# THE WEIGHT DISTRIBUTION OF CODES OVER FINITE CHAIN RINGS 

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#### Abstract

In this work, we determine new linear equations for the weight distribution of linear codes over finite chain rings. The identities are determined by counting the number of some special submatrices of the parity-check matrix of the code. Thanks to these relations we are able to compute the full weight distribution of codes with small Singleton defects, such as MDS, MDR and AMDR codes.


## 1. Introduction

Ring-linear coding theory has been widely studied because of its theoretical and practical interest. On one hand, ring-linear codes are relevant from an algebraic perspective: as shown in [5], some optimal but non-linear binary codes can be represented as linear codes over $\mathbb{Z} / 4 \mathbb{Z}$ endowed with the Lee metric. On the other hand, ring-linear codes have received attention in cryptographic community. The recent effort among cryptographers to obtain secure post-quantum ciphers [1, 2, 4, 11, 12, 15] led to an increase in the interest in computationally hard algebraic problems, and an interested reader can refer to [3, 13, 20] for more details. Code-based cryptography is one of the most studied and promising areas in the post-quantum framework. However, due to the necessity of reducing the public key size associated to a code-based cryptosystem, there has been interest in exploring different ambient spaces and metrics other then vector spaces over finite fields equipped with the Hamming metric. For example, codes over finite rings equipped with the Lee metric may decrease thepublic key size of the cryptosystems; for further details see [7, 21, 22].
Understanding the minimum distance of a code is computationally hard and it is one of the main problem in Coding Theory. In 1997 Vardy proved that, given a basis of a code, determining precisely the minimum distance of a linear code is NP-hard [20]. Hence this computational problem, as well as several related questions, is linked to the security of post-quantum cryptographic protocols.
Even the problem of calculating the weight distribution of a linear code, which implies the determination of the minimum distance, is NP-hard. In this paper we treat the problem of computing the weight distribution of linear codes over finite chain rings equipped with the Hamming metric. In classical coding theory the most fundamental result about weight distributions are the MacWilliams identities, which express how the weight enumerators of a linear code and its dual relate to each other. Several authors have generalized this work in different directions. For example, a MacWilliams theorem for codes over finite Frobenius rings was given by Wood in 1999 [23].

[^0]Here, we provide new linear equations for the weight distribution of ring-linear codes by counting the number of some special submatrices of the parity-check matrix of the code. This task is certainly as difficult as the original one; however, it allows to investigate codes having special structure in their parity-check matrix. The provided equations and MacWilliams identities seems to be independent, but there could be a possible link in between this equations and some variant of MacWilliams identities. This paper is organized as follows. In Section 2 we recall some basics on linear codes over finite fields. In Section 3 we introduce ring-linear codes; we investigate the structure of the parity-check matrix and the weight distribution of a linear code. In Section 4 we derive new relations for the weight distribution of ring-linear codes; we discuss the optimality of the result. The obtained formula is a modification of the formula given in [14, Proposition 5] for linear codes over finite fields and specialised in [17] for Hermitian codes. In Section [5 we apply our formula to verify the known results about the distribution of MDS codes. Moreover we derive the weight distribution formula for MDR and AMDR codes. Finally, in Section 6, we discuss the connection between MacWilliams identities and the provided relations.

## 2. Preliminaries on Linear codes over finite fields

In its most general setting, Coding Theory is the study of discrete sets equipped with a metric. The most studied case is that of algebraic varieties living in vector spaces over finite fields, and the metric is the Hamming metric. In this framework, a (linear) code $C$ is a vector subspace of dimension $k$ of $\left(\mathbb{F}_{q}\right)^{n}$, where the elements of the code are called codewords and the parameters $n$ and $k$ are respectively known as the length and the dimension of $C$. The Hamming metric, also known as Hamming distance, is a discrete metric counting the number of non-zero coordinates, namely,

$$
\mathrm{d}(v, w)=\left|\left\{i \mid v_{i} \neq w_{i}, 1 \leq i \leq n\right\}\right|
$$

for any $v=\left(v_{1}, \ldots, v_{n}\right)$ and $w=\left(w_{1}, \ldots, w_{n}\right)$ in $\left(\mathbb{F}_{q}\right)^{n}$.
Notice that in this work we consider the elements of vector spaces and modules to be row vectors, a standard notation in Coding Theory. If $v$ is any (row) vector, then its transpose $v^{\top}$ is a column vector.
The third most important parameter of a code is the so-called minimum distance $d$, which is the minimum among the Hamming distances of any pair of distinct codewords, i.e.

$$
d=\min _{c_{1}, c_{2} \in C} \mathrm{~d}\left(c_{1}, c_{2}\right),
$$

and it coincides with the minimum weight of a codeword. The importance of the minimum distance is related to the capability of codes to correct errors. If we are presented with a vector $v$ that should be a codeword $c$ of a given code $C$, even if its coordinates are corrupted (hence $v \notin C$ ), then we can safely reconstruct $c$ from $v$ provided that the number of erroneous coordinates of $v$ is bounded by $\left\lfloor\frac{d-1}{2}\right\rfloor$.
The weight distribution of a code of length $n$ specifies the number of codewords of each possible weight $0,1, \ldots, n$. Even if the weight distribution does not in general uniquely determine a code, it gives important information: in addition to providing the correction capability of a code, it allows to calculate the probability of undetected errors (see [19, Chapter 2]).

