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NONSTANDARD MODELS IN MEASURE THEORY AND IN FUNCTIONAL ANALYSIS

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Foreword

At a first glance, the research carried out during my PhD might seem heterogeneous: the study of an axiomatic approach to measure-theory and the study of generalized solutions to partial differential equations do not share many contact points. However, there are at least two common ideas behind my work in these different fields: on the one hand, there is the study of the mathematical continuum by means of discrete models; on the other hand, there is the attempt at a unifying approach to seemingly different phenomena.

In nonstandard measure theory, elementary numerosities generalize the properties of finite cardinality, a function with a discrete range, to arbitrary sets. Numerosities generalize also measures, as every measure can be obtained from an elementary numerosity. In functional analysis, the grid functions are functions defined on a hyperfinite domain that generalize the distributions and the Young measures. The distributional derivative is represented by finite difference operators, and time-dependent partial differential equations can be coherently formulated as hyperfinite systems of ordinary differential equations. Moreover, it turned out that the applications of grid functions go way beyond the study of partial differential equations, and that the grid functions share some common ground with other theories of generalized functions beyond distributions.

But, to me, there is another idea underlying the work of these years: simplicity. And this idea has two distinct facets. The first one stems from this consideration: sometimes, in the quest for a unifying theory, simplicity is sacrificed for generality. An example is given by the theory of Young measures: they are a very powerful, very general notion of generalized function, but they are rather nontrivial to understand – or, possibly, to get used to. Instead, to my eyes, the numerosities and the grid functions have a certain simplicity: numerosities are characterized by just four axioms that they share with finite cardinalities, and what could be simpler than a function with a finite domain? It might be argued that this apparent simplicity is offset by working with the notoriously "hard" and "esoteric" theory of nonstandard analysis, but this is up to debate.

The second aspect of the simplicity of these approaches to measure theory and to functional analysis is of a very personal nature: by taking these nonstandard routes, I have been able to work on some mathematical theories that I would have found difficult to approach by any other way. Here, perhaps, lies the source of my fascination with nonstandard methods in mathematics.

Trento, November 7, 2016,
Emanuele Bottazzi

Structure of the thesis

This thesis is divided in two parts. The first part, consisting of Chapter 1, is devoted to the study of elementary numerosities. Throughout the chapter, it is proved that elementary numerosities simultaneously generalize the notion of cardinality for finite sets and of non-atomic measures. Moreover, in Theorem 1.3.2 and in Theorem 1.3.3 it is shown that any non-atomic measure can be obtained from an elementary numerosity, and that the coherence between the original measure and the numerosity is rather sharp. The chapter closes with a discussion of several applications of these results in various areas of measure theory and probability. This part of the thesis is based upon joint work with Vieri Benci and Mauro di Nasso [6, 7].

The second part of the thesis, consisting of Chapters 2 through 4, focuses on applications of nonstandard analysis to various areas of functional analysis. Chapter 2 is devoted to the study of grid functions, a space of functions from nonstandard analysis that simultaneously generalizes the space of distributions and the space of Young measures. The results presented in this chapter are of a more theoretical nature, and are mostly geared towards the study of partial differential equations within the framework of grid functions. However, by the end of the chapter, there is also a discussion of some applications of the theory of grid functions to problems from various areas of functional analisys. Chapter 3 consists of the study of a class of ill-posed partial differential equations within the theoretical framework developed through Chapter 2. By working with grid functions, it is defined a notion of generalized solution for the class of ill-posed problems, and it is shown that this generalized solution always exists and it is unique. Moreover, it is proved that the generalized solution is coherent with other standard notions of solutions for the family of ill-posed pdes. The Chapter concludes with a study of the properties of the generalized solutions. Chapter 4 briefly outlines some research perspectives for the theory of grid functions. The second part of the thesis is based upon original research carried out under the supervision of professor Vieri Benci.

Part I

Measure theory

Chapter 1

Elementary numerosities and measures

In mathematics there are essentially two main ways to estimate the size of a set, depending on whether one is working in a discrete or in a continuous setting. In the continuous case, one uses the notion of (finitely) additive measure, namely a real-valued function (possibly taking also the value $+\infty$) which satisfies the following properties:

1. $m(\emptyset) = 0$
2. $m(A) \geq 0$
3. $m(A \cup B) = m(A) + m(B)$ whenever $A \cap B = \emptyset$.

In the discrete case, one uses the notion of cardinality $\mathfrak{n}$ that strengthens the three properties itemized above as follows:

($\mathfrak{n}$.1) $\mathfrak{n}(\emptyset) = 0$

($\mathfrak{n}$.2) $\mathfrak{n}(A) \geq 0$

($\mathfrak{n}$.3) $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B)$ whenever $A \cap B = \emptyset$

($\mathfrak{n}$.4) $\mathfrak{n}(\{x\}) = 1$ for all singletons.

Our goal is to investigate the relationships between these two notions. To this end, we will introduce the concept of *elementary numerosity* as a special function defined on *all* subsets of a given set Ω that takes values in a suitable ordered field $\mathbb{F}$ and satisfies the four properties of finite cardinalities itemized above (see Definition 1.2.1). We remark that if Ω is infinite, then the range of such a function $\mathfrak{n}$ necessarily contains infinite numbers, and hence the field $\mathbb{F}$ must be non-Archimedean. Notice that also Cantorian cardinality satisfies properties $(\mathfrak{n}.1), (\mathfrak{n}.2), (\mathfrak{n}.3), (\mathfrak{n}.4)$, the fundamental difference being that "numerosities" are required to be elements of an ordered field.

The idea of numerosity as a notion of measure for the size of infinite sets was introduced by Benci in [4], and then given sound logical foundations by Benci and Di Nasso in [8]. A theory of numerosities have been then developed in a sequel of papers (see for instance [10, 36]). The main feature of numerosities is that they preserve the spirit of the ancient Euclidean principle that "the whole is larger than the part"; indeed, the numerosity of a proper subset is strictly smaller than the numerosity of the whole set. This has to be contrasted with Cantorian cardinalities, where every infinite set have proper subsets of the same cardinality.

Inspired by the same idea, elementary numerosities refine the notion of finitely additive measure in such a way that also single points count. Elementary numerosities, developed by Benci, Bottazzi and Di Nasso in [6, 7], are functions defined on all subsets of a given set Ω which take values in a suitable non-Archimedean field, and satisfy the same formal properties as finite cardinality. By improving a classic result by C. W. Henson in nonstandard analysis, we prove a general compatibility result between such elementary numerosities and measures. Afterwards, we will present three applications of numerosity in topics of measure theory. The first one is about the existence of "inner measures" associated to any given non-atomic pre-measure. The second application is focused on sets of real numbers. We show that elementary numerosities provide a useful tool with really strong compatibility properties with respect to the Lebesgue measure. For instance, intervals of equal length can be given the same numerosity, and any interval of rational length p/q has a numerosity which is exactly p/q. We derive consequences about the existence of totally defined finitely additive measures that extend the Lebesgue measure. Finally, the third application is about non-Archimedean probability. Following ideas from [11], we consider a model for infinite sequences of coin tosses which is coherent with the original view of Laplace. Indeed, probability of an event is defined as the numerosity of positive outcomes divided by the numerosity of all possible outcomes; moreover, the probability of cylindrical sets exactly coincides with the usual Kolmogorov probability.

We will assume the reader to be familiar with the basics of nonstandard analysis; a classic reference is Davis [33] (see also the more recent book by Goldblatt [44]). For the used terminology of measure theory, we refer to Yeh [90]. A comprehensive exposition of nonstandard measure theory and probability theory is given in Ross [73].

1.1 Terminology and preliminary notions

We fix here our terminology, and recall a few basic facts from measure theory and numerosity theory that will be used in the sequel.

Let us first agree on notation. We write $A \subseteq B$ to mean that A is a subset of B, and we write $A \subset B$ to mean that A is a *proper* subset of B. The *complement* of a set A is denoted by A^c, and its *powerset* is denoted by $\mathcal{P}(A)$. We use the symbol $\sqcup$ to denote *disjoint unions*. By $\mathbb{N}$ we denote the set of *positive integers*. For an ordered field $\mathbb{F}$, we denote by $[0,\infty)_{\mathbb{F}} = \{x \in \mathbb{F} \mid x \geq 0\}$ the set of its non-negative elements. We will write $[0,+\infty]_{\mathbb{R}}$ to denote the set of non-negative real numbers plus the symbol $+\infty$, where we agree that $x + \infty = +\infty + x = +\infty + \infty = +\infty$ for all $x \in \mathbb{R}$. If A is a hyperfinite internal set, by $\| A \|$ we will denote its hyperfinite *internal cardinality*.

