CLASSIFICATION OF FOUR-REBIT STATES

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Abstract. We classify states of four rebits, that is, we classify the orbits of the group $\widehat{G}(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})^4$ in the space $(\mathbb{R}^2)^{\otimes 4}$. This is the real analogon of the well-known SLOCC operations in quantum information theory. By constructing the $\widehat{G}(\mathbb{R})$-module $(\mathbb{R}^2)^{\otimes 4}$ via a $\mathbb{Z}/2\mathbb{Z}$-grading of the simple split real Lie algebra of type D_4, the orbits are divided into three groups: semisimple, nilpotent and mixed. The nilpotent orbits have been classified in Dietrich et al. (2017), yielding applications in theoretical physics (extremal black holes in the STU model of $\mathcal{N}=2, D=4$ supergravity, see Ruggeri and Trigiante (2017)). Here we focus on the semisimple and mixed orbits which we classify with recently developed methods based on Galois cohomology, see Borovoi et al. (2021). These orbits are relevant to the classification of non-extremal (or extremal over-rotating) and two-center extremal black hole solutions in the STU model.

1. Introduction

In a recent paper [27], we obtained a complete and irredundant classification of the orbits of the group $\widehat{G} = \mathrm{SL}(2,\mathbb{C})^4$ acting on the space $(\mathbb{C}^2)^{\otimes 4} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. This is relevant to Quantum Information Theory because it amounts to the classification of the entanglement states of four pure multipartite quantum bits (*qubits*) under the group $\widehat{G}$ of reversible Stochastic Local Quantum Operations assisted by Classical Communication (SLOCC). Here we obtain the classification of the orbits of the real group $\widehat{G}(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})^4$ on the space $(\mathbb{R}^2)^{\otimes 4}$. This is relevant to real quantum mechanics, where the elements of $(\mathbb{R}^2)^{\otimes 4}$ are called four-rebit states. Via the *"black hole /qubit correspondence"* our classification has also applications to high-energy theoretical physics. We refer to Section 2 for a short introduction into rebits and their relevance to extremal black holes in string theory.

The main idea behind the complex classification is to construct the representation of $\widehat{G}$ on $(\mathbb{C}^2)^{\otimes 4}$ using a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the simple Lie algebra $\mathfrak{g}$ of type D_4 (see Section 3 for more details). In this construction the spaces $(\mathbb{C}^2)^{\otimes 4}$ and $\mathfrak{g}_1$ are identified, yielding a Jordan decomposition of the elements of $(\mathbb{C}^2)^{\otimes 4}$. This way the elements of (and hence the orbits in) the space $(\mathbb{C}^2)^{\otimes 4}$ are divided into three groups: nilpotent, semisimple, and mixed. The main result of [27] is a classification of the semisimple and mixed elements; the classification of the corresponding nilpotent orbits was already completed a decade earlier by Borsten et al. [11].

Analogously to the complex case, the representation of $\widehat{G}(\mathbb{R})$ on $(\mathbb{R}^2)^{\otimes 4}$ can be constructed using a $\mathbb{Z}/2\mathbb{Z}$-grading $\mathfrak{g}(\mathbb{R}) = \mathfrak{g}_0(\mathbb{R}) \oplus \mathfrak{g}_1(\mathbb{R})$ of the split real form $\mathfrak{g}(\mathbb{R})$ of $\mathfrak{g}$. Also here the nilpotent orbits have been classified in previous work, see Dietrich et al. [28]: There are 145 nilpotent orbits, and 101 of these turned out to be relevant to the study of (possibly multi-center) extremal black holes (BHs) in the STU model (see [6,32]); this application was discussed in full detail in a subsequent paper by Ruggeri and Trigiante [53]. While in various papers, such as Bossard et al. [15,16], the classification of extremal BH solutions had been essentially based on the *complex* nilpotent $\widehat{G}$-orbits in $\mathfrak{g}_1$, a more intrinsic, accurate and detailed treatment was provided in [28,53].

The present paper deals with the real $\widehat{G}(\mathbb{R})$-orbits of semisimple and mixed elements in $(\mathbb{R}^2)^{\otimes 4}$. Such *non-nilpotent* orbits are relevant for the classification of *non-extremal* (or extremal *over-rotating*) as well as of two-center extremal BH solutions in the STU model of $\mathcal{N}=2, D=4$ supergravity, see Section 2. A detailed discussion of the application of our classification to the study of BHs goes beyond the scope of the present investigation and we leave it for future work. From now on we focus on the mathematical side of this research.

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The methods that we use to classify the $\widehat{G}(\mathbb{R})$-orbits in $(\mathbb{R}^2)^{\otimes 4}$ are based on [8, 9] and employ the theory of Galois cohomology. One of the main implications of this theory is the following: Let $v \in (\mathbb{R}^2)^{\otimes 4}$ and consider its complex orbit $\widehat{G}v \subset (\mathbb{C}^2)^{\otimes 4}$. Then the $\widehat{G}(\mathbb{R})$-orbits contained in $\widehat{G}v \cap (\mathbb{R}^2)^{\otimes 4}$ are in bijection with the Galois cohomology set $H^1(Z_{\widehat{G}}(v))$, where $Z_{\widehat{G}}(v) = \{g \in \widehat{G} : gv = v\}$ is the stabiliser of v in $\widehat{G}$. So in principle the only thing one has to do is to compute $H^1(Z_{\widehat{G}}(v))$ for each $\widehat{G}$-orbit in $(\mathbb{C}^2)^{\otimes 4}$ that has a real representative v. This works well for the nilpotent orbits because they are finite in number and all have real representatives (we do not discuss the nilpotent case here since a classification is already given in [28]). However, for the orbits of semisimple and mixed elements this is not straightforward: firstly, there is an infinite number of them and, secondly, it is a problem to decide whether a given complex orbit has a real representative or not.

Our approach to classifying semisimple elements is described in Section 5 and analogous to the method developed in [9]. However, the work in [9] relies on some specific preliminary results that do not apply to our case; as a first step, we therefore need to establish the corresponding results for the situation discussed here. Mixed elements are considered in Section 6; also here the methods are similar to those in [9]. The main difference is that in the case treated in [9], the stabilisers of semisimple elements all have trivial Galois cohomology. This is far from being the case in the situation discussed here, which requires significant amendments. For example, we will work with sets of 4-tuples (p, h, e, f), which do not explicitly appear in [9].

In the course of our research we have made frequent use of the computer algebra system GAP4 [41]. This system makes it possible to compute with the simple Lie algebra $\mathfrak{g}$ of type D_4. We have used additional GAP programs of our own, for example to compute defining equations of the stabilisers of elements in the group $\widehat{G}$.

Our main result is summarised by the following theorem, it is proved with Theorems 5.9 and 6.3.

Theorem 1.1. *The following is established.*

a) *Up to $\widehat{G}(\mathbb{R})$-conjugacy, the nonzero semisimple elements in $(\mathbb{R}^2)^{\otimes 4}$ are the elements in Tables 6–11.*

b) *Up to $\widehat{G}(\mathbb{R})$-conjugacy, the mixed elements in $(\mathbb{R}^2)^{\otimes 4}$ are the elements in Tables 13– 27*

c) *Up to $\widehat{G}(\mathbb{R})$-conjugacy, the nilpotent elements in $(\mathbb{R}^2)^{\otimes 4}$ are given in* [28, Table I].

The notation used in these tables is explained in Definition 1.

Structure of this paper. In Section 2 we briefly comment on applications to real quantum mechanics and non-extremal black holes. In Section 3 we introduce more notation and recall the known classifications over the complex field; these classifications are the starting point for the classifications over the real numbers. In Section 4 we discuss some results from Galois cohomology that will be useful for splitting a known complex orbit into real orbits. Section 5 presents our classification of real semisimple elements; we prove our first main result Theorem 5.9. In Section 6 we prove Theorem 6.3, which completes the classification of the real mixed elements. The Appendix contains tables listing our classifications.

2. Rebits and the black hole/qubit correspondence

2.1. **On rebits.** *Real* quantum mechanics (that is, quantum mechanics defined over real vector spaces) dates back to Stückelberg [55]. It provides an interesting theory whose study may help to discriminate among the aspects of *quantum entanglement* which are unique to standard quantum theory and those aspects which are more generic over other physical theories endowed with this phenomenon [17]. Real quantum mechanics is based on the *rebit*, a quantum bit with *real* coefficients for probability amplitudes of a two-state system, namely a two-state quantum state that may be expressed as a real linear combination of $|0\rangle$ and $|1\rangle$ (which can also be considered as restricted states that are known to lie on a longitudinal great circle of the Bloch sphere corresponding to only real state vectors). In other words, the density matrix of the processed quantum state ρ is real; that is, at each point in the quantum computation, it holds that $\langle x|\rho|y\rangle \in \mathbb{R}$ for all $|x\rangle$ and $|y\rangle$ in the computational basis.

As discussed in [1], [44], and [10, Appendix B], quantum computation based on rebits is qualitatively different from the complex case. Following [17], some entanglement properties of two-rebit systems have been discussed in [4], also exploiting quaternionic quantum mechanics. Moreover, as recalled in [22], rebits were shown in [52] to be sufficient for universal quantum computation; in that scheme, a quantum state of n qubits

$$|\psi\rangle = \sum\nolimits_{\mathbf{v}\in\mathbb{Z}_2^n} r_{\mathbf{v}} e^{i\theta_{\mathbf{v}}} \,|\mathbf{v}\rangle \quad (r_{\mathbf{v}} \in \mathbb{R}^+,\ \theta_{\mathbf{v}} \in \mathbb{R})$$

can be encoded into a state of $n+1$ rebits,

$$\overline{|\psi\rangle} = \sum\nolimits_{\mathbf{v}\in\mathbb{Z}_2^n} \left(r_{\mathbf{v}} \cos\theta_{\mathbf{v}} |\mathbf{v}\rangle \otimes |R\rangle + r_{\mathbf{v}} \sin\theta_{\mathbf{v}} |\mathbf{v}\rangle \otimes |I\rangle\right),$$

where the additional rebit (which has been also named *universal rebit* or *ubit* [2]), with basis states $|R\rangle = |0\rangle$ and $|I\rangle = |1\rangle$, allows one to keep track of the real and imaginary parts of the unencoded n-qubit state.

It should also be remarked that in [59] the three-tangle for three rebits has been defined and evaluated, resulting to be expressed by the same formula as in the complex case, but *without* an overall absolute value sign: thus, unlike the usual three-tangle, the rebit three-tangle can be negative. In other words, by denoting the pure three rebits state as

$$|\phi\rangle = \sum\nolimits_{i,j,k\in\mathbb{Z}_2} a_{ijk} |ijk\rangle,$$

where the binary indices i, j, k correspond to rebits A, B, C, respectively, the three-tangle is simply four times the *Cayley's hyperdeterminant* [18] of the cubic $2\times2\times2$ matrix a_{ijk}, see [59].

2.2. **Rebits and black holes.** In recent years, the relevance of rebits in high-energy theoretical physics was highlighted by the determination of striking relations between the entanglement of pure states of two and three qubits and extremal BHs holes in string theory. In this framework, which has been subsequently dubbed as the "black hole / qubit correspondence" (see for example [12–14] for reviews and references), rebits acquire the physical meaning of the electric and magnetic charges of the extremal BH, and they linearly transform under the generalised electric-magnetic duality group (named U-duality group in string theory) $\mathcal{G}(\mathbb{R})$ of the Maxwell-Einstein (super)gravity theory under consideration.[1] This development started with the seminal paper [29], in which Duff pointed out that the entropy of the so-called extremal BPS STU BHs can be expressed in a very compact way in terms of Cayley's hyperdeterminant [18], which, as mentioned above, plays a prominent role as the three-tangle in studies of three-qubit entanglement [59]. Crucially, the electric and magnetic charges of the extremal BH, which are conserved due to the underlying Abelian gauge invariance, are forced to be real because they are nothing but the fluxes of the two-form field strengths of the Abelian potential one-forms, as well as of their dual forms, which are real. Later on, for example in [43], [45, 46], [30, 31] and subsequent developments, Duff's observation was generalised and extended to non-BPS BHs (which thus break all supersymmetries), also in ($\mathcal{N} > 2$)-extended supergravity theories in four and five space-time dimensions. Further mathematical similarities were thoroughly investigated by Lévay, which for instance showed that the frozen values of the moduli in the calculation of the macroscopic, Bekenstein-Hawking BH entropy in the STU model are related to finding the canonical form for a pure three-qubit entangled state, whereas the extremisation of the BPS mass with respect to the moduli is connected to the problem of finding the so-called optimal local *distillation protocol* [48, 49].

Another application of rebits concerns extremal BHs with two-centers. Multi-center BHs are a natural generalisation of single-center BHs. They occur as solutions to Maxwell-Einstein equations in $4D$, regardless of the presence of local supersymmetry, and they play a prominent role within the dynamics of high-energy theories whose ultra-violet completion aims at describing Quantum Gravity, such as $10D$ superstrings and $11D$ M-theory. In multi-center BHs the *attractor mechanism* [33–36, 56] is generalised by the so-called *split attractor flow* [5, 23, 24], concerning the existence of a co-dimension-one region - the *marginal stability* wall - in the target space of scalar fields, where a stable multi-center BH may decay into its various single-center constituents, whose scalar flows then separately evolve according to the corresponding attractor dynamics.

In this framework, the aforementioned real fluxes of the two-form Abelian field strengths and of their duals, which are usually referred to as *electric* and *magnetic* charges of the BH, fit into a representation $\mathbf{R}$ of the $4D$ U-duality group $\mathcal{G}(\mathbb{R})$. In the STU model of $\mathcal{N}=2, D=4$ supergravity, $\mathcal{G}(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})^3$ and $\mathbf{R} = \left(\mathbb{R}^2\right)^{\otimes 3}$, and each $\mathrm{SL}(2,\mathbb{R})^3$-orbit supports a *unique* class of single-center BH solutions. In general, in presence of a multi-center BH solution with p centers, the dimension I_p of the ring of $\mathcal{G}(\mathbb{R})$-invariant homogeneous polynomials constructed with p distinct copies of the $\mathrm{SL}(2,\mathbb{R})^3$-representation charge $\mathbf{R}$ is given by the general formula [37]

$$p \dim_{\mathbb{R}} \mathbf{R} = \dim_{\mathbb{R}} \mathcal{O}_p + I_p, \tag{2.1}$$

where $\mathcal{O}_p = \mathcal{G}(\mathbb{R}) / \mathcal{H}_p(\mathbb{R})$ is a generally non-symmetric coset describing the generic, open $\mathcal{G}(\mathbb{R})$-orbit, spanned by the p copies of the charge representation $\mathbf{R}$, each pertaining to one center of the multi-center solution. A crucial feature of *multi-center* ($p>1$) BHs is that the various ($I_p > 1$) $\mathcal{G}(\mathbb{R})$-invariant polynomials arrange into

[1] In supergravity, the approximation of *real* (rather than integer) electric and magnetic charges of the BH is often considered, thus disregarding the charge quantization.

multiplets of a global, *"horizontal"* symmetry group[2] $\mathrm{SL}_{\mathrm{hor}}(p,\mathbb{R})$ [37], encoding the combinatoric structure of the p-center solutions of the theory, and commuting with $\mathcal{G}(\mathbb{R})$ itself. Thus, by considering two-center BHs (that is, $p=2$ – an assumption which does not imply any loss of generality due to tree structure of split attractor flows in the STU model), it holds that $\dim_{\mathbb{R}}\mathbf{R}=8$ and the stabiliser of $\mathcal{O}_{p=2}$ has trivial identity connected component. The two-center version of formula (2.1) in the STU model yields

$$\text{STU}: I_{p=2} = 2\dim_{\mathbb{R}}\left(\left(\mathbb{R}^2\right)^{\otimes 3}\right) - \dim_{\mathbb{R}}\left(\mathrm{SL}(2,\mathbb{R})^3\right) = 2\cdot 8 - 9 = 7, \tag{2.2}$$

implying that the ring of $\mathrm{SL}(2,\mathbb{R})^3$-invariant homogeneous polynomials built out of two copies of the tri-fundamental representation $\left(\mathbb{R}^2\right)^{\otimes 3}$ has dimension 7. As firstly discussed in [37] and then investigated in [3,19,38,39], the seven $\mathrm{SL}(2,\mathbb{R})^3$-invariant generators of the aforementioned polynomial ring arrange into one quintuplet (in the spin-2 irreducible representation $\mathbf{5}$) and two singlets $\mathbf{1}\oplus\mathbf{1}'$ of the "horizontal" symmetry group $\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})$:

$$I_{p=2} = 7 = \underset{\text{under } \mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})}{\underset{\text{deg } 4}{\mathbf{5}} \oplus \underset{\text{deg } 2}{\mathbf{1}} \oplus \underset{\text{deg } 4}{\mathbf{1}'}}, \tag{2.3}$$

where the degrees of each term (corresponding to one or more homogeneous polynomials) has been reported. The overall semisimple global group providing the action of the U-duality as well as of the "horizontal" symmetry on two-center BHs is

$$\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})^3 \simeq \mathrm{SL}(2,\mathbb{R})^4 = \widehat{G}(\mathbb{R}), \tag{2.4}$$

acting on the $\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})$-doublet of G-representations $\mathbf{R}$'s, namely

$$\mathbb{R}^2\otimes\mathbf{R} = \mathbb{R}^2\otimes\left(\mathbb{R}^2\right)^{\otimes 3} \simeq \left(\mathbb{R}^2\right)^{\otimes 4}. \tag{2.5}$$

Since the "horizontal" factor $\mathrm{SL}_{\mathrm{hor}}$ stands on a different footing than the U-duality group $\mathrm{SL}(2,\mathbb{R})^3$, only the discrete group Sym_3 of permutations of the three tensor factors in $\mathbf{R}=\left(\mathbb{R}^2\right)^{\otimes 3}$ should be taken into account when considering two-center BH solutions in the STU model, to which a classification invariant under $\mathrm{Sym}_3\ltimes\mathrm{SL}(2,\mathbb{R})^3$ thus pertains. Clearly, the two singlets in the right hand side of (2.3) are invariant under the whole $\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})^3$; on the other hand, when enforcing the symmetry also under the "horizontal" $\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})$, one must consider its non-transitive action on the quintuplet $\mathbf{5}$ occurring in the right hand side of (2.3). As explicitly computed (for example, in [37]) and as known within the classical theory of invariants (see for example [58] as well as the Tables of [42]), the spin-2 SL_2-representation $\mathbf{5}$ has a two-dimensional ring of invariants, *finitely generated* by a *quadratic* and a *cubic* homogeneous polynomial :

$$I_{\text{spin-2}} = \dim_{\mathbb{R}}(\mathbf{5}) - \dim_{\mathbb{R}}\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R}) = 5-3 = 2 = \underset{\text{under } \mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})}{\underset{\text{deg } 2}{\mathbf{1}''} \oplus \underset{\text{deg } 3}{\mathbf{1}'''}}. \tag{2.6}$$

This results into a four-dimensional basis of $\left(\mathrm{Sym}_3\ltimes\left(\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})^3\right)\right)$-invariant homogeneous polynomials, respectively of degree 2, 4, 8 and 12 in the elements of the two-center BH charge representation space $\left(\mathbb{R}^2\right)^{\otimes 4}$. However, as discussed in [37], a lower degree invariant polynomial of degree 6 can be introduced and related to the degree-12 polynomial, giving rise to a 4-dimensional basis of $\left(\mathrm{Sym}_3\ltimes\left(\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})^3\right)\right)$-invariant homogeneous polynomials with degrees 2, 4, 6 and 8, respectively, see [37].

We recall that the enforcement of the whole discrete permutation symmetry Sym_4 (as done in Quantum Information Theory applications) allows for the degrees of the four $\left(\mathrm{Sym}_4\ltimes\left(\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})^3\right)\right)$-invariant polynomial generators to be further lowered down to 2, 4, 4 and 6; this is explicitly computed in [50, 57] and then discussed in [47] in relation to two-center extremal BHs in the STU model. In all cases, the lowest-order element of the invariant basis, namely the homogeneous polynomial *quadratic* in the BH charges, is nothing but the *symplectic product* of the two copies of the single-center charge representation $\mathbf{R}=\left(\mathbb{R}^2\right)^{\otimes 3}$; such a symplectic product is constrained to be non-vanishing in non-trivial and regular two-center BH solutions with *mutually non-local* centers [37]. This implies that regular two-center extremal BHs are related to non-nilpotent orbits of the whole symmetry $\mathrm{SL}_{\mathrm{hor}}(2,\mathbb{R})\otimes\mathrm{SL}(2,\mathbb{R})^3$ (with a discrete factor Sym_3 or Sym_4, as just specified) on $\left(\mathbb{R}^2\right)^{\otimes 4}$. The application of the classification of such orbits (which are the object of interest in this paper) to the study of two-center extremal BHs in the prototypical STU model goes beyond the scope of the present investigation, and we leave it for further future work.

[2]Actually, the "horizontal" symmetry group is $\mathrm{GL}(p,\mathbb{R})$, where the additional scale symmetry with respect to $\mathrm{SL}(p,\mathbb{R})$ is encoded by the homogeneity of the $\mathcal{G}(\mathbb{R})$-invariant polynomials in the BH charges. The subscript "hor" stands for "horizontal" throughout.

3. Notation and classifications over the complex field

Ottaviani and Reichenbach [51, Section 2] provide an overview of results related to the action of the group $H = \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_r)$ on the space $V = V_1 \otimes \cdots \otimes V_r$, where each V_i is a finite dimensional complex vector space. We refer to [51] and the references therein for a detailed discussion of these *tensor classifications* and applications.

If $r = 2$, then V can be identified with the matrix space $\mathbb{C}^{\dim V_1 \times \dim V_2}$, and G-orbits on V correspond to equivalence classes of matrices (which are parametrised by matrix rank), see [51, Theorem 1.1]. In particular, there are only finitely many G-orbits in V. For $r \geqslant 3$ the situation changes drastically and a full classification of orbits is, in general, a *wild problem*, see [51, Section 2]. However, for certain V explicit classifications exist, the easiest being $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$ acting on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$; see [10], [51, Section 3.1] for a discussion of the latter.

The classification of the orbits of $H = \mathrm{SL}(2,\mathbb{C})^4$ on $V = (\mathbb{C}^2)^{\otimes 4}$ described in [27] uses a different approach and employs Lie algebras: the simple Lie algebra $\mathfrak{g}$ of type D_4 has a $\mathbb{Z}/2\mathbb{Z}$ grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that there is an algebraic group G_0 with Lie algebra isomorphic to $\mathfrak{g}_0$, acting as H on the space $\mathfrak{g}_1 \cong V$ (see Section 3.1 for details). The identification $\mathfrak{g}_1 \cong V$ allows us to partition the elements of V into semisimple, nonzero nilpotent, and mixed elements. Having described the H-action on V via a so-called *symmetric pair*, the rich theory of and powerful computational methods for so-called θ-groups can be used to approach the orbit classification; we refer to [25, 26] for more details on orbit classifications in complex θ-groups. The determination of the orbits of the group $\mathrm{SL}(2,\mathbb{R})^4$ on $(\mathbb{R}^2)^{\otimes 4}$ is the aim of the present paper; similar to the complex case, our approach is to use a suitable description in terms of symmetric pairs. These classifications (over $\mathbb{C}$ and $\mathbb{R}$) have applications in theoretical physics [28, 53], quantum information theory [27], and are of interest in the general context of tensor rank and classifications, see [51].

We note that for $m \leqslant 4$ the representations of the groups $\mathrm{SL}(2,\mathbb{C})^m$ on the spaces $(\mathbb{C}^2)^{\otimes m}$ are *visible*: This means that the nullcone has finitely many orbits. Here we do not go into this concept, but refer to [42], where the visible representations of reductive complex algebraic groups are classified. From this classification it also follows that the representations are no longer visible for $m \geqslant 5$. Therefore, for $m \geqslant 5$ it becomes much more complicated to classify the orbits; to the best of our knowledge, no detailed orbit classifications exist for these cases.

3.1. **The grading.**

There are several ways to construct a suitable $\mathbb{Z}/2\mathbb{Z}$-grading of the simple Lie algebra of type D_4. Here d we briefly describe two of them.

3.1.1. *First construction.*

Let $U = W = \mathbb{C}^2 \otimes \mathbb{C}^2$ and set $V = U \oplus W$. Let Q denote the bilinear form on V such that U and W are orthogonal, and such that the matrix of Q restricted to U and W (with respect to fixed bases of these spaces) is

$$\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}.$$

As shown in [40, p. 303], it is possible to define a Lie bracket on $\wedge^2 V$ such that the resulting Lie algebra $\mathfrak{g}$ is isomorphic to

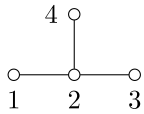
$$\mathfrak{so}(Q) = \{X \in \mathfrak{gl}(V) : Q(Xv_1, v_2) + Q(v_1, Xv_2) = 0 \text{ for all } v_1, v_2 \in V\},$$

which is the simple Lie algebra of type D_4. Furthermore we have the vector space isomorphism

$$\mathfrak{g} \cong \wedge^2 V \cong (\wedge^2 U) \oplus (\wedge^2 W) \oplus (U \otimes W).$$

Because of the choice of Q, we have that $\wedge^2 U$ and $\wedge^2 W$ are subalgebras of $\wedge^2 V$ isomorphic to $\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. Let $\mathfrak{g}_0$ denote the subalgebra $\wedge^2 U \oplus \wedge^2 W$, which is isomorphic to the direct sum of four copies of $\mathfrak{sl}(2,\mathbb{C})$. Let $\mathfrak{g}_1 = U \otimes W$; since U and W are orthogonal under Q, the space $\mathfrak{g}_1$ is stable under left multiplication by $\mathfrak{g}_0$. There is an isomorphism $\mathfrak{g}_1 \cong (\mathbb{C}^2)^{\otimes 4}$ of vector spaces, and it turns out that this isomorphism is also an isomorphism of $\mathfrak{g}_0$-modules. The analogous construction can be done over $\mathbb{R}$ with $U = W = \mathbb{R}^2 \otimes \mathbb{R}^2$. Our choice of Q implies that the subalgebras $\wedge^2 U, \wedge^2 W$ are both isomorphic to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, and the module $U \otimes W$ is isomorphic to $(\mathbb{R}^2)^{\otimes 4}$.