## 3. GENERALITIES ON LINEAR CODES OVER FINITE CHAIN RINGS

A finite ring with unity $1 \neq 0$ is called a left (resp. right) chain ring if its left (resp. right) ideals are linearly ordered by inclusion. Note that a finite chain ring is a local ring where all the ideals are principal.
Throughout the paper let $R$ be a finite commutative chain ring. Let $\gamma$ be the generator of the maximal ideal and let $s$ be its the nilpotency index. Let $\mathbb{K}$ denote the residue field with $p$ elements $R / \gamma R$.

Definition 3.1. A linear code of length $n$ in the alphabet $R$ is a submodule of $R^{n}$. The free module $R^{n}$ is called the ambient space of the code.

Definition 3.2. The Hamming weight of an element $c=\left(c_{1}, \ldots, c_{n}\right) \in R^{n}$ is the number $w(c)$ of non-zero entries of $c$.

Definition 3.3. A matrix $G$ is called a generator matrix for the code $C$ over $R$ if the rows of $G$ span $C$ and none of them can be written as a linear combination of the other rows of $G$.

As shown in [16], any linear code over a finite chain ring has a generator matrix. In our framework it is convenient to work with a generator matrix in standard form.

Proposition 3.4. 16, Proposition 3.2] Let $C$ be a linear code in $R^{n}$. $C$ is permutation equivalent to a code having the following generator matrix in standard form:

$$
G=\left[\begin{array}{ccccccc}
I_{k_{0}} & A_{0,1} & A_{0,2} & A_{0,3} & \ldots & A_{0, s-1} & A_{0, s} \\
0 & \gamma I_{k_{1}} & \gamma A_{1,2} & \gamma A_{1,3} & \ldots & \gamma A_{1, s-1} & \gamma A_{1, s} \\
0 & 0 & \gamma^{2} I_{k_{2}} & \gamma^{2} A_{2,3} & \ldots & \gamma^{2} A_{2, s-1} & \gamma^{2} A_{2, s} \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & \gamma^{s-1} I_{k_{s-1}} & \gamma^{s-1} A_{s-1, s}
\end{array}\right]
$$

where $A_{i, s} \in M_{k_{i} \times n-K}\left(R / \gamma^{s-i} R\right)$ and $A_{i, j} \in M_{k_{i} \times k_{j}}\left(R / \gamma^{s-i} R\right)$ for $j<s$.
For all $0 \leq i \leq s$ the $k_{i}$ 's denote the number of rows of $G$ that are divisible by $\gamma^{i}$ but not by $\gamma^{i+1}$. The parameters $k_{0}, \ldots, k_{s-1}$ are the same for all generator matrices in systematic form, and $C$ is said to be of type $\left(k_{0}, k_{1}, \ldots, k_{s-1}\right)$. The rank of $C$ is defined as $K=\sum_{i=0}^{s-1} k_{i}$.
Definition 3.5. The free rank of $C$ is defined to be the maximum of the rank of the free submodules of $C$ and it coincides with $k_{0}$.

Definition 3.6. A linear code is said to be free if its rank coincides with its free rank. In this case, the code is a free $R$-submodule which is isomorphic to $R^{k_{0}}$.

If $C$ is a free code, then any systematic generator matrix has the form

$$
G=\left[\begin{array}{ll}
I_{k_{0}} & A
\end{array}\right] \in\left(\mathbb{Z} / p^{s} \mathbb{Z}\right)^{k_{0} \times n}
$$

Since for all $0 \leq j \leq s-1$ we have $\left|\gamma^{j} R\right|=p^{s-j}$ (see [16, Lemma 2.4]), it is possible to compute the cardinality of a linear code.

Theorem 3.7. [16, Theorem 3.5] A linear code $C$ over $R$ of type $\left(k_{0}, \ldots, k_{s-1}\right)$ has cardinality $|C|=p^{\sum_{i=0}^{s-1}(s-i) k_{i}}$.

We attach the standard inner product to the ambient space i.e. $v \cdot w=\sum v_{i} w_{i}$. The dual code $C^{\perp}$ of $C$ is defined, as in the classical case, by

$$
C^{\perp}=\left\{v \in R^{n} \mid v \cdot w=0 \text { for all } w \in C\right\}
$$

In [23], Wood proved that the dual code of a code over a Frobenius ring, and hence over a finite chain ring, is well defined (i.e. $\left.\left(C^{\perp}\right)^{\perp}=C\right)$. The dual code $C^{\perp}$ has the following parameters:

Theorem 3.8. Let $C \subseteq R^{n}$ be a linear code of rank $K$ and type $\left(k_{0}, \ldots, k_{s-1}\right)$. Then $C^{\perp}$ is a linear code of rank $n-k_{0}$ and type $\left(n-K, k_{s-1}, \ldots, k_{1}\right)$.