Definition 1.1.1. *A* finitely additive measure *is a triple* $(\Omega, \mathfrak{A}, \mu)$ *where:*

- *The* space Ω *is a nonempty set;*
- $\mathfrak{A}$ *is an* algebra of sets *over* Ω*, i.e. a nonempty family of subsets of* Ω *which is closed under finite unions and intersections, and under relative complements, i.e.* $A, B \in \mathfrak{A} \Rightarrow A \cup B, A \cap B, A \setminus B \in \mathfrak{A}$*. (Actually, the closure under intersections follow from the other two properties, since* $A \cap B = A \setminus (A \setminus B)$*.)*
- $\mu : \mathfrak{A} \to [0,+\infty]_{\mathbb{R}}$ *is an* additive function*, i.e.* $\mu(A \cup B) = \mu(A) + \mu(B)$ *whenever* $A, B \in \mathfrak{A}$ *are disjoint. (Such functions* μ *are sometimes called* contents *in the literature.) We also assume that* $\mu(\emptyset) = 0$*.*

The measure $(\Omega, \mathfrak{A}, \mu)$ *is called* non-atomic *when all finite sets in* $\mathfrak{A}$ *have measure zero. We say that* $(\Omega, \mathfrak{A}, \mu)$ *is a* probability measure *when* $\mu : \mathfrak{A} \to [0,1]_{\mathbb{R}}$ *takes values in the unit interval, and* $\mu(\Omega) = 1$*.*

For simplicity, in the following we will often identify the triple $(\Omega, \mathfrak{A}, \mu)$ with the function μ.

Remark that a finitely additive measure μ is necessarily *monotone*, i.e.

- $\mu(A) \leq \mu(B)$ for all $A, B \in \mathfrak{A}$ with $A \subseteq B$.

Definition 1.1.2. *A finitely additive measure* μ *defined on an algebra of sets* $\mathfrak{A}$ *is called a* pre-measure *if it is* σ-additive*, i.e. if for every countable family* $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathfrak{A}$ *of pairwise disjoint sets whose union lies in* $\mathfrak{A}$*, it holds:*

$$\mu\left(\bigsqcup_{n\in\mathbb{N}} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A measure *is a pre-measure which is defined on a* σ-algebra, *i.e. on an algebra of sets which is closed under countable unions and intersections.*

Definition 1.1.3. *An* outer measure *on a set* Ω *is a function*

$$M : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$$

defined on all subsets of Ω *which is* monotone *and* σ-subadditive, *i.e.*

$$M\left(\bigcup_{n\in\mathbb{N}} A_n\right) \;\leq\; \sum_{n\in\mathbb{N}} M(A_n).$$

It is also assumed that $M(\emptyset) = 0$.

Definition 1.1.4. *Given an outer measure* M *on* Ω*, the following family is called the* Caratheodory σ-algebra *associated to* M*:*

$$\mathfrak{C}_M \;=\; \{X \subseteq \Omega \mid M(Y) = M(X \cap Y) + M(X \setminus Y) \textit{ for all } Y \subseteq \Omega\}\,.$$

A well known theorem of Caratheodory states that the above family is indeed a σ-algebra, and that the restriction of M to $\mathfrak{C}_M$ is a *complete* measure, i.e. a measure where $M(X) = 0$ implies $Y \in \mathfrak{C}_M$ for all $Y \subseteq X$. This result is usually combined with the property that every pre-measure μ over an algebra $\mathfrak{A}$ of subsets of Ω is canonically extended to the outer measure $\overline{\mu} : \mathcal{P}(\Omega) \to [0, \infty]_{\mathbb{R}}$ defined by putting:

$$\overline{\mu}(X) \;=\; \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) \,\middle|\, \{A_n\}_n \subseteq \mathfrak{A} \;\&\; X \subseteq \bigcup_{n\in\mathbb{N}} A_n\right\}.$$

Indeed, a fundamental result in measure theory is that the above function $\overline{\mu}$ is actually an outer measure that extends μ, and that the associated Caratheodory σ-algebra $\mathfrak{C}_{\overline{\mu}}$ includes $\mathfrak{A}$. Moreover, such an outer measure $\overline{\mu}$ is *regular*, i.e. for all $X \in \mathcal{P}(\Omega)$ there exists $C \in \mathfrak{C}_{\overline{\mu}}$ such that $X \subseteq C$ and $\overline{\mu}(X) = \overline{\mu}(C)$. (See e.g. [90] Prop. 20.9.)

Next, we will recall the notion of *elementary numerosity*, a variant of the notion of numerosity that was introduced in [6]. The underlying idea is that of refining the notion of finitely additive measure in such a way that also single points count. To this end, one needs to consider ordered fields that extend the real line.

Recall that every ordered field $\mathbb{F}$ that properly extend $\mathbb{R}$ is necessarily *non-Archimedean*, in that it contains *infinitesimal numbers* $\epsilon \neq 0$ such that $-1/n < \epsilon < 1/n$ for all $n \in \mathbb{N}$. Two elements $\xi, \zeta \in \mathbb{F}$ are called *infinitely close* if $\xi - \zeta$ is infinitesimal; in this case, we write $\xi \approx \zeta$. A number $\xi \in \mathbb{F}$ is called *finite* if $-n < \xi < n$ for some $n \in \mathbb{N}$, and it is called *infinite* otherwise. Clearly, a number ξ is infinite if and only if its reciprocal $1/\xi$ is infinitesimal. We remark that every finite $\xi \in \mathbb{F}$ is infinitely close to a unique real number r, namely $r = \inf\{x \in \mathbb{R} \mid x > \xi\}$. Such a number r is called the *standard part* of ξ, and is denoted by $r = {}^\circ\xi$. Notice that ${}^\circ(\xi + \zeta) = {}^\circ\xi + {}^\circ\zeta$ and ${}^\circ(\xi \cdot \zeta) = {}^\circ\xi \cdot {}^\circ\zeta$ for all finite ξ, ζ. The notion of standard part can be extended to the infinite elements $\xi \in \mathbb{F}$ by setting ${}^\circ\xi = +\infty$ when ξ is positive, and ${}^\circ\xi = -\infty$ when ξ is negative.

1.2 Elementary numerosity

We will now define the notion of elementary numerosity.

Definition 1.2.1. *An* elementary numerosity *on the set* Ω *is a function*

$$\mathfrak{n} : \mathcal{P}(\Omega) \longrightarrow [0, +\infty)_{\mathbb{F}}$$

defined on all subsets of Ω*, taking values in an ordered field* $\mathbb{F} \supseteq \mathbb{R}$ *that extends the real line, and that satisfies the following two properties:*

1. Additivity*:* $\mathfrak{n}(A \cup B) = \mathfrak{n}(A) + \mathfrak{n}(B)$ *whenever* $A \cap B = \emptyset$*;*
2. Unit size*:* $\mathfrak{n}(\{x\}) = 1$ *for every point* $x \in \Omega$*.*

Notice that if Ω is a finite set, then the only elementary numerosity is the finite cardinality. On the other hand, when Ω is infinite, then the numerosity function must also take "infinite" values, and so the field $\mathbb{F}$ must be non-Archimedean. It is worth remarking that also Cantorian cardinality satisfies the above properties (1), (2), but the sum operation between cardinals is really far from being a ring operation. (Recall that for infinite cardinals κ, ν it holds $\kappa + \nu = \max\{\kappa, \nu\}$.)

Elementary numerosities satisfy the same basic properties as finite cardinalities. Indeed:

Proposition 1.2.2. *Let* $\mathfrak{n}$ *be an elementary numerosity. Then:*

1. $\mathfrak{n}(A) = 0$ *if and only if* $A = \emptyset$*;*
2. *If* $A \subset B$ *is a proper subset, then* $\mathfrak{n}(A) < \mathfrak{n}(B)$*;*
3. *If* F *is a finite set of cardinality* n*, then* $\mathfrak{n}(F) = n$*.*

Proof. Notice that $\mathfrak{n}(\emptyset) = \mathfrak{n}(\emptyset \cup \emptyset) = \mathfrak{n}(\emptyset) + \mathfrak{n}(\emptyset)$, and $x = 0$ is the only number $x \in \mathbb{F}$ such that $x + x = x$. If $A \subseteq B$ then $\mathfrak{n}(B) = \mathfrak{n}(A) + \mathfrak{n}(B \setminus A) \geq \mathfrak{n}(A)$. Moreover, if $A \subset B$ is a proper subset and $x \in B \setminus A$, then $\mathfrak{n}(B) \geq \mathfrak{n}(A \cup \{x\}) = \mathfrak{n}(A) + \mathfrak{n}(\{x\}) = \mathfrak{n}(A) + 1 > \mathfrak{n}(A)$. In consequence, $\mathfrak{n}(A) > 0$ for all nonempty sets A. Finally, the last property directly follows by additivity and the fact that every singleton has numerosity 1. $\square$

Remark 1.2.3. *If one takes $\mathbb{F} = \mathbb{R}$, then elementary numerosities $\mathfrak{n}$ exist on a set Ω if and only if Ω is finite; and in this case, the only numerosity $\mathfrak{n}$ is given by the finite cardinality.*

We will show in the sequel that by taking suitable non-Archimedean fields that properly extend the real line, elementary numerosities exist on every infinite set.

Given an elementary numerosity and a "measure unit" $\beta \in \mathbb{F}$, there is a canonical way to construct a (real-valued) finitely additive measure.

