Hybrid Formulations of Eddy Current Problems

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Abstract

In this paper we examine the well-known magneto-quasistatic eddy current model for the behaviour of low-frequency electromagnetic fields. We restrict ourselves to formulations in the frequency domain and linear materials, but admit rather general topological arrangements.

The generic eddy current model allows two dual formulations, which may be dubbed **E**-based and **H**-based. We investigate so-called hybrid approaches that combine both formulations by means of coupling conditions across the boundaries of conducting regions. The resulting continuous and discrete variational formulations will be discussed. It is worthy to note that for this approach no difficulties arise from the topology of the conducting regions.

Keywords. Time-harmonic eddy current problems, hybrid formulation, mixed finite element approximation.

MSC. 65N30, 35Q60, 78M10

1 Introduction

The typical setting for an eddy current model distinguishes between a bounded conducting region $\Omega^C \subset \mathbb{R}^3$ and its complement, the non-conducting region $\Omega^I := \mathbb{R}^3 \setminus \overline{\Omega}^C$, which is also referred to as "air region". Here we assume that all bounded domains are curvilinear Lipschitz polyhedra in the parlance of [18]. In addition, for the sake of simplicity both Ω^C and Ω^I are assumed to be connected, so that $\Gamma := \partial\Omega^C$ consists of only one connected component (the general case can be treated as in [6]).

The spatio-temporal evolution of the electromagnetic fields and currents $\mathbf{E}, \mathbf{H}, \mathbf{B}, \mathbf{D}, \mathbf{J}$ is governed by Maxwell's equations. The interaction with matter is modelled through material laws. In what follows we will only deal with linear materials, for which the material laws become

$$\mathbf{B} = \boldsymbol{\mu}\mathbf{H} \quad , \quad \mathbf{D} = \boldsymbol{\epsilon}\mathbf{E} \quad , \quad \mathbf{J} = \boldsymbol{\sigma}\mathbf{E} + \mathbf{J}_e \; .$$

Here $\boldsymbol{\mu}$ is the magnetic permeability, $\boldsymbol{\epsilon}$ the dielectric tensor, and $\boldsymbol{\sigma}$ stands for conductivity, and $\mathbf{J}_e$ is a generator current [26]. Physics teaches that $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ have to be uniformly positive definite symmetric 3×3-matrices, whereas $\boldsymbol{\sigma}$ vanishes in Ω^I, but is supposed to be symmetric

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and uniformly positive definite inside Ω^C. All the material parameters are functions of the spatial variable $\mathbf{x}$ only. Under these circumstances time-harmonic excitations will imply the same harmonic dependence on time of all quantities. In short, we can switch to the frequency domain and end up with linear equations connecting complex amplitudes (phasors) of electromagnetic fields. From now on, the temporal behaviour of all fields will be assumed to be time-harmonic with a fixed angular frequency $\omega \neq 0$. Moreover, all symbols for field quantities will refer to complex amplitudes.

In many situations the full Maxwell's equations can be traded for simplified models that still offer a sufficiently accurate description of electromagnetic phenomena. The most popular among these simplified models is the so-called eddy current model. Formally, it can be derived from Maxwell's equations by setting $\boldsymbol{\epsilon}$ to zero. Physically speaking, this amounts to completely neglecting the electromagnetic energy contained in the electric field $\mathbf{E}$. As a consequence, the propagation of electromagnetic waves is suppressed. Therefore, the eddy current model is rated as *magneto-quasistatic*. Another consequence of $\boldsymbol{\epsilon} = 0$ is that no space charges can occur.

A crude criterion for the applicability of the eddy current model relies on the following non-dimensional problem parameters being very small

$$\bar{L}|\omega|\sqrt{\bar{\boldsymbol{\epsilon}}\bar{\boldsymbol{\mu}}} \ll 1 \quad \text{and} \quad |\omega|\bar{\boldsymbol{\epsilon}}\bar{\boldsymbol{\sigma}}^{-1} \ll 1 \; .$$

Here, an overline marks "typical values" of the material parameters and $\bar{L}$ stands for the diameter of Ω^C. In addition, all geometric details of Ω^C have to be of scale $\bar{L}$. A qualitative discussion can be found in [19] and an aymptotic analysis in terms of ω is given in [1] and [7].

The following strong transmission formulation of the eddy current model can be directly obtained from Maxwell's equations: it comprises Faraday's law

$$\mathbf{curl}\,\mathbf{E}_C = -i\omega\boldsymbol{\mu}\mathbf{H}_C \quad \text{in } \Omega^C \quad , \quad \mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}\mathbf{H}_I \quad \text{in } \Omega^I \; , \tag{1}$$

and a reduced Ampère's law

$$\mathbf{curl}\,\mathbf{H}_C = \boldsymbol{\sigma}\mathbf{E}_C + \mathbf{J}_{e,C} \quad \text{in } \Omega^I \quad , \quad \mathbf{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I} \quad \text{in } \Omega^I \; . \tag{2}$$

(Here and in the sequel we denote by $\mathbf{v}_L$ the restriction of $\mathbf{v}$ to Ω^L, $L = I, C$.) These have to be complemented by requiring the characteristic tangential continuity of $\mathbf{E}$ and $\mathbf{H}$ across the boundary Γ of Ω^C:

$$\mathbf{E}_C \times \mathbf{n} - \mathbf{E}_I \times \mathbf{n} = \mathbf{0} \quad , \quad \mathbf{H}_C \times \mathbf{n} - \mathbf{H}_I \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma \; , \tag{3}$$

where $\mathbf{n}$ denotes the unit normal vector on Γ pointing towards Ω^I. Compliance with Ampère's law (2) entails

$$\operatorname{div}\mathbf{J}_{e,I} = 0 \quad \text{in } \Omega^I \quad , \quad \int_\Gamma \mathbf{J}_{e,I}\cdot\mathbf{n}\,dS = 0 \; . \tag{4}$$

Putting it differently, $\mathbf{J}_{e,I}$ has to be compatible with vanishing space charges.

Obviously, we cannot expect a solution for $\mathbf{E}$ to be unique, because it can be altered by any gradient supported in Ω^I and will still satisfy the equations. Imposing the constraints

$$\operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_I) = 0 \quad \text{in } \Omega^I \quad , \quad \int_\Gamma \boldsymbol{\epsilon}\mathbf{E}_I\cdot\mathbf{n}\,dS = 0 \; , \tag{5}$$

will restore uniqueness of the solution for $\mathbf{E}$. Please note that (5) represent *gauge conditions* in the sense of [28], that is, a recipe for selecting a representative of an equivalence class that is obtained as a solution of the basic model. It might sound odd that the electric field is subjected to gauging. Yet, recall that in the case of the eddy current model in the frequency domain no distinction can be made between a suitably chosen magnetic vector potential $\mathbf{A}$ and the electric field in Ω^I. We point out that the gauge (5) is particular, as it selects a meaningful electric field that makes all space charges vanish in Ω^I.

Remark 1.1. A fundamental difference of the eddy current model and the full Maxwell's concerns the broken symmetry between $\mathbf{E}$ and $\mathbf{H}$. This is clearly reflected by the above-mentioned non-uniqueness of $\mathbf{E}$, and also by the fact that $\mathbf{H}$ has to satisfy the *algebraic constraint* $\mathbf{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I}$ in Ω^I. △

The discrete models investigated in this paper will be based on finite element techniques. They can only be applied on bounded computational domains. Therefore, we introduce an artificial computational domain $\Omega \subset \mathbb{R}^3$, which is to be a box containing $\overline{\Omega}^C$, and we now set $\Omega^I := \Omega \setminus \overline{\Omega}^C$. This approach is justified in light of the decay properties of the fields [7, Proposition 3.1]

$$\mathbf{H}(\mathbf{x}) = O(|\mathbf{x}|^{-2}) \quad , \quad \mathbf{E}(\mathbf{x}) = O(|\mathbf{x}|^{-2}) \quad \text{for } |\mathbf{x}| \to \infty \; .$$

On $\partial\Omega$ homogeneous boundary conditions for either $\mathbf{H}$ or $\mathbf{E}$ are imposed: below we demand

$$\mathbf{H} \times \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega \; .$$

This implies another compatibility condition for $\mathbf{J}_{e,I}$, namely

$$\mathbf{J}_{e,I} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \; , \tag{6}$$

and another *gauge condition*

$$\epsilon \mathbf{E}_I \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \; . \tag{7}$$

(therefore, due to the divergence theorem, in (4) and in (5) the integral condition on Γ can be dropped).

This cut-off technique is widely used in engineering and will lead to moderate errors provided that $\partial\Omega$ is sufficiently far from Ω^C and the support of $\mathbf{J}_{e,I}$.

Let us introduce some notation. As usual, we denote by $H^s(\Omega)$ or $H^s(\partial\Omega)$, $s \in \mathbb{R}$, the Sobolev space of order s of real or complex measurable functions defined on Ω or $\partial\Omega$, respectively. If $\Sigma \subset \partial\Omega$ we denote by $H^1_{0,\Sigma}(\Omega)$ the subspace of $H^1(\Omega)$ consisting of those functions φ satisfying $\varphi_{|\Sigma} = 0$. As usual $H^1_0(\Omega) := H^1_{0,\partial\Omega}(\Omega)$.

Given a differential operator D in the sense of distributions and a generic domain $\Omega \subset \mathbb{R}^3$, we will consider

$$\begin{aligned} H(D,\Omega) &:= \{u \in L^2(\Omega) \,|\, Du \in L^2(\Omega)\} \; , \\ H(D0,\Omega) &:= \{u \in H(D,\Omega) \,|\, Du = 0\} \; , \end{aligned}$$

Hilbert spaces equipped with the graph norm. Particular examples are $\boldsymbol{H}(\mathbf{curl};\Omega)$ and $\boldsymbol{H}(\mathrm{div};\Omega)$, see [21, Ch. 1]. If $\Sigma \subset \partial\Omega$, by $\boldsymbol{H}_{0,\Sigma}(\mathbf{curl};\Omega)$ we denote the subspace of $\boldsymbol{H}(\mathbf{curl};\Omega)$

of those functions $\mathbf{v}$ satisfying $(\mathbf{v}\times\mathbf{n})_{|\Sigma}=\mathbf{0}$. We set $\boldsymbol{H}_0(\mathbf{curl};\Omega):=\boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega)$. We also recall the trace space for $\boldsymbol{H}(\mathbf{curl};\Omega)$

$$\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma):=\{\mathbf{v}\times\mathbf{n}\,|\,\mathbf{v}\in\boldsymbol{H}(\mathbf{curl};\Omega)\}\ ,$$

see, e.g., [16], [2], [14]. For ease of reading, in the sequel we always express duality pairings by (surface) integrals.

The plan of the paper is as follows. In Sect. 2 we present the two dual formulations of the eddy current problem that emerge by retaining either $\mathbf{H}$ or $\mathbf{E}$ as principal unknown. In Sect. 3 we elucidate the main idea of hybrid coupling in the case of scalar second-order elliptic problems. This idea, in two different versions, will be applied to the eddy current problem in Sections 4 and 5. Appropriate finite element discretizations of the hybrid variational problems will be also proposed and examined.

2 One-field variational formulations

In the $\mathbf{H}$-based model Ampère's law in Ω^C is used to replace $\mathbf{E}$ in Faraday's law. This results in the following second order differential equation for the magnetic field in the conductor

$$\mathbf{curl}(\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C)+i\omega\boldsymbol{\mu}\mathbf{H}_C=\mathbf{curl}(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C})\qquad\text{in }\Omega^C.$$

In the insulator Ampère's law and the magnetic form of Gauss' law are considered:

$$\mathbf{curl}\,\mathbf{H}_I=\mathbf{J}_{e,I}\ ,\quad \mathrm{div}(\boldsymbol{\mu}\mathbf{H}_I)=0\qquad\text{in }\Omega^I.$$

Concerning the transmission conditions for the magnetic field across Γ we impose the tangential continuity of the magnetic field $\mathbf{H}$ and the normal continuity of the magnetic induction $\boldsymbol{\mu}\mathbf{H}$. For an interface of general geometry these two conditions are insufficient to determinate uniquely the magnetic field and the following non local transmission condition must be also imposed (see [5]):

$$\int_{\Omega^I} i\omega\boldsymbol{\mu}\mathbf{H}_I\cdot\boldsymbol{\rho}_l+\int_{\Gamma}[\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C-\mathbf{J}_{e,C})]\times\mathbf{n}\cdot\boldsymbol{\rho}_l=0\quad\forall\, l=1,\ldots,p\ ,$$

where $p=\beta_1(\Omega^I)$, the first Betti number of Ω^I, and $\boldsymbol{\rho}_l$ are the basis functions of the finite dimensional space of harmonic fields

$$\begin{aligned}\mathcal{H}_\mu(\partial\Omega;\Gamma):=\{\mathbf{v}_I\in(L^2(\Omega^I))^3\,|\,&\mathbf{curl}\,\mathbf{v}_I=\mathbf{0},\mathrm{div}(\boldsymbol{\mu}\mathbf{v}_I)=0,\\ &\mathbf{v}_I\times\mathbf{n}=\mathbf{0}\ \text{ on }\partial\Omega,\boldsymbol{\mu}\mathbf{v}_I\cdot\mathbf{n}=0\ \text{ on }\Gamma\}\,.\end{aligned}\tag{8}$$

In fact, the independent interface conditions implicitly contained in the Maxwell system are the tangential continuity of both magnetic and electric fields. The normal continuity of $\boldsymbol{\mu}\mathbf{H}$, that is a consequence of the tangential continuity of the electric field and Faraday's law, is weaker than the tangential continuity of the electric field. For an interface Γ of general geometry it must be strengthened in a suitable way. If the magnetic field $\mathbf{H}$ does not satisfy the non local transmission condition then no electric field exists that satisfies both Ampère's law an Faraday's law in the whole Ω.

