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The Mereological Basis of Truthmaker Semantics

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Abstract
This articles explores the mereological foundation of truthmaker semantics. Building upon Kit Fine's abstract theory of part in Fine [J Philos 107(11):559–589, 2010], we engage in an exploration of the mereological assumptions that determine the construction of truthmaker semantics. Our approach yields semantics for a diverse range of logics, including substructural logics such as the associative Lambek calculus, as well as the logics of analytic containment. Furthermore, we elucidate the philosophical implications that arise from this pioneering approach.

Keywords Mereology · Composition operators · Truthmaker semantics · Non-standard mereology

Introduction

Truthmaker semantics (TS) has garnered significant attention from philosophers and logicians, who are increasingly drawn to its potential for interpreting established logics or forging novel ones. Fine traces the truthmaker approach back to van Fraassen's fact-based semantics in Van Fraassen (1969) that interprets the exact content of classical logic formulas. But TS are also valuable for designing non-classical logics, particularly for analytic containment, cf. Fine (2016).

Although there are many variants of TS, a number of common traits can be highlighted (see Deigan 2020; Fine 2014, 2017a, b, c; Jago 2020; Leitgeb 2019). TS are compositional semantics that adopt a mereological structure, given by a *state space* $(S, \sqsubseteq)$, where S represents a set of 'states' and $\sqsubseteq$ is a parthood relation. Ultimately, the underlying philosophical intuition behind TS is that propositions should be construed as sets of states, imbued with a fundamental mereological essence (for philosophical discourse, see Jago 2017). However, the ontology of states and the characterisation of parthood are somehow abstract to allow for various interpretations. Parthood is typically a partial order, i.e., a reflexive, antisymmetric, and transitive relation, cf. Fine (2014),[1] while the state space is often assumed to be *complete*, ensuring the existence of least upper bounds (l.u.b.s) for sets of states. Given this technical assumptions, the clauses of TS associate sets of verifiers (falsifiers) to each formula of the language at issue, producing a relation of *truthmaking* (*falsemaking*) between a formula and a set of verifiers (falsifiers) which entertain mereological relations. In particular, depending on the chosen relation of verification we want to model, TS may be proposed in exact and inexact versions: the exact verification (falsification) of a formula provides the sole states that verify (falsify) it, whereas the inexact verification includes larger states, w.r.t. the parthood relation.[2]

This article embarks on an exploration of TS where the relation of parthood and the composition operation between states are not the standard ones. Specifically, we delve into semantics that are rooted in spaces featuring distinct properties of state composition, deviating from the ones dictated by standard mereology. Thus, by embracing this approach, we unlock the capacity to model a plethora of non-classical logics, extending the boundaries of TS and accommodating a broader range of logical systems. As we anticipated, the flexibility of TS for capturing various logical systems has been demonstrated in numerous studies, spanning from intu-

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[1] This assumptions correspond to the "ground mereology", which is termed "standard" in Varzi (2014).

[2] In this article, we will limit our attention only to TS based on an exact form of verification. But see Deigan (2020) for a plea of inexact truthmaking.

itionistic logic (Fine 2014) and logics of containment (Fine 2017a) to, most recently, substructural logics (Jago 2020; Majer et al. 2023). However, we are interested in anchoring this flexibility to mereology.

Hence, our aim, in particular, is to connect two very lively fields: TS and non-standard mereologies. Today many philosophers are advancing alternative mereological frameworks to address purportedly counterintuitive consequences of the standard theory. Exploring the interplay between these debates and the evolution of TS promises novel insights.

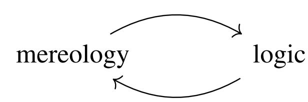
The article is organised as follows. Section 1 gives an overview of the principles of TS, while Sect. 2 explains our application of non-standard mereologies to generate new TS. Sections 3, 4, and 5, focus on an in-depth exploration of Fine's theory of part advanced in Fine (2010), which serves as the basis for studying TS based on standard as well as non-standard mereologies. We adapt Fine's approach in the third section by introducing the concept of a *pre-mereological space*, i.e., a set of states equipped with an abstract composition operation, and defining corresponding parthood relations. Then, we develop two ways of modeling non-standard composition operations: (i) as "vertical operations" that, given a bunch of objects, yields their summation as the output; and (ii) as "horizontal" binary ones that, given two bunches of objects, generate their fusion.[3] Next, in Sect. 6, we apply this abstract framework to construct TS. We take a pre-mereological space with its composition operation and part-whole relation, and incorporate them into a state space along with a valuation function. This allows us to generate TS based on standard and non-standard mereologies for various logics, including substructural logics based on their monoidal frames. Examples of the logics for analytic containment and associative Lambek calculus are presented in this section. Finally, Sect. 7 summarises the philosophical outputs of our work. We end the article by wrapping up the conclusions and indicating future work.

1 Truthmaker Semantics

From a technical viewpoint, like any mathematical structure that serves as model theory, TS can be viewed as a set of mathematical definitions that characterise a certain class of structures. However, at least from our standpoint, the true character of TS, which distinguishes them with respect to various other semantic theories [like Frame Semantics (Kripke 1963), Routley-Meyer Semantics (Beall et al. 2012), or Phase Semantics (Girard 1987)], lies in the genuinely mereological understanding of the semantic structures and the implications it carries. The state space, which is the structure required for TS, is indeed intended as a mereological space, not just any relational or algebraic structure. The mereological character of the formal structure is aligned with the philosophical understanding of the endeavor of providing semantics to statements. The TS framework is an *objectual* approach to semantics: it regards the statements' truth-conditions as entities—not just as clauses, as in Davidson (1967) (cf. also Fine 2017c, pp. 557–558). In particular, according to TS, truth-conditions are *states*, i.e., fact-like entities, which compose the worlds we live, and stand in the relation of truthmaking to the statements they make true.

These states are gathered within the state space, and, crucially, they engage in mereological relations of parthood, composition, or fusion. Moreover, the space must meet certain requirements (closure or convexity, cf. Fine 2017a, pp. 628–631).

This theoretical shift in semantics has a fundamental consequence: mereology and logic become closely tied together by setting a correspondence whose core is TS:



That is, the properties of the mereological relations affect the definitions of the logical constants and *vice versa*.

To see this more precisely, let's focus for a moment on the standard (exact) semantic clause for the verification of conjunctions given by Fine (see Fine 2014, p. 552):

($\wedge$) A state verifies $\phi \wedge \psi$ if and only if it is the fusion of a state that verifies ϕ and a state that verifies ψ.

Now suppose that we abstract from the definition of fusion (which is defined as the least upper bounds with respect to parthood in Fine 2014) and consider two distinct operations of composition: a commutative one $\star_1$ and a non-commutative one $\star_2$. With these two operators at hand, the order of the conjuncts becomes important in formulating the clause. Indeed, for the non-commutative operation $\star_2$, the clause can be rewritten in two non-equivalent forms:

($\wedge_1^{\star_2}$) A state verifies $\phi \wedge \psi$ if it is the $\star_2$-composition of a state that verifies ϕ and a state that verifies ψ;

($\wedge_2^{\star_2}$) A state verifies $\phi \wedge \psi$ if it is the $\star_2$-composition a state that verifies ψ and of a state that verifies ϕ.

The two clauses are not equivalent since $\star_2$ is not commutative. Instead, they are equivalent if we use the commutative operator $\star_1$. This illustrates how the choice of a composition operator directly impacts the semantics and its logical features.

[3] The vertical/horizontal jargon was introduced by Fine in Fine (2010).

This perspective extends to the definition of other logical constants and affects the understanding of the relation of logical consequence and of the concept of validity.

These remarks underscore how the TS project entangles the two fields of logic and mereology. For these reasons, applying non-standard mereologies to TS appears natural and promises to make some advancements from both a logical and philosophical point of view.

2 Non-standard Mereologies

There are many formal theories of parthood in the literature [for a brief overview, see Varzi (2014); for a long one, see Cotnoir and Varzi (2021)]. The most famous one is the so-called classical mereology [see Cotnoir and Varzi (2021, ch. 2) for some axiomatisations], but it is worthless to say that it has several rivals [cf. Cotnoir (2013) to consider some counterexamples to its principles].

Presently, there is considerable interest in exploring non-standard mereologies that abandon or revise some standard principles. These theories strive to explain the part-whole relation, typically positing a primitive parthood relation and specifying its assumed properties.

Recall that TS are usually exhibited by providing a state space, i.e., a mereological frame constituted by a set of verifiers and falsifiers ordered by the standard parthood relation (i.e. a partial order) and equipped with a composition operation. Because the standard mereological space is a semi-lattice,[4] it is easy to define the composition operator, called *fusion*, by taking the least upper bounds of any (non-empty and finite) set of states (w.r.t parthood). Then, based on the different logical needs, this structure is enriched with other relations (or operations) and new conditions over the set of states to capture the syntax well and to prove the usual meta-theorems.

As we shall see in detail in Sect. 6.2, for non-standard mereologies, parthood does not always ensure a straightforward definition of composition as the least upper bound. That is, in principle, parthood and composition can be taken as two separate ingredients of a TS. Thus, to work well as a TS, a structure should include at least the following features: it should include a parthood relation, which enables the definition of a number of verification and entailment relations (e.g. exact, inexact, or loose, Fine 2017a), and a binary composition operator, e.g. the fusion of two states, which serves to provide a compositional semantics of the connectives, as suggested by Fine's mereological analysis of logical constants that we encountered in the previous Section.

Now, take Cotnoir' and Bacon's non-wellfounded mereologies advanced in Cotnoir and Bacon (2012). How can we generate a TS from them? Immediately, we face a challenge due to the nature of their parthood relation, which is a *preorder*.[5] In Cotnoir and Bacon (2012), this preorder is compatible with three alternative compositions (cf. pp. 9–11). In this case, we cannot define the fusion operator by simply taking the least upper bound, as it is done in the standard case (lacking antisymmetry, there is no unique least upper bound). Instead, we are compelled to deciding the composition operator among the available ones within the specific mereology, and construct the TS later (the technical details of this construction are not pertinent to our present goals).

Note that this does not mean that Cotnoir' and Bacon's mereologies are not well-suited to generate TS. They do meet the condition of enabling composition operators and parthood, thus allowing for the construction of a new TS. However, we do so by delving into the specificity of the mereological theory at issue. A more general and uniform method that enable us to construct TS from standard as well as non-standard mereologies would be welcome.

Luckily, such a general and alternative approach to overcome this challenge exists: we can take a step back and look for a more general setting to define mereologies. Kit Fine proposed to model an abstract mereology by considering an abstract composition operation as primitive, rather than the parthood relation itself (cf. Fine 2010, §3). Fine's primary motivation for this approach was rooted in mereological pluralism and its inherent generality makes it well-suited to address our specific task.

In the subsequent sections, we will demonstrate how this operationalist framework allows us to encompass a wide range of composition operations and parthood relations that deviate from the classical ones. Instead of focusing on a particular non-standard mereology, we embrace Fine's comprehensive account and leverage it to generate TS based on standard and non-standard mereologies in a uniform way. Thus, with the remarkable flexibility offered by this general theory of part, we can explore TS more expansively and overcome our initial hurdle.

To conclude, for our goals, to properly function as a TS a parthood relation and a binary composition operator are required. We shall embrace Fine's abstract mereological approach, which allows for defining both and to investigate their interconnections.

