# POISSON STRUCTURES ON THE CONIFOLD AND LOCAL CALABI-YAU THREEFOLDS 

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#### Abstract

We describe bivector fields and Poisson structures on local Calabi-Yau threefolds which are total spaces of vector bundles on a contractible rational curve. In particular, we calculate all possible holomorphic Poisson structures on the conifold.


## 1. Motivation and Results

We are interested in holomorphic Poisson structures on Calabi-Yau threefolds that contain a contractible rational curve. Here we consider the local situation. Hence, we study Poisson structures on Calabi-Yau threefolds that are the total space of a rank 2 vector bundle on $\mathbb{P}^{1}$. A result of Jiménez [Jim] says that the contraction of a smooth rational curve on a threefold may happen in exactly 3 cases, namely when the normal bundle to such a curve is one of

$$
\mathcal{O}_{\mathbb{P}^{1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-1), \quad \mathcal{O}_{\mathbb{P}^{1}}(-2) \oplus \mathcal{O}_{\mathbb{P}^{1}}(0), \quad \text { or } \quad \mathcal{O}_{\mathbb{P}^{1}}(-3) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1),
$$

although only in the first case it can contract to an isolated singularity. In this work we describe completely the local case, that is, we classify all isomorphism classes of holomorphic Poisson structures on the local Calabi-Yau threefolds

$$
W_{k}:=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}(-k) \oplus \mathcal{O}_{\mathbb{P}^{1}}(k-2)\right), \quad k=1,2,3,
$$

calculate their Poisson cohomology, describe their symplectic foliations and some properties of their moduli.

Polishchuk shows a correspondence between Poisson structures on a scheme $X$ and a blow-up $\tilde{X}$, which applies to the cases we study [Po, Thm. 8.2, 8.4]. Hence, describing Poisson structures on $W_{k}$ is equivalent to describing Poisson structures on the singular threefolds obtained from them by contracting the rational curve to a point.

Poisson structures are parametrized by those elements of $H^{0}\left(W_{k}, \Lambda^{2} T W_{k}\right)$ which are integrable. We briefly recall some basic definitions from Poisson cohomology, for details see [LPV, Ch.4]. Let $(M, \pi)$ be a Poisson Manifold. The graded algebra $\mathfrak{X}^{\bullet}(M)=\Gamma\left(\wedge^{\bullet} T M\right)$ and the degree-1 differential operator $d_{\pi}=[\pi, \cdot]$ define the Poisson Cohomology of $(M, \pi)$. The first cohomology groups have clear geometric meaning:
$\mathrm{H}^{0}(M, \pi)=\operatorname{ker}[\pi, \cdot]=\operatorname{Cas}(\pi)=$ holomorphic functions on $M$ which are constant along symplectic leaves. These are the Casimir functions of $(M, \pi)$.
$\mathrm{H}^{1}(M, \pi)=\frac{\operatorname{Poiss}(\pi)}{\operatorname{Ham}(\pi)}$ is the quotient of Poisson vector fields by Hamiltonian vector fields.
We compute Poisson cohomology groups and use them to distinguish Poisson structures, identifying their degeneracy loci.

The $r^{\text {th }}$ degeneracy locus of a holomorphic Poisson structure $\sigma$ on a complex manifold or algebraic variety $X$ is defined as

$$
D_{2 r}(\sigma):=\{x \in X \mid \operatorname{rank} \sigma(x) \leq 2 r\},
$$

where $\sigma$ is viewed as a map $\mathcal{T}_{X}^{*} \rightarrow \mathcal{T}_{X}$ by contracting a 1 -form with the bivector field $\sigma$. At a given point on a complex threefold a holomorphic Poisson structure has either rank 2, or rank 0 . Therefore, for the threefolds $W_{k}$ we name

$$
D(\sigma):=D_{0}(\sigma)
$$

the degeneracy locus of $\sigma$, hence it consists of points where $\sigma$ has rank 0 .

A nondegenerate holomorphic Poisson structure $\sigma$ is called a holomorphic symplectic structure, given that $\sigma$ determines a nondegenerate closed holomorphic 2-form $\omega$ by setting

$$
\omega\left(X_{f}, X_{g}\right)=\{f, g\}_{\sigma},
$$

where $X_{f}$ denotes the Hamiltonian vector field associated to a function $f$.
Remark 1.1. Each Poisson structure determines a symplectic foliation, whose leaves consist of maximal symplectic submanifolds. In particular, in the case of threefolds, the degeneracy locus $D(\sigma)$ is formed by leaves consisting of a single point each, and all other leaves have complex dimension 2.

Describing holomorphic Poisson structures on the Calabi-Yau threefolds $W_{k}$ can also be regarded as describing their first-order noncommutative deformations. The commutative deformation theory of these threefolds $W_{k}$ and the structure of moduli of vector bundles on them is described in detail in [BGS]. The surfaces $Z_{k}:=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}(-k)\right)$, which were discussed in [BG1, BG2] and [BeG], occur here as useful tools.

Motivated by the definition of moduli space of Poisson structures given in [Pym, Sec. 1.2], namely the quotient $\operatorname{Poiss}(X) / \operatorname{Aut}(X)$, we also describe some isomorphisms among Poisson structures. We note that the space of Poisson structures on a threefold can be seen as a cone over global functions in the following sense:
Proposition 1.2. Let $X$ be a smooth complex threefold and $u$ a Poisson structure on $X$, i.e. an integrable bivector field. Then $f u$ is integrable for all $f \in \mathcal{O}(X)$.

Proof. Since $f u$ is a holomorphic bivector, we only need to check that $f u$ is integrable. This is a local condition. Locally $u$ is the product of an element of $K_{X}^{-1}$ and a holomorphic one-form $w$, and the integrability of $u$ is equivalent to $w \wedge d w=0$ [Pym, Eq. 4]. We have $(f w) \wedge d(f w)=$ $f^{2} w \wedge d w+w \wedge d f \wedge w=0+0$.

Among our local threefolds, the most famous is certainly $W_{1}$, known in the physics literature as the resolved conifold, since it occurs as the crepant resolution of the double point singularity $x y-z w=0$ in $\mathbb{C}^{4}$ known as the conifold. The conifold singularity is extremely popular in string theory because it can be resolved in two different ways, by a 2 -sphere (resolution) or a 3 -sphere (deformation). This leads to what is known as a geometric transition and establishes dualities between distinct theories in physics, such as gauge-gravity and open-closed string duality, see [ BBR ] and references therein. We start with bivector fields on $W_{1}$ :
Lemma (3.1). The space $M_{1}=H^{0}\left(W_{1}, \Lambda^{2} T W_{1}\right)$ parametrizing all holomorphic bivector fields on $W_{1}$ has the following structure as a module over global holomorphic functions:

$$
M_{1}=\left\langle e_{1}, e_{2}, e_{3}, e_{4}\right\rangle /\left\langle z u_{2} e_{1}-z u_{1} e_{2}-u_{2} e_{3}+u_{1} e_{4}\right\rangle
$$

Describing obstructions to integrability, we obtain an explicit description of Poisson structures on $W_{1}$. For $\mathbf{p}=\left(p^{1}, p^{2}, p^{3}, p^{4}\right) \in M_{1}$ we describe a differential operator $B(\mathbf{p})=\mathbf{p}^{t} Q \mathbf{p}$ (Note 3.10).

Theorem (3.2). Every holomorphic Poisson structure on $W_{1}$ has the form $\sum_{i=1}^{4} p^{i} e_{i}$ where $\left(p^{1}, p^{2}, p^{3}, p^{4}\right) \in B^{-1}(0)$.
Theorem (3.12). The Poisson structures $e_{1}, e_{2}, e_{3}, e_{4}$ are all pairwise isomorphic.
Since the generators give isomorphic Poisson structures, it is enough to describe the foliation corresponding to one of them, we choose $e_{2}=\partial_{0} \wedge \partial_{1}$.

Theorem (3.15). The symplectic foliation for $\left(W_{1}, \partial_{0} \wedge \partial_{1}\right)$ is given by:

- $\partial_{0} \wedge \partial_{1}$ has degeneracy locus on the line $\left\{v_{2}=\xi=0\right\}$, where the leaves are 0 -dimensional, consisting of single points, and
- 2-dimensional symplectic leaves cut out on the $U$ chart by $u_{2}$ constant.

We summarize this result in a small table.

## $W_{1}$ Poisson structures

| $\pi$ | degeneracy | Casimir |
| :---: | :---: | :--- |
| $e_{2}$ | $\longrightarrow$ | $f\left(u_{2}\right)$ |

We then see the Poisson structures as determined by surface embeddings.
Theorem (3.14). The 4 principal embeddings of the Poisson surface $\left(Z_{1}, \pi_{0}\right)$ generate all Poisson structures on $W_{1}$.

From the viewpoint of Poisson structures, $W_{2}$ is the best of our local Calabi-Yaus, since it is the only one that admits a nondenegerate Poisson structure, see Lemma 4.3, although this comes as no surprise since $W_{2} \simeq T^{*} \mathbb{P}^{1} \times \mathbb{C}$ is a product of symplectic manifolds.

Lemma (4.1). The space $M_{2}$ of holomorphic bivector fields on $W_{2}$ has the following structure as a module over global holomorphic functions:

$$
M_{2}=\left\langle e_{1}, e_{2}, e_{3}, e_{4}, e_{5}\right\rangle /\left\langle u_{1} e_{3}-z u_{1} e_{1}, u_{2} e_{5}-z u_{2} e_{3}-2 z u_{2} e_{2}\right\rangle
$$

By Lemma 4.2, $e_{1}$ and $e_{5}$ give isomorphic Poisson structures, whereas the others are distinct, giving interesting symplectic foliations, as follows.

Theorem (4.11). The symplectic foliations on $W_{2}$ have 0 -dimensional leaves consisting of single points over each of their corresponding degeneracy loci described in Lemma 4.3, and their generic leaves, which are 2-dimensional, are as follows:

- surfaces of constant $u_{1}$ for $e_{1}$ and $e_{3}$, one of them isomorphic to $\mathbb{P}^{1} \times \mathbb{C}$.
- isomorphic to $\mathbb{C}-\{0\} \times \mathbb{C}$ for $e_{2}$ (contained in the fibers of the projection to $\mathbb{P}^{1}$ ).
- isomorphic to the surface $Z_{2}$ and cut out by $u_{2}=v_{2}$ constant for $e_{4}$.