As a consequence, the dual code of a free code is again free.
We call any matrix $H$ a parity-check matrix for $C$ if its kernel is $C$.
Proposition 3.9. 16, Theorem 3.10] Let $C$ be a linear code of type $\left(k_{0}, \ldots, k_{s-1}\right)$. Then $C$ is permutation equivalent to a code having a parity-check matrix in systematic form:

$$
H=\left[\begin{array}{ccccc}
B_{0, s} & B_{0, s-1} & \cdots & B_{0,1} & I_{n-K}  \tag{3.1}\\
\gamma B_{1, s} & \gamma B_{1, s-1} & \ldots & \gamma I_{k_{s-1}} & 0 \\
\vdots & \vdots & & \vdots & \vdots \\
\gamma^{s-1} B_{s-1, s} & \gamma^{s-1} I_{k_{1}} & \cdots & 0 & 0
\end{array}\right]=\left[\begin{array}{c}
H^{(0)} \\
\gamma H^{(1)} \\
\vdots \\
\gamma^{s-1} H^{(s-1)}
\end{array}\right]
$$

where, for $0 \leq i, j \leq s, B_{i, j}=-\sum_{k=i+1}^{j-1} B_{i, k} A_{s-j, s-k}^{T}-A_{s-j, s-i}^{T}$.
Clearly, $H$ is a generator matrix for $C^{\perp}$.
3.1. Hamming weight distribution and Singleton-like bounds. As in the classical case, the Hamming-weight distribution of a ring-linear code is a vector $A=\left(A_{i}\right)_{i=0,1 \ldots, n}$, where $A_{i}$ denotes the number of codewords of $C$ of weight $i$. The weight distribution can be encoded as coefficients in a polynomial.

Definition 3.10. The (Hamming)-weight enumerator polynomial of a ring-linear code $C$ of length $n$ is the bivariate polynomial

$$
W_{C}(X, Y)=\sum_{c \in C} X^{n-w(c)} Y^{w(c)}=\sum_{i=0}^{n} A_{i} X^{n-i} Y^{i}
$$

The Hamming-weight enumerators of a code and its dual are related by the MacWilliams identities.

Theorem 3.11. [23, Theorem 8.3] For linear codes over a finite chain ring $R$ with $p^{s}$ elements, the MacWilliams identities hold:

$$
W_{C^{\perp}}(X, Y)=\frac{1}{|C|} W_{C}\left(X+\left(p^{s}-1\right) Y, X-Y\right)
$$

Moreover, for the Hamming metric over $R$, Singleton-like bounds are known.
Remark 3.12. The Singleton bound for codes over any alphabet of size $p^{s}$ states that

$$
d \leq n-\log _{p^{s}}(|C|)+1
$$

(see for example [10]). In the framework of codes over finite chain rings, only free codes meet this bound and they are said maximum distance separable (MDS) codes. As shown in [6], for codes over principal ideal rings

$$
\begin{equation*}
d \leq n-K+1 \tag{3.2}
\end{equation*}
$$

This bound is in general tighter than the Singleton bound and they coincides if and only if the code is free. A linear code over a finite chain ring meeting this bound is said to be maximum distance with respect to rank (MDR). In particular a code $C$ is MDS if and only if it is MDR and free.

It is well-known ( 18 , Corollary 1]) that the MDS property is invariant under duality (i.e. the dual of an MDS code over a finite chain ring is again MDS). In general, the dual code of an MDR code does not preserve the property.

Example 3.13. Let $C=\langle(1,0,1),(0,2,0),(0,0,2)\rangle \subset(\mathbb{Z} / 4 \mathbb{Z})^{3}$ be a linear code. $C$ is MDR since $d(C)=1=n-K+1$. However, its dual code $C^{\perp}=\langle(2,0,2),(0,2,0)\rangle$ has minimum distance $d\left(C^{\perp}\right)=1<2=(n-K+1)$.

As in the classical case of linear codes over finite fields, we can measure how far away a linear code $C$ is from being MDR.

Definition 3.14. Let $C$ be a linear code of length $n$ and rank $K$. The defect $s(C)$ of $C$ is defined as $s(C):=n+1-K-d$.
3.2. On the parity-check matrix and its submatrices. The parity-check matrix of a code give important information on some structural properties of the code, such as the minimum distance.
Analogously to linear codes over finite fields (see [8, Theorem 1.4.13] ), given a code $C$ over a finite chain ring there is a link between the weights of the codewords of $C$ and its parity-check matrix $H$.

Theorem 3.15. Let $C$ be a linear code over $R$ with parity-check matrix $H$. If $c \in C$, the columns of $H$ corresponding to the non-zero coordinates of $c$ are linearly dependent. Conversely, if a linear dependence relation with only non-zero coefficients exists among $w$ columns of $H$, then there is a codeword in $C$ of weight $w$ whose nonzero coordinates correspond to these columns.

Proof. If $c \in C$, the matrix product $0=H c^{T}=\sum_{i=1}^{n} \mathbf{h}_{\mathbf{i}} c_{i}$, where $\mathbf{h}_{\mathbf{i}}$ is the $i^{\text {th }}$ column of $H$, is a linear combination of the columns of $H$ with coefficients provided by $c$. Conversely, if there are $w$ linearly dependent columns in $H$, then $\sum_{i=0}^{n} \alpha_{i} \mathbf{h}_{\mathbf{i}}=0, \alpha_{i} \in R$ and $w$ of them are non-zero . If $c=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, then $H c^{\top}=0$ and $\mathrm{w}(c)=w$ and $c$ is the desired codeword.