[4] A semi-lattice is a reflexive, transitive, and anti-symmetric relation, with least upper bound for each non-empty finite subset.

[5] A preorder is a reflexive and transitive binary relation, lacking antisymmetry.

3 Pre-mereological Spaces

In the previous section, we suggested implementing Fine's theory of part to study TS. Now, it is time to present it in a more detailed way. To achieve this, it is indispensable to introduce the notion of *pre-mereological space*. We will work on this in the present section, mainly following Fine's treatise.

Fine's framework does not explicitly address the concept of pre-mereological spaces (but see its germs in Fine 2010, Sects. 3,5 especially). We introduce this terminology to emphasise that not every structure of this nature necessarily represents a legitimate parthood relation or a composition operator comparable to the standard mereological frameworks. These relations often exhibit weaker properties than those studied in standard mereology. Consequently, the question of whether these structures qualify as genuine mereological systems becomes a philosophical matter concerning the metaphysics of parts. Delving into this complex debate is beyond the scope of our current discussion (but see Cotnoir 2013; Schaffer 2010; Simons 2000 for some contributions on the topic).

A pre-mereological space can be understood as a generalisation achieved by considering objects along with an operation of composition among them. To illustrate this concept, let's examine the Latin alphabet and the words that we can compose. Intuitively, we can perceive a sense in which the letters M and O combine to form the word MOM. However, this composition differs from standard mereological fusion, as the order and multiplicity of the letters play crucial roles in determining the result. For instance, MOM is distinct from MMO or MO. While one could argue that this intuitive understanding aligns with a genuine mereological relationship (as discussed by Fine 2010), it is important to note that this assertion is not immediately evident. Consequently, for the purposes of our discussion, it is more appropriate to designate these relationships as pre-mereological rather than making an absolute claim about their mereological nature.

Informally, a pre-mereological space is a set of objects equipped with a composition operation (i.e. a function) Σ that takes a "bunch" of objects as input and gives an object, their composition, as an output. To make this idea technically approachable, our first challenge is to elucidate our informal discourse about "bunches" of objects. While it would be philosophically desirable to treat the concept of a "bunch" as primitive, such an approach is formally unsuitable for our goals, as the formal properties of a bunch are significant to define the composition operations as functions, with explicit domains and codomains, and to characterise their properties.

For the purpose of this paper, we may think of a bunch as a *list* or a *string* of entities, i.e. the order and the repetitions of the entities within a bunch counts. Therefore, a composition operator is a function that takes the list of entities and yields their composition.

Before introducing our definition of pre-mereological space, let's revisit the definition of Kleene algebra, or the algebra of strings, over a set of elements.

Definition 1 (*Kleene algebra*) A **Kleene algebra** on the set of generators A is the set of all finite strings of elements of A. Formally, $A^\star$ (the Kleene star of A) is the defined by $A^\star = \bigcup_{i \geqslant 0} A^i$, where $A^0 = \{\langle\rangle\}$, i.e. the set containing the empty string $\langle\rangle$, $A^1 = A$, and $A^{i+1} = \{a \mid a = a^i a' \text{ s. t. } a^i \in A^i \text{ and } a' \in A\}$. That is, $A^\star$ includes the empty string and all finite strings of elements of A.[6]

For example, we can take the set of letters $A = \{M, O\}$. $A^\star$ includes all the finite string (words) composed with letters M and O (e.g. M, and M, M, and M, O, M, and O, M, M). Throughout this article, we shall use letters a, b, etc. to denote elements of A and we use letters s, s', r, t, etc. to denote strings of $A^\star$. Moreover, we use "," to separate the elements of a string, e.g. a, b, c, d denotes a string.

Hence, a definition of pre-mereological space is this:

Definition 2 (*Pre-mereological space*) Let A be a set of objects, termed the *generators* of the pre-mereological space, X a subset of the Kleene algebra $A^\star$, and Y a subset of A. Let also $\Sigma : X \longrightarrow Y$ be a composition operator that takes a string $a_1, \ldots, a_n$ in X and generates an element y in Y. The pair $(A, \Sigma : X \longrightarrow Y)$ is the ***pre-mereological space*** generated by Σ over A.

The role of the set A is to include all the entities, both composing and composed, that we take into account.

Let's instantiate the pre-mereological space with a number of examples.

Example 1 (*Strings*) Consider a set of letters L and the strings (aka words) over L, i.e. $L^\star$. We can take $L^\star$ as our set of generators, so that the Kleene algebra required by the pre-mereological space is $L^\star$ itself (since $L^{\star\star} = L^\star$). We can define a composition operator Σ_1: it takes a string of letters and generates a word, the string itself. For instance, suppose $M, O \in L$. Then, $M, O, M \in L^\star$ and $\Sigma_1(M, O, M) = M, O, M$. The resulting structure $(L, \Sigma_1 : L^\star \longrightarrow L^\star)$ is a trivial pre-mereological space.

Example 2 (*Multisets*) Consider a set A of ordinary objects and all the multisets whose elements are those objects. E.g. $A = \{a, b, \{a, a, b\}, \ldots\}$. Let X be the subset of $A^\star$ whose elements are lists of ordinary objects but not multisets. Also, let Y be the set of multisets in A. Now, define the composition operator $\Sigma_2 : X \longrightarrow Y$ to be the function that, given a list of ordinary objects, yields the multiset with

[6] An important property of the Kleene star $\star$ is its idempotency, i.e., for any set S, $S^\star = (S^\star)^\star$.

those objects as elements (keeping fixed the exact multiplicity for each object). Roughly, this function 'forgets' the order of the letters in the string and produces their multiset. E.g. $\Sigma_2(a, b, b) = \Sigma_2(b, b, a) = \{a, b, b\}$. The structure (A, Σ_2) is a pre-mereological space.

Example 3 (*Sets*) Consider a set A of ordinary objects and all the sets whose elements are those objects. Take X to be the subset of $A^\star$ whose elements are lists of ordinary objects but not sets. The codomain Y will be the set of sets in A. Again, let's define the composition operator $\Sigma_3 : X \longrightarrow Y$ as the function that, given a list of ordinary objects, yields the set with those objects as members. This function 'forgets' both the order of the letters in the string and their multiplicity and produces their set. E.g. $\Sigma_3(a, a, b) = \Sigma_3(a, b, b) = \{a, b\}$. The structure (A, Σ_3) is a pre-mereological space.

Example 4 (*Pre-mereological loops*) Consider a set A with elements a and b. Now, we can define the composition operator $\Sigma_4 : A \longrightarrow A$, since $A \subseteq A^\star$, such that $\Sigma_4(a) = b$ and $\Sigma_4(b) = a$. The resulting structure (A, Σ_4) is a pre-mereological space, where composing a *via* Σ_4 yields b, and *vice versa*. This means that, in this space, an interesting phenomenon arises: a "pre-mereological loop" is formed, where b is composed of a and a is composed of b, yet $a \neq b$.

This series of examples shows the flexibility of Definition 2. By forcing some conditions on Σs, as we shall see, we can easily model different ways things can be composed.

Before discussing some of the most natural and useful properties that Σs can instantiate, let's list a few definitions indispensable for our work.

Definition 3 Let $\mathcal{P}$ be a pre-mereological space. We say that:

- A state $y \in Y$ is Σ ***-simple*** iff there is no element $x \in X$ such that $\Sigma(x) = y$.
- A state $y \in Y$ is Σ ***-composite*** iff it is not Σ-simple.
- A state $y \in Y$ is Σ ***-composable*** iff $y \in X$.
- A state $y \in Y$ is Σ ***-incomposable*** iff $y \notin X$.

Roughly put: Σ-simple states are not composed by anything; Σ-composable states are in X, hence they can in turn be composed by Σ; Σ-incomposable states are not in the domain of Σ. Incomposable elements may occur when $X \subset Y$. In particular, to facilitate our subsequent discussion, let us state the following definition:

Definition 4 (Σ-*composable states*) Let $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ be a pre-mereological space. The ***set of*** Σ***-composable states of*** $\mathcal{P}$, denoted $\mathsf{Comp}(\mathcal{P})$, is the intersection of the domain and the codomain of Σ, i.e., $\mathsf{Comp}(\mathcal{P}) = X \cap Y$.

Since the domain X of the composition operation Σ is a subset of $A^\star$, we are not committing to including the empty list, denoted by $\langle\rangle$, into it. If it is there, since Σ is a function, $\Sigma(\langle\rangle)$ is an element of Y, let's denote it by e. Notice that e is Σ-composite.[7]

Now, equipped with these definitions, we begin our examination of the properties of pre-mereological spaces and Σs. Three fundamental conditions not explicitly explored in Fine (2010) are the following ones:

Condition 1 (String-closure) *A pre-mereological space* $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ *is* ***string-closed*** *if the domain of* Σ *is the Kleene algebra of itself, i.e.,* $X = X^\star$.[8]

In our setting, since we are working with strings, string-closure is a fundamental condition on a pre-mereological space, as it is essential to prove the completeness of standard state space (cf. Sect. 6.1).

Example 5 The pre-mereological spaces of Examples 1, 2, and 3 are all string-complete. In contrast, the pre-mereological space of Example 4 is not.

Condition 2 (Injectivity) Σ *is* ***injective*** *iff for all* $x, x' \in X$, *if* $\Sigma(x) = \Sigma(x')$, *then* $x = x'$.

Injectivity entails that each element of Y that is composed by *something*, it is only composed by that *something*.[9]

Condition 3 (Surjectivity) Σ *is* ***surjective*** *iff for every* $y \in Y$, *there exist a* $x \in X$ *such that* $y = \Sigma(x)$.

By imposing the surjectivity, we can swiftly establish that each element of Y is composed of some string; that is, each y is Σ-composite.

Other properties of Σ that are instead introduced in Fine (2010) are the ones reported below:

Condition 4 (Permutation) Σ *is* ***permutative*** *iff for all* $\ldots, a, \ldots, a', \ldots$ *and* $\ldots, a', \ldots, a, \ldots \in X$,

$$\Sigma(\ldots, a, \ldots, a', \ldots) = \Sigma(\ldots, a', \ldots, a, \ldots) \qquad \text{(PER)}$$

In simpler words, when Σ is permutative, it ensures that for any pair of strings of X that differ only in the order of elements a and a', Σ will yield the same result.

[7] Embracing Fine's operationalist interpretation, $\Sigma(\langle\rangle) = e$ amounts to stating that e is the object that we get out of the empty string, so that e is an object made out of "nothing". "Nothing" in mereology is, of course, a delicate concept, cf., for instance, Casati and Fujikawa (2019) and Carrara et al. (2022).

[8] Alternatively, this condition can be expressed by stating that X is string-closed if there exists a subset A' of A such that $X = (A')^\star$.

[9] In particular, this entails that the "special" object $e \in Y$ is composed only by $\langle\rangle$.

Condition 5 (Absorption) Σ *is* ***absorbing*** *iff for all* $\ldots, a, a \ldots, a', a' \ldots$ *and* $\ldots, a, \ldots, a', \ldots \in X$,

$$\Sigma(\ldots, a, a, \ldots, a', a', \ldots) = \Sigma(\ldots, a, \ldots, a', \ldots) \tag{ABS}$$

This condition entails that Σ does not count repetitions of elements of A in the composition. The set-builder satisfies absorption, aa and a shall compose the same object.[10]

Condition 6 (Collapse) Σ *satisfies* ***collapse*** *iff*

$$\text{For all } a \in A \cap X,\ \Sigma(a) = a \tag{COL}$$

This condition is *prima facie* reasonable because the composition of an object, which is essentially a generator of A, is identical to that object itself. However, some composition operators, like the set-builder $\{\ldots\}$, do not respect it. The set-builder $\{\ldots\}$ applied to x returns the singleton $\{x\}$, that is different from x (see also Fine 2010).