We depict their degeneracy loci and Casimir functions in the following table.

$$
W_{2} \text { Poisson structures }
$$



We then continue onto the case of $W_{3}$, obtaining:
Lemma (5.1). The space $M_{3}$ of holomorphic bivector fields on $W_{3}$ has the following structure as a module over global holomorphic functions:

$$
M_{3}=\mathbb{C}\left\langle e_{1}, \ldots, e_{13}\right\rangle / R
$$

with the set of relations $R$ is the ideal generated by the expressions

$$
\begin{array}{llll}
u_{1} e_{2}-u_{1} u_{2} e_{1} & z u_{1} e_{12}-u_{1} u_{2} e_{6} & u_{1} e_{4}-z u_{1} e_{3} & u_{1} e_{6}-z u_{1} e_{5}-3 z^{2} u_{1} e_{1} \\
u_{1} e_{10}-u_{1} u_{2} e_{3} & z u_{1} e_{13}-u_{1} u_{2} e_{8} & u_{1} e_{5}-z u_{1} e_{4} & u_{1} e_{9}-z u_{1} e_{8}+z u_{1} e_{2} \\
u_{1} e_{13}-u_{1} u_{2} e_{7} & u_{1} e_{11}-z u_{1} e_{10} & u_{1} e_{8}-z u_{1} e_{7} & u_{1} e_{12}-z u_{1} e_{11}-3 z u_{1} e_{1} .
\end{array}
$$

Subsequently, we describe some features of the symplectic foliations corresponding to the generating Poisson structures on $W_{3}$.

Theorem (5.10). The symplectic foliations on $W_{3}$ have 0 -dimensional leaves consisting of single points over each of their corresponding degeneracy loci described in the proofs of Lemmata 5.7, $5.8,5.9$, and their generic leaves, which are 2-dimensional, are as follows:

- Isomorphic to $\mathbb{C}^{*} \times \mathbb{C}$ for $e_{1}$.
- Isomorphic to $\mathbb{C}^{*} \times \mathbb{C}^{*}$ for $e_{2}$.
- Surfaces of constant $u_{1}$, for $e_{3}, e_{4}, e_{5}, e_{10}, e_{11}$ and $e_{13}$.
- Surfaces of constant $u_{2}$, for $e_{7}$ and $e_{8}$.

Geometrically, these structures can be obtained from surface embeddings.
Theorem (5.5). The embeddings of Poisson surfaces $j_{0}\left(\mathbb{C}^{2}, \pi_{0}\right), j_{1}\left(Z_{3}, \pi_{i}\right)$ with $i=0,1,2$ and $j_{2}\left(Z_{-1}, \pi_{i}\right)$ with $i=0,1,2,3$ generate all Poisson structures on $W_{3}$.

We finish by showing that except for $e_{6}, e_{9}, e_{12}$, the Poisson structures $e_{i}$ are all pairwise non-isomorphic, which can be seen from their degeneracy loci.
$W_{3}$ Poisson structures

| $\pi$ | degeneracy | Casimir |
| :---: | :---: | :---: |
| $e_{1}$ |  | $f(z)$ |
| $e_{2}$ |  | $f(z)$ |
| $e_{3}$ | $\stackrel{ }{\square}$ | $f\left(u_{1}\right)$ |
| $e_{4}$ |  | $f\left(u_{1}\right)$ |
| $e_{5}$ | $\uparrow \cup \uparrow$ | $f\left(u_{1}\right)$ |
| $e_{7}$ | $\bigcup$ <br> $\stackrel{\square}{\longrightarrow}$ | $f\left(u_{2}\right)$ |
| $e_{8}$ | $\cup$ <br> $\stackrel{\bullet}{\longrightarrow} \cup$ | $f\left(u_{2}\right)$ |
| $e_{10}$ | $\cup$ | $f\left(u_{1}\right)$ |
| $e_{11}$ |  | $f\left(u_{1}\right)$ |
| $e_{13}$ |  <br> $\cup$ | $f\left(u_{2}\right)$ |

Remark 1.3. Lemma 3.1 (resp. 4.1, 5.1) shows that the space of holomorphic bivector fields on $W_{1}$ (resp. $W_{2}, W_{3}$ ) is generated as a module over global holomorphic functions by 4 (resp. 5, 13) holomorphic bivectors. Moreover, we find that $\mathbb{C}$-linear combinations of the basis vectors are integrable, and by Proposition 1.2 any multiples thereof by holomorphic functions are integrable, too. In the case of $W_{1}$, Theorem 3.2 describes the space of integrable bivectors as the kernel of an explicit differential operator.

However, we note that general combinations of the basis vectors with global functions as coefficients may not be integrable. For example, on $W_{1}$ the expression $z u_{2} e_{1}+e_{3}$ gives a nonintegrable bivector field, despite the fact that both $e_{1}$ and $e_{3}$ are integrable.

## 2. VECTOR FIELDS ON $W_{k}$

Definition 2.1. For integers $k_{1}$ and $k_{2}$, we set

$$
W_{k_{1}, k_{2}}=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}\left(-k_{1}\right) \oplus \mathcal{O}_{\mathbb{P}^{1}}\left(-k_{2}\right)\right) .
$$

The complex manifold structure can be described by gluing the open sets

$$
U=\mathbb{C}_{\left\{z, u_{1}, u_{2}\right\}}^{3} \quad \text { and } \quad V=\mathbb{C}_{\left\{\xi, v_{1}, v_{2}\right\}}^{3}
$$

by the relation

$$
\begin{equation*}
\left(\xi, v_{1}, v_{2}\right)=\left(z^{-1}, z^{k_{1}} u_{1}, z^{k_{2}} u_{2}\right) \tag{2.2}
\end{equation*}
$$

whenever $z$ and $\xi$ are not equal to 0 . We call (2.2) the canonical coordinates for $W_{k_{1}, k_{2}}$.
Lemma 2.3. The threefold $W_{k_{1}, k_{2}}$ is Calabi-Yau if and only if $k_{1}+k_{2}=2$.
Proof. The canonical bundle is given by the transition $-z^{k_{1}+k_{2}-2}$, so it is trivial if and only if $k_{1}+k_{2}=2$.

Notation 2.4. We denote by $W_{k}$ the Calabi-Yau threefold

$$
W_{k}:=W_{k,-k+2}=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}(-k) \oplus \mathcal{O}_{\mathbb{P}^{1}}(k+2)\right)
$$

Let $U \subset W_{k}$ be our usual chart with coordinates $\left\{z, u_{1}, u_{2}\right\}$. As a module over the ring of functions $H^{0}(U ; \mathcal{O})$, the module of global sections of vector fields over $U, H^{0}(U ; T U)$ is spanned by the coordinate partial derivatives, which we relabel for convenience:

$$
\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x^{0}} \equiv \partial_{0}, \quad \frac{\partial}{\partial u_{1}} \equiv \frac{\partial}{\partial x^{1}} \equiv \partial_{1}, \text { and } \frac{\partial}{\partial u_{2}} \equiv \frac{\partial}{\partial x^{2}} \equiv \partial_{2}
$$

The exterior powers are spanned by the appropriate wedge products:

$$
\begin{aligned}
H^{0}\left(U ; \Lambda^{1} T U\right) & =\left\langle\left\{\partial_{i}\right\}_{i=0}^{2}\right\rangle \\
H^{0}\left(U ; \Lambda^{2} T U\right) & =\left\langle\left\{b_{0} \equiv \partial_{1} \wedge \partial_{2}, \quad b_{1} \equiv \partial_{2} \wedge \partial_{0}, \quad b_{2} \equiv \partial_{0} \wedge \partial_{1}\right\}\right\rangle \\
H^{0}\left(U ; \Lambda^{3} T U\right) & =\left\langle\left\{\partial_{0} \wedge \partial_{1} \wedge \partial_{2}\right\}\right\rangle
\end{aligned}
$$

We are interested in bivectors, i.e. elements of $H^{0}\left(U ; \Lambda^{2} T U\right)$. We write a bivector field as

$$
\begin{equation*}
q=q^{i} b_{i}=\frac{1}{2} q^{i} \varepsilon_{i}^{j k} \partial_{j} \wedge \partial_{k} \tag{2.5}
\end{equation*}
$$

where the coefficients $q^{i}$ are functions on $U$. We are using Einstein summation convention throughout, and we write $f_{, i}$ for $\frac{\partial f}{\partial x^{i}}$.

We collect a few useful identities involving Lie brackets, and some preliminary expressions used to compute the Schouten-Nijenhuis brackets. Let $X, Y$ be vector fields and $f, g$ be functions. Then:

- For the coordinate partial derivatives, the Lie bracket vanishes: $\left[\partial_{j}, \partial_{k}\right]=0$ for all $j, k$.
- $[X, g Y]=X(g) Y+g[X, Y]$, so in particular, $\left[\partial_{j}, g \partial_{k}\right]=\frac{\partial g}{\partial x^{j}} \partial_{k}$ and $\left[f \partial_{j}, \partial_{k}\right]=-\frac{\partial f}{\partial x^{k}} \partial_{j}$.
- The SN-bracket of two bivectors is commutative and results in a degree-3 trivector, i.e. a scalar multiple of $\partial_{0} \wedge \partial_{1} \wedge \partial_{2}$. On basis elements, it is given by:

$$
\begin{align*}
{\left[f \partial_{j} \wedge \partial_{k}, g \partial_{m} \wedge \partial_{n}\right]=} & {\left[f \partial_{j}, g \partial_{m}\right] \wedge \partial_{k} \wedge \partial_{n}-\left[f \partial_{j}, \partial_{n}\right] \wedge \partial_{k} \wedge g \partial_{m} } \\
& -\left[\partial_{k}, g \partial_{m}\right] \wedge f \partial j \wedge \partial_{n}+\left[\partial_{k}, \partial_{n}\right] \wedge f \partial_{j} \wedge g \partial_{m} \\
= & f g_{, j} \partial_{m} \wedge \partial_{k} \wedge \partial_{n}-g f_{, m} \partial_{j} \wedge \partial_{k} \wedge \partial_{n} \\
& +g f_{, n} \partial_{j} \wedge \partial_{k} \wedge \partial_{m}-f g_{, k} \partial_{m} \wedge \partial_{j} \wedge \partial_{n} . \tag{2.6}
\end{align*}
$$

We are now in a position to compute the self-bracket of a general bivector field $q$ :

$$
[q, q]=\left[q^{0} \partial_{1} \wedge \partial_{2}+q^{1} \partial_{2} \wedge \partial_{0}+q^{2} \partial_{0} \wedge \partial_{1}, q^{0} \partial_{1} \wedge \partial_{2}+q^{1} \partial_{2} \wedge \partial_{0}+q^{2} \partial_{0} \wedge \partial_{1}\right]
$$

Consider distributing the sums out of this expression. From the basis expression in Equation (2.6) we see that terms vanish unless the indices in the triple wedge product are pairwise distinct, so that self-brackets of individual summands vanish. Furthermore, commutativity of the SN-bracket on bivectors means that the cross terms group in pairs, so we have:

$$
[q, q]=2 \times\left(\left[q^{0} \partial_{1} \wedge \partial_{2}, q^{1} \partial_{2} \wedge \partial_{0}\right]+\left[q^{1} \partial_{2} \wedge \partial_{0}, q^{2} \partial_{0} \wedge \partial_{1}\right]+\left[q^{2} \partial_{0} \wedge \partial_{1}, q^{0} \partial_{1} \wedge \partial_{2}\right]\right)
$$