For any $0 \leq j \leq s-1$, let $H^{(j)}$ be submatrices of $H$ defined according to (3.1). Since the $H^{(j)}$ s are all full rank, the following is immediate.

Proposition 3.16. Let $H$ be a parity-check matrix for the code $C$, and hence a generator matrix for $C^{\perp}$. A message $v \in R^{n-k_{0}}$ is encoded as the zero codeword in $C^{\perp}$ if and only if it is of the form

$$
v=[\underbrace{0, \ldots, 0}_{n-K}\|\gamma^{s-1} \cdot \underbrace{v_{1}}_{k_{s-1}}\| \gamma^{s-2} \cdot \underbrace{v_{2}}_{k_{s-2}}\|\ldots\| \gamma \cdot \underbrace{v_{s}}_{k_{1}}] \text {. }
$$

Notation 3.17. Let $M \in M_{t \times n}(R)$ be a matrix. According to the notation of Proposition 3.4, we say that $M$ is of type $\left(t_{0}, \ldots, t_{s-1}\right)$ if $t_{i}$ rows of $M$ are divisible by $\gamma^{i}$ but not by $\gamma^{i+1}, 0 \leq i \leq s-1$.

Definition 3.18. Let $M \in M_{t \times n}(R)$ be a matrix of type $\left(t_{0}, \ldots, t_{s-1}\right)$.

- For any subset $\mathcal{I} \subseteq\{1, \ldots, n\}$ of size $\nu, \mathcal{I}=\left\{i_{1}, \ldots i_{\nu}\right\}$ with $i_{1}<i_{2}<\cdots<i_{\nu}$, we define $M_{[\mathcal{I}]}$ as the $t \times \nu$ submatrix of $M$ identified by the columns indices $\mathcal{I}$.
- We define $N_{M}\left(\nu, r_{0}, r_{1}, \ldots, r_{t}\right)$ to be the number of $t \times \nu$ submatrices of $M$ of type $\left(r_{0}, r_{1}, \ldots, r_{t}\right)$

Let $C$ be an $R$-linear code of type $\left(k_{0}, k_{1}, \ldots, k_{s-1}\right)$, and let $H \in M_{\left(n-k_{0}\right) \times n}(R)$ be a parity-check matrix in standard form for $C$. For any fixed $\mathcal{I}$ of size $\nu, H_{[\mathcal{I}]}$ is a $\left(n-k_{0}\right) \times \nu$ submatrix of $H$ and, as in (3.1), we can write:

$$
H_{[\mathcal{I}]}=\left[\begin{array}{c}
H_{[\mathcal{I}]}^{(0)}  \tag{3.3}\\
\gamma H_{[\mathcal{I}]}^{(1)} \\
\vdots \\
\gamma^{s-1} H_{[\mathcal{I}]}^{(s-1)}
\end{array}\right]
$$

where each $H_{[\mathcal{I}]}^{(j)}$ is obtained from $H^{(j)}$ by removing the unnecessary columns. Since we are selecting $\nu<n$ columns from $H$, the type of $H_{[\mathcal{I}]}$ and $H$ may differ. First of all, note that the dimension of the first block may decrease. Indeed:
(1) Some rows of $H_{[\mathcal{I}]}^{(0)}$ can be written as linear combinations of the others, so they can be removed from the parity-check matrix;
(2) Some rows of $H_{[\mathcal{I}]}^{(0)}$ are multiples of $\gamma^{l}$ for some $1 \leq l \leq s-1$. If this is the case, the rows can be moved in one of the subsequent blocks.
On the other hand, the dimension of the second block can either increase, decrease or remain unchanged. One or more of the following may occur:
(1) Some rows of $H_{[\mathcal{I}]}^{(0)}$ are added to $H_{[\mathcal{I}]}^{(1)}$;
(2) Some rows of of $H_{[\mathcal{I}]}^{(1)}$ are multiples of $\gamma^{l}, 2 \leq l \leq s-1$. In this case, collecting $\gamma^{l}$, the row can be moved in one of the subsequent blocks;
(3) Some rows of $H_{[\mathcal{I}]}^{(1)}$ are linear combinations of the others, and thus they can be removed from the matrix.
The procedure can be iterated for any other block $H_{[\mathcal{I}]}^{(j)}, 2 \leq j \leq s-1$. Therefore, the type of $H_{[\mathcal{I}]}$, being different from the type of $H$, can not be studied in its full generality. However, if $\nu$ satisfies certain conditions, the structure of $H_{[\mathcal{I}]}$ become more clear: we will focus on this problem in the next section.

## 4. Weight distribution for linear codes

From now on, let $C$ be a linear code over a finite chain ring $R$ of length $n$, rank $K$ and type $\left(k_{0}, k_{1}, \ldots, k_{s-1}\right)$.
Lemma 4.1. Let $H \in M_{\left(n-k_{0}\right) \times n}(R)$ be a parity-check matrix for the code $C$. If $n-d^{\perp}<\nu \leq n$, then all the $\left(n-k_{0}\right) \times \nu$ submatrices of $H$ are of type $(n-$ $K, k_{s-1}, k_{s-2}, \ldots, k_{1}$ ). In particular they all have rank $n-K$.