Hence the **H**-based formulation of the eddy current problem reads:

$$
\begin{array}{ll}
\mathbf{curl}(\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C) + i\omega\boldsymbol{\mu}\mathbf{H}_C = \mathbf{curl}(\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) & \text{in } \Omega^C \\
\mathbf{curl}\,\mathbf{H}_I = \mathbf{J}_{e,I} \quad , \quad \operatorname{div}(\boldsymbol{\mu}\mathbf{H}_I) = 0 & \text{in } \Omega^I \\
\mathbf{H}_I \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega \\
\int\limits_{\Omega^I} i\omega\boldsymbol{\mu}\mathbf{H}_I \cdot \boldsymbol{\rho}_l + \int\limits_{\Gamma} [\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C})] \times \mathbf{n} \cdot \boldsymbol{\rho}_l = 0 & \forall\, l = 1, \ldots, p \\
\boldsymbol{\mu}\mathbf{H}_I \cdot \mathbf{n} - \boldsymbol{\mu}\mathbf{H}_C \cdot \mathbf{n} = 0 & \text{on } \Gamma \\
\mathbf{H}_I \times \mathbf{n} - \mathbf{H}_C \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma\,.
\end{array}
\tag{9}
$$

In order to obtain the weak formulation of (9) one introduces the Hilbert space of *complex-valued* vector functions

$$
\mathbf{V}^{\mathbf{0}} := \{\mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{0} \text{ in } \Omega^I\}\ .
$$

and the space

$$
\mathbf{V}^{\mathbf{J}_{e,I}} := \{\mathbf{v} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \,|\, \mathbf{curl}\,\mathbf{v}_I = \mathbf{J}_{e,I} \text{ in } \Omega^I\}\ .
$$

The **H**-based weak formulation of eddy current problem reads (see [10], [5], [9])

$$
\left\{
\begin{array}{l}
\text{Find } \mathbf{H} \in \mathbf{V}^{\mathbf{J}_{e,I}} \text{ such that:} \\[2mm]
\int\limits_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C + i\omega\boldsymbol{\mu}\mathbf{H}_C \cdot \overline{\mathbf{v}}_C) + \int\limits_{\Omega^I} i\omega\boldsymbol{\mu}\mathbf{H}_I \cdot \overline{\mathbf{v}}_I \\[2mm]
\qquad\qquad = \int\limits_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C \qquad \forall\, \mathbf{v} \in \mathbf{V}^{\mathbf{0}}\ .
\end{array}
\right.
\tag{10}
$$

The existence and uniqueness of solution of (10) follows from Lax–Milgram lemma since the bilinear form

$$
a(\mathbf{u},\mathbf{v}) := \int\limits_{\Omega^C} \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{u}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C + \int\limits_{\Omega} i\omega\boldsymbol{\mu}\mathbf{u} \cdot \overline{\mathbf{v}}
$$

is coercive in $\mathbf{V}^{\mathbf{0}}$.

We have now to determine the electric field in Ω. From Ampère's law

$$
\mathbf{E}_C = \boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C - \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}\ . \tag{11}
$$

In order to determine $\mathbf{E}_I$ we must take into account Faraday's law in Ω^I

$$
\mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}\mathbf{H}_I \quad \text{in } \Omega^I\ ,
$$

the interface condition

$$
\mathbf{E}_C \times \mathbf{n} - \mathbf{E}_I \times \mathbf{n} = \mathbf{0} \quad \text{on } \Gamma\ ,
$$

and the gauge conditions (5) and (7).

So we must solve:

$$
\begin{array}{c}
\mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}\mathbf{H}_I \quad , \quad \operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_I) = 0 \quad \text{in } \Omega^I \\
\boldsymbol{\epsilon}\mathbf{E}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \quad , \quad \mathbf{E}_I \times \mathbf{n} = \mathbf{E}_C \times \mathbf{n} \quad \text{on } \Gamma\ .
\end{array}
\tag{12}
$$

As shown in [5], if the magnetic field $\mathbf{H}$ is the solution to (9) and $\mathbf{E}_C$ is given by (11), this problem has a unique solution.

The other one-field formulation we can consider is in term of the electric field $\mathbf{E}$. Computing the magnetic field from the Faraday's equation (1) and inserting it in the Ampère's equation (2) one finds

$$\begin{array}{ll} \mathbf{curl}(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_C) + i\omega\boldsymbol{\sigma}\mathbf{E}_C = -i\omega\mathbf{J}_{e,C} & \text{in } \Omega^C \\ \mathbf{curl}(\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I) = -i\omega\mathbf{J}_{e,I} & \text{in } \Omega^I \\ \mathbf{E}_C \times \mathbf{n} - \mathbf{E}_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_C \times \mathbf{n} - \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I \times \mathbf{n} = \mathbf{0} & \text{on } \Gamma \\ \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I \times \mathbf{n} = \mathbf{0} & \text{on } \partial\Omega\,. \end{array} \tag{13}$$

Clearly, uniqueness of $\mathbf{E}_I$ is not guaranteed, and one has to add the gauge conditions (5) and (7):

$$\operatorname{div}(\boldsymbol{\epsilon}\mathbf{E}_I) = 0 \text{ in } \Omega^I \quad , \quad \boldsymbol{\epsilon}\mathbf{E}_I \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\,. \tag{14}$$

Let us now consider the vector space

$$\mathbf{Z} := \{\mathbf{z} \in \boldsymbol{H}(\mathbf{curl};\Omega)\,|\,\mathbf{z}_I \text{ satisfies (14)}\}\,,$$

that is a closed subspace of $\boldsymbol{H}(\mathbf{curl};\Omega)$, therefore a Hilbert space with respect to the scalar product of $\boldsymbol{H}(\mathbf{curl};\Omega)$. Therefore the weak formulation reads:

$$\left\{\begin{array}{l} \text{Find } \mathbf{E} \in \mathbf{Z} \text{ such that:} \\ \\ \int_\Omega \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E} \cdot \mathbf{curl}\,\overline{\mathbf{z}} + i\omega \int_{\Omega^C} \boldsymbol{\sigma}\mathbf{E}_C \cdot \overline{\mathbf{z}}_C = -i\omega \int_\Omega \mathbf{J}_e \cdot \overline{\mathbf{z}} \qquad \forall\, \mathbf{z} \in \mathbf{Z}\,. \end{array}\right. \tag{15}$$

The proof of the existence and uniqueness of a solution to (15) can be done as in [3], Theorem 3.1. In fact, it can be proved that the bilinear form

$$b(\mathbf{w},\mathbf{z}) := \int_\Omega \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{w} \cdot \mathbf{curl}\,\overline{\mathbf{z}} + i\omega \int_{\Omega^C} \boldsymbol{\sigma}\mathbf{w}_C \cdot \overline{\mathbf{z}}_C$$

is coercive in $\mathbf{Z}$. The existence and uniqueness is therefore a consequence of Lax–Milgram's lemma.

Remark 2.1. A finite element method for the weak $\mathbf{H}$-based model or the weak $\mathbf{E}$-based model would have to deal with the constrained space $\mathbf{V}^{\mathbf{J}_{e,I}}$ or $\mathbf{Z}$, respectively. Alternatively, an equivalent saddle-point variational formulation of (10) has been considered in [6], and of (15) has been proposed in [27] (for a domain of simple topology). Finite element discretizations are available for both these saddle-point $\mathbf{H}$-based and $\mathbf{E}$-based models. Yet, they invariably require meshes that are conforming across Γ. △

3 Hybrid coupling

There is a general principle underlying the two different formulations of the eddy current problem presented in the previous section. It is the principle that many second order elliptic boundary value problems allow two variational formulations that can be viewed as dual to each other. As in [30] let us consider the standard second order scalar elliptic boundary value problem

$$-\Delta u = f \quad \text{in } \Omega \quad , \quad u = 0 \quad \text{on } \partial\Omega \; , \tag{16}$$

for some $f \in L^2(\Omega)$. This way to write the boundary value problem is called the *primal formulation.* The corresponding *dual formulation* relies on the first order system

$$\mathbf{j} - \mathbf{grad}\, u = 0, \quad \operatorname{div} \mathbf{j} = -f \quad \text{in } \Omega \quad , \quad u = 0 \quad \text{on } \partial\Omega \; . \tag{17}$$

This is sometimes referred to as the *mixed formulation* of (16). Primal and dual formulations give rise to different variational problems. It is the gist of the hybrid coupling approach to employ these two variational formulations on different subdomains of a partition of the computational domain. An important restriction is that the same formulation must not be used on adjacent sub-domains.

Let us assume that Ω is partitioned into two connected Lipschitz subdomains Ω^P ("primal") and Ω^D ("dual") such that $\partial\Omega^P \cap \partial\Omega = \emptyset$. Let Γ be their common interface, with the unit normal $\mathbf{n}$ on Γ pointing into Ω^D. On Ω^P we resort to the primal variational formulation obtained from integration by parts applied to (16): seek $u^P \in H^1(\Omega^P)$ such that

$$\int_{\Omega^P} \mathbf{grad}\, u_P \cdot \mathbf{grad}\, v - \int_{\Gamma} v \; \mathbf{grad}\, u_P \cdot \mathbf{n} = \int_{\Omega^P} f v \quad \forall\, v \in H^1(\Omega^P) \; . \tag{18}$$

On Ω^D the mixed variational formulation will be used, which emerges from casting the first equation of (17) into weak form and retaining the second strongly: seek $\mathbf{j}_D \in \boldsymbol{H}(\operatorname{div};\Omega^D)$, $u^D \in L^2(\Omega^D)$ such that

$$\begin{array}{lllll} \int_{\Omega^D} \mathbf{j}_D \cdot \mathbf{z} & + \int_{\Omega^D} \operatorname{div} \mathbf{z}\, u_D & + \int_{\Gamma} u_D\, \mathbf{z} \cdot \mathbf{n} & = & 0 \qquad \forall\, \mathbf{z} \in \boldsymbol{H}(\operatorname{div};\Omega^D) \\ \int_{\Omega^D} \operatorname{div} \mathbf{j}_D\, q & & & = & -\int_{\Omega^D} f q \quad \forall\, q \in L^2(\Omega^D) \end{array} \tag{19}$$

(remember that $\mathbf{n}$ is the unit *inward* normal vector on $\Gamma = \partial\Omega^D$).

Both problems can be linked by the *transmission conditions* on Γ:

$$u_{P|\Gamma} = u_{D|\Gamma} \quad , \quad \mathbf{grad}\, u_P \cdot \mathbf{n} = \mathbf{j}_D \cdot \mathbf{n} \; . \tag{20}$$

We can use these to express the interface terms in both (18) and (19) through quantities from the other subdomain. Subsequently merging the variational formulations we arrive at the final coupled problem:

$$\left\{ \begin{array}{l} \text{Find } (u_P, \mathbf{j}_D, u_D) \in H^1(\Omega^P) \times \boldsymbol{H}(\operatorname{div};\Omega^D) \times L^2(\Omega^D) \text{ such that:} \\ \begin{array}{lllllll} \int_{\Omega^P} \mathbf{grad}\, u_P \cdot \mathbf{grad}\, v & - & \int_{\Gamma} \mathbf{j}_D \cdot \mathbf{n}\, v & & & = \int_{\Omega^P} f v & \forall\, v \in H^1(\Omega^P) \\ \int_{\Gamma} u_P\, \mathbf{z} \cdot \mathbf{n} & + & \int_{\Omega^D} \mathbf{j}_D \cdot \mathbf{z} & + & \int_{\Omega^D} \operatorname{div} \mathbf{z}\, u_D & = 0 & \forall\, \mathbf{z} \in \boldsymbol{H}(\operatorname{div};\Omega^D) \\ & & \int_{\Omega^D} \operatorname{div} \mathbf{j}_D\, q & & & = -\int_{\Omega^D} f q & \forall\, q \in L^2(\Omega^D) \; . \end{array} \end{array} \right. \tag{21}$$

Let us denote by $\mathcal{B}$ the bilinear form on $H^1(\Omega^P) \times \boldsymbol{H}(\mathrm{div};\Omega^D)$ corresponding to the left-upper 2×2 block of (21). The crucial observation is that $\mathcal{B}$ itself features a *block-skew-symmetric* structure, which involves

$$\mathcal{B}((v,\mathbf{z}),(v,\mathbf{z})) = \int_{\Omega^P} |\mathbf{grad}\, v|^2 + \int_{\Omega_D} |\mathbf{z}|^2 \;, \quad v \in H^1(\Omega^P),\; \mathbf{z} \in \boldsymbol{H}(\mathrm{div};\Omega^D) \;, \tag{22}$$

because the interface contributions cancel. This makes it possible to show that $\mathcal{B}$ satisfies an inf–sup condition on $H^1(\Omega^P) \times \boldsymbol{H}(\mathrm{div}\,0;\Omega^D)$. Beware, that this is not straightforward from (22), because the one-dimensional *topological* subspace of constants on Ω^P requires a special treatment. Then the theory of saddle-point problems confirms existence and uniqueness of solutions of (22). Moreover, Galerkin approximations by the standard conforming finite element spaces for $H^1(\Omega^P)$, $\boldsymbol{H}(\mathrm{div};\Omega^D)$ and $L^2(\Omega^D)$ will enjoy quasi-optimality [12], [13].