Definition 1.2.4. *If $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{\mathbb{F}}$ is an elementary numerosity, and $\beta \in \mathbb{F}$ is a positive number, the map $\mathfrak{n}_\beta : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$ is defined by setting*

$$\mathfrak{n}_\beta(A) \;=\; {}^\circ\left(\frac{\mathfrak{n}(A)}{\beta}\right).$$

Proposition 1.2.5. *$\mathfrak{n}_\beta$ is a finitely additive measure. Moreover, $\mathfrak{n}_\beta$ is non-atomic if and only if β is an infinite number.*

Proof. For all disjoint $A, B \subseteq \Omega$, one has:

$$\begin{aligned} \mathfrak{n}_\beta(A \cup B) \;&=\; {}^\circ\left(\frac{\mathfrak{n}(A \cup B)}{\beta}\right) \;=\; {}^\circ\left(\frac{\mathfrak{n}(A)}{\beta} + \frac{\mathfrak{n}(B)}{\beta}\right) \\ &=\; {}^\circ\left(\frac{\mathfrak{n}(A)}{\beta}\right) + {}^\circ\left(\frac{\mathfrak{n}(B)}{\beta}\right) \;=\; \mathfrak{n}_\beta(A) + \mathfrak{n}_\beta(B). \end{aligned}$$

Notice that the measure $\mathfrak{n}_\beta$ is non-atomic if and only if $\mathfrak{n}_\beta(\{x\}) = {}^\circ(1/\beta) = 0$, and this holds if and only if β is infinite. □

The class of measures that we just introduced turns out to be really general. Indeed, we will prove a strong version of the following

- **Claim.** *Every finitely additive non-atomic measure is a restriction of a suitable $\mathfrak{n}_\beta$.*

1.3 The main result

In the 60s and early 70s of the last century, researchers in nonstandard analysis deeply investigated the possibility of representing finitely additive measures as counting measures on suitable hyperfinite samples. Notice that, after the introduction of the *Loeb measure* [59] in 1975, this line of research has been almost abandoned; however, see D.A. Ross' paper [74] for a survey of alternative nonstandard approaches to measure theory. The starting point was the following key observation, pointed out by A.R. Bernstein and F. Wattenberg in [19]:

- *For every nonempty hyperfinite set $F \subseteq {}^*\Omega$,*

$$\mu_F(A) \;=\; {}^\circ\left(\frac{\|{}^*A \cap F\|}{\|F\|}\right)$$

is a finitely additive probability measure defined on all subsets of Ω.

In that paper, the tool of hyperfinite counting measures was used to give a nonstandard proof of the existence of a totally defined translation-invariant extension of the Lebesgue measure on $\mathbb{R}$. In 1972, extending a previous result obtained by A. Robinson [72], C.W. Henson proved the following general representation theorem:

- (Theorem 1 of [47]) *Let Ω be an infinite set and assume that the nonstandard extension $*$ satisfies the property of $\left(2^{|\Omega|}\right)^+$-enlargement. If m is a non-atomic finitely additive probability measure defined on all subsets of Ω, then there exists a nonempty hyperfinite set $F \subseteq {}^*\Omega$ such that $m = \mu_F$.*

Remark 1.3.1. *In any model of nonstandard analysis, every hyperfinite set $F \subseteq {}^*\Omega$ such that ${}^*x \in F$ for all $x \in \Omega$ determines an elementary numerosity $\mathfrak{n}_F : \mathcal{P}(\Omega) \to {}^*\mathbb{R}$ simply by letting:*

$$\mathfrak{n}_F(A) \;=\; \|{}^*A \cap F\|.$$

In consequence, by taking ratios of the elementary numerosity $\mathfrak{n} = \mathfrak{n}_F$ to the fixed "measure unit" $\beta = \|F\| > 0$, the above Henson's Theorem yields the following corollary:

- *For every non-atomic finitely additive probability measure $(\Omega, \mathcal{P}(\Omega), \mu)$ defined on all subsets of a set Ω, there exist an ordered field of hyperreals ${}^*\mathbb{R}$, an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{{}^*\mathbb{R}}$, and a positive number $\beta \in {}^*\mathbb{R}$, such that $\mu = \mathfrak{n}_\beta$.*

The above result shows that elementary numerosities can be found which are compatible with any given non-atomic finitely additive probability measure, *provided* one takes ratios over a suitable measure unit, and identities are taken only up to infinitesimals. Here, we investigate the possibility of a stronger coherence of elementary numerosities with measures. Most notably, a natural requirement would be to have equal numerosity for sets of equal measure.

- *Given a non-atomic finitely additive measure μ defined on an algebra $\mathfrak{A} \subseteq \mathcal{P}(\Omega)$, is there an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_\mathbb{F}$ which is "coherent" with μ, in the following strong sense?*

  1. *There exists a positive β such that $\mathfrak{n}_\beta(A) = \mu(A)$ for all $A \in \mathfrak{A}$;*

2. $\mu(A) = \mu(A') \Longleftrightarrow \mathfrak{n}(A) = \mathfrak{n}(A')$ *for all* $A, A' \in \mathfrak{A}$.

Unfortunately, in the presence of nonempty null sets or of sets of infinite measure, it is readily seen that (2) cannot hold in general. (Recall that proper subsets have a strictly smaller numerosity.) However, these are basically the only obstacles; indeed, for any given measure, we will be able to find elementary numerosities that satisfy (1), and that satisfy also the "strong" coherence property (2) on suitable subalgebras. To this end, we will prove an improvement of Henson's Theorem about nonstandard representation of measures, as given by the Theorem below.

We remark that our proof is grounded on a combinatorial lemma, and uses different arguments with respect to the ones used in the original proofs of the classic results by Bernstein, Wattenberg and Henson.

Since the proof is rather long, it is put off to Section 1.5.

Theorem 1.3.2. *Let* $(\Omega, \mathfrak{A}, \mu)$ *be a non-atomic finitely additive measure on the infinite set* Ω*, and let* $\mathfrak{B} \subseteq \mathfrak{A}$ *be a subalgebra that does not contain nonempty null sets. Then in any model of nonstandard analysis that satisfies the property of* $\left(2^{|\Omega|}\right)^+$*-enlargement there exists a hyperfinite set* $F \subseteq {}^*\Omega$ *such that:*

1. ${}^*x \in F$ *for every* $x \in \Omega$*;*
2. $\|F \cap {}^*B\| = \|F \cap {}^*B'\| \Leftrightarrow \mu(B) = \mu(B')$ *for all* $B, B' \in \mathfrak{B}$ *of finite measure;*
3. *for every hyperreal number of the form* $\beta = \frac{\|F \cap {}^*Z\|}{\mu(Z)}$ *where* $Z \in \mathfrak{A}$ *has positive finite measure, and for every* $A \in \mathfrak{A}$*:*

$$\mu(A) \;=\; {}^\circ\!\left(\frac{\|F \cap {}^*A\|}{\beta}\right).$$

Let us see now the relevant corollary about elementary numerosities.

Theorem 1.3.3. *Let* $(\Omega, \mathfrak{A}, \mu)$ *and* $\mathfrak{B} \subseteq \mathfrak{A}$ *satisfy the hypotheses of Theorem 1.3.2. Then there exists an elementary numerosity* $\mathfrak{n}$ *on* Ω *such that:*

1. $\mu(B) = \mu(B') \Leftrightarrow \mathfrak{n}(B) = \mathfrak{n}(B')$ *for all* $B, B' \in \mathfrak{B}$ *of finite measure;*
2. *For every* $\beta = \mathfrak{n}(Z)/\mu(Z)$ *where* $Z \in \mathfrak{A}$ *has positive finite measure,* $\mu = (\mathfrak{n}_\beta)_{|\mathfrak{A}}$ *is the restriction of* $\mathfrak{n}_\beta$ *to* $\mathfrak{A}$*.*

Proof. Let $F \subseteq {}^*\Omega$ be the hyperfinite set as given by Theorem 1.3.2, and let $\mathfrak{n} = \mathfrak{n}_F$. □

Remark 1.3.4. *We stress the fact that in* (2)*, the measure* $\mathfrak{n}_\beta$ *that represents* μ does not *depend on the choice of* β*, as long as* $\beta = \mathfrak{n}(Z)/\mu(Z)$ *for some* Z *with* $0 < \mu(Z) < +\infty$*. In fact, let* $\beta = \mathfrak{n}(Z)/\mu(Z)$ *and* $\beta' = \mathfrak{n}(Z')/\mu(Z')$ *for some* $Z, Z' \in \mathfrak{A}$ *with* $0 < \mu(Z)$*,* $\mu(Z') < +\infty$*. Then*

$$\circ\left(\frac{\mathfrak{n}(Z')\cdot\mu(Z)}{\mathfrak{n}(Z)\cdot\mu(Z')}\right) = \frac{\mathfrak{n}_\beta(Z')}{\mu(Z')} = 1$$

and, for all $A \subseteq \Omega$*,*

$$\begin{aligned}
\mathfrak{n}_{\beta'}(A) &= \circ\left(\frac{\mathfrak{n}(A)\cdot\mu(Z')}{\mathfrak{n}(Z')}\right)\cdot\circ\left(\frac{\mathfrak{n}(Z')\cdot\mu(Z)}{\mathfrak{n}(Z)\cdot\mu(Z')}\right) \\
&= \circ\left(\frac{\mathfrak{n}(A)\cdot\mu(Z')}{\mathfrak{n}(Z')}\cdot\frac{\mathfrak{n}(Z')\cdot\mu(Z)}{\mathfrak{n}(Z)\cdot\mu(Z')}\right) \\
&= \circ\left(\frac{\mathfrak{n}(A)\cdot\mu(Z)}{\mathfrak{n}(Z)}\right) = \mathfrak{n}_\beta(A)
\end{aligned}$$

In several examples, one naturally finds subalgebras $\mathfrak{B}$ with the property that every nonempty $B \in \mathfrak{B}$ has positive measure. For instance, if one considers the *Lebesgue measure* m on $\mathbb{R}$ then one can take $\mathfrak{B}$ as the algebra of the finite unions of half-open intervals $[a, b)$ where possibly $b = +\infty$, and intervals of the form $(-\infty, b)$ where possibly $b = +\infty$. In this case, the above theorem guarantees the existence of an elementary numerosity $\mathfrak{n}$ defined on all subsets of $\mathbb{R}$ such that $m(A) \approx \mathfrak{n}(A)/\mathfrak{n}([0, 1))$ for all Lebesgue measurable $A \subseteq \mathbb{R}$, and with the strong translation-invariant property that $\mathfrak{n}([a, a + \ell)) = \mathfrak{n}([b, b + \ell))$ for every a, b and for every length $\ell > 0$. This example, along with others, is studied in Benci, Bottazzi and Di Nasso [6].