Condition 7 (Levelling) Σ *satisfies* ***levelling*** *iff the following condition holds:*

$$\text{For all } a'_1, \ldots, a'_n \in X,\ \text{if } y = \Sigma(a'_1, \ldots, a'_m) \text{ then,} \tag{LEV}$$

$$\text{if } a_1, \ldots, y, \ldots, a_n \text{ and } a_1, \ldots, a'_1, \ldots, a'_m, \ldots, a_n \in X,$$
$$\Sigma(a_1, \ldots, y, \ldots, a_n) = \Sigma(a_1, \ldots, a'_1, \ldots, a'_m, \ldots, a_n)$$

A composition that satisfies levelling is not hierarchical. As Fine points out in Fine (2010), the standard mereological fusion meets levelling. There are, however, composition operators that do not satisfy this property. One of them, for instance, is the set-builder mentioned earlier, e.g. $\{\{x\}\} \neq \{x\}$.

Finally, Fine describes the next condition (anti-cyclicity) as "if x can be built up from x itself, then any intermediate whole $\Sigma(..., x, ...)$ involved in the construction must itself be x (here, $\Sigma(..., x, ...)$ can occur at *any* depth within $\Sigma(..., \Sigma(..., x, ...), ...)$)" (Fine 2010, p. 568, emphasis ours). Before stating anti-cyclicity, it is worthwhile to provide a compact notation to write down precisely the compositions nested at a certain depth.

Definition 5 (*Nested compositions*) Given the strings $s, s', s^j, s'^j \in A^\star, y, y^j \in A$ for $j \in \{1, \ldots, n\}$, we define recursively the following set of expressions to denote composition Σs.

$$\text{(base)}\Sigma^1(s, y, s') := \Sigma(s, y, s') \quad \text{i.e., no } \Sigma \text{ occurs in } s, y, s'$$
$$\text{(step)}\Sigma^j(s^{j-1}, y^{j-1}, s'^{j-1}) := \Sigma(s^{j-1}, \Sigma^{j-1}(s^{j-2}, y^{j-2}, s'^{j-2}), s'^{j-1})$$
$$\text{if } y^{j-1} = \Sigma^{j-1}(s^{j-2}, y^{j-2}, s'^{j-2})$$

Hence, by writing $\Sigma^n(s^{n-1}, y^{n-1}, s'^{n-1})$, we are supposing that Σ is defined on all the required strings. To compute the denotation of the expression Σ^n, we use the two clauses of the inductive definition. For example, $\Sigma(a_1, \ldots, \Sigma(b_1, \ldots, \Sigma(c_1, \ldots, c_n), \ldots, b_m), \ldots, a_k)$ is obtained from $\Sigma^3(a_1, \ldots, y, \ldots, a_k)$, where $y = \Sigma^2(b_1, \ldots, y', \ldots b_m)$ and $y' = \Sigma^1(c_1, \ldots, c_n)$. The notation also indicates the depth of the nesting, e.g. 3 in the previous example.

Condition 8 (Anti-cyclicity) Σ *is* ***anti-cyclic*** *iff whenever there is a natural number* $m \in \mathbb{N}$ *and strings* $s, s', s^j, s'^j \in A^\star$, $y^j \in A$, *for* $j \in \{1, \ldots, m\}$, *such that* $\Sigma^m(s^{m-1}, y^{m-1}, s'^{m-1}) = y$ *and* $\Sigma^1(s, y, s') = \Sigma(s, y, s')$, *then the following condition holds:*

$$\text{For all } 2 < j < m,\ \Sigma^j(s^{j-1}, y^{j-1}, s'^{j-1}) = y \tag{ACY}$$

When the context is clear, to simplify the notation, we shall write $\Sigma(s, \ldots, \Sigma(r, \ldots, \Sigma(..., y, ...), \ldots, r'), \ldots, s')$ to indicate a Σ^n term. This simplification also eases the comparison with the notation in Fine (2010).

The notion of anti-cyclicity plays a significant role in preventing pre-mereological loops within a composition. In essence, it states that if an object y can be derived from itself through compositions nested at any level, then each intermediate composition $\Sigma(\ldots, y, \ldots)$ within this nesting must ultimately result in the same individual y (cf. Fine 2010, §3). In Fine (2010), anti-cyclicity is indeed closely connected to the well-foundedness of the corresponding parthood relation. In a well-founded theory, the antecedent of anti-cyclicity cannot be satisfied, rendering the condition trivially true. Furthermore, assuming that the composition operator is anti-cyclical has the added consequence of supporting the establishment of anti-symmetry within the parthood relation, as we will explore further in subsequent sections.

3.1 From Pre-mereology to Mereologies

It is time to elucidate how to define the parthood relations from a pre-mereological space. A strategy proposed by Fine (cf. Fine 2010, pp. 567–568), provides a compelling technique that always yields a preorder — a reflexive and transitive relation. In fact, one could argue that it is almost too powerful, as there are cases where we require weaker parthood relations that do not even satisfy the properties of

[10] One can consider a stronger condition of absorption, where many occurrences of substrings reduce to a single occurrence, e.g. when ...s...s... reduces to ...s.... The strong version is equivalent to ABS in presence of PER.

a preorder [as discussed in works such as Kearns (2011), Rescher (1955), Varzi (2006)]. To handle this case, Fine revises his definition of parthood in Fine (2010, p. 580).

We restate Fine's definition of component and parthood. Given any pre-mereological space $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$, we define component and parthood as relations on $Y \times Y$, as follows.

Definition 6 (*Component*) We say that y in Y is a ***component*** of y' in Y iff $\Sigma(\ldots, y, \ldots) = y'$, for at least one string $\ldots, y, \ldots$ in X that includes y.

Examples of components are the elements of a set w.r.t. the set-builder or the parts of a whole w.r.t. mereological fusion. It is worth noting that the component relation is not necessarily reflexive. For instance, consider the set builder Σ_3 of Example 3 as a composition operator. In that case, given a string $s = o_1, \ldots, o_n$ in X, $\Sigma_3(s) = \{o_1, \ldots, o_n\}$. However, due to our definition of Σ_3, there is no list in X that contains $\{o_1, \ldots, o_n\}$, as X contains only lists of ordinary objects, not sets. Consequently, $\{o_1, \ldots, o_n\}$ is not a component of itself, as there is no list containing $\{o_1, \ldots, o_n\}$ in X such that $\Sigma_3(\ldots, \{o_1, \ldots, o_n\}, \ldots) = \{o_1, \ldots, o_n\}$.

By means of components, parthood is defined.

Definition 7 (*Part*) We say that y in Y is a ***part*** of y' in Y iff there exists a sequence $y_1, y_2, \ldots, y_n$ of elements of Y (with $n \geqslant 1$) such that, $y = y_1$, $y' = y_n$ and for each y_i and y_{i+1}, $i \leqslant n$, y_i is a component of y_{i+1}. If y is part of y', we write $y \sqsubseteq y'$.

Definition 7 produces parthoods that are always preorders, i.e. (i) reflexive and (ii) transitive. (i) They are reflexive because the parthood relation is trivially reflexive: notice that y is a part of y because there exists a sequence of length 1 such that $y = y_1$, namely $y = y$; in this case, there is no y_{i+1}, so the condition about components holds trivially (cf. Fine 2010, p. 569). (ii) Parthood relations are transitive, simply because the composition of sequences of elements is transitive.

By means of Definition 7, other mereological concepts can be introduced.

Definition 8 (*Proper part*) We say that y in Y is a ***proper part*** of y' in Y, $y \sqsubset y'$, iff $y \sqsubseteq y'$ and $y \neq y$.[11]

Definition 9 (*Atom*) We say that y in Y is an ***atom*** iff there is no y' in Y such that $y' \sqsubset y$.

By introducing the definition of atoms, we can discuss the correspondence between Σ-simple and atomic objects.

Proposition 1 *Let $\mathcal{P}$ be a pre-mereological space and $\sqsubseteq$ a parthood relation on $Y \times Y$ defined by Definitions 6 and 7. If $y \in Y$ is Σ-simple, then y is atomic (w.r.t. $\sqsubseteq$).*[12]

Proof If $y \in Y$ is Σ-simple, then there is no $x \in X$ such that $\Sigma(x) = y$. Then, y has no proper parts. Otherwise, there would exist a sequence $y_1, \ldots, y_n$ of elements of Y such that $y_1 = y$ and $y_2 \neq y$ and y_1 is a component of y_2, that is $y = \Sigma(\ldots, y_2, \ldots)$, against the simplicity of $y = y_1$. □

Notice that the reflexivity of parthood comes from the limit case of Definition 7. A condition that would force the component relation to be reflexive is Condition COL. However, there is a weaker one.

Condition 9 *For every $y \in Y$, there exists a string $\ldots, y, \ldots$ that includes y in X such that $\Sigma(\ldots, y, \ldots) = y$.*

This condition directly entails reflexivity of components. However, it has two drawbacks. Firstly, it makes Σ surjective: every object is composed by something that includes the object. Secondly, if we also assume that Σ is injective, the condition prevents genuinely composed object, e.g. it prevents that $y = \Sigma(a, b)$, for $a \neq b \neq y$.

We could easily abandon the reflexivity of parthood, by tinkering with Definition 7 and imposing that the sequence of compositions must have length greater than 2 (cf. Fine 2010, p. 580).

Traditionally, "standard" parthood relations, cf. Varzi (2014) are modeled *via* partial orders: it is often acknowledged that these relations are reflexive, transitive, and anti-symmetric. The importance of anti-symmetry for understanding parthood is also discussed in Fine (2010, p. 580). In general, our framework does not always yield anti-symmetric parthood relations, as shown by this example:

Example 6 Consider the pre-mereological space $(A = \{a, b\}, \Sigma_4 : X \longrightarrow X)$, where $a \neq b$, from Example 4. In this example, we have the set $X = A = \{a, b\}$ and the composition operator Σ_4 defined as $\Sigma_4(a) = b$ and $\Sigma_4(b) = a$.

Now, let's define the parthood relation $\sqsubseteq$ as suggested by Definition 7. According to this definition, we can observe that $a \sqsubseteq b$ and $b \sqsubseteq a$, as b is a component of a, and *vice versa*. In other words, each element can be seen as part of the other within the context of the composition Σ_4. However, a and b are not identical, resulting in a simple case of a pre-mereological loop.

Concerning anti-symmetry, Fine notices that, if Σ is anti-cyclical, the obtained parthood relation is anti-symmetric (cf. Fine 2010, p. 568). Thus, the outcome can be ensured by considering anti-cyclical Σ operators.

[11] For a discussion of the choice of the definition of proper parts in case of preorders, see Varzi (2014).

[12] The other direction does not hold in general. It suffices to define an injective Σ such that, for every $a \in A \cap X$, $\Sigma(a) = a$ (cf. COL). In this case, if a is atomic, it has no proper parts (all of the parts of a are a), but a is not Σ-simple.