It is important to note that the coupling of the two subdomains in (22) is purely variational, because none of the transmission conditions (20) shows up in the definition of the spaces. This makes (22) attractive, if unrelated ("*non-matching*") finite element meshes are to be used on Ω^P and Ω^D, *cf.* the rationale behind the so-called mortar finite element schemes [31].

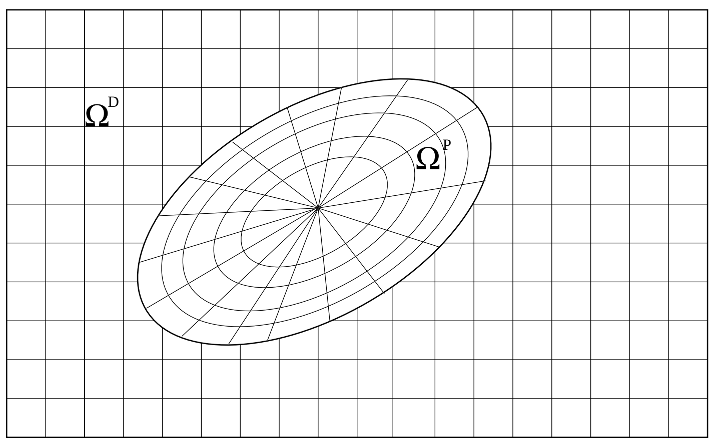

Figure 1: Example of two subdomains equipped with non-matching grids

To what extent do these considerations apply to eddy currents? The answer lies in the intimate relationship of the eddy current equations with scalar second order problems: just look at

$$\begin{aligned} -\mathrm{div}\,\mathbf{grad}\, u \;&+\; \sigma u \;&=\; f \;, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{u} \;&+\; i\omega\sigma\mathbf{u} \;&=\; \mathbf{f} \;. \end{aligned}$$

In fact, both second-order problems are members of a larger family of problems, for which the differential operators arise from the so-called exterior derivative in the calculus of differential forms. In fact, the above boundary value problems just differ in the degree of the differential form, see [23].

Thus, many of the preceding considerations will also apply to the eddy current problem. However, new difficulties are introduced due to the non-trivial kernel of **curl** (compared to the very simple kernel of **grad**) along with the fact that $\sigma = 0$ in parts of Ω.

4 Hybrid $\mathbf{E}_C/\mathbf{H}_I$ formulation

4.1 Variational formulations

In addition to the **H**-based and **E**-based models presented before, we want to consider a primal-dual coupling, in which the magnetic field $\mathbf{H}_I$ in the air region Ω^I is an unknown, while the other one is the electric field $\mathbf{E}_C$ in the conductor Ω^C. Since for these two vector fields we need not impose any matching condition on Γ, we can use non-matching grids in the finite element scheme we are going to introduce for numerical approximation (see Sect. 3).

The variational formulation is derived as follows: from the first equation in (13), for each $\mathbf{z}_C \in \boldsymbol{H}(\mathbf{curl};\Omega^C)$ one finds by integration by parts

$$\int_{\Omega^C} \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}}_C + i\omega \int_{\Omega^C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}}_C - \int_{\Gamma} \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_C = -i\omega \int_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \,.$$

Therefore, from Faraday's law and from the matching condition for the magnetic field we have

$$\int_{\Omega^C} \boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}}_C + i\omega \int_{\Omega^C} \boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}}_C + i\omega \int_{\Gamma} \mathbf{H}_I \times \mathbf{n} \cdot \overline{\mathbf{z}}_C = -i\omega \int_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \,.$$

On the other hand, integrating by part the Faraday's equation in Ω^I and using the matching condition for the electric field one has

$$i\omega \int_{\Omega^I} \boldsymbol{\mu} \mathbf{H}_I \cdot \overline{\mathbf{v}}_I = -\int_{\Omega^I} \mathbf{curl}\, \mathbf{E}_I \cdot \overline{\mathbf{v}}_I = -\int_{\Gamma} \mathbf{E}_C \times \mathbf{n} \cdot \overline{\mathbf{v}}_I$$

for each test function $\mathbf{v}_I$ such that $\mathbf{curl}\, \mathbf{v}_I = \mathbf{0}$ in Ω^I and $\mathbf{v}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

We have thus arrived at the following formulation:

$$\left\{ \begin{array}{llll} \text{Find } (\mathbf{E}_C, \mathbf{H}_I) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \mathbf{V}_I^{\mathbf{J}_{e,I}} \ : & & & \\ \int_{\Omega^C} (\boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{E}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}}_C + i\omega\boldsymbol{\sigma} \mathbf{E}_C \cdot \overline{\mathbf{z}}_C) & - \quad i\omega \int_{\Gamma} \overline{\mathbf{z}}_C \times \mathbf{n} \cdot \mathbf{H}_I & = & -i\omega \int_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \\ -i\omega \int_{\Gamma} \mathbf{E}_C \times \mathbf{n} \cdot \overline{\mathbf{v}}_I & + \quad \omega^2 \int_{\Omega^I} \boldsymbol{\mu} \mathbf{H}_I \cdot \overline{\mathbf{v}}_I & = & 0 \\ \text{for all } \mathbf{z}_C \in \boldsymbol{H}(\mathbf{curl};\Omega^C), \mathbf{v}_I \in \mathbf{V}_I^{\mathbf{0}} \,, & & & \end{array} \right. \tag{23}$$

where

$$\mathbf{V}_I^{\mathbf{J}_{e,I}} := \{ \mathbf{v}_I \in \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \mid \mathbf{curl}\, \mathbf{v}_I = \mathbf{J}_{e,I} \text{ in } \Omega^I \} \,,$$

and similarly for $\mathbf{V}_I^{\mathbf{0}}$.

Problem (23) is associated to the sesquilinear form

$$\begin{array}{rcl} c((\mathbf{w}_C, \mathbf{u}_I), (\mathbf{z}_C, \mathbf{v}_I)) & := & \int_{\Omega^C} (\boldsymbol{\mu}^{-1} \mathbf{curl}\, \mathbf{w}_C \cdot \mathbf{curl}\, \overline{\mathbf{z}}_C + i\omega\boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{z}}_C) \\ & & -i\omega \int_{\Gamma} \overline{\mathbf{z}}_C \times \mathbf{n} \cdot \mathbf{u}_I - i\omega \int_{\Gamma} \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{v}}_I + \omega^2 \int_{\Omega^I} \boldsymbol{\mu} \mathbf{u}_I \cdot \overline{\mathbf{v}}_I \,, \end{array}$$

which can be seen to be coercive in $\boldsymbol{H}(\mathbf{curl};\Omega^C) \times \mathbf{V}_I^{\mathbf{0}}$. In fact, we have

$$|c((\mathbf{w}_C, \mathbf{u}_I), (\mathbf{w}_C, \mathbf{u}_I))|^2 = \Big(\int_{\Omega^C} \boldsymbol{\mu}^{-1} \,\mathbf{curl}\, \mathbf{w}_C \cdot \mathbf{curl}\, \overline{\mathbf{w}}_C + \omega^2 \int_{\Omega^I} \boldsymbol{\mu} \mathbf{u}_I \cdot \overline{\mathbf{u}}_I\Big)^2$$

$$+\omega^2\Big(\int_{\Omega^C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{u}}_I\Big)^2 .$$

Taking into account that $\mathbf{curl}\, \mathbf{u}_I = \mathbf{0}$, from the continuity estimate

$$2\left|\int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{u}}_I\right| \leq k_0 \left(\int_{\Omega^I} |\mathbf{u}_I|^2\right)^{1/2} \left(\int_{\Omega^C} (|\mathbf{w}_C|^2 + |\,\mathbf{curl}\, \mathbf{w}_C|^2)\right)^{1/2}$$

and the inequality $(A+B)^2 \geq A^2/2 - B^2$, we find

$$\Big(\int_{\Omega^C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{u}}_I\Big)^2$$

$$\geq \Big(\int_{\Omega^C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C\Big)^2/2 - k_0^2 \left(\int_{\Omega^I} |\mathbf{u}_I|^2\right) \left(\int_{\Omega^C} (|\mathbf{w}_C|^2 + |\,\mathbf{curl}\, \mathbf{w}_C|^2)\right)$$

$$\geq \Big(\int_{\Omega^C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C\Big)^2/2 - \delta^{-1} k_0^2 \Big(\int_{\Omega^I} |\mathbf{u}_I|^2\Big)^2/2$$

$$-\delta k_0^2 \left(\int_{\Omega^C} |\mathbf{w}_C|^2\right)^2 - \delta k_0^2 \left(\int_{\Omega^C} |\,\mathbf{curl}\, \mathbf{w}_C|^2\right)^2 ,$$

for each $\delta > 0$. Finally, for each $0 < \gamma \leq 1/2$ we also have

$$\omega^2 \left(\int_{\Omega^C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{u}}_I\right)^2$$

$$\geq 2\gamma\omega^2 \left(\int_{\Omega^C} \boldsymbol{\sigma} \mathbf{w}_C \cdot \overline{\mathbf{w}}_C - 2\mathrm{Re} \int_\Gamma \mathbf{w}_C \times \mathbf{n} \cdot \overline{\mathbf{u}}_I\right)^2 ,$$

so that the proof of the coerciveness of $c(\cdot, \cdot)$ follows by taking at first δ small enough and then γ small enough.

Therefore, the existence and uniqueness of a solution to problem (23) follow from the Lax–Milgram lemma. However, in view of numerical approximation, we would like to reformulate the problem in a non-constrained vector space.

A first possible approach is to introduce a scalar magnetic potential in Ω^I (see, for instance, [3], [11]): this needs the construction of a suitable set of cutting surfaces and of the related basis functions of the space of harmonic fields (see Sect. 5). Algorithms for this construction have been proposed by [22] and [29]. In complicated geometrical configurations, this approach can be computationally expensive. A second approach, the one we will adopt in the sequel, relies on Lagrangian multipliers.

Consider the saddle-point problem

$$\left\{\begin{array}{l}
\text{Find } (\mathbf{E}_C, \mathbf{H}_I, \mathbf{A}_I) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \times (L^2(\Omega^I))^3 \text{ such that:} \\
c((\mathbf{E}_C,\mathbf{H}_I),(\mathbf{z}_C,\mathbf{v}_I)) + \int_{\Omega^I} \mathbf{curl}\,\overline{\mathbf{v}}_I \cdot \mathbf{A}_I = -i\omega \int_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \\
\qquad\qquad\qquad\qquad \forall\, (\mathbf{z}_C, \mathbf{v}_I) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \\
\int_{\Omega^I} \mathbf{curl}\,\mathbf{H}_I \cdot \overline{\mathbf{N}}_I = \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_I \qquad \forall\, \mathbf{N}_I \in (L^2(\Omega^I))^3 \,.
\end{array}\right. \tag{24}$$

This problem has no unique solution, since it is possible to add to $\mathbf{A}_I$ any function of $\boldsymbol{H}_{0,\Gamma}(\mathbf{curl};\Omega^I) \cap \boldsymbol{H}(\mathbf{curl}\,0;\Omega^I)$. It is not difficult to see that any solution of problem (24) satisfies $\mathbf{curl}\,\mathbf{A}_I = -\omega^2\boldsymbol{\mu}\mathbf{H}_I (= -i\omega\,\mathbf{curl}\,\mathbf{E}_I)$ and $\mathbf{A}_I \times \mathbf{n} = -i\omega\mathbf{E}_C \times \mathbf{n} (= -i\omega\mathbf{E}_I \times \mathbf{n})$ on Γ, so in order to deal with a well-posed saddle-point problem it is natural to look for $\mathbf{A}_I$ in the constrained space $\mathbf{W}_I$, given by

$$\mathbf{W}_I := \{\mathbf{N}_I \in (L^2(\Omega^I))^3 \,|\, \mathbf{N}_I \text{ satisfies (14)}\} \,.$$

In this way $\mathbf{A}_I = -i\omega\mathbf{E}_I$.