Remark 1.3.5. *Notice that Theorem 1.3.3 still does not provide a full proof to our claim (made at the end of 1.2) that every finitely additive non-atomic measure* $(\Omega, \mathfrak{A}, \mu)$ *be a restriction of a measure of the form* $\mathfrak{n}_\beta$*. Indeed, if* μ *only takes the values* 0 *and* $+\infty$*, then there are no suitable "measure units"* β*, because there are no sets* $Z \in \mathfrak{A}$ *with positive finite measure. Nevertheless, we remark that even such measures are restrictions of suitable* $\mathfrak{n}_\beta$*. To see this, pick any non-atomic finitely additive* probability *measure* $(\Omega', \mathcal{P}(\Omega'), \mu')$ *where* $\Omega' \cap \Omega = \emptyset$*. Then let*

$$\mathfrak{C} \;=\; \{A \cup B \mid A \in \mathfrak{A}, B \in \mathcal{P}(\Omega')\},$$

and define $\nu : \mathfrak{C} \to [0, +\infty]_{\mathbb{R}}$ *by putting*

$$\nu(C) \;=\; \mu(C \cap \Omega) + \mu'(C \cap \Omega').$$

It is easily verified that ν *is a non-atomic finitely additive measure over* $\Omega \cup \Omega'$*; notice also that* $\nu(\Omega') = \mu'(\Omega') = 1$*. So, Theorem 1.3.3 can be applied to* ν *and we obtain the existence of an elementary numerosity* $\mathfrak{n}$ *and of a number* β *(e.g.,* $\beta = \mathfrak{n}(\Omega')/\nu(\Omega') = \mathfrak{n}(\Omega')$*) such that* $\mathfrak{n}_\beta(C) = \nu(C)$ *for all* $C \in \mathfrak{C}$*. In particular,* $\mathfrak{n}_\beta(A) = \mu(A)$ *for all* $A \in \mathfrak{A}$*, as desired.*

1.4 Some applications of elementary numerosities

We will now discuss some consequences of Theorems 1.3.2 and 1.3.3. In particular, in the proofs of Theorem 1.4.1, Theorem 1.4.3 and Theorem 1.4.5, we will work with elementary numerosities taking values in $\mathbb{F} = {}^*\mathbb{R}$, a field of hyperreal numbers of a model of nonstandard analysis that satisfies the property of $\left(2^{|\Omega|}\right)^+$-enlargement.

1.4.1 Numerosities and inner measures

In this section we will use elementary numerosities to prove a general existence result about "inner" measures.

Theorem 1.4.1. *Let $\mathfrak{A}$ be an algebra of subsets of Ω and let $\mu : \mathfrak{A} \to [0, +\infty]_{\mathbb{R}}$ be a non-atomic pre-measure. Assume that μ is non-trivial, in the sense that there are sets $Z \in A$ with $0 < \mu(Z) < +\infty$. Then, along with the associated outer measure $\overline{\mu}$, there exists an "inner" finitely additive measure*

$$\underline{\mu} : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$$

such that:

1. *$\underline{\mu}(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$, the Caratheodory σ-algebra associated to μ. In particular, $\underline{\mu}(A) = \mu(A) = \overline{\mu}(A)$ for all $A \in \mathfrak{A}$.*

2. *$\underline{\mu}(X) \leq \overline{\mu}(X)$ for all $X \subseteq \Omega$.*

Proof. By *Caratheodory extension theorem*, the restriction of $\overline{\mu}$ to $\mathfrak{C}_\mu$ is a measure that agrees with μ on $\mathfrak{A}$. Now we apply Theorem 1.3.3 to the measure $(\mathfrak{C}_\mu, \mathfrak{A}, \overline{\mu})$, and obtain the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0, +\infty)_{^*\mathbb{R}}$. By property (2) in the Theorem, if we pick any number $\beta = \frac{\mathfrak{n}(Z)}{\mu(Z)}$ where $0 < \mu(Z) < +\infty$, then $\mathfrak{n}_\beta(C) = \overline{\mu}(C)$ for all $C \in \mathfrak{C}_\mu$. We claim that $\underline{\mu} = \mathfrak{n}_\beta : \mathcal{P}(\Omega) \to [0, +\infty]_{\mathbb{R}}$ is the desired "inner" finitely additive measure.

Property (1) is trivially satisfied by our definition of $\underline{\mu}$, so let us turn to (2). For every $X \subseteq \Omega$, by definition of outer measure we have that for every $\epsilon > 0$ there exists a countable union $A = \bigcup_{n=1}^{\infty} A_n$ of sets $A_n \in \mathfrak{A}$ such that $A \supseteq X$ and $\sum_{n=1}^{\infty} \mu(A_n) \leq \overline{\mu}(X) + \epsilon$. Notice that A belongs to the σ-algebra generated by $\mathfrak{A}$, and hence $A \in \mathfrak{C}_\mu$. In consequence, $\underline{\mu}(A) = \mathfrak{n}_\beta(A) = \overline{\mu}(A)$. Finally, by monotonicity of the finitely additive measure $\underline{\mu}$, and by σ-subadditivity of the outer measure $\overline{\mu}$, we obtain:

$$\underline{\mu}(X) \ \leq \ \underline{\mu}(A) \ = \ \overline{\mu}(A) \ \leq \ \sum_{n=1}^{\infty} \overline{\mu}(A_n) \ = \ \sum_{n=1}^{\infty} \mu(A_n) \ \leq \ \overline{\mu}(X) + \epsilon.$$

As $\epsilon > 0$ is arbitrary, the desired inequality $\underline{\mu}(X) \leq \overline{\mu}(X)$ follows. $\square$

It seems of some interest to investigate the properties of the extension of the Caratheodory algebra given by family of all sets for which the outer measure coincides with the above “inner measure”:

$$\mathfrak{C}(\mathfrak{n}_\beta) \;=\; \left\{X \subseteq \Omega \mid \underline{\mu}(X) = \overline{\mu}(X)\right\}.$$

Clearly, the properties of $\mathfrak{C}(\mathfrak{n}_\beta)$ may depend on the choice of the elementary numerosity $\mathfrak{n}$.

Theorem 1.4.1 ensures that the inclusion $\mathfrak{C}_\mu \subseteq \mathfrak{C}(\mathfrak{n}_\beta)$ always holds. Moreover, this inclusion is an equality if and only if every $X \notin \mathfrak{C}_\mu$ satisfies the inequality $\underline{\mu}(X) < \overline{\mu}(X)$. It turns out that, when $\mu(\Omega) < +\infty$, this property is equivalent to a number of other statements.

Proposition 1.4.2. *If $\mu(\Omega) < +\infty$, then the following are equivalent:*

1. $\mathfrak{C}_\mu = \mathfrak{C}(\mathfrak{n}_\beta)$.
2. $X \notin \mathfrak{C}_\mu \Rightarrow \underline{\mu}(X) < \overline{\mu}(X)$ *and* $\underline{\mu}(X^c) < \overline{\mu}(X^c)$.
3. $\underline{\mu}(X) = \overline{\mu}(X) \Longleftrightarrow \underline{\mu}(X^c) = \overline{\mu}(X^c)$.
4. $\underline{\mu}(X) = 0 \Longleftrightarrow \overline{\mu}(X) = 0$.

If $\mu(\Omega) = +\infty$, then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. We have already seen that (1) and (2) are equivalent.

(2) $\Rightarrow$ (3). Suppose towards a contradiction that (2) holds but (3) is false. The latter hypothesis ensures the existence of a set X such that $\underline{\mu}(X) = \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$. Thanks to Theorem 1.4.1, we deduce that $X \notin \mathfrak{C}_\mu$. By (2) we get the contradiction $\underline{\mu}(X) < \overline{\mu}(X)$.

(3) $\Rightarrow$ (4). The implication $\overline{\mu}(X) = 0 \Rightarrow \underline{\mu}(X) = 0$ is always true. On the other hand, if $\underline{\mu}(X) = 0$, then $\underline{\mu}(X^c) = \underline{\mu}(\Omega) = \overline{\mu}(\Omega)$. By the inequality $\underline{\mu}(X^c) \leq \overline{\mu}(X^c)$, we deduce $\overline{\mu}(X^c) = \overline{\mu}(\Omega) = \underline{\mu}(X^c)$ and, thanks to (3), also $\overline{\mu}(X) = 0$ follows.