Proposition 2 *Let $\mathcal{P}$ be a pre-mereological space, if $\Sigma : X \longrightarrow Y$ is anti-cyclical, then $\sqsubseteq$ is antisymmetic.*

Proof Suppose, for y and y' in Y, that $y \sqsubseteq y'$ and $y' \sqsubseteq y$. Then, there exist two series of components: (i) $y_1, \dots, y_n$ such that $y_1 = y$, $y_n = y'$ and, for $i \in \{1, \dots, n\}$, y_i is a component of y_{i+1}, and (ii) $y'_1, \dots, y'_m$ such that $y'_1 = y'$, $y'_m = y$ and, for $i \in \{1, \dots, m\}$, y'_i is a component of y'_{i+1}.

The assumptions $y \sqsubseteq y'$ and $y' \sqsubseteq y$ entail that y and y' are of the following forms:

$$y = \Sigma(s^1, \dots, \Sigma(s^2, \dots, \Sigma(s, y', s'), \dots, s'^2), \dots, s'^1) \tag{1}$$

$$y' = \Sigma(r^1, \dots, \Sigma(r^2, \dots, \Sigma(r, y, r'), \dots, r'^2), \dots, r'^1) \tag{2}$$

By substituting y' in (4) with $\Sigma(r^1, \dots, \Sigma(r^2, \dots, \Sigma(r, y, r'), \dots, r'^2), \dots, r'^1)$, we obtain that

$$y = \Sigma(\dots, \Sigma(s, \Sigma(r^1, \dots, \Sigma(r^2, \dots, \Sigma(r, y, r'), \dots, r'^2), \dots, r'^1), s'), \dots) \tag{3}$$

Equation (5) meets the assumption of Condition ACY, thus, by anti-cyclicity, any whole in (5) that includes y returns y. In particular, $\Sigma(r^1, \dots, \Sigma(r^2, \dots, \Sigma(r, y, r'), \dots, r'^2), \dots, r'^1) = y$. However, by (5), $\Sigma(r^1, \dots, \Sigma(r^2, \dots, \Sigma(r, y, r'), \dots, r'^2), \dots, r'^1) = y'$. Therefore, by functionality of Σ, $y = y'$. □

Notice that the operator of Example 6 is not anti-cyclical, e.g. $\Sigma_4(\Sigma_4(a)) = a$, as $\Sigma_4(a) = b$.

We have only touched upon a few connections between pre-mereological spaces and mereologies. The main take-away from this approach is that it provides the freedom to define mereologies using a range of different types of relations, such as preorders, partial orders, and even weaker relations obtained by relaxing reflexivity or transitivity.

In Sect. 6, we will apply pre-mereological spaces to obtain two distinct mereological systems: a quite standard one, serving as the semantic foundation for logics for analytic containment as discussed in Fine (2016), and a quite non-standard one to provide semantics for Associative Lambek Calculus.

4 From Vertical to Horizontal Compositions

In Fine (2010), the composition operators Σ discussed in the previous section are referred to as "vertical" operators, since they combine a collection of objects to form a new composite object. But we are also familiar with different composition operators that, given two objects, blend them and generate a new one. The set-theoretic union gives a simple example: $\{a, b, c\} \cup \{a, d, e\} = \{a, b, c, d, e\}$. Fine calls these binary composition operators "flat" or "horizontal" and outlines a method to define them from vertical ones (cf. Fine 2010, §7). Given our specific goals, we will dig into this topic further in this section.

If the composition is injective, to define a horizontal operation, it is sufficient to offer the following definition.

Definition 10 (*Horizontalisation*) Let $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ be a pre-mereological space with an injective Σ. The ***horizontal composition obtained from*** Σ, i.e., its horizontalisation, is the partial function $\oplus : \mathsf{Comp}(\mathcal{P}) \cup \{e\} \times \mathsf{Comp}(\mathcal{P}) \cup \{e\} \longrightarrow Y : (y_1, y_2) \mapsto y_1 \oplus y_2$ where $y_1 \oplus y_2$ is defined as[13]

$$\begin{cases} y_1 \text{ if } y_1 \text{ is } \Sigma\text{-simple and } y_2 = e \\ y_2 \text{ if } y_2 \text{ is } \Sigma\text{-simple and } y_1 = e \\ \Sigma(y_1, y_2) \text{ if } y_1, y_2 \text{ are } \Sigma\text{-simple} \\ \Sigma(y_1, a_1, \dots, a_m) \text{ if } y_1 \text{ is } \Sigma\text{-simple and } y_2 \\ \qquad = \Sigma(a_1, \dots, a_m) \\ \Sigma(a_1, \dots, a_n, y_2) \text{ if } y_1 = \Sigma(a_1, \dots, a_n) \text{ and } y_2 \\ \qquad \text{is } \Sigma\text{-simple} \\ \Sigma(a_1, \dots, a_n, a'_1, \dots, a'_m) \text{ if } y_1 = \\ \qquad \Sigma(a_1, \dots, a_n), y_2 = \Sigma(a'_1, \dots, a'_m) \end{cases}$$

It is important to highlight a few key points. Firstly, the definition is well-posed because $\oplus$ is defined over $\mathsf{Comp}(\mathcal{P})$, cf. Definition 4, ensuring that its inputs are elements in both X and Y.

Second, the definition assumes the injectivity of Σ. In cases where Σ is not injective, a more sophisticated procedure is required for horizontalisation, as described in Fact 1. It is also worth observing that when the vertical composition is surjective, since there are no Σ-simple states in $\mathsf{Comp}(\mathcal{P})$, the definition simplifies to a single case:

$$y_1 \oplus y_2 := \Sigma(a_1, \dots, a_n, a'_1, \dots, a'_m)$$

where $y_1 = \Sigma(a_1, \dots, a_n)$ and $y_2 = \Sigma(a'_1, \dots, a'_m)$. Therefore, the number of cases of our definition is motivated by the flexibility of our framework, which aims to horizontalise general Σs.

In particular, the first four cases outlined in Definition 10 are driven by the goal of horizontalising Σs that operate on both simple and composite elements. However, one might find this definition somewhat unconventional, assuming that horizontal compositions should only apply to composite elements, similar to set union (standard set union does not apply

[13] It is worth noting that the presence of a neutral element e in the definition of horizontal composition is not mandatory. In fact, many standard mereologies do not have a neutral element for fusion. If you prefer to exclude it, you can define the function as $\oplus : \mathsf{Comp}(\mathcal{P}) \times \mathsf{Comp}(\mathcal{P}) \longrightarrow Y$, ignoring the first two cases of the definition.

to non-sets). An associated concern is whether the specified conditions genuinely capture horizontal compositions of simple elements or not, as alternative ways of defining horizontalisations on them are possible.

For example, an alternative approach might be that 'simples cannot blend together at all'. In that case, Σ is not defined on simples and Definition 10 reduces to the final case. Anyway, for the purpose of generating TS from non-standard mereologies, addressing this matter can be legitimately deferred, maintaining our definition as a starting point in this inquiry.

Additionally, it is important to note that, due to the generality of our framework, it is possible to have two strings s_1 and s_2 in X, while $s_1, s_2 \notin X$ (i.e. $X \subseteq A^\star$). To allow for these cases, we have defined $\oplus$ as a partial function. However, in many contexts (e.g., in standard mereology when Σ is conceived as fusion), $\oplus$ will be a total function. More precisely, the following result holds.

Proposition 3 *Let $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ be a string-complete pre-mereological space. The corresponding horizontal composition obtained from Σ* via *horizontalisation is a total function.*

Proof The totalness of the horizontal composition is ensured when, for every pair of composable states y_1 and y_2, their binary composition $y_1 \oplus y_2$ is defined in Y. Specifically, $y_1 \oplus y_2 \notin Y$ only when $\Sigma(s)$ is not defined, i.e., $s \notin X$, where s is a string of elements in $A \cap X$ that is the result of applying Definition 10. However, due to the string-completeness of the space, such a string shall belong to X by hypothesis. Consequently, $\oplus$ is total. □

Finally, it is worth noting that this definition forces that e serves as a neutral element for $\oplus$. In other words, e plays a role similar to that of $\langle\rangle$ for vertical operators Σ.

Fact 1 (Horizontalising non-injective Σs) *Consider a pre-mereological space $(A, \Sigma : X \longrightarrow Y)$, where Σ is a non-injective function. Given Σ, we define an equivalence relation over X:*

$$\text{For all } s, s' \in X, \quad s \sim_\Sigma s' \leftrightarrow_{def} \Sigma(s) = \Sigma(s').$$

We use this equivalence relation to build a quotient set $X/\sim_\Sigma$ whose elements are equivalence classes of strings with identical images. To choose a representative from each equivalence class, we invoke the Axiom of Choice, which guarantees the existence of a choice function in the form:

$$[s]_{\sim_\Sigma} \mapsto s$$

This function selects exactly one string s from each equivalence class $[s]_{\sim_\Sigma}$. Let us denote the image of this function as $\tilde{X}$. It is important to note that $\tilde{X}$ is a subset of the Kleene algebra $A^\star$ of the pre-mereological space. Furthermore, it can be easily shown that the restriction $\tilde{\Sigma}$ of Σ to $\tilde{X}$, defined as:

$$\tilde{\Sigma} : \tilde{X} \longrightarrow Y : s \mapsto \Sigma(s)$$

is injective.

Now, let's define the horizontal composition using $\tilde{\Sigma}$. Note that if $x \in \tilde{X} \cap Y$, then $x \in \mathsf{Comp}(\mathcal{P})$ since $\tilde{X} \subseteq X$. Therefore, we can proceed by applying Definition 10 and define a new function $\tilde{\oplus} : (\tilde{X} \cap Y) \cup \{e\} \times (\tilde{X} \cap Y) \cup \{e\} \longrightarrow Y$ by replacing Σ with $\tilde{\Sigma}$:

$$y_1 \tilde{\oplus} y_2 := \begin{cases} y_1 & \text{if } y_1 \text{ is } \tilde{\Sigma}\text{-simple and } y_2 = e \\ y_2 & \text{if } y_2 \text{ is } \tilde{\Sigma}\text{-simple and } y_1 = e \\ \tilde{\Sigma}(y_1, y_2) & \text{if } y_1, y_2 \text{ are } \tilde{\Sigma}\text{-simple} \\ \tilde{\Sigma}(y_1, a_1, \ldots, a_m) & \text{if } y_1 \text{ is } \tilde{\Sigma}\text{-simple and } y_2 = \tilde{\Sigma}(a_1, \ldots, a_m) \\ \tilde{\Sigma}(a_1, \ldots, a_n, y_2) & \text{if } y_1 = \tilde{\Sigma}(a_1, \ldots, a_n) \text{ and } y_2 \text{ is } \tilde{\Sigma}\text{-simple} \\ \tilde{\Sigma}(a_1, \ldots, a_n, a'_1, \ldots, a'_m) & \text{if } y_1 = \tilde{\Sigma}(a_1, \ldots, a_n), y_2 = \tilde{\Sigma}(a'_1, \ldots, a'_m) \end{cases}$$

Then, let's extend $\tilde{\oplus}$ to $\oplus : \mathsf{Comp}(\mathcal{P}) \cup \{e\} \times \mathsf{Comp}(\mathcal{P}) \cup \{e\} \longrightarrow Y : (y_1, y_2) \mapsto y_1 \oplus y_2$ as follows:

$$\begin{cases} y_1 \tilde{\oplus} y_2 & \text{if } y_1, y_2 \in \tilde{X} \cup \{e\} \\ y_1 \tilde{\oplus} z & \text{if } y_1 \in \tilde{X} \cup \{e\}, y \notin \tilde{X} \cup \{e\} \text{ and } z \sim_\Sigma y_2 \text{ with } z \in \tilde{X} \\ z \tilde{\oplus} y_2 & \text{if } y_1 \notin \tilde{X} \cup \{e\}, y_2 \in \tilde{X} \cup \{e\} \text{ and } z \sim_\Sigma y_1 \text{ with } z \in \tilde{X} \\ z \tilde{\oplus} z' & \text{if } y_1, y_2 \notin \tilde{X} \cup \{e\} \text{ and } z \sim_\Sigma y_1, z' \sim_\Sigma y_2 \text{ with } z, z' \in \tilde{X} \end{cases}$$

This new operation is a function because $\tilde{\oplus}$ was a function and because we have extended it simply using the representatives of the equivalence classes. By this, we have just defined the horizontal counterpart of our non-injective Σ.