Noting also that

$$\mathbf{V}_I^{\mathbf{J}_{e,I}} = \{\mathbf{v}_I \in \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \mid \int_{\Omega^I} \mathbf{curl}\,\mathbf{v}_I \cdot \overline{\mathbf{N}}_I = \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_I \qquad \forall\, \mathbf{N}_I \in \mathbf{W}_I \,\} \,,$$

as can be seen by taking $\mathbf{N}_I = \boldsymbol{\epsilon}^{-1}(\mathbf{curl}\,\mathbf{v}_I - \mathbf{J}_{e,I})$, we are led to consider the following problem:

$$\left\{\begin{array}{l}
\text{Find } (\mathbf{E}_C, \mathbf{H}_I, \mathbf{A}_I) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \times \mathbf{W}_I \text{ such that:} \\
c((\mathbf{E}_C,\mathbf{H}_I),(\mathbf{z}_C,\mathbf{v}_I)) + \int_{\Omega^I} \mathbf{curl}\,\overline{\mathbf{v}}_I \cdot \mathbf{A}_I = -i\omega \int_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \\
\qquad\qquad\qquad\qquad \forall\, (\mathbf{z}_C, \mathbf{v}_I) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) \\
\int_{\Omega^I} \mathbf{curl}\,\mathbf{H}_I \cdot \overline{\mathbf{N}}_I = \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_I \qquad \forall\, \mathbf{N}_I \in \mathbf{W}_I \,.
\end{array}\right. \tag{25}$$

It can be proved that this problem has a unique solution $(\mathbf{E}_C, \mathbf{H}_I, \mathbf{A}_I)$, but again the space $\mathbf{W}_I$ is not easily approximated by finite element spaces. As for equation (15), we introduce another Lagrange multiplier. We note that conditions (14) can be rewritten in a weak sense, namely,

$$\mathbf{W}_I = \{\mathbf{N}_I \in (L^2(\Omega^I))^3 \mid \int_{\Omega^I} \boldsymbol{\epsilon}\mathbf{N}_I \cdot \mathbf{grad}\,\overline{\psi}_I = 0 \text{ for all } \psi_I \in H^1_{0,\Gamma}(\Omega^I)\}.$$

Now we deal with the following problem

$$\begin{cases} \text{Find } (\mathbf{E}_C, \mathbf{H}_I, \phi_I, \mathbf{A}_I) \text{ in } \boldsymbol{H}(\mathbf{curl}; \Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl}; \Omega^I) \times H^1_{0,\Gamma}(\Omega^I) \times (L^2(\Omega^I))^3 : \\ c((\mathbf{E}_C, \mathbf{H}_I), (\mathbf{z}_C, \mathbf{v}_I)) + \int\limits_{\Omega^I} \mathbf{curl}\, \overline{\mathbf{v}}_I \cdot \mathbf{A}_I + \int\limits_{\Omega^I} \boldsymbol{\epsilon} \mathbf{A}_I \cdot \mathbf{grad}\, \overline{\psi}_I = -i\omega \int\limits_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \\ \qquad\qquad \forall\, (\mathbf{z}_C, \mathbf{v}_I, \psi_I) \in \boldsymbol{H}(\mathbf{curl}; \Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl}; \Omega^I) \times H^1_{0,\Gamma}(\Omega^I) \\ \int\limits_{\Omega^I} \mathbf{curl}\, \mathbf{H}_I \cdot \overline{\mathbf{N}}_I + \int\limits_{\Omega^I} \boldsymbol{\epsilon} \overline{\mathbf{N}}_I \cdot \mathbf{grad}\, \phi_I = \int\limits_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_I \qquad \forall\, \mathbf{N}_I \in (L^2(\Omega^I))^3 \,. \end{cases} \tag{26}$$

Setting $\Lambda := \boldsymbol{H}(\mathbf{curl}; \Omega^C) \times \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl}; \Omega^I) \times H^1_{0,\Gamma}(\Omega^I)$,

$$r(\cdot,\cdot) : \Lambda \times (L^2(\Omega^I))^3 \to \mathbb{C}$$

$$r((\mathbf{z}_C, \mathbf{v}_I, \psi_I), \mathbf{N}_I) := \int\limits_{\Omega^I} (\mathbf{curl}\, \mathbf{v}_I \cdot \overline{\mathbf{N}}_I + \boldsymbol{\epsilon} \overline{\mathbf{N}}_I \cdot \mathbf{grad}\, \psi_I) \,,$$

and

$$\widehat{c}(\cdot,\cdot) : \Lambda \times \Lambda \to \mathbb{C}$$

$$\widehat{c}((\mathbf{w}_C, \mathbf{u}_I, \varphi_I), (\mathbf{z}_C, \mathbf{v}_I, \psi_I)) := c((\mathbf{w}_C, \mathbf{u}_I), (\mathbf{z}_C, \mathbf{v}_I)) \,,$$

problem (26) reads

$$\begin{cases} \text{Find } [(\mathbf{E}_C, \mathbf{H}_I, \phi_I), \mathbf{A}_I] \text{ in } \Lambda \times (L^2(\Omega^I))^3 : \\ \widehat{c}((\mathbf{E}_C, \mathbf{H}_I, \phi_I), (\mathbf{z}_C, \mathbf{v}_I, \psi_I)) + \overline{r((\mathbf{z}_C, \mathbf{v}_I, \psi_I), \mathbf{A}_I)} = -i\omega \int\limits_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_C \\ \qquad\qquad \forall\, (\mathbf{z}_C, \mathbf{v}_I, \psi_I) \in \Lambda \\ r((\mathbf{E}_C, \mathbf{H}_I, \phi_I), \mathbf{N}_I) = \int\limits_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_I \qquad \forall\, \mathbf{N}_I \in (L^2(\Omega^I))^3 \,. \end{cases} \tag{27}$$

Theorem 4.1 *Problem* (27) *has a unique solution.*

Proof. It is a consequence of the general theory of variational saddle-point problems. We note that, if $r((\mathbf{z}_C, \mathbf{v}_I, \psi_I), \mathbf{N}_I) = 0$ for all $\mathbf{N}_I \in (L^2(\Omega^I))^3$, then $\mathbf{v}_I \in \mathbf{V}_I^{\mathbf{0}}$ and $\psi_I = 0$ (it follows taking in (27) $\mathbf{N}_I = \boldsymbol{\epsilon}^{-1}\, \mathbf{curl}\, \mathbf{v}_I$ and $\mathbf{N}_I = \mathbf{grad}\, \psi_I$, respectively). Then the bilinear form $\widehat{c}(\cdot,\cdot)$ is coercive on the space

$$\{(\mathbf{z}_C, \mathbf{v}_I, \psi_I) \in \Lambda \mid r((\mathbf{z}_C, \mathbf{v}_I, \psi_I), \mathbf{N}_I) = 0\ \forall\, \mathbf{N}_I \in (L^2(\Omega^I))^3\} = \boldsymbol{H}(\mathbf{curl}; \Omega^C) \times \mathbf{V}_I^{\mathbf{0}} \times \{0\}.$$

On the other hand we recall that any given vector function $\mathbf{N}_I \in (L^2(\Omega^I))^3$ can be decomposed into the following sum

$$\mathbf{N}_I = \boldsymbol{\epsilon}^{-1}\, \mathbf{curl}\, \mathbf{q}_I + \mathbf{grad}\, \varphi_I \,, \tag{28}$$

where $\mathbf{q}_I \in \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl}; \Omega^I)$, $\varphi_I \in H^1_{0,\Gamma}(\Omega^I)$, and that there exists a positive constant C such that

$$\|\mathbf{N}_I\|_{L^2(\Omega^I)} \geq C(\|\mathbf{q}_I\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)} + \|\varphi_I\|_{H^1(\Omega^I)})$$

(see, e.g., [6]).

Hence, it is easy to see that the following inf–sup condition is satisfied

$$\sup_{\substack{(\mathbf{z}_C,\mathbf{v}_I,\psi_I)\in\Lambda\\(\mathbf{z}_C,\mathbf{v}_I,\psi_I)\neq(\mathbf{0},\mathbf{0},0)}} \frac{r((\mathbf{z}_C,\mathbf{v}_I,\psi_I),\mathbf{N}_I)}{\|\mathbf{z}_C\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)}+\|\mathbf{v}_I\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)}+\|\psi_I\|_{H^1(\Omega^I)}} \geq C_1\|\mathbf{N}_I\|_{L^2(\Omega^I)} .$$

In fact, it follows by taking $(\mathbf{z}_C,\mathbf{v}_I,\psi_I)=(\mathbf{0},\mathbf{q}_I,\varphi_I)$, being $\mathbf{N}_I=\boldsymbol{\epsilon}^{-1}\,\mathbf{curl}\,\mathbf{q}_I+\mathbf{grad}\,\varphi_I$ as in (28). □

4.2 Finite element discretization

Our aim is to find a Galerkin finite element discretization of (27). We want to verify the assumptions of the theory of discrete saddle-point problem [13, Chap. 2].

We assume that Ω, Ω^C, Ω^I are Lipschitz polyhedral domains and consider two families of regular tetrahedral meshes $\mathcal{T}_{C,h}$ and $\mathcal{T}_{I,h}$ of Ω^C and Ω^I, respectively.

We employ the (complex valued) Nédélec curl-conforming edge elements of the lowest order $\mathbf{X}_{L,h}$ to approximate the functions in $\boldsymbol{H}(\mathbf{curl};\Omega^L)$, $L=I,C$. The homogeneous boundary conditions on $\partial\Omega$ are incorporated by setting degrees of freedom on $\partial\Omega$ to zero. So we consider the space

$$\mathbf{V}_{I,h} := \mathbf{X}_{I,h}\cap\boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I) .$$

For additional information about edge elements the reader is referred to [24, Chap. 3], and [21, Chap. III, Sect. 5.3].

In order to approximate the space $H^1_{0,\Gamma}(\Omega^I)$, let us start with denoting by P_k the standard space of complex polynomials of total degree less than or equal to k with respect to the real variable $\mathbf{x}$. The nonconforming Crouzeix-Raviart elements are defined as follows:

$$\begin{aligned} U_{I,h} \;:=\; & \{\psi_{I,h}\in L^2(\Omega^I)\mid \psi_{I,h|K}\in P_1\ \forall\,K\in\mathcal{T}_{I,h} \text{ and } \psi_{I,h} \text{ is continuous at the} \\ & \text{centroid of any face } f \text{ common to two elements in } \mathcal{T}_{I,h}\}. \end{aligned}$$

Then the discrete $\psi_{I,h}$ will belong to

$$M_{I,h} \;:=\; \{\psi_{I,h}\in U_{I,h}\,|\psi_{I,h}(\mathbf{p})=0 \text{ for all centroids } \mathbf{p} \text{ of faces of } \Gamma\} .$$

Note that, since functions in $U_{I,h}$ are no longer continuous, they are no longer in $H^1(\Omega^I)$. Therefore we must define a modified sesquilinear form r_h acting also on $H^1_{0,\Gamma}(\Omega^I)+M_{I,h}$, and a norm on $H^1_{0,\Gamma}(\Omega^I)+M_{I,h}$. This can be done as follows: first, for each $\psi_I\in[H^1_{0,\Gamma}(\Omega^I)+M_{I,h}]$ we denote by $\widetilde{\mathbf{grad}}\psi_I$ the function in $(L^2(\Omega^I))^3$ defined as

$$(\widetilde{\mathbf{grad}}\psi_I)_{|K} := \mathbf{grad}(\psi_{I|K}) \quad \forall\,K\in\mathcal{T}_{I,h}.$$

Note that if $\psi_I\in H^1_{0,\Gamma}(\Omega^I)$, then $\widetilde{\mathbf{grad}}\psi_I=\mathbf{grad}\,\psi_I$. Then, we define the norm in $H^1_{0,\Gamma}(\Omega^I)+M_{I,h}$ as

$$\|\psi_I\|_h^2 := \sum_K\int_K|\,\mathbf{grad}\,\psi_I|^2 = \|\widetilde{\mathbf{grad}}\psi_I\|^2_{L^2(\Omega^I)} .$$

For all $\mathbf{z}_C \in \boldsymbol{H}(\mathbf{curl};\Omega^C)$, $\mathbf{u}_I \in \boldsymbol{H}_{0,\partial\Omega}(\mathbf{curl};\Omega^I)$, $\psi_I \in [H^1_{0,\Gamma}(\Omega^I)+M_{I,h}]$ and $\mathbf{N}_I \in (L^2(\Omega^I))^3$ we set

$$r_h((\mathbf{z}_C,\mathbf{u}_I,\psi_I)) := \int_{\Omega^I} \mathbf{curl}\,\mathbf{u}_I \cdot \overline{\mathbf{N}}_I + \sum_K \int_K \boldsymbol{\epsilon}\overline{\mathbf{N}}_I \cdot \mathbf{grad}\,\psi_I = \int_{\Omega^I} (\mathbf{curl}\,\mathbf{u}_I \cdot \overline{\mathbf{N}}_I + \boldsymbol{\epsilon}\overline{\mathbf{N}}_I \cdot \widetilde{\mathbf{grad}}\psi_I).$$

To discretize the Lagrangian multiplier $\mathbf{A}_I \in (L^2(\Omega^I))^3$ we choose piecewise constant vector functions in the space

$$\mathbf{Q}_{I,h} := \{\mathbf{N}_{I,h} \in (L^2(\Omega^I))^3 \mid \mathbf{N}_{I,h|K} \in (P_0)^3\ \forall K \in \mathcal{T}_{I,h}\}\,.$$

So, defining $\Lambda_h := \mathbf{X}_{C,h} \times \mathbf{V}_{I,h} \times M_{I,h}$ the finite elements approximation of (27) can be formulated as follows:

$$\begin{cases} \text{Find } [(\mathbf{E}_{C,h},\mathbf{H}_{I,h},\phi_{I,h}),\mathbf{A}_{I,h}] \text{ in } \Lambda_h \times \mathbf{Q}_{I,h}: \\ \widehat{c}((\mathbf{E}_{C,h},\mathbf{H}_{I,h},\phi_{I,h}),(\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h})) + \overline{r_h((\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h}),\mathbf{A}_{I,h})} \\ \qquad\qquad = -i\omega \int_{\Omega^C} \mathbf{J}_{e,C} \cdot \overline{\mathbf{z}}_{C,h} \quad \forall\,(\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h}) \in \Lambda_h \\ r_h((\mathbf{E}_{C,h},\mathbf{H}_{I,h},\phi_{I,h}),\mathbf{N}_{I,h}) = \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{N}}_{I,h} \qquad \forall\,\mathbf{N}_{I,h} \in \mathbf{Q}_{I,h}\,. \end{cases} \tag{29}$$

The following results will be crucial in the proof of existence and uniqueness of solution of problem (29). For the proofs see [6].