Now we apply Equation (2.6) to each term and group the results. Since the four indices $j$, $k, m, n$ are always three distinct numbers and $k=m$, only two terms are non-zero, namely $-g f_{, m} \partial_{j} \wedge \partial_{k} \wedge \partial_{n}-f g_{, k} \partial_{m} \wedge \partial_{j} \wedge \partial_{n}=\left(f g_{, k}-g f_{, k}\right) \partial_{j} \wedge \partial_{k} \wedge \partial_{n}$. We find:

$$
[q, q]=2 \partial_{0} \wedge \partial_{1} \wedge \partial_{2}\left(q^{1} q_{, 0}^{2}-q^{2} q_{, 0}^{1}+q^{2} q_{, 1}^{0}-q^{0} q_{, 1}^{2}+q^{0} q_{, 2}^{1}-q^{1} q_{, 2}^{0}\right)
$$

A bivector field $q$ is called a Poisson bivector if it is integrable, which happens if and only if its SN -self-bracket vanishes, $[q, q]=0$. If $q$ is given in coordinates by Equation (2.5), with $q^{0}, q^{1}, q^{2} \in H^{0}(U ; \mathcal{O})$, then the integrability condition is:

$$
\begin{align*}
0 & =q^{1} \frac{\partial q^{2}}{\partial x^{0}}-q^{2} \frac{\partial q^{1}}{\partial x^{0}}+q^{2} \frac{\partial q^{0}}{\partial x^{1}}-q^{0} \frac{\partial q^{2}}{\partial x^{1}}+q^{0} \frac{\partial q^{1}}{\partial x^{2}}-q^{1} \frac{\partial q^{0}}{\partial x^{2}} \\
& =q^{1} \frac{\partial q^{2}}{\partial z}-q^{2} \frac{\partial q^{1}}{\partial z}+q^{2} \frac{\partial q^{0}}{\partial u_{1}}-q^{0} \frac{\partial q^{2}}{\partial u_{1}}+q^{0} \frac{\partial q^{1}}{\partial u_{2}}-q^{1} \frac{\partial q^{0}}{\partial u_{2}} . \tag{2.7}
\end{align*}
$$

Note that by Proposition 1.2, $\mathcal{O}$-multiples of Poisson bivectors are themselves Poisson, which we can also see directly from the above explicit expressions.

Some Poisson structures on local surfaces will be useful. We summarize a few results.
Remark 2.8 (surfaces). Using canonical coordinate charts $Z_{k}=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}(-k)\right)$ [BG2, Lem. 2.8] calculated all their Poisson structures, obtaining generators as: $(1,-\xi),(z,-1)$ for $k=1 ;(1,-1)$ for $k=2 ;\left(u,-\xi^{2} v\right),(z u,-\xi v),\left(z^{2} u,-v\right)$ for $k \geq 3$, written in the basis $\left(\partial_{z} \wedge \partial_{u}, \partial_{\xi} \wedge \partial_{v}\right)$. We will also use the generators for Poisson structures on $Z_{0}$ which are $\left(1,-\xi^{2}\right),(z,-\xi),\left(z^{2},-1\right)$, and for $Z_{-1}$ which are $\left(1,-\xi^{3}\right),\left(z,-\xi^{2}\right),\left(z^{2},-\xi\right),\left(z^{3}, 1\right)$.

## 3. Poisson structures on $W_{1}$

Let $\imath: U \hookrightarrow W_{1}$ denote the inclusion. We actually demand that the coefficients of $q$ are functions on all of $W_{1}$, i.e. that they should be in the image of $\imath^{*}: R:=H^{0}\left(W_{1} ; \mathcal{O}_{W_{1}}\right) \rightarrow$ $H^{0}\left(U ; \mathcal{O}_{U}\right)$. (We will not distinguish between $R$ and its image over $U$ : we are only working in local coordinates on $U$, but with the understanding that we are describing global objects on $W_{1}$.) In local coordinates on $U, R$ consists of convergent power series in

$$
\left\{1, u_{1}, z u_{1}, u_{2}, z u_{2}\right\} .
$$

This imposes additional conditions on the coefficients $q^{i}$.
Lemma 3.1. The space $M_{1}=H^{0}\left(W_{1}, \Lambda^{2} T W_{1}\right)$ parametrizing all holomorphic bivector fields on $W_{1}$ has the following structure as a module over global holomorphic functions:

$$
M_{1}=\left\langle e_{1}, e_{2}, e_{3}, e_{4}\right\rangle /\left\langle z u_{2} e_{1}-z u_{1} e_{2}-u_{2} e_{3}+u_{1} e_{4}\right\rangle
$$

Proof. Since $M_{1}$ is given by global holomorphic sections of $\Lambda^{2} T W_{1}$, using Čech cohomology, we search for $a, b, c$ holomorphic functions on $U$ such that

$$
\left[\begin{array}{ccc}
z^{2} & -z u_{1} & -z u_{2} \\
0 & z^{-1} & 0 \\
0 & 0 & z^{-1}
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

is holomorphic on $V$. To start with $a=\sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} a_{l i s} z^{l} u_{1}^{i} u_{2}^{s}$ and similar for $b$ and $c$. Direct calculation by formal neighborhoods of $\mathbb{P}^{1} \subset W_{1}$ gives the expression of the sections. It turns out that all generators we need already appear on the second formal neighborhood, where we have:

$$
\begin{aligned}
& {\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]=b_{000}\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]+c_{000}\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]+b_{100}\left[\begin{array}{c}
u_{1} \\
z \\
0
\end{array}\right]+c_{100}\left[\begin{array}{c}
u_{2} \\
0 \\
z
\end{array}\right]+a_{020}\left[\begin{array}{c}
u_{1}^{2} \\
0 \\
0
\end{array}\right]+a_{002}\left[\begin{array}{c}
u_{2}^{2} \\
0 \\
0
\end{array}\right]+a_{011}\left[\begin{array}{c}
u_{1} u_{2} \\
0 \\
0
\end{array}\right]} \\
& \quad+b_{010}\left[\begin{array}{c}
0 \\
u_{1} \\
0
\end{array}\right]+b_{110}\left[\begin{array}{c}
0 \\
z u_{1} \\
0
\end{array}\right]+b_{210}\left[\begin{array}{c}
z u_{1}^{2} \\
z^{2} u_{1} \\
0
\end{array}\right]+b_{001}\left[\begin{array}{c}
0 \\
u_{2} \\
0
\end{array}\right]+b_{101}\left[\begin{array}{c}
0 \\
z u_{2} \\
0
\end{array}\right]+b_{201}\left[\begin{array}{c}
z u_{1} u_{2} \\
z^{2} u_{2} \\
0
\end{array}\right]
\end{aligned}
$$

At this point we have 13 generators of $M_{1}$ as a vector space over $\mathbb{C}$ :

$$
\begin{aligned}
& e_{1}:=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], e_{2}:=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right], e_{3}:=\left[\begin{array}{c}
u_{1} \\
z \\
0
\end{array}\right], e_{4}:=\left[\begin{array}{c}
u_{2} \\
0 \\
z
\end{array}\right], e_{5}:=\left[\begin{array}{c}
u_{1}^{2} \\
0 \\
0
\end{array}\right], e_{6}:=\left[\begin{array}{c}
u_{2}^{2} \\
0 \\
0
\end{array}\right], e_{7}:=\left[\begin{array}{c}
u_{1} u_{2} \\
0 \\
0
\end{array}\right], \\
& e_{8}:=\left[\begin{array}{c}
0 \\
u_{1} \\
0
\end{array}\right], e_{9}:=\left[\begin{array}{c}
0 \\
z u_{1} \\
0
\end{array}\right], e_{10}:=\left[\begin{array}{c}
z u_{1}^{2} \\
z^{2} u_{1} \\
0
\end{array}\right], e_{11}:=\left[\begin{array}{c}
0 \\
u_{2} \\
0
\end{array}\right], e_{12}:=\left[\begin{array}{c}
0 \\
z u_{2} \\
0
\end{array}\right], e_{13}:=\left[\begin{array}{c}
z u_{1} u_{2} \\
z^{2} u_{2} \\
0
\end{array}\right] .
\end{aligned}
$$

These satisfy the set of relations:

$$
\begin{gathered}
z u_{2} e_{1}-z u_{1} e_{2}-u_{2} e_{3}+u_{1} e_{4}=0, \quad u_{2}^{2} e_{5}-u_{1}^{2} e_{6}=0 \\
e_{5}-u_{1} e_{3}+z u_{1} e_{1}=0 \quad e_{6}-u_{2} e_{4}+z u_{2} e_{2}=0 \\
e_{7}-u_{2} e_{3}+z u_{2} e_{1}=0 \quad e_{7}-u_{1} e_{4}+z u_{1} e_{2}=0 \\
u_{2} e_{5}-u_{1} e_{7}=0, \quad u_{1} e_{6}-e_{7} u_{2}=0
\end{gathered}
$$

We then proceed to obtain simpler presentations for $M_{1}$. For instance, clearly the relations on lines 2 and 3 may be used to remove $e_{5}, e_{6}, e_{7}$, simplifying the presentation of $M_{1}$ to a set of 10 generators with 4 relations, and so on.

After a long series of reductions, or else, using a computer algebra, we arrive at a far simpler presentation: $M_{1}=\left\langle e_{1}, e_{2}, e_{3}, e_{4}\right\rangle$ with the single relation.

$$
z u_{2} e_{1}-z u_{1} e_{2}-u_{2} e_{3}+u_{1} e_{4}=0
$$

Theorem 3.2. Every holomorphic Poisson structure on $W_{1}$ has the form $\sum_{i=1}^{4} p^{i} e_{i}$ where $\left(p^{1}, p^{2}, p^{3}, p^{4}\right) \in \operatorname{ker} B$.