Before focusing on other topics, it is worth noting some results pertaining to horizontalisations.

Proposition 4 *For each vertical composition operator, there is a unique horizontal operator defined from it by horizontalisation (cf. Definition 10), which is associative and has a neutral element.*

Proof The existence and uniqueness of such binary compositions is assured by Definition 10 and Fact 1. The neutral element is e because of Definition 10. To prove that $\oplus$ is associative, let's consider the generic elements $x, y, z \in \mathsf{Comp}(\mathcal{P}) \cup \{e\}$. We can distinguish two cases: (i) when x, y, z are all Σ-simple, and (ii) when they are all Σ-composite. The other cases are trivial and can be omitted.

Let's begin by assuming that x, y, z are all Σ-composite and that the horizontal composition is well-defined over them. We posit the following:

$$x = \Sigma(s_1), \qquad y = \Sigma(s_2), \qquad z = \Sigma(s_3)$$

We have that the operation is associative:

$$\begin{aligned} x \oplus (y \oplus z) &= \Sigma(s_1) \oplus (\Sigma(s_2) \oplus \Sigma(s_3)) \\ &= \Sigma(s_1) \oplus \Sigma(s_2, s_3) \\ &= \Sigma(s_1, s_2, s_3) \\ &= \Sigma(s_1, s_2) \oplus \Sigma(s_3)) \\ &= (\Sigma(s_1) \oplus \Sigma(s_2)) \oplus \Sigma(s_3)) = (x \oplus y) \oplus z \end{aligned}$$

Similarly, if x, y, z are all Σ-simple, we obtain that the operation is associative as well:

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus (\Sigma(y, z)) \\ &= \Sigma(x, y, z) \\ &= \Sigma(x, y) \oplus z) = (x \oplus y) \oplus z) \end{aligned}$$

In conclusion, this proves that the horizontal operation is associative as wanted. □

Proposition 5 *Let $\mathcal{P}$ be a pre-mereological space and $\oplus$ the horizontalisation obtained as in Definition 10. If Σ satisfies COL and is defined on $\langle\rangle$, then $\Sigma(\langle\rangle)$ is the neutral element of $\oplus$ (consequently, the first two cases of Definition 10 can be dropped).*

Proof Take $y \in \mathsf{Comp}(\mathcal{P})$. Let's suppose $\mathcal{P}$ satisfies COL. Then, we gain $y = \Sigma(y) = \Sigma(y, \langle\rangle, \langle\rangle) = y \oplus \Sigma(\langle\rangle, \langle\rangle) = y \oplus \Sigma(\langle\rangle)$. Similarly, if we consider $y = \Sigma(y) = \Sigma(\langle\rangle, \langle\rangle, y)$, we obtain $\Sigma(\langle\rangle) \oplus y = y$. Hence, $\Sigma(\langle\rangle)$ is the neutral element for $\oplus$. □

Proposition 6 *Let $\mathcal{P}$ be a pre-mereological space with a permutative PER and absorbing ABS composition operator Σ well-defined on $\langle\rangle$. In such cases, the horizontal composition is both commutative and idempotent. This means that for any elements x and y in $\mathsf{Comp}(\mathcal{P})$, we have $x \oplus y = y \oplus x$ and $x \oplus x = x$.*

Proof To simplify the proof, let's consider the case where the elements x and y in $\mathsf{Comp}(\mathcal{P})$ are both simple. To prove the proposition for non-simple states, the proof follows a similar approach.

- *Commutativity:* we want to show that $x \oplus y = y \oplus x$. By Definition 10, we have: $x \oplus y = \Sigma(x, y)$ and $y \oplus x = \Sigma(y, x)$. Since Σ is permutative, it satisfies $\Sigma(x, y) = \Sigma(y, x)$. Therefore, we have $x \oplus y = y \oplus x$.
- *Idempotency:* We want to show that $x \oplus x = x$. By Definition 10, we have: $x \oplus x = \Sigma(x, x)$. Since Σ is an absorbing operator, $\Sigma(x, x) = \Sigma(x) = \Sigma(x, \langle\rangle, \langle\rangle)$. Therefore, we have $x \oplus x = x \oplus \Sigma(\langle\rangle, \langle\rangle) = x \oplus \Sigma(\langle\rangle) = x$ for a simple x.

□

Before moving on to the next section, it is important to emphasise that not every horizontal composition operator defined in this manner has the features of a standard mereological sum, as it is clear from the following example:

Example 7 Consider the pre-mereological space with the list-builder Σ_1 of Example 1. By horizontalising it, this list-builder is converted into the list concatenation operator. This operator is non-idempotent, non-commutative, and associative, with the empty string as the neutral element.

By Proposition 4, our horizontalisation always produces associative operators $\oplus$. This is motivated in this paper as a simplification. Our intention is to construct TS for the logics of Sect. 6, which have associative conjunctions and disjunctions. Therefore, we have opted to refrain from delving into the technicalities of non-associative logics.

A natural question is the following: Can we introduce an alternative technique of horizontalisation that generates non-associative binary compositions? Exploring this question is intriguing because it opens up possibilities for axiomatising new forms of mereological composition that are suitable for non-associative logics (e.g. non-associative Lambek calculus).

For example, we can restate Definition 10 by putting $x \oplus y = \Sigma(\Sigma(y_1), \Sigma(y_2))$, where $x = \Sigma(y_1)$ and $y = \Sigma(y_2)$, for x and y non-simple. Associativity, in this case, amounts to proving the following equality, which follows if Σ satisfies LEV:

$$\begin{aligned} &\Sigma(\Sigma(y_1), \Sigma(\Sigma(y_2), \Sigma(y_3))) \\ &\quad = \Sigma(\Sigma(\Sigma(y_1), \Sigma(y_2)), \Sigma(y_3)) \end{aligned} \tag{4}$$

While this question holds significance in the study of non-standard mereologies and non-associative logics, we defer this inquiry to a future dedicated research.

5 From Horizontal to Vertical Compositions

In the previous section, we showed that, given a vertical composition, it is always possible to define a corresponding associative horizontal operation with a neutral element. In this section, we prove that we can also define a vertical operation given any horizontal one. This result is significant in providing TS for substructural logics with monoidal models, as we will see in Sect. 6.

To prove this result, consider any set M equipped with a binary associative operation $\oplus$. We also assume that $\oplus$ has a neutral element e. Because of these very minimal conditions, $(M \cup \{e\}, \oplus, e)$ is in fact a *monoid*. In every monoid, it is possible to generalise $\oplus$ to operate on finite strings of elements of $M \cup \{e\}$. This new function is the verticalisation of the binary operation of the monoid.

Definition 11 (*Verticalisation*) Given a monoid $(M \cup \{e\}, \oplus, e)$, we define the vertical operation $\Sigma_\oplus : (M \cup \{e\})^\star \longrightarrow M \cup \{e\}$ corresponding to $\oplus$ as

$$\Sigma_\oplus : (M \cup \{e\})^\star \longrightarrow M \cup \{e\} : m_1, \ldots, m_n \mapsto m_1 \oplus \ldots \oplus m_n$$

This function takes a string s of length n of elements of $M \cup \{e\}$ as input and generates an element of M by applying, $n-1$ times, $\oplus$ to the elements of s (notice that this definition is well posed because $\oplus$ is associative).

By this simple procedure, we have just defined a composition operator Σ for $M \cup \{e\}$ that inherits all the algebraic properties of $\oplus$.

Example 8 Consider the commutative monoid $(M \cup \{\varnothing\}, \cup, \varnothing)$, where $M = \{A, B, A \cup B\}$ and A and B are sets. In this case, $\cup$ functions as the horizontal composition. Now, let's proceed to verticalise it. We define the following:

$$\Sigma_\cup : (M \cup \{\varnothing\})^\star \longrightarrow M \cup \{\varnothing\} : s \mapsto \begin{cases} \varnothing & \text{if } s = \langle\rangle \\ m & \text{if } s = m, \text{ i.e. } l = 1 \\ m_1 \cup m_2 \cup \cdots \cup m_n & \text{if } s = m_1, m_2, \ldots, m_n \end{cases}$$

While the second case always returns either A or B, the last case always results in the union of the inputs. This clearly indicates that $\Sigma_\cup$ is a vertical composition, and consequently, $(M \cup \varnothing, \Sigma_\cup)$ forms a pre-mereological space.

Observe that the vertical operations defined by Definition 11 always satisfy COL and LEV. You gain COL by the second condition of the definition. Instead, LEV is yielded by the following reasoning. Wlog. consider a string $s = m_1, \Sigma_\oplus(m_2, m_3, \ldots)$. We have that

$$\begin{aligned} \Sigma_\oplus(m_1, \Sigma_\oplus(m_2, m_3, \ldots)) &= m_1 \oplus \Sigma_\oplus(m_2, m_3, \ldots) \\ &= m_1 \oplus (m_2 \oplus \Sigma_\oplus (m_3, m_4, \ldots)) \\ &= m_1 \oplus (m_2 \oplus (m_3 \oplus \Sigma_\oplus (m_4, \ldots))) \ldots \\ &= m_1 \oplus (m_2 \oplus (m_3 \cdots) \cdots) \\ {}^\dagger &= \Sigma_\oplus(m_1 \oplus m_2 \oplus m_3 \oplus \cdots) \end{aligned}$$

Notice that LEV follows because $\oplus$ is associative. If $\oplus$ were not associative, then step † is not legit. Other properties of $\Sigma_\oplus$ can be inherited from $\oplus$, e.g. if $\oplus$ is commutative, then $\Sigma_\oplus$ is permutative.

An interesting question, that we leave for future work, is natural to investigate whether the constructions presented in Sect. 4 and Sect. 5 are invertible.

In fact, the procedures of horizontalisation and verticalisation presented in this article are not generally inverse operations. To illustrate this, let's imagine to horizontalise a vertical composition Σ that satisfies neither LEV nor COL. This process yields the horizontal operator $\oplus$. Subsequently, if we verticalise $\oplus$, we obtain a second vertical composition $\Sigma_\oplus$. By definition, $\Sigma_\oplus$ satisfies both LEV and COL, showing that $\Sigma \neq \Sigma_\oplus$. This implies that the two procedures are not inverses of each other.[14]

We conclude this section by observing that the construction of Definition 11 suggests that any monoid has a (pre-)mereological interpretation: it can always be associated to a pre-mereological space, by means of which we can define a mereological structure, according to Sect. 3.1. Moreover, by means of horisontalisation, cf. Definition 10, any pre-mereological space returns a monoidal structure, when the horizontalisation of Σ is a total function.