Lemma 4.2 *We have the $L^2(\Omega^I)$-orthogonal decomposition $\mathbf{Q}_{I,h} = \mathbf{curl}\,\mathbf{V}_{I,h} \oplus \widetilde{\mathbf{grad}} M_{I,h}$.*

Lemma 4.3 *There exists a positive constant C_0, independent of h, such that for all $\mathbf{p}_{I,h} \in (\mathbf{V}^0_{I,h})^\perp$*

$$\|\mathbf{p}_{I,h}\|_{L^2(\Omega_I)} \le C_0 \|\,\mathbf{curl}\,\mathbf{p}_{I,h}\|_{L^2(\Omega_I)}\,.$$

Here, the following notation has been used:

$$\mathbf{V}^{\mathbf{0}}_{I,h} := \mathbf{V}_{I,h} \cap \mathbf{V}^{\mathbf{0}}_I \quad , \quad (\mathbf{V}^{\mathbf{0}}_{I,h})^\perp := \{\mathbf{p}_{I,h} \in \mathbf{V}_{I,h} \mid \int_{\Omega_I} \mathbf{p}_{I,h} \cdot \overline{\mathbf{v}}_{I,h} = 0\ \forall\,\mathbf{v}_{I,h} \in \mathbf{V}^{\mathbf{0}}_{I,h}\}\,.$$

Now we are in a position to proof the main result of this section.

Theorem 4.4 *Given a triangulation $\mathcal{T}_{I,h}$ of Ω^I, assume that the entries of the matrix $\boldsymbol{\epsilon}$ are piecewise constants in Ω^I. Then problem (29) has a unique solution.*

Proof. We can proceed as in the continuous case. Using Lemma 4.2 it is easy to see that if $(\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h}) \in \Lambda_h$ is such that $r_h((\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h}),\mathbf{N}_{I,h}) = 0$ for all $\mathbf{N}_{I,h} \in \mathbf{Q}_{I,h}$, then $\mathbf{curl}\,\mathbf{v}_{I,h} = \mathbf{0}$ and $\psi_{I,h} = 0$. Therefore, it is clear that the bilinear form $\widehat{c}(\cdot,\cdot)$ is coercive in the space

$$\{(\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h}) \in \Lambda_h \mid r_h((\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h}),\mathbf{N}_{I,h}) = 0\ \forall\,\mathbf{N}_{I,h} \in \mathbf{Q}_{I,h}\}.$$

Concerning the discrete inf–sup condition, by Lemma 4.2 for any given $\mathbf{N}_{I,h} \in \mathbf{Q}_{I,h}$ there exist $\mathbf{q}_{I,h} \in \mathbf{V}_{I,h}$ and $\varphi_{I,h} \in M_{I,h}$ such that $\mathbf{N}_{I,h} = \mathbf{curl}\,\mathbf{q}_{I,h} + \widetilde{\mathbf{grad}}\varphi_{I,h}$. Moreover it is possible to choose $\mathbf{q}_{I,h} \in (\mathbf{V}^0_{I,h})^\perp$ and then, by Lemma 4.3,

$$\|\mathbf{N}_{I,h}\|^2_{L^2(\Omega^I)} = \|\,\mathbf{curl}\,\mathbf{q}_{I,h}\|^2_{L^2(\Omega^I)} + \|\widetilde{\mathbf{grad}}\varphi_{I,h}\|^2_{L^2(\Omega^I)} \geq C(\|\mathbf{q}_{I,h}\|^2_{\boldsymbol{H}(\mathbf{curl};\Omega^I)} + \|\varphi_{I,h}\|^2_h).$$

Now taking $(\mathbf{0}, \mathbf{q}_{I,h}, \boldsymbol{\epsilon}^{-1}\varphi_{I,h}) \in \Lambda_h$ one has

$$\sup_{\substack{(\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h})\in\Lambda_h \\ (\mathbf{z}_{C,h},\mathbf{v}_{I,h},\psi_{I,h})\neq(\mathbf{0},\mathbf{0},0)}} \frac{r_h((\mathbf{z}_{C,h}, \mathbf{v}_{I,h}, \psi_{I,h}), \mathbf{N}_{I,h})}{\|\mathbf{z}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)} + \|\mathbf{v}_{I,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)} + \|\psi_{I,h}\|_h} \geq C_2\|\mathbf{N}_{I,h}\|_{L^2(\Omega^I)}.$$

□

The convergence theory is now standard (see [13] and [6]), and yields the quasi-optimality of the Galerkin finite element approximation.

5 Hybrid $\mathbf{H}_C/\mathbf{E}_I$ formulation

5.1 Topological prerequisites

As already hinted in Sect. 3, certain topological aspects will enter into the theoretical treatment of hybrid coupled variational formulations. Let $p := \beta_1(\Omega^I)$ stand for the first Betti number of Ω^I, a topological invariant measuring the number of holes drilled through Ω^I. Recall that in Ω^I there will be p mutually disjoint, orientable two-dimensional surfaces called *cuts* such that $\Omega^I \setminus (\Sigma_1 \cup \ldots \cup \Sigma_p)$ has trivial first homology group.

Let us denote by $\boldsymbol{\mathcal{H}}^I$ the space of harmonic Neumann vector fields in Ω^I:

$$\boldsymbol{\mathcal{H}}^I := \boldsymbol{H}(\mathbf{curl}\,0; \Omega^I) \cap \boldsymbol{H}(\mathrm{div}\,0; \Omega^I) \cap \boldsymbol{H}_0(\mathrm{div}; \Omega^I)\ .$$

It is well known [8] that $\dim \boldsymbol{\mathcal{H}}^I = p$ and that it possesses a representation based on the cuts: denote by κ_k a harmonic function in $H^1(\Omega^I \setminus \Sigma_k)$ that has a jump of height 1 across Σ_k, and such that $\mathbf{grad}\,\kappa_k \cdot \mathbf{n} = 0$ on $\partial\Omega \setminus \Sigma_k$. Then $\{\widetilde{\mathbf{grad}}\kappa_1, \ldots, \widetilde{\mathbf{grad}}\kappa_p\}$ will be a basis of $\boldsymbol{\mathcal{H}}^I$, $\widetilde{\mathbf{grad}}\kappa_k$ denoting the extension to Ω^I of $\mathbf{grad}\kappa_k$ computed in $\Omega^I \setminus \Sigma_k$.

On Γ we can find $2p$ non-bounding homologically independent *fundamental cycles* $\gamma_1, \ldots, \gamma_{2p}$ that represent generators of the homology group $H_1(\Gamma)$. They can be chosen such that $\gamma_k = \partial\Sigma_k$, $k = 1, \ldots, p$ (see [25]). Moreover, $\gamma_{p+1}, \ldots, \gamma_{2p}$ can be chosen dual to $\gamma_1, \ldots, \gamma_p$, which implies

$$\int_{\gamma_{p+k}} \widetilde{\mathbf{grad}}\kappa_j \cdot \mathrm{d}\vec{s} = \delta_{kj}, \quad k, j \in \{1, \ldots, p\}\ . \tag{30}$$

5.2 Variational formulation

Now, we are ready to consider the second primal-dual coupling, namely the $\mathbf{H}_C/\mathbf{E}_I$ formulation, in which the unknows are the magnetic field in the conductor Ω^C and the electric field in the air region Ω^I.

From the first equation in (9), for each $\mathbf{v}_C \in \boldsymbol{H}(\mathbf{curl};\Omega^C)$ one finds by integration by parts

$$\int_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C + i\omega\boldsymbol{\mu}\mathbf{H}_C \cdot \overline{\mathbf{v}}_C) - \int_{\Gamma} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C) \times \mathbf{n} \cdot \overline{\mathbf{v}}_C$$
$$= \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C - \int_{\Gamma} (\boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C}) \times \mathbf{n} \cdot \overline{\mathbf{v}_C}\,. \quad (31)$$

Using the Maxwell transmission conditions for the electric field and Ampère's law in Ω^C

$$\int_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C + i\omega\boldsymbol{\mu}\mathbf{H}_C \cdot \overline{\mathbf{v}}_C) - \int_{\Gamma} \mathbf{E}_I \times \mathbf{n} \cdot \overline{\mathbf{v}}_C$$
$$= \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C. \quad (32)$$

Analogously, from the second equation in (13) for each $\mathbf{z}_I \in \boldsymbol{H}(\mathbf{curl};\Omega^I)$ one finds by integration by parts

$$\int_{\Omega^I} \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{z}}_I + \int_{\Gamma} (\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I) \times \mathbf{n} \cdot \overline{\mathbf{z}}_I = -i\omega \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{z}}_I, \quad (33)$$

and using the Maxwell transmission conditions and Faraday's law in Ω^I

$$\int_{\Omega^I} \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{z}}_I - i\omega \int_{\Gamma} \mathbf{H}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I = -i\omega \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{z}}_I. \quad (34)$$

Taking into account the gauge conditions (14) for $\mathbf{E}_I$, and setting

$$\mathbf{Z}_I := \{\mathbf{z}_I \in \boldsymbol{H}(\mathbf{curl};\Omega^I) \,|\, \mathbf{z}_I \text{ satisfies } (14)\}\,,$$

the weak formulation of the hybrid $\mathbf{H}_C/\mathbf{E}_I$ formulation reads

$$\begin{cases} \text{Find } (\mathbf{H}_C, \mathbf{E}_I) \in \boldsymbol{H}(\mathbf{curl};\Omega^C) \times \mathbf{Z}_I \ : \\ \int_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{H}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C + i\omega\boldsymbol{\mu}\mathbf{H}_C \cdot \overline{\mathbf{v}}_C) + \int_{\Gamma} \overline{\mathbf{v}}_C \times \mathbf{n} \cdot \mathbf{E}_I = f(\mathbf{v}_C) \\ \int_{\Gamma} \mathbf{H}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I + i\omega^{-1} \int_{\Omega^I} \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I \cdot \mathbf{curl}\,\overline{\mathbf{z}}_I = g(\mathbf{z}_I) \\ \text{for all } \mathbf{v}_C \in \boldsymbol{H}(\mathbf{curl};\Omega^C), \mathbf{z}_I \in \mathbf{Z}_I\,, \end{cases} \quad (35)$$

where

$$f(\mathbf{v}_C) := \int_{\Omega^C} \boldsymbol{\sigma}^{-1}\mathbf{J}_{e,C} \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C\,,$$
$$g(\mathbf{z}_I) := \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \overline{\mathbf{z}}_I\,.$$

It can be shown, via the standard theory for saddle-point problems, that this problem has a unique solution. Unfortunately, the proof relies on the stability of the pairing $(\mathbf{u}, \mathbf{w}) \mapsto \int_\Gamma (\mathbf{u} \times \mathbf{n}) \cdot \overline{\mathbf{w}}$ on $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \times \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. However, this stability is very hard to preserve in the discrete setting. For a detailed account of this problem see [17, Sect. 3].

A remedy is offered by working on a smaller constrained space. Assuming for simplicity that $\mathrm{supp}\,\mathbf{J}_e \cap \Gamma = \emptyset$, one sees that $\mathbf{H}_I$ is **curl**-free in an exterior neighborhood of Γ. By tangential continuity of $\mathbf{H}$ we can infer that

$$\mathrm{div}_\Gamma(\mathbf{H}_C \times \mathbf{n}) = \mathbf{curl}\,\mathbf{H}_I \cdot \mathbf{n} = 0 \quad \text{on } \Gamma\,.$$

Let us define the spaces

$$\widetilde{\mathbf{X}}_C := \{\mathbf{v}_C \in \boldsymbol{H}(\mathbf{curl}; \Omega^C) \,|\, \mathrm{div}_\Gamma(\mathbf{v}_C \times \mathbf{n}) = 0 \text{ on } \Gamma\}$$

and

$$\widetilde{\mathbf{Z}}_I := \{\mathbf{z}_I \in \boldsymbol{H}(\mathbf{curl}; \Omega^I) \,|\, \int_{\Omega^I} \mathbf{z}_I \cdot \mathbf{grad}\,\overline{\psi}_I = 0 \text{ for all } \psi_I \in H^1(\Omega^I)\}\,.$$

Note that they are closed subspaces of $\boldsymbol{H}(\mathbf{curl}; \Omega^C)$ and $\boldsymbol{H}(\mathbf{curl}; \Omega^I)$, respectively. Moreover, $\widetilde{\mathbf{Z}}_I$ is the space of $\mathbf{z}_I \in \boldsymbol{H}(\mathbf{curl}; \Omega^I)$ such that $\mathrm{div}\,\mathbf{z}_I = 0$ in Ω^I and $\mathbf{z}_I \cdot \mathbf{n} = 0$ on $\partial\Omega \cup \Gamma$.