Proof. Being the parity-check matrix of a linear code of type $\left(k_{0}, k_{1}, \ldots, k_{s-1}\right), H$ is of type $\left(n-K, k_{s-1}, k_{s-2}, \ldots, k_{1}\right)$. Let $H_{[\mathcal{I}]}, \mathcal{I}=\left\{i_{1}, \ldots, i_{\nu}\right\}$, be a $\left(n-k_{0}\right) \times \nu$ submatrix of $H$. Without loss of generality $H_{[\mathcal{I}]}$ can be written as in (3.3). By contradiction, assume that a row in $H_{[\mathcal{I}]}^{(0)}$ is multiple of $\gamma$. Without loss of generality we may assume it is the first one. If $\bar{v}=[\gamma, 0, \ldots, 0] \in R^{n-K}$, then $\bar{v} \cdot H_{[\mathcal{I}]}^{(1)}=0$. The vector $\bar{v}$ can be used as a first brick for constructing a new vector $v$. By Proposition 3.16, $v$ is not encoded as the zero codeword. Therefore $c:=v \cdot H$ is a codeword in $C^{\perp}$ of weight $w(c) \leq n-\nu$, contradicting the hypothesis $d^{\perp}>n-\nu$. So, since no row in $H_{[\mathcal{I}]}^{(0)}$ is a multiple of $\gamma$, in $H_{[\mathcal{I}]}$ there are at most $k_{s-1}$ rows which are multiple of $\gamma$ but not of $\gamma^{2}$. Applying to $H_{[\mathcal{I}]}^{(1)}$ the procedure described above, it is possible to show they are exactly $k_{s-1}$. Iterating the process for all the remaining blocks $H_{[\mathcal{I}]}^{(2)}, \ldots, H_{[\mathcal{I}]}^{(s-1)}$ we get the thesis.

Corollary 4.2. Let $C$ be a linear code with parity-check matrix $H$ and $n-d^{\perp}<\nu \leq n$.

$$
N_{H}\left(\nu, t_{0}, t_{1}, \ldots t_{s-1}\right)=\left\{\begin{array}{cl}
\binom{n}{\nu} & \text { if } t_{0}=n-K \text { and } t_{i}=k_{s-i}, 1 \leq i \leq s-1 \\
0 & \text { otherwise }
\end{array}\right.
$$

Proposition 4.3. Let $C$ be a code of type $\left(k_{0}, \ldots, k_{s-1}\right)$ with parity-check matrix $H \in M_{n-k_{0} \times n}(R)$. Let $\left\{A_{i}\right\}_{i=0, \ldots, n}$ be the weight distribution of $C$. If $n-d^{\perp}<\nu \leq n$, then

$$
\begin{equation*}
\sum_{l=0}^{\nu}\binom{n-l}{\nu-l} A_{l}=\binom{n}{\nu} \frac{|C|}{p^{s(n-\nu)}} \tag{4.1}
\end{equation*}
$$

Proof. Let $V_{[\mathcal{I}]}$ be the kernel of $H_{[\mathcal{I}]}$. Consider the map

$$
\varphi_{[\mathcal{I}]}: V_{[\mathcal{I}]} \rightarrow R^{n}, \quad \varphi_{[\mathcal{I}]}(v)=\left(\bar{v}_{1}, \ldots, \bar{v}_{n}\right), \quad \bar{v}_{j}=\left\{\begin{array}{l}
v_{j} \text { if } j \in\{1, \ldots, \nu\} \\
0 \text { otherwise }
\end{array}\right.
$$

$\varphi_{[\mathcal{I}]}$ is the restriction of the map

$$
\varphi_{\nu}: \bigsqcup_{\mathcal{I}:|\mathcal{I}|=\nu} V_{[\mathcal{I}]} \rightarrow C
$$

to $V_{[\mathcal{I}]}$. The elements in the domain of $\varphi_{\nu}$ can be counted in two different ways:
(1) For any choice of $\mathcal{I}=\left\{i_{1}, \ldots, i_{\nu}\right\}$, by Lemma 4.1, the matrix $H_{[\mathcal{I}]}$ is of length $\nu$ and of type $\left(n-K, k_{s-1}, k_{s-2}, \ldots, k_{1}\right)$. Therefore $H_{[\mathcal{I}]}$ is a paritycheck matrix of a linear code $C^{\prime}$ of type $\left(\nu-n+k_{0}, k_{1}, \ldots, k_{s-1}\right)$ and length $\nu$. Thus, by Theorem 3.7, $C^{\prime}$ has with $p^{s\left(\nu-n+k_{0}\right)+\sum_{i=1}^{s-1}(s-i) k_{i}}$ elements. Hence, by Corollary 4.2,

$$
\begin{align*}
\left|\bigsqcup_{\mathcal{I}:|\mathcal{I}|=\nu} V_{[\mathcal{I}]}\right| & =N_{H}\left(\nu, n-K, k_{s-1}, k_{s-2}, \ldots, k_{1}\right)\left|V_{[\mathcal{I}]}\right|=  \tag{4.2}\\
& =\binom{n}{\nu} p^{s\left(\nu-n+k_{0}\right)+\sum_{i=1}^{s-1}(s-i) k_{i}} .
\end{align*}
$$