Specifically, by Lemma 3.1 global bivectors on $W_{1}$ are generated by four elements over $R$, given on the $U$ chart by

$$
e_{1}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], e_{2}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right], e_{3}=\left[\begin{array}{c}
u_{1} \\
z \\
0
\end{array}\right], e_{4}=\left[\begin{array}{c}
u_{2} \\
0 \\
z
\end{array}\right] .
$$

Now write $p=\sum_{h=1}^{4} p^{h} e_{h}$ for a bivector field $p$ that extends to all of $W_{1}, p \in H^{0}\left(W_{1} ; \Lambda^{2} T W_{1}\right)$, that is,

$$
\begin{aligned}
p & =p^{1} b_{1}+p^{2} b_{2}+p^{3}\left(u_{1} b_{0}+z b_{1}\right)+p^{4}\left(u_{2} b_{0}+z b_{2}\right) \\
& =(\underbrace{u_{1} p^{3}+u_{2} p^{4}}_{q^{0}}) b_{0}+(\underbrace{p^{1}+z p^{3}}_{q^{1}}) b_{1}+(\underbrace{p^{2}+z p^{4}}_{q^{2}}) b_{2}
\end{aligned}
$$

where $p^{h} \in R$ for $h=1,2,3,4$. We consider the integrability condition (2.7) with:

$$
\begin{aligned}
& q^{0}\left(z, u_{1}, u_{2}\right)=u_{1} p^{3}\left(z, u_{1}, u_{2}\right)+u_{2} p^{4}\left(z, u_{1}, u_{2}\right) \\
& q^{1}\left(z, u_{1}, u_{2}\right)=p^{1}\left(z, u_{1}, u_{2}\right)+z p^{3}\left(z, u_{1}, u_{2}\right) \\
& q^{2}\left(z, u_{1}, u_{2}\right)=p^{2}\left(z, u_{1}, u_{2}\right)+z p^{4}\left(z, u_{1}, u_{2}\right)
\end{aligned}
$$

The condition becomes:

$$
\begin{align*}
0= & \left(p^{1}+z p^{3}\right) \frac{\partial\left(p^{2}+z p^{4}\right)}{\partial z}-\left(p^{2}+z p^{4}\right) \frac{\partial\left(p^{1}+z p^{3}\right)}{\partial z} \\
& +\left(p^{2}+z p^{4}\right) \frac{\partial\left(u_{1} p^{3}+u_{2} p^{4}\right)}{\partial u_{1}}-\left(u_{1} p^{3}+u_{2} p^{4}\right) \frac{\partial\left(p^{2}+z p^{4}\right)}{\partial u_{1}} \\
& -\left(p^{1}+z p^{3}\right) \frac{\partial\left(u_{1} p^{3}+u_{2} p^{4}\right)}{\partial u_{2}}+\left(u_{1} p^{3}+u_{2} p^{4}\right) \frac{\partial\left(p^{1}+z p^{3}\right)}{\partial u_{2}} \\
= & \left(p^{1}+z p^{3}\right)\left(p_{, 0}^{2}+p^{4}+z p_{, 0}^{4}\right)-\left(p^{2}+z p^{4}\right)\left(p_{, 0}^{1}+p^{3}+z p_{, 0}^{3}\right) \\
& +\left(p^{2}+z p^{4}\right)\left(p^{3}+u_{1} p_{, 1}^{3}+u_{2} p_{, 1}^{4}\right)-\left(u_{1} p^{3}+u_{2} p^{4}\right)\left(p_{, 1}^{2}+z p_{, 1}^{4}\right) \\
& -\left(p^{1}+z p^{3}\right)\left(p^{4}+u_{2} p_{, 2}^{4}+u_{1} p_{, 2}^{3}\right)+\left(u_{1} p^{3}+u_{2} p^{4}\right)\left(p_{, 2}^{1}+z p_{, 2}^{3}\right) \\
= & p^{1} p_{, 0}^{2}-p^{2} p_{, 0}^{1}+z\left(p^{1} p_{, 0}^{4}-p^{4} p_{, 0}^{1}+p^{3} p_{, 0}^{2}-p^{2} p_{, 0}^{3}\right)+z^{2}\left(p^{3} p_{, 0}^{4}-p^{4} p_{, 0}^{3}\right) \\
& +u_{1}\left(p^{2} p_{,, 1}^{3}-p^{3} p_{, 1}^{2}+p^{3} p_{, 2}^{1}-p^{1} p_{, 2}^{3}\right)+z u_{1}\left(p^{4} p_{, 1}^{3}-p^{3} p_{, 1}^{4}\right) \\
& +u_{2}\left(p^{2} p_{, 1}^{4}-p^{4} p_{, 1}^{2}+p^{4} p_{, 2}^{1}-p^{1} p_{, 2}^{4}\right)+z u_{2}\left(p^{4} p_{, 2}^{3}-p^{3} p_{, 2}^{4}\right) \tag{3.3}
\end{align*}
$$

for $p^{h}$ has $(n+1)^{2}$ terms on the $n^{\text {th }}$ infinitesimal neighbourhood (more precisely: in the kernel of $\mathcal{O}_{\ell^{(n)}} \rightarrow \mathcal{O}_{\ell^{(n-1)}}$, where $n=s+t$.

Note 3.9. The expression in Equation (3.3) is an element of $R$, i.e. a globally holomorphic function. This is to be expected, since $p=\sum_{h=1}^{4} p^{h} e_{h}$ is (the restriction to $U$ of) a global bivector field, and the NS-bracket maps global (multi)vector fields to global (multi)vector fields (being a composition of Lie brackets, which map vector fields to vector fields). We can also verify this in local coordinates: Let $[p, p]=f\left(p^{i}, p_{, j}^{i}\right) \partial_{0} \wedge \partial_{1} \wedge \partial_{2}$, so that $f$ is the right-hand side of Equation (3.3). Note that $\partial_{0} \wedge \partial_{1} \wedge \partial_{2}=\partial_{\tilde{0}} \wedge \partial_{\tilde{1}} \wedge \partial_{\tilde{2}}$ on $U \cap V$ (after all, $W_{1}$ is Calabi-Yau); we show that $f$ is globally holomorphic on $W_{1}$ : If $p^{h} \in R$, then $p_{, 0}^{h}$ and $z p_{, 0}^{h}$ are in $R$, too, as is clear from considering (3.6). Terms $u_{1} p_{, 1}^{h}$ and $u_{2} p_{, 1}^{h}$ are holomorphic, as can be seen from (3.7), and similarly for $u_{1} p_{, 2}^{h}$ and $u_{2} p_{, 2}^{h}$. The remaining terms are not individually globally holomorphic, but they group as follows:

$$
p^{3}\left(z^{2} p_{, 0}^{4}-z u_{1} p_{, 1}^{4}-z u_{2} p_{, 2}^{4}\right)-p^{4}\left(z^{2} p_{, 0}^{3}-z u_{1} p_{, 1}^{3}-z u_{2} p_{, 2}^{3}\right)
$$

By considering (3.6), (3.7), and (3.8), we see that the only non-holomorphic terms are $z^{s+t+1} u_{1}^{s} u_{2}^{t}$, and those appear with coefficient $((s+t)-s-t)\left(p_{s+t, s, t}^{4}-p_{s+t, s, t}^{3}\right)=0$.

Note 3.10. The quasi-linear differential operator $B$ defined above can be written as follows:

$$
\begin{aligned}
& B\left(p^{1}, p^{2}, p^{3}, p^{4}\right)=\mathbf{p}^{T} Q \mathbf{p}= \\
& {\left[p^{1} p^{2} p^{3} p^{4}\right]\left[\begin{array}{cccc}
0 & \partial_{0} & -u_{1} \partial_{2} & z \partial_{0}-u_{2} \partial_{2} \\
-\partial_{0} & 0 & -z \partial_{0}+u_{1} \partial_{1} & u_{2} \partial_{1} \\
u_{1} \partial_{2} & z \partial_{0}-u_{1} \partial_{1} & 0 & z^{2} \partial_{0}-z u_{1} \partial_{1}-z u_{2} \partial_{2} \\
-z \partial_{0}+u_{2} \partial_{2} & -u_{2} \partial_{1} & -z^{2} \partial_{0}+z u_{1} \partial_{1}+z u_{2} \partial_{2} & 0
\end{array}\right]\left[\begin{array}{l}
p^{1} \\
p^{2} \\
p^{3} \\
p^{4}
\end{array}\right]}
\end{aligned}
$$

where have expressed $f$ using the quadratic form $Q$. We may linearize this differential equation around a fixed solution $\mathbf{p}^{T}=\left[p^{1} p^{2} p^{3} p^{4}\right]$ :

$$
\lim _{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}(f(\mathbf{p}+\varepsilon \Delta \mathbf{p})-f(\mathbf{p}))=\Delta \mathbf{p}^{T} Q \mathbf{p}+\mathbf{p}^{T} Q \Delta \mathbf{p}
$$

3.1. Symmetries and embeddings. We now give isomorphism between the Poisson structures on $W_{1}$.

Remark 3.11. Note that there are two clear symmetries of $W_{1}$ :

- exchanging the radial directions $s_{0}\left(z, u_{1}, u_{2}\right)=\left(z, u_{2}, u_{1}\right)$ and
- exchanging the charts $U$ and $V$, that is, $s_{1}\left(z, u_{1}, u_{2}\right)=\left(\xi, v_{1}, v_{2}\right)$.

These symmetries are automorphisms of $W_{1}$ and are also Poisson isomorphisms between some structures on $W_{1}$ as shown in the diagram below:


In other words, we have that $e_{1}=s_{0}^{*}\left(e_{2}\right), e_{4}=s_{1}^{*}\left(e_{2}\right)$, and $e_{4}=s_{1}^{*} s_{0}^{*} e_{2}$.
Theorem 3.12. The Poisson structures $e_{1}, e_{2}, e_{3}, e_{4}$ are all pairwise isomorphic.
There are 2 obvious inclusions of the surface $Z_{1}$ into the threefold $W_{1}$.
Notation 3.13. We denote by $\pi_{i}$ the Poisson structure on $Z_{k}$ that is given on the $U$-chart by $z^{i} u^{\varepsilon}$ where $\varepsilon=0$ if $i \leq 2$ and $\varepsilon=1$ if $i \geq 3$. We denote by $j_{1}: Z_{1} \rightarrow W_{1}$ (resp. $j_{2}$ ) the inclusion into the first (resp. second) fiber coordinate, that is, on the $U$-chart $j_{1}(z, u)=(z, u, 0)$ (resp. $\left.j_{2}(z, u)=(z, 0, u)\right)$. We call $j_{1}, s_{0} j_{1}, s_{1} j_{1}, s_{1} s_{0} j_{1}$ the principal embeddings of $Z_{1}$ into $W_{1}$.

Theorem 3.14. The 4 principal embeddings of the Poisson surface $\left(Z_{1}, \pi_{0}\right)$ generate all Poisson structures on $W_{1}$.

Proof. Let $j_{1}\left(Z_{1}\right)$ (resp. $j_{2}\left(Z_{1}\right)$ ) be the embedding of the surface $Z_{1}$ into $W_{1}$ by $u_{2}=0$ and $v_{2}=0$ (resp. $u_{1}=0$ and $v_{1}=0$ ). Then Poisson structures induced by the first embedding are:

$$
\left(j_{1}\right)_{*}(1)_{U}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]_{U}, \quad\left(j_{1}\right)_{*}(-\xi)_{V}=\left[\begin{array}{c}
0 \\
0 \\
-\xi
\end{array}\right]_{V}, \quad \text { hence }\left.\quad e_{2}\right|_{j_{1}\left(Z_{1}\right)}=\left(j_{1}\right)_{*} \pi_{0},
$$

analogously, $\left.\gamma_{1}\right|_{j_{1}\left(Z_{1}\right)}=j_{1}\left(s \pi_{0}\right)$. The induced Poisson structures by the second embedding are

$$
\left(j_{2}\right)_{*}(1)_{U}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]_{U}, \quad\left(j_{2}\right)_{*}(-\xi)_{V}=\left[\begin{array}{c}
0 \\
-\xi \\
0
\end{array}\right]_{V}, \quad \text { hence }\left.\quad e_{3}\right|_{j_{2}\left(Z_{1}\right)}=\left(j_{2}\right)_{*} \pi_{0},
$$

analogously $\left.e_{1}\right|_{j_{2}\left(Z_{1}\right)}=\left(j_{2}\right)_{*}\left(s \pi_{0}\right)$.
3.2. Symplectic foliations on $\boldsymbol{W}_{\mathbf{1}}$. Since $e_{1}, e_{2}, e_{3}, e_{4}$ are all isomorphic, to understand their corresponding symplectic foliations, it is enough to describe the symplectic foliation in one case. We consider $e_{2}$ whose expression in canonical coordinates is $\left.\{f, g\}_{e_{2}}=(d f \wedge d g)\right\lrcorner\left(\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u_{1}}\right)=$ $(d f \wedge d g)\lrcorner\left(\partial_{0} \wedge \partial_{1}\right)$.
Theorem 3.15. The symplectic foliation for $\left(W_{1}, \partial_{0} \wedge \partial_{1}\right)$ is given by:

- $\partial_{0} \wedge \partial_{1}$ has degeneracy locus on the line $\left\{v_{2}=\xi=0\right\}$, where the leaves are 0-dimensional, consisting of single points, and
- 2-dimensional symplectic leaves cut out on the $U$ chart by $u_{2}$ constant.