Before analyzing problem (35) in $\widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$, we need some preliminary results. First of all, we recall that the following orthogonal decomposition holds (see, e.g., [2, 20]):

$$(L^2(\Omega^I))^3 = (\boldsymbol{H}(\mathbf{curl}\,0; \Omega^I))^\perp \oplus \boldsymbol{\mathcal{H}}^I \oplus \mathbf{grad}\,H^1(\Omega^I)\,, \tag{36}$$

where $\perp$ denotes the $\boldsymbol{L}^2(\Omega^I)$-orthogonal complements, and $\boldsymbol{\mathcal{H}}^I$ is the finite-dimensional space of harmonic Neumann vector fields in Ω^I. Therefore, it is easy to see that the following Poincaré-type inequality will hold:

Theorem 5.1 *There exists a positive constant* C_1 *such that*

$$\|\mathbf{w}_I\|_{L^2(\Omega^I)} \leq C_1 \,\|\mathbf{curl}\,\mathbf{w}_I\|_{L^2(\Omega^I)} \;\; \forall\, \mathbf{w}_I \in (\boldsymbol{H}(\mathbf{curl}\,0; \Omega^I))^\perp \tag{37}$$

We are now in a position to prove:

Theorem 5.2 *The variational problem* (35) *has a unique solution in* $\widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$*.*

Proof. To prove that (35) has a unique solution in $\widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$ we appeal to saddle-point theory. Since we have the direct sum decomposition

$$\widetilde{\mathbf{Z}}_I = (\boldsymbol{H}(\mathbf{curl}\,0; \Omega^I))^\perp \oplus \boldsymbol{\mathcal{H}}^I\,, \tag{38}$$

we can rewrite (35) in equivalent form as: seek $\mathbf{H}_C \in \widetilde{\mathbf{X}}_C$, $\mathbf{E}_I^\perp \in (\boldsymbol{H}(\mathbf{curl}\,0; \Omega^I))^\perp$, $\mathbf{E}_I^{\mathcal{H}} \in \boldsymbol{\mathcal{H}}^I$ such that

$$\begin{aligned}
\mathcal{A}((\mathbf{H}_C, \mathbf{E}_I^\perp), (\mathbf{v}_C, \mathbf{z}_I^\perp)) + \int_\Gamma \overline{\mathbf{v}}_C \times \mathbf{n} \cdot \mathbf{E}_I^{\mathcal{H}} &= f(\mathbf{v}_C) + g(\mathbf{z}_I^\perp) \\
\int_\Gamma \mathbf{H}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I^{\mathcal{H}} &= g(\mathbf{z}_I^{\mathcal{H}})
\end{aligned}$$

for all $\mathbf{v}_C \in \widetilde{\mathbf{X}}_C$, $\mathbf{z}_I^\perp \in (\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I))^\perp$, and $\mathbf{z}_I^{\mathcal{H}} \in \boldsymbol{\mathcal{H}}^I$. The abbreviation $\mathcal{A}$ is used for sum of the two left-hand sides of (35), namely, the sesquilinear form

$$\mathcal{A}((\mathbf{u}_C,\mathbf{w}_I),(\mathbf{v}_C,\mathbf{z}_I)) := \int_{\Omega^C} (\boldsymbol{\sigma}^{-1}\,\mathbf{curl}\,\mathbf{u}_C \cdot \mathbf{curl}\,\overline{\mathbf{v}}_C + i\omega\boldsymbol{\mu}\mathbf{u}_C \cdot \overline{\mathbf{v}}_C) + \int_\Gamma \overline{\mathbf{v}}_C \times \mathbf{n} \cdot \mathbf{w}_I + \int_\Gamma \mathbf{u}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I + i\omega^{-1} \int_{\Omega^I} \boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{w}_I \cdot \mathbf{curl}\,\overline{\mathbf{z}}_I \,.$$

It can be proved that it is coercive in $\widetilde{\mathbf{X}}_C \times (\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I))^\perp$: in fact, based on (37), the proof is analogous to that presented for the sesquilinear form $c(\cdot,\cdot)$ in Sect. 4.

Now, we only need to check the inf–sup condition

$$\exists \beta > 0 : \qquad \sup_{\substack{\mathbf{v}_C \in \widetilde{\mathbf{X}}_C \\ \mathbf{v}_C \neq \mathbf{0}}} \frac{\left|\int_\Gamma \mathbf{v}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I^{\mathcal{H}}\right|}{\|\mathbf{v}_C\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)}} \geq \beta \left\|\mathbf{z}_I^{\mathcal{H}}\right\|_{L^2(\Omega^I)} \quad \forall\, \mathbf{z}_I^{\mathcal{H}} \in \boldsymbol{\mathcal{H}}^I \,. \tag{39}$$

For the sake of simplicity, we will assume that Ω^C is a simple loop, i.e., it has the first Betti number equal to 1. A typical example is a torus. Then for $L = I, C$ we can find an orientable two-dimensional surface Σ^L such the $\Omega^L \setminus \Sigma^L$ has trivial first homology group.

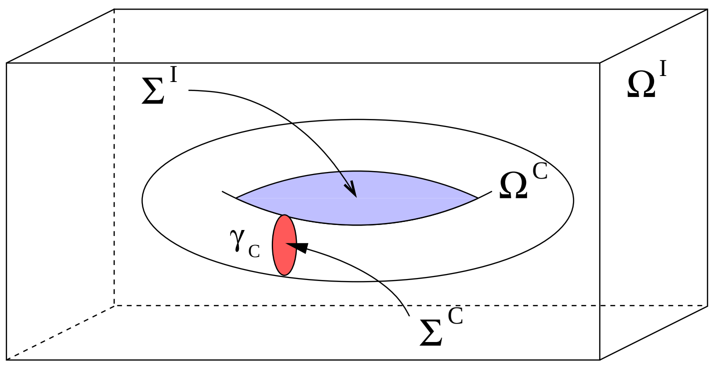

Figure 2: The cutting surfaces Σ^C and Σ^I.

Recalling the results presented in Sect. 5.1, we have $\mathbf{z}_I^{\mathcal{H}} = c\,\widetilde{\mathbf{grad}}\,\kappa_I$, where κ_I has a jump of height 1 across Σ^I, and $c \in \mathbb{C}$. Then we can use as special candidate for $\mathbf{v}_C \in \widetilde{\mathbf{X}}_C$ the Neumann harmonic vector field $c\,\widetilde{\mathbf{grad}}\,\kappa_C$, where now κ_C has a jump of height 1 across Σ^C. We have (remember that $\partial\Sigma^C = \gamma_C$)

$$\begin{aligned}\int_\Gamma c\,\widetilde{\mathbf{grad}}\,\kappa_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I^{\mathcal{H}} &= c\int_\Gamma \widetilde{\mathbf{curl}}_\Gamma \kappa_C \cdot \overline{\mathbf{z}}_I^{\mathcal{H}} = c \int_{\Gamma\setminus\gamma_C} \kappa_C\ \mathbf{curl}\,\overline{\mathbf{z}}_I^{\mathcal{H}} \cdot \mathbf{n} + c\int_{\gamma_C} \overline{\mathbf{z}}_I^{\mathcal{H}} \cdot \mathrm{d}\vec{s} \\ &= c\int_{\gamma_C} \overline{\mathbf{z}}_I^{\mathcal{H}} \cdot \mathrm{d}\vec{s} = |c|^2 \int_{\gamma_C} \widetilde{\mathbf{grad}}\,\kappa_I \cdot \mathrm{d}\vec{s} = |c|^2 \,,\end{aligned} \tag{40}$$

having used (30). Since $\left\|\mathbf{z}_I^{\mathcal{H}}\right\|_{L^2(\Omega^I)} = |c| \left\|\widetilde{\mathbf{grad}}\,\kappa_I\right\|_{L^2(\Omega^I)}$, we arrive at

$$\sup_{\substack{\mathbf{v}_C \in \widetilde{\mathbf{X}}_C \\ \mathbf{v}_C \neq \mathbf{0}}} \frac{\left|\int_\Gamma \mathbf{v}_C \times \mathbf{n} \cdot \overline{\mathbf{z}}_I^{\mathcal{H}}\right|}{\|\mathbf{v}_C\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)}} \geq \frac{|c|}{\left\|\widetilde{\mathbf{grad}}\,\kappa_C\right\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)}} = \frac{\left\|\mathbf{z}_I^{\mathcal{H}}\right\|_{L^2(\Omega^I)}}{\left\|\widetilde{\mathbf{grad}}\,\kappa_C\right\|_{L^2(\Omega^C)} \left\|\widetilde{\mathbf{grad}}\,\kappa_I\right\|_{L^2(\Omega^I)}} \,.$$

If the first Betti number of Ω^C is equal to $p > 1$, by proceeding in a similar way one can easily see that the constant β in the inf–sup condition is given by

$$\beta = \min_{\substack{\mathbf{c} \in \mathbb{C}^p \\ \mathbf{c} \neq \mathbf{0}}} \frac{|\mathbf{c}|^2}{(M^C \mathbf{c} \cdot \overline{\mathbf{c}})^{1/2} (M^I \mathbf{c} \cdot \overline{\mathbf{c}})^{1/2}} ,$$

where M_L, $L = I, C$, is the matrix given by $M^L_{kj} = \int_{\Omega^L} \widetilde{\mathbf{grad}}\, \kappa_{k,L} \cdot \widetilde{\mathbf{grad}}\, \kappa_{j,L}$, $k, j = 1, \dots, p$. □

In order to get rid of the constrained space $\widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$, we can make use of the fact that both the constraints can be included in an augmented variational problem as extra linear conditions. Let us define the space

$$\mathbf{X}^*_C := \{\mathbf{v}_C \in \boldsymbol{H}(\mathbf{curl}; \Omega^C) \,|\, \mathrm{div}_\Gamma(\mathbf{v}_C \times \mathbf{n}) \in L^2(\Gamma)\} ,$$

endowed with the graph norm

$$\|\mathbf{v}_C\|_{\mathbf{X}^*_C} := \|\mathbf{v}_C\|_{\boldsymbol{H}(\mathbf{curl}; \Omega^C)} + \|\mathrm{div}_\Gamma(\mathbf{v}_C \times \mathbf{n})\|_{L^2(\Gamma)}$$

(which coincides with $\|\mathbf{v}_C\|_{\boldsymbol{H}(\mathbf{curl}; \Omega^C)}$ when $\mathbf{v}_C \in \widetilde{\mathbf{X}}_C$).

The unconstrained variational problem that we consider is:

$$\left\{ \begin{array}{lcl} \text{Find } (\mathbf{H}_C, \mathbf{E}_I, Q, \phi_I) \in \mathbf{X}^*_C \times \boldsymbol{H}(\mathbf{curl}; \Omega^I) \times L^2(\Gamma)/\mathbb{C} \times H^1(\Omega^I)/\mathbb{C} \text{ such that:} & & \\ \mathcal{A}((\mathbf{H}_C, \mathbf{E}_I), (\mathbf{v}_C, \mathbf{z}_I)) - \int_\Gamma \mathrm{div}_\Gamma(\overline{\mathbf{v}}_C \times \mathbf{n})\, Q - \int_{\Omega^I} \overline{\mathbf{z}}_I \cdot \mathbf{grad}\, \phi_I & = & f(\mathbf{v}_C) + g(\mathbf{z}_I) \\ \int_\Gamma \mathrm{div}_\Gamma(\mathbf{H}_C \times \mathbf{n})\, \overline{P} & = & 0 \\ \int_{\Omega^I} \mathbf{E}_I \cdot \mathbf{grad}\, \overline{\psi}_I & = & 0 \\ \text{for all } \mathbf{v}_C \in \mathbf{X}^*_C, \mathbf{z}_I \in \boldsymbol{H}(\mathbf{curl}; \Omega^I), P \in L^2(\Gamma)/\mathbb{C}, \psi_I \in H^1(\Omega^I)/\mathbb{C}. & & \end{array} \right. \tag{41}$$

Theorem 5.3 *The variational problem* (41) *has a unique solution in* $\mathbf{X}^*_C \times \boldsymbol{H}(\mathbf{curl}; \Omega^I) \times L^2(\Gamma)/\mathbb{C} \times H^1(\Omega^I)/\mathbb{C}$.

