac{2}{\sqrt{\tilde{h}}}\frac{\delta(\sqrt{\tilde{h}}\,\tilde{L})}{\delta\tilde{h}_{\mu\nu}}, \tag{A.10}$$

then the conformal transformation of the brane stress tensor reads

$$\tilde{\tau}^{\mu\nu} = \Omega^{(q-2)}\tau^{\mu\nu}. \tag{A.11}$$

Alternatively [2], we can define the brane stress tensor in Einstein frame according to the conformal metric $\tilde{\gamma}_{\alpha\beta}$ (often called the Jordan frame) to which brane matter fields couple to, i.e.,

$$^{(\gamma)}\tilde{\tau}^{\mu\nu} = \frac{2}{\sqrt{\tilde{\gamma}}}\frac{\delta(\sqrt{\tilde{h}}\,\tilde{L})}{\delta\tilde{\gamma}_{\mu\nu}}. \tag{A.12}$$

Then, according to Eq. (A.5), in Einstein frame the two stress tensors (A.10) and (A.12) are related by

$$^{(\gamma)}\tilde{\tau}^{\mu}{}_{\nu} = e^{-4\tilde{\xi}(\phi)}\tilde{\tau}^{\mu}{}_{\nu}. \tag{A.13}$$

In string frame, we have the analogous equivalence

$$^{(\gamma)}\tau^{\mu}{}_{\nu} = e^{-4\xi(\phi)}\tau^{\mu}{}_{\nu}, \tag{A.14}$$

provided that we impose

$$\gamma_{\alpha\beta} = e^{2\xi(\phi)}h_{\alpha\beta}. \tag{A.15}$$

Hence, consistency requires that

$$\tilde{\gamma}_{\alpha\beta} = e^{2\tilde{\xi}(\phi)}\Omega^2\gamma_{\alpha\beta}e^{-2\xi(\phi)}. \tag{A.16}$$

If we make the choice $\tilde{\xi}(\phi) = \xi(\phi)$, then $\tilde{\gamma}_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$ transform in the same way under the rescaling (A.5), i.e.,

$$\tilde{\gamma}_{\mu\nu} = \Omega^2\gamma_{\mu\nu} \iff \tilde{h}_{\mu\nu} = \Omega^2 h_{\mu\nu}. \tag{A.17}$$

With these settings, it is then easy to show that Eqs. (A.2) and (A.4) in string frame read

$$K_{\alpha\beta} = -\frac{1}{2}\Omega^{(q+1)}\left(\tau_{\alpha\beta} - \frac{\tau}{3}h_{\alpha\beta}\right) - h_{\alpha\beta}\Omega^{-1}n^{\mu}\partial_{\mu}\Omega, \tag{A.18}$$

$$n^{\mu}\partial_{\mu} = -\frac{1}{2}\Omega^{(q+1)}\frac{\partial\xi}{\partial\phi}\tau. \tag{A.19}$$

If we set $q = -4$ and we define the extrinsic curvature as $K^{\mu\nu} = -\nabla^{\mu}n^{\nu}$, we then recover the junction conditions found in [14]. Alternatively, for $q = -1$ and L replaced with $-L/2$, our results match with the ones obtained in [15].

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