3.1.2. *Second construction.* The second construction is more abstract and starts with the simple Lie algebra $\mathfrak{g}$ of type D_4 defined over the complex numbers. Let Ψ denote its root system with respect to a fixed Cartan subalgebra $\mathfrak{t}$. Let $\gamma_1, \ldots, \gamma_4$ be a fixed choice of simple roots such that the Dynkin diagram of Ψ is labelled as follows



We now construct a $\mathbb{Z}/2\mathbb{Z}$-grading of $\mathfrak{g}$: let $\mathfrak{g}_0$ be spanned by $\mathfrak{t}$ along with the root spaces $\mathfrak{g}_\gamma$, where $\gamma = \sum_i k_i\gamma_i$ has k_2 even, and let $\mathfrak{g}_1$ be spanned by those $\mathfrak{g}_\gamma$ where $\gamma = \sum_i k_i\gamma_i$ has k_2 odd. Let $\gamma_0 = \gamma_1 + 2\gamma_2 + \gamma_3 + \gamma_4$ be the highest root of Ψ. The root system of $\mathfrak{g}_0$ is $\{\pm\gamma_0, \pm\gamma_1, \pm\gamma_3, \pm\gamma_4\}$, hence

$$\mathfrak{g}_0 \cong \mathfrak{sl}(2,\mathbb{C})^4 = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}).$$

Taking $-\gamma_0, \gamma_1, \gamma_3, \gamma_4$ as basis of simple roots of $\mathfrak{g}_0$ we have that $-\gamma_2$ is the highest weight of the $\mathfrak{g}_0$-module $\mathfrak{g}_1$, which therefore is isomorphic to $(\mathbb{C}^2)^{\otimes 4}$. We fix a basis $\{e_0, e_1\}$ of $\mathbb{C}^2$ and denote the basis elements of $(\mathbb{C}^2)^{\otimes 4}$ by

$$|i_1i_2i_3i_4\rangle = e_{i_1} \otimes e_{i_2} \otimes e_{i_3} \otimes e_{i_4}.$$

Mapping any nonzero root vector in $\mathfrak{g}_{-\gamma_2}$ to $|0000\rangle$ extends uniquely to an isomorphism $\mathfrak{g}_1 \to (\mathbb{C}^2)^{\otimes 4}$ of $\mathfrak{sl}(2,\mathbb{C})^4$-modules. We denote by G the adjoint group of $\mathfrak{g}$, and we write G_0 for the connected algebraic subgroup of G with Lie algebra $\mathrm{ad}_{\mathfrak{g}}\mathfrak{g}_0 \cong \mathfrak{sl}(2,\mathbb{C})^4$. The isomorphism $\mathfrak{sl}(2,\mathbb{C})^4 \to \mathfrak{g}_0$ lifts to a surjective morphism $\pi\colon \widehat{G} \to G_0$ of algebraic groups, which makes $\mathfrak{g}_1$ into a $\widehat{G}$-module isomorphic to $(\mathbb{C}^2)^{\otimes 4}$.

In order to define a similar grading over $\mathbb{R}$ we take a basis of $\mathfrak{g}$ consisting of root vectors and basis elements of $\mathfrak{t}$, whose real span is a real Lie algebra (for example, we can take a Chevalley basis of $\mathfrak{g}$). We denote this real Lie algebra by $\mathfrak{g}(\mathbb{R})$. We set $\mathfrak{g}_0(\mathbb{R}) = \mathfrak{g}_0 \cap \mathfrak{g}(\mathbb{R})$ and $\mathfrak{g}_1(\mathbb{R}) = \mathfrak{g}_1 \cap \mathfrak{g}(\mathbb{R})$, so that

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{g}_0(\mathbb{R}) \oplus \mathfrak{g}_1(\mathbb{R}).$$

If $G_0(\mathbb{R})$ denotes the group of real points of G_0, then π restricts to a morphism $\pi\colon \widehat{G}(\mathbb{R}) \to G_0(\mathbb{R})$ that makes $\mathfrak{g}_1(\mathbb{R})$ a $\widehat{G}(\mathbb{R})$-module isomorphic to $(\mathbb{R}^2)^{\otimes 4}$.

3.1.3. *Nilpotent, semisimple, and mixed elements.* A first consequence of these constructions is the existence of a Jordan decomposition of the elements of the modules $(\mathbb{C}^2)^{\otimes 4}$ and $(\mathbb{R}^2)^{\otimes 4}$. Indeed, the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}(\mathbb{R})$ have such decompositions as every element x can be written uniquely as $x = s + n$ where $\mathrm{ad}s$ is semisimple, $\mathrm{ad}n$ is nilpotent, and $[s, n] = 0$. It is straightforward to see that if x lies in $\mathfrak{g}_1$ or $\mathfrak{g}_1(\mathbb{R})$, then the same holds for its semisimple and nilpotent parts. Thus, the elements of $(\mathbb{C}^2)^{\otimes 4}$ and $(\mathbb{R}^2)^{\otimes 4}$ are divided into three groups: semisimple, nilpotent and mixed. Since the actions of $\widehat{G}$ and $\widehat{G}(\mathbb{R})$ respect the Jordan decomposition, also the orbits of these groups in their respective modules are divided into the same three groups.

We note that nilpotent and semisimple orbits in $(\mathbb{C}^2)^{\otimes 4}$ can also be characterized differently. A $\mathrm{SL}(2,\mathbb{C})^4$-orbit in $(\mathbb{C}^2)^{\otimes 4}$ is nilpotent if and only if its (Zariski-) closure contains 0. For example, $u = |0000\rangle$ is nilpotent as it is straightforward to find $g_t \in \mathrm{SL}(2,\mathbb{C})^4$ (for $t \in \mathbb{C}^\times$) such that $g_t \cdot u = tu$, which converges to 0 for $t \to 0$. Furthermore, an orbit is semisimple if and only if it is closed. However, this characterization does not yield straightforward examples of semisimple elements.

A second consequence is that we can consider $\mathfrak{sl}_2$-triples instead of nilpotent elements; we use these in Section 6 when considering mixed elements: the classification of the orbits of mixed elements with a fixed semisimple part p reduces to the classification of the nilpotent orbits in the centraliser of p, which in turn reduces to the classification of orbits of certain $\mathfrak{sl}_2$-triples. We provide more details in Section 6.

3.2. **Notation.** We now we recall the notation used in [27] to describe the classification of $\widehat{G}$-orbits in $(\mathbb{C}^2)^{\otimes 4}$.

A *Cartan subspace* of $\mathfrak{g}_1$ is a maximal space consisting of commuting semisimple elements. A Cartan subspace $\mathfrak{h}$ of $\mathfrak{g}_1$ (and in fact a Cartan subalgebra of $\mathfrak{g}$) is spanned by

$$u_1 = |0000\rangle + |1111\rangle, \quad u_2 = |0110\rangle + |1001\rangle, \quad u_3 = |0101\rangle + |1010\rangle, \quad u_4 = |0011\rangle + |1100\rangle.$$

We denote by Φ the corresponding root system with Weyl group W. This group acts on Φ and $\mathfrak{h}$ in the following way. For $\alpha \in \Phi$ let $s_\alpha \in W$ be the corresponding reflection. If $\beta \in \Phi$ and $h \in \mathfrak{h}$, then $s_\alpha(\beta) = \beta - \beta(h_\alpha)\alpha$ and $s_\alpha(h) = h - \alpha(h)h_\alpha$ where h_α is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \leqslant \mathfrak{h}$ with $\alpha(h_\alpha) = 2$. This defines a W-action on $\mathfrak{h}$ and we write $W_p = \{\alpha \in W : \alpha(p) = p\}$ for the stabiliser of $p \in \mathfrak{h}$ in W; the latter is generated by all s_α with $\alpha \in \Phi_p$ where $\Phi_p = \{\alpha \in \Phi : \alpha(p) = 0\}$, see [27, Lemma 2.4]. For a root subsystem $\Pi \subseteq \Phi$ define

$$\mathfrak{h}_\Pi = \{p \in \mathfrak{h} : \alpha(p) = 0 \text{ for all } \alpha \in \Pi\}, \qquad W_\Pi = \langle s_\alpha : \alpha \in \Pi\rangle,$$
$$\mathfrak{h}_\Pi^\circ = \{p \in \mathfrak{h}_\Pi : \alpha(p) \neq 0 \text{ for all } \alpha \in \Phi \setminus \Pi\}, \qquad \Gamma_\Pi = N_W(W_\Pi)/W_\Pi.$$

Let ζ be a fixed primitive 8-th root of unity; for $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and $u \in \mathbb{C}^\times$ we write

$$\begin{aligned} A^\# &= \begin{pmatrix} d & c \\ b & a\end{pmatrix}, & D(u) &= \begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}, & I &= \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}, & J &= \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, & K &= \begin{pmatrix} 0 & \imath \\ \imath & 0\end{pmatrix}, \\ F &= \begin{pmatrix} 1/2 & \imath/2 \\ \imath & 1\end{pmatrix}, & L &= \begin{pmatrix} \imath & 0 \\ 0 & -\imath\end{pmatrix}, & M &= \begin{pmatrix} \zeta^3 & 0 \\ 0, & -\zeta\end{pmatrix}, & N &= \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta^3\end{pmatrix}. \end{aligned} \tag{3.1}$$

Throughout this paper, we freely identify the spaces $\mathfrak{g}_1 \cong (\mathbb{C}^2)^{\otimes 4}$ and $\mathfrak{g}_1(\mathbb{R}) \cong (\mathbb{R}^2)^{\otimes 4}$ and we write elements of $\widehat{G}$ and $\widehat{G}(\mathbb{R})$ as 4-tuples (A, B, C, D), with A, B, C, D in $\mathrm{SL}(2, \mathbb{C})$, respectively in $\mathrm{SL}(2, \mathbb{R})$.

3.3. **Complex classifications.** In [27, Section 3.1] we have determined 11 subsystems $\Pi_1, \ldots, \Pi_{11}$ to classify the semisimple $\widehat{G}$-orbits in $\mathfrak{g}_1$; these sets are also described in Table 1. The following result summarises [27, Proposition 2.5, Lemma 2.9, Theorem 3.2, Lemma 3.5, Proposition 3.6].

Theorem 3.1 (Complex classification of semisimple elements [27])**.**

a) *Each semisimple $\widehat{G}$-orbit in $(\mathbb{C}^2)^{\otimes 4}$ intersects exactly one of the sets $\mathfrak{h}^\circ_{\Pi_i}$ nontrivially. Two elements of $\mathfrak{h}^\circ_{\Pi_i}$ are $\widehat{G}$-conjugate if and only if they are Γ_{Π_i}-conjugate. Each Γ_{Π_i} can be realised as complement subgroup to W_{Π_i} in $N_W(W_{\Pi_i})$, so as a matrix group relative to the basis $u_1, \ldots, u_4$ of $\mathfrak{h}$. The group $\Gamma_{\Pi_2} \cong (\mathbb{Z}/2\mathbb{Z})^3$ is generated by all 4×4 diagonal matrices that have two 1s and two -1s on the diagonal; the groups $\Gamma_{\Pi_4}, \Gamma_{\Pi_5}, \Gamma_{\Pi_6} \cong \mathrm{Dih}_4$ are isomorphic to the dihedral group of order 8 and defined as*

$$\Gamma_{\Pi_4} = \left\langle \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}, \begin{pmatrix} 0&0&0&-1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\right\rangle,\ \Gamma_{\Pi_5} = \left\langle \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 0&0&-1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\right\rangle,\ \Gamma_{\Pi_6} = \left\langle \begin{pmatrix} -1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \begin{pmatrix} 0&-1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}\right\rangle.$$

Furthermore, $\Gamma_{\Pi_1} = W$ and the remaining Γ_{Π_i} are equal to $\{\pm 1\}$.

b) *If $x, y \in \mathfrak{h}^\circ_{\Pi_i}$, then $Z_{\widehat{G}}(x) = Z_{\widehat{G}}(y)$, and the group $Z_{\widehat{G}}(x)$ is given in Row i of Table 2.*

See [27, Remark 3.3] for a comment on the Γ_{Π_i}-orbits in $\mathfrak{h}^\circ_{\Pi_i}$; this yields a complete and irredundant classification of the semisimple $\widehat{G}$-orbits in $\mathfrak{g}_1$, see Table 1. The zero orbit is covered by the case $i = 11$. The next theorem is [27, Theorem 3.7].

Theorem 3.2 (Complex classification of mixed elements [27])**.** *For $i \in \{2, \ldots, 10\}$ let Σ_i be a set of $\widehat{G}$-conjugacy representatives of semisimple elements in $\mathfrak{h}^\circ_{\Pi_i}$ as specified in Table 1. Up to $\widehat{G}$-conjugacy, the mixed elements in $\mathfrak{g}_1$ are the elements $s + n_{i,j}$ where $i \in \{2, \ldots, 10\}$, $s \in \Sigma_i$, and $n_{i,j}$ as specified in Table 3.*

The nilpotent $\widehat{G}$-orbits in $\mathfrak{g}_1$ and the nilpotent $\widehat{G}(\mathbb{R})$-orbits in $\mathfrak{g}_1(\mathbb{R})$ are determined in [11] and [28], respectively, see also [27, Table 7]; therefore we do not recall these classifications here. We conclude this section by mentioning [27, Remark 3.1]; the symmetries described in this remark allow us to simplify our classifications.

Remark 3.3. If $\sigma \in \mathrm{Sym}_4$, then the linear map $\pi_\sigma \colon \mathfrak{g}_1 \to \mathfrak{g}_1$ that maps each $|i_1 i_2 i_3 i_4\rangle$ to $|i_{1^\sigma} i_{2^\sigma} i_{3^\sigma} i_{4^\sigma}\rangle$ extends to a Lie algebra automorphism of $\mathfrak{g}$ that preserves $\mathfrak{g}_0$ and $\mathfrak{g}_1$. The group generated by all these π_σ fixes u_1 and permutes $\{u_2, u_3, u_4\}$ as Sym_3. Specifically, $\pi_{(2,3)}$ swaps u_3 and u_4, and $\pi_{(2,4)}$ swaps u_2 and u_4.

4. Galois cohomology

We describe some results from Galois cohomology that we use for determining the real orbits within a complex orbit; see [8, Section 3] for a recent treatment of Galois cohomology in the context of orbit classifications.

In this section only we consider the following notation. Let G be a group with *conjugation* $\sigma \colon G \to G$, that is, an automorphism of G of order 2; often σ is the complex conjugation of a complex group. An element $c \in G$ is a *cocycle* (with respect to σ) if $c\sigma(c) = 1$; write $Z^1(G, \sigma)$ for the set of all cocycles. Two cocycles c, c' are *equivalent* if $c' = ac\sigma(a)^{-1}$ for some $a \in G$; the equivalence class of c is denoted $[c]$, and the set of equivalence classes is denoted $H^1(G, \sigma)$. We also write $Z^1(G)$ and $H^1(G)$ if it is clear which conjugation is used; these definitions are an adaption of the definitions in [54, I.§5.1] to the special case of an acting group $\langle\sigma\rangle$ of size 2. We now list a few results that help to determine $H^1(G, \sigma)$. In the following we write $G^\sigma = \{g \in G : \sigma(g) = g\}$.

Let X be a set on which G acts. We suppose that X has a conjugation, also denoted σ (that is, a map $\sigma\colon X \to X$ with $\sigma^2 = \mathrm{Id}_X$), such that $\sigma(gx) = \sigma(g)\sigma(x)$ for all $x \in X$ and $g \in G$. Let $\mathcal{O}$ be a G-orbit in X that has a real point, that is, there is $x_0 \in \mathcal{O}$ with $\sigma(x_0) = x_0$. In this situation, $\mathcal{O}$ is stable under σ, and we are interested in listing the G^σ-orbits in $\mathcal{O}^\sigma = \{y \in \mathcal{O} : \sigma(y) = y\}$. For this we consider the stabiliser

$$Z_G(x_0) = \{g \in G : gx_0 = x_0\}$$

and the exact sequence

$$1 \to Z_G(x_0) \xrightarrow{i} G \xrightarrow{j} \mathcal{O} \to 1$$

resulting from the orbit-stabiliser theorem; here j maps g to gx_0. This sequence gives rise to the exact sequence

$$1 \to (Z_G(x_0))^\sigma \xrightarrow{i} G^\sigma \xrightarrow{j} \mathcal{O}^\sigma \xrightarrow{\delta} H^1(Z_G(x_0), \sigma) \xrightarrow{i^*} H^1(G, \sigma),$$

see [54, Proposition 36]: the map i^* sends the class defined by $g \in Z^1(Z_G(x_0), \sigma)$ to its class in $H^1(G, \sigma)$; moreover, $\delta(gx_0)$ is the class of the cocycle $g^{-1}\sigma(g)$. The following is one of the main theorems in Galois cohomology, see [54, §5.4, Corollary 1 to Proposition 36].

Theorem 4.1. *The map δ induces a bijection between the orbits of G^σ in $\mathcal{O}^\sigma$ and the set* $\ker i^*$.

Remark 4.2. It is known for the usual complex conjugation $\bar{*}$ that $H^1(\mathrm{GL}(n,\mathbb{C})) = 1 = H^1(\mathrm{SL}(n,\mathbb{C}))$ for all $n \geqslant 1$, see for example [7, Proposition III.8.24 and Corollary III.8.26], in particular, $H^1(\mathbb{C}^\times) = 1$. Moreover, if $\bar{*}$ acts entry-wise on a complex matrix group $G = X \times Y$, then $H^1(X \times Y) \cong H^1(X) \times H^1(Y)$. Since a torus $T \leqslant G$ is a direct product of copies of $\mathbb{C}^\times$, we have that $H^1(T, \bar{*}) = 1$.

4.1. **Cartan subspaces.** Recall that $\mathfrak{h}$ is the fixed Cartan subspace spanned by $\{u_1, \ldots, u_4\}$. Semisimple elements that lie in $\mathfrak{h}$ are represented as a linear combination of these basis elements, however, most of our real orbit representatives lie in a Cartan subspace different to $\mathfrak{h}$. To simplify the notation in our classification tables, we classify all Cartan subspaces and then represent our semisimple orbit representatives with respect to fixed bases of these spaces.

It follows from Galois cohomology that the real Cartan subspaces in $\mathfrak{g}_1$ are, up to $\widehat{G}$-conjugacy, in bijection with $H^1(N)$ where $N = N_{\widehat{G}}(\mathfrak{h})$, see [8, Theorem 4.4.9]. The group N fits into an exact sequence

$$1 \to Z_{\widehat{G}}(\mathfrak{h}) \to N \to W \to 1.$$

Since $Z_{\widehat{G}}(\mathfrak{h})$ and W are finite groups of orders 32 and 192, respectively, N is a finite group of order $32 \cdot 192 = 6144$. Because we know $Z_{\widehat{G}}(\mathfrak{h})$ and W (for the former see the first line of Table 2), we can determine N. Since N is finite, a brute force calculation determines $H^1(N)$, and we obtain $|H^1(N)| = 7$.

For a fixed $[n] \in H^1(N)$ define $\tau\colon \mathfrak{h} \to \mathfrak{h}$ by $\tau(u) = n\bar{u}$. Since τ is an anti-involution of $\mathfrak{h}$, the $\mathbb{R}$-dimension of the fixed space $\mathfrak{h}^\tau = \{u \in \mathfrak{h} : \tau(u) = u\}$ equals the $\mathbb{C}$-dimension of $\mathfrak{h}$. Let $g \in \widehat{G}$ be such that $g^{-1}\bar{g} = n$; if $u \in \mathfrak{h}^\tau$, then $u = n\bar{u}$, and the element $\overline{gu} = gnn^{-1}u = gu$ is real. Thus, the real span of all gu with $u \in \mathfrak{h}^\tau$ gives a real Cartan subspace. Iterating this procedure for all $[n] \in H^1(N)$ gives all real Cartan subspaces up to $G(\mathbb{R})$-conjugacy; we fix the notation in the following definition.

Definition 1. There are seven classes in $H^1(N)$ corresponding to cocycles $n_1^*, \ldots, n_7^* \in Z^1(N)$; for each $i \in \{1, \ldots, 7\}$ choose $g_i^* \in \widehat{G}$ such that $(g_i^*)^{-1}\bar{g}_i^* = n_i^*$ and $\mathfrak{c}_i = g_i^*(\mathfrak{h}^\tau)$. Specifically, using the notation introduced in (3.1), we choose

$$\begin{array}{llll} g_1^* = (I, I, I, I), & g_2^* = (L, I, I, I), & g_3^* = (D(\eta^5), D(\eta^5), -D(\eta^3), -D(\eta^7)), & g_4^* = (M, I, I, M) \\ g_5^* = (I, M, I, M), & g_6^* = (I, I, M, M), & g_7^* = (D(\eta^5), D(\eta^5), D(\eta^5), D(\eta^5)), & \end{array}$$

where η is a primitive 16-th root of unity with $\eta^2 = \zeta$. Moreover, we fix the following bases for the seven Cartan subspaces $\mathfrak{c}_1, \ldots, \mathfrak{c}_7$ constructed above:

$$\begin{array}{c} \{u_1 = |0000\rangle + |1111\rangle, \quad u_2 = |0110\rangle + |1001\rangle, \quad u_3 = |0101\rangle + |1010\rangle, \quad u_4 = |0011\rangle + |1100\rangle\} \\ \{v_1 = |0000\rangle - |1111\rangle, \quad v_2 = |0110\rangle - |1001\rangle, \quad v_3 = |0101\rangle - |1010\rangle, \quad v_4 = |0011\rangle - |1100\rangle\} \\ \{w_1 = |0000\rangle - |1111\rangle, \quad w_2 = |0110\rangle - |1001\rangle, \quad w_3 = |0101\rangle - |1010\rangle, \quad w_4 = |0011\rangle + |1100\rangle\} \\ \{x_1 = |0000\rangle - |1111\rangle, \quad x_2 = |0110\rangle - |1001\rangle, \quad x_3 = |0101\rangle + |1010\rangle, \quad x_4 = |0011\rangle + |1100\rangle\} \\ \{y_1 = |0000\rangle - |1111\rangle, \quad y_2 = |0110\rangle + |1001\rangle, \quad y_3 = |0101\rangle - |1010\rangle, \quad y_4 = |0011\rangle + |1100\rangle\} \\ \{z_1 = |0000\rangle - |1111\rangle, \quad z_2 = |0110\rangle + |1001\rangle, \quad z_3 = |0101\rangle + |1010\rangle, \quad z_4 = |0011\rangle - |1100\rangle\} \\ \{t_1 = |0000\rangle - |1111\rangle, \quad t_2 = |0110\rangle + |1001\rangle, \quad t_3 = |0101\rangle + |1010\rangle, \quad t_4 = |0011\rangle + |1100\rangle\}. \end{array}$$

5. Real semisimple elements

Throughout this section, we fix one of the subsystems $\Pi = \Pi_i$ of Theorem 3.1 and abbreviate $\mathcal{C} = \mathfrak{h}_\Pi^\circ$. We fix a complex $\widehat{G}$-orbit $\mathcal{O} = \widehat{G}t$ for some nonzero $t \in \mathcal{C}$. We now discuss the following problems related to the orbit $\mathcal{O}$:

1) Decide whether $\mathcal{O} \cap \mathfrak{g}_1(\mathbb{R})$ is nonempty, that is, whether $\mathcal{O}$ has a real point.
2) If $\mathcal{O}$ has real points, how can we find one?
3) Determine representatives of the real $\widehat{G}(\mathbb{R})$-orbits contained in $\mathcal{O}$.

We prove a number of results that help to decide these questions. These results as well as the proofs are similar to material found in [9]. However, the results in [9] concern a specific $\mathbb{Z}/3\mathbb{Z}$-grading of the Lie algebra of type E_8. Since here we consider a different situation, we have included the new proofs.

In the following, the centraliser and normaliser of $\mathcal{C}$ in $\widehat{G}$ are denoted by

$$\begin{aligned} Z_{\widehat{G}}(\mathcal{C}) &= \{g \in \widehat{G} : gx = x \text{ for all } x \in \mathcal{C}\} \\ N_{\widehat{G}}(\mathcal{C}) &= \{g \in \widehat{G} : gx \in \mathcal{C} \text{ for all } x \in \mathcal{C}\}. \end{aligned}$$

Lemma 5.1. *Let $t_1, t_2 \in \mathcal{C}$. If $gt_1 = t_2$ for some $g \in \widehat{G}$, then $g \in N_{\widehat{G}}(\mathcal{C})$.*

Proof. Theorem 3.1a) shows that $w(t_1) = t_2$ for some $w \in N_W(W_\Pi)$. If $\hat{w} \in N_{\widehat{G}}(\mathfrak{h})$ is a preimage of w, then $g^{-1}\hat{w} \in Z_{\widehat{G}}(t_1)$. Theorem 3.1b) shows that $g^{-1}\hat{w} \in Z_{\widehat{G}}(x)$ for every $x \in \mathcal{C}$, so $gx = \hat{w}x = w(x)$ for all $x \in \mathcal{C}$. Since $w \in N_W(W_\Pi)$, we have $\Pi = w\Pi$ and hence $w(\mathcal{C}) = \mathcal{C}$, see [27, Lemma 2.3 and Proposition 2.5]. □

Now we define a map $\varphi\colon N_{\widehat{G}}(\mathcal{C}) \to \Gamma_\Pi$: If $g \in N_{\widehat{G}}(\mathcal{C})$, then $gq = x \in \mathcal{C}$, hence $w(q) = x$ for some $w \in N_W(W_\Pi)$, and we define $\varphi(g) = wW_\Pi \in \Gamma_\Pi$. This is well-defined: if $w' \in N_W(W_\Pi)$ satisfies $w'(q) = x$, then $w^{-1}w' \in W_\Pi$, so $w'W_\Pi = wW_\Pi$.

Lemma 5.2. *The map $\varphi\colon N_{\widehat{G}}(\mathcal{C}) \to \Gamma_\Pi$ is a surjective group homomorphism with kernel $Z_{\widehat{G}}(\mathcal{C})$. Moreover, if $g \in N_{\widehat{G}}(\mathcal{C})$ and $x \in \mathcal{C}$, then $gx = \varphi(g)x$.*

Proof. We start with a preliminary observation. If $g \in N_{\widehat{G}}(\mathcal{C})$ and $w \in N_W(W_\Pi)$ such that $gq = w(q)$, then $gy = w(y)$ for all $y \in \mathcal{C}$: indeed, if $\hat{w} \in N_{\widehat{G}}(\mathfrak{h})$ is a preimage of w, then $\hat{w}^{-1}g \in Z_{\widehat{G}}(q)$ and $\hat{w}^{-1}g \in Z_{\widehat{G}}(y)$ by Theorem 3.1b). Now let $g_1, g_2 \in N_{\widehat{G}}(\mathcal{C})$ and let $w_1, w_2 \in N_W(W_\Pi)$ be such that each $w_i(q) = g_iq$. By the made observation, $g_1g_2q = w_1(w_2(q))$; this implies that that φ is a group homomorphism. If $wW_\Pi \in \Gamma_\Pi$ with preimage $\hat{w} \in N_{\widehat{G}}(\mathfrak{h})$, then $\hat{w} \in N_{\widehat{G}}(\mathcal{C})$ and $\varphi(\hat{w}) = wW_\Pi$, which shows that φ is surjective. If $g \in \ker(\varphi)$, then $gq = q$ and the first part of the proof shows that $g \in Z_{\widehat{G}}(\mathcal{C})$. □

By abuse of notation, we also write

$$\varphi\colon N_{\widehat{G}}(\mathcal{C})/Z_{\widehat{G}}(\mathcal{C}) \to \Gamma_\Pi$$

for the induced isomorphism. The next theorem provides solutions to Problems 1) and 2); it is similar to [9, Proposition 5.2.4]. Recall that we fixed $\Pi = \Pi_i$ and $\mathcal{O} = \widehat{G}t$ with $t \in \mathcal{C}$.