Proof. It is clear that the two last equations in (41) imply that $(\mathbf{H}_C, \mathbf{E}_I) \in \widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$. Therefore, from Theorem 5.2 and the classical theory for saddle-point problems ([12], [13]), we only need to prove the inf–sup condition

$$\exists \beta^* > 0 : \quad \sup_{\substack{(\mathbf{v}_C, \mathbf{z}_I) \in \mathbf{X}^*_C \times \boldsymbol{H}(\mathbf{curl}; \Omega^I) \\ (\mathbf{v}_C, \mathbf{z}_I) \neq (\mathbf{0}, \mathbf{0})}} \frac{\left| \int_\Gamma \mathrm{div}_\Gamma(\mathbf{v}_C \times \mathbf{n})\, \overline{P} + \int_{\Omega^I} \mathbf{z}_I \cdot \mathbf{grad}\, \overline{\psi}_I \right|}{\|\mathbf{v}_C\|_{\mathbf{X}^*_C} + \|\mathbf{z}_I\|_{\boldsymbol{H}(\mathbf{curl}; \Omega^I)}} \tag{42}$$

$$\geq \beta^* (\|P\|_{L^2(\Gamma)} + \|\psi_I\|_{H^1(\Omega^I)}) \quad \forall\, P \in L^2(\Gamma)/\mathbb{C}, \psi_I \in H^1(\Omega^I)/\mathbb{C} .$$

We choose $\mathbf{z}_I = \mathbf{grad}\, \psi_I$ and $\mathbf{v}_C$ such that $\mathrm{div}_\Gamma(\mathbf{v}_C \times \mathbf{n}) = P$. This can be done since there exists $\lambda \in H^1(\Gamma)/\mathbb{C}$ such that $\Delta_\Gamma \lambda = P$ and $\|\mathbf{grad}_\Gamma\, \lambda\|_{L^2(\Gamma)} \leq \|P\|_{L^2(\Gamma)}$, and we can take for $\mathbf{v}_C$ a continuous extension in $\boldsymbol{H}(\mathbf{curl};\Omega^C)$ of $\mathbf{grad}_\Gamma \lambda \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$, namely, $\mathbf{v}_C \times \mathbf{n} = \mathbf{grad}_\Gamma \lambda$. In particular $\mathbf{v}_C \in \mathbf{X}_C^*$ and $\|\mathbf{v}_C\|_{\mathbf{X}_C^*} \leq C_1(\|\mathbf{grad}_\Gamma \lambda\|_{H^{-\frac{1}{2}}(\Gamma)} + \|P\|_{L^2(\Gamma)}) \leq C_2 \|P\|_{L^2(\Gamma)}$. Hence

$$\sup_{\substack{(\mathbf{v}_C, \mathbf{z}_I) \in \mathbf{X}_C^* \times \boldsymbol{H}(\mathbf{curl};\Omega^I) \\ (\mathbf{v}_C, \mathbf{z}_I) \neq (\mathbf{0}, \mathbf{0})}} \frac{\left| \int_\Gamma \mathrm{div}_\Gamma(\mathbf{v}_C \times \mathbf{n}) \overline{P} + \int_{\Omega^I} \mathbf{z}_I \cdot \mathbf{grad}\, \overline{\psi}_I \right|}{\|\mathbf{v}_C\|_{\mathbf{X}_C^*} + \|\mathbf{z}_I\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)}}$$

$$\geq \frac{\int_\Gamma |P|^2 + \int_{\Omega^I} |\mathbf{grad}\, \psi_I|^2}{C_2 \|P\|_{L^2(\Gamma)} + \|\mathbf{grad}\, \psi_I\|_{L^2(\Omega^I)}} ,$$

and the inf–sup condition follows using the Poincaré inequality in $H^1(\Omega^I)/\mathbb{C}$. □

Remark 5.1. It is worthy to note that the solution $\mathbf{E}_I$ to (35) (considered in $\widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$) or to (41) is not the physical electric field we are looking for. In fact, what we have determined satisfies the interface condition $\mathbf{E}_I \cdot \mathbf{n} = 0$ on Γ, which is not the case for the correct electric field (see (3)). Therefore, it has to be interpreted as a vector potential for the magnetic field $\mathbf{H}_I = i\omega^{-1}\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I$.

In order to be sure that we have really solved the eddy-current problem, we have thus to check that the magnetic field $(\mathbf{H}_C, \mathbf{H}_I)$ satisfies (9).

First of all, choosing $P = \mathrm{div}_\Gamma(\mathbf{H}_C \times \mathbf{n})$ one has at once $\mathrm{div}_\Gamma(\mathbf{H}_C \times \mathbf{n}) = 0$ on Γ. Then, choosing as $\mathbf{v}_C$ a smooth vector function with compact support we find $(9)_1$, the Faraday equation in Ω^C. Moreover, $(9)_3$ is trivial from the definition of $\mathbf{H}_I$. In order to verify $(9)_2$, the Ampère equation in Ω^I, we first have to show that $\mathbf{grad}\,\phi_I = \mathbf{0}$. Taking $\mathbf{z}_I = \mathbf{grad}\,\phi_I$, it follows

$$-\int_{\Omega^I} |\mathbf{grad}\,\phi_I|^2 = \int_{\Omega^I} \mathbf{J}_{e,I} \cdot \mathbf{grad}\,\overline{\phi}_I - \int_\Gamma \mathbf{H}_C \times \mathbf{n} \cdot \mathbf{grad}\,\overline{\phi}_I = 0 ,$$

having taken into account that $\mathrm{div}\,\mathbf{J}_{e,I} = 0$ in Ω^I, $\mathbf{J}_{e,I} \cdot \mathbf{n} = 0$ on $\partial\Omega \cup \Gamma$ and $\mathrm{div}_\Gamma(\mathbf{H}_C \times \mathbf{n}) = 0$ on Γ; therefore $\mathbf{grad}\,\phi_I = \mathbf{0}$ in Ω^I. Taking now as $\mathbf{z}_I$ a smooth vector function with compact support we obtain $(9)_2$, and then a similar choice with $\mathbf{z}_I$ vanishing only in the neighborhood of Γ gives $\mathbf{H}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$.

Concerning the interface conditions, the choice $\mathbf{v}_C = \mathbf{grad}\,\eta_C$, where η_C is an arbitrary function in $H^1(\Omega^C)$ (so that $\mathrm{div}_\Gamma(\mathbf{grad}\,\eta_C \times \mathbf{n}) = \mathbf{curl}\,\mathbf{grad}\,\eta_C \cdot \mathbf{n} = 0$ on Γ), gives easily that $i\omega\boldsymbol{\mu}\mathbf{H}_C \cdot \mathbf{n} + \mathbf{curl}\,\mathbf{E}_I \cdot \mathbf{n} = 0$ on Γ, hence $(9)_6$. On the other hand, choosing $\mathbf{z}_I \in \boldsymbol{H}(\mathbf{curl};\Omega^I)$ and using the Ampère equation in Ω^I and the boundary condition $\mathbf{H}_I \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$ one finds at once $\mathbf{H}_C \times \mathbf{n} - i\omega^{-1}\boldsymbol{\mu}^{-1}\,\mathbf{curl}\,\mathbf{E}_I \times \mathbf{n} = \mathbf{0}$ on Γ, namely, $(9)_7$.

The last condition $(9)_5$ follows by taking $\mathbf{v}_C = \mathbf{R}_l$, an extension of the trace $\boldsymbol{\rho}_l \times \mathbf{n}$ in Ω^C. Clearly, $\mathbf{R}_l \in \mathbf{X}_C^*$, as $\mathrm{div}_\Gamma(\mathbf{R}_l \times \mathbf{n}) = \mathrm{div}_\Gamma(\boldsymbol{\rho}_l \times \mathbf{n}) = \mathbf{curl}\,\boldsymbol{\rho}_l \cdot \mathbf{n} = 0$ on Γ. This choice and the Faraday equation in Ω^C yield

$$\begin{aligned} 0 &= \int_{\Omega^C} [\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C}) \cdot \mathbf{curl}\,\mathbf{R}_l + i\omega\boldsymbol{\mu}\mathbf{H}_C \cdot \mathbf{R}_l] + \int_\Gamma \mathbf{R}_l \times \mathbf{n} \cdot \mathbf{E}_I \\ &= \int_\Gamma [\boldsymbol{\sigma}^{-1}(\mathbf{curl}\,\mathbf{H}_C - \mathbf{J}_{e,C}) - \mathbf{E}_I] \times \mathbf{n} \cdot \mathbf{R}_l , \end{aligned}$$

and therefore $(9)_5$, recalling that $\mathbf{R}_l \times \mathbf{n} = \boldsymbol{\rho}_l \times \mathbf{n}$ on Γ and $\mathbf{curl}\,\mathbf{E}_I = -i\omega\boldsymbol{\mu}\mathbf{H}_I$ in Ω^I. △

5.3 Finite Element Discretization

Let us start by approximating the variational problem (35) in $\widetilde{\mathbf{X}}_C \times \widetilde{\mathbf{Z}}_I$. We introduce the finite element spaces

$$\widetilde{\mathbf{X}}_{C,h} := \{\mathbf{v}_{C,h} \in \mathbf{X}_{C,h} \mid \mathrm{div}_\Gamma(\mathbf{v}_{C,h} \times \mathbf{n}) = 0 \text{ on } \Gamma\} \tag{43}$$

and

$$\widetilde{\mathbf{Z}}_{I,h} := \{\mathbf{z}_{I,h} \in \mathbf{X}_{I,h} \mid \int_{\Omega^I} \mathbf{z}_{I,h} \cdot \mathbf{grad}\, \psi_{I,h} = 0 \;\forall\, \psi_{I,h} \in H_{I,h}\}\,, \tag{44}$$

where $\mathbf{X}_{C,h}$ and $\mathbf{X}_{I,h}$ have been defined in Sect. 4.2 and

$$H_{I,h} := \{\psi_{I,h} \in C^0(\Omega^I) \mid \psi_{I,h|K} \in P_1,\ \forall\, K \in \mathcal{T}_{I,h}\}\,.$$

We consider the following finite element approximation of the hybrid $\mathbf{H}_C/\mathbf{E}_I$ formulation:

$$\begin{cases} \text{Find } (\mathbf{H}_{C,h}, \mathbf{E}_{I,h}) \in \widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h} \ : \\ \int_{\Omega^C} (\boldsymbol{\sigma}^{-1}\, \mathbf{curl}\, \mathbf{H}_{C,h} \cdot \mathbf{curl}\, \overline{\mathbf{v}}_{C,h} + i\omega\boldsymbol{\mu}\mathbf{H}_{C,h} \cdot \overline{\mathbf{v}}_{C,h}) \ + \ \int_\Gamma \overline{\mathbf{v}}_{C,h} \times \mathbf{n} \cdot \mathbf{E}_{I,h} \\ \qquad = f(\mathbf{v}_{C,h}) \\ \int_\Gamma \mathbf{H}_{C,h} \times \mathbf{n} \cdot \overline{\mathbf{z}}_{I,h} \ + \ i\omega^{-1} \int_{\Omega^I} \boldsymbol{\mu}^{-1}\, \mathbf{curl}\, \mathbf{E}_{I,h} \cdot \mathbf{curl}\, \overline{\mathbf{z}}_{I,h} \\ \qquad = g(\mathbf{z}_{I,h}) \\ \text{for all } \mathbf{v}_{C,h} \in \widetilde{\mathbf{X}}_{C,h}, \mathbf{z}_{I,h} \in \widetilde{\mathbf{Z}}_{I,h}\,. \end{cases} \tag{45}$$

Theorem 5.4 *The variational problem* (45) *has a unique solution in* $\widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h}$.

Proof. First we note that the space $\widetilde{\mathbf{Z}}_{I,h}$ can be decomposed similarly to $\widetilde{\mathbf{Z}}_I$ in (38). In fact, we have $\mathbf{grad}\, H_{I,h} \subset \boldsymbol{H}(\mathbf{curl}\, 0; \Omega^I) \cap \mathbf{X}_{I,h}$ (note that in general topology we do not have the equality). Consequently, $[\boldsymbol{H}(\mathbf{curl}\, 0; \Omega^I) \cap \mathbf{X}_{I,h}]^\perp \subset \widetilde{\mathbf{Z}}_{I,h}$. Denoting

$$\boldsymbol{\mathcal{H}}_h^I := \boldsymbol{H}(\mathbf{curl}\, 0; \Omega^I) \cap \widetilde{\mathbf{Z}}_{I,h}$$

(in the discrete case this corresponds to the space of harmonic vector fields $\boldsymbol{\mathcal{H}}^I$, though it is not a subspace of it), we finally have

$$\widetilde{\mathbf{Z}}_{I,h} = [\boldsymbol{H}(\mathbf{curl}\, 0; \Omega^I) \cap \mathbf{X}_{I,h}]^\perp \oplus \boldsymbol{\mathcal{H}}_h^I\,. \tag{46}$$

It is also checked easily that $\dim \boldsymbol{\mathcal{H}}^I = \dim \boldsymbol{\mathcal{H}}_h^I$.

Based on (46) we can rewrite (45) in equivalent form as: seek $\mathbf{H}_{C,h} \in \widetilde{\mathbf{X}}_{C,h}$, $\mathbf{E}_{I,h}^\perp \in [\boldsymbol{H}(\mathbf{curl}\, 0; \Omega^I) \cap \mathbf{X}_{I,h}]^\perp$, $\mathbf{E}_{I,h}^{\mathcal{H}} \in \boldsymbol{\mathcal{H}}_h^I$ such that

$$\begin{aligned} \mathcal{A}((\mathbf{H}_{C,h}, \mathbf{E}_{I,h}^\perp), (\mathbf{v}_{C,h}, \mathbf{z}_{I,h}^\perp)) + \int_\Gamma \overline{\mathbf{v}}_{C,h} \times \mathbf{n} \cdot \mathbf{E}_{I,h}^{\mathcal{H}} &= f(\mathbf{v}_{C,h}) + g(\mathbf{z}_{I,h}^\perp) \\ \int_\Gamma \mathbf{H}_{C,h} \times \mathbf{n} \cdot \overline{\mathbf{z}}_{I,h}^{\mathcal{H}} &= g(\mathbf{z}_{I,h}^{\mathcal{H}}) \end{aligned}$$

for all $\mathbf{v}_{C,h} \in \widetilde{\mathbf{X}}_{C,h}$, $\mathbf{z}_{I,h}^{\perp} \in [\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I) \cap \mathbf{X}_{I,h}]^{\perp}$, and $\mathbf{z}_{I,h}^{\mathcal{H}} \in \boldsymbol{\mathcal{H}}_h^I$.

To prove that $\mathcal{A}(\cdot,\cdot)$ is uniformly coercive on $\widetilde{\mathbf{X}}_{C,h} \times [\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I) \cap \mathbf{X}_{I,h}]^{\perp}$ we can proceed as in the continuous case, since there exists a constant C_2, independent of h, such that

$$\|\mathbf{w}_{I,h}\|_{L^2(\Omega^I)} \leq C_2 \|\mathbf{curl}\,\mathbf{w}_{I,h}\|_{L^2(\Omega^I)} \ \forall\, \mathbf{w}_{I,h} \in [\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I) \cap \mathbf{X}_{I,h}]^{\perp}$$

(see [24, Theorem 4.7] for a proof).