(4) $\Rightarrow$ (2), under the hypothesis that $\mu(\Omega) < +\infty$. Suppose towards a contradiction that (4) holds but (2) is false. The latter hypothesis ensures the existence of a set $X \notin \mathfrak{C}_\mu$ satisfying $\underline{\mu}(X) = \overline{\mu}(X)$ and $\underline{\mu}(X^c) < \overline{\mu}(X^c)$. Thanks to Propositions 20.9 and 20.11 of [90], we can find a set $A \in \mathfrak{C}_\mu$ satisfying $A \supset X$, $\overline{\mu}(A) = \overline{\mu}(X)$ and $\overline{\mu}(A \setminus X) > 0$. From the hypothesis $\underline{\mu}(X) = \overline{\mu}(X)$ we obtain the following equalities:

$$\underline{\mu}(X) \;=\; \overline{\mu}(X) \;=\; \overline{\mu}(A) \;=\; \underline{\mu}(A).$$

The above equalities and the hypothesis $\mu(\Omega) < +\infty$ imply $\underline{\mu}(A \setminus X) = 0$. By (4), we obtain the contradiction $\overline{\mu}(A \setminus X) = 0$. $\square$

1.4.2 Numerosities and Lebesgue measure

In this section, we show that elementary numerosities exist which are consistent with Lebesgue measure in a strong sense. Precisely, the following result holds:

Theorem 1.4.3. *Let* $(\mathbb{R}, \mathfrak{L}, \mu_L)$ *be the* Lebesgue measure *over* $\mathbb{R}$. *Then there exists an elementary numerosity* $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \to [0, +\infty)_{^*\mathbb{R}}$ *such that:*

1. $\mathfrak{n}([x, x+a)) = \mathfrak{n}([y, y+a))$ *for all* $x, y \in \mathbb{R}$ *and for all* $a > 0$.

2. $\mathfrak{n}([x, x+a)) = a \cdot \mathfrak{n}([0, 1))$ *for all rational numbers* $a > 0$.

3. $^\circ\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right) = \mu_L(X)$ *for all* $X \in \mathfrak{L}$.

4. $^\circ\left(\frac{\mathfrak{n}(X)}{\mathfrak{n}([0,1))}\right) \leq \overline{\mu}_L(X)$ *for all* $X \subseteq \mathbb{R}$.

Proof. Notice that the family of half-open intervals

$$\mathfrak{I} \; = \; \{[x, x+a) \mid x \in \mathbb{R} \;\&\; a > 0\}$$

generates a subalgebra $\mathfrak{B} \subset \mathfrak{L}$ whose nonempty sets have all finite positive measure. Then, by combining Theorems 1.3.3 and 1.4.1, we obtain the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\mathbb{R}) \to [0, +\infty)_{\mathbb{F}}$ such that, for $\beta = \mathfrak{n}([0,1)) = \frac{\mathfrak{n}([0,1))}{\mu_L([0,1))}$, one has:

(*i*) $\mathfrak{n}(X) = \mathfrak{n}(Y)$ for all $X, Y \in \mathfrak{B}$ with $\mu_L(X) = \mu_L(Y)$;

(*ii*) $\mathfrak{n}_\beta(X) = \mu_L(X)$ for all $X \in \mathfrak{L}$;

(*iii*) $\mathfrak{n}_\beta(X) \leq \overline{\mu}_L(X)$ for all $X \subseteq \mathbb{R}$.

Since $[x, x+a) \in \mathfrak{B}$ for all $x \in \mathbb{R}$ and for all $a > 0$, property (1) directly follows from (*i*). In order to prove (2), it is enough to show that $\mathfrak{n}([0,a)) = a \cdot \mathfrak{n}([0,1))$ for all positive $a \in \mathbb{Q}$. Given $p, q \in \mathbb{N}$, by (1) and additivity we have that

$$\mathfrak{n}\left(\left[0, \frac{p}{q}\right)\right) \; = \; \mathfrak{n}\left(\bigsqcup_{i=0}^{p-1}\left[\frac{i}{q}, \frac{i+1}{q}\right)\right) \; = \; \sum_{i=0}^{p-1}\mathfrak{n}\left(\left[\frac{i}{q}, \frac{i+1}{q}\right)\right) \; = \; p \cdot \mathfrak{n}\left(\left[0, \frac{1}{q}\right)\right).$$

In particular, for $p = q$ we get that $\mathfrak{n}([0,1)) = q \cdot \mathfrak{n}([0, 1/q))$, and hence property (2) follows:

$$\mathfrak{n}\left(\left[0, \frac{p}{q}\right)\right) \; = \; \frac{p}{q} \cdot \mathfrak{n}([0,1)).$$

Finally, (*ii*) and (*iii*) directly correspond to properties (3) and (4), respectively. $\square$

Remark 1.4.4. *Let* $\{X_n \mid n \in \mathbb{N}\}$ *be a countable family of isometric, pairwise disjoint, non-Lebesgue measurable sets such that the union* $A = \bigcup_{n\in\mathbb{N}} X_n$ *is measurable with positive finite measure. (For instance, one can consider a* Vitali set *on* $[0,1)$ *and take the countable family of its rational translations modulo 1.) Let* $\mathfrak{n}$ *be an elementary numerosity as given by the above theorem, and consider the finitely additive measure* $\mathfrak{n}_\beta$ *with* $\beta = \mathfrak{n}(A)/\mu(A)$*. Then, one and only one of the following holds:*

- $\mathfrak{n}_\beta(X_n) = 0$ *for all* $n \in \mathbb{N}$*. In this case, the measure* $\mathfrak{n}_\beta$ *is not* σ*-additive because* $\mathfrak{n}_\beta(A) = \mu_L(A) > 0$*.*
- $\mathfrak{n}_\beta(X_n) = \epsilon > 0$ *for some* $n \in \mathbb{N}$*. In this case,* $\mathfrak{n}_\beta$ *is not invariant with respect to isometries, as otherwise one would get the contradiction* $\mu_L(A) = \mathfrak{n}_\beta(A) \geq \sum_{n\in\mathbb{N}} \mathfrak{n}_\beta(X_n) = \sum_{n\in\mathbb{N}} \epsilon = +\infty$*.*

1.4.3 Numerosities and probability of infinite coin tosses

The last application of elementary numerosities that we present is about the existence of a non-Archimedean probability for infinite sequences of coin tosses, which we propose as a sound mathematical model for Laplace's original ideas.

Recall the *Kolmogorovian framework*:

- The *sample space*

$$\Omega \;=\; \{H,T\}^{\mathbb{N}} \;=\; \{\omega \mid \omega : \mathbb{N} \to \{H,T\}\}$$

  is the set of sequences which take either H ("head") or T ("tail") as values.

- A *cylinder set* of codimension n is a set of the following form, where we agree that $i_1 < \ldots < i_n$.

$$C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)} \;=\; \{\omega \in \Omega \mid \omega(i_s) = t_s \;\text{ for } s = 1,\ldots,n\}$$

From the probabilistic point of view, the cylinder set $C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}$ represents the event that for every $s = 1,\ldots,n$, the i_s-th coin toss gives t_s as outcome. Notice that the family $\mathfrak{C}$ of all finite disjoint unions of cylinder sets is an algebra of sets over Ω.

- The function $\mu_C : \mathfrak{C} \to [0,1]$ is defined by setting:

$$\mu_C\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}\right) \;=\; 2^{-n}$$

  for all cylindrical sets, and then it is extended to a generic element of $\mathfrak{C}$ by finite additivity:

$$\mu_C\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)} \cup \ldots \cup C^{(j_1,\ldots,i_m)}_{(u_1,\ldots,u_m)}\right) = \mu_C\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}\right) + \ldots + \mu_C\left(C^{(j_1,\ldots,i_m)}_{(u_1,\ldots,u_m)}\right).$$

It is shown that μ_C is a probability pre-measure on the algebra $\mathfrak{C}$.

Let $\mathfrak{A}$ be the σ-algebra generated by the algebra of cylinder sets $\mathfrak{C}$, and let $\mu : \mathfrak{A} \to [0,1]$ be the unique probability measure that extends μ_C, as guaranteed by *Caratheodory extension theorem.*

The triple $(\Omega, \mathfrak{A}, \mu)$ is named the *Kolmogorovian probability* for *infinite sequences of coin tosses.*

In [11] it is proved the existence of an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{\mathbb{F}}$ which is coherent with the pre-measure μ_C. Namely, by considering the ratio $P(E) = \mathfrak{n}(E)/\mathfrak{n}(\Omega)$ between the numerosity of the given event E and the numerosity of the whole space Ω, then one obtains a *non-Archimedean* finitely additive probability

$$P : \mathcal{P}(\Omega) \longrightarrow [0,1]_{\mathbb{F}}$$

that satisfies the following properties:

1. If $F \subset \Omega$ is finite, then for all $E \subseteq \Omega$, the conditional probability
$$P(E|F) \;=\; \frac{|E \cap F|}{|F|}.$$

2. P agrees with μ_C over all cylindrical sets:
$$P\left(C^{(i_1,\dots,i_n)}_{(t_1,\dots,t_n)}\right) \;=\; \mu_C\left(C^{(i_1,\dots,i_n)}_{(t_1,\dots,t_n)}\right) \;=\; 2^{-n}.$$

We are now able to refine this result by showing that, up to infinitesimals, we can take P to agree with μ on the whole σ-algebra $\mathfrak{A}$.