Proof. To find the symplectic leaves we compute Poisson cohomology $H^{0}\left(W_{1}, e_{2}\right)$, obtaining that $e_{2}=\partial_{0} \wedge \partial_{1}$ has 2 dimensional symplectic leaves cut out on the $U$ chart by $u_{2}$ constant (the Casimir functions), and next changing coordinates

$$
\left[\begin{array}{ccc}
z^{2} & -z u_{1} & -z u_{2} \\
0 & z^{-1} & 0 \\
0 & 0 & z^{-1}
\end{array}\right]\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]=\left[\begin{array}{c}
-z u_{2} \\
0 \\
z^{-1}
\end{array}\right]=\left[\begin{array}{c}
-v_{2} \\
0 \\
\xi
\end{array}\right]
$$

so, we see that the expression of $e_{2}$ on the $V$-coordinate is $-v_{2} \frac{\partial}{\partial v_{1}} \wedge \frac{\partial}{\partial v_{2}}+\xi \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial v_{1}}$, which vanishes when $\xi=v_{2}=0$. Hence $e_{2}$ has degeneracy locus on the line $D\left(e_{2}\right)=\left\{v_{2}=\xi=0\right\}$, where the leaves are 0 dimensional, consisting of each of the points in the line $\xi=v_{2}=0$.

## 4. Poisson structures on $W_{2}$

The Calabi-Yau threefold we consider in this section is

$$
W_{2}:=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}(-2) \oplus \mathcal{O}_{\mathbb{P}^{1}}\right)=Z_{2} \otimes \mathbb{C} .
$$

We will carry out calculations using the canonical coordinates $W_{1}=U \cup V$ where $U \simeq \mathbb{C}^{3} \simeq V$ with coordinates $U=\left\{z, u_{1}, u_{2}\right\}, V=\left\{\xi, v_{1}, v_{2}\right\}$, and change of coordinates on $U \cap V \simeq$ $\mathbb{C}^{*} \times \mathbb{C} \times \mathbb{C}$ given by

$$
\left\{\xi=z^{-1}, \quad v_{1}=z^{2} u_{1}, \quad v_{2}=u_{2}\right\},
$$

so that $z=\xi^{-1}, u_{1}=\xi^{2} v_{1}$, and $u_{2}=v_{2}$.
The transition matrix for the tangent bundle is the Jacobian matrix of the change of coordinates, and taking the second exterior power we obtain the transition matrix for $\Lambda^{2} T W_{2}$ :

$$
\left[\begin{array}{ccc}
z^{2} & -2 z u_{1} & 0 \\
0 & -z^{-2} & 0 \\
0 & 0 & -1
\end{array}\right] .
$$

Let $\imath: U \hookrightarrow W_{2}$ denote the inclusion. We actually demand that the coefficients of $q$ are functions on all of $W_{2}$, i.e. that they should be in the image of $\imath^{*}: R:=H^{0}\left(W_{2} ; \mathcal{O}_{W_{2}}\right) \rightarrow$ $H^{0}\left(U ; \mathcal{O}_{U}\right)$. (We will not distinguish between $R$ and its image over $U$ : we are only working in local coordinates on $U$, but with the understanding that we are describing global objects on $W_{2}$.) In local coordinates on $U, R$ consists of convergent power series in

$$
\left\{1, u_{1}, z u_{1}, z^{2} u_{1}, u_{2}\right\} .
$$

Now write $p=\sum_{h=1}^{5} p^{h} e_{h}$ for a bivector field $p$ that extends to all of $W_{2}, p \in H^{0}\left(W_{2} ; \Lambda^{2} T W_{2}\right)$.
Lemma 4.1. The space $M_{2}=H^{0}\left(W_{2}, \Lambda^{2} T W_{2}\right)$ parametrizing all holomorphic bivector fields on $W_{2}$ has the following structure as a module over global holomorphic functions:

$$
M_{2}=\left\langle e_{1}, e_{2}, e_{3}, e_{4}, e_{5}\right\rangle /\left\langle u_{1} e_{3}-z u_{1} e_{1}, u_{2} e_{5}-z u_{2} e_{3}-2 z u_{2} e_{2}\right\rangle .
$$

Proof. To find $H^{0}\left(W_{2}, \Lambda^{2} T W_{2}\right)$ we need global holomorphic sections, that is, we must find $a, b, c$ holomorphic on $U$ such that

$$
\left[\begin{array}{ccc}
z^{2} & -2 z u_{1} & 0 \\
0 & -z^{-2} & 0 \\
0 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

is holomorphic on $V$. To start with $a=\sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} a_{l i s} z^{l} u_{1}^{i} u_{2}^{s}$ and similar for $b$ and $c$. We proceed by calculations on formal neighborhoods of the $\mathbb{P}^{1} \subset W_{2}$ and verify that generators for all global sections are already found on the first formal neighborhood, where the general expression of a section of $\Lambda^{2} T W_{2}$ is:

$$
\begin{aligned}
& {\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]=c_{000}\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]+c_{010}\left[\begin{array}{c}
0 \\
0 \\
u_{1}
\end{array}\right]+c_{110}\left[\begin{array}{c}
0 \\
0 \\
z u_{1}
\end{array}\right]+c_{210}\left[\begin{array}{c}
0 \\
0 \\
z^{2} u_{1}
\end{array}\right]+c_{001}\left[\begin{array}{c}
0 \\
0 \\
u_{2}
\end{array}\right]+a_{010}\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right] } \\
&+b_{000}\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]+b_{100}\left[\begin{array}{l}
0 \\
z \\
0
\end{array}\right]+b_{200}\left[\begin{array}{c}
2 z u_{1} \\
z^{2} \\
0
\end{array}\right]+b_{010}\left[\begin{array}{c}
0 \\
u_{1} \\
0
\end{array}\right]+b_{110}\left[\begin{array}{c}
0 \\
z u_{1} \\
0
\end{array}\right]+b_{210}\left[\begin{array}{c}
0 \\
z^{2} u_{1} \\
0
\end{array}\right]+b_{310}\left[\begin{array}{c}
0 \\
z^{3} u_{1} \\
0
\end{array}\right] .
\end{aligned}
$$

We then need the structure of $M=H^{0}\left(W_{2}, T W_{2}\right)$ as a module over global functions. At first this gives us potentially 13 generators, but since $u_{1}, z u_{1}, z^{2} u_{1}, u_{2}$ are global functions, we obtain that in fact all sections can be obtained from the smaller set of generators:

$$
e_{1}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], e_{2}=\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right], e_{3}=\left[\begin{array}{l}
0 \\
z \\
0
\end{array}\right], e_{4}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right], e_{5}=\left[\begin{array}{c}
2 z u_{1} \\
z^{2} \\
0
\end{array}\right] .
$$

To describe the module structure over global sections, we write relations among the generators. We have the equations:

$$
\begin{gathered}
e_{3}-z e_{1}=0 \\
e_{5}-z e_{3}-2 z e_{2}=0
\end{gathered}
$$

Note that there are no equations involving $e_{4}$. This corresponds to the fact that the geometry of $W_{2}=Z_{2} \times \mathbb{C}$ is that of a surface product $\mathbb{C}$. Accordingly, we shall not involve $u_{2}$ in the relations to be obtained from these equations. To get relations as an $\mathcal{O}\left(W_{2}\right)$-module, we multiply the equations by $u_{1}$, obtaining:

$$
\begin{gathered}
u_{1} e_{3}-z u_{1} e_{1}=0 \\
u_{2} e_{5}-z u_{3} e_{4}-2 z u_{2} e_{2}=0 .
\end{gathered}
$$

Next we discuss which of these bivector fields give isomorphic Poisson structures.
Lemma 4.2. The Poisson manifolds $\left(W_{2}, e_{1}\right)$ and ( $W_{2}, e_{5}$ ) are isomorphic.
Proof. Note that by writing $e_{1}$ in $V$-coordinates we get:

$$
\left[\begin{array}{ccc}
z^{2} & -2 z u_{1} & 0 \\
0 & -z^{-2} & 0 \\
0 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]=\left[\begin{array}{c}
-2 z u_{1} \\
-z^{-2} \\
0
\end{array}\right]=\left[\begin{array}{c}
-2 \xi v_{1} \\
-\xi^{2} \\
0
\end{array}\right]
$$

So we get an isomorphism between $\left(W_{2}, e_{1}\right)$ and $\left(W_{2},-e_{5}\right)$ by mapping the $U$-chart of one to the $V$-chart of the other and vice-versa. Then the desired isomorphism follows from the fact that ( $W_{2}, e_{5}$ ) and ( $W_{2},-e_{5}$ ) are isomorphic.

We now describe the loci where Poisson structures on $W_{2}$ degenerate.

Lemma 4.3. The degeneracy loci of Poisson structures on $W_{2}$ are:

- isomorphic to $\mathbb{C}^{2}$ for $e_{1}$, and
- isomorphic to $\mathbb{P}^{1} \times \mathbb{C}$ for $e_{2}$,
- isomorphic to $\mathbb{C}^{2} \cup \mathbb{C}$ for $e_{3}$, and
- empty for $e_{4}$.