Now we need to check the discrete inf–sup condition

$$\exists \widetilde{\beta} > 0 : \qquad \sup_{\substack{\mathbf{v}_{C,h} \in \widetilde{\mathbf{X}}_{C,h} \\ \mathbf{v}_{C,h} \neq \mathbf{0}}} \frac{\left| \int_\Gamma \mathbf{v}_{C,h} \times \mathbf{n} \cdot \overline{\mathbf{z}}_{I,h}^{\mathcal{H}} \right|}{\|\mathbf{v}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)}} \geq \widetilde{\beta} \left\|\mathbf{z}_{I,h}^{\mathcal{H}}\right\|_{L^2(\Omega^I)} \quad \forall\, \mathbf{z}_{I,h}^{\mathcal{H}} \in \boldsymbol{\mathcal{H}}_h^I \,. \tag{47}$$

Let us assume that the families of triangulations $\mathcal{T}_{C,h}$ and $\mathcal{T}_{I,h}$ are obtained as a refinement of coarse triangulations $\mathcal{T}_{C,h^*}$ and $\mathcal{T}_{I,h^*}$ for a fixed h^*.

For the sake of simplicity, we will assume that Ω^C is a torus. Let us denote by Π_L^*, $L = I, C$, the piecewise linear function taking value 1 at the nodes on one side of Σ^L and 0 at all the other nodes. Then $\widetilde{\mathbf{grad}}\,\Pi_I^*$ belongs to $\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I) \cap \mathbf{X}_{I,h}$, but clearly $\widetilde{\mathbf{grad}}\,\Pi_I^* \notin \mathbf{grad}\,H_{I,h}$. Any function $\mathbf{z}_{I,h}^{\mathcal{H}}$ can be written as $\mathbf{z}_{I,h}^{\mathcal{H}} = c\,\widetilde{\mathbf{grad}}\,\Pi_I^* + \mathbf{grad}\,\psi_{I,h}$ for some $c \in \mathbb{C}$ and $\psi_{I,h} \in H_{I,h}$. Therefore we choose $\mathbf{v}_{C,h} = c\,\widetilde{\mathbf{grad}}\,\Pi_C^* \in \widetilde{\mathbf{X}}_{C,h}$, and we can proceed as in the continuous case. At first, we have to note that for $\mathbf{grad}\,\psi_{I,h}$ the line integral on a closed cycle is always vanishing. Moreover, we have

$$\left\|\mathbf{z}_{I,h}^{\mathcal{H}}\right\|_{L^2(\Omega^I)}^2 = \int_{\Omega^I} \mathbf{z}_{I,h}^{\mathcal{H}} \cdot \overline{\mathbf{z}}_{I,h}^{\mathcal{H}} = \overline{c} \int_{\Omega^I} \mathbf{z}_{I,h}^{\mathcal{H}} \cdot \widetilde{\mathbf{grad}}\,\Pi_I^* \leq |c| \left\|\widetilde{\mathbf{grad}}\,\Pi_I^*\right\|_{L^2(\Omega^I)} \left\|\mathbf{z}_{I,h}^{\mathcal{H}}\right\|_{L^2(\Omega^I)} \,,$$

so that $\left\|\mathbf{z}_{I,h}^{\mathcal{H}}\right\|_{L^2(\Omega^I)} \leq |c| \left\|\widetilde{\mathbf{grad}}\,\Pi_I^*\right\|_{L^2(\Omega^I)}$. Therefore, the proof ends as in the continuous case.

For the Betti number $p > 1$, one arrives at the inf–sup constant

$$\widetilde{\beta} = \min_{\substack{\mathbf{c} \in \mathbb{C}^p \\ \mathbf{c} \neq \mathbf{0}}} \frac{|\mathbf{c}|^2}{(\widetilde{M}^C \mathbf{c} \cdot \overline{\mathbf{c}})^{1/2} (\widetilde{M}^I \mathbf{c} \cdot \overline{\mathbf{c}})^{1/2}} \,,$$

where $\widetilde{M}^L$, $L = I, C$, is the matrix given by $\widetilde{M}^L_{kj} = \int_{\Omega^L} \widetilde{\mathbf{grad}}\,\Pi_{k,L}^* \cdot \widetilde{\mathbf{grad}}\,\Pi_{j,L}^*$, $k, j = 1, \ldots, p$. □

Remark 5.2. From the arguments of Theorem 5.4 we readily derive that for all $(\mathbf{F}_h, \mathbf{G}_h) \in (\widetilde{\mathbf{X}}_{C,h})' \times (\widetilde{\mathbf{Z}}_{I,h})'$ there exists a unique solution of the problem

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{u}_{C,h}, \mathbf{w}_{I,h}) \in \widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h} \text{ such that:} \\[2ex] \mathcal{A}((\mathbf{u}_{C,h}, \mathbf{w}_{I,h}), (\mathbf{v}_{C,h}, \mathbf{z}_{I,h})) = \langle \mathbf{F}_h, \mathbf{v}_{C,h} \rangle + \langle \mathbf{G}_h, \mathbf{z}_{I,h} \rangle \\[2ex] \text{ for all } (\mathbf{v}_{C,h}, \mathbf{z}_{I,h}) \in \widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h} \,. \end{array} \right. \tag{48}$$

Moreover the solution is bounded as follows:

$$\|\mathbf{u}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)} + \|\mathbf{w}_{I,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)} \leq C(\|\mathbf{F}_h\|_{(\widetilde{\mathbf{X}}_{C,h})'} + \|\mathbf{G}_h\|_{(\widetilde{\mathbf{Z}}_{I,h})'}) \,.$$

The constant C depends on the continuity constant of the bilinear form $\mathcal{A}(\cdot,\cdot)$ in $\widetilde{\mathbf{X}}_C \times (\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I))^\perp$, on its coerciveness constant in $\widetilde{\mathbf{X}}_{C,h} \times [\boldsymbol{H}(\mathbf{curl}\,0;\Omega^I) \cap \mathbf{X}_{I,h}]^\perp$, and on the constant $\widetilde{\beta}$ in (47), hence it is independent of h. As a consequence, it is easily shown that there exists a constant α, independent of h, such that

$$\sup_{\substack{(\mathbf{v}_{C,h},\mathbf{z}_{I,h}) \in \widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h} \\ (\mathbf{v}_{C,h},\mathbf{z}_{I,h}) \neq (\mathbf{0},\mathbf{0})}} \frac{\mathcal{A}((\mathbf{u}_{C,h},\mathbf{w}_{I,h}),(\mathbf{v}_{C,h},\mathbf{z}_{I,h}))}{\|\mathbf{v}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)} + \|\mathbf{z}_{I,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)}} \geq \alpha(\|\mathbf{u}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)} + \|\mathbf{w}_{I,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)}) \tag{49}$$

for all $(\mathbf{u}_{C,h},\mathbf{w}_{I,h}) \in \widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h}$. △

For devising a suitable conforming finite element approximation of problem (41) we need another discrete space, namely,

$$Y_{\Gamma,h} := \{P_h \in L^2(\Gamma) \mid P_{h|T} \in P_0 \; \forall T \in \mathcal{T}_{\Gamma,h}\} \,,$$

where $\mathcal{T}_{\Gamma,h}$ is the restriction to Γ of the mesh $\mathcal{T}_{C,h}$.

We consider the problem

$$\begin{cases} \text{Find } (\mathbf{H}_{C,h},\mathbf{E}_{I,h},Q_h,\phi_{I,h}) \in \mathbf{X}_{C,h} \times \mathbf{X}_{I,h} \times Y_{\Gamma,h}/\mathbb{C} \times H_{I,h}/\mathbb{C} \text{ such that:} \\ \mathcal{A}((\mathbf{H}_{C,h},\mathbf{E}_{I,h}),(\mathbf{v}_{C,h},\mathbf{z}_{I,h})) - \int_\Gamma \operatorname{div}_\Gamma(\overline{\mathbf{v}}_{C,h} \times \mathbf{n})\, Q_h - \int_{\Omega^I} \overline{\mathbf{z}}_{I,h} \cdot \mathbf{grad}\, \phi_{I,h} = f(\mathbf{v}_{C,h}) + g(\mathbf{z}_{I,h}) \\ \int_\Gamma \operatorname{div}_\Gamma(\mathbf{H}_{C,h} \times \mathbf{n})\, \overline{P}_h = 0 \\ \int_{\Omega^I} \mathbf{E}_{I,h} \cdot \mathbf{grad}\, \overline{\psi}_{I,h} = 0 \\ \text{for all } \mathbf{v}_{C,h} \in \mathbf{X}_{C,h}, \mathbf{z}_{I,h} \in \mathbf{X}_{I,h}, P_h \in Y_{\Gamma.h}/\mathbb{C}, \psi_{I,h} \in H_{I,h}/\mathbb{C}. \end{cases} \tag{50}$$

Theorem 5.5 *The variational problem* (50) *has a unique solution in* $\mathbf{X}_{C,h} \times \mathbf{X}_{I,h} \times Y_{\Gamma,h}/\mathbb{C} \times H_{I,h}/\mathbb{C}$.

Proof. Since $\operatorname{div}_\Gamma(\mathbf{H}_{C,h} \times \mathbf{n}) \in Y_{\Gamma,h}/\mathbb{C}$ we have that $(\mathbf{H}_{C,h},\mathbf{E}_{I,h}) \in \widetilde{\mathbf{X}}_{C,h} \times \widetilde{\mathbf{Z}}_{I,h}$. Therefore estimate (49) holds, and, as in the proof of Theorem 5.3, we only need to verify the uniform discrete inf–sup condition

$$\exists \beta_* > 0 : \sup_{\substack{(\mathbf{v}_{C,h},\mathbf{z}_{I,h}) \in \mathbf{X}_{C,h} \times \mathbf{X}_{I,h} \\ (\mathbf{v}_{C,h},\mathbf{z}_{I,h}) \neq (\mathbf{0},\mathbf{0})}} \frac{\left| \int_\Gamma \operatorname{div}_\Gamma(\mathbf{v}_{C,h} \times \mathbf{n})\, \overline{P}_h + \int_{\Omega^I} \mathbf{z}_{I,h} \cdot \mathbf{grad}\, \overline{\psi}_{I,h} \right|}{\|\mathbf{v}_{C,h}\|_{\mathbf{X}_C^*} + \|\mathbf{z}_{I,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^I)}} \geq \beta_*(\|P_h\|_{L^2(\Gamma)} + \|\psi_{I,h}\|_{H^1(\Omega^I)}) \quad \forall\, P_h \in Y_{\Gamma,h}/\mathbb{C}, \psi_{I,h} \in H_{I,h}/\mathbb{C}\,. \tag{51}$$

We choose $\mathbf{z}_{I,h} = \mathbf{grad}\,\psi_{I,h}$ and $\mathbf{v}_{C,h}$ such that $\mathrm{div}_\Gamma(\mathbf{v}_{C,h} \times \mathbf{n}) = P_h$. More precisely, let us denote by $\mathbf{R}_h$ the space of tangential traces on Γ of $\mathbf{X}_{C,h}$ (namely, the Raviart-Thomas finite elements on Γ) and by $\mathbf{R}_h^0$ the kernel of the div_Γ operator in $\mathbf{R}_h$. Since $\mathrm{div}_\Gamma \mathbf{R}_h = Y_{\Gamma,h}/\mathbb{C}$, there exists a function $\mathbf{r}_h \in (\mathbf{R}_h^0)^\perp$ such that $\mathrm{div}_\Gamma \mathbf{r}_h = P_h$. We can take for $\mathbf{v}_{C,h}$ a uniformly continuous extension in $\mathbf{X}_{C,h}$ of $\mathbf{r}_h$, so that $\mathbf{v}_{C,h} \times \mathbf{n} = \mathbf{r}_h$ (see [4]). By proceeding as in [15], Theorem 4.2, it can be shown that there exists a positive constant C, independent of h, such that

$$\|\mathbf{s}_h\|_{L^2(\Gamma)} \leq C \, \|\mathrm{div}_\Gamma \mathbf{s}_h\|_{L^2(\Gamma)} \quad \forall\, \mathbf{s}_h \in (\mathbf{R}_h^0)^\perp \,.$$

Hence

$$\begin{aligned} \|\mathbf{v}_{C,h}\|_{\mathbf{X}_C^*} &= \|\mathbf{v}_{C,h}\|_{\boldsymbol{H}(\mathbf{curl};\Omega^C)} + \|\mathrm{div}_\Gamma(\mathbf{v}_{C,h} \times \mathbf{n})\|_{L^2(\Gamma)} \\ &\leq C_1 \, \|\mathbf{r}_h\|_{H^{-\frac{1}{2}}(\Gamma)} + \|P_h\|_{L^2(\Gamma)} \\ &\leq C_2 \, \|P_h\|_{L^2(\Gamma)} \,, \end{aligned}$$

and, by proceeding as in the continuous case, we have (51). □

The convergence of the solution of problem (50) to the solution of problem (41) is a consequence of the standard theory of saddle-point problems (see [12], [13]). The quasi-optimality of the discrete solution is a consequence of (49) and (51).

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