Theorem 1.4.5. *Let* $(\Omega, \mathfrak{A}, \mu)$ *be the Kolmogorovian probability for infinite coin tosses. Then there exists an elementary numerosity* $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{^*\mathbb{R}}$ *such that the corresponding non-Archimedean probability* $P(E) = \mathfrak{n}(E)/\mathfrak{n}(\Omega)$ *satisfies the above properties* (1) *and* (2)*, along with the additional condition:*

(3) $^\circ(P(E)) = \mu(E)$ *for all* $E \in \mathfrak{A}$.

Proof. Recall that the family $\mathfrak{C} \subset \mathfrak{A}$ of finite disjoint unions of cylinder sets is an algebra whose nonempty sets have all positive measure. So, by applying Theorems 1.3.3 and 1.4.1, we obtain an elementary numerosity $\mathfrak{n} : \mathcal{P}(\Omega) \to [0,+\infty)_{\mathbb{F}}$ such that for every positive number of the form $\beta = \frac{\mathfrak{n}(Z)}{\mu(Z)}$ (where $0 < \mu(Z) < +\infty$), one has:

(i) $\mathfrak{n}(C) = \mathfrak{n}(C')$ whenever $C, C' \in \mathfrak{C}$ are such that $\mu(C) = \mu(C')$;

(*ii*) $\mathfrak{n}_\beta(E) = \mu(E)$ for all $E \in \mathfrak{A}$.

Property (1) trivially follows by recalling that elementary numerosities of finite sets agree with cardinality:

$$P(E|F) \;=\; \frac{P(E\cap F)}{P(F)} \;=\; \frac{\frac{\mathfrak{n}(E\cap F)}{\mathfrak{n}(\Omega)}}{\frac{\mathfrak{n}(F)}{\mathfrak{n}(\Omega)}} \;=\; \frac{\mathfrak{n}(E\cap F)}{\mathfrak{n}(F)} \;=\; \frac{|E\cap F|}{|F|}.$$

Let us now turn to condition (2). Notice that for any fixed n-tuple of indices $(i_1,\ldots,i_n)$:

- There are exactly 2^n-many different n-tuples $(t_1,\ldots,t_n)$ of heads and tails;
- The associated cylinder sets $C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}$ are pairwise disjoint and their union equals the whole sample space Ω.

By (*i*), all those cylinder sets of codimension n have the same numerosity $\eta = \mathfrak{n}\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_n)}\right)$ and so, by additivity, it must be $\mathfrak{n}(\Omega) = 2^n\cdot\eta$. We conclude that

$$P\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_k)}\right) \;=\; \frac{\mathfrak{n}\left(C^{(i_1,\ldots,i_n)}_{(t_1,\ldots,t_k)}\right)}{\mathfrak{n}(\Omega)} \;=\; \frac{\eta}{2^n\cdot\eta} \;=\; 2^{-n}.$$

We are left to prove (3). By taking as $\beta = \frac{\mathfrak{n}(\Omega)}{\mu(\Omega)} = \mathfrak{n}(\Omega)$, property (*ii*) ensures that for every $E \in \mathfrak{A}$:

$$\mu(E) \;=\; \mathfrak{n}_\beta(E) \;=\; {}^\circ\!\left(\frac{\mathfrak{n}(E)}{\beta}\right) \;=\; {}^\circ\!\left(\frac{\mathfrak{n}(E)}{\mathfrak{n}(\Omega)}\right) \;=\; {}^\circ(P(E)).$$

□

1.5 Proof of the Main Theorem 1.3.2

The proof of the main theorem is grounded on the following combinatorial property.

Lemma 1.5.1. *Let $(\Omega,\mathfrak{A},\mu)$ be a non-atomic finitely additive measure, and let $\mathfrak{B} \subseteq \mathfrak{A}$ be a subalgebra of subsets of Ω whose nonempty sets have all positive measure. Denote by $\mathfrak{A}_f$ (by $\mathfrak{B}_f$) the family of sets in $\mathfrak{A}$ (in $\mathfrak{B}$, respectively) which have finite measure. Given $m \in \mathbb{N}$, finitely many points $x_1,\ldots,x_k \in \Omega$, and finitely many nonempty sets $A_1,\ldots,A_v \in \mathfrak{A}$, there exists a finite subset $\lambda \subset \Omega$ that satisfies the following properties:*

1. $x_1,\ldots,x_k \in \lambda$;

2. If $A_i, A_j \in \mathfrak{B}_f$ satisfy $\mu(A_i) = \mu(A_j)$, then $|\lambda \cap A_i| = |\lambda \cap A_j|$;

3. If $A_i \in \mathfrak{A}_f$ and $\mu(A_i) \neq 0$, then for all j such that $A_j \in \mathfrak{A}_f$:

$$\left| \frac{|\lambda \cap A_j|}{|\lambda \cap A_i|} - \frac{\mu(A_j)}{\mu(A_i)} \right| \; < \; \frac{1}{m};$$

4. If $A_i \in \mathfrak{A}_f$ and $\mu(A_i) \neq 0$, then for all j such that $A_j \in \mathfrak{A} \setminus \mathfrak{A}_f$:

$$\frac{|\lambda \cap A_j|}{|\lambda \cap A_i|} > m.$$

Proof. Without loss of generality, we can assume that the given sets A_i are arranged in such a way that $A_1, \ldots, A_l \in \mathfrak{B}_f$, $A_{l+1}, \ldots, A_n \in \mathfrak{A}_f \setminus \mathfrak{B}_f$ and $A_{n+1}, \ldots, A_v \in \mathfrak{A} \setminus \mathfrak{A}_f$ for suitable l and n. It will be convenient in the sequel that the considered elements in $\mathfrak{B}_f$ be pairwise disjoint. To this end, consider the partition $\{B_1, \ldots, B_h\}$ induced by $\{A_1, \ldots, A_l\}$.[1] Notice that, by the algebra properties of $\mathfrak{B}$, every piece B_s belongs to $\mathfrak{B}_f$. Finally, let

$$\bigcup_{i=1}^{n} A_i \;=\; C_1 \sqcup \ldots \sqcup C_p \sqcup D_1 \sqcup \ldots \sqcup D_q$$

be the partition of $\bigcup_{i=1}^{n} A_i$ induced by $\{B_1, \ldots, B_h, A_{l+1}, \ldots, A_v\}$, where $0 < \mu(C_s) < +\infty$ for $s = 1, \ldots, p$ and $\mu(D_t) = 0$ for $t = 1, \ldots, q$. (The union $\bigcup_{i=1}^{n} A_i$ has finite measure because it is a finite union of sets of finite measure.) For every $s = 1, \ldots, h$, the set B_s includes at least one piece C_j. Moreover, since $B_1, \ldots, B_h$ are pairwise disjoint, by re-arranging if necessary, we can also assume that $C_s \subseteq B_s$ for $s = 1, \ldots, h$.

We now need the following result:

- *"Given finitely many real numbers $y_s > 0$, for every $\varepsilon > 0$ there exists an arbitrarily large $N \in \mathbb{N}$ such that every fractional part $\{N \cdot y_s\} = N \cdot y_s - [N \cdot y_s] < \varepsilon$".*

Recall that by *Dirichlet's simultaneous approximation theorem* (see e.g. Hardy and Wright [46, S 11.12]), the above property holds if at least one of the y_s is irrational. On the other hand, when all $y_s \in \mathbb{Q}$, if N is any multiple of the greatest common denominator of the numbers y_s, then trivially all fractional parts $\{N \cdot y_s\} = 0$.

Let

- $\alpha = \mu\left(\bigcup_{i=1}^{n} A_i\right)$;

[1] Recall that the *partition induced* by a finite family $\{A_1, \ldots, A_n\}$ is the partition on $A_1 \cup \ldots \cup A_n$ given by the nonempty intersections $\bigcap_{i=1}^{n} A_i^{\chi(i)}$ for $\chi : \{1, \ldots, n\} \to \{-1, 1\}$, where $A_j^1 = A_j$ and $A_j^{-1} = \left(\bigcup_{i=1}^{n} A_i\right) \setminus A_j$.

- $c = \min\{\mu(C_s) \mid s = 1, \dots, p\}$.

By the above property we can pick a natural number N such that:

(a) $N > \frac{\alpha\,(2m+1)\,(k+1)}{c^2}$;

(b) $e_s = \{N \cdot \mu(C_s)\} < \frac{1}{p}$ for all $s = 1, \dots, p$.

Denote by

- $C = \bigsqcup_{s=1}^{p} C_s$ the "relevant part" of the partition;
- $D = \bigsqcup_{t=1}^{q} D_t$ the "negligible part" of the partition;
- $F = \{x_1, \dots, x_k\}$.

Then, set

- $N_s = [N \cdot \mu(C_s)]$ for $s = 1, \dots, p$;
- $M_s = |B_s \cap D \cap F|$ for $s = 1, \dots, h$.

