# Robin-Robin Domain Decomposition Methods for the Stokes-Darcy Coupling

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**Abstract.** In this paper we consider a coupled system made of the Stokes and Darcy equations, and we propose some iteration-by-subdomain methods based on Robin conditions on the interface. We prove the convergence of these algorithms, and for suitable finite element approximations we show that the rate of convergence is independent of the mesh size $h$. A special attention is paid to the optimization of the performance of the methods when both the kinematic viscosity $\nu$ of the fluid and the hydraulic conductivity tensor $K$ of the porous medium are very small.

## 1   Introduction and problem setting

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain, decomposed in two non intersecting subdomains $\Omega_f$ and $\Omega_p$ separated by an interface $\Gamma$, i.e., $\overline{\Omega} = \overline{\Omega_f} \cup \overline{\Omega_p}$, $\Omega_f \cap \Omega_p = \emptyset$ and $\overline{\Omega_f} \cap \overline{\Omega_p} = \Gamma$.

We are interested in the case in which $\Gamma$ is a surface separating an upper domain $\Omega_f$ filled by a fluid, from a lower domain $\Omega_p$ formed by a porous medium. We assume that the fluid contained in $\Omega_f$ has an upper fixed surface (i.e., we do not consider the free surface fluid case) and can filtrate through the porous medium beneath.

The motion of the fluid in $\Omega_f$ is modeled by the Stokes equations:

$$-\boldsymbol{\nabla} \cdot \mathsf{T}(\mathbf{u}_f, p_f) = \mathbf{f}, \quad \nabla \cdot \mathbf{u}_f = 0 \qquad \text{in } \Omega_f , \tag{1}$$

where $\mathsf{T}(\mathbf{u}_f, p_f) = 2\nu\,\mathsf{D}(\mathbf{u}_f) - p_f\mathsf{I}$ is the stress tensor, and $\mathsf{D}(\mathbf{u}_f) = \frac{1}{2}(\nabla \mathbf{u}_f + \nabla^T \mathbf{u}_f)$ is the deformation tensor; as usual, $\nabla$ and $\nabla\cdot$ denote the gradient operator and the divergence operator, respectively, with respect to the space coordinates. The parameter $\nu > 0$ is the kinematic viscosity of the fluid, while $\mathbf{u}_f$ and $p_f$ denote the fluid velocity and pressure, respectively. We suppose $\nu$ to be constant in the whole domain $\Omega_f$.

In the lower domain $\Omega_p$ we define the piezometric head $\varphi = z + p_p/(\rho g)$, where $z$ is the elevation from a reference level, $p_p$ the pressure of the fluid in $\Omega_p$, $\rho > 0$ the density of the fluid (assumed to be constant in the whole domain $\Omega$) and $g > 0$ the gravity acceleration.

The flow in $\Omega_p$ is modeled by the equations:

$$\mathbf{u}_p = -\frac{K}{n}\nabla\varphi, \quad \nabla \cdot \mathbf{u}_p = 0 \qquad \text{in } \Omega_p \, , \tag{2}$$

where $\mathbf{u}_p$ is the fluid velocity, $n > 0$ is the volumetric porosity. The tensor $K$ is the hydraulic conductivity $K = \text{diag}\,(K_1, \ldots, K_d)$, and we suppose that $K_i \in L^\infty(\Omega_p)$ and $\inf_{\Omega_p} K_i > 0$, $i = 1, \ldots, d$. In the following we shall denote $\mathsf{K} = K/n = \text{diag}\,(K_i/n)$ $(i = 1, \ldots, d)$. The first equation in (2) is Darcy's law. For the sake of simplicity, we adopt homogenous boundary conditions. We impose the no-slip condition $\mathbf{u}_f = \mathbf{0}$ on $\Gamma_f = \partial\Omega_f \setminus \Gamma$ for the Stokes problem (1), while, for the Darcy problem (2), we set the piezometric head $\varphi = 0$ on the lateral surface $\Gamma_p$, and we require a slip condition on $\Gamma_p^b$: $\mathbf{u}_p \cdot \mathbf{n}_p = 0$ on $\Gamma_p$, where $\partial\Omega_p = \Gamma \cup \Gamma_p^b \cup \Gamma_p$ (see Fig. 1). The vectors $\mathbf{n}_p$ and $\mathbf{n}_f$ denote the unit outward normal vectors to the surfaces $\partial\Omega_p$ and $\partial\Omega_f$, respectively; in particular, we have $\mathbf{n}_f = -\mathbf{n}_p$ on $\Gamma$. In the following we shall indicate $\mathbf{n} = \mathbf{n}_f$ for simplicity of notation. We also assume that the boundary $\partial\Omega$ and the interface $\Gamma$ are piecewise smooth manifolds.

Other boundary conditions (see, e.g., [8, 9], [13] and [10, 11]) could also be considered, and all the results in this paper would remain true without essential changes in the proofs.
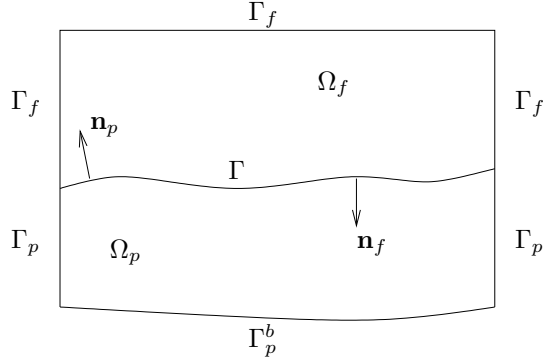


Figure 1: Schematic representation of a 2D vertical section of the computational domain.

We supplement the Stokes and Darcy problems with the following matching conditions on $\Gamma$ (see [12]):

$$\mathbf{u}_p \cdot \mathbf{n} = \mathbf{u}_f \cdot \mathbf{n} \, , \tag{3}$$

$$-\varepsilon\boldsymbol{\tau}_j \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) = \nu\mathbf{u}_f \cdot \boldsymbol{\tau}_j, \qquad j = 1, \ldots, d-1 \, , \tag{4}$$

$$-\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) = g\varphi_{|\Gamma} \, , \tag{5}$$

where $\boldsymbol{\tau}_j$ $(j = 1, \ldots, d-1)$ are linear independent unit tangential vectors to the interface $\Gamma$, and $\varepsilon$ represents the characteristic length of the pores of the porous medium.

Conditions (3)-(5) impose the continuity of the normal velocity on $\Gamma$, as well as that of the normal component of the normal stress, but they allow the pressure to be discontinuous across the interface.

This problem has been studied in several works. In [6, 8, 9] the mathematical and numerical analysis of the coupled problem was carried out, in the case in which the Darcy equation is replaced by a scalar elliptic problem for the sole piezometric head $\varphi$. The analysis of the coupled problem in its original form (1)-(2) has been considered in [13] and [10], and the recent works [18] and [11] address the analysis and preconditioning of mortar discretizations of the Stokes-Darcy problem.

A domain decomposition method of Dirichlet-Neumann type based on the choice of the fluid normal velocity across $\Gamma$ as interface variable was proposed and analyzed in [8, 9]. A similar approach, using the trace of $\varphi$ on $\Gamma$ as interface variable, has been studied in [6]. After proving that this method is equivalent to a preconditioned Richardson algorithm for the Steklov-Poincaré interface equation associated to the Stokes-Darcy problem, it was proved that the convergence rate of the algorithm is independent of the mesh parameter $h$, for suitable conforming finite element approximations of the coupled problem. An extension to the time-dependent case has been presented in [7].

The previous results indicate that, in the steady case, preconditioners of Dirichlet-Neumann type may be sensitive to the variation of the viscosity $\nu$ and of the entries of the hydraulic conductivity $\mathsf{K}$, downgrading the convergence rate of the algorithm.

In this work we extend some preliminary results contained in [6], by presenting improved domain decomposition methods based on Robin interface conditions. The aim is twofold: first, to propose an algorithm whose rate of convergence does not deteriorate as $\nu$ and the entries of $\mathsf{K}$ become smaller and smaller; secondly, devise an algorithm that is more "symmetric" with respect to the treatment of either $\Omega_f$ and $\Omega_p$, namely, being based on solvers that treat simultaneously (i.e., in parallel) the two subdomains.

After having presented in Sect. 2 the weak formulation of the coupled problem, in Sect. 3 we introduce two methods, based on a multiplicative and on an additive paradigm, respectively. Then, in Sect. 4 the convergence analysis of the algoritms is developed. Finally, some numerical results are presented in Sect. 5. The first algorithm has optimal convergence properties with respect to $\nu$ and $\mathsf{K}$. On the other hand, the second algorithm, which indeed for small values of $\nu$ and $\mathsf{K}$ does not outperform the Dirichlet-Neumann scheme, is interesting for its parallel nature. Moreover, its convergence analysis is rather simple, and is based on the fact that the so-called Robin-to-Dirichlet and Robin-to-Neumann maps are symmetric and positive, uniformly with respect to the mesh size $h$. These important properties seem to be yet overlooked in the literature, and could reveal very useful also in different contexts.

## 2 Weak form of the coupled problem

From now on, instead of (2), we will take the following scalar formulation of the Darcy problem:
$$-\nabla \cdot (\mathsf{K}\nabla\varphi) = 0 \quad \text{in } \Omega_p. \tag{6}$$

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Accordingly, (3) becomes

$$-\mathsf{K}\nabla\varphi \cdot \mathbf{n} = \mathbf{u}_f \cdot \mathbf{n} \quad \text{on } \Gamma. \tag{7}$$

We define the following functional spaces:

$$H_f = \{\mathbf{v} \in (H^1(\Omega_f))^d | \ \mathbf{v} = \mathbf{0} \text{ on } \Gamma_f\}, \quad Q = L^2(\Omega_f), \tag{8}$$

$$H_p = \{\psi \in H^1(\Omega_p) | \ \psi = 0 \text{ on } \Gamma_p^b\}, \tag{9}$$

and the bilinear forms

$$a_f(\mathbf{v}, \mathbf{w}) = 2\nu \int_{\Omega_f} \mathsf{D}(\mathbf{v}) : \mathsf{D}(\mathbf{w}) \qquad \forall \mathbf{v}, \mathbf{w} \in (H^1(\Omega_f))^d, \tag{10}$$

$$b_f(\mathbf{v}, q) = -\int_{\Omega_f} q\, \nabla \cdot \mathbf{v} \qquad \forall \mathbf{v} \in (H^1(\Omega_f))^d, \quad \forall q \in Q, \tag{11}$$

$$a_p(\varphi, \psi) = \int_{\Omega_p} \nabla\psi \cdot \mathsf{K}\nabla\varphi \qquad \forall \varphi, \psi \in H^1(\Omega_p). \tag{12}$$

The coupling conditions (4), (5), (7) can be incorporated in the weak formulation of the global problem as natural conditions on $\Gamma$. In particular, we can write the following weak saddle-point formulation of the coupled Stokes-Darcy problem:
Find $(\mathbf{u}_f, p_f) \in H_f \times Q$, $\varphi \in H_p$ such that

$$a_f(\mathbf{u}_f, \mathbf{v}) + b_f(\mathbf{v}, p_f) + g\, a_p(\varphi, \psi) + \int_\Gamma g\, \varphi(\mathbf{v} \cdot \mathbf{n}) - \int_\Gamma g\, \psi(\mathbf{u}_f \cdot \mathbf{n})$$

$$+ \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{u}_f \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \qquad \forall \mathbf{v} \in H_f, \ \psi \in H_p \tag{13}$$

$$b_f(\mathbf{u}_f, q) = 0 \qquad \forall q \in Q. \tag{14}$$

Using Brezzi's theory of saddle-point problems [2], we can guarantee that the coupled problem (13)-(14) has a unique solution (see [6, 8] and also [13]).
In the rest of the paper, instead of (4) we shall adopt the following simplified condition on the interface:

$$\mathbf{u}_f \cdot \boldsymbol{\tau}_j = 0 \quad \text{on } \Gamma \qquad (j = 1, \dots, d-1), \tag{15}$$

and, consequently, we will use the functional space:

$$H_f^\tau = \{\mathbf{v} \in H_f | \ \mathbf{v} \cdot \boldsymbol{\tau}_j = 0 \text{ on } \Gamma, \ j = 1, \dots, d-1\}. \tag{16}$$

This simplification is acceptable from the physical viewpoint since the term in (4) involving the normal derivative of $\mathbf{u}_f$ is multiplied by $\varepsilon$ and the velocity itself can be supposed at least of order $O(\varepsilon)$ in the neighborhood of $\Gamma$, so that the left hand side can be approximated by zero. We point out that this simplification does not dramatically influence the coupling of the two subproblems since (4) is not strictly speaking a coupling condition, but only a boundary condition for the fluid problem in $\Omega_f$. In any case, all the results in the paper are still true for the more general interface condition (4), provided $H_f^\tau$ is replaced by $H_f$ and the bilinear form $a_f(\mathbf{w}, \mathbf{v})$ by $a_f(\mathbf{w}, \mathbf{v}) + \int_\Gamma \sum_{j=1}^{d-1} \frac{\nu}{\varepsilon}(\mathbf{w} \cdot \boldsymbol{\tau}_j)(\mathbf{v} \cdot \boldsymbol{\tau}_j)$.