(2) We consider a codeword $c \in C$ of weight $l \leq \nu$. Let $\mathcal{I}_{1}=\operatorname{supp}(c)$. Any choice of $\nu-l$ indices $\mathcal{I}_{2} \subset\{1, \ldots, n\} \backslash \mathcal{I}_{1}$ identifies uniquely an element in $\sqcup_{\mathcal{I}:|\mathcal{I}|=\nu} V_{[\mathcal{I}]}$. More precisely, $\mathcal{I}_{1} \cup \mathcal{I}_{2}$ determines uniquely $H_{\left[\mathcal{I}_{1} \cup \mathcal{I}_{2}\right]}$, clearly $c_{\left[\mathcal{I}_{1} \cup \mathcal{I}_{2}\right]} \in V_{\left[\mathcal{I}_{1} \cup \mathcal{I}_{2}\right]}$, and so there is an unique element $v \in V_{\left[\mathcal{I}_{1} \cup \mathcal{I}_{2}\right]}$ such that $\varphi_{\left[\mathcal{I}_{1} \cup \mathcal{I}_{2}\right]}(v)=c$, that is $v=c_{\left[\mathcal{I}_{1} \cup \mathcal{I}_{2}\right]}$. In order to determine the size of $\varphi_{\nu}^{-1}(c)$, the fiber of $c$ under the map $\varphi_{\nu}$, it is enough to count all possible subsets of $\{1, \ldots, n\} \backslash \mathcal{I}_{1}$ with size $\nu-l$. It follows that the fiber of each codeword of weight $l$ has $\binom{n-l}{\nu-l}$ elements, and we observe that all the fibers of such codewords form a partition of $\sqcup_{\mathcal{I}:|\mathcal{I}|=\nu} V_{[\mathcal{I}]}$. Since there are $A_{l}$ codeword of weight $l$, we obtain

$$
\begin{equation*}
\left|\bigsqcup_{\mathcal{I}:|\mathcal{I}|=\nu} V_{[\mathcal{I}]}\right|=\sum_{l=0}^{\nu}\binom{n-l}{\nu-l} A_{l} . \tag{4.3}
\end{equation*}
$$

Putting together (4.2) and (4.3) we get (4.1).
Corollary 4.4. For a free code $C$ of length $n$ and rank $k_{0}=K$ over $R$ the weight distribution formula reads:

$$
\sum_{l=0}^{\nu}\binom{n-l}{\nu-l} A_{l}=\binom{n}{\nu} p^{s(\nu+K-n)}
$$

Theorem 4.5. Let $\sigma$ be the sum of the Singleton defects of $C$ and $C^{\perp}$. the knowledge of $\sigma+d+K-k_{0}-1$ elements of the weight distribution $\left\{A_{0}, \ldots, A_{n}\right\}$ is enough to compute the full weight distribution of $C$ and $C^{\perp}$. In particular the knowledge of $d$ and of any $\sigma+K-k_{0}-1$ elements of $\left\{A_{d}, \ldots, A_{n}\right\}$ is enough to compute the entire weight distribution of $C$ and $C^{\perp}$.

Proof. Consider equation (4.1) with $\nu$ varying in range $\left\{n-d^{\perp}+1, \ldots, n\right\}$. We obtain a linear system of the form

$$
\mathcal{P} \cdot A(C)=b,
$$

where $\mathcal{P}$ is a truncated Pascal matrix with $d^{\perp}$ rows and $n$ columns. By [9], all the minors of $\mathcal{P}$ of order $d^{\perp}$ are non-zero. Hence, the knowledge of $n-d^{\perp}+1=K-\sigma-$ $1+d-\left(K-k_{0}\right)$ elements in the weight distribution allow us to obtain a linear system that admit a unique solution. Finally, the knowledge of $d$ implies the knowledge of $A_{0}, \ldots, A_{d-1}$. Hence, it is sufficient to know other $n-d^{\perp}+1-d=\sigma+K-k_{0}-1$ elements in $\left\{A_{d}, \ldots A_{n}\right\}$ to determine the full weight distribution of the code.

We now show that, at least in some cases, Theorem 4.5 is optimal. Indeed, in general, it is not possible to deterministically deduce the weight distribution of a linear code with less then $\sigma+K-k_{0}-1$ elements in $\left\{A_{0}, \ldots, A_{n}\right\}$.

Definition 4.6. A linear code is said to be Almost-MDR code(AMDR) if it has Singleton defect equal to 1 . An AMDR code whose dual is still AMDR is called Near-MDR.