Notice that $N_s > k$ for all s. In fact, by the above conditions (a) and (b):

$$\begin{aligned} N_s \;&=\; N \cdot \mu(C_s) - e_s \;>\; \frac{\alpha\,(2m+1)\,(k+1)}{c^2} \cdot \mu(C_s) - e_s \;> \\ &>\; \frac{\alpha \cdot \mu(C_s)}{c^2} \cdot (k+1) - e_s \;>\; 1 \cdot (k+1) - 1 \;=\; k. \end{aligned}$$

For $s = 1, \dots, h$, pick a finite subset $\lambda_s \subset C_s$ containing exactly $(N_s - M_s)$-many elements, and such that $C_s \cap F \subseteq \lambda_s$. Observe that this is possible because

$$|C_s \cap F| \;\le\; |B_s \cap C \cap F| \;=\; |B_s \cap F| - M_s \;\le\; k - M_s \;<\; N_s - M_s.$$

For $s = h+1, \dots, p$, pick a finite subset $\lambda_s \subset C_s$ containing exactly N_s-many elements. Finally, define

$$\lambda \;=\; F \cup \bigcup_{s=1}^{p} \lambda_s.$$

We claim that λ satisfies the desired properties (1), (2), (3). Since $F \subseteq \lambda$, condition (1) is trivially satisfied. For every $i = 1, \dots, n$ let:

$$G(i) \;=\; \{s \le h \mid C_s \subseteq A_i\} \quad \text{and} \quad G'(i) \;=\; \{s > h \mid C_s \subseteq A_i\}.$$

With the above definitions, we obtain:

$$\begin{aligned}
|\lambda\cap A_i| &= \sum_{s\in G(i)}|\lambda_s| + \sum_{s\in G'(i)}|\lambda_s| \;+\; |A_i\cap D\cap F| \\
&= \sum_{s\in G(i)}(N_s - M_s) \;+\; \sum_{s\in G'(i)} N_s \;+\; |A_i\cap D\cap F| \\
&= \sum_{s\in G(i)\cup G'(i)} N_s \;-\; \sum_{s\in G(i)} M_s \;+\; |A_i\cap D\cap F| \\
&= N\cdot\left(\sum_{s\in G(i)\cup G'(i)}\mu(C_s)\right) - \varepsilon_i - \eta_i + \vartheta_i \\
&= N\cdot\mu(A_i) - \varepsilon_i - \eta_i + \vartheta_i
\end{aligned}$$

where:

- $\varepsilon_i = \sum_{s\in G(i)\cup G'(i)} e_s \;\le\; \sum_{s=1}^{p} e_s \;<\; 1$ by condition (b);
- $\eta_i = \sum_{s\in G(i)} M_s \;\le\; \sum_{s=1}^{h}|B_s\cap D\cap F| \;\le\; |F| = k$;
- $\vartheta_i = |A_i\cap D\cap F| \;\le\; k$.

If $A_i \in \mathfrak{B}_f$, i.e. if $i \le l$, recall that $A_i = \bigsqcup_{s\in S(i)} B_s$ for a suitable $S(i)\subseteq\{1,\dots,h\}$. Since $C_s\subseteq B_s$ for all $s=1,\dots,h$, it must be $G(i)=S(i)$. So, for $i\le l$ we have

$$\begin{aligned}
\eta_i &= \sum_{s\in S(i)} M_s \;=\; \sum_{s\in S(i)}|B_s\cap D\cap F| \;=\; \left|\left(\bigsqcup_{s\in S(i)} B_s\right)\cap D\cap F\right| \\
&= |A_i\cap D\cap F| \;=\; \vartheta_i,
\end{aligned}$$

and hence $|\lambda\cap A_i| = N\cdot\mu(A_i)-\varepsilon_i$. In consequence, for every $i,j\le l$ such that $\mu(A_i)=\mu(A_j)$, one has that

$$\big||\lambda\cap A_i| - |\lambda\cap A_j|\big| \;=\; |N\cdot\mu(A_i)-\varepsilon_i - N\cdot\mu(A_j)+\varepsilon_j| \;=\; |\varepsilon_j-\varepsilon_i|.$$

Now notice that $|\varepsilon_j-\varepsilon_i|\le\max\{\varepsilon_i,\varepsilon_j\}<1$, and so the natural numbers $|\lambda\cap A_i| = |\lambda\cap A_j|$ necessarily coincide. This completes the proof of (2).

As for (3), notice that $|\lambda\cap A_i| = N\cdot\mu(A_i)+\zeta_i$ where $\zeta_i = (\vartheta_i-\eta_i)-\varepsilon_i$ is such that $-(k+1)<\zeta_i\le k$. For every i,j such that $\mu(A_j)\neq 0$, we have that

$$\frac{N\cdot\mu(A_i)+\zeta_i}{N\cdot\mu(A_j)+\zeta_j} - \frac{\mu(A_i)}{\mu(A_j)} \;=\; \frac{\mu(A_j)\cdot\zeta_i - \mu(A_i)\cdot\zeta_j}{N\cdot\mu(A_j)^2+\mu(A_j)\cdot\zeta_j}.$$

Now, the absolute value of the numerator

$$|\mu(A_j)\cdot\zeta_i-\mu(A_i)\cdot\zeta_j| \;<\; (\mu(A_i)+\mu(A_j))\cdot(k+1) \;\le\; 2\,\alpha\,(k+1);$$

and the absolute value of the denominator

$$\begin{aligned}|N\cdot\mu(A_j)^2+\mu(A_j)\cdot\zeta_j| \;&>\; N\,c^2-\alpha\,(k+1)\\ &\geq\; \alpha\,(2m+1)\,(k+1)-\alpha\,(k+1) \;=\; 2m\,\alpha\,(k+1).\end{aligned}$$

So, we reach the thesis:

$$\left|\frac{|\lambda\cap A_i|}{|\lambda\cap A_j|}-\frac{\mu(A_i)}{\mu(A_j)}\right| \;<\; \frac{2\,\alpha\,(k+1)}{2m\,\alpha\,(k+1)} \;=\; \frac{1}{m}.$$

In order to get a finite subset λ' satisfying also property (4), for $t = n+1,\ldots,v$ pick a finite subset $\eta_t\subset A_t\setminus\bigcup_{i=1}^n A_i$ with $|\eta_t|>m\cdot|\lambda|$, and set $\lambda' = \lambda\cup\bigcup_{t=n+1}^v \eta_t$. It is clear that such a λ' still satisfies properties (1), (2), (3), and it is readily checked that it also satisfies property (4). $\square$

Proof of 1.3.2. Let $\Lambda = \mathrm{Fin}(\Omega)$ be the family of all finite subsets of Ω, and define the following subsets of Λ:

- For all $x\in\Omega$, let
$$\widehat{x} \;=\; \{\lambda\in\Lambda : x\in\lambda\}\,.$$
- For all $A,A'\in\mathfrak{A}_f$ with $\mu(A')>0$ and for all $n\in\mathbb{N}$, let
$$\Gamma(A,A',n) \;=\; \left\{\lambda\in\Lambda \;:\; \lambda\cap A'\neq\emptyset \text{ and } \left|\frac{|\lambda\cap A|}{|\lambda\cap A'|}-\frac{\mu(A)}{\mu(A')}\right| \;<\; \frac{1}{n}\right\}.$$
- For all nonempty $B,B'\in\mathfrak{B}_f$, let
$$\Theta(B,B') \;=\; \big\{\lambda\in\Lambda \;:\; |B\cap\lambda| = |B'\cap\lambda|\big\}\,.$$
- For all $C\in\mathfrak{A}\setminus\mathfrak{A}_f$, for all $C'\in\mathfrak{A}_f$ with $\mu(C')>0$, and for all $n\in\mathbb{N}$, let
$$\Xi(C,C',n) \;=\; \left\{\lambda\in\Lambda \;:\; \lambda\cap C'\neq\emptyset \text{ and } \frac{|\lambda\cap C|}{|\lambda\cap C'|}>n\right\}.$$

Then consider the following family:

$$\begin{aligned}\mathcal{G} \;=\; &\{\widehat{x}\mid x\in\Omega\}\bigcup\big\{\Gamma(A,A',n)\mid A,A'\in\mathfrak{A}_f,\ \mu(A')>0,\ n\in\mathbb{N}\big\}\\ &\bigcup\big\{\Theta(B,B')\mid B,B'\in\mathfrak{B}_f,\ \mu(B)=\mu(B')>0\big\}\\ &\bigcup\big\{\Xi(C,C',n)\mid C\in\mathfrak{A}\setminus\mathfrak{A}_f,\ C'\in\mathfrak{A}_f,\ \mu(C')>0,\ n\in\mathbb{N}\big\}\,.\end{aligned}$$

As a consequence of the Lemma, the family $\mathcal{G}$ has the *finite intersection property*. Indeed, let finitely many elements of $\mathcal{G}$ be given, say

$$\widehat{x}_1,\ldots,\widehat{x}_k;\ \Gamma(A_1,A'_1,n_1),\ldots,\Gamma(A_u,A'_u,n_u);$$
$$\Theta(B_1,B'_1),\ldots,\Theta(B_w,B'_w);\ \Xi(C_1,C'_1,m_1),\ldots,\Xi(C_s,C'_s,m_s)\,.$$