**Remark 2.1** *In [8, 9] we considered another simplified form of (4), i.e., $\boldsymbol{\tau}_j \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) = 0$ on $\Gamma$. Although not completely precise from the physical point of view, this simplified condition is perfectly acceptable from the mathematical viewpoint for the set-up and analysis of solution methods for the coupled problem.*

# 3 Iterative domain decomposition methods for solving the coupled problem

In this section we propose new iterative methods to compute the solution of the coupled problem which exploit the decoupled structure of the problem, thus requiring at each step to solve independently the fluid and the groundwater subproblems, i.e., using as building blocks a Stokes solver and an elliptic solver. As we have already remarked, the numerical performances of the domain decomposition methods of Dirichlet-Neumann type presented in [8, 9] strongly depend on the fluid viscosity $\nu$ and on the entries of the hydraulic conductivity $\mathsf{K}$. More precisely, the convergence rate of the algorithm deteriorates as $\nu$ and the entries of $\mathsf{K}$ decrease. The following numerical example illustrates the situation.

**Example 3.1** *We consider the computational domain $\Omega \subset \mathbb{R}^2$ with $\Omega_f = (0,1) \times (1,2)$, $\Omega_p = (0,1) \times (0,1)$ and $\Gamma = (0,1) \times \{1\}$, and choose the parameter $g = 1$; moreover, we assume that the hydraulic conductivity tensor $\mathsf{K}$ is a multiple of the identity tensor, namely, a scalar function. Boundary conditions and the right hand side $\mathbf{f}$ are chosen in such a way that the exact solution of the coupled Stokes-Darcy problem is $\mathbf{u}_f = (y^2 - 2y + 1, x^2 - x)^T$, $p_f = 2\nu(x+y-1) + 1/(3\mathsf{K})$, $\varphi = (x(1-x)(y-1) + y^3/3 - y^2 + y)/\mathsf{K} + 2x\nu$, with $\nu$ and $\mathsf{K}$ constant in $\Omega_f$ and $\Omega_p$, respectively. Table 1 reports the number of iterations obtained for several choices of $\nu$ and $\mathsf{K}$ and four different grid sizes, using the preconditioned conjugate gradient (PCG) method on the interface equation, with the preconditioner characterized by the Dirichlet-Neumann method. A tolerance of $10^{-9}$ has been imposed on the relative increment. Taylor-Hood finite elements have been used to approximate the Stokes problem, and quadratic Lagrangian elements for the Darcy equation (6).*

| $\nu$ | $\mathsf{K}$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
|-------|------|-------|-------|-------|-------|
| 1 | 1 | 5 | 5 | 5 | 5 |
| $10^{-1}$ | $10^{-1}$ | 10 | 10 | 8 | 8 |
| $10^{-2}$ | $10^{-1}$ | 13 | 15 | 14 | 14 |
| $10^{-3}$ | $10^{-2}$ | 19 | 49 | 60 | 55 |
| $10^{-4}$ | $10^{-3}$ | 20 | 58 | 143 | 167 |
| $10^{-6}$ | $10^{-4}$ | 20 | 56 | 138 | 202 |

Table 1: Iterations using PCG with the Dirichlet-Neumann preconditioner with respect to several values of $\nu$ and $\mathsf{K}$ and of the grid parameter $h$ ($h_1 \approx 0.14$ and $h_i = h_1/2^{i-1}$, $i = 2, 3, 4$).

Such small values of $\nu$ and $\mathsf{K}$ are quite realistic for real-life physical flows. This fact motivates our interest to set up new algorithms that are more robust to parameter variations.

## 3.1 Iterative methods based on Robin interface conditions

We present two possible domain decomposition methods based on the adoption of Robin interface conditions, i.e., proper linear combinations of the coupling conditions (5) and (7).

### 3.1.1 A sequential Robin-Robin (sRR) method

We consider a sequential Robin-Robin (sRR) method which at each iteration requires to solve a Darcy problem in $\Omega_p$ followed by a Stokes problem in $\Omega_f$, both with Robin conditions on $\Gamma$. Precisely, the algorithm reads as follows. Having assigned a trace function $\eta^0 \in L^2(\Gamma)$, and two acceleration parameters $\gamma_f \geq 0$ and $\gamma_p > 0$, for each $k \geq 0$:

*i)* Find $\varphi^{k+1} \in H_p$ such that

$$\gamma_p a_p(\varphi^{k+1}, \psi) + \int_\Gamma g\varphi_{|\Gamma}^{k+1} \psi_{|\Gamma} = \int_\Gamma \eta^k \psi_{|\Gamma} \qquad \forall \, \psi \in H_p \ . \qquad (17)$$

This corresponds to imposing the following interface condition (in weak, or natural, form) for the Darcy problem:

$$-\gamma_p \mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} + g\varphi_{|\Gamma}^{k+1} = \eta^k \qquad \text{on } \Gamma \ . \qquad (18)$$

*ii)* Then, find $(\mathbf{u}_f^{k+1}, p_f^{k+1}) \in H_f^\tau \times Q$ such that

$$a_f(\mathbf{u}_f^{k+1}, \mathbf{v}) + b_f(\mathbf{v}, p_f^{k+1}) + \gamma_f \int_\Gamma (\mathbf{u}_f^{k+1} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})$$
$$= \int_\Gamma \left( \frac{\gamma_f}{\gamma_p}\eta^k - \frac{\gamma_f + \gamma_p}{\gamma_p} g\varphi_{|\Gamma}^{k+1} \right)(\mathbf{v} \cdot \mathbf{n}) + \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \quad \forall \, \mathbf{v} \in H_f^\tau, \qquad (19)$$
$$b_f(\mathbf{u}_f^{k+1}, q) = 0 \qquad \forall \, q \in Q \ .$$

This corresponds to imposing on the Stokes problem the following matching conditions on $\Gamma$ (still in natural form):

$$\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \cdot \mathbf{n}) + \gamma_f \mathbf{u}_f^{k+1} \cdot \mathbf{n} = \frac{\gamma_f}{\gamma_p}\eta^k - \frac{\gamma_f + \gamma_p}{\gamma_p}g\varphi_{|\Gamma}^{k+1}$$
$$= -g\varphi_{|\Gamma}^{k+1} - \gamma_f \mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} \qquad (20)$$
$$\mathbf{u}_f^{k+1} \cdot \boldsymbol{\tau}_j = 0, \qquad j = 1, \ldots, d-1.$$

*iii)* Finally, set

$$\eta^{k+1} = -\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \cdot \mathbf{n}) + \gamma_p \mathbf{u}_f^{k+1} \cdot \mathbf{n}$$
$$= (\gamma_f + \gamma_p)(\mathbf{u}_f^{k+1} \cdot \mathbf{n}) + \frac{\gamma_f + \gamma_p}{\gamma_p}g\varphi_{|\Gamma}^{k+1} - \frac{\gamma_f}{\gamma_p}\eta^k \in L^2(\Gamma). \ (21)$$

Concerning the solvability of problem (19), we note first that using the trace theorem and the Korn inequality (see, e.g., [3], p. 416), there exist two constants $\kappa_1, \kappa_2 > 0$ such that

$$\int_\Gamma |\mathbf{u}_f \cdot \mathbf{n}|^2 \leq \kappa_1 \left( \int_{\Omega_f} (|\mathbf{u}_f|^2 + |\nabla\mathbf{u}_f|^2) \right) \leq \kappa_2 \int_{\Omega_f} |\mathsf{D}(\mathbf{u}_f)|^2 \ . \qquad (22)$$

Therefore, the bilinear form

$$a_f(\mathbf{u}_f, \mathbf{v}) + \gamma_f \int_\Gamma (\mathbf{u}_f \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})$$

is continuous and coercive in $H_f^\tau \times H_f^\tau$. Moreover, the bilinear form $b_f(\mathbf{v}, p)$ satisfies an inf–sup condition on the space $H_f^\tau \times Q$ (see, e.g., [17], pp. 157–158). Then, for every $\mathbf{f} \in (L^2(\Omega_f))^d$, $\eta^k \in L^2(\Gamma)$ and $\varphi_{|\Gamma}^{k+1} \in L^2(\Gamma)$, there exists a unique solution of problem (19).

If the sRR method converges, in the limit we recover the solution $(\mathbf{u}_f, p_f) \in H_f^\tau \times Q$ and $\varphi \in H_p$ of the coupled Stokes-Darcy problem. Indeed, denoting by $\varphi^*$ the limit of the sequence $\varphi^k$ in $H^1(\Omega_p)$ and by $(\mathbf{u}_f^*, p_f^*)$ that of $(\mathbf{u}_f^k, p_f^k)$ in $(H^1(\Omega_f))^d \times Q$, we obtain

$$-\gamma_p \mathsf{K}\nabla\varphi^* \cdot \mathbf{n} + g\varphi_{|\Gamma}^* = -\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^*, p_f^*) \cdot \mathbf{n}) + \gamma_p \mathbf{u}_f^* \cdot \mathbf{n} \quad \text{on } \Gamma, \qquad (23)$$

so that, as a consequence of (20), we have

$$(\gamma_f + \gamma_p)\mathbf{u}_f^* \cdot \mathbf{n} = -(\gamma_f + \gamma_p)\mathsf{K}\nabla\varphi^* \cdot \mathbf{n} \quad \text{on } \Gamma,$$

yielding, since $\gamma_f + \gamma_p \neq 0$, $\mathbf{u}_f^* \cdot \mathbf{n} = -\mathsf{K}\nabla\varphi^* \cdot \mathbf{n}$ on $\Gamma$, and also, from (23), that $\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^*, p_f^*) \cdot \mathbf{n}) = -g\varphi_{|\Gamma}^*$ on $\Gamma$. Thus, the two interface conditions (5) and (7) are satisfied, and we can conclude that the limit functions $\varphi^* \in H_p$ and $(\mathbf{u}_f^*, p_f^*) \in H_f^\tau \times Q$ are the solutions of the coupled Stokes-Darcy problem.

The proof of convergence will be given in Sect. 4.1.

### 3.1.2 A parallel Robin-Robin (pRR) method

We consider now a parallel Robin-Robin (pRR) algorithm. The idea behind this new method resembles that for a Neumann-Neumann scheme. However, the latter cannot be considered straightforwardly in our case, since we would not be able to guarantee the correct regularity of the data for each subproblem, as we shall point out more precisely in Remark 3.1.
The pRR algorithm that we propose reads as follows. Let $\mu^k \in L^2(\Gamma)$ be an assigned trace function on $\Gamma$, and $\gamma_1, \gamma_2$ be two positive parameters; then, for $k \geq 0$,

*i)* Find $(\mathbf{u}_f^{k+1}, p_f^{k+1}) \in H_f^\tau \times Q$ such that

$$a_f(\mathbf{u}_f^{k+1}, \mathbf{v}) + b_f(\mathbf{v}, p_f^{k+1}) - \gamma_1 \int_\Gamma (\mathbf{u}_f^{k+1} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})$$
$$= \int_\Gamma \mu^k(\mathbf{v} \cdot \mathbf{n}) + \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in H_f^\tau, \qquad (24)$$
$$b_f(\mathbf{u}_f^{k+1}, q) = 0 \qquad \forall\, q \in Q,$$

and, at the same time, find $\varphi^{k+1} \in H_p$ such that

$$a_p(\varphi^{k+1}, \psi) + \frac{1}{\gamma_1}\int_\Gamma g\varphi_{|\Gamma}^{k+1}\psi_{|\Gamma} = -\frac{1}{\gamma_1}\int_\Gamma \mu^k \psi_{|\Gamma} \qquad \forall\, \psi \in H_p. \qquad (25)$$

Remark that on the interface $\Gamma$ we are imposing the matching conditions

$$\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^{k+1}, p^{k+1}) \cdot \mathbf{n}) - \gamma_1 \mathbf{u}_f^{k+1} \cdot \mathbf{n} = \mu^k$$
$$= -g\varphi_{|\Gamma}^{k+1} + \gamma_1 \mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} \qquad (26)$$
$$\mathbf{u}_f^{k+1} \cdot \boldsymbol{\tau}_j = 0, \quad j = 1, \dots, d-1.$$