Proof. The coefficients of the Poisson structures in coordinate charts are:

$$
e_{1}:=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-2 \xi v_{1} \\
-\xi^{2} \\
0
\end{array}\right]_{V} e_{2}:=\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
v_{1} \\
0 \\
0
\end{array}\right]_{V} e_{3}:=\left[\begin{array}{l}
0 \\
z \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-2 v_{1} \\
\xi \\
0
\end{array}\right]_{V} e_{4}:=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]_{U}=\left[\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right]_{V} .
$$

On the $V$ chart we have that $e_{1}$ degenerates when $\xi=0$ which is copy of $\mathbb{C}^{2}$. Therefore, we have that $e_{2}$ degenerates when $u_{1}=v_{1}=0$ which gives a trivial product $\mathbb{P}^{1} \times \mathbb{C}$. On the $U$ chart we have that $e_{3}$ degenerates when $z=0$ which is a copy of $\mathbb{C}^{2}$, on the $V$ chart we have that $e_{3}$ degenerates when $\xi=v_{1}=0$ which is copy of $\mathbb{C}$. For $e_{4}$ the degeneracy locus is empty.
Corollary 4.4. The brackets $e_{1}, e_{2}, e_{3}, e_{4}$ give $W_{2}$ non-isomorphic Poisson structures.
There are natural inclusions of the surfaces $\mathbb{C}, Z_{0}$, and $Z_{2}$ into $W_{2}$ :
Notation 4.5. We denote by $j_{s}$ for $s=0,1,2$ the inclusions of $\mathbb{C}^{2}, Z_{0}$, and $Z_{2}$ into the threefold $W_{2}$. Hence, in coordinates we have:

- $j_{0}: \mathbb{C}^{2} \rightarrow W_{2}$ includes $\mathbb{C}^{2}$ as the fiber $z=\xi=1$,
- $j_{1}: Z_{2} \rightarrow W_{2}$ includes $Z_{2}$ as the surface $u_{2}=v_{2}=0$,
- $j_{2}: Z_{0} \rightarrow W_{2}$ includes $Z_{0}$ as the surface $u_{1}=\xi^{2} v_{1}=0$.

Theorem 4.6. The embedded Poisson surfaces $j_{0}\left(\mathbb{C}^{2}, \pi_{0}\right), j_{1}\left(Z_{2}, \pi_{0}\right)$ and $j_{2}\left(Z_{0}, \pi_{i}\right)$ with $i=$ $0,1,2$, generate all Poisson structures on $W_{2}$.

Proof. Let $j_{1}\left(Z_{2}\right)$ (resp. $j_{2}\left(Z_{0}\right)$ ) be the embedding of the surface $Z_{2}$ (resp. $Z_{0}$ ) into $W_{2}$ cut out by $u_{2}=v_{2}=0$ (resp. $u_{1}=\xi^{2} v_{1}=0$ ). Then Poisson structure induced by the first embedding is:

$$
\left(j_{1}\right)_{*}(1)_{U}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]_{U}, \quad\left(j_{1}\right)_{*}(-\xi)_{V}=\left[\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right]_{V}, \quad \text { hence }\left.\quad e_{4}\right|_{j_{1}\left(Z_{2}\right)}=\left(j_{1}\right)_{*} \pi_{0} .
$$

The Poisson structures induced by the second embedding are

$$
\left(j_{2}\right)_{*}(1)_{U}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]_{U}, \quad\left(j_{2}\right)_{*}\left(-\xi^{2}\right)_{V}=\left[\begin{array}{c}
0 \\
-\xi^{2} \\
0
\end{array}\right]_{V}, \quad \text { hence }\left.\quad e_{1}\right|_{j_{2}\left(Z_{0}\right)}=\left(j_{2}\right)_{*} \pi_{0},
$$

and analogously $\left.e_{2}\right|_{j_{2}\left(Z_{0}\right)}=\left(j_{2}\right)_{*}\left(s \pi_{1}\right)$. Since $j_{0}$ has image at $u=\xi=1$, we obtain

$$
\left(j_{0}\right)_{*}(1)_{U}=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]_{U}, \quad\left(j_{0}\right)_{*}(1)_{V}=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]_{V} \Leftrightarrow\left(j_{0}\right)_{*}(z)_{U}=\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right]_{V}, \quad\left(j_{0}\right)_{*}(\xi)_{V}=\left[\begin{array}{c}
v_{1} \\
0 \\
0
\end{array}\right]_{V},
$$

hence $\left.e_{3}\right|_{j_{0}\left(Z_{0}\right)}=\left(j_{0}\right)_{*} \pi_{0}$.
4.1. Symplectic foliations on $\boldsymbol{W}_{\mathbf{2}}$. In this section we perform the cohomological calculations, and identify the leaves of the symplectic foliation associated to each Poisson structure on $W_{2}$.
Lemma 4.7. $H^{0}\left(W_{2}, e_{1}\right)=\left\{f \in \mathcal{O}\left(W_{2}\right) / f=f\left(u_{1}\right)\right\}$
Proof. Recall that $e_{1}=-\partial_{0} \wedge \partial_{2}$. Then we have

$$
\left[f, e_{1}\right]=-\left[f, \partial_{0} \wedge \partial_{2}\right]=-\left[f, \partial_{0}\right] \wedge \partial_{2}+\partial_{0} \wedge\left[f, \partial_{2}\right]=\frac{\partial f}{\partial u_{2}} \partial_{0}-\frac{\partial f}{\partial z} \partial_{2},
$$

so that $f \in \operatorname{ker}\left[e_{1}, \cdot\right]$ if and only if $\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u_{2}}=0$, i.e., $f$ does not depend on $z$ and $u_{2}$.
Lemma 4.8. $H^{0}\left(W_{2}, e_{2}\right)=\left\{f \in \mathcal{O}\left(W_{2}\right) / f=f(z)\right\}$

Proof. Recall that $e_{2}=u_{1} \partial_{1} \wedge \partial_{2}$. Then we have

$$
\left[f, e_{2}\right]=\left[f, u_{1} \partial_{1} \wedge \partial_{2}\right]=\left[f, u_{1} \partial_{1}\right] \wedge \partial_{2}-u_{1} \partial_{1} \wedge\left[f, \partial_{2}\right]=u_{1} \frac{\partial f}{\partial u_{1}} \partial_{2}-u_{1} \frac{\partial f}{\partial u_{2}} \partial_{1},=
$$

so that $f \in \operatorname{ker}\left[e_{3}, \cdot\right]$ if and only if $\frac{\partial f}{\partial u_{1}}=\frac{\partial f}{\partial u_{2}}=0$, i.e., $f$ does not depend on $u_{1}$ and $u_{2}$.
Lemma 4.9. $H^{0}\left(W_{2}, e_{3}\right)=\left\{f \in \mathcal{O}\left(W_{2}\right) / f=f\left(u_{1}\right)\right\}$
Proof. Recall that $e_{3}=-z \partial_{0} \wedge \partial_{2}$. Then we have

$$
\left[f, e_{3}\right]=-\left[f,-z \partial_{0} \wedge \partial_{2}\right]=-\left[f, z \partial_{0}\right] \wedge \partial_{2}+z \partial_{0} \wedge\left[f, \partial_{2}\right]=z \frac{\partial f}{\partial u_{2}} \partial_{0}-z \frac{\partial f}{\partial z} \partial_{2}
$$

so that $f \in \operatorname{ker}\left[e_{3}, \cdot\right]$ if and only if $\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u_{2}}=0$, i.e., $f$ does not depend on $z$ and $u_{2}$.
Lemma 4.10. $H^{0}\left(W_{2}, e_{4}\right)=\left\{f \in \mathcal{O}\left(W_{2}\right) / f=f\left(u_{2}\right)\right\}$.
Proof. Recall that $e_{4}=\partial_{0} \wedge \partial_{1}$. Then we have

$$
\left[f, e_{4}\right]=\left[f, \partial_{0} \wedge \partial_{1}\right]=\left[f, \partial_{0}\right] \wedge \partial_{1}-\partial_{0} \wedge\left[f, \partial_{1}\right]=\frac{\partial f}{\partial z} \partial_{1}-\frac{\partial f}{\partial u_{1}} \partial_{0}
$$

so that $f \in \operatorname{ker}\left[e_{4}, \cdot\right]$ if and only if $\frac{\partial f}{\partial z}=\frac{\partial f}{\partial u_{1}}=0$, i.e., $f$ does not depend on $z$ and $u_{1}$.
We then obtain the description of the symplectic foliations on $W_{2}$ determined by these Poisson structures.

Theorem 4.11. The symplectic foliations on $W_{2}$ have 0 -dimensional leaves consisting of single points over each of their corresponding degeneracy loci described in Lemma 4.3, and their generic leaves, which are 2-dimensional, are as follows:

- surfaces of constant $u_{1}$ for $e_{1}$ and $e_{3}$, one of them isomorphic to $\mathbb{P}^{1} \times \mathbb{C}$.
- isomorphic to $\mathbb{C}-\{0\} \times \mathbb{C}$ for $e_{2}$ (contained in the fibers of the projection to $\mathbb{P}^{1}$ ).
- isomorphic to the surface $Z_{2}$ and cut out by $u_{2}=v_{2}$ constant for $e_{4}$.


## 5. Poisson structures on $W_{3}$

The Calabi-Yau threefold we consider in this section is $W_{3}:=\operatorname{Tot}\left(\mathcal{O}_{\mathbb{P}^{1}}(-3) \oplus \mathcal{O}_{\mathbb{P}^{1}}(1)\right)$. We will carry out calculations using the canonical coordinates $W_{1}=U \cup V$ with $U \simeq \mathbb{C}^{3} \simeq V$ with coordinates $U=\left\{z, u_{1}, u_{2}\right\}$ and $V=\left\{\xi, v_{1}, v_{2}\right\}$ with change of coordinates on $U \cap V \simeq \mathbb{C}^{*} \times \mathbb{C} \times \mathbb{C}$ given by

$$
\left\{\xi=z^{-1}, \quad v_{1}=z^{3} u_{1}, \quad v_{2}=z^{-1} u_{2}\right\}
$$

so that $z=\xi^{-1}, u_{1}=\xi^{3} v_{1}$, and $u_{2}=\xi^{-1} v_{2}$.
In these coordinates, the transition matrix for the tangent bundle of $W_{3}$ is the Jacobian matrix of the change of coordinates, and taking $\Lambda^{2}$ we obtain the transition matrix for the second exterior power of the tangent bundle:

$$
\left[\begin{array}{ccc}
z^{2} & -3 z u_{1} & z u_{2} \\
0 & -z^{-3} & 0 \\
0 & 0 & -z
\end{array}\right] .
$$

Let $\imath: U \hookrightarrow W_{3}$ denote the inclusion. We actually demand that the coefficients of $q$ are functions on all of $W_{3}$, i.e. that they should be in the image of $\imath^{*}: R:=H^{0}\left(W_{3} ; \mathcal{O}_{W_{3}}\right) \rightarrow$ $H^{0}\left(U ; \mathcal{O}_{U}\right)$. (We will not distinguish between $R$ and its image over $U$ : we are only working in local coordinates on $U$, but with the understanding that we are describing global objects on $W_{2}$.) In local coordinates on $U, R$ consists of convergent power series in

$$
\left\{1, u_{1}, z u_{1}, z^{2} u_{1}, z^{3} u_{1}, u_{1} u_{2}, z u_{1} u_{2}, z^{2} u_{1} u_{2}\right\} .
$$