6 Constructing Truthmakers Semantics

In this section, we apply pre-mereological spaces and the horizontalisation of Σ to provide structures for TS. That is, we construct *state spaces*, cf. Fine (2017a), from pre-mereological spaces.

To do that, we offer a definition of an abstract state space, that merely serves as a bridge between pre-mereological spaces and actual state spaces to lists the required ingredients for a TS. As we shall see, certain abstract state spaces, where

[14] They are each other's inverses to the extent that the vertical compositions under investigation satisfy COL and LEV.

Σ meets certain conditions, can be instantiated to obtain the usual structure for TS, as those in Fine (2016). Other conditions on Σ are required to provide TS for other logics, we shall see, as an example, the case of the associative Lambek calculus (Buszkowski 2006).[15]

Definition 12 (*Abstract state space*) Given a pre-mereological space $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$, an ***abstract state space*** is a triple $\mathcal{S} = (S, \sqsubseteq, \oplus)$ such that:

1. The set of states S of $\mathcal{S}$ is $\mathsf{Comp}(\mathcal{P})$.
2. The relation $\sqsubseteq$ of $\mathcal{S}$ is defined from Σ according to Definition 6 and 7 of Sect. 3.1.[16]
3. The binary composition operator $\oplus$ is defined by taking the horizontalisation of the vertical composition Σ of $\mathcal{P}$, according to Definition 10 of Sect. 4.

Notice that, by Definitions 6 and 7, the abstract state space $\mathcal{S}$ is always a preorder.[17] Moreover, if Σ satisfies ACY, then the state space is a partial order.

6.1 Standard State Spaces

A *standard state space*, as defined in Fine (2016), exhibits two key properties: (i) $\sqsubseteq$ is a partial order, and (ii) S is *complete*, meaning that for any non-empty finite subset $T \subseteq S$, there exists a least upper bound (l.u.b.) denoted by $\bigsqcup T$. The least upper bound of a set T is an element $s \in S$ that satisfies two conditions: (i) s is an upper bound of T, meaning that for all $t \in T$, $t \sqsubseteq s$, and (ii) for every upper bound s' of T, we have $s \sqsubseteq s'$.

In the case of a standard state space, the binary composition operator, denoted by $\sqcup$ in e.g. Fine (2016), is defined as $s \sqcup t = \bigsqcup\{s, t\}$. Consequently, the standard horizontal composition $\sqcup$, also known as fusion, is commutative and idempotent, as expected.

To construct a standard state space, we need to demonstrate that every non-empty (finite) subset of $\mathsf{Comp}(\mathcal{P})$ has a least upper bound. It is worth noting that if $T = \{y_1, \ldots, y_m\} \subseteq \mathsf{Comp}(\mathcal{P})$ and if $\Sigma(y_1, \ldots, y_n) = y$ is defined, then y serves as an upper bound for T. This is because each y_i is a component of y, thus $y_i \sqsubseteq y$.

Importantly, to ensure that the required strings are in the domain of Σ, we shall impose that the pre-mereological space is *string-closed*, cf. Condition 1.[18]

Moreover, as we shall see, the existence of least upper bounds for *subsets* of $\mathsf{Comp}(\mathcal{P})$ is guaranteed if Σ is absorbing, permutative, and satisfies the leveling property.

Firstly, we prove the two following useful lemmas.

Lemma 1 *Let $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ be a string-closed pre-mereological space (cf. Condition 1). Suppose that Σ satisfies levelling LEV, absorption ABS, and permutation PER. Hence, Σ satisfies **cumulativity**: for all s and $s' \in X$,*

$$If\ \Sigma(s) = y\ and\ \Sigma(s') = y,\ then\ \Sigma(s, s') = y. \quad \text{(CUMU)}$$

Proof Let $y = \Sigma(s) = \Sigma(s')$ as in the hypotheses. The following reasoning assures that CUMU holds as true. Starting from the properties of absorption (ABS) and permutativity (PER), we have $\Sigma(s) = \Sigma(s, s)$. Because of LEV, we also have: $\Sigma(s, s) = \Sigma(s, \Sigma(s))$. Now, $\Sigma(s, \Sigma(s))$ is equal to $\Sigma(s, \Sigma(s'))$ by hypothesis, and to $\Sigma(s, s')$ as well, for LEV. As a consequence, this reasoning yields that $y = \Sigma(s, s')$ as wanted. □

Lemma 2 *Let $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ be a string-closed pre-mereological space (cf. Condition 1). Suppose Σ satisfies levelling LEV. Hence, Σ satisfies **monotonicity**: for all s, r, r', $y \in X$,*

$$If\ \Sigma(s) = y\ and\ \Sigma(r, s, r') = y',\ then\ \Sigma(r, y, r') = y' \quad \text{(MONO)}$$

Proof The following reasoning demonstrates that such a composition satisfies MONO. Starting with the equation $\Sigma(r, y, r') = \Sigma(r, \Sigma(s), r')$ as given by the hypothesis, we can proceed as follows:

1. Apply the leveling property (LEV) to obtain $\Sigma(r, \Sigma(s), r') = \Sigma(r, s, r')$;
2. According to the hypothesis, we have $\Sigma(r, s, r') = y'$.

Combining these steps, we can conclude that $\Sigma(r, y, r') = y'$, satisfying MONO. □

These two lemmas allow us to prove that the resulting state space is complete:

[15] The following definition uses the horizontal binary operation to facilitate the comparison with the models of a number of logics, e.g. substrutural logics, whose structures are indeed based on sets equipped with binary operations. However, it is intriguing to suggest a definition of an abstract state space directly using the vertical operator Σ of the pre-mereological space. Depending on the properties of Σ, our Definition 12 might return horizontal operators that are substantially equivalent to their vertical counterpart. In other cases, the equivalence fails. For instance, take the vertical set-builder, whose horizontalisation returns the set-theoretic union, cf. Fine (2010), p. 578.

[16] In particular, it is the restriction of the parthood relation defined over $Y \times Y$ to $\mathsf{Comp}(\mathcal{P}) \times \mathsf{Comp}(\mathcal{P})$. However, for the sake of readability, we will continue to denote it by $\sqsubseteq$ instead of $\sqsubseteq|_{\mathsf{Comp}(\mathcal{P}) \times \mathsf{Comp}(\mathcal{P})}$. The context will make it clear whether we are referring to the relation over $Y \times Y$ or the one defined over $\mathsf{Comp}(\mathcal{P}) \times \mathsf{Comp}(\mathcal{P})$.

[17] To get even weaker relations, cf. the discussion of Sect. 3.1.

[18] The condition is required, since we defined X, the domain of Σ, as any subset of $A^\star$, which may lack certain strings. The motivation for this approach is to make all the assumptions required to obtain a standard state space explicit.

Theorem 1 (State space completeness) *Let* $\mathcal{P} = (A, \Sigma : X \longrightarrow Y)$ *be a string-closed pre-mereological space (cf. Condition 1). Suppose* Σ *satisfies acyclicity ACY, levelling LEV, absorption ABS, and permutation PER. Hence, the state space* $(\mathsf{Comp}(\mathcal{P}), \sqsubseteq)$ *is a complete poset.*

Proof By Condition ACY, $\sqsubseteq$ is a partial order. We show that $\mathsf{Comp}(\mathcal{P})$ is complete w.r.t. $\sqsubseteq$. Suppose $T = \{y_1, \ldots, y_m\} \subseteq \mathsf{Comp}(\mathcal{P})$. By Condition 1, $\Sigma(y_1, \ldots, y_n)$ is defined. Suppose $\Sigma(y_1, \ldots, y_n) = y$. We show that y is the l.u.b. of T. Let y' be an upper bound of T, that is, for each $y_i \in T$, $y_i \sqsubseteq y'$. Thus, by definition of parthood, for all y_i, we have $\Sigma(\ldots, \Sigma(\ldots, y_i, \ldots), \ldots) = y'$ and by LEV, we have $\Sigma(\ldots, y_i, \ldots) = y'$. Then, by Condition CUMU, $y' = \Sigma(\ldots, y_1, \ldots, y_m, \ldots)$. By Condition MONO, from $y' = \Sigma(\ldots, y_1, \ldots, y_m, \ldots)$ and $y = \Sigma(y_1, \ldots, y_m)$, we have that $y' = \Sigma(\ldots, y, \ldots)$, that is $y \sqsubseteq y'$. □

Now recall that, from any vertical operator Σ in a pre-mereological space $\mathcal{P}$, we can always define a horizontal operator $\oplus$ as described in Sect. 4. So, suppose that Σ satisfies the conditions of Theorem 1 and returns a complete poset $(\mathsf{Comp}(\mathcal{P}), \sqsubseteq)$. In this context, $\oplus$ yields the least upper bounds of any set of elements in $\mathsf{Comp}(\mathcal{P})$. Consequently, in this case, we meet the approach of Fine (2014), where $s \sqcup t = \bigsqcup\{s, t\}$.

To see that $\oplus$ yields the l.u.b.s, we illustrate only two cases, as the others are similar:

- If y_1 and y_2 are Σ-simple, then $y_1 \oplus y_2 = \Sigma(y_1, y_2)$, so $\oplus$ is the l.u.b. of $\{y_1, y_2\}$, cf. Theorem 1;
- If y_1 and y_2 are non-simple, then $y_1 = \Sigma(a_1, \ldots, a_n)$, $y_2 = \Sigma(a'_1, \ldots, a'_m)$, and $y_1 \oplus y_2 = \Sigma(a_1, \ldots, a_n, a'_1, \ldots, a'_m)$. By leveling LEV, we have that $\oplus$ is the l.u.b. of $\{y_1, y_2\}$. This can be demonstrated through the following reasoning:

$$\begin{aligned}&\Sigma(a_1, \ldots, a_n, a'_1, \ldots, a'_m)\\ &\quad = \Sigma(\Sigma(a_1, \ldots, a_n), \Sigma(a'_1, \ldots, a'_m))\\ &\quad = \Sigma(y_1, y_2)\end{aligned}$$

In particular, since $\oplus$ is always associative (cf. Proposition 4), we can see that for Σ-simple elements, according to LEV, $y_1 \oplus \cdots \oplus y_n = \Sigma(y_1, \ldots, y_n)$, which is the l.u.b. of $y_1, \ldots, y_n$, by Theorem 1.

With the foundation we have established, we can now apply pre-mereological spaces to develop TS for the logics of analytic containment, e.g. those in Fine (2016). Let $\mathcal{A}$ be a set of atomic letters, the language $\mathcal{L}$ of the logic of analytic containment is inductively defined as follows.

$$\phi := p \in \mathcal{A} \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi$$

Consider any (string-complete) $\mathcal{P}$ that satisfies the assumptions of Theorem 1. The abstract state space obtained from $\mathcal{P}$ by Definition 12, i.e. $\mathcal{S} = (S, \sqsubseteq, \oplus)$, can be used as semantics for $\mathcal{L}$.