*ii)* As a second step, find $(\widehat{\boldsymbol{\omega}}^{k+1}, \widehat{\pi}^{k+1}) \in H_f^\tau \times Q$ such that

$$a_f(\widehat{\boldsymbol{\omega}}^{k+1}, \mathbf{v}) + b_f(\mathbf{v}, \widehat{\pi}^{k+1}) + \gamma_2 \int_\Gamma (\widehat{\boldsymbol{\omega}}^{k+1} \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})$$
$$= \gamma_2 \int_\Gamma \widehat{\sigma}^{k+1} (\mathbf{v} \cdot \mathbf{n}) \qquad \forall \, \mathbf{v} \in H_f^\tau, \tag{27}$$
$$b_f(\widehat{\boldsymbol{\omega}}^{k+1}, q) = 0 \qquad \forall \, q \in Q \; ,$$

and find $\widehat{\chi}^{k+1} \in H_p$ such that

$$a_p(\widehat{\chi}^{k+1}, \psi) + \frac{1}{\gamma_2} \int_\Gamma g \widehat{\chi}^{k+1}_{|\Gamma} \psi_{|\Gamma} = \int_\Gamma \widehat{\sigma}^{k+1} \psi_{|\Gamma} \qquad \forall \, \psi \in H_p \; , \tag{28}$$

where

$$\widehat{\sigma}^{k+1} = \mathbf{u}_f^{k+1} \cdot \mathbf{n} + \mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} = \mathbf{u}_f^{k+1} \cdot \mathbf{n} + \frac{1}{\gamma_1}(g\varphi^{k+1}_{|\Gamma} + \mu^k) \in L^2(\Gamma) \; . \tag{29}$$

Note that on the interface $\Gamma$ we are now imposing the matching conditions

$$\mathbf{n} \cdot (\mathsf{T}(\widehat{\boldsymbol{\omega}}^{k+1}, \widehat{\pi}^{k+1}) \cdot \mathbf{n}) + \gamma_2 \widehat{\boldsymbol{\omega}}^{k+1} \cdot \mathbf{n} = \gamma_2 \widehat{\sigma}^{k+1}$$
$$= g\widehat{\chi}^{k+1}_{|\Gamma} - \gamma_2 \mathsf{K}\nabla\widehat{\chi}^{k+1} \cdot \mathbf{n} \tag{30}$$
$$\widehat{\boldsymbol{\omega}}^{k+1} \cdot \boldsymbol{\tau}_j = 0, \quad j = 1, \dots, d-1.$$

*iii)* Finally, set

$$\begin{aligned} \mu^{k+1} &= \mu^k - \theta[\mathbf{n} \cdot (\mathsf{T}(\widehat{\boldsymbol{\omega}}^{k+1}, \widehat{\pi}^{k+1}) \cdot \mathbf{n}) + g\widehat{\chi}^{k+1}_{|\Gamma}] \\ &= \mu^k - \theta[\gamma_2(\widehat{\sigma}^{k+1} - \widehat{\boldsymbol{\omega}}^{k+1} \cdot \mathbf{n}) + g\widehat{\chi}^{k+1}_{|\Gamma}] \in L^2(\Gamma) \; , \end{aligned} \tag{31}$$

where $\theta > 0$ is a further acceleration parameter.

Before moving to the convergence analysis of the pRR method (24)-(31) a few remarks are in order.

Concerning the well-posedness of problem (24), since the inf–sup condition is satisfied (see [17], pp. 157–158), and thanks to (22), the bilinear form

$$a_f(\mathbf{u}_f, \mathbf{v}) - \gamma_1 \int_\Gamma (\mathbf{u}_f \cdot \mathbf{n})(\mathbf{v} \cdot \mathbf{n})$$

is coercive in $H_f^\tau \times H_f^\tau$ provided

$$\gamma_1 < \frac{2\nu}{\kappa_2} \; . \tag{32}$$

As regards the consistency of the algorithm, note that if we find a fixed point $\mu^*$, from (31) we have (again denoting the limit functions by an upper $*$):

$$\gamma_2(\widehat{\boldsymbol{\omega}}^* \cdot \mathbf{n} - \widehat{\sigma}^*) = g\widehat{\chi}^*_{|\Gamma} \qquad \text{on } \Gamma \; , \tag{33}$$

and also, equivalently,

$$\frac{1}{\gamma_2} g\widehat{\chi}^*_{|\Gamma} - \widehat{\sigma}^* = \frac{2}{\gamma_2} g\widehat{\chi}^*_{|\Gamma} - \widehat{\boldsymbol{\omega}}^* \cdot \mathbf{n} \qquad \text{on } \Gamma \; . \tag{34}$$

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Therefore, if we multiply (28) by $g$, sum the resulting equation to (27) and use relations (33) and (34), we obtain

$$a_f(\widehat{\boldsymbol{\omega}}^*, \mathbf{v}) + b_f(\mathbf{v}, \widehat{\pi}^*) + \int_\Gamma g\widehat{\chi}^*_{|\Gamma}(\mathbf{v} \cdot \mathbf{n}) + ga_p(\widehat{\chi}^*, \psi)$$

$$- \int_\Gamma g(\widehat{\boldsymbol{\omega}}^* \cdot \mathbf{n})\psi_{|\Gamma} + \int_\Gamma \frac{2g^2}{\gamma_2}\widehat{\chi}^*_{|\Gamma}\psi_{|\Gamma} = 0 \qquad \forall(\mathbf{v}, \psi) \in H_f^\tau \times H_p.$$

Taking $\mathbf{v} = \widehat{\boldsymbol{\omega}}^*$ and $\psi = \widehat{\chi}^*$, we find

$$a_f(\widehat{\boldsymbol{\omega}}^*, \widehat{\boldsymbol{\omega}}^*) + ga_p(\widehat{\chi}^*, \widehat{\chi}^*) + \int_\Gamma \frac{2g^2}{\gamma_2}(\widehat{\chi}^*_{|\Gamma})^2 = 0 \ ,$$

hence $\widehat{\chi}^* = 0$ in $\Omega_p$, and $\widehat{\boldsymbol{\omega}}^* = \mathbf{0}$ in $\Omega_f$ thanks to the Korn inequality.
The interface equation (30) gives $\widehat{\sigma}^* = 0$ on $\Gamma$, hence $\mathbf{u}_f^* \cdot \mathbf{n} = -\mathsf{K}\nabla\varphi^* \cdot \mathbf{n}$ on $\Gamma$. Moreover, using (26), we obtain $\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^*, p_f^*) \cdot \mathbf{n}) = -g\varphi^*_{|\Gamma}$ on $\Gamma$. Thus, the two interface conditions (5) and (7) are fulfilled, so that the solutions $(\mathbf{u}_f^*, p_f^*) \in H_f^\tau \times Q$ and $\varphi^* \in H_p$ (corresponding to the fixed point $\mu^*$) satisfy the coupled Stokes-Darcy problem.

Our aim is now to prove that the map generating the sequence $\mu^k$ is a contraction in $L^2(\Gamma)$. We shall address this point in Sect. 4.2.

**Remark 3.1** *A Neumann-Neumann method corresponding to the choice of the normal velocity $\mathbf{u}_f \cdot \mathbf{n}$ as interface variable would involve the following steps. For an assigned function $\lambda^k \in H_{00}^{1/2}(\Gamma)$ with $\int_\Gamma \lambda^k = 0$ (we refer to [14] for a definition of the trace space $H_{00}^{1/2}(\Gamma)$), first solve a Stokes problem in $\Omega_f$ with boundary conditions $\mathbf{u}_f^{k+1} \cdot \mathbf{n} = \lambda^k$, $\mathbf{u}_f^{k+1} \cdot \boldsymbol{\tau}_j = 0$ on $\Gamma$, and a Darcy problem in $\Omega_p$ imposing $-\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} = \lambda^k$ on $\Gamma$. Then, similarly to (29), we have to compute $\widehat{\sigma}^{k+1} = -\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \cdot \mathbf{n}) - g\varphi_{|\Gamma}^{k+1}$ on $\Gamma$. Here, we would have $\widehat{\sigma}^{k+1} \in H^{-1/2}(\Gamma)$. Therefore, this regularity of $\widehat{\sigma}^{k+1}$ would not be enough to guarantee the solvability of the subsequent Darcy problem, which would demand to impose $g\widehat{\chi}_{|\Gamma}^{k+1} = \widehat{\sigma}^{k+1}$ as boundary condition on $\Gamma$. Thus, a Neumann-Neumann method does not guarantee that the regularity of the interface data is preserved at each iteration, and that the sequence $\lambda^k$ generated by the algorithm is in $H_{00}^{1/2}(\Gamma)$.*
*Of course one may speculate that this issue of lack of regularity is not relevant at the finite dimensional level, for instance for finite element approximation. However, the difficulty is only hidden, and we should expect that it will show up as the mesh parameter $h$ goes to 0.*

# 4 Convergence analysis

In the sequel, for either an open set or a manifold $D$, we denote the norm in the Sobolev space $H^s(D)$, $s \geq -1$, by $\|\cdot\|_{s,D}$.

## 4.1 Convergence of the sRR method

We prove that the sequences $\varphi^k$ and $(\mathbf{u}_f^k, p_f^k)$ generated by the sRR method (17)-(21) converge in $H^1(\Omega_p)$ and $(H^1(\Omega_f))^d \times Q$, respectively. As a consequence,

the sequence $\eta^k$ is convergent in the dual space $H^{-1/2}(\Gamma)$ and weakly convergent in $L^2(\Gamma)$.

The proof of convergence that we are presenting follows the guidelines of the theory by P.-L. Lions [15] for the Robin-Robin method (see also [17], Sect. 4.5). We denote by $\mathbf{e}_u^k = \mathbf{u}_f^k - \mathbf{u}_f$, $e_p^k = p_f^k - p_f$ and $e_\varphi^k = \varphi^k - \varphi$ the errors at the $k$-th step. Remark that, thanks to the linearity, the functions $(\mathbf{e}_u^k, e_p^k)$ satisfy problem (19) with $\mathbf{f} = \mathbf{0}$, while $e_\varphi^k$ is a solution to (17). Moreover, we assume that $\gamma_p = \gamma_f$, and we denote by $\gamma$ their common value.

Finally, let us point out that the solutions $(\mathbf{u}_f, p_f) \in H_f^\tau \times Q$ and $\varphi \in H_p$ of the coupled Stokes-Darcy problem satisfy $\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) \in H^{1/2}(\Gamma)$ (as it is equal to $-g\varphi_{|\Gamma}$ on $\Gamma$), and $\nabla\varphi \cdot \mathbf{n} \in L^2(\Gamma)$ (as it is equal to $-\mathsf{K}^{-1}\mathbf{u}_f \cdot \mathbf{n}$ on $\Gamma$), i.e., these functions enjoy a better regularity than one might usually expect. Therefore, the interface conditions (18) and (20) for the error functions hold in $L^2(\Gamma)$.