Definition 4.7. A free code is said to be Almost-MDS code(AMDS) if it has Singleton defect equal to 1. An AMDS code whose dual is still AMDS is called Near-MDS

Consider two Near-MDS codes with the same parameters. For a Near-MDS only $\sigma+K-k_{0}-1=1$ weight is necessary to determine the full weight distribution. If the sole knowledge of $d$ and $n$ was enough to compute the entire weight distribution of the code, then any two near-MDS would be formally equivalent. But this is in general false:

Example 4.8. Let $C_{1}$ and $C_{2}$ be two near-MDS codes over $\mathbb{Z}_{5^{3}}$ of length $n=4$, rank $K=2$ and minimum Hamming distance $d=2$, generated respectively by

$$
G_{1}=\left[\begin{array}{cccc}
1 & 0 & 57 & 0 \\
0 & 1 & 0 & 68
\end{array}\right],
$$

and

$$
G_{2}=\left[\begin{array}{cccc}
1 & 0 & 5 & 43 \\
0 & 1 & 82 & 5
\end{array}\right] .
$$

Their weight distributions are respectively:

$$
\mathcal{A}_{1}=(1,0,248,0,15376),
$$

and

$$
\mathcal{A}_{2}=(1,0,8,480,15136) .
$$

Therefore $C_{1}$ and $C_{2}$ are not formally equivalent.

## 5. Weight distribution of codes with small Singleton defects

Theorem 4.5 lead us to further investigate codes having a small number of Singleton defects.

We start focusing on codes meeting the Singleton bound: from Remark 3.12 follows that $\sigma=0$ for any MDS code.
The weight distribution of MDS codes is well known (see [18, Theorem 5]); however it can be directly obtained with the sole knowledge of the length and the minimum distance from Theorem 4.5
Theorem 5.1. Let $C$ be a MDS code of length $n$ and rank $k$, then

$$
A_{w}(C)=\binom{n}{w} \sum_{j=0}^{w-d}(-1)^{j}\binom{i}{j}\left(p^{s-d+1-j}-1\right)
$$

Now we move to linear codes meeting the generalized Singleton bound (3.2), the MDR codes. As shown in 3.12 the dual of an MDR code is not necessarily MDR. Therefore the weight distribution may depend on one or more parameters.
Let $C$ be an MDR code of length $n$, rank $K$, minimum distance $d=n-K+1$ and let $C^{\perp}$ be its dual having rank $n-K$ and minimum distance $k_{0}-\sigma+1$ for some $\sigma \geq 0$. According to the notation of Proposition 4.1, since $\nu<n-d^{\perp}$ and $d^{\perp}=k_{0}-\sigma+1$, we can write $\nu=n-k_{0}+\sigma+i$ with $i$ ranging in $\left\{0, \ldots, k_{0}-\sigma\right\}$. Let $q=\frac{|C|}{p^{s(n-\nu)}}$. Then equation (4.1) becomes

$$
\sum_{l=0}^{n-k_{0}+\sigma-1+i}\binom{n-l}{n-k_{0}+\sigma+i-l} A_{l}=\binom{n}{n-k_{0}+\sigma+i} q .
$$

Since $A_{0}=1$ and $A_{l}=0$ for all $1 \leq l \leq n-k$, we may write

$$
\begin{array}{r}
\binom{n}{n-k_{0}+\sigma+i}+\sum_{h=0}^{\sigma-k+k_{0}-2}\binom{k-1-h}{\sigma+i+K-k_{0}-h} A_{n+K+1+h}+ \\
\quad+\sum_{l=n+\sigma-k_{0}}^{n-k_{0}+\sigma+i}\binom{n-l}{n-k_{0}+\sigma+i-l} A_{l}=\binom{n}{n-k_{0}+\sigma+i} q
\end{array}
$$

Let $l=n+\sigma-k_{0}+j$, then

$$
\begin{gathered}
\sum_{j=0}^{i}\binom{k_{0}-\sigma-j}{i-j} A_{n+\sigma-k_{0}+j}= \\
=\binom{n}{n-k_{0}+\sigma+i}(q-1)-\sum_{h=0}^{\sigma+K-k_{0}-2}\binom{K-1-h}{K-k_{0}+\sigma+i+h} A_{n+j+1+h} .
\end{gathered}
$$

We can re-write the linear system in matrix form, as $\mathcal{P} \cdot A=b$ with $\mathcal{P}$ Pascal ma-$\operatorname{trix}\left[\binom{k_{0}-\sigma-j}{i-j}\right]_{i, j=0, \ldots, k_{0}-\sigma}$. Therefore $\mathcal{P}^{-1}=(-1)^{i-j}\left[\binom{k_{0}-\sigma-j}{i-j}\right]_{i, k=0, \ldots, k_{0}-\sigma}$. More explicitly:

Proposition 5.2. Let $C$ be an MDR code of length $n$, rank $K$, minimum distance $d=n-K+1$ and let $C^{\perp}$ be its dual having rank $n-k_{0}$ and minimum distance $k_{0}-$ $\sigma+1$ for some $\sigma \geq 0$. Let $q=\frac{|C|}{p^{s(n-\nu)}}$. The knowledge of $\left\{A_{n-K+1}, \ldots, A_{n-k_{0}+\sigma-1}\right\}$ in the weight distribution of $C$ is enough to compute the entire weight distribution of $C$.
In particular, for all $0 \leq i \leq k_{0}-\sigma-K$, we have:

$$
\left.\left.\begin{array}{rl}
A_{n-k_{0}+\sigma+i}=\sum_{j=0}^{i}(-1)^{i-j} & \binom{k_{0}-\sigma-j}{i-j} \\
& -\sum_{h=0}^{\sigma+K-k_{0}-2}\binom{n}{n-k_{0}+\sigma+i}(q-1)- \\
k-k_{0}+\sigma+i-h-1
\end{array}\right) A_{n+k+1+h}\right] .
$$