A *Truthmakers model* for $\mathcal{L}$ is obtained by adding to $\mathcal{S}$ two evaluation functions (in the bilateral approach) from the set of atomic letters $\mathcal{A}$ of $\mathcal{L}$, v_0^+ and v_0^-, both of type $\mathcal{A} \longrightarrow \mathbb{P}(\mathsf{Comp}(\mathcal{P}))$. The atomic valuations extend to the full $\mathcal{L}$, as follows.

i^+) $y \Vdash p$ iff $y \in v_0^+(p)$
i^-) $y \dashv\mid p$ iff $y \in v_0^-(p)$
ii^+) $y \Vdash \neg\phi$ iff $y \dashv\mid \phi$
ii^-) $y \dashv\mid \neg\phi$ iff $y \Vdash \phi$
iii^+) $y \Vdash \phi \wedge \psi$ iff for some $y_1, y_2 \in Y$ such that $y = y_1 \oplus y_2$, $y_1 \Vdash \phi$ and $y_2 \Vdash \psi$.
iii^-) $y \dashv\mid \phi \vee \psi$ iff $y \dashv\mid \phi$ or $y \dashv\mid \psi$
iv^+) $y \Vdash \phi \vee \psi$ iff $y \Vdash \phi$ or $y \Vdash \psi$.
iv^-) $y \dashv\mid \phi \vee \psi$ iff for some $y_1, y_2 \in Y$ such that $y = y_1 \oplus y_2$, $y_1 \dashv\mid \phi$ and $y_2 \dashv\mid \psi$.

Notice that, we require that the composition of $\mathcal{P}$ satisfies PER and ABS (cf. Theorem 1). Thus, by Proposition 6, $\oplus$ is idempotent and commutative. For this reason, the semantic conditions for $\wedge$ and $\vee$ are well-posed. E.g. there is a single commutative conjunction and a single commutative disjunction in this logic.

An important property in TS is to establish that there exist truthmakers (or falsemakers) of any formula of $\mathcal{L}$. This fact, cf. Fine (2016), is readily generalisable to abstract state space. In any truthmaker model based on abstract state spaces, if $v_0^+(p)$ ($v_0^-(p)$) is non-empty for any $p \in \mathcal{A}$, then the sets of verifiers (falsifiers) of formulas ϕ, i.e. $|\phi|^+ = \{y \mid y \Vdash \phi\}$, are non-empty. It is interesting to notice that this result requires that the pre-mereological is string-closed. To see this, consider a truthmaker model where $y \Vdash p$, $y' \Vdash q$ and they are the sole states that verify those proposition. If $\mathcal{P}$ is not string-closed, $y \oplus y'$ may be undefined (see Sect. 4), resulting in $|p \wedge q|^+ = \varnothing$.

Any structure $(\mathsf{Comp}(\mathcal{P}), \sqsubseteq, \oplus)$ is then isomorphic to a standard state space. If we define a relation of logical entailment by posing that ϕ entails ψ, $\phi \Vdash \psi$, iff for truthmaker every model, $|\phi|^+ \subseteq |\psi|^+$, then any abstract state space that meet the conditions of Theorem 1 validates axioms E1–E15 in cf. Fine (2016), p. 201.[19]

[19] For the sake of example, we are only defining the *exact entailment* $\phi \Rightarrow \psi$, cf. Fine (2016, p. 202). So axioms E1–15, which are written in terms of $\leftrightarrow$, are interpreted by $\phi \leftrightarrow \psi$ iff $(\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$. *Analytic entailment* (i.e. $\rightarrow$) can simply be defined by considering complete contents. For an exhaustive treatment of this logics in terms of sequents, that handle consequences from sets of hypotheses, we refer to Jago (2017). For our illustrative purpose, the treatment of Fine (2016) suffices.

Further conditions on state spaces are required to properly model various notions of analytic containment (e.g. convexity, cf. Fine 2016). Other important constructions of TS relate state spaces and valuations functions, e.g. *possible states* and *world states* (cf. Fine 2017c). While we cannot examine this topics here, we notice that those aspects of TS are all build on top of a state space, so in principle they can be restated in our setting.[20]

To conclude this section, we have established that standard truthmakers model can be obtained from any pre-mereological space that meets the required conditions.

6.2 State Space for the Associative Lambek Calculus

A direct consequence of our discourse of Sect. 5 is this: Any substructural logic whose semantics is faithfully represented by monoidal structures can be faithfully represented by an abstract state space.

The current section makes this result apparent by writing down the general recipe for cooking up TS and applying it to one specific quite representative case: the associative Lambek calculus, **L**, see Lambek (1958). Our motivation for focussing on **L** is that it is the weakest substructural logic: besides associativity[21], it lacks Exchange, Contraction, and Weakening (cf. Buszkowski 2006). Thus, **L** lays quite distantly from the logics obtained from standard state spaces, which enable Exchange and Contraction (and, sometimes, a mild form of Weakening[22]).

An important example of structures that provide models of **L** are the *language frames*, cf. Pentus (1994).[23] They are based on non-commutative monoids, in our notation $(L^\star, \oplus)$, where $L^\star$ is the string algebra constructed from L, $\oplus$ is the concatenation of strings, and $\langle\rangle$ is the null string in $L^\star$.

The language of the Lambek calculus $\mathcal{L}_{\mathbf{L}}$ is defined as follows. Let $\mathcal{A}$ be a set of atomic letters, we have:

$$\phi := p \in \mathcal{A} \mid \phi \bullet \phi \mid \phi \to \phi \mid \phi \leftarrow \phi$$

That is, the language contains a non-commutative conjunction and two order-sensitive implications.

A *model* of the formulas of $\mathcal{L}_{\mathbf{L}}$ is a language frame $(L^\star, \oplus)$ together with a valuation $v_0 : \mathcal{A} \longrightarrow \mathbb{P}(L^\star)$. v_0 associates a propositions p to a set of strings, which can be interpreted as its "verifiers". From this perspective, falsifiers are not present in **L**.

The valuation v_0 extends to the full language, by means of the following semantic clauses:

(i) $y \Vdash p$ iff $y \in v_0(p)$
(ii) $y \Vdash \phi \bullet \psi$ iff for some $y_1, y_2 \in Y$ such that $y = y_1 \oplus y_2$, $y_1 \Vdash \phi$ and $y_2 \Vdash \psi$.
(iii) $y \Vdash \phi \to \psi$ iff for every y' such that $y' \Vdash \phi$, $y' \oplus y \Vdash \psi$.
(iv) $y \Vdash \phi \leftarrow \psi$ iff for every y' such that $y' \Vdash \psi$, $y \oplus y' \Vdash \psi$.

Intuitively, a state y verifies $\phi \bullet \psi$ iff y is composed by two states y_1 and y_2, where y_1 "precedes" y_2 and the first verifies ϕ while the second verifies ψ. In particular, $y_1 \oplus y_2$ may verify $\phi \bullet \psi$, while $y_2 \oplus y_1$ does not. Moreover, a state y verifies the right implication $\phi \to \psi$ iff every state y that verifies ϕ can be "followed" by a state y' so that $y \oplus y'$ verifies ψ. By contrast, y verifies $\psi \leftarrow \phi$ if every state y that verifies ϕ can be "preceded" by a state y' such that $y' \oplus y$ verifies ψ.

We denote by $|\phi| = \{y \in L^\star \mid y \Vdash \phi\}$. We say that ϕ logically entails ψ in **L**, $\phi \Vdash_{\mathbf{L}} \psi$, iff in every model, $|\phi| \subseteq |\psi|$.[24]

The resulting logic **L** is non-commutative, i.e. $\phi \bullet \psi$ is not equivalent to $\psi \bullet \phi$, non-idempotent, $\phi \bullet \phi$ is not equivalent to ϕ, while it is associative. For the presentation of **L** in terms of sequents, we refer to Buszkowski (2006).

To suggest an interpretation of the language frames of **L** as a truthmaker semantics for the Lambek calculus, we propose to interpret them as abstract state spaces obtained from a pre-mereological space (Definition 12). In particular, we start to show that any language frame can be defined from a suitable pre-mereological space.

Lemma 3 *Any language frame* $(L^\star, \oplus)$ *can be defined from a suitable pre-mereological space* $\mathcal{P} = (L, \Sigma : L \longrightarrow L)$ *through horizontalisation.*

Proof Consider the pre-mereological space of strings as illustrated in Example 1. Choose the set of letters and strings, denoted as L, as the domain for the composition operator, resulting in $\mathcal{P} = (L, \Sigma : L \longrightarrow L)$ where Σ is the identity

[20] For instance, TS theorists often introduce a special subset of X called the *possible states* ($X^\diamond$), which satisfies specific conditions: (i) closure under parts, (ii) *exhaustivity*, $X^\diamond$ contains either a verifier or a falsifier of every proposition, and (iii) *exclusivity*: $X^\diamond$ does not contain any state which is a verifier and a falsifier of the same proposition. This conditions are readily available also for abstract state spaces.

[21] The only implicit structural rule that **L** permits is associativity. Lambek calculus has a non-associative version, which is capable of accounting for the way in which linguistic resources are combined, cf. De Groote and Lamarche (2002). We leave a treatment dedicated to non-associative Lambek calculus for future work.

[22] See axiom $A \wedge B \to B$ for the logic $AC_{\to}$ in Fine (2016).

[23] For language frames, see Pentus (1994). The most general models of **L** are *residuated semigroups*, cf. Buszkowski (2006), Sect. 2. Language frames are suitable structure for **L** since a completeness theorem of **L** can be established with respect to language frames.

[24] The general definition of truth in a model for substructural logics, appeals to the neutral element of the structure: ϕ is true in $(L^\star, \oplus, \langle\rangle)$ iff $\langle\rangle \Vdash \phi$, cf. Restall (2000) Sect. 3.2.2. However, we present this definition to enhance the comparison with the logics of analytic containment, which often lack a special point that evaluates logical truths. This definition of true in a model for **L** is from Buszkowski (2006).

function on strings. Notably, $\mathcal{P}$ is string-complete, evident from $L^\star = (L^\star)^\star = L$.

By applying the horizontalisation procedure (cf. Definition 10), we derive a binary, total, and associative composition operator $\oplus : L \times L \longrightarrow L$ with a neutral element $e = \langle\rangle$. This binary operation corresponds to the concatenation operator. Therefore, the structure $(L, \oplus)$ forms a monoid and, more precisely, a language frame, as desired. □

In this particular case, it is remarkable that we also have that:

Proposition 7 *Each language frame $(L^\star, \oplus)$ is associated by the procedure of verticalisation to a specific pre-mereological space $(L, \Sigma_\oplus : L^\star \longrightarrow L^\star)$. Moreover, the horizontalisation of $\Sigma_\oplus$ returns precisely $\oplus$.*

Proof Given a language frame $(L^\star, \oplus)$ (i.e. a non-commutative monoid), through the strategy given in Sect. 5, we define the vertical composition $\Sigma_\oplus$ corresponding to $\oplus$.[25]

Specifically, given $\oplus$, the verticalisation $\Sigma_\oplus : (L^\star)^\star \longrightarrow L^\star$ is then defined by

$$s \mapsto \Sigma_\oplus(s) := \begin{cases} \langle\rangle & \text{if } s = \langle\rangle \\ a_1 & \text{if } s = a_1 \\ a_1 \oplus (\Sigma_\oplus(a_2, \ldots, a_n)) & \text{if } s = a_1, a_2, \ldots, a_n \end{cases}$$

Since $(L^\star)^\star = L^\star$, the vertical composition is simply the identity of strings (because $\oplus$ is string concatenation). Thus, every language frame $(L^\star, \oplus)$ can be associated to a pre-mereological space $(L, \Sigma_\oplus : L^\star \longrightarrow L^\star)$. In particular, notice also that the horizontalisation of $\Sigma_\oplus$ returns, in this case, precisely $\oplus$, as expected by considering the proof of Lemma 3. □

The pre-mereological space obtained from a language frame in Proposition 7 is string-closed and it is easy to see that $\Sigma_\oplus$ satisfies levelling LEV, acyclicity ACY, and collapse COL, while it fails permutativity PER and absorption ABS.