Let us come to the proof of convergence. Choosing $\psi = e_\varphi^{k+1}$ in (17), and using the identity

$$AB = \frac{1}{4}[(A+B)^2 - (A-B)^2] \; ,$$

we have

$$
\begin{aligned}
g\,a_p(e_\varphi^{k+1}, e_\varphi^{k+1}) &= \frac{1}{\gamma}\int_\Gamma (\eta^k - g e_{\varphi|\Gamma}^{k+1}) g e_{\varphi|\Gamma}^{k+1} \\
&= \frac{1}{4\gamma}\int_\Gamma (\eta^k)^2 - \frac{1}{4\gamma}\int_\Gamma (\eta^k - 2g e_{\varphi|\Gamma}^{k+1})^2 \; .
\end{aligned}
\tag{35}
$$

Similarly, taking $\mathbf{v} = \mathbf{e}_u^{k+1}$ in (19) and using (21) we have:

$$
\begin{aligned}
a_f(\mathbf{e}_u^{k+1}, \mathbf{e}_u^{k+1}) &= \frac{1}{\gamma}\int_\Gamma (\eta^k - 2g e_{\varphi|\Gamma}^{k+1} - \gamma\mathbf{e}_u^{k+1} \cdot \mathbf{n})(\gamma\mathbf{e}_u^{k+1} \cdot \mathbf{n}) \\
&= \frac{1}{4\gamma}\int_\Gamma (\eta^k - 2g e_{\varphi|\Gamma}^{k+1})^2 - \frac{1}{4\gamma}\int_\Gamma (\eta^k - 2g e_{\varphi|\Gamma}^{k+1} - 2\gamma\mathbf{e}_u^{k+1} \cdot \mathbf{n})^2 \\
&= \frac{1}{4\gamma}\int_\Gamma (\eta^k - 2g e_{\varphi|\Gamma}^{k+1})^2 - \frac{1}{4\gamma}\int_\Gamma (\eta^{k+1})^2 \; .
\end{aligned}
\tag{36}
$$

Adding (35) and (36) we find

$$g\,a_p(e_\varphi^{k+1}, e_\varphi^{k+1}) + a_f(\mathbf{e}_u^{k+1}, \mathbf{e}_u^{k+1}) + \frac{1}{4\gamma}\int_\Gamma (\eta^{k+1})^2 = \frac{1}{4\gamma}\int_\Gamma (\eta^k)^2 \; .$$

Summing over $k$ from $k = 0$ to $k = N$, with $N \geq 1$, we finally obtain

$$\sum_{k=0}^{N} \left(g\,a_p(e_\varphi^{k+1}, e_\varphi^{k+1}) + a_f(\mathbf{e}_u^{k+1}, \mathbf{e}_u^{k+1})\right) + \frac{1}{4\gamma}\int_\Gamma (\eta^{N+1})^2 = \frac{1}{4\gamma}\int_\Gamma (\eta^0)^2 .$$

Thus, the series

$$\sum_{k=0}^{\infty} \left(g\,a_p(e_\varphi^{k+1}, e_\varphi^{k+1}) + a_f(\mathbf{e}_u^{k+1}, \mathbf{e}_u^{k+1})\right)$$

is convergent, and the errors $e_\varphi^k$ and $\mathbf{e}_u^k$ tend to zero in $H^1(\Omega_p)$ and $(H^1(\Omega_f))^d$, respectively. The convergence of the pressure error $e_p^k$ to 0 in $Q$ is then a well-known consequence of the convergence of the velocity.

### 4.1.1 Interpretation of the sRR method as an alternating direction scheme

The sRR method can be interpreted as an alternating direction scheme (see [1]; see also [6]). For technical reasons, to make precise this statement let us assume that the a flux boundary condition $\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n} = \mathbf{g}$ is imposed on the top of the fluid domain $\Omega_f$, $\mathbf{g}$ being a given vector function. Moreover, we assume that the interface $\Gamma$ is smooth, say, a $\mathcal{C}^2$-manifold with boundary.
Then, introduce the spaces

$$\widehat{H}_f = \{\mathbf{v} \in (H^1(\Omega_f))^d | \ \mathbf{v} = \mathbf{0} \text{ on the lateral boundary of } \Omega_f\}$$
$$\widehat{H}_f^\tau = \{\mathbf{v} \in \widehat{H}_f | \ \mathbf{v} \cdot \boldsymbol{\tau}_j = 0 \text{ on } \Gamma, \ j = 1, \dots, d-1\}$$
$$\widehat{H}_f^{\tau,n} = \{\mathbf{v} \in \widehat{H}_f^\tau | \ \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ on } \Gamma\}, \quad H_p^0 = \{\psi \in H_p | \ \psi = 0 \ \text{ on } \Gamma_p\} \, ,$$

and define the operator $S_f$ as

$$S_f : H_{00}^{1/2}(\Gamma) \to (H_{00}^{1/2}(\Gamma))', \quad \chi \to S_f\chi = \mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{n}),$$

where $(\mathbf{u}_\chi, p_\chi) \in \widehat{H}_f^\tau \times Q$ satisfies

$$a_f(\mathbf{u}_\chi, \mathbf{v}) + b_f(\mathbf{v}, p_\chi) = 0 \qquad \forall \ \mathbf{v} \in \widehat{H}_f^{\tau,n}(\Omega_f) \, ,$$
$$b_f(\mathbf{u}_\chi, q) = 0 \qquad \forall \ q \in Q \, ,$$

with $\mathbf{u}_\chi \cdot \mathbf{n} = \chi$ on $\Gamma$.
In a similar way, for each $\eta \in (H_{00}^{1/2}(\Gamma))'$ define the operator $S_p$ as

$$S_p : (H_{00}^{1/2}(\Gamma))' \to H_{00}^{1/2}(\Gamma), \quad \eta \to S_p\eta = g\varphi_{\eta|\Gamma},$$

where $\varphi_\eta \in H_p^0$ is the solution to

$$a_p(\varphi_\eta, \psi) = \langle \eta, \psi_{|\Gamma} \rangle_\Gamma \qquad \forall \ \psi \in H_p^0 \, ,$$

where $\langle \cdot, \cdot \rangle_\Gamma$ denotes the duality pairing between $(H_{00}^{1/2}(\Gamma))'$ and $H_{00}^{1/2}(\Gamma)$. As a consequence, we have $-\mathsf{K}\nabla\varphi_\eta \cdot \mathbf{n} = \eta$ on $\Gamma$.
Since for each $\varphi \in H_p^0$ we have $S_p(-\mathsf{K}\nabla\varphi \cdot \mathbf{n}) = g\varphi_{|\Gamma}$, the first step (19) of our procedure corresponds to imposing on $\Gamma$

$$
\begin{aligned}
-\gamma_p\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} + g\varphi_{|\Gamma}^{k+1} &= -\gamma_p\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} + S_p(-\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n}) \\
&= (\gamma_p I + S_p)(-\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n}) = \eta^k \, ,
\end{aligned}
$$

hence

$$-\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} = (\gamma_p I + S_p)^{-1}\eta^k \, . \tag{37}$$

On the other hand, the right hand side in (20) can be written as

$$
\begin{aligned}
-g\varphi_{|\Gamma}^{k+1} - \gamma_f\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} &= S_p(\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n}) - \gamma_f\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} \\
&= -(\gamma_f I - S_p)\mathsf{K}\nabla\varphi^{k+1} \cdot \mathbf{n} \\
&= (\gamma_f I - S_p)(\gamma_p I + S_p)^{-1}\eta^k \, . \tag{38}
\end{aligned}
$$

In an analogous way, still denoting by $(\mathbf{u}_f^{k+1}, p_f^{k+1})$ the solution to (19) with $\mathbf{f} = \mathbf{0}$ and $H_f^\tau$ replaced by $\widehat{H}_f^\tau$, one has $S_f(\mathbf{u}_f^{k+1} \cdot \mathbf{n}) = \mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f^{k+1}, p_f^{k+1}) \cdot \mathbf{n})$. Then, the left hand side in (20) can be written as

$$\begin{aligned}
\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}^{k+1}, p^{k+1}) \cdot \mathbf{n}) + \gamma_f \mathbf{u}^{k+1} \cdot \mathbf{n} &= S_f(\mathbf{u}^{k+1} \cdot \mathbf{n}) + \gamma_f \mathbf{u}^{k+1} \cdot \mathbf{n} \\
&= (\gamma_f I + S_f)(\mathbf{u}^{k+1} \cdot \mathbf{n}) .
\end{aligned} \tag{39}$$

Using (38) and (39), the interface condition (20) becomes

$$\mathbf{u}^{k+1} \cdot \mathbf{n} = (\gamma_f I + S_f)^{-1}(\gamma_f I - S_p)(\gamma_p I + S_p)^{-1}\eta^k . \tag{40}$$

In conclusion, our iterative procedure (with homegeneous data $\mathbf{f}$ and $\mathbf{g}$) can be written as

$$\begin{aligned}
\eta^{k+1} &= -\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}^{k+1}, p^{k+1}) \cdot \mathbf{n}) + \gamma_p \mathbf{u}^{k+1} \cdot \mathbf{n} \\
&= -S_f(\mathbf{u}^{k+1} \cdot \mathbf{n}) + \gamma_p \mathbf{u}^{k+1} \cdot \mathbf{n} \\
&= (\gamma_p I - S_f)\mathbf{u}^{k+1} \cdot \mathbf{n} \\
&= (\gamma_p I - S_f)(\gamma_f I + S_f)^{-1}(\gamma_f I - S_p)(\gamma_p I + S_p)^{-1}\eta^k .
\end{aligned} \tag{41}$$

This is an alternating direction scheme, *à la* Peaceman–Rachford (see [16]), that has been deeply analyzed. Sufficient conditions for convergence are that $\gamma_f = \gamma_p$ and the operators $S_f$ and $S_p$ are bounded and strictly positive in a given Hilbert space. These do not apply in the present situation, as the operators $S_f$ and $S_p$ act from a space into its dual. In fact, we can only prove that the iteration operator is non-expansive, but not a contraction in $(H_{00}^{1/2}(\Gamma))'$.

On the other hand, it is worthy to note that the convergence of this alternating direction scheme can be easily proved in the discrete case, as the matrices that correspond to the finite dimensional Steklov–Poincaré operators $S_f$ and $S_p$ are in fact symmetric and positive definite.

To illustrate how the proof of convergence works, we consider a suitable modification of the iteration scheme. Let us introduce the operators $J_- : H_{00}^{1/2}(\Gamma) \to (H_{00}^{1/2}(\Gamma))'$ and $J_+ : (H_{00}^{1/2}(\Gamma))' \to H_{00}^{1/2}(\Gamma)$ defined as follows:

$$\begin{aligned}
(J_- \chi, \mu)_{-1/2,00,\Gamma} &= \langle \mu, \chi \rangle_\Gamma \qquad \forall \chi \in H_{00}^{1/2}(\Gamma) , \; \mu \in (H_{00}^{1/2}(\Gamma))' \\
(J_+ \eta, \xi)_{1/2,00,\Gamma} &= \langle \eta, \xi \rangle_\Gamma \qquad \forall \eta \in (H_{00}^{1/2}(\Gamma))' , \; \xi \in H_{00}^{1/2}(\Gamma) .
\end{aligned}$$

(Here and in the sequel we are denoting by $(\cdot, \cdot)_{1/2,00,\Gamma}$ and $(\cdot, \cdot)_{-1/2,00,\Gamma}$ the scalar products in $H^{1/2}(\Gamma)$ and $(H_{00}^{1/2}(\Gamma))'$, respectively, and by $\| \cdot \|_{1/2,00,\Gamma}$ and $\| \cdot \|_{-1/2,00,\Gamma}$ the associated norms.)

The existence of these operators is guaranteed by the Riesz representation theorem. Moreover, it is easily verified that $\|J_- \chi\|_{-1/2,00,\Gamma} = \|\chi\|_{1/2,00,\Gamma}$, $\|J_+ \eta\|_{1/2,00,\Gamma} = \|\eta\|_{-1/2,00,\Gamma}$ (so that the operator norms are $\|J_-\| = \|J_+\| = 1$), and $(J_- \chi, \eta)_{-1/2,00,\Gamma} = (\chi, J_+ \eta)_{1/2,00,\Gamma}$.

We consider the following iterative scheme:

$$\eta^{k+1} = (\gamma J_- - S_f)(\gamma J_- + S_f)^{-1} J_- (\gamma J_+ - S_p)(\gamma J_+ + S_p)^{-1} J_+ \eta^k . \tag{42}$$

This represents a slight modification of (41) in which we have inserted the operators $J_-$ and $J_+$ instead of the identity $I$, and we have taken $\gamma_p = \gamma_f = \gamma$. The convergence of (42) is a consequence of the contraction mapping theorem (see the Appendix).