Holomorphic Poisson structures on $W_{3}$ are parametrized by elements of $M_{3}:=H^{0}\left(W_{2}, \Lambda^{2} T W_{3}\right)$, which is infinite dimensional as a vector space over $\mathbb{C}$. We will describe the structure of $M_{3}$ as a module over global functions.
Lemma 5.1. The space $M_{3}=H^{0}\left(W_{3}, \Lambda^{2} T W_{3}\right)$ parametrizing all holomorphic bivector fields on $W_{3}$ has the following structure as a module over global holomorphic functions:

$$
M_{3}=\mathbb{C}<e_{1}, \ldots, e_{13}>/ R
$$

with the set of relations $R$ given by

$$
\begin{array}{llll}
u_{1} e_{2}-u_{1} u_{2} e_{1} & z u_{1} e_{12}-u_{1} u_{2} e_{6} & u_{1} e_{4}-z u_{1} e_{3} & u_{1} e_{6}-z u_{1} e_{5}-3 z^{2} u_{1} e_{1} \\
u_{1} e_{10}-u_{1} u_{2} e_{3} & z u_{1} e_{13}-u_{1} u_{2} e_{8} & u_{1} e_{5}-z u_{1} e_{4} & u_{1} e_{9}-z u_{1} e_{8}+z u_{1} e_{2} \\
u_{1} e_{13}-u_{1} u_{2} e_{7} & u_{1} e_{11}-z u_{1} e_{10} & u_{1} e_{8}-z u_{1} e_{7} & u_{1} e_{12}-z u_{1} e_{11}-3 z u_{1} e_{1} .
\end{array}
$$

Remark 5.2. There is no natural way to simplify the presentation of $M_{3}$, in fact, computer algebra calculations (for example in Macaulay2) also give the same expression for the minimal presentation of $M_{3}$. So, we really need all 13 generators and 13 relations to describe the space of Poisson structures on $W_{3}$ as a module over global functions. As a complex vector space it is infinite dimensional.
Proof of Lemma 5.1. To find $H^{0}\left(W_{3}, \Lambda^{2} T W_{3}\right)$ we need global holomorphic sections so that we must find $a, b, c$ holomorphic on $U$ such that

$$
\left[\begin{array}{ccc}
z^{2} & -3 z u_{1} & z u_{2} \\
0 & -z^{-3} & 0 \\
0 & 0 & -z
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]
$$

is holomorphic on $V$. To start with $a=\sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} a_{l i s} z^{l} u_{1}^{i} u_{2}^{s}$ and similar for $b$ and $c$.
We will give a presentation of $M:=H^{0}\left(W_{3}, \Lambda^{2} T W_{3}\right)$ as a module over global sections. Here we will need to perform calculations up to at least neighborhood 2, unlike the case of $W_{2}$ where neighborhood 1 was enough. Thus, to calculate the module structure here, we need the expressions of sections on the second formal neighborhood, which consist of linear combinations of the following 42 terms:
$\left[\begin{array}{c}0 \\ 0 \\ z^{l} u_{1}\end{array}\right]_{0 \leq l \leq 1} ;\left\{\left[\begin{array}{c}0 \\ z^{l} \\ 0\end{array}\right] ;\left[\begin{array}{c}0 \\ z^{l} u_{2} \\ 0\end{array}\right]\right\}_{0 \leq l \leq 2} ;\left\{\left[\begin{array}{c}z^{l} u_{1}^{2} \\ 0 \\ 0\end{array}\right] ;\left[\begin{array}{c}0 \\ 0 \\ z^{l} u_{1}^{2}\end{array}\right] ;\left[\begin{array}{c}0 \\ z^{l} u_{1} u_{2} \\ 0\end{array}\right]\right\}_{0 \leq l \leq 4} ;\left[\begin{array}{c}0 \\ z^{l} u_{1} \\ 0\end{array}\right]_{0 \leq l \leq 5} ;\left[\begin{array}{c}0 \\ z^{l} u_{1}^{2} \\ 0\end{array}\right]_{0 \leq l \leq 8} ;$
$\left[\begin{array}{c}u_{1} \\ 0 \\ 0\end{array}\right] ;\left[\begin{array}{c}u_{1} u_{2} \\ 0 \\ 0\end{array}\right] ;\left[\begin{array}{c}0 \\ 0 \\ u_{1} u_{2}\end{array}\right] ;\left[\begin{array}{c}3 z^{2} u_{1} \\ z^{3} \\ 0\end{array}\right] ;\left[\begin{array}{c}3 z u_{1} u_{2} \\ z^{2} u_{2} \\ 0\end{array}\right] ;\left[\begin{array}{c}3 z^{5} u_{1}^{2} \\ z^{6} u_{1} \\ 0\end{array}\right] ;\left[\begin{array}{c}3 z^{8} u_{1}^{3} \\ z^{9} u_{1}^{2} \\ 0\end{array}\right] ;\left[\begin{array}{c}3 z^{4} u_{1}^{2} u_{2} \\ z^{5} u_{1} u_{2} \\ 0\end{array}\right] ;\left[\begin{array}{c}-z u_{1} u_{2} \\ 0 \\ z^{2} u_{1}\end{array}\right] ;\left[\begin{array}{c}-u_{1} u_{2}^{2} \\ 0 \\ z u_{1} u_{2}\end{array}\right]$.

But, upon removing all vectors that can be obtained from others by multiplying by a global function we reduce the expression of a global section to:

$$
\begin{aligned}
{\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right] } & =a_{010}\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right]+a_{011}\left[\begin{array}{c}
u_{1} u_{2} \\
0 \\
0
\end{array}\right]+b_{000}\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]+b_{100}\left[\begin{array}{l}
0 \\
z \\
0
\end{array}\right]+b_{200}\left[\begin{array}{c}
0 \\
z^{2} \\
0
\end{array}\right]+b_{300}\left[\begin{array}{c}
3 z^{2} u_{1} \\
z^{3} \\
0
\end{array}\right]+b_{001}\left[\begin{array}{c}
0 \\
u_{2} \\
0
\end{array}\right]+ \\
& +b_{101}\left[\begin{array}{c}
0 \\
z u_{2} \\
0
\end{array}\right]+b_{201}\left[\begin{array}{c}
3 z u_{1} u_{2} \\
z^{2} u_{2} \\
0
\end{array}\right]+c_{010}\left[\begin{array}{l}
0 \\
0 \\
u_{1}
\end{array}\right]+c_{110}\left[\begin{array}{c}
0 \\
0 \\
z u_{1}
\end{array}\right]+c_{210}\left[\begin{array}{c}
-z u_{1} u_{2} \\
0 \\
z^{2} u_{1}
\end{array}\right]+c_{011}\left[\begin{array}{c}
0 \\
0 \\
u_{1} u_{2}
\end{array}\right] .
\end{aligned}
$$

We now need the module structure of $M=H^{0}\left(W_{3}, T W_{3}\right)$ as a module over global functions. So, we first write the generators and relations among them. We establish the notation for the generators:

$$
e_{1}=\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right], e_{2}=\left[\begin{array}{c}
u_{1} u_{2} \\
0 \\
0
\end{array}\right], e_{3}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], e_{4}=\left[\begin{array}{l}
0 \\
z \\
0
\end{array}\right], e_{5}=\left[\begin{array}{c}
0 \\
z^{2} \\
0
\end{array}\right], e_{6}=\left[\begin{array}{c}
3 z^{2} u_{1} \\
z^{3} \\
0
\end{array}\right], e_{7}=\left[\begin{array}{c}
0 \\
0 \\
u_{1}
\end{array}\right],
$$

$$
e_{8}=\left[\begin{array}{c}
0 \\
0 \\
z u_{1}
\end{array}\right], e_{9}=\left[\begin{array}{c}
-z u_{1} u_{2} \\
0 \\
z^{2} u_{1}
\end{array}\right], e_{10}=\left[\begin{array}{c}
0 \\
u_{2} \\
0
\end{array}\right], e_{11}=\left[\begin{array}{c}
0 \\
z u_{2} \\
0
\end{array}\right], e_{12}=\left[\begin{array}{c}
3 z u_{1} u_{2} \\
z^{2} u_{2} \\
0
\end{array}\right], e_{13}=\left[\begin{array}{c}
0 \\
0 \\
u_{1} u_{2}
\end{array}\right] .
$$

These then satisfy the equations:

$$
\begin{array}{llll}
e_{2}-u_{2} e_{1}=0 & z e_{12}-u_{2} e_{6}=0 & e_{4}-z e_{3}=0 & e_{6}-z e_{5}-3 z^{2} e_{1}=0 \\
e_{10}-u_{2} e_{3}=0 & z e_{13}-u_{2} e_{8}=0 & e_{5}-z e_{4}=0 & e_{9}-z e_{8}+z e_{2}=0 \\
e_{13}-u_{2} e_{7}=0 & e_{11}-z e_{10}=0 & e_{8}-z e_{7}=0 & e_{12}-z e_{11}-3 z e_{1}=0
\end{array}
$$

Since neither $z$ nor $u_{2}$ are global functions, we multiply the equations by $u_{1}$ to obtain relations over $\mathcal{O}\left(W_{3}\right)$, obtaining the claimed module structure.

We now proceed to investigate the question of isomorphism of Poisson structures.
Lemma 5.3. There are isomorphisms $e_{3} \simeq e_{6}, e_{7} \simeq e_{9}$, and $e_{10} \simeq e_{12}$.
Proof. For each isomorphism use the transition function of $\Lambda^{2} T W_{3}$ and then exchange the $U$ and $V$ charts as in the proof of Lemma 4.2.

There are natural inclusions of the surfaces $\mathbb{C}, Z_{-1}$, and $Z_{3}$ into $W_{3}$ :
Notation 5.4. We denote by $j_{s}$ for $s=0,1,2$ the inclusions of $\mathbb{C}^{2}, Z_{-1}$, and $Z_{3}$ into the threefold $W_{3}$. Hence, in coordinates we have:

- $j_{0}: \mathbb{C}^{2} \rightarrow W_{3}$ includes $\mathbb{C}^{2}$ as the fiber $z=0$, taking $\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \mapsto \partial_{1} \wedge \partial_{2}$
- $j_{1}: Z_{3} \rightarrow W_{3}$ includes $Z_{3}$ as $u_{2}=0$, taking $\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} \mapsto \partial_{0} \wedge \partial_{1}$
- $j_{2}: Z_{-1} \rightarrow W_{3}$ includes $Z_{-1}$ as $u_{1}=0$, taking $\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} \mapsto \partial_{0} \wedge \partial_{2}$.

Theorem 5.5. The embeddings of Poisson surfaces $j_{0}\left(\mathbb{C}^{2}, \pi_{0}\right), j_{1}\left(Z_{3}, \pi_{i}\right)$ with $i=0,1,2$ and $j_{2}\left(Z_{-1}, \pi_{i}\right)$ with $i=0,1,2,3$ generate all Poisson structures on $W_{3}$.