Theorem 5.3. *Write $H^1(\Gamma_\Pi) = \{[\gamma_1], \ldots, [\gamma_s]\}$. Suppose that for each $\gamma_i \in Z^1(\Gamma_\Pi)$ there is $n_i \in Z^1(N_{\widehat{G}}(\mathcal{C}))$ with $\varphi(n_i) = \gamma_i$. Then $\mathcal{O}$ has a real point if and only if there exist $q' \in \mathcal{O} \cap \mathcal{C}$ and $i \in \{1, \ldots, s\}$ with $\bar{q}' = \gamma_i^{-1}q'$. If the latter holds, then gq' is a real point of $\mathcal{O}$, where $g \in \widehat{G}$ is such that $g^{-1}\bar{g} = n_i$.*

Proof. The elements n_i exist by Lemma 5.2. If $\mathcal{O}$ has a real point, say $p = gt$ for some $g \in \widehat{G}$, then $\overline{gt} = gt$, and so $\bar{t} = n^{-1}t$ for $n = g^{-1}\bar{g}$; note that n is a cocycle since $n\bar{n} = 1$. Because $t, \bar{t} \in \mathcal{C}$, Lemma 5.1 shows that $n \in N_{\widehat{G}}(\mathcal{C})$, so we can define $\gamma = \varphi(n)$. Since $\gamma \in Z^1(\Gamma_\Pi)$, there is $i \in \{1, \ldots, s\}$ and $\beta \in \Gamma_\Pi$ with $\gamma = \beta^{-1}\gamma_i\bar{\beta}$. Now Lemma 5.2 shows that $\bar{t} = \gamma^{-1}t = \bar{\beta}^{-1}\gamma_i^{-1}\beta t$, so if we set $q' = \beta t$, then $q' \in \mathcal{O} \cap \mathcal{C}$ and $\bar{q}' = \gamma_i^{-1}q'$, as claimed. Conversely, let $q' \in \mathcal{O} \cap \mathcal{C}$ and γ_i be such that $\bar{q}' = \gamma_i^{-1}q'$. By hypothesis there is $n_i \in Z^1(N_{\widehat{G}}(\mathcal{C}))$ with $\varphi(n_i) = \gamma_i$, hence $\bar{q}' = n_i^{-1}q'$ by Lemma 5.2. Because $n_i\bar{n}_i = 1$ in $\widehat{G}$ and $H^1(\widehat{G}) = 1$, there is a $g \in \widehat{G}$ with $n_i = g^{-1}\bar{g}$. Now $p = gq'$ is a real point of $\mathcal{O}$, as can be seen from $\bar{p} = \bar{g}\bar{q}' = gn_in_i^{-1}q' = p$. □

Remark 5.4. The real point gq' mentioned in Theorem 5.3 might lie in a Cartan subspace different to $\mathfrak{h}$. The real points corresponding to the class $[\gamma_1] = [1]$ can be chosen to lie in the Cartan subspace $\mathfrak{h}$.

Remark 5.5. One of the hypotheses of the theorem is that for each γ_i there is a cocycle $n_i \in Z^1(N_{\widehat{G}}(\mathcal{C}))$ such that $\varphi(n_i) = \gamma_i$. We cannot prove this a priori, but for the cases that are relevant to the classification given in this paper we have verified it.

Galois cohomology also comes in handy for a solution to Problem 3): the next theorem follows from Theorem 4.1, taking into account that $H^1(\widehat{G}) = 1$, see Remark 4.2.

Theorem 5.6. *Let $p \in \mathcal{O}$ be a real representative. There is a 1-to-1 correspondence between the elements of $H^1(Z_{\widehat{G}}(p))$ and the $\widehat{G}(\mathbb{R})$-orbits of semisimple elements in $\mathcal{O}$ that are $\widehat{G}$-conjugate to p: the real orbit corresponding to $[z] \in H^1(Z_{\widehat{G}}(p))$ has representative bp where $b \in \widehat{G}$ is chosen with $z = b^{-1}\bar{b}$.*

5.1. **Classification approach.** Now we explain the classification procedure in full detail. For $i \in \{1, \dots, 10\}$ we compute some information related to Row i of Table 1; recall that the case $i = 11$ corresponds to the zero orbit: First, we construct the cohomology sets $H^1(\Gamma_{\Pi_i})$; the Γ_{Π_i} are finite groups with trivial conjugation, so the cohomology classes coincide with the conjugacy classes of elements of order dividing 2. These can be computed brute-force. If the complex orbit of a semisimple element q has a real point, then Theorem 5.3 shows that there is some γ_j and some q' in the orbit of q such that $\overline{q'} = \gamma_j^{-1}q'$; now gq' is a real point in the orbit of q, where g is defined in Theorem 5.3. We therefore proceed by looking at each $[\gamma_j] \in H^1(\Gamma_{\Pi_i})$ and determining all q' such that $\overline{q'} = \gamma_j^{-1}q'$; this will eventually determine the orbits of elements in $\mathfrak{h}^\circ_{\Pi_i}$ that have real points, along with a real point in each such orbit. The next lemma clarifies that the elements determined for different j yield real orbit representatives of different orbits.

Lemma 5.7. *With the above notation, if $j \neq k$, then the real orbit representatives obtained for $[\gamma_j]$ are not $\widehat{G}$-conjugate to those representatives obtained for $[\gamma_k]$.*

Proof. Suppose in the above process we construct $q_j, q_k \in \mathfrak{h}_{\Pi_i^\circ}$ such that $\bar{q}_j = \gamma_j^{-1}q_j$ and $\bar{q}_k = \gamma_k^{-1}q_k$. If q_j and q_k lie in the same $\widehat{G}$-orbit, then Theorem 3.1 shows that $q_k = \beta q_j$ for some $\beta \in \Gamma_{\Pi_i}$, that is, $\bar{q}_k = \gamma_k^{-1}\beta q_j$, and solving for q_j yields $q_j = \beta^{-1}\gamma_k\bar{\beta}\bar{q}_j = \beta^{-1}\gamma_k\bar{\beta}\gamma_j^{-1}q_j$. Since $q_j, q_k \in \mathfrak{h}^\circ_{\Pi_i}$, we have $W_{q_j} = W_{q_k}$, see Section 3.2, and so $\beta^{-1}\gamma_k\bar{\beta}\gamma_j^{-1} \in W_{\Pi_i}$. Since $\Gamma_{\Pi_i} = N_W(W_{\Pi_i})/W_{\Pi_i}$, it follows that $[\gamma_j] = [\gamma_k]$ in $H^1(\Gamma_{\Pi_i})$. □

Our algorithm now proceeds as follows; recall that each of our Γ_{Π_i} is realised as a subgroup of W:

(A) For each component $\mathcal{C} = \mathfrak{h}^\circ_{\Pi_i}$ and each cohomology class $[\gamma_j] \in H^1(\Gamma_{\Pi_i})$ with $\gamma_j \in W$, we determine all $q' \in \mathcal{C}$ that satisfy $\overline{q'} = \gamma_j^{-1}q'$ (using Table 4, this condition on q' is easily obtained). We then determine $g_j \in \widehat{G}$ such that $n_j = g_j^{-1}\bar{g}_j$ (using Table 5), and set $p = g_jq'$. Theorem 5.3 shows that p is a real representative in the complex orbit $\widehat{G}q'$ of q'. We note that in this approach we do not fix a complex orbit $\mathcal{O}$ and look for $q' \in \mathcal{O} \cap \mathcal{C}$ as in Theorem 5.3, but we first look for suitable $q' \in \mathcal{C}$ and then consider the reduction up to Γ_{Π_i}-conjugacy.

(B) Next, we determine the real orbits contained in $\widehat{G}p = \widehat{G}q'$. Using Theorem 5.6, we need to consider $Z = Z_{\widehat{G}}(p)$ and determine $H^1(Z)$ with respect to the usual complex conjugation $\bar{*}$. Note that Z is one of the centralisers in Table 2 and $Z = g_jZ_{\widehat{G}}(q)g_j^{-1}$. It will turn out that in most cases we can decompose $Z = \widetilde{Z} \times H$ where H is abelian of finite order; then $H^1(Z) = H^1(\widetilde{Z}) \times H^1(H)$ by Remark 4.2, which is useful for determining $H^1(Z)$. We will see that the component group $\widetilde{Z}/Z^\circ$ is of order at most 2; if it is nontrivial, then is generated by the class of (J, J, J, J) where J is as in (3.1). The connected component Z° is in most cases parametrised by a torus or by $\mathrm{SL}(2, \mathbb{C})$, and we show that $H_1(Z^\circ)$ is trivial (see Remark 4.2). To compute $H^1(Z)$ it remains to consider the cohomology classes of elements of the form $u = w(J, J, J, J) \in \widetilde{Z} \setminus Z^\circ$ where $w \in Z^\circ$. We determine conditions on w such that u is a 1-cocycle, and then solve the equivalence problem. All these calculations can be done by hand, but we have also verified them computationally with the system GAP.

(C) Finally, given γ_j, n_j, g_j and $H^1(Z)$, Theorem 5.6 shows that a real orbit representative corresponding to $[z_k] \in H^1(Z)$ is b_kp where $b_k \in \widehat{G}$ is chosen such that $b_k^{-1}\overline{b_k} = z_k$ (using the elements $\varepsilon(M)$ given in Table 5); specifically, we obtain the following real orbits representatives, recall that $p = g_jq'$:

$$\{(\varepsilon(A), \varepsilon(B), \varepsilon(C), \varepsilon(D))(g_jq') : [(A, B, C, D)] \in H^1(Z) \text{ and } q' \in \mathfrak{h}^\circ_{\Pi_i} \text{ with } \overline{q'} = \gamma_j^{-1}q'\}. \tag{5.1}$$

Remark 5.8. Some real points and some real orbit representatives computed in (A) and (C) lie outside our fixed Cartan subspaces $\mathfrak{c}_1, \dots, \mathfrak{c}_7$ as defined in Definition 1. If this is the case, then we rewrite these elements by following steps (A') and (C') after (A) and (C), respectively:

(A') If the real point p in (A) is not in one of our fixed Cartan spaces $\mathfrak{c}_1,\dots,\mathfrak{c}_7$, then we search for $g\in g_k^*N_{\widehat{G}}(\mathfrak{h})$ for some $k\in\{1,\dots,7\}$ such that $n_j=g^{-1}\bar{g}$; recall the definition of g_k^* and n_k^* from Definition 1; we then replace g_j by g. This is indeed possible: by construction, there is $g_0\in N_{\widehat{G}}(\mathfrak{h})$ and $k\in\{1,\dots,7\}$ with $g_0n_j\overline{g}_0^{-1}=n_k^*=(g_k^*)^{-1}\bar{g}_k^*$, so $g=g_k^*g_0\in g_k^*N_{\widehat{G}}(\mathfrak{h})$ is a suitable element, and $p=gq'$ lies in $\mathfrak{c}_k$.

(C') If one of the b_kp in Step (C) is not in our Cartan spaces $\mathfrak{c}_1,\dots,\mathfrak{c}_7$, then we proceed as follows (using Definition 1). Note that b_kp is $\widehat{G}(\mathbb{R})$-conjugate to one of our Cartan spaces, say $b_k'b_kp\in\mathfrak{c}_j$ for some $b_k'\in\widehat{G}(\mathbb{R})$ and $j\in\{1,\dots,7\}$. Since b_k' is real, $(b_k'b_k)^{-1}\overline{b_k'b_k}=z_k$, and it follows that $b_0=b_k'b_k\in\widehat{G}$ satisfies $b_0^{-1}\bar{b}_0=z_k$ and $b_0p\in\mathfrak{c}_j$, as required. Since $(g_j^*)^{-1}b_0p\in\mathfrak{h}$ is $\widehat{G}$-conjugate to $q\in\mathfrak{h}$, there exists $w\in N_{\widehat{G}}(\mathfrak{h})$ such that $(g_j^*)^{-1}b_0p=wq=wg^{-1}p$, where $p=gq$ as in (A'); we always succeed finding b_0 in $g_j^*N_{\widehat{G}}(\mathfrak{h})g^{-1}$.

To simplify the exposition, in our proof below we do not comment on the rewriting process (A') and (C'), but only describe the results for (A), (B), and (C).

5.2. **Classification results.** The procedure detailed in Section 5.1 leads to the following result; we prove it in this section.

Theorem 5.9. *Up to $\widehat{G}(\mathbb{R})$-conjugacy, the nonzero semisimple elements in $\mathfrak{g}_1(\mathbb{R})$ are the elements in Tables 6–11 in Appendix A.2; see Definition 1 for the notation used in these tables.*

It follows from Theorem 3.1 and Theorem 5.9 that there are many complex semisimple orbits that have no real points. For example, consider the Case $i=10$ and let us fix $q=\lambda u_1$ with $\lambda=a+\imath b$ and $a,b\neq 0$. According to the description given in Case 10 of the proof of Theorem 5.9, there exists a real point in that orbit if and only if there is q' with $\overline{q'}=\gamma_{10}^{-1}q'$; this is equivalent to requiring that λ is either a real or a purely imaginary number, which is a contradiction; thus the complex semisimple orbit determined by q has no real points.

We now prove Theorem 5.9 by considering each case $i\in\{1,2,3,4,7,10\}$ individually; due to Remark 3.3, the classifications for the cases $i\in\{5,6,8,9\}$ can be deduced from those for $i\in\{4,7\}$. For each i we comment on the classification steps (A), (B), (C) as explained in the previous section; we do not comment on the rewriting process (A') and (C'). In Case i below we write $Z=Z_{\widehat{G}}(p_i)$ as in Table 2. Throughout, we use the notation introduced in (3.1).

Case $i=1$. There are 7 equivalence classes of cocycles in $\Gamma_{\Pi_1}=W$, with representatives $\gamma_1=I$, $\gamma_2=-I$, $\gamma_3=\operatorname{diag}(-1,-1,-1,1)$, $\gamma_4=\operatorname{diag}(-1,-1,1,1)$, $\gamma_5=\operatorname{diag}(-1,1,-1,1)$, $\gamma_6=\operatorname{diag}(-1,1,1,-1)$, and $\gamma_7=\operatorname{diag}(-1,1,1,1)$. The centraliser Z is finite so the cohomology can be easily computed, and $H^1(Z)$ has 12 classes with representatives

$$\begin{array}{llll}
z_1=(I,I,I,I), & z_2=(I,I,-I,-I), & z_3=(I,-I,I,-I), & z_4=(I,-I,-I,I),\\
z_5=(K,K,K,K), & z_6=(K,K,-K,-K), & z_7=(K,-K,K,-K), & z_8=(K,-K,-K,K),\\
z_9=(L,L,L,L), & z_{10}=(L,L,-L,-L), & z_{11}=(L,-L,L,-L), & z_{12}=(L,-L,-L,L).
\end{array}$$

We follow the procedure outlined in Section 5.1 and consider the various $[c]\in H^1(\Gamma_{\Pi_1})$. We note that in all cases $(1,j)$ below we start with a complex semisimple element $\lambda_1u_1+\ldots+\lambda_4u_4$ as in Table 1, with $(\lambda_1,\dots,\lambda_4)$ reduced up to Γ_{Π_1}-conjugacy, where $\Gamma_{\Pi_1}=W$ is the Weyl group.

$(i,j)=(1,1)$: Let $[c]=[\gamma_1]=[\operatorname{diag}(1,1,1,1)]$. We first determine all $q'\in\mathfrak{h}_{\Pi_1}^{\circ}$ with $\overline{q'}=q'$; by Theorem 3.1, these are the elements $q'=\lambda_1u_1+\ldots+\lambda_4u_4\in\mathfrak{h}_{\Pi_1}$ with $\lambda_1,\dots,\lambda_4\in\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$. Since $\gamma_1=I$, we can choose $n_1=g_1=(I,I,I,I)$, and obtain $p=g_1q'=q'$ as real point in the complex $\widehat{G}$-orbit of q'. Since the first cohomology group of $Z_{\widehat{G}}(q')=Z_{\widehat{G}}(p)$ has 12 elements, it follows from Theorem 5.6 that $\widehat{G}q'$ splits into 12 real orbits with representatives determined as in (5.1). For $z_1=(I,I,I,I)$ we have $b_1=(I,I,I,I)$, with real orbit representative $b_1p=b_1q'=q'$. For $z_2=(I,I,-I,-I)$ we choose $b_2=(I,I,L,L)$, with real orbit representative $b_2p=b_2q'=-\lambda_1u_1+\lambda_2u_2+\lambda_3u_3-\lambda_4u_4\in\mathfrak{h}_{\Pi_1}$. In the same way we obtain the orbit representatives for $z_3,\dots,z_{12}$, which we summarise in Table 11 in the block $j=1$.

$(i,j)=(1,2)$: Now let $[c]=[\gamma_2]=[\operatorname{diag}(-1,-1,-1,-1)]$. As before we determine all $q'\in\mathfrak{h}_{\Pi_1}^{\circ}$ with $\overline{q'}=-q'$; these are the elements $q'=\lambda_1u_1+\ldots+\lambda_4u_4\in\mathfrak{h}_{\Pi_1}$ with $\lambda_1,\dots,\lambda_4\in\imath\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$. Table 4 shows that $\operatorname{diag}(-1,-1,-1,-1)\in W$ is induced by $n_2=(-I,I,I,I)$. An element g_2 with $g_2^{-1}\overline{g_2}=n_2$ is $g_2=(L,I,I,I)$. Now $p=g_2q'$ is a real point in the complex orbit of q'. Since $Z_{\widehat{G}}(p)=g_2Z_{\widehat{G}}(q')g_2^{-1}$ is finite, we can directly compute $H^1(Z_{\widehat{G}}(p))$ and obtain 12 classes with the following representatives

$$\begin{array}{llll} z_1=(I,I,I,I), & z_2=(-I,-I,I,I), & z_3=(I,-I,I,-I), & z_4=(-I,I,I,-I), \\ z_5=(K,K,K,-K), & z_6=(-K,-K,-K,K), & z_7=(K,-K,K,K), & z_8=(-K,K,K,K), \\ z_9=(-L,-L,-L,-L), & z_{10}=(L,L,-L,-L), & z_{11}=(-L,L,-L,L), & z_{12}=(L,-L,-L,L). \end{array}$$

Real orbits representatives are now determined as in Equation (5.1), see Table 11. Note that here every λ_i is purely imaginary, but each product $(\varepsilon(A),\varepsilon(B),\varepsilon(C),\varepsilon(D))g_2(\lambda_1u_1+\ldots+\lambda_4u_4)$ is a real point.

We repeat the same procedure for $\gamma_3,\cdots,\gamma_7$; for each case we only summarise the important data, and we refer to Table 11 for the list of real orbits representatives.

$\boldsymbol{(i,j)=(1,3)}$: If $[c]=[\gamma_3]$, then $q'\in\mathfrak{h}^\circ_{\Pi_1}$ satisfies $\overline{q'}=\gamma_3^{-1}q'$ if and only if $q'=\lambda_1u_1+\ldots+\lambda_4u_4$ with $\imath\lambda_1,\imath\lambda_2,\imath\lambda_3,\lambda_4\in\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$; we have $n_3=(M,M,-N,N)$ and $g_3=(D(\eta^5),D(\eta^5),-D(\eta^3),-D(\eta^7))$; the first cohomology of $Z_{\widehat{G}}(p)=g_3Z_{\widehat{G}}(q')g_3^{-1}$ consists of four classes defined by the representatives

$$z_1=(-I,-I,-I,-I),\quad z_2=(-I,-I,I,I),\quad z_3=(-I,I,-I,I),\quad z_4=(-I,I,I,-I).$$

$\boldsymbol{(i,j)=(1,4)}$: If $[c]=[\gamma_4]$, then $q'\in\mathfrak{h}^\circ_{\Pi_1}$ satisfies $\overline{q'}=\gamma_4^{-1}q'$ if and only if $q'=\lambda_1u_1+\ldots+\lambda_4u_4$ with $\imath\lambda_1,\imath\lambda_2,\lambda_3,\lambda_4\in\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$. We have $n_4=(L,I,I,L)$, $g_4=(M,I,I,M)$, and the first cohomology of $Z_{\widehat{G}}(p)=g_4Z_{\widehat{G}}(q')g_4^{-1}$ consists of four classes defined by the representatives

$$z_1=(I,I,I,I),\quad z_2=(I,I,-I,-I),\quad z_3=(L,L,L,L),\quad z_4=(L,L,-L,-L).$$

$\boldsymbol{(i,j)=(1,5)}$: If $[c]=[\gamma_5]$, then $q'\in\mathfrak{h}^\circ_{\Pi_1}$ satisfies $\overline{q'}=\gamma_5^{-1}q'$ if and only if $q'=\lambda_1u_1+\ldots+\lambda_4u_4$ with $\imath\lambda_1,\lambda_2,\imath\lambda_3,\lambda_4\in\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$. We get $n_5=(I,L,I,L)$, $g_5=(I,M,I,M)$, and the first cohomology of $Z_{\widehat{G}}(p)=g_5Z_{\widehat{G}}(q')g_5^{-1}$ consists of four classes defined by the representatives

$$z_1=(I,I,I,I),\quad z_2=(I,I,-I,-I),\quad z_3=(L,L,L,L),\quad z_4=(L,L,-L,-L).$$

$\boldsymbol{(i,j)=(1,6)}$: If $[c]=[\gamma_6]$, then $q'\in\mathfrak{h}^\circ_{\Pi_1}$ satisfies $\overline{q'}=\gamma_6^{-1}q'$ if and only if $q'=\lambda_1u_1+\ldots+\lambda_4u_4$ with $\imath\lambda_1,\lambda_2,\lambda_3,\imath\lambda_4\in\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$. Now $n_6=(I,I,L,L)$, $g_6=(I,I,M,M)$, and the first cohomology of $Z_{\widehat{G}}(p)=g_6Z_{\widehat{G}}(q')g_6^{-1}$ consists of four classes defined by the representatives

$$z_1=(I,I,I,I),\quad z_2=(I,-I,I,-I),\quad z_3=(L,L,L,L),\quad z_4=(L,-L,L,-L).$$

$\boldsymbol{(i,j)=(1,7)}$: If $[c]=[\gamma_7]$, then $q'\in\mathfrak{h}^\circ_{\Pi_1}$ satisfies $\overline{q'}=\gamma_7^{-1}q'$ if and only if $q'=\lambda_1u_1+\ldots+\lambda_4u_4$ with $\imath\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{R}\setminus\{0\}$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$. We have $n_7=(M,M,M,M)$ and $g_7=(D(\eta^5),D(\eta^5),D(\eta^5),D(\eta^5))$; the first cohomology of $Z_{\widehat{G}}(p)=g_7Z_{\widehat{G}}(q')g_7^{-1}$ consists of four classes defined by the representatives

$$z_1=(I,I,I,I),\quad z_2=(I,I,-I,-I),\quad z_3=(I,-I,I,-I),\quad z_4=(I,-I,-I,I).$$

Case $\boldsymbol{i=2}$. Since Γ_{Π_2} is elementary abelian of order 8, there are 8 equivalence classes of cocycles in $H^1(\Gamma_{\Pi_2})$, with representatives

$$\begin{array}{llll} \gamma_1=\mathrm{diag}(1,1,1,1), & \gamma_2=\mathrm{diag}(1,-1,-1,1), & \gamma_3=\mathrm{diag}(1,-1,1,-1), & \gamma_4=\mathrm{diag}(-1,-1,1,1), \\ \gamma_5=\mathrm{diag}(1,1,-1,-1), & \gamma_6=\mathrm{diag}(-1,1,-1,1), & \gamma_7=\mathrm{diag}(-1,1,1,-1), & \gamma_8=\mathrm{diag}(-1,-1,-1,-1). \end{array}$$

The centraliser decomposes as $Z=\widetilde{Z}\times H$ where H is abelian of order 4, generated by $(-I,-I,I,I)$ and $(-I,I,-I,I)$. Furthermore $\widetilde{Z}/Z^\circ$ has size 2, generated by the class of (J,J,J,J), and Z° is a 1-dimensional torus consisting of elements $T_1(a)=(D(a^{-1}),D(a^{-1}),D(a),D(a))$ with $a\in\mathbb{C}\setminus\{0\}$. The main difference to the Case $i=1$ is that here Z is not finite; we include some details to explain our computations. First, a direct calculation shows that $H^1(H)$ consists of four classes defined by the representatives

$$z_1=(I,I,I,I),\quad z_2=(-I,-I,I,I),\quad z_3=(-I,I,-I,I),\quad z_4=(I,-I,-I,I).$$

Next, we look at $H^1(\widetilde{Z})$. Since Z° is a 1-dimensional torus, a direct computation (together with Remark 4.2) shows that $H^1(Z^\circ)$ is trivial. It remains to consider the cohomology classes of elements in $\widetilde{Z}/Z^\circ$. Let $u=wj^*$ where $w\in Z^\circ$ and

$$j^*=(J,J,J,J).$$

A short calculation shows that u is a 1-cocycle if and only if there is $a \in \mathbb{R} \setminus \{0\}$ with

$$u = \begin{pmatrix} 0 & -\imath a^{-1} \\ -\imath a & 0 \end{pmatrix} \times \begin{pmatrix} 0 & -\imath a^{-1} \\ -\imath a & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \imath a \\ \imath a^{-1} & 0 \end{pmatrix} \times \begin{pmatrix} 0 & \imath a \\ \imath a^{-1} & 0 \end{pmatrix}.$$

Moreover, every such u is equivalent to $k = (-K, -K, K, K)$, thus $H^1(\tilde{Z}) = \{[1], [k]\}$. Indeed, we can verify (by a short calculation or with the help of GAP) that two 1-cocycles u, u' satisfying $u' = gu\overline{g}^{-1}$ where $g = (D(c^{-1}), D(c^{-1}), D(c), D(c)) \in Z^\circ$ for some $c \in \mathbb{C}^\times$ if and only if $a' = a|c|^2$, thus we can assume $a = \pm 1$. Now $u' = gj^* u(\overline{gj^*})^{-1}$ for some $g \in Z^\circ$ if and only if $aa' = (c/|c|)^2$, then the 1-cocycles corresponding to $a = 1$ and $a' = -1$ are equivalent. Since $H^1(Z) = H^1(\widetilde{Z}) \times H^1(H)$, representatives of the classes in $H^1(Z_{\widehat{G}}(p))$ are $z_1, \ldots, z_4$ and

$$z_5 = z_1 k = (-K, -K, K, K), \quad z_6 = z_2 k = (K, K, K, K),$$
$$z_7 = z_3 k = (K, -K, -K, K), \quad z_8 = z_4 k = (-K, K, -K, K).$$

With the same approach we obtain $H^1(g_j Z g_j^{-1})$ for all γ_j for $j = 1, \ldots, 8$. Representatives of the real orbits are listed in Table 10; below we only list the important data.