**Remark 4.1** *One could argue that the iterative scheme (42) is not relevant with the problem at hand, since it is not equivalent to (41). Indeed, (42) converges to our original problem with slightly modified interface conditions, which read*

$$\gamma J_-(\mathbf{u}_f \cdot \mathbf{n}) + \mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n}) = -\gamma J_- J_+(\mathsf{K}\nabla\varphi \cdot \mathbf{n}) - J_-(g\varphi_{|\Gamma}) \quad \text{on } \Gamma$$
$$\gamma J_+ J_-(\mathbf{u}_f \cdot \mathbf{n}) - J_+(\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_f, p_f) \cdot \mathbf{n})) = -\gamma J_+(\mathsf{K}\nabla\varphi \cdot \mathbf{n}) + g\varphi_{|\Gamma} \quad \text{on } \Gamma \;.$$

*The operators $J_-$ and $J_+$ have the role of assuring that the functions on either sides are in the same trace space.*
*The problem of equalization of trace spaces can be encountered in other domain decompositions of heterogeneous problems as well. For these cases, the procedure that we have advocated here (and the associated convergence proof) might be useful.*

## 4.2 Convergence of the pRR method

We turn now to the proof of convergence of the parallel method (24)-(31). Our aim is to prove that the map $\mu^k \to \mu^{k+1}$ defined through (24)–(31) is a contraction in $L^2(\Gamma)$. As a consequence of linearity, in the whole section we can assume without restriction that $\mathbf{f} = \mathbf{0}$. In order to introduce a suitable representation of this map, we define several interface operators.
Let $\mathcal{H}_S$ be the *Robin-to-Dirichlet* map for the *Stokes problem*,

$$\mathcal{H}_S : L^2(\Gamma) \to L^2(\Gamma), \qquad \mu \to \mathcal{H}_S\mu = \mathbf{u}_\mu \cdot \mathbf{n}, \tag{43}$$

where $(\mathbf{u}_\mu, p_\mu) \in H_f^\tau \times Q$ is the solution to (24) with $\mathbf{f} = \mathbf{0}$ and the Robin boundary datum $\mu$.
Define $\mathcal{H}_D$ as the *Robin-to-Neumann* operator for the *Darcy scalar problem*,

$$\mathcal{H}_D : L^2(\Gamma) \to L^2(\Gamma), \qquad \mu \to \mathcal{H}_D\mu = \frac{1}{\gamma_1}(g\varphi_{\mu|\Gamma} + \mu), \tag{44}$$

where $\varphi_\mu \in H_p$ is the solution to (25) corresponding to the Robin boundary datum $\mu$.
Moreover, let $\mathcal{K}_S$ be the *Robin-to-Neumann* operator for the *Stokes problem*,

$$\mathcal{K}_S : L^2(\Gamma) \to L^2(\Gamma), \qquad \sigma \to \mathcal{K}_S\sigma = \gamma_2(\sigma - \boldsymbol{\omega}_\sigma \cdot \mathbf{n}), \tag{45}$$

where $(\boldsymbol{\omega}_\sigma, \pi_\sigma) \in H_f^\tau \times Q$ is the solution to (27) with the Robin boundary datum $\sigma$.
Finally, $\mathcal{K}_D$ denotes the *Robin-to-Dirichlet* operator for the *Darcy scalar problem*,

$$\mathcal{K}_D : L^2(\Gamma) \to L^2(\Gamma), \qquad \sigma \to \mathcal{K}_D\sigma = g\chi_{\sigma|\Gamma}, \tag{46}$$

$\chi_\sigma \in H_p$ being the solution to (28) with the Robin boundary datum $\sigma$.
By means of these operators, we reformulate (29) as:

$$\widehat{\sigma}^{k+1} = \mathcal{H}_S\mu^k + \mathcal{H}_D\mu^k = (\mathcal{H}_S + \mathcal{H}_D)\mu^k,$$

and the relaxation step (31) as:

$$\begin{aligned}
\mu^{k+1} &= \mu^k - \theta(\mathcal{K}_S\widehat{\sigma}^{k+1} + \mathcal{K}_D\widehat{\sigma}^{k+1}) = \mu^k - \theta(\mathcal{K}_S + \mathcal{K}_D)(\mathcal{H}_S + \mathcal{H}_D)\mu^k \\
&= [I - \theta(\mathcal{K}_S + \mathcal{K}_D)(\mathcal{H}_S + \mathcal{H}_D)]\mu^k.
\end{aligned}$$

**Proposition 4.1** *The operators defined in (43)-(46) enjoy the following properties:*

1. $\mathcal{H}_S$ *and* $\mathcal{K}_D$ *are symmetric, continuous and non-negative in* $L^2(\Gamma)$*;*

2. $\mathcal{H}_D$ *and* $\mathcal{K}_S$ *are symmetric, continuous and coercive in* $L^2(\Gamma)$*.*

*Proof.* 1. We consider first the operator $\mathcal{H}_S$. For every $\eta$ and $\mu$, letting $\mathbf{u}_\eta \cdot \mathbf{n} = \mathcal{H}_S\eta$ and $\mathbf{u}_\mu \cdot \mathbf{n} = \mathcal{H}_S\mu$, we have:

$$
\begin{aligned}
\int_\Gamma (\mathcal{H}_S\mu)\eta &= \int_\Gamma \mathbf{u}_\mu \cdot \mathbf{n}\,\eta = a_f(\mathbf{u}_\eta, \mathbf{u}_\mu) - \gamma_1 \int_\Gamma (\mathbf{u}_\eta \cdot \mathbf{n})(\mathbf{u}_\mu \cdot \mathbf{n}) \\
&= \int_\Gamma \mu\, \mathbf{u}_\eta \cdot \mathbf{n} = \int_\Gamma \mu\,(\mathcal{H}_S\eta),
\end{aligned}
$$

therefore $\mathcal{H}_S$ is symmetric.
Now, taking $\mathbf{v} = \mathbf{u}_\mu$ in (24) (with $\mathbf{f} = \mathbf{0}$), thanks to (22) we have

$$
\begin{aligned}
2\nu \int_{\Omega_f} |\mathsf{D}(\mathbf{u}_\mu)|^2 &= a_f(\mathbf{u}_\mu, \mathbf{u}_\mu) = \gamma_1 \int_\Gamma |\mathbf{u}_\mu \cdot \mathbf{n}|^2 + \int_\Gamma \mu\, \mathbf{u}_\mu \cdot \mathbf{n} \\
&\leq \gamma_1 \kappa_2 \int_{\Omega_f} |\mathsf{D}(\mathbf{u}_\mu)|^2 + \kappa_2^{1/2} \|\mu\|_{0,\Gamma} \|\mathsf{D}(\mathbf{u}_\mu)\|_{0,\Omega_f} .
\end{aligned}
$$

Therefore, for $\gamma_1 < (2\nu)/\kappa_2$, one has $\|\mathsf{D}(\mathbf{u}_\mu)\|_{0,\Omega_f} \leq \kappa_3 \|\mu\|_{0,\Gamma}$, with $\kappa_3 = \kappa_2^{1/2}/(2\nu - \gamma_1\kappa_2)$. Hence, from (22), $\mathcal{H}_S$ is a continuous operator.
Finally, for $\gamma_1 < (2\nu)/\kappa_2$ we have

$$
\int_\Gamma (\mathcal{H}_S\mu)\,\mu = 2\nu \int_{\Omega_f} |\mathsf{D}(\mathbf{u}_\mu)|^2 - \gamma_1 \int_\Gamma |\mathbf{u}_\mu \cdot \mathbf{n}|^2 \geq (2\nu - \gamma_1\kappa_2) \int_{\Omega_f} |\mathsf{D}(\mathbf{u}_\mu)|^2 \geq 0 ,
$$

hence $\mathcal{H}_S$ is a non-negative operator.

We consider now the operator $\mathcal{K}_D$. We denote by $\chi_\sigma$ and $\chi_\xi$ the solutions to (28) with data $\sigma$ and $\xi$, respectively. Thus, $\mathcal{K}_D\sigma = g\chi_{\sigma|\Gamma}$ and $\mathcal{K}_D\xi = g\chi_{\xi|\Gamma}$. Then, using (28) we have

$$
\begin{aligned}
\int_\Gamma (\mathcal{K}_D\sigma)\xi &= \int_\Gamma g\chi_{\sigma|\Gamma}\xi = g\,a_p(\chi_\xi, \chi_\sigma) + \frac{g^2}{\gamma_2} \int_\Gamma \chi_{\xi|\Gamma}\,\chi_{\sigma|\Gamma} \\
&= \int_\Gamma g\sigma\chi_{\xi|\Gamma} = \int_\Gamma \sigma(\mathcal{K}_D\xi) ,
\end{aligned}
$$

which proves the symmetry of $\mathcal{K}_D$.
Now, if we take in (28) the test function $\psi = \chi_\sigma$, we find

$$
a_p(\chi_\sigma, \chi_\sigma) + \frac{1}{\gamma_2} \int_\Gamma g\chi_{\sigma|\Gamma}^2 = \int_\Gamma \sigma\chi_{\sigma|\Gamma} \leq \left(\int_\Gamma \sigma^2\right)^{1/2} \left(\int_\Gamma \chi_{\sigma|\Gamma}^2\right)^{1/2} ;
$$

consequently, since $a_p(\chi_\sigma, \chi_\sigma) \geq 0$, we have $g\|\chi_{\sigma|\Gamma}\|_{0,\Gamma} \leq \gamma_2\|\sigma\|_{0,\Gamma}$, i.e., $\mathcal{K}_D$ is a continuous operator.
Finally, $\mathcal{K}_D$ is non-negative, since

$$
\int_\Gamma (\mathcal{K}_D\sigma)\,\sigma = \int_\Gamma g\chi_{\sigma|\Gamma}\,\sigma = g\,a_p(\chi_\sigma, \chi_\sigma) + \frac{g^2}{\gamma_2} \int_\Gamma \chi_{\sigma|\Gamma}^2 \geq 0 \quad \forall \sigma \in L^2(\Gamma).
$$

2. Consider now the operator $\mathcal{H}_D$. For all $\mu$ and $\eta$ we denote by $\varphi_\mu$ and $\varphi_\eta$ the solutions of (25) corresponding to the data $\mu$ and $\eta$, respectively, so that $\mathcal{H}_D\mu = (g\varphi_{\mu|\Gamma} + \mu)/\gamma_1$ and $\mathcal{H}_D\eta = (g\varphi_{\eta|\Gamma} + \eta)/\gamma_1$. Then, proceeding as we did for the operator $\mathcal{K}_D$, we have

$$
\begin{aligned}
\int_\Gamma (\mathcal{H}_D\mu)\,\eta &= \frac{1}{\gamma_1}\int_\Gamma (\mu\,\eta + g\varphi_{\mu|\Gamma}\,\eta) \\
&= \frac{1}{\gamma_1}\int_\Gamma \mu\,\eta - \frac{g^2}{\gamma_1}\int_\Gamma \varphi_{\eta|\Gamma}\varphi_{\mu|\Gamma} - g\,a_p(\varphi_\eta, \varphi_\mu) \\
&= \frac{1}{\gamma_1}\int_\Gamma \mu\,\eta + \frac{g}{\gamma_1}\int_\Gamma \mu\,\varphi_{\eta|\Gamma} = \int_\Gamma \mu\,(\mathcal{H}_D\eta)\,,
\end{aligned}
$$

thus $\mathcal{H}_D$ is symmetric.

Moreover, taking $\psi = \varphi_\mu$ in (25), the continuity of $\mathcal{H}_D$ easily follows from the estimate:

$$
a_p(\varphi_\mu, \varphi_\mu) + \frac{g}{\gamma_1}\int_\Gamma \varphi_{\mu|\Gamma}^2 = -\frac{1}{\gamma_1}\int_\Gamma \mu\varphi_{\mu|\Gamma} \leq \frac{1}{\gamma_1}\left(\int_\Gamma \mu^2\right)^{1/2}\left(\int_\Gamma \varphi_{\mu|\Gamma}^2\right)^{1/2},
$$

that yields $\|\varphi_{\mu|\Gamma}\|_{0,\Gamma} \leq g^{-1}\|\mu\|_{0,\Gamma}$, as $a_p(\varphi_\mu, \varphi_\mu) \geq 0$.

Finally, let us show that $\mathcal{H}_D$ is a coercive operator. Recalling its definition, we have

$$
\begin{aligned}
a_p(\varphi_\mu, \varphi_\mu) &= -\frac{1}{\gamma_1}\int_\Gamma g\varphi_{\mu|\Gamma}^2 - \frac{1}{\gamma_1}\int_\Gamma \mu\varphi_{\sigma|\Gamma} = -\int_\Gamma (\mathcal{H}_D\mu)\,\varphi_{\mu|\Gamma} \\
&= -\frac{1}{g}\int_\Gamma (\mathcal{H}_D\mu)(\gamma_1\mathcal{H}_D\mu - \mu) = \frac{1}{g}\int_\Gamma (\mathcal{H}_D\mu)\,\mu - \frac{\gamma_1}{g}\int_\Gamma (\mathcal{H}_D\mu)^2\,.
\end{aligned}
$$

Consequently, since $a_p(\varphi_\mu, \varphi_\mu) \geq \kappa_3 \int_{\Omega_p} |\nabla\varphi_\mu|^2$ for a suitable constant $\kappa_3 > 0$, there exists a constant $q_1 > 0$ such that

$$
\int_\Gamma (\mathcal{H}_D\mu)\,\mu \geq q_1\left(\int_\Gamma (\mathcal{H}_D\mu)^2 + \int_{\Omega_p} |\nabla\varphi_\mu|^2\right)\,.
$$

On the other hand, using the trace inequality and the Poincaré inequality,

$$
\begin{aligned}
\int_\Gamma \mu^2 &= \int_\Gamma (\gamma_1\mathcal{H}_D\mu - g\varphi_{\mu|\Gamma})^2 \leq 2\gamma_1^2\int_\Gamma (\mathcal{H}_D\mu)^2 + 2g^2\int_\Gamma \varphi_{\mu|\Gamma}^2 \\
&\leq Q_1\left(\int_\Gamma (\mathcal{H}_D\mu)^2 + \int_{\Omega_p} |\nabla\varphi_\mu|^2\right),
\end{aligned}
$$

where $Q_1 > 0$ is a suitable constant. The coerciveness of $\mathcal{H}_D$ now follows.