This shows that we can start from a suitable pre-mereological space and obtain a monoidal structures which is a semantics for the Lambek calculus.

Now, deriving the abstract state space suitable for **L** from this pre-mereological space is straightforward:

Lemma 4 *Given the pre-mereological $(L, \Sigma_\oplus : L^\star \longrightarrow L^\star)$ the abstract state space $(L^\star, \sqsubseteq, \oplus)$ obtained through horizontalisation is a suitable semantic structure for **L**.*

Proof As we discussed in Sect. 3.1, in any pre-mereological space, from $\Sigma_\oplus$, we can introduce a parthood relation $\sqsubseteq$, defined on the set of composable objects, in this case $L^\star = L^\star \cap L^\star$.

Thus, by Definition 12, from the pre-mereological space $(L, \Sigma_\oplus : L^\star \longrightarrow L^\star)$, we obtain an abstract state space $(L^\star, \sqsubseteq, \oplus)$ which is the same structure as the language frame $(L^\star, \oplus)$ from which we have started, now equipped with a parthood relation. □

This results make clear that language frames for **L** can be construed as mereological structures, apt to produce TS, at least in the sense that they can be obtained from pre-mereological spaces by means of the very same construction as the one that we have used to obtain standard state spaces.

Let's discuss the mereological features of the abstract state space for **L** that we have obtained and how they affect the TS.

First, note that the parthood relation $\sqsubseteq$ obtained for language frames is a poset, due to acyclicity ACY of $\Sigma_\oplus$, but it is non-standard. In fact, it is the sub-string relation. To see why, recall that s_1 is a substring of s_2 if and only if there exists two strings $s, t \in L^\star$ such that $s_2 = s, s_1, t$. Since the vertical composition $\Sigma_\oplus$ satisfies LEV and COL, we may simplify this definition by saying that s_1 is a part of s_2 if and only if s_1 is a component of s_2. Since the vertical composition is the identity function and by the definition of a component, s_1 is part of s_2 if and only if s_1 is a sub-string of s_2.

Second, observe that the abstract state space corresponding to a language frame is not always *complete*. Given a set of strings (e.g. of lenght 1) $\{a, b\} \subseteq L^\star$, both strings a, b and b, a are upper bounds of $\{a, b\}$ wrt. $\sqsubseteq$. a and b are parts of a, b (because of $\Sigma_\oplus(a, b) = a, b$) and of b, a (because of $\Sigma_\oplus(b, a) = a, b$). However, $a, b \not\sqsubseteq b, a$ nor $a, b \not\sqsubseteq b, a$.

Therefore, we have an example of a TS where the $\oplus$ operation is not defined by means of the l.u.b ($s \oplus t \neq \bigsqcup\{s, t\}$).

Third, it is interesting to notice how the mereology affects key aspects of the TS project, notably the definition of inexact (as well as loose) truthmaking (Fine 2017c). A state s *inexactly* verifies ϕ iff there exists s' such that $s' \sqsubseteq s$ and s' exactly verifies ϕ ($s' \Vdash \phi$). When parthood is construed as substring, for instance, if $a, b \Vdash \phi$, then a, b, c and c, a, b are inexact verifiers of ϕ, whereas a, c, b is not. Namely, the internal structure of the states is crucial for inexact (as well as loose) truthmaking.[26]

We conclude this section by noticing that this interpretation of the models of the Lambek calculus can be easily adapted to other substructural logics, at least those that have monoidal models or Urquart frames, cf. Urquhart (1972) (e.g. the multiplicative fragment of intuitionistic linear logic and

[25] As we shall see, this construction returns a string builder, cf. Examples 7 e 1.

[26] The mereological structure of this abstract state space fails the principle of *extensionality of composition* (Cotnoir 2010, p. 402).

disjunction-free fragments of relevant logic, Restall, 2000). That is, the claim with which we commenced this section can be substantiated by the following theorem.

Theorem 2 *The following facts hold:*

1. *Any monoidal structure returns a pre-mereological space (through verticalisation);*
2. *Any pre-mereological* $(A, \Sigma : X \longrightarrow Y)$ *space that satisfies string-closure* (*Condition* 1) *returns a monoidal structure (through horizontalisation).*

Proof Point (1) is simply an application of the verticalisation construction of Sect. 5.
Point (2) is established by the following reasoning. By Proposition 3, if $\mathcal{P}$ is string-closed, then $\oplus$ is a total function. Moreover, by Proposition 4, $\oplus$ is associative and has neutural-element. Thus the abstract state space is a monoid, therefore also a language frame, cf. Pentus (1994). Moreover, by Proposition 6, if Σ satisfies PER, then the monoid is commutative. □

6.3 The Logic of Pre-mereological Spaces

Theorems 1 and 2 provide conditions on pre-mereological spaces that are sufficient to obtain standard state spaces and monoidal state spaces (respectively). Therefore, such conditions enable us to use pre-merelogical spaces to ground truthmaker semantics for a number of logics.

An interesting question arises when we contemplate the logic corresponding to general pre-mereological spaces.

By Definition 12, any pre-mereological space returns an abstract state space $(S, \sqsubseteq, \oplus)$. At this level of generality, $\oplus$ needs not to be total (cf. Proposition 4). Moreover, it is non-commutative. By contrast, following the horizontalisation that we have defined in 4, $\oplus$ is always associative. The general mathematical structure corresponding to the abstract state space is thus a *partial semi-group* (Pentus 1994, Sect. 1.2).[27] Quite strikingly, Pentus (1994) proved that partial-semigroups are also models for the (associative) Lambek calculus. Therefore, it is reasonable to conjecture that the general logic for pre-mereological spaces is the (associative) Lambek calculus.

Notice that, at the end of Sect. 4, we discussed a strategy of horizontalisation that might return non-associative $\oplus$, cf. Equation 4. In this case, abstract state spaces do not need to be partial semi-groups. To approach this case, we have to delve into the models of the non-associative Lambek calculus. We leave this for a future dedicated work.[28]

7 Philosophical Outputs

We suggest an application of our technical contribution to the philosophical project of TS. Indeed, our framework opens up new opportunities and dilemmas for philosophers working in the field and in related areas.

Firstly, note that our perspective does not impose an interpretation of pre-mereological spaces as inherently mereological. Philosophers who maintain a substantive view of parthood can reject certain compositions as non-legitimate by requiring conditions that are more demanding than those that we listed. In this context, our discussion emphasises the value of pre-mereological spaces as manners to obtain TS for weak logics, such as the Lambek Calculus.

Our framework holds philosophical interest for proponents of TS as it provides support in advocating for a comprehensive account of logic and semantics based on TS. Indeed, TS are widely recognised for their flexibility, accommodating different logics and enabling the modelling of various semantic notions, including analytic containment, subject matter, and counterfactuals (see Fine 2017c). This flexibility is often leveraged to demonstrate the applicability of TS. Fine himself, in Fine (2017a), employed an implicit argument in favour of TS to defend the notion of truthmaker content and propose a revision of more traditional approaches (p. 626).

The technical contribution of this article present TS supporters with new grounds to argue for the superiority of TS over alternative approaches like possible world semantics.[29]

For example, possible world semantics, based on relational frames, face challenges in accommodating substructural logics. On the other hand, by allowing for non-standard compositions and parthood relations, TS demonstrate their aptitude for providing semantics to these logics effectively.

Exploring TS for various logics suggests novel applications and interpretations of TS. For instance, by introducing state spaces for fragments of relevant logic, we are enabling an interpretation of the states in terms of information states, cf. Mares (1996). Moreover, states can be temporally ordered by using a non-commutative composition to render the rela-

[27] Partial semi-groups do not need the existence of a neutral element. Thus, pre-merelogical spaces returns partial semi-group also if we omit the construction of the neutral element in Definition 10.

[28] Truthmaker Semantics for the non-associative Lambek calculus, as well as for a number of substructural logics, have been recently introduced in Majer et al. (2023). Notably, their approach differs from ours, as the authors retain standard mereology and subsequently develop the new TS. In contrast, our approach would involve modifying the mereological assumptions to generate the novel TS.

[29] Interestingly, a relational frame is in fact a partial semi-group, see e.g. Pentus (1994), Example 1. So, in principle, relational frames can be construed as pre-mereological.

tion "before" of Allen's logic, cf. Allen (1984). Also, causal dependencies between states can be introduced by selecting state spaces that are adequate for contraction-less logic Girard (1995). Finally, if states are to be intended as spatially and temporally located entities, non-standard mereologies may come into play, cf. Kleinschmidt (2011).

The appeal of the generality of the TS project, of course, depends on the individual philosophical perspective. Some philosophers who reject non-standard mereologies may find it uninteresting. Or they could use our constructions to argue that substructural logics have no genuine mereological foundation. At end, as we have made explicit, the state space for the Lambek calculus is mereologically quite non-standard (cf. Sect. 6.2).

Additionally, skeptics of TS may see our approach as highlighting potential theoretical flaws within the TS project. In particular, these philosophers may question the philosophical significance of the notion of verifiers and falsifiers within TS. The mereological nature of states suggests that their identity and compositional behaviour are determined by their mereological properties. However, the metaphysical nature of verifier states is not entirely clear and has been the subject of debate [Jago discusses three alternative metaphysical accounts in Jago (2017)]. Consequently, the exploration of non-standard mereologies introduces further indeterminacy regarding the philosophical foundations of states, on which TS are built. To sum up, the TS project is facing a dilemma: the more hospitable it is to various logics, the less determinate its mereological foundation can be.

Conclusions

This article applied Fine's theory of composition and parthood in Fine (2010) to explore the framework of TS and to expand it, by developing new TS whose underlying mereology is non-standard. The application of this framework proves to be fruitful, thanks to its generality. We showed how to create new TS from composition operators and we designed some of them for substructural logics. Let us stress that, in this regard, our exploration of associative Lambek Calculus is driven by the observation that it could potentially be the logic emerging from our representation of pre-mereological spaces—as highlighted in Sect. 6.3. However, by modifying our framework to handle Σs that deal with less structured objects than strings, we might expect that pre-mereological spaces could generate even weaker logics, including non-associative ones. Anyway, this remains a subject for future investigations.

Finally, in the previous section, we pointed out the novelties that the investigation of TS based on non-standard mereologies brings to the general TS framework, we highlighted the benefits in terms of generality, and we outlined a possible philosophical challenge to the TS project.

Future work is planned along two directions. Our pre-mereological interpretation of substructural logics is confined to logics with monoidal semantics. Future work shall face the case of logics such as (full) Linear Logic and Relevant Logic, that require sophisticated semantic structures.

On the philosophical side, we have seen how the TS project tightens the connection between logic and mereology. An interesting application of our framework shall be dedicated to exploring the relationships between logical and mereological pluralism.

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