$\boldsymbol{(i,j) = (2,1)}$**:** If $[c] = [\gamma_1]$, then $q' \in \mathfrak{h}_{\Pi_2}$ satisfies $\overline{q'} = q'$ if and only if $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ are nonzero and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$. We have $n_1 = g_1 = (I, I, I, I)$, and $H^1(Z)$ was computed above.

$\boldsymbol{(i,j) = (2,2)}$**:** If $[c] = [\gamma_2]$, then $q' \in \mathfrak{h}_{\Pi_2}$ satisfies $\overline{q'} = \gamma_2^{-1} q'$ if and only if $\lambda_1, \imath\lambda_2, \imath\lambda_3 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$. We have $n_2 = (I, I, L, -L)$ and $g_2 = (I, I, M, D(\zeta))$, with real point $p = g_2 q'$. A direct calculation shows that $u = g_2 w j^* g_2^{-1}$ with $w = (D(a)^{-1}, D(a)^{-1}, D(a), D(a)) \in Z^\circ$ for some $a \in \mathbb{C}^\times$ is a 1-cocycle if and only if $a = -\overline{a}$ and $a = \overline{a}$, which is a contradiction. This shows that there is no 1-cocycle with representative $u = g_2 w j^* g_2^{-1}$, so $H^1(Z) = H^1(H)$.

For $\gamma_3, \ldots, \gamma_7$ we also deduce that $H^1(Z) = H^1(H)$; the case of γ_8 is similar to γ_1.

$\boldsymbol{(i,j) = (2,3)}$**:** If $[c] = [\gamma_3]$, then the condition on $q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}$ is $\lambda_1, \imath\lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$, and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$. In this case $n_3 = (I, L, I, -L)$ and $g_3 = (I, M, I, D(\zeta))$.

$\boldsymbol{(i,j) = (2,4)}$**:** If $[c] = [\gamma_4]$, then the condition on $q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}$ is $\imath\lambda_1, \imath\lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$. We have $n_4 = (L, I, I, L)$ and $g_4 = (M, I, I, M)$.

$\boldsymbol{(i,j) = (2,5)}$**:** If $[c] = [\gamma_5]$, then the condition on $q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}$ is $\lambda_1, \lambda_2, \imath\lambda_3 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$; we have $n_5 = (L, I, I, -L)$ and $g_5 = (M, I, I, N)$.

$\boldsymbol{(i,j) = (2,6)}$**:** If $[c] = [\gamma_6]$, then the condition on $q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}$ is $\imath\lambda_1, \lambda_2, \imath\lambda_3 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$; we have $n_6 = (I, L, I, L)$ and $g_6 = (I, M, I, M)$.

$\boldsymbol{(i,j) = (2,7)}$**:** If $[c] = [\gamma_7]$, then the condition on $q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}$ is $\imath\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$; we have $n_7 = (I, I, L, L)$ and $g_7 = (I, I, M, M)$.

$\boldsymbol{(i,j) = (2,8)}$**:** If $[c] = [\gamma_8]$, then the condition on $q' = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \in \mathfrak{h}_{\Pi_2}$ is $\imath\lambda_1, \imath\lambda_2, \imath\lambda_3 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$; we have $n_8 = (-I, I, I, I)$ and $g_8 = (L, I, I, I)$. A short calculation shows that $u = wj^*$ with $w = (D(a)^{-1}, D(a)^{-1}, D(a), D(a)) \in Z^\circ$ is a 1-cocycle if and only if a is purely imaginary. Moreover, every such u is equivalent to $k = (K, -K, K, K)$, thus $|H^1(\tilde{Z})| = 2$. In conclusion, $H^1(Z)$ has 8 classes with representatives $z_1, \ldots, z_4$ and

$$z_5 = z_1 k = (K, -K, K, K), \qquad z_6 = z_2 k = (-K, K, K, K),$$
$$z_7 = z_3 k = (-K, -K, -K, K), \quad z_8 = z_4 k = (K, K, -K, K).$$

Case $\boldsymbol{i = 3}$. Since $H^1(\Gamma_{\Pi_3}) = \{[I], [-I]\}$, there are 2 equivalence classes of cocycles with representatives $\gamma_1 = \mathrm{diag}(1, 1, 1, 1)$ and $\gamma_2 = \mathrm{diag}(-1, -1, -1, -1)$. We decompose $Z = Z^\circ \times H$, where H is the same as in the case $i = 2$ and

$$Z^\circ = \{(A^\#, A^\#, A, A) : A \in \mathrm{SL}(2, \mathbb{C})\}.$$

A short calculation and Remark 4.2 show that $H^1(Z^\circ) = 1$, so $H^1(Z) = H^1(H) = \{[z_1], [z_2], [z_3], [z_4]\}$ as determined for $i = 2$. Representatives of the real orbits are listed in Table 9; below we give some details.

$\boldsymbol{(i,j) = (3,1)}$**:** If $[c] = [\gamma_1]$, then $q' = \lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3) \in \mathfrak{h}^\circ_{\Pi_3}$ satisfies $\overline{q'} = q'$ if and only if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2(\lambda_1 + \lambda_2) \neq 0$; we have $n_1 = g_1 = (I, I, I, I)$.

$(i,j) = (3,2)$: If $[c] = [\gamma_2]$, then the condition on $q' = \lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3) \in \mathfrak{h}_{\Pi_3}$ is $\lambda_1, \lambda_2 \in \imath\mathbb{R}$ and $\lambda_1\lambda_2(\lambda_1 + \lambda_2) \neq 0$; we have $n_2 = (-I, I, I, I)$ and $g_2 = (L, I, I, I)$. A short calculation shows that $H^1(g_2 Z^\circ g_2^{-1})$ is in bijection with $H^1(\mathrm{SL}(2,\mathbb{C})) = 1$; in conclusion, $H^1(Z_{\widehat{G}}(p)) = H^1(H)$.

Case $i = 4$. This case is similar to $i = 2$. Here $H^1(\Gamma_{\Pi_4})$ consists of the classes of

$$\gamma_1 = \mathrm{diag}(1,1,1,1), \quad \gamma_2 = \mathrm{diag}(-1,1,1,1), \quad \gamma_3 = \mathrm{diag}(-1,1,1,-1), \quad \gamma_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We have $Z = \widetilde{Z} \times H$ where H is abelian of order 2 and generated by $(-I, I, -I, I)$. Furthermore $\widetilde{Z}/Z^\circ$ has order 2, generated by the class of (J, J, J, J), and Z° is a 2-dimensional torus consisting of elements $T_2(a,b) = \{(D(a)^{-1}, D(a), D(b)^{-1}, D(b)) \ : \ a, b \in \mathbb{C}^\times\}$. First, $H^1(H)$ consists of 2 classes defined by the representatives

$$z_1 = (I, I, I, I), \quad z_2 = (-I, I, -I, I).$$

Since Z° is parametrised by a 2-dimensional torus, a direct computation and Remark 4.2) shows that $H^1(Z^\circ)$ is trivial. Now consider the cohomology classes of elements in $\widetilde{Z}/Z^\circ$, that is, $u = wj^*$ where $w \in T_2(a,b)$. Computations similar to the ones in Case $i = 2$ show that u is a 1 cocycle if and only if $a, b \in \imath\mathbb{R}$. Moreover, such a 1-cocycle u is equivalent to either $(-K, K, -K, K)$ or $(-K, K, K, -K)$, thus $|H^1(\widetilde{Z})| = 3$. In conclusion, $H^1(Z)$ has 6 classes with representatives z_1, z_2 and

$$z_3 = (-K, K, -K, K), \quad z_4 = (-K, K, K, -K), \quad z_5 = (K, K, K, K), \quad z_6 = (K, K, -K, -K).$$

Real orbit representatives are listed in Table 8; we summarise the important data below.

$(i,j) = (4,1)$: If $[c] = [\gamma_1]$, then $q' = \lambda_1 u_1 + \lambda_4 u_4 \in \mathfrak{h}_{\Pi_4}$ satisfies $\overline{q'} = q'$ if and only if $\lambda_1, \lambda_4 \in \mathbb{R}$ and $\lambda_1\lambda_4(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_4) \neq 0$. We have $n_1 = g_1 = (I, I, I, I)$ and $p = g_1 q' = q'$

$(i,j) = (4,2)$: If $[c] = [\gamma_2]$, then the condition on $q' = \lambda_1 u_1 + \lambda_4 u_4 \in \mathfrak{h}_{\Pi_4}$ is $\imath\lambda_1, \lambda_4 \in \mathbb{R} \setminus \{0\}$; we have $n_2 = (M, M, M, M)$ and $g_2 = (D(\eta^5), D(\eta^5), D(\eta^5), D(\eta^5))$. Let $u = g_2 w j^* g_2^{-1}$ with $w \in Z^\circ$. As in Case $i = 2$, a short calculation shows that there is no 1-cocycle with representative u, so $H^1(Z) = H^1(H)$ has size 2.

$(i,j) = (4,3)$: If $[c] = [\gamma_3]$, then the condition on q' is $q' = \lambda_1 u_1 + \lambda_4 u_4 \in \mathfrak{h}_{\Pi_4}$ with $\lambda_1, \lambda_4 \in \imath\mathbb{R}$ and $\lambda_1\lambda_4(\lambda_1 + \lambda_4)(\lambda_1 - \lambda_4) \neq 0$; we have $n_3 = (I, I, L, L)$ and $g_3 = (I, I, M, M)$. A direct calculation shows that a 1-cocycle $u = g_3 w j^* g_3^{-1}$ with $w \in Z^\circ$ is equivalent to $(-K, K, -K, -K)$ or $(-K, K, K, K)$. In conclusion $H^1(g_3 Z g_3^{-1})$ consists of 6 classes with representatives z_1, z_2 and

$$z_3 = (-K, K, -K, -K), \quad z_4 = (K, K, K, -K), \quad z_5 = (-K, K, K, K), \quad z_6 = (K, K, -K, K).$$

$(i,j) = (4,4)$: Let $[c] = [\gamma_4]$, then the condition on $q' = \lambda_1 u_1 + \lambda_4 u_4 \in \mathfrak{h}_{\Pi_4}$ is $\overline{\lambda_1} = -\lambda_4$; we have $\lambda_1 - \lambda_4 \in \mathbb{R}$ and $\lambda_1 + \lambda_4 \in \imath\mathbb{R}$. We have $g_4 = (M, M, F, LF)$, and a direct calculation (assisted by GAP) shows that $H^1(g_4 Z g_4^{-1})$ has size 4 with representatives z_1, z_2 and $z_3 = (-K, -K, -L, -L)$ and $z_4 = (K, -K, L, -L)$.

Case $i = 7$. Since $H^1(\Gamma_{\Pi_7}) = \{[I], [-I]\}$, we have the same γ_1, γ_2 as in the Case $i = 3$. We decompose $Z = Z^\circ \times H$ where H has order 2, generated by $(-I, I, -I, I)$, and Z° is parametrised by $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. As before, $H^1(Z^\circ) = 1$, so $H^1(Z) = H^1(H)$ consists of the classes of $z_1 = (I, I, I, I)$ and $z_2 = (-I, I, -I, I)$. Table 7 lists the real orbit representatives.

$(i,j) = (7,1)$: If $[c] = [\gamma_1]$, then $q' \in \mathfrak{h}^\circ_{\Pi_7}$ satisfies $\overline{q'} = q'$ if and only if $q' = \lambda(u_1 - u_4)$ with $\lambda \in \mathbb{R}$ and $\lambda \neq 0$; we have $n_1 = g_1 = (I, I, I, I)$.

$(i,j) = (7,2)$: If $[c] = [\gamma_2]$, then the condition on $q' = \lambda(u_1 - u_4)$ is $\lambda \in \imath\mathbb{R} \setminus \{0\}$; we have $n_2 = (-I, I, I, I)$ and $g_2 = (L, I, I, I)$. A direct computation shows that $H^1(g_2 Z^\circ g_2^{-1}) = 1$, so $H^1(g_2 Z g_2^{-1}) = H^1(H)$ determines 2 real orbits.

Case $i = 10$. Here we have $H^1(\Gamma_{\Pi_{10}}) = \{[1], [-1]\}$ and $Z = \widetilde{Z}$, where Z° is a 3-dimensional torus consisting of elements $T_3(a,b,c) = \{(D(abc)^{-1}, D(a), D(b), D(c)) \ : \ a, b, c \in \mathbb{C}^\times\}$ and $\widetilde{Z}/Z^\circ$ is of order 2, generated by the class of (J, J, J, J). As before, $H^1(Z^\circ) = 1$, and elements of the form $u = wj^*$ with $w = T_3(a,b,c) \in Z^\circ$ are 1-cocycles if and only if $a, b, c \in \imath\mathbb{R}$. Moreover, every such 1-cocycle u is equivalent to (K, K, K, K), $(-K, K, K, -K)$, $(-K, K, -K, K)$, or $(K, K, -K, -K)$, thus $H^1(Z)$ has 5 classes with representatives

$$z_1 = (I, I, I, I), z_2 = (K, K, K, K), z_3 = (-K, K, K, -K), z_4 = (-K, K, -K, K), z_5 = (K, K, -K, -K).$$

Table 6 lists the real orbits representatives.

$(i, j) = (10, 1)$: If $[c] = [\gamma_1]$, then the condition on $q' = \lambda u_1$ is $\lambda \in \mathbb{R} \setminus \{0\}$; we have $n_1 = g_1 = (I, I, I, I)$.

$(i, j) = (10, 2)$: If $[c] = [\gamma_2]$, then the condition on $q' = \lambda u_1 \in \mathfrak{h}_{\Pi_{10}}$ is $\lambda \in \imath\mathbb{R} \setminus \{0\}$; we have $n_2 = (-I, I, I, I)$ and $g_2 = (L, I, I, I)$. Since L commutes with diagonal matrices, $H^1(g_2 Z^\circ g_2^{-1}) = H^1(Z^\circ)$. On the other hand, every 1-cocycle $u = wj^*$ with $w \in Z^\circ$ is equivalent to (K, K, K, K), $(-K, K, K, -K)$, $(-K, K, -K, K)$, or $(K, K, -K, -K)$; thus $H^1(Z)$ has 5 classes with representatives

$$z_1 = (I, I, I, I), z_2 = (-K, K, K, K), z_3 = (K, K, K, -K), z_4 = (K, K, -K, K), z_5 = (-K, K, -K, -K).$$

6. Real elements of mixed type

An element of mixed type is of the form $p + e$ where p is semisimple, e is nilpotent and $[p, e] = 0$. From the uniqueness of the Jordan decomposition it follows that two elements $p + e$ and $p' + e'$ of mixed type are $\widehat{G}$-conjugate if and only if there is a $g \in \widehat{G}$ with $gp = p'$ and $ge = e'$. So if we want to classify orbits of mixed type then we may assume that the semisimple part is one of a fixed set of orbit representatives of semisimple elements. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we define

$$\mathcal{M}(\mathbb{K}) = \{p + e \colon p + e \text{ is of mixed type in } \mathfrak{g}_1(\mathbb{K})\}.$$

We know the $\widehat{G}$-orbits in $\mathcal{M}(\mathbb{C})$ and we want to classify the $\widehat{G}(\mathbb{R})$-orbits in $\mathcal{M}(\mathbb{R})$. Applying the general Galois cohomology approach will lead to additional challenges. To avoid these, instead of working with $\mathcal{M}(\mathbb{K})$, we will consider 4-tuples (p, h, e, f), where $p + e$ is a mixed element and (h, e, f) is a suitable $\mathfrak{sl}_2$-triple. This has the advantage that the stabiliser of such a 4-tuple in $\widehat{G}$ is smaller than the stabiliser of $p + e$, and secondly it is reductive. This makes it easier to compute the Galois cohomology sets. We now explain the details of this approach.

Let $p \in \mathfrak{g}_1(\mathbb{K})$ be a semisimple element. The nilpotent parts of mixed elements with semisimple part p lie in the subalgebra

$$\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}(\mathbb{K})}(p) = \{x \in \mathfrak{g}(\mathbb{K}) : [x, p] = 0\}.$$

This subalgebra inherits the grading from $\mathfrak{g}$, that is, if we set $\mathfrak{a}_i = \mathfrak{a} \cap \mathfrak{g}_i(\mathbb{K})$ then $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$. Moreover, the possible nilpotent parts of mixed elements with semisimple part p correspond, up to $\widehat{G}(\mathbb{K})$-conjugacy, to the $Z_{\widehat{G}(\mathbb{K})}(p)$-orbits of nilpotent elements in $\mathfrak{a}_1$. The latter are classified using *homogeneous $\mathfrak{sl}_2$-triples*, which are triples (h, e, f) with $h \in \mathfrak{a}_0$ and $e, f \in \mathfrak{a}_1$ such that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

By the Jacobson-Morozov Theorem (see [8, Proposition 4.2.1]), every nonzero nilpotent $e \in \mathfrak{a}_1$ lies in some homogeneous $\mathfrak{sl}_2$-triple. Moreover, if $e, e' \in \mathfrak{a}_1$ lie in homogeneous $\mathfrak{sl}_2$-triples (h, e, f) and (h', e', f'), then e and e' are $Z_{\widehat{G}(\mathbb{K})}(p)$-conjugate if and only if the triples (h, e, f) and (h', e', f') are $Z_{\widehat{G}(\mathbb{K})}(p)$-conjugate. For this reason we consider the set of quadruples

$$\mathcal{Q}(\mathbb{K}) = \{(p, h, e, f) \colon p \in \mathfrak{g}_1(\mathbb{K}) \text{ is semisimple and } (h, e, f) \text{ is a homogeneous } \mathfrak{sl}_2\text{-triple in } \mathfrak{z}_{\mathfrak{g}(\mathbb{K})}(p)\}.$$

We have just shown that there is a surjective map $\mathcal{Q}(\mathbb{K}) \to \mathcal{M}(\mathbb{K})$, $(p, h, e, f) \mapsto p + e$. By the next lemma, this map defines a bijection between the $\widehat{G}(\mathbb{K})$-orbits in the two sets.

Lemma 6.1. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $p, \hat{p} \in \mathfrak{g}_1(\mathbb{K})$ be semisimple and let (h, e, f) and $(\hat{h}, \hat{e}, \hat{f})$ be homogeneous $\mathfrak{sl}_2$-triples in $\mathfrak{z}_{\mathfrak{g}(\mathbb{K})}(p)$ and $\mathfrak{z}_{\mathfrak{g}(\mathbb{K})}(\hat{p})$, respectively. Then $p + e$ and $\hat{p} + \hat{e}$ are $\widehat{G}(\mathbb{K})$-conjugate if and only if (p, h, e, f) and $(\hat{p}, \hat{h}, \hat{e}, \hat{f})$ are $\widehat{G}(\mathbb{K})$-conjugate.*

Proof. Only one direction needs proof. If $g(p + e) = \hat{p} + \hat{e}$ with $g \in \widehat{G}(\mathbb{K})$, then $gp = \hat{p}$ and $ge = \hat{e}$ by uniqueness of the Jordan decomposition. Now $(gh, \hat{e}, gf)$ is a homogeneous $\mathfrak{sl}_2$-triple in $\mathfrak{z}_{\mathfrak{g}(\mathbb{K})}(\hat{p})$. By [8, Proposition 4.2.1], there is $g_1 \in Z_{\widehat{G}(\mathbb{K})}(\hat{p})$ such that $g_1(gh, \hat{e}, gf) = (\hat{h}, \hat{e}, \hat{f})$, so $(g_1 g)(p, h, e, f) = (\hat{p}, \hat{h}, \hat{e}, \hat{f})$. □

Our approach now is to classify the $\widehat{G}(\mathbb{R})$-orbits in $\mathcal{Q}(\mathbb{R})$; the main tool for this is the following theorem which follows directly from Theorem 4.1 and the fact that $\widehat{G}$ has trivial cohomology.

Theorem 6.2. *Let (p', h', e', f') be a real point in the $\widehat{G}$-orbit of $(p, h, e, f) \in \mathcal{Q}(\mathbb{C})$. There is a 1-to-1 correspondence between $H^1(Z_{\widehat{G}}(p', h', e', f'))$ and the $\widehat{G}(\mathbb{R})$-orbits in $\widehat{G}(p, h, e, f)$: the orbit corresponding to the class $[z] \in H^1(Z_{\widehat{G}}(p', h', e', f'))$ has representative $b(p', h', e', f')$ where $b \in \widehat{G}$ satisfies $z = b^{-1}\overline{b}$.*

The complex semisimple and mixed orbits are parametrised as follows. For each $i \in \{1, \ldots, 10\}$ let Σ_i be a set of $\widehat{G}$-orbit representatives of semisimple elements in $\mathfrak{h}^{\circ}_{\Pi_i}$ as specified in Table 1. By Theorem 3.2, up to $\widehat{G}$-conjugacy, the complex elements in $\mathfrak{g}_1$ of mixed type are $s + n$ where $s \in \Sigma_i$ for some i and $n = n_{i,r}$ for some r, as specified in Table 3. In the following we write

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_{10}.$$

The first problem is to decide which orbits in $\mathcal{Q}(\mathbb{C})$ have real representatives, but we know already which semisimple orbits have real representative. So let us consider $p \in \Sigma$ such that $p' \in \mathfrak{g}_1(\mathbb{R})$ is a real element in its $\widehat{G}$-orbit. We define $\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}(\mathbb{C})}(p')$ as above, with the induced grading $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$. It remains to determine which nilpotent $Z_{\widehat{G}(\mathbb{C})}(p')$-orbits in $\mathfrak{a}_1$ have real representatives. In the case that the real point p' also lies in Σ, this is straightforward; we discuss this case in Section 6.1. We treat the case $p' \notin \Sigma$ in Section 6.2.

In conclusion, our efforts lead to the following theorem.

Theorem 6.3. *Up to $\widehat{G}(\mathbb{R})$-conjugacy, the mixed elements in $\mathfrak{g}_1(\mathbb{R})$ are the elements in Tables 13– 27 in Appendices A.4 and A.5; see Definition 1 for the notation used in all tables.*

6.1. **Classification for the case $p' \in \Sigma$.** Here we suppose that the $\widehat{G}$-orbit of p has a real point p' in Σ. As before set $\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}(\mathbb{C})}(p')$. If $p' \in \Sigma_i$, then we can assume that p' corresponds to an element in the first row of block $j = 1$ in the table for Case i (see Tables 6–10). With this assumption, it follows from Theorem 3.2 that every nilpotent $Z_{\widehat{G}(\mathbb{C})}(p')$-orbit in $\mathfrak{a}_1$ has a real representative; in particular, we can assume that $e' = n_{i,r}$ for some r, as specified in Table 3. This then yields a real 4-tuple $(p', h', e', f') \in \mathcal{Q}(\mathbb{R})$.

We start by computing the centralisers $Z_{\widehat{G}}(p', h', e', f')$, similarly to how we computed $Z_{\widehat{G}}(p')$ before; the result is listed in Table 12. Due to Theorem 3.1b), we can always take one explicit element for our computations; for example, in Case $i = 2$, we can always take $p' = u_1 + u_2 + u_3$.

We now consider the different cases $i = 2, \ldots, 10$; for $i = 1$ and $i = 11$ there are no elements of mixed type. (Recall that $i = 11$ covers the zero orbit.) As before, cases $i \in \{5, 6\}$ and $i \in \{8, 9\}$ follow from $i = 4$ and $i = 7$, respectively. We compute the first cohomology of each centraliser by using the same approach as described in Section 5.2; all the centralisers in this section can be found in Table 12.

Case $i = 2$. We can assume p' corresponds to the first element in block $j = 1$ in Table 10. By Theorem 3.2, there is only one nilpotent element $e' = |0011\rangle$ such that $p' + e'$ has mixed type; this is the real point we use. First, we compute a real $\mathfrak{sl}_2$-triple associated to e'. A direct calculation shows that $H^1(Z)$ for $Z = Z_{\widehat{G}}(p', h', e', f')$ has 8 classes with representatives

$$\begin{array}{llll} z_1 = (I, I, I, I), & z_2 = (-I, -I, I, I), & z_3 = (-I, I, -I, I), & z_4 = (-I, I, I, -I), \\ z_5 = (L, L, L, L), & z_6 = (L, L, -L, -L), & z_7 = (-L, L, -L, L), & z_8 = (L, -L, -L, L). \end{array} \tag{6.1}$$

This shows that the complex orbit $\widehat{G}(p, h, e, f)$ splits into 8 real orbits: each $[z] \in H^1(Z_i)$ determines some $b \in \widehat{G}$ with $z = b^{-1}\overline{b}$, and then $b(p' + e')$ is the real representative of the mixed type orbit corresponding to $[z]$; the resulting orbit representatives are listed in Table 13.