Turning now to the operator $\mathcal{K}_S$, its symmetry can be proved as we did for $\mathcal{H}_S$. Moreover, taking $\mathbf{v} = \boldsymbol{\omega}_\sigma$ in (27) (where $\boldsymbol{\omega}_\sigma$ is the solution with datum $\sigma$), one has

$$
a_f(\boldsymbol{\omega}_\sigma, \boldsymbol{\omega}_\sigma) + \gamma_2\int_\Gamma (\boldsymbol{\omega}_\sigma\cdot\mathbf{n})^2 = \gamma_2\int_\Gamma \sigma\,\boldsymbol{\omega}_\sigma\cdot\mathbf{n}\,.
$$

Since $a_f(\boldsymbol{\omega}_\sigma, \boldsymbol{\omega}_\sigma) \geq 0$, this yields

$$
\int_\Gamma (\boldsymbol{\omega}_\sigma\cdot\mathbf{n})^2 \leq \int_\Gamma \sigma\,\boldsymbol{\omega}_\sigma\cdot\mathbf{n} \leq \left(\int_\Gamma \sigma^2\right)^{1/2}\left(\int_\Gamma (\boldsymbol{\omega}_\sigma\cdot\mathbf{n})^2\right)^{1/2}\,,
$$

and this proves that the operator $\mathcal{K}_S$ is continuous.

Finally, using the definition (45) of $\mathcal{K}_S$, we have

$$
\begin{aligned}
a_f(\boldsymbol{\omega}_\sigma, \boldsymbol{\omega}_\sigma) &= -\gamma_2 \int_\Gamma (\boldsymbol{\omega}_\sigma \cdot \mathbf{n})^2 + \gamma_2 \int_\Gamma \sigma \, \boldsymbol{\omega}_\sigma \cdot \mathbf{n} = \int_\Gamma (\mathcal{K}_S \sigma) \, \boldsymbol{\omega}_\sigma \cdot \mathbf{n} \\
&= \int_\Gamma (\mathcal{K}_S \sigma) \, (\sigma - \gamma_2^{-1} \mathcal{K}_S \sigma) = \int_\Gamma (\mathcal{K}_S \sigma) \, \sigma - \gamma_2^{-1} \int_\Gamma (\mathcal{K}_S \sigma)^2 \; .
\end{aligned}
$$

Therefore, since $a_f(\boldsymbol{\omega}_\sigma, \boldsymbol{\omega}_\sigma) = 2\nu \int_{\Omega_f} |\mathsf{D}(\boldsymbol{\omega}_\sigma)|^2$, there exists a constant $q_2 > 0$ such that

$$
\int_\Gamma (\mathcal{K}_S \sigma) \, \sigma \geq q_2 \left( \int_{\Omega_f} |\mathsf{D}(\boldsymbol{\omega}_\sigma)|^2 + \int_\Gamma (\mathcal{K}_S \sigma)^2 \right) \; .
$$

On the other hand, by the trace and the Korn inequalities, we have

$$
\begin{aligned}
\int_\Gamma \sigma^2 &= \int_\Gamma (\boldsymbol{\omega}_\sigma \cdot \mathbf{n} + \gamma_2^{-1} \mathcal{K}_S \sigma)^2 \leq 2 \int_\Gamma (\boldsymbol{\omega}_\sigma \cdot \mathbf{n})^2 + 2\gamma_2^{-2} \int_\Gamma (\mathcal{K}_S \sigma)^2 \\
&\leq Q_2 \left( \int_{\Omega_f} |\mathsf{D}(\boldsymbol{\omega}_\sigma)|^2 + \int_\Gamma (\mathcal{K}_S \sigma)^2 \right)
\end{aligned}
$$

for a suitable constant $Q_2 > 0$. Thus, the operator $\mathcal{K}_S$ is coercive. $\qquad \square$

It follows from Proposition 4.1 that the operators $\mathcal{H} = \mathcal{H}_S + \mathcal{H}_D$ and $\mathcal{K} = \mathcal{K}_S + \mathcal{K}_D$ are both *symmetric, continuous and coercive* on $L^2(\Gamma)$.

To prove the convergence of the pRR iterative scheme, we shall apply the following abstract result whose proof is similar to that of Theorem 4.2.5 in [17].

**Theorem 4.1** *Let $X$ be a (real) Hilbert space and $X'$ its dual. We consider a linear invertible continuous operator $\mathcal{Q} \colon X \to X'$, which can be split as $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$, where both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are linear operators. Taken $\mathcal{Z} \in X'$, let $x \in X$ be the unknown solution to the equation $\mathcal{Q}x = \mathcal{Z}$, and consider for its solution the preconditioned Richardson method*

$$
x^{k+1} = x^k + \theta \mathcal{N}(\mathcal{Z} - \mathcal{Q}x^k), \quad k \geq 0, \tag{47}
$$

*$\theta$ being a positive relaxation parameter, and $\mathcal{N} : X' \to X$ a suitable scaling operator. Suppose that the following conditions are satisfied:*

1. *$\mathcal{Q}_i$ $(i = 1, 2)$ are continuous and coercive;*

2. *$\mathcal{N}$ is symmetric, continuous and coercive.*

*Then, there exists $\theta_{max} > 0$ such that for each $\theta \in (0, \theta_{max})$ and for any given $x^0 \in X$ the sequence (47) converges in $X$ to the solution of problem $\mathcal{Q}x = \mathcal{Z}$.*

We can now prove the main result of this section.

**Corollary 4.1** *Under the constraint (32), the pRR iterative method (24), (25), (27), (28), (31) converges to the solution $(\mathbf{u}_f, p_f) \in H_f^\tau \times Q$, $\varphi \in H_p$ of the coupled Stokes-Darcy problem, for any choice of the initial guess $\mu^0 \in L^2(\Gamma)$, and for suitable values of the relaxation parameter $\theta$.*

*Proof.* It follows from Theorem 4.1 whose hypotheses are satisfied thanks to Proposition 4.1. $\qquad \square$

# 5 Finite element approximation and numerical results

We consider a regular family of triangulations $\mathcal{T}_h$ of the domain $\overline{\Omega_f} \cup \overline{\Omega_p}$ depending on a positive parameter $h > 0$, made up of triangles if $d = 2$, of tetrahedra in the 3-dimensional case. We assume that the triangulations $\mathcal{T}_{fh}$ and $\mathcal{T}_{ph}$ induced on the subdomains $\Omega_f$ and $\Omega_p$ are compatible on $\Gamma$, i.e., they share the same edges (if $d = 2$) or faces (if $d = 3$) therein. The family of triangulations induced on $\Gamma$ will be denoted by $\mathcal{B}_h$.

Several choices of finite element spaces can be made to approximate the coupled problem (13)–(14). For the sake of exposition, we will consider the following conforming spaces ($d = 2, 3$):

$$H_{fh} = \{\mathbf{v}_h \in (X_{fh})^d | \, \mathbf{v}_h = \mathbf{0} \text{ on } \Gamma_f\}$$

with

$$X_{fh} = \{v_h \in \mathcal{C}^0(\overline{\Omega_f})| \, v_{h|T} \in \mathbb{P}_2(T) \, \forall T \in \mathcal{T}_{fh}\} \ ,$$

and

$$Q_h = \{q_h \in \mathcal{C}^0(\overline{\Omega_f})| \, q_{h|T} \in \mathbb{P}_1(T), \, \forall T \in \mathcal{T}_{fh}\} \ ;$$

moreover, $H_{fh}^\tau$ will be an internal approximation of $H_f^\tau$.

On the other hand, we set

$$H_{ph} = \{\psi_h \in X_{ph}| \, \psi_h = 0 \text{ on } \Gamma_p^b\}$$

with

$$X_{ph} = \{\psi_h \in \mathcal{C}^0(\overline{\Omega_p})| \, \psi_{h|T} \in \mathbb{P}_2(T) \, \forall T \in \mathcal{T}_{ph}\} \ .$$

Finally, define

$$\Lambda_h = \{\eta_h \in L^2(\Gamma) \, | \, \eta_{h|\tau} \in \mathbb{P}_2(\tau) \, \forall \tau \in \mathcal{B}_h\} \ ;$$

in particular, we have that $\mathbf{v}_h \cdot \mathbf{n} \in \Lambda_h$ for each $\mathbf{v}_h \in H_{fh}$ and $\psi_{h|\Gamma} \in \Lambda_h$ for each $\psi_h \in H_{ph}$.

We will now present the discrete counterpart of the sRR and pRR algorithms.

## 5.1 The discrete sRR method

The finite element discretization of the coupled Stokes–Darcy problem (13)–(16) reads:

Find $(\mathbf{u}_{fh}, p_{fh}) \in H_{fh}^\tau \times Q_h$, $\varphi_h \in H_{ph}$ such that

$$a_f(\mathbf{u}_{fh}, \mathbf{v}_h) + b_f(\mathbf{v}_h, p_{fh}) + g \, a_p(\varphi_h, \psi_h) + \int_\Gamma g \, \varphi_h (\mathbf{v}_h \cdot \mathbf{n})$$

$$- \int_\Gamma g \, \psi_h (\mathbf{u}_{fh} \cdot \mathbf{n}) = \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}_h \qquad \forall \mathbf{v}_h \in H_{fh}^\tau, \ \psi_h \in H_{ph} \quad (48)$$

$$b_f(\mathbf{u}_{fh}, q_h) = 0 \qquad \forall q_h \in Q_h \ . \tag{49}$$

The sRR algorithm on the discrete problem (48)–(49) becomes: taking a trace function $\eta_h^0 \in \Lambda_h$, and considering two acceleration parameters $\gamma_f \geq 0$ and $\gamma_p > 0$, for each $k \geq 0$,

*i)* Find $\varphi_h^{k+1} \in H_{ph}$ such that

$$\gamma_p a_p(\varphi_h^{k+1}, \psi_h) + \int_\Gamma g\varphi_{h|\Gamma}^{k+1}\psi_{h|\Gamma} = \int_\Gamma \eta_h^k \psi_{h|\Gamma} \qquad \forall \, \psi_h \in H_{ph} \; . \qquad (50)$$

*ii)* Then, find $(\mathbf{u}_{fh}^{k+1}, p_{fh}^{k+1}) \in H_{fh}^\tau \times Q_h$ such that

$$
\begin{aligned}
a_f(\mathbf{u}_{fh}^{k+1}, \mathbf{v}_h) &+ b_f(\mathbf{v}_h, p_{fh}^{k+1}) + \gamma_f \int_\Gamma (\mathbf{u}_{fh}^{k+1} \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n}) \\
&= \int_\Gamma \left( \frac{\gamma_f}{\gamma_p} \eta_h^k - \frac{\gamma_f + \gamma_p}{\gamma_p} g\varphi_{h|\Gamma}^{k+1} \right)(\mathbf{v}_h \cdot \mathbf{n}) + \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}_h \quad \forall \, \mathbf{v}_h \in H_{fh}^\tau, \\
b_f(\mathbf{u}_{fh}^{k+1}, q_h) &= 0 \qquad \forall \, q_h \in Q_h \; .
\end{aligned}
$$

$$(51)$$

*iii)* Finally, set

$$\eta_h^{k+1} = (\gamma_f + \gamma_p)(\mathbf{u}_{fh}^{k+1} \cdot \mathbf{n}) + \frac{\gamma_f + \gamma_p}{\gamma_p} g\varphi_{h|\Gamma}^{k+1} - \frac{\gamma_f}{\gamma_p}\eta_h^k \in \Lambda_h \; . \qquad (52)$$

For $\gamma_p = \gamma_f$, the convergence of this algorithm to the solution of (48)–(49) can be proved as we did in Section 4.1 to show the convergence of (17)–(20) to the solution of problem (13)–(16). Moreover, it is also possible to prove the convergence of the alternating direction scheme (see Section 4.1.1), as the discrete Steklov–Poincaré operators are positive definite (however, in principle the proof of convergence cannot assure that the rate of convergence is independent of the mesh size $h$).