Proof. Let $j_{1}\left(Z_{3}\right)$ (resp. $j_{2}\left(Z_{-1}\right)$ ) be the embedding of the surface $Z_{3}$ (resp. $Z_{-1}$ ) into $W_{3}$ by $u_{2}=v_{2}=0$ (resp. $u_{1}=v_{1}=0$ ). Then Poisson structures induced by the first embedding are:

$$
\left(j_{1}\right)_{*}(1)_{U}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]_{U}, \quad\left(j_{1}\right)_{*}\left(-\xi^{2} v\right)_{V}=\left[\begin{array}{c}
0 \\
0 \\
-\xi^{2} v
\end{array}\right]_{V}, \quad \text { hence }\left.\quad e_{7}\right|_{j_{1}\left(Z_{3}\right)}=\left(j_{1}\right)_{*} \pi_{0} .
$$

Analogously, $\left.e_{8}\right|_{j_{1}\left(Z_{3}\right)}=j_{1}\left(\pi_{0}\right)$. The Poisson structures induced by the second embedding are

$$
\left(j_{2}\right)_{*}(1)_{U}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]_{U}, \quad\left(j_{2}\right)_{*}\left(-\xi^{3}\right)_{V}=\left[\begin{array}{c}
0 \\
-\xi^{3} \\
0
\end{array}\right]_{V}, \quad \text { hence }\left.\quad e_{3}\right|_{j_{2}\left(Z_{-1}\right)}=\left(j_{2}\right)_{*} \pi_{0}
$$

Analogously $\left.e_{4}\right|_{j_{2}\left(Z_{-1}\right)}=\left(j_{2}\right)_{*} \pi_{1},\left.e_{5}\right|_{j_{2}\left(Z_{-1}\right)}=\left(j_{2}\right)_{*} \pi_{2}$.
Next, take $g: Z_{-1} \rightarrow \mathbb{C}$ defined by $\left.g\right|_{U}(z, u)=u$ and $\left.g\right|_{V}(\xi, v)=\xi^{-1} v$, then $\left.e_{10}\right|_{j_{2}\left(Z_{-1}\right)}=$ $\left(j_{2}\right)_{*}\left(g .\left(1,-\xi^{3}\right)\right)$ and $\left.e_{11}\right|_{j_{2}\left(Z_{-1}\right)}=\left(j_{2}\right)_{*}\left(g .\left(z,-\xi^{2}\right)\right)$. Finally, $\left(j_{0}\right)_{*}(u)=e_{1}=u_{1} \partial_{1} \wedge \partial_{2}$ and $\left(j_{0}\right)_{*}(u v)=e_{2}=u_{1} u_{2} \partial_{1} \wedge \partial_{2}$ give $e_{1}$ and $e_{2}$.

### 5.1. Poisson cohomology for $\boldsymbol{W}_{\mathbf{3}}$.

Lemma 5.6. The generators of Poisson structures on $W_{3}$ are divided up into 3 groups according to their Casimir functions: $\left\{e_{1}, e_{2}\right\}, \quad\left\{e_{3}, e_{4}, e_{5}, e_{10}, e_{11}\right\}, \quad\left\{e_{7}, e_{8}, e_{13}\right\}$.

Proof. By Lemma 5.1 all isomorphism classes of Poisson structures on $W_{3}$ are generated by $e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{7}, e_{8}, e_{10}, e_{11}, e_{13}$ and calculating 0 -th Poisson cohomology, we obtain:

$$
\operatorname{Cas}(\pi)=\left\{\begin{array}{lll}
f \in \mathcal{O}\left(W_{3}\right) / f=f(z) & \text { if } & \pi=e_{1}, e_{2}, \\
f \in \mathcal{O}\left(W_{3}\right) / f=f\left(u_{1}\right) & \text { if } & \pi=e_{3}, e_{4}, e_{5}, e_{10}, e_{11} \\
f \in \mathcal{O}\left(W_{3}\right) / f=f\left(u_{2}\right) & \text { if } & \pi=e_{7}, e_{8}, e_{13} .
\end{array}\right.
$$

Lemma 5.7. The Poisson manifolds ( $W_{3}, e_{1}$ ) and ( $W_{3}, e_{2}$ ) are not isomorphic.

Proof. We have that $e_{1}=u_{1} \partial u_{1} \wedge \partial u_{2}$ and $e_{2}=u_{2} e_{1}$, or written as section of $\Lambda^{2} T W_{3}$ we have

$$
e_{1}=\left[\begin{array}{c}
u_{1} \\
0 \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
\xi v_{1} \\
0 \\
0
\end{array}\right]_{V}, \quad e_{2}=\left[\begin{array}{c}
u_{1} u_{2} \\
0 \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
v_{1} v_{2} \\
0 \\
0
\end{array}\right]_{V}
$$

Therefore the degeneracy loci have the following irreducible components:

$$
\begin{gathered}
D\left(e_{1}\right)=\left\{u_{1}=0\right\} \cup\{\xi=0\} \cup\left\{v_{1}=0\right\}, \\
D\left(e_{2}\right)=\left\{u_{1}=0\right\} \cup\left\{u_{2}=0\right\} \cup\left\{v_{1}=0\right\} \cup\left\{v_{2}=0\right\} .
\end{gathered}
$$

These have different number of irreducible components, implying $\alpha_{0}$ is not isomorphic to $\alpha_{1}$.
Lemma 5.8. The Poisson manifolds $\left(W_{3}, e_{3}\right),\left(W_{3}, e_{4}\right),\left(W_{3}, e_{5}\right),\left(W_{3}, e_{10}\right)$ and $\left(W_{3}, e_{11}\right)$ are pairwise nonisomorphic.

Proof. We compute the degeneracy loci of the Poisson structures. We have that

$$
\begin{gathered}
e_{3}=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-3 \xi^{2} v_{1} \\
-\xi^{3} \\
0
\end{array}\right]_{V}, \quad e_{4}=\left[\begin{array}{l}
0 \\
z \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-3 \xi v_{1} \\
-\xi^{2} \\
0
\end{array}\right]_{V}, e_{5}=\left[\begin{array}{c}
0 \\
z^{2} \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-3 v_{1} \\
-\xi \\
0
\end{array}\right]_{V} \\
e_{10}=\left[\begin{array}{c}
0 \\
u_{2} \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-3 \xi v_{1} v_{2} \\
-\xi^{2} v_{2} \\
0
\end{array}\right]_{V}, \quad e_{11}=\left[\begin{array}{c}
0 \\
z u_{2} \\
0
\end{array}\right]_{U}=\left[\begin{array}{c}
-3 v_{1} v_{2} \\
-\xi v_{2} \\
0
\end{array}\right]_{V} .
\end{gathered}
$$

The degeneracy loci are:

$$
\begin{aligned}
D\left(e_{3}\right) & =\{\xi=0\}, \\
D\left(e_{4}\right) & =\{z=0\} \cup\{\xi=0\}, \\
D\left(e_{5}\right) & =\{z=0\} \cup\left\{\xi=v_{1}=0\right\}, \\
D\left(e_{10}\right) & =\left\{u_{2}=0\right\} \cup\{\xi=0\} \cup\left\{v_{2}=0\right\}, \\
D\left(e_{11}\right) & =\{z=0\} \cup\left\{u_{2}=0\right\} \cup\left\{v_{2}=0\right\} \cup\left\{\xi=v_{1}=0\right\},
\end{aligned}
$$

these are pairwise nonisomorphic, and thus also their corresponding Poisson structures.
Lemma 5.9. The Poisson manifolds $\left(W_{3}, e_{7}\right),\left(W_{3}, e_{8}\right)$ and ( $W_{3}, e_{13}$ ) are pairwise nonisomorphic.

Proof. We compute the degeneracy loci of the Poisson structures. We have that

$$
e_{7}=\left[\begin{array}{c}
0 \\
0 \\
u_{1}
\end{array}\right]_{U}=\left[\begin{array}{c}
\xi v_{1} v_{2} \\
0 \\
-\xi^{2} v_{1}
\end{array}\right]_{V}, e_{8}=\left[\begin{array}{c}
0 \\
0 \\
z u_{1}
\end{array}\right]_{U}=\left[\begin{array}{c}
v_{1} v_{2} \\
0 \\
-\xi v_{1}
\end{array}\right]_{V}, e_{13}=\left[\begin{array}{c}
0 \\
0 \\
u_{1} u_{2}
\end{array}\right]_{U}=\left[\begin{array}{c}
v_{1} v_{2}^{2} \\
0 \\
-\xi v_{1} v_{2}
\end{array}\right]_{V} .
$$

So, the degeneracy loci:

$$
\begin{aligned}
D\left(e_{7}\right) & =\left\{u_{1}=0\right\} \cup\{\xi=0\} \cup\left\{v_{1}=0\right\}, \\
D\left(e_{8}\right) & =\{z=0\} \cup\left\{u_{1}=0\right\} \cup\left\{\xi=v_{2}=0\right\} \cup\left\{v_{1}=0\right\}, \\
D\left(e_{13}\right) & =\left\{u_{1}=0\right\} \cup\left\{u_{2}=0\right\} \cup\left\{v_{1}=0\right\} \cup\left\{v_{2}=0\right\},
\end{aligned}
$$

are pairwise nonisomorphic, and therefore the corresponding Poisson structures are distinct.
This concludes the lemmata needed to prove Lemma 5.1, showing that all 10 listed isomorphism classes of Poisson structures are indeed distinct.
5.2. Symplectic foliations on $\boldsymbol{W}_{\mathbf{3}}$. We have seen that all possible Poisson structures on $W_{3}$ can be obtained from $e_{1}, e_{2}, e_{3}, e_{4}, e_{5}, e_{7}, e_{8}, e_{10}, e_{11}, e_{13}$ given in Lemma 5.1. Their corresponding symplectic foliations have 0-dimensional leaves consisting of single points inside their degeneracy loci described in Lemma 5.6, and outside these loci, all symplectic leaves are 2-dimensional and can be described as follows.

Theorem 5.10. The symplectic foliations on $W_{3}$ have 0-dimensional leaves consisting of single points over each of their corresponding degeneracy loci described in the proofs of Lemmata 5.7, $5.8,5.9$, and their generic leaves, which are 2-dimensional, are as follows:

- Isomorphic to $\mathbb{C}^{*} \times \mathbb{C}$ for $e_{1}$.
- Isomorphic to $\mathbb{C}^{*} \times \mathbb{C}^{*}$ for $e_{2}$.
- Surfaces of constant $u_{1}$, for $e_{3}, e_{4}, e_{5}, e_{10}, e_{11}$ and $e_{13}$.
- Surfaces of constant $u_{2}$, for $e_{7}$ and $e_{8}$.

Proof. Combine the Casimir functions given in the proof of Lemma 5.6 with Remark 1.1.
Acknowledgements. We are grateful to Brent Pym for kindly explaining to us some of the fundamental notions of Poisson geometry. E. Ballico is a member of GNSAGA of INdAM (Italy). E. Gasparim thanks the Department of Mathematics of the University of Trento for the support and hospitality. B. Suzuki was supported by the ANID-FAPESP cooperation 2019/13204-0.

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