Case $i = 3$. We proceed as before and the resulting real orbit representatives are exhibited in Table 14. Here we have to consider the nilpotent elements $n_{3,1}$ and $n_{3,2}$. The case $e' = n_{3,1}$ yields the same centraliser $Z_1 = Z$ as in Case $i = 2$. The centraliser Z_2 for the case $e' = n_{3,2}$ leads to a first cohomology $H^1(Z_2)$ with 8 classes, given by representatives

$$\begin{array}{llll} z_1 = (I, I, I, I), & z_2 = (-I, -I, I, I), & z_3 = (-I, I, -I, I), & z_4 = (-I, I, I, -I), \\ z_5 = (I, -I, -I, I), & z_6 = (I, -I, I, -I), & z_7 = (I, I, -I, -I), & z_8 = (-I, -I, -I, -I). \end{array} \tag{6.2}$$

Case $i = 4$. Here we have four nilpotent elements $n_{4,r}$ with $r \in \{1, 2, 3, 4\}$; the real orbit representatives for this case are given in Table 15. The centralisers for $r = 1$ and $r = 2$ coincide with Z as in Case $i = 2$; if $r \in \{3, 4\}$, then the centraliser is infinite; its first cohomology has 4 classes with representatives

$$z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I), \quad z_3 = (-I, I, -I, I), \quad z_4 = (-I, I, I, -I). \tag{6.3}$$

Case $i = 7$. There are six nilpotent elements $n_{7,r}$ with $r \in \{1, \dots, 6\}$; the real orbit representatives are exhibited in Table 16. The centraliser for $r = 1, 2, 3$ is the same as Z_2 as in Case $i = 3$; the centraliser for $r = 6$ is as in Case $i = 4$ and $n_{4,4}$. It therefore remains to consider $r \in \{4, 5\}$; we use the approach described in Section 5.1. For $r = 4$, a short calculation shows that $k = (-L, L, K, K)$ is the only 1-cocycle arising from $(-L, L, -J, J)$; it follows that the first cohomology has 8 classes with representatives z_i as in (6.3) along with $z_i k$, that is

$$\begin{array}{llll} z_1 = (I, I, I, I), & z_2 = (-I, -I, I, I), & z_3 = (-I, I, -I, I), & z_4 = (-I, I, I, -I), \\ z_5 = (-L, L, K, K), & z_6 = (L, -L, K, K), & z_7 = (L, L, -K, K), & z_8 = (L, L, K, -K). \end{array} \tag{6.4}$$

For $r = 5$, the first cohomology has representatives $z_1, \dots, z_4$ as above along with $z_i(K, K, -L, L)$:

$$\begin{array}{llll} z_1 = (I, I, I, I), & z_2 = (-I, -I, I, I), & z_3 = (-I, I, -I, I), & z_4 = (-I, I, I, -I), \\ z_5 = (K, K, -L, L), & z_6 = (-K, -K, -L, L), & z_7 = (-K, K, L, L), & z_8 = (-K, K, -L, -L). \end{array} \tag{6.5}$$

Case $i = 10$. There are 12 nilpotent elements; the real orbit representatives are exhibited in Tables 17 and 18. Cases $r = 1, 3, 7, 9$ lead to centralisers that have the same first cohomology as in Case $i = 2$; Cases $r = 2, 4, 6, 8, 10, 12$ yield the same first cohomology as Case $(i, r) = (4, 4)$, see (6.3). For $r \in \{5, 11\}$ we obtain the following cohomology representatives:

$$z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I). \tag{6.6}$$

For $j = 13$ the fist cohomology has 2 classes with representatives

$$z_1 = (I, I, I, I), \quad z_2 = (-I, I, -I, I). \tag{6.7}$$

Remark 6.4. The semisimple parts of the real orbit representatives arising from cocycles involving K or $-K$ are not in our fixed Cartan subspaces $\mathfrak{c}_1, \dots, \mathfrak{c}_7$, and we use Remark 5.8 to replace these elements by elements in our spaces. For example, consider the cocycle $z = (-L, L, K, K)$ in Case $(i, r) = (7, 4)$. There is $b \in \widehat{G}$ with $b^{-1}\bar{b} = z$, and we compute bp', so that $b(p' + e')$ is a real point; in this case, $bp' \notin \mathfrak{c}_1 \cup \ldots \cup \mathfrak{c}_7$. However, bp' is $\widehat{G}(\mathbb{R})$-conjugate to an element in some space $\mathfrak{c}_j$, so there is $b' \in \widehat{G}(\mathbb{R})$ such that $(b'b)p' \in \mathfrak{c}_j$. Since b' is real, $(b'b)^{-1}\overline{b'b} = z$. Thus, there is $b_0 \in \widehat{G}$ with $b_0^{-1}\bar{b}_0 = z$ and $b_0 p' \in \mathfrak{c}_j$. Now $(b'b)e'$ is real and $b_0(p' + e')$ is a real point.

6.2. **Classification for the case $p' \notin \Sigma$.** As before, we consider a complex semisimple element p. By Theorem 3.2, we can assume that $p \in \Sigma_i$. Let p' be a real point in $\widehat{G}p$ as in Theorem 5.9. This time we consider the case $p' \notin \Sigma$, so if $\mathfrak{a} = \mathfrak{z}_{\mathfrak{g}(\mathbb{C})}(p') = \mathfrak{a}_0 \oplus \mathfrak{a}_1$, then we do not know which nilpotent $Z_{\widehat{G}}(p')$-orbits in $\mathfrak{a}_1$ have real points. We now discuss how decide this question. The method that we describe is borrowed from [9, Section 5.3]. However, some difficulties that occurred in the case considered in [9] do not appear here, see Remark 6.8.

Recall that our proof of Theorem 5.9 has exhibited an explicit $g \in \widehat{G}$ such that $p' = gp$; this construction used Theorem 5.6 and a 1-cocycle $n = g^{-1}\bar{g} \in N_{\widehat{G}}(\mathfrak{h}_p^\circ)$. In the following write

$$U_{p'} = \mathfrak{z}_{\mathfrak{g}}(p') \cap \mathfrak{g}_1 \quad \text{and} \quad U_p = \mathfrak{z}_{\mathfrak{g}}(p) \cap \mathfrak{g}_1.$$

Since $Z_{\widehat{G}}(p') = gZ_{\widehat{G}}(p)g^{-1}$, the next lemma allows us to determine the nilpotent $Z_{\widehat{G}}(p')$-orbits in $U_{p'}$ from the known $Z_{\widehat{G}}(p)$-orbits in U_p, cf. Theorem 3.2.

Lemma 6.5. *The map $\varphi\colon U_p \to U_{p'}$, $x \mapsto gx$, is a bijection that maps $Z_{\widehat{G}}(p)$-orbits to $Z_{\widehat{G}}(p')$-orbits.*

Having determined the $Z_{\widehat{G}}(p')$-orbits in $U_{p'}$, it remains to decide when such a complex orbit has a real point. Note that if e' is a real nilpotent element in $\mathfrak{a}_1$ then $e' = \varphi(x)$ for some $x \in U_p$ lying in the $Z_{\widehat{G}}(p)$-orbit of some $e = n_{i,r}$. Motivated by this observation, we proceed as follows: we fix $p \in \Sigma_i$ and $p' = gp$, and for each $e = n_{i,r}$ we look for x in the complex $Z_{\widehat{G}}(p)$-orbit of e such that $\varphi(x)$ is real. Note that the condition that $\varphi(x)$ is real is equivalent to $n\bar{x} = x$ where $n = g^{-1}\bar{g}$ as above. Thus, we define

$$\mu\colon U_p \to U_p, \quad x \mapsto n\bar{x}; \tag{6.8}$$

note that $\mu^2 = 1$ since $n\bar{n} = 1$ and, by construction, $\varphi(x)$ is real if and only if $\mu(x) = x$. The following lemma is analogous to [8, Lemma 5.3.1].

Lemma 6.6. *Let $Y = Z_{\widehat{G}}(p)e$. Then $\mu(Y) = Y$ if and only if $\mu(y) \in Y$ for some $y \in Y$.*

Proof. It follows from Theorem 3.1b) that $Z_{\widehat{G}}(p) = Z_{\widehat{G}}(\mathfrak{h}^\circ_\Pi)$, where $\mathfrak{h}^\circ_\Pi$ is the component containing p. Suppose there is $y \in Y$ such that $\mu(y) \in Y$; we have to show $\mu(Y) = Y$. Write $y = he$ with $h \in Z_{\widehat{G}}(p)$. We know that $\mu(y) = n\overline{y} = n\overline{he} = ke$ for some $k \in Z_{\widehat{G}}(p)$; note that $\overline{he} = \overline{h}e$ because e is real. Since $\overline{h} \in Z_{\widehat{G}}(p) = Z_{\widehat{G}}(\mathfrak{h}^\circ_\Pi)$ and the latter is normal in $N_{\widehat{G}}(\mathfrak{h}^\circ_\Pi)$, we have $n\overline{h} = zn$ for some $z \in Z_{\widehat{G}}(p)$, and the previous equation yields $ne = z^{-1}ke$. Now let $w \in Y$, say $w = te$ with $t \in Z_{\widehat{G}}(p)$. There is some $s \in Z_{\widehat{G}}(p)$ such that $n\overline{t} = sn$. We have $\mu(w) = n\overline{t}e = sne = sz^{-1}ke$ and, since $sz^{-1}k \in Z_{\widehat{G}}(p)$, we deduce that $\mu(w) \in Y$, so $\mu(Y) = Y$. The other implication is trivial. □

Corollary 6.7. *If there is $x \in Z_{\widehat{G}}(p)e$ such that $\varphi(x)$ is real, then $\mu(e)$ is $Z_{\widehat{G}}(p)$-conjugate to e.*

If $e = n_{i,r}$ does not satisfy $\mu(e) \in Z_{\widehat{G}}(p)e$, then we can discard the pair (p, e). If $\mu(e)$ is $Z_{\widehat{G}}(p)$-conjugate to e, then we attempt to compute $x \in Z_{\widehat{G}}(p)e$ such that $\mu(x) = x$; then $e' = \varphi(x)$ is a real nilpotent element commuting with p'. Once this real point is found, we construct a real 4-tuple (p', h', e', f') and apply Theorem 6.2 to compute the $\widehat{G}(\mathbb{R})$-orbits in $\widehat{G}(p', h', e', f')$.

Remark 6.8. In our classification, using ad hoc methods, we always found suitable elements x as above. We note that if such ad hoc methods would not have worked, then we could have used a method described in [9, Section 5.3] for finding such elements (or for deciding that none exists). This method is based on computations with the second cohomology set $H^2(Z_{\widehat{G}}(p))$.

Classification approach. We summarise our approach for classifying the real mixed orbits in $\widehat{G}(p+e)$, where the real point p' in $\widehat{G}p$ (as in Theorem 5.9) does not lie in Σ.

(1) Recall that Tables 6–11 list our real semisimple orbits; each table corresponds to a case $i \in \{1, \ldots, 10\}$ and has subcases $j = 1, 2, \ldots$, where $j = 1$ lists elements in Σ. For each $i \in \{2, \ldots, 10\}$ and each $j > 1$ listed in the corresponding table, choose the first real element p' in the block labelled j. The proof of Theorem 5.9 shows that $p' = gp$ where $p \in \Sigma_i$ as given in Table 1; in particular, the element g is determined by our classification, and we define $n = g^{-1}\overline{g}$.

(2) For each $p \in \Sigma_i$ as determined in (1) we consider each $e = n_{i,r}$ $(r = 1, 2, \ldots)$ such that the elements $p + n_{i,r}$ are the mixed orbit representatives as determined in Theorem 3.2. Using (6.8), we then define μ with respect to n, and check whether $\mu(e) \in Z_{\widehat{G}}(p)e$. If true, then we compute $x \in Z_{\widehat{G}}(p)e$ with $\mu(x) = x$ by computing the 1-eigenspace $U_p^\mu = \{u \in U_p : \mu(u) = u\}$ and looking for some $x \in U_p^\mu$ that is $Z_{\widehat{G}}(p)$-conjugate to e. Note that $\dim_\mathbb{R} U_p^\mu = \dim_\mathbb{C} U_p$; this follows from the fact that μ is an $\mathbb{R}$-linear map of order 2, so U_p is the direct sum of the ± 1-eigenspaces. Since multiplication by $\imath$ is a bijective $\mathbb{R}$-linear map swapping these eigenspaces, they have equal dimension. In our classification, this search is always successful, and we set $e' = \varphi(x)$.

(3) We determine a real 4-tuple (p', h', e', f') and apply Theorem 6.2 to find the real $\widehat{G}(\mathbb{R})$-orbits in the $\widehat{G}$-orbit of this 4-tuple. If (p'', h'', e'', f'') is a representative of such an orbit then $p'' + e''$ is the corresponding element of mixed type. This is a representative of a $\widehat{G}(\mathbb{R})$-orbit contained in $\widehat{G}(p + e)$.

We now discuss the individual cases in detail. The following case distinction determines the relevant real points $n_{i,j,r}$ for Case i and the cohomology class $[\gamma_j]$ of $H^1(\Gamma_{p_i})$ with γ_j as determined in the proof of Theorem 5.9; note that we do not have to consider the trivial class $[\gamma_1]$ because this class produces elements in Σ.

Case $\boldsymbol{i = 2}$. We have to consider cocycles $\gamma_2, \ldots, \gamma_8$, with corresponding elements n_j given by (see Section 6)

$$\begin{aligned} &n_2 = (I, I, L, -L), \quad n_3 = (I, L, I, -L), \quad n_4 = (L, I, I, L),\\ &n_5 = (L, I, I, -L), \quad n_6 = (I, L, I, L), \qquad n_7 = (I, I, L, L), \quad n_8 = (-I, I, I, I). \end{aligned}$$

For this case there is only one nilpotent element $e = n_{2,1}$. After a short calculation we can see that $\mu(e) = n_j e$ is conjugate to e for all $j = 2, \ldots, 8$. For $n_2 = (I, I, L, -L)$ we obtain $g_2 = (I, I, M, D(\zeta))$; following Remark 5.8, we replace g_2 by $g_2 = (-I, -J, -M, -MJ)$. We then compute U_p^μ and verify that $e \in U_p^\mu$, thus $x = e$ is the element we are looking for, and therefore we set $e' = \varphi(x) = g_2 e = -|0110\rangle$; we denote the latter by $n_{2,2,1}$. For n_3 we obtain $g_3 = (I, M, I, D(\zeta))$. Again, to get elements in one of the seven Cartan subspaces, it is necessary to replace it by $g_3 = (-I, -M, -J, -MJ)$. After computing U_q^μ we have that $x = \imath e \in U_p^\mu$, which is conjugate to e under the action of $Z_{\widehat{G}}(p)$ via $g = (D(\imath), D(\imath), D(\imath^{-1}), D(\imath^{-1}))$. Therefore we set $e' = \varphi(x) = g_3 e = |0000\rangle$; the latter is denoted by $n_{2,3,1}$. The other cases $j > 3$ are computed along the same way.

Case $\boldsymbol{i = 3}$. We have to consider the cocycle γ_2 with $n_2 = (-I, I, I, I)$. There are two nilpotent elements $n_{3,1}$ and $n_{3,2}$. We compute U_p^μ and see that $x = \imath|0011\rangle$ is a nilpotent element in U_p^μ conjugate to $n_{3,1}$ via

$(D(a)^{-1}, D(a)^{-1}, D(a), D(a))$ with $a^{-4} = \imath$. Thus, the real nilpotent element is $e' = \varphi(x) = -|0011\rangle$, denoted $n_{3,2,1}$. Similarly, we find $x \in U_p^\mu$ such that x is conjugate to $n_{3,2}$, see $\varphi(x) = n_{3,2,2}$ in Table 19.

Case $i = 4$. We have to consider cocycles $\gamma_2, \dots, \gamma_4$, with corresponding elements

$$n_2 = (M, M, M, M), \quad n_3 = (I, I, L, L), \quad n_4 = (L, L, -K, K).$$

For each of them, we have to consider four nilpotent elements $n_{4,1}, \dots, n_{4,4}$. After computing U_p^μ for n_2, we observe that each $n_{4,\ell} \in U_p^\mu$, thus the real representatives are $n_{4,2,1}, \dots, n_{4,2,4}$, defined as $n_{4,2,\ell} = g_2 n_{4,\ell} = n_{4,\ell}$ with $g_2 = (D(\eta^5), D(\eta^5), D(\eta^5), D(\eta^5))$. The case n_3 is similar, so now let us consider n_4. We obtain that $\mu(n_{4,\ell})$ is not conjugate to $n_{4,\ell}$ for $\ell \in \{3, 4\}$, thus we have to only consider $n_{4,1}$ and $n_{4,2}$. For $n_{4,1}$ there is no $x \in U_q^\mu$ such that x is conjugate to $n_{4,1}$. This is seen by acting on $n_{4,1}$ by a general element g of $Z_{\widehat{G}}(p)$ which is $g = h(D(a^{-1}), D(a), D(b^{-1}), D(b))$ where $a, b \in \mathbb{C}^*$ and h lies in the component group (see Table 2). From the expressions obtained it is straightforward to see that the image can never lie in U_p^μ.

On the other hand, we observe that $n_{4,2}$ lies in U_p^μ and therefore we set $e' = \varphi(n_{4,2}) = g_4 n_{4,2} = \frac{1}{2}(-|1110\rangle - |1101\rangle + |1010\rangle + |1001\rangle + |0110\rangle + |0101\rangle - |0010\rangle - |0001\rangle)$. In this case it is necessary to replace g_4 in order to get elements in one of the seven Cartan subspaces.

Cases $i \in \{7, 10\}$. The nilpotent elements for the remaining cases $i = 7$ and $i = 10$ are computed analogously.

The next step is to compute the centralisers $Z = Z_{\widehat{G}}(p', h', e', f')$ of the 4-tuples (p', h', e', f'), and then the first cohomology $H^1(Z)$ as we did in Table 12. The difference here is that $p' \notin \Sigma$ and e' is a nilpotent element from Table 19, with corresponding $\mathfrak{sl}_2$-triple (h', e', f') in $\mathfrak{z}_{\mathfrak{g}}(p')$. We exemplify the details with a few examples:

Example 6.9. For $(i, j) = (2, 2)$ let p' be the first element in the second block of Table 10 and let $e' = n_{2,2,1} = -|0011\rangle$. We compute an $\mathfrak{sl}_2$-triple with nilpotent element e' in the whole Lie algebra and then check that it centralises p'. Since the centralisers of semisimple elements in the same component are equal, in order to compute $Z_{\widehat{G}}(p')$ we can assume that the parameters defining p' are $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = \imath$. Using Groebner basis techniques, we determine that $Z_{\widehat{G}}(p', h', e', f')$ is the same as in the first row of Table 12, hence $H^1(Z)$ has 8 classes with representatives given by (6.1). The results are listed in the first block of Table 20.

In fact, in most of the cases, the centraliser $Z = Z_{\widehat{G}}(p', h', e', f')$ for Case i, cohomology class $[\gamma_j]$, and nilpotent element $e' = n_{i,j,r}$ is exactly the same given in Table 12 for parameters i and r. For example, if $i = 2$ and $e' = n_{2,j,1}$ with $j \in \{2, 3, 4, 5, 6, 7, 8\}$, then $Z_{\widehat{G}}(p', h', e', f')$ is always given in the first row of Table 12. The only exceptions where this behaviour was not observed are the following two cases.

Example 6.10. For $(i, j) = (4, 4)$ let p' be the first element in the second block of Table 8 and let $e' = n_{4,4,1}$ as in Table 19. Here the centraliser $Z = Z_{\widehat{G}}(p', h', e', f')$ is generated by $(-I, -I, I, I)$, $(-I, I, -I, I)$, (K, K, J, J), therefore different from the centraliser given in Table 12 for parameters $(i, r) = (4, 1)$. The first cohomology $H^1(Z)$ has 4 classes with representatives

$$z_1 = (I, I, I, I), \quad z_2 = (-I, -I, I, I), \quad z_3 = (-I, I, -I, I), \quad z_4 = (-I, I, I, -I). \tag{6.9}$$

For $(i, j) = (7, 2)$ and $e' = n_{7,2,5}$ the centraliser is generated by $(-I, I, -I, I)$, $(-I, I, I, -I)$, $(-J, J, L, L)$, and $\{(D(a)^{-1}, D(a), I, I) \ : \ a \in \mathbb{C}^\times\}$, and therefore is different from the centraliser given in Table 12 for parameters $(i, r) = (7, 5)$; its first cohomology has 8 classes with representatives

$$\begin{array}{llll} z_1 = (I, I, I, I), & z_2 = (-I, I, -I, I), & z_3 = (-I, I, I, -I), & z_4 = (I, I, -I, -I), \\ z_5 = (K, K, L, L), & z_6 = (-K, K, -L, L), & z_7 = (-K, K, L, -L), & z_8 = (K, K, -L, -L). \end{array} \tag{6.10}$$

Having found all real points and having determined the corresponding cohomology class representatives, the last step is to employ the usual Galois theory approach to obtain a real representative for each of the mixed orbits obtained above. We exhibit the results in Tables 13– 27. This proves Theorem 6.3.

Appendix A. Tables

A.1. **Complex classification.**

i	type of Π_i	roots of Π_i	elements of $\mathfrak{h}_{\Pi_i}$	condition for being in $\mathfrak{h}_{\Pi_i}^\circ$	Γ_{Π_i}	$\mathfrak{z}_{\mathfrak{g}}(p_i)'$
1	$\emptyset$		$\lambda_1 u_1 + \cdots + \lambda_4 u_4$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3 \pm \lambda_4\}$	W	0
2	A_1	α_4	$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$	$(\mathbb{Z}/2\mathbb{Z})^3$	$\mathfrak{sl}(2,\mathbb{C})$
3	A_2	α_2, α_4	$\lambda_1(u_1 - u_2) + \lambda_2(u_1 - u_3)$	$\lambda_u \neq 0$ and $\lambda_1 \neq -\lambda_2$	$\langle -I_4 \rangle$	$\mathfrak{sl}(3,\mathbb{C})$
4	$2\mathrm{A}_1$	α_1, α_3	$\lambda_1 u_1 + \lambda_2 u_4$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2\}$	Dih_4	$\mathfrak{sl}(2,\mathbb{C})^2$
5	$2\mathrm{A}_1$	α_1, α_4	$\lambda_1 u_1 + \lambda_2 u_3$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2\}$	Dih_4	$\mathfrak{sl}(2,\mathbb{C})^2$
6	$2\mathrm{A}_1$	α_3, α_4	$\lambda_1 u_1 + \lambda_2 u_2$	$\lambda_i \neq 0$ and $\lambda_1 \notin \{\pm\lambda_2\}$	Dih_4	$\mathfrak{sl}(2,\mathbb{C})^2$
7	A_3	$\alpha_1, \alpha_2, \alpha_3$	$\lambda_1(u_1 - u_4)$	$\lambda_1 \neq 0$	$\langle -I_4 \rangle$	$\mathfrak{sl}(4,\mathbb{C})$
8	A_3	$\alpha_1, \alpha_2, \alpha_4$	$\lambda_1(u_1 - u_3)$	$\lambda_1 \neq 0$	$\langle -I_4 \rangle$	$\mathfrak{sl}(4,\mathbb{C})$
9	A_3	$\alpha_2, \alpha_3, \alpha_4$	$\lambda_1(u_1 - u_2)$	$\lambda_1 \neq 0$	$\langle -I_4 \rangle$	$\mathfrak{sl}(4,\mathbb{C})$
10	$3\mathrm{A}_1$	$\alpha_1, \alpha_3, \alpha_4$	$\lambda_1 u_1$	$\lambda_1 \neq 0$	$\langle -I_4 \rangle$	$\mathfrak{sl}(2,\mathbb{C})^3$
11	D_4	$\alpha_1, \ldots, \alpha_4$	0	0	1	$\mathfrak{so}(4,\mathbb{C})$

Table 1. This is [27, Table 2]: Complete root subsystems Π_i of Φ and related data, with parameters $\lambda_1, \ldots, \lambda_4 \in \mathbb{C}$; the last column displays the derived algebra of the centraliser $\mathfrak{z}_{\mathfrak{g}}(p_i)$ for $p_i \in \mathfrak{h}_{\Pi_i}^\circ$. Elements in $\mathfrak{h}_{\Pi_i}^\circ$ are not $\widehat{G}$-conjugate to elements in $\mathfrak{h}_{\Pi_j}^\circ$ if $i \neq j$; two elements in the same component $\mathfrak{h}_{\Pi_i}^\circ$ are $\widehat{G}$-conjugate if and only if they are Γ_{Π_i}-conjugate.

i	identity component $Z_{\widehat{G}}(s)^\circ$	preimages of generators of $Z_{\widehat{G}}(s)/Z_{\widehat{G}}(s)^\circ$
1	1	$(J,J,J,J), (-I,-I,I,I), (-I,I,-I,I), (K,K,K,K)$
2	$\{(D(a)^{-1}, D(a)^{-1}, D(a), D(a)) \;:\; a \in \mathbb{C}^\times\}$	$(-I,-I,I,I), (-I,I,-I,I), (J,J,J,J)$
3	$\{(A^\#, A^\#, A, A) \;:\; A \in \mathrm{SL}(2,\mathbb{C})\}$	$(-I,-I,I,I), (-I,I,-I,I)$
4	$\{(D(a)^{-1}, D(a), D(b)^{-1}, D(b)) \;:\; a,b \in \mathbb{C}^\times\}$	$(-I,I,-I,I), (J,J,J,J)$
5	$\{(D(a)^{-1}, D(b)^{-1}, D(a), D(b)) \;:\; a,b \in \mathbb{C}^\times\}$	$(-I,-I,I,I), (J,J,J,J)$
6	$\{(D(a)^{-1}, D(b), D(b)^{-1}, D(a)) \;:\; a,b \in \mathbb{C}^\times\}$	$(-I,I,-I,I), (J,J,J,J)$
7	$\{(A^\#, A, B^\#, B) : A, B \in \mathrm{SL}(2,\mathbb{C})\}$	$(-I,I,-I,I)$
8	$\{(A^\#, B^\#, A, B) : A, B \in \mathrm{SL}(2,\mathbb{C})\}$	$(-I,-I,I,I)$
9	$\{(A^\#, B, B^\#, A) : A, B \in \mathrm{SL}(2,\mathbb{C})\}$	$(-I,I,-I,I)$
10	$\{(D(abc)^{-1}, D(a), D(b), D(c)) \;:\; a,b,c \in \mathbb{C}^\times\}$	(J,J,J,J)

Table 2. This is [27, Table 3]; the groups $Z_{\widehat{G}}(s)$: the entry i is the label of the canonical semisimple set $\mathfrak{h}_{\Pi_i}^\circ$ that contains s, as in Table 1; the notation is explained in (3.1).

i	**nilpotent elements**
2	$n_{2,1} = \lvert 0011\rangle$
3	$n_{3,1} = \lvert 0011\rangle,\ n_{3,2} = \lvert 0111\rangle + \lvert 1011\rangle + \lvert 0010\rangle + \lvert 0001\rangle$
4	$n_{4,1} = \lvert 0110\rangle + \lvert 1010\rangle,\ n_{4,2} = \lvert 0110\rangle + \lvert 0101\rangle,\ n_{4,3} = \lvert 0110\rangle,\ n_{4,4} = \lvert 0101\rangle$
5	$n_{5,1} = \lvert 0110\rangle + \lvert 1100\rangle,\ n_{5,2} = \lvert 0110\rangle + \lvert 0011\rangle,\ n_{5,3} = \lvert 0110\rangle,\ n_{5,4} = \lvert 0011\rangle$
6	$n_{6,1} = \lvert 0011\rangle + \lvert 1010\rangle,\ n_{6,2} = \lvert 0011\rangle + \lvert 0101\rangle,\ n_{6,3} = \lvert 0011\rangle,\ n_{6,4} = \lvert 0101\rangle$
7	$n_{7,1} = \lvert 1101\rangle + \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0001\rangle,\ n_{7,2} = \lvert 1101\rangle + \lvert 1010\rangle + \lvert 0001\rangle,$ $n_{7,3} = \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0101\rangle, n_{7,4} = \lvert 1011\rangle + \lvert 1000\rangle,\ n_{7,5} = \lvert 1101\rangle + \lvert 0001\rangle,\ n_{7,6} = \lvert 1001\rangle$
8	$n_{8,1} = \lvert 1011\rangle + \lvert 1101\rangle + \lvert 1000\rangle + \lvert 0001\rangle,\ n_{8,2} = \lvert 1011\rangle + \lvert 1100\rangle + \lvert 0001\rangle,$ $n_{8,3} = \lvert 1101\rangle + \lvert 1000\rangle + \lvert 0011\rangle,\ n_{8,4} = \lvert 1101\rangle + \lvert 1000\rangle,\ n_{8,5} = \lvert 1011\rangle + \lvert 0001\rangle,\ n_{8,6} = \lvert 1001\rangle$
9	$n_{9,1} = \lvert 1101\rangle + \lvert 1110\rangle + \lvert 1000\rangle + \lvert 0100\rangle,\ n_{9,2} = \lvert 1101\rangle + \lvert 1010\rangle + \lvert 0100\rangle,$ $n_{9,3} = \lvert 1110\rangle + \lvert 1000\rangle + \lvert 0101\rangle,\ n_{9,4} = \lvert 1110\rangle + \lvert 1000\rangle,\ n_{9,5} = \lvert 1101\rangle + \lvert 0100\rangle,\ n_{9,6} = \lvert 1100\rangle$
10	$n_{10,1} = \lvert 1100\rangle + \lvert 1010\rangle + \lvert 0110\rangle,\ n_{10,2} = \lvert 1010\rangle + \lvert 0110\rangle,\ n_{10,3} = \lvert 1010\rangle + \lvert 0110\rangle + \lvert 0011\rangle,$ $n_{10,4} = \lvert 1100\rangle + \lvert 0110\rangle,\ n_{10,5} = \lvert 0110\rangle,\ n_{10,6} = \lvert 0110\rangle + \lvert 0011\rangle,\ n_{10,7} = \lvert 1100\rangle + \lvert 0110\rangle + \lvert 0101\rangle,$ $n_{10,8} = \lvert 0110\rangle + \lvert 0101\rangle,\ n_{10,9} = \lvert 0110\rangle + \lvert 0101\rangle + \lvert 0011\rangle,\ n_{10,10} = \lvert 1100\rangle + \lvert 1010\rangle,$ $n_{10,11} = \lvert 1010\rangle,\ n_{10,12} = \lvert 1010\rangle + \lvert 0011\rangle,\ n_{10,13} = \lvert 0011\rangle.$

TABLE 3. This is from [27, Theorem 3.7]: The nilpotent elements $n_{i,j}$ used in Theorem 3.2.