For the numerical tests we have exploited the interpretation of the method in terms of ADI iterations (Sect. 4.1.1) in order to obtain some guidelines for the choice of the relaxation parameters, at least for the case of our interest, that is, when $\nu$ and the entries of $\mathsf{K}$ are very small (we recall that in this case the convergence rate of the Dirichlet-Neumann method deteriorates).
In particular, considering (41), we are led to investigate the behaviour of the eigenvalues, say $\delta_f^j$ and $\delta_p^j$, of the operators $S_f$ and $S_p$, respectively; in fact, if we can estimate

$$\max_j \left| \frac{\gamma_p - \delta_f^j}{\gamma_f + \delta_f^j} \right| \cdot \max_j \left| \frac{\gamma_f - \delta_p^j}{\gamma_p + \delta_p^j} \right| \; , \qquad (53)$$

this could be taken as a rough estimate of the convergence rate of the algorithm. Assuming that $\mathsf{K}$ is a constant multiple of the identity, we proved that in the limit $\nu \to 0$ and $\mathsf{K} \to 0$ (for a fixed mesh size $h$), $\delta_f^j \to 0$ while $\delta_p^j \to \infty$ ([6]). Thus, for small values of $\nu$ and $\mathsf{K}$ the ratio (53) behaves like $\gamma_p/\gamma_f$. This provides a first indication for the choice of the relaxation parameters, i.e., one should take $\gamma_f > \gamma_p > 0$. Moreover, $\gamma_f$ and $\gamma_p$ should not be taken too large to avoid possible increases of the condition numbers of the Stokes and Darcy stiffness matrices in (50) and (51), respectively. A reasonable trade-off is to choose both parameters approximately equal to $10^{-1}$.
For the numerical tests, we take the same setting as in Example 3.1. In Table 2 we report the number of iterations obtained using the sRR method for some small values of $\nu$ and $\mathsf{K}$ and for four different computational grids. A convergence test based on the relative increment of the trace of the discrete normal velocity on the interface $\mathbf{u}_{fh}^k \cdot \mathbf{n}_{|\Gamma}$ has been considered with tolerance $10^{-9}$. In all computations we have taken $\gamma_f = 0.3$ and $\gamma_p = 0.1$.

| $\nu$ | K | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
|---|---|---|---|---|---|
| $10^{-4}$ | $10^{-3}$ | 19 | 19 | 19 | 19 |
| $10^{-6}$ | $10^{-4}$ | 20 | 20 | 20 | 20 |
| $10^{-6}$ | $10^{-7}$ | 20 | 20 | 20 | 20 |

Table 2: Number of iterations using the sRR method with respect to $\nu$, K and four different grid sizes $h$ ($h_1 \approx 0.14$ and $h_i = h_1/2^{i-1}$, $i = 2, 3, 4$); the acceleration parameters are $\gamma_f = 0.3$ and $\gamma_p = 0.1$.

## 5.2 The discrete pRR method

The pRR algorithm designed on (48)–(49) reads as follows. Let $\mu_h^0 \in \Lambda_h$ be a discrete trace function on $\Gamma$, and $\gamma_1, \gamma_2 > 0$ two positive relaxation parameters; then, for $k \geq 0$,

*i)* Find $(\mathbf{u}_{fh}^{k+1}, p_{fh}^{k+1}) \in H_{fh}^\tau \times Q_h$ such that

$$a_f(\mathbf{u}_{fh}^{k+1}, \mathbf{v}_h) + b_f(\mathbf{v}_h, p_{fh}^{k+1}) - \gamma_1 \int_\Gamma (\mathbf{u}_{fh}^{k+1} \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n})$$
$$= \int_\Gamma \mu_h^k(\mathbf{v}_h \cdot \mathbf{n}) + \int_{\Omega_f} \mathbf{f} \cdot \mathbf{v}_h \qquad \forall\, \mathbf{v}_h \in H_{fh}^\tau, \qquad (54)$$
$$b_f(\mathbf{u}_{fh}^{k+1}, q_h) = 0 \qquad \forall\, q_h \in Q_h,$$

and find $\varphi_h^{k+1} \in H_{ph}$ such that

$$a_p(\varphi_h^{k+1}, \psi_h) + \frac{1}{\gamma_1} \int_\Gamma g \varphi_{h|\Gamma}^{k+1} \psi_{h|\Gamma} = -\frac{1}{\gamma_1} \int_\Gamma \mu_h^k \psi_{h|\Gamma} \qquad \forall\, \psi_h \in H_{ph} \,. \quad (55)$$

*ii)* Then, find $(\widehat{\boldsymbol{\omega}}_h^{k+1}, \widehat{\pi}_h^{k+1}) \in H_{fh}^\tau \times Q_h$ such that

$$a_f(\widehat{\boldsymbol{\omega}}_h^{k+1}, \mathbf{v}_h) + b_f(\mathbf{v}_h, \widehat{\pi}_h^{k+1}) + \gamma_2 \int_\Gamma (\widehat{\boldsymbol{\omega}}_h^{k+1} \cdot \mathbf{n})(\mathbf{v}_h \cdot \mathbf{n})$$
$$= \gamma_2 \int_\Gamma \widehat{\sigma}_h^{k+1}(\mathbf{v}_h \cdot \mathbf{n}) \qquad \forall\, \mathbf{v}_h \in H_{fh}^\tau, \qquad (56)$$
$$b_f(\widehat{\boldsymbol{\omega}}_h^{k+1}, q_h) = 0 \qquad \forall\, q_h \in Q_h \,,$$

and find $\widehat{\chi}_h^{k+1} \in H_{ph}$ such that

$$a_p(\widehat{\chi}_h^{k+1}, \psi_h) + \frac{1}{\gamma_2} \int_\Gamma g \widehat{\chi}_{h|\Gamma}^{k+1} \psi_{h|\Gamma} = \int_\Gamma \widehat{\sigma}_h^{k+1} \psi_{h|\Gamma} \qquad \forall\, \psi_h \in H_{ph} \,, \quad (57)$$

where

$$\widehat{\sigma}_h^{k+1} = \mathbf{u}_{fh}^{k+1} \cdot \mathbf{n} + \frac{1}{\gamma_1}(g \varphi_{h|\Gamma}^{k+1} + \mu_h^k) \in \Lambda_h \,. \qquad (58)$$

*iii)* Finally, update $\mu_h^k$ as follows

$$\mu_h^{k+1} = \mu_h^k - \theta[\gamma_2(\widehat{\sigma}_h^{k+1} - \widehat{\boldsymbol{\omega}}_h^{k+1} \cdot \mathbf{n}) + g \widehat{\chi}_{h|\Gamma}^{k+1}] \in \Lambda_h \,, \qquad (59)$$

where $\theta > 0$ is an acceleration parameter.

As for the continuous case, this iterative scheme can be reformulated in terms of suitable interface operators on $\Lambda_h$. Precisely, let $\mathcal{H}_{Sh}$ and $\mathcal{K}_{Dh}$ be the discrete Robin-to-Dirichlet maps:

$$\mathcal{H}_{Sh} : \Lambda_h \to \Lambda_h, \qquad \mu_h \to \mathcal{H}_{Sh}\mu_h = \mathbf{u}_{\mu_h} \cdot \mathbf{n} \; ,$$

$$\mathcal{K}_{Dh} : \Lambda_h \to \Lambda_h, \qquad \sigma_h \to \mathcal{K}_{Dh}\sigma_h = g\chi_{\sigma_h|\Gamma} \; ,$$

where $(\mathbf{u}_{\mu_h}, p_{\mu_h}) \in H_{fh}^\tau \times Q_h$ is the solution to (54) with $\mathbf{f} = \mathbf{0}$ and Robin boundary datum $\mu_h$, while $\chi_{\sigma_h} \in H_{ph}$ is the solution of (57) with boundary datum $\sigma_h$ on $\Gamma$.

Then, consider the discrete Robin-to-Neumann operators

$$\mathcal{H}_{Dh} : \Lambda_h \to \Lambda_h, \qquad \mu_h \to \mathcal{H}_{Dh}\mu_h = \frac{1}{\gamma_1}(g\varphi_{\mu_h|\Gamma} + \mu_h) \; ,$$

$$\mathcal{K}_{Sh} : \Lambda_h \to \Lambda_h, \qquad \sigma_h \to \mathcal{K}_{Sh}\sigma_h = \gamma_2(\sigma_h - \boldsymbol{\omega}_{\sigma_h} \cdot \mathbf{n}) \; ,$$

where $\varphi_{\mu_h} \in H_{ph}$ is the solution of (55) with boundary datum $\mu_h$, and $(\boldsymbol{\omega}_{\sigma_h}, \pi_{\sigma_h})$ is the solution of (56) with boundary datum $\sigma_h$.

Finally, we denote by $(\widetilde{\mathbf{u}}_h, \widetilde{p}_h) \in H_{fh}^\tau \times Q_h$ the solution of (54) with null boundary conditions, so that $\mathbf{u}_{fh}^{k+1} \cdot \mathbf{n} = \mathcal{H}_{Sh}\mu_h^k + \widetilde{\mathbf{u}}_h \cdot \mathbf{n}$, for all $k \geq 0$.

Then (58) becomes

$$\widehat{\sigma}_h^{k+1} = \mathcal{H}_{Sh}\mu_h^k + \mathcal{H}_{Dh}\mu_h^k + \widetilde{\mathbf{u}}_h \cdot \mathbf{n} \; .$$

Problem (48)–(49) can be associated with the discrete interface problem

$$\text{Find } \mu_h \in \Lambda_h : \quad (\mathcal{H}_{Sh} + \mathcal{H}_{Dh})\mu_h = -\widetilde{\mathbf{u}}_h \cdot \mathbf{n} \qquad \text{on } \Gamma \; . \qquad (60)$$

Thus the discrete pRR method can be interpreted as the following preconditioned Richardson scheme to solve (60):

$$\mu_h^{k+1} = \mu_h^k - \theta(\mathcal{K}_{Sh} + \mathcal{K}_{Dh})[\widetilde{\mathbf{u}}_h \cdot \mathbf{n} + (\mathcal{H}_{Sh} + \mathcal{H}_{Dh})\mu_h^k], \qquad k \geq 0 \; , \qquad (61)$$

the preconditioner being

$$P = (\mathcal{K}_{Sh} + \mathcal{K}_{Dh})^{-1} \; . \qquad (62)$$

The convergence of (61) is proved as done in Section 4.2 for the infinite dimensional case; besides, its rate of convergence is independent of the mesh size $h$, as it only depends on the continuity and coerciveness constants of the operators $\mathcal{H}_{Sh}, \mathcal{H}_{Dh}, \mathcal{K}_{Sh}, \mathcal{K}_{Dh}$, which are all independent of $h$.

Moreover, since the operators $\mathcal{H}_{Sh}$ and $\mathcal{H}_{Dh}$ are symmetric, we can use the PCG method to compute the solution of (60) using the same preconditioner (62).

More generally, we consider the following (variable) preconditioner:

$$P_k = (\sigma_1^k \mathcal{K}_{Sh} + \sigma_2^k \mathcal{K}_{Dh})^{-1} \; , \qquad (63)$$

where $\sigma_1^k$ and $\sigma_2^k$ are two suitable acceleration coefficients (possibly depending on the iteration $k$).

The choice of the coefficients $\gamma_1$, $\gamma_2$, $\sigma_1^k$ and $\sigma_2^k$ to accelerate convergence is not straightforward. In our numerical experiments we have adopted two different strategies. First, we have used the PCG method with $P^{-1} = \sigma_1 \mathcal{K}_{Sh} + \sigma_2 \mathcal{K}_{Dh}$

with a suitable choice of the acceleration coefficients. Secondly, we have considered the preconditioner $P_k^{-1}$ as in (63) in the framework of a Richardson method and we have computed $\sigma_1^k$ and $\sigma_2^k$ according to an Aitken acceleration procedure (see, e.g., [4, 5]).

More precisely, the algorithm reads: let $r_h^0$ be the residual of (60) computed with respect to an initial datum $\mu_h^0 \in \Lambda_h$, and let $z_h^0 = P_0^{-1} r_h^0$. Then, for $k \geq 0$,

1. compute the local preconditioned residuals $z_{Dh}^k = \mathcal{K}_{Dh} r_h^k$, $z_{Sh}^k = \mathcal{K}_{Sh} r_h^k$;

2. solve the linear system

$$A_k^T A_k \begin{pmatrix} \sigma_1^k \\ \sigma_2^k \end{pmatrix} = -A_k^T (\mu_h^k - \mu_h^{k-1})$$

where $A_k$ is the two column matrix $A_k = (z_{Sh}^k - z_{Sh}^{k-1}; \; z_{Dh}^k - z_{Dh}^{k-1})$.