In a similar fashion we can derive the weight distribution of an AMDR code:
Proposition 5.3. Let $C$ be an AMDR code of length $n$, rank $K$, minimum distance $d=n-K$ and let $C^{\perp}$ be its dual having rank $n-k_{0}$ and minimum distance $k_{0}-\sigma+1$ for some $\sigma \geq 0$. Let $q=\frac{|C|}{p^{s(n-\nu)}}$. The knowledge of $\left\{A_{n-K+1}, \ldots, A_{n-k_{0}+\sigma-1}\right\}$ is enough to compute the entire weight distribution of $C$.
In particular, for all $0 \leq i \leq k_{0}-\sigma-K$ we have:

$$
\begin{gathered}
A_{n-k_{0}+\sigma+i}=\sum_{j=0}^{i}(-1)^{i-j}\binom{k_{0}-\sigma-j+1}{i-j}\left[\binom{n}{n-k_{0}+\sigma+i-1}(q-1)-\right. \\
\left.\sum_{h=0}^{\sigma+K-k_{0}-2}\binom{k-h}{k-k_{0}+\sigma+i-h+1} A_{n+k+h}\right] .
\end{gathered}
$$

Clearly, by specializing the previous formula, we also get the weight distributions of Near-MDS and Near-MDR codes.

## 6. Relation with MacWilliams identities

Both in classical and ring-linear coding theory, the most fundamental result about weight distributions are the MacWilliams identities (Theorem 3.11). They relate the weight enumerator polynomial of a linear code and its dual. However, in our framework it is more convenient to work with other equivalent set of equations in place of the polynomial form of 3.11. Following the outline of 10, Chapter 5, Section 2], and combining it with 3.11 we can deduce the following equality:

$$
\sum_{j=0}^{n-\nu}\binom{n-j}{\nu} A_{j}=\frac{|C|}{p^{s \nu}} \sum_{j=0}^{\nu}\binom{n-j}{n-\nu} A_{j}^{\perp}, \quad \text { for } \quad 0 \leq \nu \leq n
$$

Moreover, in a similar fashion to [8, Theorem 7.2.3], we get:

$$
\begin{equation*}
\sum_{j=0}^{n}\binom{j}{\nu} A_{j}=\frac{|C|}{p^{s \nu}} \sum_{j=0}^{\nu}(-1)^{j}\binom{n-j}{n-\nu}\left(p^{s}-1\right)^{\nu-j} A_{j}^{\perp}, \quad \text { for } \quad 0 \leq \nu \leq n \tag{6.1}
\end{equation*}
$$

If $\nu<d^{\perp}$, each $A_{j}^{\perp}$ of the right hand side of (6.1) is equal to zero except for $A_{0}^{\perp}$ which is equal to 1 . Therefore we get a ring-variant of Pless' equations.

Proposition 6.1. For any $\nu<d^{\perp}$

$$
\begin{equation*}
\sum_{j=0}^{n}\binom{j}{\nu} A_{j}=\frac{|C|}{p^{s \nu}}\binom{n}{n-\nu}\left(p^{s}-1\right)^{\nu}, \quad \text { for } \quad 0 \leq \nu<d^{\perp} \tag{6.2}
\end{equation*}
$$

Hence, when enough terms of the weight distributions are known, systems in 4.3 and 6.1 are equivalent.

Corollary 6.2. Let $\sigma$ be the sum of the Singleton defects of $C$ and $C^{\perp}$. Using equation (6.2), the knowledge of $\sigma+d+K-k_{0}-1$ elements of the weight distribution $\left\{A_{0}, \ldots, A_{n}\right\}$ is enough to compute the full weight distribution of $C$ and $C^{\perp}$. In particular the knowledge of $d$ and of any $\sigma+K-k_{0}-1$ elements of $\left\{A_{d}, \ldots, A_{n}\right\}$ is enough to compute the entire weight distribution of $C$ and $C^{\perp}$.

Proof. The proof follows the same outline of Proposition 4.5
Therefore, the two systems of equations (4.3) and (6.2) are equivalent provided the existence of the code.

## 7. Conclusion

In analogy to linear codes over finite fields, the minors of the parity-check matrix of a ring-linear code enable us to determine linear relations between the weights of the codes. Our formulae enable to verify the weight distribution of MDS codes. Moreover this result allows to determine the full weight distributions of MDR, NearMDR, AMDR codes.
The number of parameters necessary to derive the full weight distribution of a code and its dual depends on the the sum of the Singleton defects of the code and its dual, an in particular it is bounded by $n+K-k_{0}-1$.
An interesting extension of this work would be the study of more classes of codes, either by considering the case of non-AMDR codes or families obtained via structured parity check matrices. A second promising line of research would be the derivation
of formulas for weight distribution related to different metrics, e.g. Lee metric or Rank metric.

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