A.2. **Semisimple elements.**

$w \in W$	cocycle in $\widehat{G}$
$\mathrm{diag}(-1,-1,-1,-1)$	$(-I,I,I,I)$
$\mathrm{diag}(-1,-1,-1,1)$	$(M,M,-N,N)$
$\mathrm{diag}(-1,-1,1,1)$	(L,I,I,L)
$\mathrm{diag}(-1,1,-1,1)$	(I,L,I,L)
$\mathrm{diag}(-1,1,1,-1)$	(I,I,L,L)
$\mathrm{diag}(1,-1,-1,1)$	$(I,I,L,-L)$
$\mathrm{diag}(1,-1,1,-1)$	$(I,L,I,-L)$
$\mathrm{diag}(1,1,-1,-1)$	$(L,I,I,-L)$
$\mathrm{diag}(-1,1,1,1)$	(M,M,M,M)
$\mathrm{diag}(1,-1,1,1)$	(N,M,M,N)
$\mathrm{diag}(1,1,-1,1)$	(M,N,M,N)
$\mathrm{diag}(1,1,1,-1)$	(M,M,N,N)
$\mathrm{diag}(1,-1,-1,-1)$	$(-N,N,N,N)$

$w \in W$	cocycle in $\widehat{G}$
$\begin{pmatrix} 0&0&0&-1\\ 0&0&1&0\\ 0&1&0&0\\ -1&0&0&0 \end{pmatrix}$	$(L,L,-K,K)$
$\begin{pmatrix} 0&0&-1&0\\ 0&0&0&-1\\ -1&0&0&0\\ 0&-1&0&0 \end{pmatrix}$	(I,K,I,K)
$\begin{pmatrix} 0&-1&0&0\\ -1&0&0&0\\ 0&0&0&-1\\ 0&0&-1&0 \end{pmatrix}$	(K,I,I,K)

TABLE 4. Cocycles in $\widehat{G}$ that induce the cocycles in W whose equivalence classes form the various sets $H^1(\Gamma_{\Pi_i})$: these will be the elements n_i that map under φ to γ_i as described in Theorem 5.3. Matrices I, J, K, L, M, N are from (3.1) and elements in $H^1(\Gamma_{\Pi_i})$ are considered as elements in W.

$\boldsymbol{A}$	$-I$	M	$-M$	N	$-N$	L	$-L$	K	$-K$
$\boldsymbol{\varepsilon(A)}$	L	$D(\eta^5)$	$D(\eta)$	$D(\eta^7)$	$-D(\eta^3)$	M	$D(\zeta)$	LF	F

TABLE 5. Matrices F, I, J, K, L, M, N are from (3.1), and η is a primitive 16-th root of unity with $\eta^2 = \zeta$; if $A \in \mathrm{SL}(2,\mathbb{C})$, then $\varepsilon(A) \in \mathrm{SL}(2,\mathbb{C})$ satisfies $\varepsilon(A)^{-1}\overline{\varepsilon(A)} = A$.

j	k	**real orbit representatives (semisimple)**
1	$1,2$	$\lambda_1(1,0,0,0) \quad (-2,2,2,2)/\lambda_1$
	Coefficients:	with respect to the basis $\{u_1,u_2,u_3,u_4\}$
	Conditions:	λ_1 up to $\Gamma_{\Pi_{10}}$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$
2	$1,2$	$\imath\lambda_1(1,0,0,0) \quad (-2,2,2,2)/(\imath\lambda_1)$
	Coefficients:	with respect to the basis $\{v_1,v_2,v_3,v_4\}$
	Conditions:	λ_1 up to $\Gamma_{\Pi_{10}}$-conjugacy, $\lambda_1 \in \imath\mathbb{R}^\times$

TABLE 6. **Case $i = 10$:** The table lists real orbit representatives corresponding to γ_j and $[z_k] \in H^1(Z)$.

j	k	**real orbit representatives (semisimple)**
1	$1,2$	$\lambda_1(1,-1) \quad -\lambda_1(1,1)$
	Coefficients:	with respect to the basis $\{u_1,u_4\}$
	Conditions:	λ_1 up to Γ_{Π_7}-conjugacy, $\lambda_1 \in \mathbb{R}^\times$
2	$1,2$	$\imath\lambda_1(1,-1) \quad -\imath\lambda_1(1,1)$
	Coefficients:	with respect to the basis $\{v_1,v_4\}$
	Conditions:	λ_1 up to Γ_{Π_7}-conjugacy, $\lambda_1 \in \imath\mathbb{R}^\times$

TABLE 7. **Cases $i = 7, 8, 9$:** The table lists real orbit representatives corresponding to γ_j and $[z_k] \in H^1(Z)$ for $\mathfrak{h}^\circ_{\Pi_7}$; the real representatives for Cases $i \in \{8,9\}$ are obtained via Remark 3.3

j	k	**real orbit representatives (semisimple)**	
1	$1,2$	$(\lambda_1,0,0,\lambda_4)$	$(-\lambda_1,0,0,\lambda_4)$
	3	$(-\lambda_1+\lambda_4,-\lambda_1-\lambda_4,\lambda_1+\lambda_4,-\lambda_1+\lambda_4)/2$	
	4	$(-\lambda_1+\lambda_4,\lambda_1+\lambda_4,-\lambda_1-\lambda_4,-\lambda_1+\lambda_4)/2$	
	5	$(-\lambda_1-\lambda_4,\lambda_1-\lambda_4,\lambda_1-\lambda_4,\lambda_1+\lambda_4)/2$	
	6	$(-\lambda_1-\lambda_4,-\lambda_1+\lambda_4,-\lambda_1+\lambda_4,\lambda_1+\lambda_4)/2$	
	Coefficients:	with respect to the basis $\{u_1,u_2,u_3,u_4\}$	
	Conditions:	(λ_1,λ_4) up to Γ_{Π_4}-conjugacy, $\lambda_1,\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm\lambda_4\}$	
2	$1,2$	$(\imath\lambda_1,\lambda_4)$	$(-\imath\lambda_1,\lambda_4)$
	Coefficients:	with respect to the basis $\{t_1,t_4\}$	
	Conditions:	(λ_1,λ_4) up to Γ_{Π_4}-conjugacy, $\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \in \imath\mathbb{R}^\times$	
3	$1,2$	$\imath(-\lambda_1,0,0,\lambda_4)$	$\imath(\lambda_1,0,0,\lambda_4)$
	3	$\imath(-\lambda_1-\lambda_4,-\lambda_1+\lambda_4,-\lambda_1+\lambda_4,\lambda_1+\lambda_4)/2$	
	4	$\imath(-\lambda_1+\lambda_4,\lambda_1+\lambda_4,-\lambda_1-\lambda_4,-\lambda_1+\lambda_4)/2$	
	5	$\imath(-\lambda_1-\lambda_4,\lambda_1-\lambda_4,\lambda_1-\lambda_4,\lambda_1+\lambda_4)/2$	
	6	$\imath(-\lambda_1+\lambda_4,-\lambda_1-\lambda_4,\lambda_1+\lambda_4,-\lambda_1+\lambda_4)/2$	
	Coefficients:	with respect to the basis $\{v_1,v_2,v_3,v_4\}$	
	Conditions:	(λ_1,λ_4) up to Γ_{Π_4}-conjugacy, $\lambda_1,\lambda_2 \in \imath\mathbb{R}^\times$, $\lambda_1 \notin \{\pm\lambda_2\}$	
4	1	$\frac{1}{2}(-\imath(\lambda_1+\lambda_4),\lambda_1-\lambda_4,-\lambda_1+\lambda_4,-\imath(\lambda_1+\lambda_4))$	
	2	$\frac{1}{2}(\imath(\lambda_1+\lambda_4),\lambda_1-\lambda_4,\lambda_1-\lambda_4,-\imath(\lambda_1+\lambda_4))$	
	3	$\frac{1}{2}(\imath(\lambda_1+\lambda_4),\lambda_1-\lambda_4,-\lambda_1+\lambda_4,-\imath(\lambda_1+\lambda_4))$	
	4	$\frac{1}{2}(-\imath(\lambda_1+\lambda_4),\lambda_1-\lambda_4,\lambda_1-\lambda_4,-\imath(\lambda_1+\lambda_4))$	
	Coefficients:	with respect to the basis $\{z_1,z_2,z_3,z_4\}$	
	Conditions:	(λ_1,λ_4) up to Γ_{Π_4}-conjugacy, $\imath(\lambda_1+\lambda_4),\lambda_1-\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm\lambda_4\}$	

TABLE 8. **Cases $i = 4, 5, 6$:** The table lists real orbit representatives corresponding to γ_j and $[z_k] \in H^1(Z)$ for $\mathfrak{h}^\circ_{\Pi_4}$; the real representatives for Cases $i \in \{5,6\}$ are obtained via Remark 3.3.

j	k	**real orbit representatives (semisimple)**			
1	$1,\ldots,4$	$(\lambda_1+\lambda_2,-\lambda_1,-\lambda_2)$	$(-\lambda_1-\lambda_2,-\lambda_1,-\lambda_2)$	$(-\lambda_1-\lambda_2,-\lambda_1,\lambda_2)$	$(-\lambda_1-\lambda_2,\lambda_1,-\lambda_2)$
	Coefficients:	with respect to the basis $\{u_1,u_2,u_3\}$			
	Conditions:	(λ_1,λ_2) up to Γ_{Π_3}-conjugacy, each $\lambda_i\in\mathbb{R}^\times$, $\lambda_1\neq-\lambda_2$			
2	$1,\ldots,4$	$\imath(\lambda_1+\lambda_2,-\lambda_1,-\lambda_2)$	$-\imath(\lambda_1+\lambda_2,\lambda_1,\lambda_2)$	$-\imath(\lambda_1+\lambda_2,\lambda_1,-\lambda_2)$	$-\imath(\lambda_1+\lambda_2,-\lambda_1,\lambda_2)$
	Coefficients:	with respect to the basis $\{v_1,v_2,v_3\}$			
	Conditions:	(λ_1,λ_2) up to Γ_{Π_3}-conjugacy, each $\lambda_i\in\imath\mathbb{R}^\times$, $\lambda_1\neq-\lambda_2$			

TABLE 9. **Case $i=3$:** The table lists real orbit representatives corresponding to γ_j and $[z_k]\in H^1(Z)$.

j	k	**real orbit representatives (semisimple)**			
1	$1,\ldots,4$	$(\lambda_1,\lambda_2,\lambda_3,0)$	$(-\lambda_1,\lambda_2,\lambda_3,0)$	$(-\lambda_1,\lambda_2,-\lambda_3,0)$	$(-\lambda_1,-\lambda_2,\lambda_3,0)$
	5	$(-\lambda_1+\lambda_2+\lambda_3,-\lambda_1+\lambda_2-\lambda_3,-\lambda_1-\lambda_2+\lambda_3,\lambda_1+\lambda_2+\lambda_3)/2$			
	6	$(-\lambda_1-\lambda_2-\lambda_3,\lambda_1+\lambda_2-\lambda_3,\lambda_1-\lambda_2+\lambda_3,\lambda_1-\lambda_2-\lambda_3)/2$			
	7	$(-\lambda_1-\lambda_2+\lambda_3,\lambda_1+\lambda_2+\lambda_3,-\lambda_1+\lambda_2+\lambda_3,-\lambda_1+\lambda_2-\lambda_3)/2$			
	8	$(-\lambda_1+\lambda_2-\lambda_3,-\lambda_1+\lambda_2+\lambda_3,\lambda_1+\lambda_2+\lambda_3,-\lambda_1-\lambda_2+\lambda_3)/2$			
	Coefficients:	with respect to the basis $\{u_1,u_2,u_3,u_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, each $\lambda_i\in\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\}$			
2	$1,\ldots,4$	$(-\imath\lambda_3,0,\lambda_1,-\imath\lambda_2)$	$(\imath\lambda_3,0,\lambda_1,\imath\lambda_2)$	$(\imath\lambda_3,0,-\lambda_1,-\imath\lambda_2)$	$(\imath\lambda_3,0,\lambda_1,-\imath\lambda_2)$
	Coefficients:	with respect to the basis $\{z_1,z_2,z_3,z_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_1\in\mathbb{R}^\times$, $\lambda_2,\lambda_3\in\imath\mathbb{R}^\times$			
3	$1,\ldots,4$	$(0,-\lambda_3,-\imath\lambda_2,\lambda_1)$	$(0,-\lambda_3,-\imath\lambda_2,-\lambda_1)$	$(0,-\lambda_3,\imath\lambda_2,\lambda_1)$	$(0,\lambda_3,-\imath\lambda_2,\lambda_1)$
	Coefficients:	with respect to the basis $\{y_1,y_2,y_3,y_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_1,\lambda_3\in\mathbb{R}^\times$, $\lambda_2\in\imath\mathbb{R}^\times$, and $\lambda_1\notin\{\pm\lambda_3\}$			
4	$1,\ldots,4$	$(-\imath\lambda_1,-\imath\lambda_2,\lambda_3,0)$	$(\imath\lambda_1,-\imath\lambda_2,\lambda_3,0)$	$(\imath\lambda_1,-\imath\lambda_2,-\lambda_3,0)$	$(\imath\lambda_1,\imath\lambda_2,\lambda_3,0)$
	Coefficients:	with respect to the basis $\{x_1,x_2,x_3,x_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_3\in\mathbb{R}^\times$, $\lambda_1,\lambda_2\in\imath\mathbb{R}^\times$, and $\lambda_1\notin\{\pm\lambda_2\}$			
5	$1,\ldots,4$	$(0,\imath\lambda_3,-\lambda_2,\lambda_1)$	$(0,\imath\lambda_3,-\lambda_2,-\lambda_1)$	$(0,\imath\lambda_3,\lambda_2,\lambda_1)$	$(0,-\imath\lambda_3,-\lambda_2,\lambda_1)$
	Coefficients:	with respect to the basis $\{x_1,x_2,x_3,x_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_1,\lambda_2\in\mathbb{R}^\times$, $\lambda_3\in\imath\mathbb{R}^\times$, and $\lambda_1\notin\{\pm\lambda_2\}$			
6	$1,\ldots,4$	$(-\imath\lambda_1,\lambda_2,\imath\lambda_3,0)$	$(\imath\lambda_1,\lambda_2,\imath\lambda_3,0)$	$(\imath\lambda_1,\lambda_2,-\imath\lambda_3,0)$	$(\imath\lambda_1,-\lambda_2,\imath\lambda_3,0)$
	Coefficients:	with respect to the basis $\{y_1,y_2,y_3,y_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_2\in\mathbb{R}^\times$, $\lambda_1,\lambda_3\in\imath\mathbb{R}^\times$, and $\lambda_1\notin\{\pm\lambda_3\}$			
7	$1,\ldots,4$	$(-\imath\lambda_1,\lambda_2,\lambda_3,0)$	$(\imath\lambda_1,\lambda_2,\lambda_3,0)$	$(\imath\lambda_1,\lambda_2,-\lambda_3,0)$	$(\imath\lambda_1,-\lambda_2,\lambda_3,0)$
	Coefficients:	with respect to the basis $\{y_1,y_2,y_3,y_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_2,\lambda_3\in\mathbb{R}^\times$, $\lambda_1\in\imath\mathbb{R}^\times$, and $\lambda_1\notin\{\pm\lambda_3\}$			
8	$1,\ldots,4$	$\imath(-\lambda_1,-\lambda_2,-\lambda_3,0)$	$\imath(\lambda_1,-\lambda_2,-\lambda_3,0)$	$\imath(\lambda_1,-\lambda_2,\lambda_3,0)$	$\imath(\lambda_1,\lambda_2,-\lambda_3,0)$
	5	$\imath(\lambda_1-\lambda_2-\lambda_3,\lambda_1-\lambda_2+\lambda_3,\lambda_1+\lambda_2-\lambda_3,-\lambda_1-\lambda_2-\lambda_3)/2$			
	6	$\imath(\lambda_1+\lambda_2+\lambda_3,-\lambda_1-\lambda_2+\lambda_3,-\lambda_1+\lambda_2-\lambda_3,-\lambda_1+\lambda_2+\lambda_3)/2$			
	7	$\imath(\lambda_1+\lambda_2-\lambda_3,-\lambda_1-\lambda_2-\lambda_3,\lambda_1-\lambda_2-\lambda_3,\lambda_1-\lambda_2+\lambda_3)/2$			
	8	$\imath(\lambda_1-\lambda_2+\lambda_3,\lambda_1-\lambda_2-\lambda_3,-\lambda_1-\lambda_2-\lambda_3,\lambda_1+\lambda_2-\lambda_3)/2$			
	Coefficients:	with respect to the basis $\{v_1,v_2,v_3,v_4\}$			
	Conditions:	$(\lambda_1,\lambda_2,\lambda_3)$ up to Γ_{Π_2}-conjugacy, each $\lambda_i\in\imath\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\}$			

TABLE 10. **Case $i=2$:** The table lists real orbit representatives corresponding to γ_j and $[z_k]\in H^1(Z)$.

j	k	**real orbit representatives (semisimple)**
1	$1,\dots,4$	$(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $(-\lambda_1,\lambda_2,\lambda_3,-\lambda_4)$ $(-\lambda_1,\lambda_2,-\lambda_3,\lambda_4)$ $(-\lambda_1,-\lambda_2,\lambda_3,\lambda_4)$
	5	$(-\lambda_1-\lambda_2-\lambda_3-\lambda_4,\lambda_1+\lambda_2-\lambda_3-\lambda_4,\lambda_1-\lambda_2+\lambda_3-\lambda_4,\lambda_1-\lambda_2-\lambda_3+\lambda_4)/2$
	6	$(-\lambda_1+\lambda_2+\lambda_3-\lambda_4,-\lambda_1+\lambda_2-\lambda_3+\lambda_4,-\lambda_1-\lambda_2+\lambda_3+\lambda_4,\lambda_1+\lambda_2+\lambda_3+\lambda_4)/2$
	7	$(-\lambda_1+\lambda_2-\lambda_3+\lambda_4,-\lambda_1+\lambda_2+\lambda_3-\lambda_4,\lambda_1+\lambda_2+\lambda_3+\lambda_4,-\lambda_1-\lambda_2+\lambda_3+\lambda_4)/2$
	8	$(-\lambda_1-\lambda_2+\lambda_3+\lambda_4,\lambda_1+\lambda_2+\lambda_3+\lambda_4,-\lambda_1+\lambda_2+\lambda_3-\lambda_4,-\lambda_1+\lambda_2-\lambda_3+\lambda_4)/2$
	$9,\dots,12$	$(-\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $(\lambda_1,\lambda_2,\lambda_3,-\lambda_4)$ $(\lambda_1,\lambda_2,-\lambda_3,\lambda_4)$ $(\lambda_1,-\lambda_2,\lambda_3,\lambda_4)$
	Coefficients:	with respect to the basis $\{u_1,u_2,u_3,u_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, each $\lambda_i\in\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$
2	$1,\dots,4$	$\imath(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $\imath(-\lambda_1,\lambda_2,\lambda_3,-\lambda_4)$ $\imath(-\lambda_1,\lambda_2,-\lambda_3,\lambda_4)$ $\imath(-\lambda_1,-\lambda_2,\lambda_3,\lambda_4)$
	5	$\imath(-\lambda_1-\lambda_2+\lambda_3+\lambda_4,\lambda_1+\lambda_2+\lambda_3+\lambda_4,-\lambda_1+\lambda_2+\lambda_3-\lambda_4,+-\lambda_1+\lambda_2-\lambda_3+\lambda_4)/2$
	6	$\imath(-\lambda_1+\lambda_2-\lambda_3+\lambda_4,-\lambda_1+\lambda_2+\lambda_3-\lambda_4,\lambda_1+\lambda_2+\lambda_3+\lambda_4,-\lambda_1-\lambda_2+\lambda_3+\lambda_4)/2$
	7	$\imath(-\lambda_1+\lambda_2+\lambda_3-\lambda_4,-\lambda_1+\lambda_2-\lambda_3+\lambda_4,-\lambda_1-\lambda_2+\lambda_3+\lambda_4,\lambda_1+\lambda_2+\lambda_3+\lambda_4)/2$
	8	$\imath(-\lambda_1-\lambda_2-\lambda_3-\lambda_4,\lambda_1+\lambda_2-\lambda_3-\lambda_4,\lambda_1-\lambda_2+\lambda_3-\lambda_4,\lambda_1-\lambda_2-\lambda_3+\lambda_4)/2$
	$9,\dots,12$	$\imath(-\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $\imath(\lambda_1,\lambda_2,\lambda_3,-\lambda_4)$ $\imath(\lambda_1,\lambda_2,-\lambda_3,\lambda_4)$ $\imath(\lambda_1,-\lambda_2,\lambda_3,\lambda_4)$
	Coefficients:	with respect to the basis $\{v_1,v_2,v_3,v_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, each $\lambda_i\in\imath\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\pm\lambda_4\}$
3	$1,\dots,4$	$(\imath\lambda_1,\imath\lambda_2,-\imath\lambda_3,\lambda_4)$ $(-\imath\lambda_1,\imath\lambda_2,-\imath\lambda_3,-\lambda_4)$ $(-\imath\lambda_1,\imath\lambda_2,\imath\lambda_3,\lambda_4)$ $(-\imath\lambda_1,-\imath\lambda_2,-\imath\lambda_3,\lambda_4)$
	Coefficients:	with respect to the basis $\{w_1,w_2,w_3,w_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, $\lambda_1,\lambda_2,\lambda_3\in\imath\mathbb{R}^\times$ and $\lambda_4\in\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_2\pm\lambda_3\}$
4	$1,\dots,4$	$(-\imath\lambda_1,-\imath\lambda_2,\lambda_3,\lambda_4)$ $(\imath\lambda_1,-\imath\lambda_2,\lambda_3,-\lambda_4)$ $(\imath\lambda_1,-\imath\lambda_2,\lambda_3,\lambda_4)$ $(-\imath\lambda_1,-\imath\lambda_2,\lambda_3,-\lambda_4)$
	Coefficients:	with respect to the basis $\{x_1,x_2,x_3,x_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, $\lambda_1,\lambda_2\in\imath\mathbb{R}^\times$ and $\lambda_3,\lambda_4\in\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_2\}$
5	$1,\dots,4$	$(-\imath\lambda_1,\lambda_2,\imath\lambda_3,\lambda_4)$ $(\imath\lambda_1,\lambda_2,\imath\lambda_3,-\lambda_4)$ $(\imath\lambda_1,\lambda_2,\imath\lambda_3,\lambda_4)$ $(-\imath\lambda_1,\lambda_2,\imath\lambda_3,-\lambda_4)$
	Coefficients:	with respect to the basis $\{y_1,y_2,y_3,y_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, $\lambda_1,\lambda_3\in\imath\mathbb{R}^\times$ and $\lambda_2,\lambda_4\in\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_3\}$
6	$1,\dots,4$	$(-\imath\lambda_1,\lambda_2,\lambda_3,\imath\lambda_4)$ $(\imath\lambda_1,\lambda_2,-\lambda_3,\imath\lambda_4)$ $(\imath\lambda_1,\lambda_2,\lambda_3,\imath\lambda_4)$ $(-\imath\lambda_1,\lambda_2,-\lambda_3,\imath\lambda_4)$
	Coefficients:	with respect to the basis $\{z_1,z_2,z_3,z_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, $\lambda_1,\lambda_4\in\imath\mathbb{R}^\times$ and $\lambda_2,\lambda_3\in\mathbb{R}^\times$ and $\lambda_1\notin\{\pm\lambda_4\}$
7	$1,\dots,4$	$(\imath\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ $(-\imath\lambda_1,\lambda_2,\lambda_3,-\lambda_4)$ $(-\imath\lambda_1,\lambda_2,-\lambda_3,\lambda_4)$ $(-\imath\lambda_1,-\lambda_2,\lambda_3,\lambda_4)$
	Coefficients:	with respect to the basis $\{t_1,t_2,t_3,t_4\}$
	Conditions:	$(\lambda_1,\dots,\lambda_4)$ up to W-conjugacy, $\lambda_1\in\imath\mathbb{R}^\times$ and $\lambda_2,\lambda_3,\lambda_4\in\mathbb{R}^\times$

TABLE 11. **Case $i=1$:** The table lists real orbit representatives corresponding to γ_j and $[z_k]\in H^1(Z)$.