This corresponds to minimize

$$\|(\mu_h^k - \mu_h^{k-1}) + \sigma_1(z_{Sh}^k - z_{Sh}^{k-1}) + \sigma_2(z_{Dh}^k - z_{Dh}^{k-1})\|$$

over all possible values of $\sigma_1$ and $\sigma_2$.

3. Finally, update $z_h^{k+1} = \sigma_1^k z_{Sh}^k + \sigma_2^k z_{Dh}^k$, $r_h^{k+1} = r_h^k - (\mathcal{H}_{Sh} + \mathcal{H}_{Dh}) z_h^{k+1}$ and $\mu_h^{k+1} = \mu_h^k + z_h^{k+1}$.

For the numerical tests, we have considered the same settings as in Example 3.1. A tolerance of $10^{-9}$ has been imposed on the relative increment, and a maximal number of iterations maxit = 300 has been required.

Table 3 reports the number of iterations obtained using the PCG method for three values of $\nu$ and four different grids. It is apparent that the rate of convergence deteriorates as $\nu$ goes to 0. We have noticed a similar behaviour for small values of K as well.

The Richardson-Aitken strategy gives better results, as shown in Table 4. However, the Dirichlet-Neumann algorithm still turns out to be more efficient in this respect (see Table 1).

| $\nu$ | K | $\gamma_1$ | $\gamma_2$ | $\sigma_1$ | $\sigma_2$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ |
|-------|---|-----------|-----------|-----------|-----------|-------|-------|-------|-------|
| 1 | 1 | 0.5 | 0.5 | 1 | 1 | 11 | 12 | 11 | 12 |
| $10^{-1}$ | 1 | $10^{-1}$ | 1 | 1 | 1 | 27 | 28 | 29 | 28 |
| $10^{-2}$ | 1 | $10^{-2}$ | 1 | 1 | 1 | 68 | 76 | 72 | 64 |

Table 3: Number of iterations using the PCG method with the pRR preconditioner $P$ as in (62), with respect to $\nu$ and the grid size $h$ ($h_1 \approx 0.14$ and $h_i = h_1/2^{i-1}$, $i = 2, 3, 4$).

# 6   Appendix

We present here the proof of the convergence of the (modified) sRR scheme (42).

**Theorem 6.1** *Let us assume that the interface $\Gamma$ is smooth, say, a $\mathcal{C}^2$-manifold with boundary. Then, for each $\gamma > 0$, the operator $(\gamma J_- - S_f)(\gamma J_- + S_f)^{-1}$ is a contraction in $(H_{00}^{1/2}(\Gamma))'$, and the operator $(\gamma J_+ - S_p)(\gamma J_+ + S_p)^{-1}$ is a contraction in $H_{00}^{1/2}(\Gamma)$.*

| $\nu$ | K | $\gamma_1$ | $\gamma_2$ | grid size | iter. | $|\bar{\sigma}_1|$ | $|\bar{\sigma}_2|$ |
|---|---|---|---|---|---|---|---|
| | | | | $h_1$ | 10 | 2.68 | 0.64 |
| 1 | 1 | 0.5 | 0.5 | $h_2$ | 10 | 2.67 | 0.66 |
| | | | | $h_3$ | 10 | 2.66 | 0.67 |
| | | | | $h_4$ | 10 | 2.66 | 0.68 |
| | | | | $h_1$ | 12 | 1.53 | 0.13 |
| $10^{-1}$ | 1 | $10^{-1}$ | 1 | $h_2$ | 11 | 1.50 | 0.13 |
| | | | | $h_3$ | 11 | 1.54 | 0.13 |
| | | | | $h_4$ | 12 | 1.50 | 0.12 |
| | | | | $h_1$ | 23 | 0.90 | 0.06 |
| $10^{-2}$ | 1 | $10^{-2}$ | 1 | $h_2$ | 23 | 0.95 | 0.04 |
| | | | | $h_3$ | 23 | 0.96 | 0.06 |
| | | | | $h_4$ | 23 | 0.94 | 0.06 |
| | | | | $h_1$ | 47 | 0.33 | 0.07 |
| $10^{-3}$ | 1 | $10^{-3}$ | 1 | $h_2$ | 47 | 0.38 | 0.04 |
| | | | | $h_3$ | 50 | 0.37 | 0.03 |
| | | | | $h_4$ | 52 | 0.38 | 0.03 |
| | | | | $h_1$ | 23 | 0.90 | 0.06 |
| $10^{-1}$ | $10^{-1}$ | $10^{-1}$ | 10 | $h_2$ | 23 | 0.95 | 0.04 |
| | | | | $h_3$ | 23 | 0.96 | 0.06 |
| | | | | $h_4$ | 23 | 0.94 | 0.06 |
| | | | | $h_1$ | 40 | 0.25 | 0.02 |
| $10^{-2}$ | $10^{-1}$ | $10^{-2}$ | $10^2$ | $h_2$ | 39 | 0.26 | 0.01 |
| | | | | $h_3$ | 40 | 0.30 | 0.01 |
| | | | | $h_4$ | 44 | 0.27 | 0.01 |

Table 4: Number of iterations using the Aitken-accelerated Richardson method with the pRR preconditioner $P_k$ as in (63); in the last two columns we indicate the mean value of the absolute values of the parameters $\sigma_1^k$ and $\sigma_2^k$ generated by the method. The $h_i$ are as in Table 3.

*Proof.* We have:

$$\|(\gamma J_- - S_f)(\gamma J_- + S_f)^{-1}\|^2 = \sup_{\mu \neq 0} \frac{\|(\gamma J_- - S_f)(\gamma J_- + S_f)^{-1}\mu\|^2_{-1/2,00,\Gamma}}{\|\mu\|^2_{-1/2,00,\Gamma}}$$

$$= \sup_{\chi \neq 0} \frac{\|(\gamma J_- - S_f)\chi\|^2_{-1/2,00,\Gamma}}{\|(\gamma J_- + S_f)\chi\|^2_{-1/2,00,\Gamma}}$$

$$= \sup_{\chi \neq 0} \frac{\gamma^2 \|J_-\chi\|^2_{-1/2,00,\Gamma} - 2\gamma(S_f\chi, J_-\chi)_{-1/2,00,\Gamma} + \|S_f\chi\|^2_{-1/2,00,\Gamma}}{\gamma^2 \|J_-\chi\|^2_{-1/2,00,\Gamma} + 2\gamma(S_f\chi, J_-\chi)_{-1/2,00,\Gamma} + \|S_f\chi\|^2_{-1/2,00,\Gamma}}$$

$$= \sup_{\chi \neq 0} \frac{\gamma^2 \|\chi\|^2_{1/2,00,\Gamma} - 2\gamma\langle S_f\chi, \chi\rangle_\Gamma + \|S_f\chi\|^2_{-1/2,00,\Gamma}}{\gamma^2 \|\chi\|^2_{1/2,00,\Gamma} + 2\gamma\langle S_f\chi, \chi\rangle_\Gamma + \|S_f\chi\|^2_{-1/2,00,\Gamma}} \, .$$

We prove now that $S_f$ is positive and bounded, that is there exist two positive constants $C_1$ and $C_2$ such that

$$\langle S_f\chi, \chi\rangle_\Gamma \geq C_1 \|\chi\|^2_{1/2,00,\Gamma} \, , \qquad \|S_f\chi\|^2_{-1/2,00,\Gamma} \leq C_2 \|\chi\|^2_{1/2,00,\Gamma} \, . \qquad (64)$$

In fact, using the Korn and the trace inequality in $H_{00}^{1/2}(\Gamma)$ we have

$$
\begin{aligned}
\langle S_f \chi, \chi \rangle_\Gamma &= \langle \mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{n}), \mathbf{u}_\chi \cdot \mathbf{n} \rangle_\Gamma \\
&= \langle \mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{n}, \mathbf{n}(\mathbf{u}_\chi \cdot \mathbf{n}) + \sum_{j=1}^{d-1} \boldsymbol{\tau}_j (\mathbf{u}_\chi \cdot \boldsymbol{\tau}_j) \rangle_\Gamma \\
&\qquad (\text{as } \mathbf{u}_\chi \cdot \boldsymbol{\tau}_j = 0 \text{ on } \Gamma) \\
&= \int_{\Omega_f} \boldsymbol{\nabla} \cdot [\mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{u}_\chi] = 2\nu \int_{\Omega_f} |\mathsf{D}(\mathbf{u}_\chi)|^2 \\
&\geq c_1 \|\mathbf{u}_\chi\|_{1,\Omega_f}^2 \geq c_2 \|\mathbf{u}_{\chi|\Gamma}\|_{1/2,00,\Gamma}^2 .
\end{aligned}
$$

The regularity assumption on $\Gamma$ yields $\|\mathbf{u}_\chi \cdot \mathbf{n}\|_{1/2,00,\Gamma} \leq c_3 \|\mathbf{u}_{\chi|\Gamma}\|_{1/2,00,\Gamma}$, hence

$$
\langle S_f \chi, \chi \rangle_\Gamma \geq C_1 \|\mathbf{u}_\chi \cdot \mathbf{n}\|_{1/2,00,\Gamma}^2 = C_1 \|\chi\|_{1/2,00,\Gamma}^2 .
$$

Moreover, the regularity assumption on $\Gamma$ also yields

$$
\|\mathbf{n} \cdot (\mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{n})\|_{-1/2,00,\Gamma} \leq c_4 \|\mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{n}\|_{-1/2,00,\Gamma} ,
$$

therefore the trace inequality in $(H_{00}^{1/2}(\Gamma))'$ and the a-priori estimate for the solution of the Stokes problem give

$$
\begin{aligned}
\|S_f \chi\|_{-1/2,00,\Gamma}^2 &\leq c_4^2 \|\mathsf{T}(\mathbf{u}_\chi, p_\chi) \cdot \mathbf{n}\|_{-1/2,00,\Gamma}^2 \\
&\leq c_5 \|\mathsf{T}(\mathbf{u}_\chi, p_\chi)\|_{0,\Omega_f}^2 \leq C_2 \|\chi\|_{1/2,00,\Gamma}^2 ,
\end{aligned}
$$

so that both inequalities in (64) are proved.
Consequently, setting $q_0 = (C_2 - 2\gamma C_1 + \gamma^2)/(C_2 + 2\gamma C_1 + \gamma^2)$, we can easily prove that

$$
\sup_{\chi \neq 0} \frac{\gamma^2 \|\chi\|_{1/2,00,\Gamma}^2 - 2\gamma \langle S_f \chi, \chi \rangle_\Gamma + \|S_f \chi\|_{-1/2,00,\Gamma}^2}{\gamma^2 \|\chi\|_{1/2,00,\Gamma}^2 + 2\gamma \langle S_f \chi, \chi \rangle_\Gamma + \|S_f \chi\|_{-1/2,00,\Gamma}^2} \leq q_0 < 1 .
$$

The proof that $\|(\gamma J_+ - S_p)(\gamma J_+ + S_p)^{-1}\| < 1$ can be done in a similar way. In fact, using the trace inequality in $(H_{00}^{1/2}(\Gamma))'$ we have

$$
\begin{aligned}
\langle \eta, S_p \eta \rangle_\Gamma &= -g \langle \mathsf{K} \nabla \varphi_\eta \cdot \mathbf{n}, \varphi_{\eta|\Gamma} \rangle_\Gamma \\
&= g \int_{\Omega_p} \nabla \cdot [\varphi_\eta \mathsf{K} \nabla \varphi_\eta] = g \int_{\Omega_p} \nabla \varphi_\eta \cdot \mathsf{K} \nabla \varphi_\eta \\
&= g \int_{\Omega_p} \mathsf{K}^{-1} \mathsf{K} \nabla \varphi_\eta \cdot \mathsf{K} \nabla \varphi_\eta \geq c_6 \int_{\Omega_p} |\mathsf{K} \nabla \varphi_\eta|^2 \\
&\geq C_3 \|\mathsf{K} \nabla \varphi_\eta \cdot \mathbf{n}\|_{-1/2,00,\Gamma}^2 = C_3 \|\eta\|_{-1/2,00,\Gamma}^2 .
\end{aligned}
$$

Moreover, by the trace inequality in $H_{00}^{1/2}(\Gamma)$ and the a-priori estimate for the solution of the Laplace equation, we obtain

$$
\|S_p \eta\|_{1/2,00,\Gamma}^2 = \|g \varphi_{\eta|\Gamma}\|_{1/2,00,\Gamma}^2 \leq c_7 \|\varphi_\eta\|_{1,\Omega_p}^2 \leq C_4 \|\eta\|_{-1/2,00,\Gamma}^2 .
$$

These two inequalities permit to repeat for the operator $S_p$ the same procedure used for the operator $S_f$. $\qquad\square$

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