A.3. **Mixed elements: Centralisers.**

i	r	**identity component Z°**	**preimages of generators of Z/Z°**	**H^1 ref.**
2	1	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
3	1	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
3	2	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$	(6.2)
4	1	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
4	2	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
4	3	$\{(D(a)^{-1},D(a),D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
4	4	$\{(D(a),D(a)^{-1},D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
7	1	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$	(6.2)
7	2	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$	(6.2)
7	3	1	$(-I,-I,I,I),(-I,I,-I,I),(-I,-I,-I,-I)$	(6.2)
7	4	$\{(I,I,D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I),(-L,L,-J,J)$	(6.4)
7	5	$\{(D(a)^{-1},D(a),I,I)\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I),(-J,J,-L,L)$	(6.5)
7	6	$\{(D(a)^{-1},D(a),D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	1	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
10	2	$\{(D(a)^{-1},D(a)^{-1},D(a),D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	3	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
10	4	$\{(D(a)^{-1},D(a),D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	5	$\{(D(a)^{-1},D(b)^{-1},D(b),D(a))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,-I,I,I)$	(6.6)
10	6	$\{(D(a)^{-1},D(a),D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	7	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
10	8	$\{(D(a)^{-1},D(a)^{-1},D(a),D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	9	1	$(-I,-I,I,I),(-I,I,-I,I),(-L,-L,L,L)$	(6.1)
10	10	$\{(D(a),D(a)^{-1},D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	11	$\{(D(b)^{-1},D(a)^{-1},D(b),D(a))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,-I,I,I)$	(6.6)
10	12	$\{(D(a),D(a)^{-1},D(a)^{-1},D(a))\ :\ a\in\mathbb{C}^\times\}$	$(-I,-I,I,I),(-I,I,-I,I)$	(6.3)
10	13	$\{(D(b)^{-1},D(b),D(a)^{-1},D(a))\ :\ a,b\in\mathbb{C}^\times\}$	$(-I,I,-I,I)$	(6.7)

TABLE 12. Centralisers $Z = Z_{\widehat{G}}(p,h,e,f)$ for each $p\in\Sigma_i$ and $e = n_{i,r}$ as in Theorem 3.2. The last column lists reference labels for the equations describing the representatives of the classes in $H^1(Z)$; the notation used in the third and fourth columns is from (3.1).

A.4. **Mixed elements, case $p \in \Sigma$.**

i	r	k	**semisimple part**	**nilpotent part**
2	1	1	$(\lambda_1, \lambda_2, \lambda_3, 0)$	$\|0011\rangle$
		2	$(-\lambda_1, \lambda_2, \lambda_3, 0)$	$-\|0011\rangle$
		3	$(-\lambda_1, \lambda_2, -\lambda_3, 0)$	$\|0011\rangle$
		4	$(-\lambda_1, -\lambda_2, \lambda_3, 0)$	$\|0011\rangle$
		5	$(-\lambda_1, \lambda_2, \lambda_3, 0)$	$\|0011\rangle$
		6	$(\lambda_1, \lambda_2, \lambda_3, 0)$	$-\|0011\rangle$
		7	$(\lambda_1, \lambda_2, -\lambda_3, 0)$	$\|0011\rangle$
		8	$(\lambda_1, -\lambda_2, \lambda_3, 0)$	$\|0011\rangle$
Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, each $\lambda_i \in \mathbb{R}^\times$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$				

TABLE 13. **Case $i = 2$ and $p \in \Sigma$:** Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

i	r	k	**semisimple part**	**nilpotent part**
3	1	1	$(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$	$\|0011\rangle$
		2	$(-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2, 0)$	$-\|0011\rangle$
		3	$(-\lambda_1 - \lambda_2, -\lambda_1, \lambda_2, 0)$	$\|0011\rangle$
		4	$(-\lambda_1 - \lambda_2, \lambda_1, -\lambda_2, 0)$	$\|0011\rangle$
		5	$(-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2, 0)$	$\|0011\rangle$
		6	$(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$	$-\|0011\rangle$
		7	$(\lambda_1 + \lambda_2, -\lambda_1, \lambda_2, 0)$	$\|0011\rangle$
		8	$(\lambda_1 + \lambda_2, \lambda_1, -\lambda_2, 0)$	$\|0011\rangle$
3	2	1	$(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$	$\|1011\rangle + \|0111\rangle + \|0010\rangle + \|0001\rangle$
		2	$(-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2, 0)$	$\|1011\rangle + \|0111\rangle - \|0010\rangle - \|0001\rangle$
		3	$(-\lambda_1 - \lambda_2, -\lambda_1, \lambda_2, 0)$	$-\|1011\rangle + \|0111\rangle + \|0010\rangle - \|0001\rangle$
		4	$(-\lambda_1 - \lambda_2, \lambda_1, -\lambda_2, 0)$	$-\|1011\rangle + \|0111\rangle - \|0010\rangle + \|0001\rangle$
		5	$(-\lambda_1 - \lambda_2, \lambda_1, -\lambda_2, 0)$	$\|1011\rangle - \|0111\rangle + \|0010\rangle - \|0001\rangle$
		6	$(-\lambda_1 - \lambda_2, -\lambda_1, \lambda_2,)$	$\|1011\rangle - \|0111\rangle - \|0010\rangle + \|0001\rangle$
		7	$(-\lambda_1 - \lambda_2, -\lambda_1, -\lambda_2, 0)$	$-\|1011\rangle - \|0111\rangle + \|0010\rangle + \|0001\rangle$
		8	$(\lambda_1 + \lambda_2, -\lambda_1, -\lambda_2, 0)$	$-\|1011\rangle - \|0111\rangle - \|0010\rangle - \|0001\rangle$
Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$				
Conditions: (λ_1, λ_2) up to Γ_{Π_3}-conjugacy, each $\lambda_i \in \mathbb{R}^\times$, $\lambda_1 \neq -\lambda_2$				

TABLE 14. **Case $i = 3$ and $p \in \Sigma$:** Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

i	r	k	semisimple part	nilpotent part
4	1	1	$(\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle + \|0110\rangle$
		2	$(-\lambda_1, 0, 0, -\lambda_4)$	$\|1010\rangle + \|0110\rangle$
		3	$(-\lambda_1, 0, 0, \lambda_4)$	$-\|1010\rangle + \|0110\rangle$
		4	$(-\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle - \|0110\rangle$
		5	$(-\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle + \|0110\rangle$
		6	$(\lambda_1, 0, 0, -\lambda_4)$	$\|1010\rangle + \|0110\rangle$
		7	$(\lambda_1, 0, 0, \lambda_4)$	$-\|1010\rangle + \|0110\rangle$
		8	$(\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle - \|0110\rangle$
4	2	1	$(\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle + \|0101\rangle$
		2	$(-\lambda_1, 0, 0, -\lambda_4)$	$\|0110\rangle + \|0101\rangle$
		3	$(-\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle - \|0101\rangle$
		4	$(-\lambda_1, 0, 0, \lambda_4)$	$-\|0110\rangle + \|0101\rangle$
		5	$(-\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle + \|0101\rangle$
		6	$(\lambda_1, 0, 0, -\lambda_4)$	$\|0110\rangle + \|0101\rangle$
		7	$(\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle - \|0101\rangle$
		8	$(\lambda_1, 0, 0, \lambda_4)$	$-\|0110\rangle + \|0101\rangle$
4	3	1	$(\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle$
		2	$(-\lambda_1, 0, 0, -\lambda_4)$	$\|0110\rangle$
		3	$(-\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle$
		4	$(-\lambda_1, 0, 0, \lambda_4)$	$-\|0110\rangle$
4	4	1	$(\lambda_1, 0, 0, \lambda_4)$	$\|0101\rangle$
		2	$(-\lambda_1, 0, 0, -\lambda_4)$	$\|0101\rangle$
		3	$(-\lambda_1, 0, 0, \lambda_4)$	$-\|0101\rangle$
		4	$(-\lambda_1, 0, 0, \lambda_4)$	$\|0101\rangle$

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: (λ_1, λ_4) up to Γ_{Π_4}-conjugacy, $\lambda_1, \lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm\lambda_4\}$

TABLE 15. **Case $i = 4$ and $p \in \Sigma$:** Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

i	r	k	**semisimple part**	**nilpotent part**
7	1	1	$\lambda_1(1,0,0,-1)$	$\lvert 1101\rangle + \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0001\rangle$
		2	$\lambda_1(-1,0,0,1)$	$-\lvert 1101\rangle + \lvert 1011\rangle + \lvert 1000\rangle - \lvert 0001\rangle$
		3	$\lambda_1(-1,0,0,-1)$	$\lvert 1101\rangle - \lvert 1011\rangle + \lvert 1000\rangle - \lvert 0001\rangle$
		4	$\lambda_1(-1,0,0,-1)$	$-\lvert 1101\rangle - \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0001\rangle$
		5	$\lambda_1(-1,0,0,-1)$	$\lvert 1101\rangle + \lvert 1011\rangle - \lvert 1000\rangle - \lvert 0001\rangle$
		6	$\lambda_1(-1,0,0,-1)$	$-\lvert 1101\rangle + \lvert 1011\rangle - \lvert 1000\rangle + \lvert 0001\rangle$
		7	$\lambda_1(-1,0,0,1)$	$\lvert 1101\rangle - \lvert 1011\rangle - \lvert 1000\rangle + \lvert 0001\rangle$
		8	$\lambda_1(1,0,0,-1)$	$-\lvert 1101\rangle - \lvert 1011\rangle - \lvert 1000\rangle - \lvert 1,1,1,2\rangle$
7	2	1	$\lambda_1(1,0,0,-1)$	$\lvert 1101\rangle + \lvert 1010\rangle + \lvert 0001\rangle$
		2	$\lambda_1(-1,0,0,1)$	$-\lvert 1101\rangle + \lvert 1010\rangle - \lvert 0001\rangle$
		3	$\lambda_1(-1,0,0,-1)$	$\lvert 1101\rangle - \lvert 1010\rangle - \lvert 0001\rangle$
		4	$\lambda_1(-1,0,0,-1)$	$-\lvert 1101\rangle + \lvert 1010\rangle + \lvert 0001\rangle$
		5	$\lambda_1(-1,0,0,-1)$	$\lvert 1101\rangle + \lvert 1010\rangle - \lvert 0001\rangle$
		6	$\lambda_1(-1,0,0,-1)$	$-\lvert 1101\rangle - \lvert 1010\rangle + \lvert 0001\rangle$
		7	$\lambda_1(-1,0,0,1)$	$\lvert 1101\rangle + \lvert 1010\rangle + \lvert 0001\rangle$
		8	$\lambda_1(1,0,0,-1)$	$-\lvert 1101\rangle + \lvert 1010\rangle - \lvert 0001\rangle$
7	3	1	$\lambda_1(1,0,0,-1)$	$\lvert 1011\rangle + \lvert 1000\rangle + \lvert 0101\rangle$
		2	$\lambda_1(-1,0,0,1)$	$\lvert 1011\rangle + \lvert 1000\rangle + \lvert 0101\rangle$
		3	$\lambda_1(-1,0,0,-1)$	$-\lvert 1011\rangle + \lvert 1000\rangle - \lvert 0101\rangle$
		4	$\lambda_1(-1,0,0,-1)$	$-\lvert 1011\rangle + \lvert 1000\rangle + \lvert 0101\rangle$
		5	$\lambda_1(-1,0,0,-1)$	$\lvert 1011\rangle - \lvert 1000\rangle + \lvert 0101\rangle$
		6	$\lambda_1(-1,0,0,-1)$	$\lvert 1011\rangle - \lvert 1000\rangle - \lvert 0101\rangle$
		7	$\lambda_1(-1,0,0,1)$	$-\lvert 1011\rangle - \lvert 1000\rangle + \lvert 0101\rangle$
		8	$\lambda_1(1,0,0,-1)$	$-\lvert 1011\rangle - \lvert 1000\rangle + \lvert 0101\rangle$
7	4	1	$\lambda_1(1,0,0,-1)$	$\lvert 1011\rangle + \lvert 1000\rangle$
		2	$\lambda_1(-1,0,0,1)$	$\lvert 1011\rangle + \lvert 1000\rangle$
		3	$\lambda_1(-1,0,0,-1)$	$-\lvert 1011\rangle + \lvert 1000\rangle$
		4	$\lambda_1(-1,0,0,-1)$	$\lvert 1011\rangle - \lvert 1000\rangle$
		5	$\lambda_1(0,1,1,0)$	$\frac{1}{2}(-\lvert 1111\rangle + \lvert 1100\rangle + \lvert 1011\rangle - \lvert 1000\rangle - \lvert 0111\rangle + \lvert 0100\rangle + \lvert 0011\rangle - \lvert 0000\rangle)$
		6	$\lambda_1(0,1,1,0)$	$\frac{1}{2}(\lvert 1111\rangle - \lvert 1100\rangle - \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0111\rangle - \lvert 0100\rangle - \lvert 0011\rangle + \lvert 0000\rangle)$
		7	$\lambda_1(0,1,-1,0)$	$\frac{1}{2}(\lvert 1111\rangle + \lvert 1100\rangle + \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0111\rangle + \lvert 0100\rangle + \lvert 0011\rangle + \lvert 0000\rangle)$
		8	$\lambda_1(0,-1,1,0)$	$\frac{1}{2}(\lvert 1111\rangle + \lvert 1100\rangle + \lvert 1011\rangle + \lvert 1000\rangle + \lvert 0111\rangle + \lvert 0100\rangle + \lvert 0011\rangle + \lvert 0000\rangle)$
7	5	1	$\lambda_1(1,0,0,-1)$	$-\lvert 1101\rangle - \lvert 0001\rangle$
		2	$\lambda_1(-1,0,0,1)$	$-\lvert 1101\rangle - \lvert 0001\rangle$
		3	$\lambda_1(-1,0,0,-1)$	$\lvert 1101\rangle - \lvert 0001\rangle$
		4	$\lambda_1(-1,0,0,-1)$	$-\lvert 1101\rangle + \lvert 0001\rangle$
		5	$\lambda_1(0,-1,-1,0)$	$\frac{1}{2}(-\lvert 1111\rangle - \lvert 1110\rangle + \lvert 1101\rangle + \lvert 1100\rangle + \lvert 0011\rangle + \lvert 0010\rangle - \lvert 0001\rangle - \lvert 0000\rangle)$
		6	$\lambda_1(0,1,1,0)$	$\frac{1}{2}(-\lvert 1111\rangle - \lvert 1110\rangle + \lvert 1101\rangle + \lvert 1100\rangle + \lvert 0011\rangle + \lvert 0010\rangle - \lvert 0001\rangle - \lvert 0000\rangle)$
		7	$\lambda_1(0,-1,1,0)$	$\frac{1}{2}(\lvert 1111\rangle - \lvert 1110\rangle - \lvert 1101\rangle + \lvert 1100\rangle + \lvert 0011\rangle - \lvert 0010\rangle - \lvert 0001\rangle + \lvert 0000\rangle)$
		8	$\lambda_1(0,1,-1,0)$	$\frac{1}{2}(\lvert 1111\rangle - \lvert 1110\rangle - \lvert 1101\rangle + \lvert 1100\rangle + \lvert 0011\rangle - \lvert 0010\rangle - \lvert 0001\rangle + \lvert 0000\rangle)$
7	6	1	$\lambda_1(1,0,0,-1)$	$\lvert 1001\rangle$
		2	$\lambda_1(-1,0,0,1)$	$\lvert 1001\rangle$
		3	$\lambda_1(-1,0,0,-1)$	$\lvert 1001\rangle$
		4	$\lambda_1(-1,0,0,-1)$	$-\lvert 1001\rangle$

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: λ_1 up to Γ_{Π_7}-conjugacy, $\lambda_1 \in \mathbb{R}^\times$

TABLE 16. **Case $i = 7$ and $p \in \Sigma$:** Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

$\boldsymbol{i}$	$\boldsymbol{r}$	$\boldsymbol{k}$	**semisimple part**	**nilpotent part**
10	1	1	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|1010\rangle + \|0110\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$-\|1100\rangle + \|1010\rangle + \|0110\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle - \|1010\rangle + \|0110\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|1010\rangle - \|0110\rangle$
		5	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|1010\rangle + \|0110\rangle$
		6	$(\lambda_1, 0, 0, 0)$	$-\|1100\rangle + \|1010\rangle + \|0110\rangle$
		7	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle - \|1010\rangle + \|0110\rangle$
		8	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|1010\rangle - \|0110\rangle$
10	3	1	$(\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0110\rangle + \|0011\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0110\rangle - \|0011\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$-\|1010\rangle + \|0110\rangle + \|0011\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle - \|0110\rangle + \|0011\rangle$
		5	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0110\rangle + \|0011\rangle$
		6	$(\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0110\rangle - \|0011\rangle$
		7	$(\lambda_1, 0, 0, 0)$	$-\|1010\rangle + \|0110\rangle + \|0011\rangle$
		8	$(\lambda_1, 0, 0, 0)$	$\|1010\rangle - \|0110\rangle + \|0011\rangle$
10	7	1	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|0110\rangle + \|0101\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$-\|1100\rangle + \|0110\rangle + \|0101\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|0110\rangle - \|0101\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle - \|0110\rangle + \|0101\rangle$
		5	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|0110\rangle + \|0101\rangle$
		6	$(\lambda_1, 0, 0, 0)$	$-\|1100\rangle + \|0110\rangle + \|0101\rangle$
		7	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|0110\rangle - \|0101\rangle$
		8	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle - \|0110\rangle + \|0101\rangle$
10	9	1	$(\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0101\rangle + \|0011\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0101\rangle - \|0011\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle - \|0101\rangle + \|0011\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$-\|0110\rangle + \|0101\rangle + \|0011\rangle$
		5	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0101\rangle + \|0011\rangle$
		6	$(\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0101\rangle - \|0011\rangle$
		7	$(\lambda_1, 0, 0, 0)$	$\|0110\rangle - \|0101\rangle + \|0011\rangle$
		8	$(\lambda_1, 0, 0, 0)$	$-\|0110\rangle + \|0101\rangle + \|0011\rangle$
10	5	1	$(\lambda_1, 0, 0, 0)$	$\|0110\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle$
10	11	1	$(\lambda_1, 0, 0, 0)$	$\|1010\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle$
10	13	1	$(\lambda_1, 0, 0, 0)$	$\|0011\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|0011\rangle$

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: λ_1 up to $\Gamma_{\Pi_{10}}$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$

TABLE 17. **Case $\boldsymbol{i = 10}$ and $\boldsymbol{p \in \Sigma}$ (Part I):** Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

i	r	k	semisimple part	nilpotent part
10	2	1	$(\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0110\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0110\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$-\|1010\rangle + \|0110\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle - \|0110\rangle$
10	4	1	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|0110\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$-\|1100\rangle + \|0110\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|0110\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle - \|0110\rangle$
10	6	1	$(\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0011\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle - \|0011\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0011\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$-\|0110\rangle + \|0011\rangle$
10	8	1	$(\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0101\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle + \|0101\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|0110\rangle - \|0101\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$-\|0110\rangle + \|0101\rangle$
10	10	1	$(\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|1010\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$-\|1100\rangle + \|1010\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle - \|1010\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1100\rangle + \|1010\rangle$
10	12	1	$(\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0011\rangle$
		2	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle - \|0011\rangle$
		3	$(-\lambda_1, 0, 0, 0)$	$-\|1010\rangle + \|0011\rangle$
		4	$(-\lambda_1, 0, 0, 0)$	$\|1010\rangle + \|0011\rangle$

Coefficients: with respect to the basis $\{u_1, u_2, u_3, u_4\}$

Conditions: λ_1 up to $\Gamma_{\Pi_{10}}$-conjugacy, $\lambda_1 \in \mathbb{R}^\times$

TABLE 18. **Case $i = 10$ and $p \in \Sigma$ (Part II):** Real representatives corresponding to $[z_k] \in H^1(Z)$ and $n_{i,r}$.

A.5. **Mixed elements, case $p \notin \Sigma$.**

i	j	nilpotent elements
2	2	$n_{2,2,1} = -\vert 0110\rangle$
	$4, 6, 7, 8$	$n_{2,j,1} = \vert 0011\rangle$
	$3, 5$	$n_{2,j,1} = \vert 0000\rangle$
3	2	$n_{3,2,1} = -\vert 0011\rangle,\quad n_{3,2,2} = -\vert 0111\rangle + \vert 1011\rangle - \vert 0010\rangle - \vert 0001\rangle$
4	$2, 3$	$n_{4,j,1} = \vert 0110\rangle + \vert 1010\rangle,\quad n_{4,j,2} = \vert 0110\rangle + \vert 0101\rangle,\quad n_{4,j,3} = \vert 0110\rangle,\ n_{4,j,4} = \vert 0101\rangle$
	4	$n_{4,4,1} = -\frac{1}{2}(-\vert 1110\rangle - \vert 1101\rangle + \vert 1010\rangle + \vert 1001\rangle + \vert 0110\rangle + \vert 0101\rangle - \vert 0010\rangle - \vert 0001\rangle)$
7	2	$n_{7,2,1} = -\vert 1101\rangle - \vert 1011\rangle - \vert 1000\rangle + \vert 0001\rangle,\quad n_{7,2,2} = -\vert 1101\rangle - \vert 1010\rangle + \vert 0001\rangle,$ $n_{7,2,3} = -\vert 1011\rangle - \vert 1000\rangle + \vert 0101\rangle,\quad n_{7,2,4} = -\vert 1011\rangle - \vert 1000\rangle,$ $n_{7,2,5} = -\vert 1101\rangle + \vert 0001\rangle,\quad n_{7,2,6} = -\vert 1001\rangle$
10	2	$n_{10,2,1} = -\vert 1100\rangle - \vert 1010\rangle + \vert 0110\rangle,\quad n_{10,2,2} = -\vert 1010\rangle + \vert 0110\rangle,$ $n_{10,2,3} = -\vert 1010\rangle + \vert 0110\rangle + \vert 0011\rangle,\quad n_{10,2,4} = -\vert 1100\rangle + \vert 0110\rangle,\quad n_{10,2,5} = \vert 0110\rangle,$ $n_{10,2,6} = \vert 0110\rangle + \vert 0011\rangle,\quad n_{10,2,7} = -\vert 1100\rangle + \vert 0110\rangle + \vert 0101\rangle,\quad n_{10,2,8} = \vert 0110\rangle + \vert 0101\rangle,$ $n_{10,2,9} = \vert 0110\rangle - \vert 0101\rangle - \vert 0011\rangle,\quad n_{10,2,10} = -\vert 1100\rangle - \vert 1010\rangle,\quad n_{10,2,11} = -\vert 1010\rangle,$ $n_{10,2,12} = -\vert 1010\rangle + \vert 0011\rangle,\quad n_{10,2,13} = \vert 0011\rangle.$

TABLE 19. The nilpotent elements $n_{i,j,r}$ used in the classification of mixed elements with semisimple part $p' \notin \Sigma$, sorted by Case i and cohomology class $[\gamma_j]$.

i	j	k	semisimple part	nilpotent part
2	2	1	$(-\imath\lambda_3, 0, \lambda_1, -\imath\lambda_2)$	$-\|0110\rangle$
		2	$(\imath\lambda_3, 0, \lambda_1, \imath\lambda_2)$	$-\|0110\rangle$
		3	$(\imath\lambda_3, 0, -\lambda_1, -\imath\lambda_2)$	$-\|0110\rangle$
		4	$(\imath\lambda_3, 0, \lambda_1, -\imath\lambda_2)$	$\|0110\rangle$
		5	$(\imath\lambda_3, 0, \lambda_1, -\imath\lambda_2)$	$-\|0110\rangle$
		6	$(-\imath\lambda_3, 0, \lambda_1, \imath\lambda_2)$	$-\|0110\rangle$
		7	$(-\imath\lambda_3, 0, -\lambda_1, -\imath\lambda_2)$	$-\|0110\rangle$
		8	$(-\imath\lambda_3, 0, \lambda_1, -\imath\lambda_2)$	$\|0110\rangle$
Coefficients: with respect to the basis $\{z_1, z_2, z_3, z_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_1 \in \mathbb{R}^\times$, $\lambda_2, \lambda_3 \in \imath\mathbb{R}^\times$				
2	3	1	$(0, -\lambda_3, -\imath\lambda_2, \lambda_1)$	$\|0000\rangle$
		2	$(0, -\lambda_3, -\imath\lambda_2, -\lambda_1)$	$-\|0000\rangle$
		3	$(0, -\lambda_3, \imath\lambda_2, \lambda_1)$	$-\|0000\rangle$
		4	$(0, \lambda_3, -\imath\lambda_2, \lambda_1)$	$-\|0000\rangle$
		5	$(0, -\lambda_3, -\imath\lambda_2, \lambda_1)$	$-\|0000\rangle$
		6	$(0, -\lambda_3, -\imath\lambda_2, -\lambda_1)$	$\|0000\rangle$
		7	$0, -\lambda_3, \imath\lambda_2, \lambda_1$	$\|0000\rangle$
		8	$(0, \lambda_3, -\imath\lambda_2, \lambda_1)$	$\|0000\rangle$
Coefficients: with respect to the basis $\{y_1, y_2, y_3, y_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_1, \lambda_3 \in \mathbb{R}^\times$, $\lambda_2 \in \imath\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_3\}$				
2	4	1	$(-\imath\lambda_1, -\imath\lambda_2, \lambda_3, 0)$	$\|0011\rangle$
		2	$(\imath\lambda_1, -\imath\lambda_2, \lambda_3, 0)$	$-\|0011\rangle$
		3	$(\imath\lambda_1, -\imath\lambda_2, -\lambda_3, 0)$	$\|0011\rangle$
		4	$(\imath\lambda_1, \imath\lambda_2, \lambda_3, 0)$	$\|0011\rangle$
		5	$(\imath\lambda_1, -\imath\lambda_2, \lambda_3, 0)$	$\|0011\rangle$
		6	$(-\imath\lambda_1, -\imath\lambda_2, \lambda_3, 0)$	$-\|0011\rangle$
		7	$(-\imath\lambda_1, -\imath\lambda_2, -\lambda_3, 0)$	$\|0011\rangle$
		8	$(-\imath\lambda_1, \imath\lambda_2, \lambda_3, 0)$	$\|0011\rangle$
Coefficients: with respect to the basis $\{x_1, x_2, x_3, x_4\}$				
Conditions:$(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_3 \in \mathbb{R}^\times$, $\lambda_1, \lambda_2 \in \imath\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_2\}$				

TABLE 20. **Cases $i = 2$ and $p \notin S$ (Part I):** Mixed real representatives corresponding to γ_j, $[z_k] \in H^1(Z)$, and $n_{2,j,1}$.

i	j	k	semisimple part	nilpotent part
2	5	1	$(0, \imath\lambda_3, -\lambda_2, \lambda_1)$	$\vert 0000\rangle$
		2	$(0, \imath\lambda_3, -\lambda_2, -\lambda_1)$	$-\vert 0000\rangle$
		3	$(0, \imath\lambda_3, \lambda_2, \lambda_1)$	$-\vert 0000\rangle$
		4	$(0, -\imath\lambda_3, -\lambda_2, \lambda_1)$	$-\vert 0000\rangle$
		5	$(0, \imath\lambda_3, -\lambda_2, \lambda_1)$	$-\vert 0000\rangle$
		6	$(0, \imath\lambda_3, -\lambda_2, -\lambda_1)$	$\vert 0000\rangle$
		7	$(0, \imath\lambda_3, \lambda_2, \lambda_1)$	$\vert 0000\rangle$
		8	$(0, -\imath\lambda_3, -\lambda_2, \lambda_1)$	$\vert 0000\rangle$
Coefficients: with respect to the basis $\{x_1, x_2, x_3, x_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_1, \lambda_2 \in \mathbb{R}^\times$, $\lambda_3 \in \imath\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_2\}$				
2	6	1	$(-\imath\lambda_1, \lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
		2	$(\imath\lambda_1, \lambda_2, \imath\lambda_3, 0)$	$-\vert 0011\rangle$
		3	$(\imath\lambda_1, \lambda_2, -\imath\lambda_3, 0)$	$\vert 0011\rangle$
		4	$(\imath\lambda_1, -\lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
		5	$(\imath\lambda_1, \lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
		6	$(-\imath\lambda_1, \lambda_2, \imath\lambda_3, 0)$	$-\vert 0011\rangle$
		7	$(-\imath\lambda_1, \lambda_2, -\imath\lambda_3, 0)$	$\vert 0011\rangle$
		8	$(-\imath\lambda_1, -\lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
Coefficients: with respect to the basis $\{y_1, y_2, y_3, y_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_2 \in \mathbb{R}^\times$, $\lambda_1, \lambda_3 \in \imath\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_3\}$				
2	7	1	$(-\imath\lambda_1, \lambda_2, \lambda_3, 0)$	$\vert 0011\rangle$
		2	$(\imath\lambda_1, \lambda_2, \lambda_3, 0)$	$-\vert 0011\rangle$
		3	$(\imath\lambda_1, \lambda_2, -\lambda_3, 0)$	$\vert 0011\rangle$
		4	$(\imath\lambda_1, -\lambda_2, \lambda_3, 0)$	$\vert 0011\rangle$
		5	$(\imath\lambda_1, \lambda_2, \lambda_3, 0)$	$\vert 0011\rangle$
		6	$(-\imath\lambda_1, \lambda_2, \lambda_3, 0)$	$-\vert 0011\rangle$
		7	$(-\imath\lambda_1, \lambda_2, -\lambda_3, 0)$	$\vert 0011\rangle$
		8	$(-\imath\lambda_1, -\lambda_2, \lambda_3, 0)$	$-\vert 0011\rangle$
Coefficients: with respect to the basis $\{y_1, y_2, y_3, y_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, $\lambda_2, \lambda_3 \in \mathbb{R}^\times$, $\lambda_1 \in \imath\mathbb{R}^\times$, and $\lambda_1 \notin \{\pm\lambda_3\}$				
2	8	1	$(\imath\lambda_1, \imath\lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
		2	$(-\imath\lambda_1, \imath\lambda_2, \imath\lambda_3, 0)$	$-\vert 0011\rangle$
		3	$(-\imath\lambda_1, \imath\lambda_2, -\imath\lambda_3, 0)$	$\vert 0011\rangle$
		4	$(-\imath\lambda_1, -\imath\lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
		5	$(-\imath\lambda_1, \imath\lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
		6	$(\imath\lambda_1, \imath\lambda_2, \imath\lambda_3, 0)$	$-\vert 0011\rangle$
		7	$(\imath\lambda_1, \imath\lambda_2, -\imath\lambda_3, 0)$	$\vert 0011\rangle$
		8	$(\imath\lambda_1, -\imath\lambda_2, \imath\lambda_3, 0)$	$\vert 0011\rangle$
Coefficients: with respect to the basis $\{v_1, v_2, v_3, v_4\}$				
Conditions: $(\lambda_1, \lambda_2, \lambda_3)$ up to Γ_{Π_2}-conjugacy, each $\lambda_i \in \imath\mathbb{R}^\times$ and $\lambda_1 \notin \{\pm\lambda_2 \pm \lambda_3\}$				

TABLE 21. **Cases $i = 2$ and $\notin S$ (Part II):** Mixed real representatives corresponding to γ_j, $[z_k] \in H^1(Z)$, and $n_{2,j,1}$.

i	r	k	**semisimple part**	**nilpotent part**
3	1	1	$(\imath\lambda_1 + \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$\|0011\rangle$
		2	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$-\|0011\rangle$
		3	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, \imath\lambda_2, 0)$	$\|0011\rangle$
		4	$(-\imath\lambda_1 - \imath\lambda_2, \imath\lambda_1, -\imath\lambda_2, 0)$	$\|0011\rangle$
		5	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$\|0011\rangle$
		6	$(\imath\lambda_1 + \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$-\|0011\rangle$
		7	$(\imath\lambda_1 + \imath\lambda_2, -\imath\lambda_1, \imath\lambda_2, 0)$	$\|0011\rangle$
		8	$(\imath\lambda_1 + \imath\lambda_2, \imath\lambda_1, -\imath\lambda_2, 0)$	$\|0011\rangle$
			Coefficients: with respect to the basis $\{v_1, v_2, v_3\}$	
			Conditions: (λ_1, λ_2) up to Γ_{Π_3}-conjugacy, each $\lambda_i \in \imath\mathbb{R}^\times$, $\lambda_1 \neq -\lambda_2$	
3	2	1	$(\imath\lambda_1 + \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$\|1011\rangle - \|0111\rangle - \|0010\rangle - \|0001\rangle$
		2	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$\|1011\rangle - \|0111\rangle + \|0010\rangle + \|0001\rangle$
		3	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, \imath\lambda_2, 0)$	$-\|1011\rangle - \|0111\rangle - \|0010\rangle + \|0001\rangle$
		4	$(-\imath\lambda_1 - \imath\lambda_2, \imath\lambda_1, -\imath\lambda_2, 0)$	$-\|1011\rangle - \|0111\rangle + \|0010\rangle - \|0001\rangle$
		5	$(-\imath\lambda_1 - \imath\lambda_2, \imath\lambda_1, -\imath\lambda_2, 0)$	$\|1011\rangle + \|0111\rangle - \|0010\rangle + \|0001\rangle$
		6	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, \imath\lambda_2, 0)$	$\|1011\rangle + \|0111\rangle + \|0010\rangle - \|0001\rangle$
		7	$(-\imath\lambda_1 - \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$-\|1011\rangle + \|0111\rangle - \|0010\rangle - \|0001\rangle$
		8	$(\imath\lambda_1 + \imath\lambda_2, -\imath\lambda_1, -\imath\lambda_2, 0)$	$-\|1011\rangle + \|0111\rangle + \|0010\rangle + \|0001\rangle$
			Coefficients: with respect to the basis $\{v_1, v_2, v_3\}$	
			Conditions: (λ_1, λ_2) up to Γ_{Π_3}-conjugacy, each $\lambda_i \in \imath\mathbb{R}^\times$, $\lambda_1 \neq -\lambda_2$	

TABLE 22. **Cases $i = 3$ and $p \notin S$**: Mixed real representatives corresponding to γ_2, $[z_k] \in H^1(Z)$, and $n_{3,2,r}$.

i	r	k	**semisimple part**	**nilpotent part**
7	1	1	$(\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1101\rangle - \|1011\rangle - \|1000\rangle + \|0001\rangle$
		2	$(-\imath\lambda, 0, 0, \imath\lambda)$	$\|1101\rangle - \|1011\rangle - \|1000\rangle - \|0001\rangle$
		3	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1101\rangle + \|1011\rangle - \|1000\rangle - \|0001\rangle$
		4	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\|1101\rangle + \|1011\rangle - \|1000\rangle + \|0001\rangle$
		5	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1101\rangle - \|1011\rangle + \|1000\rangle - \|0001\rangle$
		6	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\|1101\rangle - \|1011\rangle + \|1000\rangle + \|0001\rangle$
		7	$(-\imath\lambda, 0, 0, \imath\lambda)$	$-\|1101\rangle + \|1011\rangle + \|1000\rangle + \|0001\rangle$
		8	$(\imath\lambda, 0, 0, -\imath\lambda)$	$\|1101\rangle + \|1011\rangle + \|1000\rangle - \|0001\rangle$
7	2	1		$-\|1101\rangle - \|1010\rangle + \|0001\rangle$
		2		$\|1101\rangle - \|1010\rangle - \|0001\rangle$
		3		$-\|1101\rangle + \|1010\rangle - \|0001\rangle$
		4	same as $r = 1$	$\|1101\rangle - \|1010\rangle + \|0001\rangle$
		5		$-\|1101\rangle - \|1010\rangle - \|0001\rangle$
		6		$\|1101\rangle + \|1010\rangle + \|0001\rangle$
		7		$-\|1101\rangle - \|1010\rangle + \|0001\rangle$
		8		$\|1101\rangle - \|1010\rangle - \|0001\rangle$
7	3	1		$-\|1011\rangle - \|1000\rangle + \|0101\rangle$
		2		$-\|1011\rangle - \|1000\rangle + \|0101\rangle$
		3		$\|1011\rangle - \|1000\rangle - \|0101\rangle$
		4	same as $r = 1$	$\|1011\rangle - \|1000\rangle + \|0101\rangle$
		5		$-\|1011\rangle + \|1000\rangle + \|0101\rangle$
		6		$-\|1011\rangle + \|1000\rangle - \|0101\rangle$
		7		$\|1011\rangle + \|1000\rangle + \|0101\rangle$
		8		$\|1011\rangle + \|1000\rangle + \|0101\rangle$
7	4	1	$(\imath\lambda, 0, 0, -\imath\lambda)$	$\|1011\rangle + \|1000\rangle$
		2	$(-\imath\lambda, 0, 0, \imath\lambda)$	$\|1011\rangle + \|1000\rangle$
		3	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1011\rangle + \|1000\rangle$
		4	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\|1011\rangle - \|1000\rangle$
		5	$(-\imath\lambda, 0, 0, \imath\lambda)$	$\frac{1}{2}(-\|1110\rangle - \|1101\rangle + \|1010\rangle + \|1001\rangle - \|0110\rangle - \|0101\rangle + \|0010\rangle + \|0001\rangle)$
		6	$(\imath\lambda, 0, 0, -\imath\lambda)$	$\frac{1}{2}(-\|1110\rangle - \|1101\rangle + \|1010\rangle + \|1001\rangle - \|0110\rangle - \|0101\rangle + \|0010\rangle + \|0001\rangle)$
		7	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\frac{1}{2}(\|1110\rangle - \|1101\rangle + \|1010\rangle - \|1001\rangle + \|0110\rangle - \|0101\rangle + \|0010\rangle - \|0001\rangle)$
		8	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\frac{1}{2}(-\|1110\rangle + \|1101\rangle - \|1010\rangle + \|1001\rangle - \|0110\rangle + \|0101\rangle - \|0010\rangle + \|0001\rangle)$
7	5	1	$(\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1101\rangle + \|0001\rangle$
		2	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1101\rangle - \|0001\rangle$
		3	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$+\|1101\rangle + \|0001\rangle$
		4	$(-\imath\lambda, 0, 0, \imath\lambda)$	$-\|1101\rangle + \|0001\rangle$
		5	$(\imath\lambda, 0, 0, \imath\lambda)$	$\frac{1}{2}(-\|1011\rangle - \|1010\rangle + \|1001\rangle + \|1000\rangle - \|0111\rangle - \|0110\rangle + \|0101\rangle + \|0100\rangle)$
		6	$(\imath\lambda, 0, 0, -\imath\lambda)$	$\frac{1}{2}(\|1011\rangle - \|1010\rangle - \|1001\rangle + \|1000\rangle - \|0111\rangle + \|0110\rangle + \|0101\rangle - \|0100\rangle)$
		7	$(\imath\lambda, 0, 0, -\imath\lambda)$	$\frac{1}{2}(-\|1011\rangle + \|1010\rangle + \|1001\rangle - \|1000\rangle + \|0111\rangle - \|0110\rangle - \|0101\rangle + \|0100\rangle)$
		8	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\frac{1}{2}(-\|1011\rangle - \|1010\rangle + \|1001\rangle + \|1000\rangle - \|0111\rangle - \|0110\rangle + \|0101\rangle + \|0100\rangle)$
7	6	1	$(\imath\lambda, 0, 0, -\imath\lambda)$	$\|1001\rangle$
		2	$(-\imath\lambda, 0, 0, \imath\lambda)$	$\|1001\rangle$
		3	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$\|1001\rangle$
		4	$(-\imath\lambda, 0, 0, -\imath\lambda)$	$-\|1001\rangle$

Coefficients: with respect to the basis $\{v_1, v_4\}$

Conditions: λ up to Γ_{Π_7}-conjugacy, $\lambda \in \imath\mathbb{R}^\times$

TABLE 23. **Cases $i = 7$ and $p \notin S$**: Mixed real representatives corresponding to γ_2, $[z_k] \in H^1(Z)$, and $n_{7,2,r}$.

i	r	k	semisimple part	nilpotent part
10	1	1	$(\imath\lambda, 0, 0, 0)$	$-\lvert 1100\rangle - \lvert 1010\rangle + \lvert 0110\rangle$
	3			$-\lvert 1010\rangle + \lvert 0110\rangle + \lvert 0011\rangle$
	7			$-\lvert 1100\rangle + \lvert 0110\rangle + \lvert 0101\rangle$
	9			$\lvert 0110\rangle - \lvert 0101\rangle - \lvert 0011\rangle$
	1	2	$(-\imath\lambda, 0, 0, 0)$	$\lvert 1100\rangle - \lvert 1010\rangle + \lvert 0110\rangle$
	3			$-\lvert 1010\rangle + \lvert 0110\rangle - \lvert 0011\rangle$
	7			$\lvert 1100\rangle + \lvert 0110\rangle + \lvert 0101\rangle$
	9			$\lvert 0110\rangle - \lvert 0101\rangle + \lvert 0011\rangle$
	1	3	$(-\imath\lambda, 0, 0, 0)$	$-\lvert 1100\rangle + \lvert 1010\rangle + \lvert 0110\rangle$
	3			$\lvert 1010\rangle + \lvert 0110\rangle + \lvert 0011\rangle$
	7			$-\lvert 1100\rangle + \lvert 0110\rangle - \lvert 0101\rangle$
	9			$\lvert 0110\rangle + \lvert 0101\rangle - \lvert 0011\rangle$
	1	4	$(-\imath\lambda, 0, 0, 0)$	$-\lvert 1100\rangle - \lvert 1010\rangle - \lvert 0110\rangle$
	3			$-\lvert 1010\rangle - \lvert 0110\rangle + \lvert 0011\rangle$
	7			$-\lvert 1100\rangle - \lvert 0110\rangle + \lvert 0101\rangle$
	9			$-\lvert 0110\rangle - \lvert 0101\rangle - \lvert 0011\rangle$
	1	5	$(-\imath\lambda, 0, 0, 0)$	$-\lvert 1100\rangle - \lvert 1010\rangle + \lvert 0110\rangle$
	3			$-\lvert 1010\rangle + \lvert 0110\rangle + \lvert 0011\rangle$
	7			$-\lvert 1100\rangle + \lvert 0110\rangle + \lvert 0101\rangle$
	9			$\lvert 0110\rangle - \lvert 0101\rangle - \lvert 0011\rangle$
	1	6	$(\imath\lambda, 0, 0, 0)$	$\lvert 1100\rangle - \lvert 1010\rangle + \lvert 0110\rangle$
	3			$-\lvert 1010\rangle + \lvert 0110\rangle - \lvert 0011\rangle$
	7			$\lvert 1100\rangle + \lvert 0110\rangle + \lvert 0101\rangle$
	9			$\lvert 0110\rangle - \lvert 0101\rangle + \lvert 0011\rangle$
	1	7	$(\imath\lambda, 0, 0, 0)$	$-\lvert 1100\rangle + \lvert 1010\rangle + \lvert 0110\rangle$
	3			$\lvert 1010\rangle + \lvert 0110\rangle + \lvert 0011\rangle$
	7			$-\lvert 1100\rangle + \lvert 0110\rangle - \lvert 0101\rangle$
	9			$\lvert 0110\rangle + \lvert 0101\rangle - \lvert 0011\rangle$
	1	8	$(\imath\lambda, 0, 0, 0)$	$-\lvert 1100\rangle - \lvert 1010\rangle - \lvert 0110\rangle$
	3			$-\lvert 1010\rangle - \lvert 0110\rangle + \lvert 0011\rangle$
	7			$-\lvert 1100\rangle - \lvert 0110\rangle + \lvert 0101\rangle$
	9			$-\lvert 0110\rangle - \lvert 0101\rangle - \lvert 0011\rangle$

Coefficients: with respect to the basis $\{v_1, v_2, v_3, v_4\}$

Conditions: λ up to $\Gamma_{\Pi_{10}}$-conjugacy, $\lambda \in \imath\mathbb{R}^\times$

TABLE 24. **Cases $i = 10$ and $p \notin S$ (Part I):** Mixed real representatives corresponding to γ_2, $[z_k] \in H^1(Z)$, and $n_{10,2,r}$.

i	r	k	semisimple part	nilpotent part
10	$2,4,6,8,10,12$	1	$(\imath\lambda,0,0,0)$	$-\vert 1010\rangle+\vert 0110\rangle$
				$-\vert 1100\rangle+\vert 0110\rangle$
				$\vert 0110\rangle+\vert 0011\rangle$
				$\vert 0110\rangle+\vert 0101\rangle$
				$-\vert 1100\rangle-\vert 1010\rangle$
				$-\vert 1010\rangle+\vert 0011\rangle$
		2	$(-\imath\lambda,0,0,0)$	$-\vert 1010\rangle+\vert 0110\rangle$
				$\vert 1100\rangle+\vert 0110\rangle$
				$\vert 0110\rangle-\vert 0011\rangle$
				$\vert 0110\rangle+\vert 0101\rangle$
				$\vert 1100\rangle-\vert 1010\rangle$
				$-\vert 1010\rangle-\vert 0011\rangle$
		3	$(-\imath\lambda,0,0,0)$	$\vert 1010\rangle+\vert 0110\rangle$
				$-\vert 1100\rangle+\vert 0110\rangle$
				$\vert 0110\rangle+\vert 0011\rangle$
				$\vert 0110\rangle-\vert 0101\rangle$
				$-\vert 1100\rangle+\vert 1010\rangle$
				$\vert 1010\rangle+\vert 0011\rangle$
		4	$(-\imath\lambda,0,0,0)$	$-\vert 1010\rangle-\vert 0110\rangle$
				$-\vert 1100\rangle-\vert 0110\rangle$
				$-\vert 0110\rangle+\vert 0011\rangle$
				$-\vert 0110\rangle+\vert 0101\rangle$
				$-\vert 1100\rangle-\vert 1010\rangle$
				$-\vert 1010\rangle+\vert 0011\rangle$
10	$5,11$	1	$(\imath\lambda,0,0,0)$	$\vert 0110\rangle$
				$-\vert 1010\rangle$
		2	$(-\imath\lambda,0,0,0)$	$\vert 0110\rangle$
				$-\vert 1010\rangle$
10	13	1	$(\imath\lambda,0,0,0)$	$\vert 0011\rangle$
		2	$(-\imath\lambda,0,0,0)$	$\vert 0011\rangle$

Coefficients: with respect to the basis $\{v_1,v_2,v_3,v_4\}$
Conditions: λ up to $\Gamma_{\Pi_{10}}$-conjugacy, $\lambda\in\imath\mathbb{R}^{\times}$

TABLE 25. **Cases $i=10$ and $p\notin S$ (Part II):** Mixed real representatives corresponding to γ_2, $[z_k]\in H^1(Z)$, and $n_{10,2,r}$.

i	j	r	k	**semisimple part**	**nilpotent part**
4	2	1	1	$(\imath\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle + \|0110\rangle$
			2	$(-\imath\lambda_1, 0, 0, -\lambda_4)$	$\|1010\rangle + \|0110\rangle$
			3	$(-\imath\lambda_1, 0, 0, \lambda_4)$	$-\|1010\rangle + \|0110\rangle$
			4	$(-\imath\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle - \|0110\rangle$
			5	$(-\imath\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle + \|0110\rangle$
			6	$(\imath\lambda_1, 0, 0, -\lambda_4)$	$\|1010\rangle + \|0110\rangle$
			7	$(\imath\lambda_1, 0, 0, \lambda_4)$	$-\|1010\rangle + \|0110\rangle$
			8	$(\imath\lambda_1, 0, 0, \lambda_4)$	$\|1010\rangle - \|0110\rangle$
4	2	2	1		$\|0110\rangle + \|0101\rangle$
			2		$\|0110\rangle + \|0101\rangle$
			3		$\|0110\rangle - \|0101\rangle$
			4		$-\|0110\rangle + \|0101\rangle$
			5	same as $r = 1$	$\|0110\rangle + \|0101\rangle$
			6		$\|0110\rangle + \|0101\rangle$
			7		$\|0110\rangle - \|0101\rangle$
			8		$-\|0110\rangle + \|0101\rangle$
4	2	3	1	$(\imath\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle$
			2	$(-\imath\lambda_1, 0, 0, -\lambda_4)$	$\|0110\rangle$
			3	$(-\imath\lambda_1, 0, 0, \lambda_4)$	$\|0110\rangle$
			4	$(-\imath\lambda_1, 0, 0, \lambda_4)$	$-\|0110\rangle$
4	2	4	1		$\|0101\rangle$
			2		$\|0101\rangle$
			3	same as $r = 3$	$-\|0101\rangle$
			4		$\|0101\rangle$

Coefficients: with respect to the basis $\{t_1, t_4\}$

Conditions: (λ_1, λ_4) up to Γ_{Π_4}-conjugacy, $\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \in \imath\mathbb{R}^\times$

TABLE 26. **Cases $i = 4$ and $p \notin S$ (Part I):** Mixed real representatives corresponding to γ_j, $[z_k] \in H^1(Z)$, and $n_{4,j,r}$.

i	j	r	k	**semisimple part**	**nilpotent part**
4	3	1	1	$(-\imath\lambda_1, 0, 0, \imath\lambda_4)$	$\lvert 1010\rangle + \lvert 0110\rangle$
			2	$(\imath\lambda_1, 0, 0, -\imath\lambda_4)$	$\lvert 1010\rangle + \lvert 0110\rangle$
			3	$(\imath\lambda_1, 0, 0, \imath\lambda_4)$	$-\lvert 1010\rangle + \lvert 0110\rangle$
			4	$(\imath\lambda_1, 0, 0, \imath\lambda_4)$	$\lvert 1010\rangle - \lvert 0110\rangle$
			5	$(\imath\lambda_1, 0, 0, \imath\lambda_4)$	$\lvert 1010\rangle + \lvert 0110\rangle$
			6	$(-\imath\lambda_1, 0, 0, -\imath\lambda_4)$	$\lvert 1010\rangle + \lvert 0110\rangle$
			7	$(-\imath\lambda_1, 0, 0, \imath\lambda_4)$	$-\lvert 1010\rangle + \lvert 0110\rangle$
			8	$(-\imath\lambda_1, 0, 0, \imath\lambda_4)$	$\lvert 1010\rangle - \lvert 0110\rangle$
4	3	2	1		$\lvert 0110\rangle + \lvert 0101\rangle$
			2		$\lvert 0110\rangle + \lvert 0101\rangle$
			3		$\lvert 0110\rangle - \lvert 0101\rangle$
			4		$-\lvert 0110\rangle + \lvert 0101\rangle$
			5	same as $r = 1$	$\lvert 0110\rangle + \lvert 0101\rangle$
			6		$\lvert 0110\rangle + \lvert 0101\rangle$
			7		$\lvert 0110\rangle - \lvert 0101\rangle$
			8		$-\lvert 0110\rangle + \lvert 0101\rangle$
4	3	3	1	$(-\imath\lambda_1, 0, 0, \imath\lambda_4)$	$\lvert 0110\rangle$
			2	$(\imath\lambda_1, 0, 0, -\imath\lambda_4)$	$\lvert 0110\rangle$
			3	$(\imath\lambda_1, 0, 0, \imath\lambda_4)$	$\lvert 0110\rangle$
			4	$(\imath\lambda_1, 0, 0, \imath\lambda_4)$	$-\lvert 0110\rangle$
4	3	4	1		$\lvert 0101\rangle$
			2		$\lvert 0101\rangle$
			3	same as $r = 3$	$-\lvert 0101\rangle$
			4		$\lvert 0101\rangle$
Coefficients: with respect to the basis $\{z_1, z_2, z_3, z_4\}$					
Conditions: (λ_1, λ_4) up to Γ_{Π_4}-conjugacy, $\lambda_1, \lambda_2 \in \imath\mathbb{R}^\times$, $\lambda_1 \notin \{\pm\lambda_2\}$					
4	4	1	1	$\frac{1}{2}(-\imath(\lambda_1+\lambda_4), \lambda_1-\lambda_4, -\lambda_1+\lambda_4, -\imath(\lambda_1+\lambda_4))$	$\frac{1}{2}(\lvert 1110\rangle + \lvert 1101\rangle + \lvert 1010\rangle + \lvert 1001\rangle$ $+\lvert 0110\rangle + \lvert 0101\rangle + \lvert 0010\rangle + \lvert 0001\rangle)$
			2	$\frac{1}{2}(\imath(\lambda_1+\lambda_4), \lambda_1-\lambda_4, -\lambda_1+\lambda_4, \imath(\lambda_1+\lambda_4))$	$\frac{1}{2}(\lvert 1110\rangle + \lvert 1101\rangle + \lvert 1010\rangle + \lvert 1001\rangle$ $+\lvert 0110\rangle + \lvert 0101\rangle + \lvert 0010\rangle + \lvert 0001\rangle)$
			3	$\frac{1}{2}(\imath(\lambda_1+\lambda_4), \lambda_1-\lambda_4, \lambda_1-\lambda_4, -\imath(\lambda_1+\lambda_4))$	$\frac{1}{2}(\lvert 1110\rangle - \lvert 1101\rangle - \lvert 1010\rangle + \lvert 1001\rangle$ $-\lvert 0110\rangle - \lvert 0101\rangle + \lvert 0010\rangle + \lvert 0001\rangle)$
			4	$\frac{1}{2}(\imath(\lambda_1+\lambda_4), -\lambda_1+\lambda_4, -\lambda_1+\lambda_4, -\imath(\lambda_1+\lambda_4))$	$\frac{1}{2}(-\lvert 1110\rangle + \lvert 1101\rangle + \lvert 1010\rangle - \lvert 1001\rangle$ $-\lvert 0110\rangle + \lvert 0101\rangle + \lvert 0010\rangle - \lvert 0001\rangle)$
Coefficients: with respect to the basis $\{z_1, z_2, z_3, z_4\}$					
Conditions: (λ_1, λ_4) up to Γ_{Π_4}-conjugacy, $\imath(\lambda_1+\lambda_4), \lambda_1-\lambda_4 \in \mathbb{R}^\times$, $\lambda_1 \notin \{\pm\lambda_4\}$					

TABLE 27. **Cases $i = 4$ and $p \notin S$ (Part II):** Mixed real representatives corresponding to γ_j, $[z_k] \in H^1(Z)$, and $n_{4,j,r}$.

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