Without loss of generality, we can assume that the set $\{x_1, \dots, x_k\}$ includes at least one point from each of the above sets A'_i and C'_j. Pick $m = \max\{n_1, \dots, n_u, m_1, \dots, m_s\}$ and apply Lemma 1.5.1 to get the existence of a finite set $\lambda \in \Lambda$ such that:

1. $x_1, \dots, x_k \in \lambda$;
2. For all $i = 1, \dots, w$, if $\mu(B_i) = \mu(B'_i)$ then $|\lambda \cap B_i| = |\lambda \cap B'_i|$;
3. For all $i, j = 1, \dots, u$, then
$$\left| \frac{|\lambda \cap A_j|}{|\lambda \cap A'_i|} - \frac{\mu(A_j)}{\mu(A'_i)} \right| < \frac{1}{m};$$
4. for all $i, j = 1, \dots, s$:
$$\frac{|\lambda \cap C_i|}{|\lambda \cap C'_j|} > m.$$

Then it is readily verified that such a λ belongs to all considered sets of $\mathcal{G}$. Since $|\mathcal{G}| \leq 2^{|\Omega|}$, we can apply the *enlarging property* and pick a set

$$F \in \bigcap_{G \in \mathcal{G}} {}^*G.$$

Now notice that:

(a) For every $x \in \Omega$, $F \in {}^*\widehat{x}$ means that ${}^*x \in F$;

(b) For every $A, A' \in \mathfrak{A}_f$ with $\mu(A') > 0$, $F \in {}^*\Gamma(A, A', n)$ for every $n \in \mathbb{N}$ means that
$$\frac{\|F \cap {}^*A\|}{\|F \cap {}^*A'\|} \approx \frac{\mu(A)}{\mu(A')};$$

(c) For every $B, B' \in \mathfrak{B}_f$ with $\mu(B) = \mu(B') > 0$, $F \in {}^*\Theta(B, B')$ means that $\|F \cap {}^*B\| = \|F \cap {}^*B'\|$;

(d) For every $C, C' \in \mathfrak{A}$ with $\mu(C) = +\infty$ and $0 < \mu(C') < +\infty$, $F \in {}^*\Xi(C, C', n)$ for every $n \in \mathbb{N}$ means that
$$\frac{\|F \cap {}^*C\|}{\|F \cap {}^*C'\|} \text{ is infinite.}$$

Properties (1) and (2) of the thesis are directly given by (a) and (c), respectively. As for (3), let $\beta = \|F \cap {}^*Z\|/\mu(Z)$ where $Z \in \mathfrak{A}$ is such that $0 < \mu(Z) < +\infty$. If $\mu(A) < +\infty$, by property (b) where $A' = Z$, we get

$$\frac{\|F \cap {}^*A\|}{\beta} = \frac{\|F \cap {}^*A\|}{\|F \cap {}^*Z\|} \cdot \mu(Z) \approx \frac{\mu(A)}{\mu(Z)} \cdot \mu(Z) = \mu(A);$$

and if $\mu(A) = +\infty$, by property (d) where $C = A$ and $C' = Z$, we have that the following number is infinite:

$$\frac{\|F \cap {}^*A\|}{\beta} \;=\; \frac{\|F \cap {}^*A\|}{\|F \cap {}^*Z\|} \cdot \mu(Z).$$

$\square$

Part II

Functional analysis

Chapter 2

Grid functions in functional analysis

The theory of distributions, pioneered by Dirac in [37] and developed in the first half of the XX Century, has become one of the fundamental tools of functional analysis. In particular, the possibility to define the weak derivative of a non-differentiable function has allowed the formulation and the study of a wide variety of nonsmooth phenomena by the theory of partial differential equations. However, the lack of a nonlinear theory of distributions is a limiting factor both for the applications and for the theoretical study of nonlinear PDEs. On the one hand, in the description of some physical phenomena such as shock waves and relativistic fields, it arises the need to have some mathematical objects which cannot be formalized in the sense of distributions (we refer to [26] for some examples). On the other hand, the absence of a nonlinear theory of distributions poses some limitations in the study of nonlinear partial differential equations: while some nonlinear problems can be solved by studying the limit of suitable regularized problems, other problems do not allow for solutions in the sense of distributions (see for instance the discussion in [39]).

In 1954, L. Schwartz proved that the absence of a nonlinear theory for distributions is intrinsic: more formally, the main theorem of [77] entails that there is no differential algebra $(A, +, \otimes, D)$ in which the real distributions $\mathcal{D}'$ can be embedded and the following conditions are satisfied:

1. $\otimes$ extends the product over C^0 functions;
2. D extends the distributional derivative ∂;
3. the product rule holds: $D(u \otimes v) = (Du) \otimes v + u \otimes (Dv)$.

Despite this negative result, there have been many attempts at defining some notions of product between distributions (see for instance [27, 57]).

Following this line of research, Colombeau in 1983 proposed an organic approach to a theory of generalized functions [25]: Colombeau's idea is to embed the distributions in a differential algebra with a good nonlinear theory, but at the cost of sacrificing the coherence between the product of the differential algebra with the product over C^0 functions. This approach has been met with interest and has proved to be a prolific field of research. For a survey of the approach by Colombeau and for recent advances, we refer to [26].

Research about generalized functions beyond distributions is also being carried out within the setting of nonstandard analysis. Possibly the earliest result in this sense is the proof by Robinson that the distributions can be represented by smooth functions of nonstandard analysis and by polynomials of a hyperfinite degree [71]. Distributions have also been represented by functions defined on hyperfinite domains, for instance by Kinoshita in [55] and, with a different approach, by Sousa Pinto and Hoskins in [51]. Another nonstandard approach to the theory of generalized functions has been proposed by Oberbuggenberg and Todorov in [67] and further studied by Todorov et al. [85, 86]. In this approach, the distributions are embedded in an algebra of asymptotic functions defined over a Robinson field of asymptotic numbers. Moreover, this algebra of asymptotic functions can be seen as a generalized Colombeau algebra where the set of scalars is an algebraically closed field rather than a ring with zero divisors. In this setting, it is possible to study generalized solutions to differential equations, and in particular to those with nonsmooth coefficient and distributional initial data [34, 64].

Another theory of generalized functions oriented towards the applications in the field of partial differential equations and of the calculus of variations has been developed by Benci and Luperi Baglini. In [5] and subsequent papers [12, 13, 14, 15], the authors developed a theory of ultrafunctions, i.e. nonstandard vector spaces of a hyperfinite dimension that extend the space of distributions. In particular, the space of distributions can be embedded in an algebra of ultrafunctions V such that the following inclusions hold: $\mathscr{D}'(\mathbb{R}) \subset V \subset {}^*C^1(\mathbb{R})$ [14]. This can be seen as a variation on a result by Robinson and Bernstein, that in [18] showed that any Hilbert space H can be embedded in a hyperfinite dimensional subspace of ${}^*H$. In the setting of ultrafunctions, some partial differential equations can be formulated coherently by a Galerkin approximation, while the problem of finding the minimum of a functional can be turned to a minimization problem over a formally finite vector space. For a discussion of the applications of ultrafunctions to functional analysis, we refer to [5, 13, 15].

The idea of studying the solutions to a partial differential equation via a hyperfinite Galerkin approximation is not new. For instance, Capińsky and Cutland in [24] studied statistical solutions to parabolic differential equations by discretizing the equation in space by a Galerkin approximation in an hyperfinite dimension. The nonstandard model becomes then a hyperfinite system of ODEs that, by transfer, has a unique nonstandard solution. From this solution, the authors showed that it is possible to define a standard weak solution to the original problem. In the subsequent [21], the authors proved the existence of weak and statistical solutions to the Navier-Stokes equations in 3-dimensions by modelling the equations with a similar hyperfinite Galerkin discretization in space. This approach has spanned a whole line of research on the Navier-Stokes equations, concerning both the proof of the existence of solutions (see for instance [23, 30]) and the definition and the existence of attractors (see for instance [22, 31]). One of the advantages of this approach is that, by a hyperfinite discretization in space, the nonstandard models have a unique global solution, even when the original problem does not. For a discussion of the relation between the uniqueness of the solutions of the nonstandard formulation and the non-uniqueness of the weak solutions of the original problem in the case of the Navier-Stokes equations, we refer to [21].

In the theoretical study of nonlinear partial differential equations, sometimes problems do not allow even for a weak solution. However, the development of the notion of Young measures, originally introduced by L. C. Young in the field of optimal control in [91], has allowed for a synthetic characterization of the behaviour of the weak-$\star$ limit of the composition between a nonlinear continuous function and a uniformly bounded sequence in L^∞. By enlarging the class of admissible solutions to include Young measures, one can define generalized solutions for some class of nonlinear problems as the weak-$\star$ limit of the solutions to a sequence of regularized problems [35, 39, 62, 63, 70, 78, 79]. A similar approach can be carried out in the field of optimal controls, where generalized controls in the sense of Young measures can be defined as the measure-valued limit points of a minimizing sequence of controls. For an in-depth discussion of the role of Young measures as generalized solution to PDEs and as generalized controls, we refer to [2, 39, 81, 88].

In [28, 29, 32], Cutland showed that Young measures can be interpreted also as the standard part of internal controls of nonstandard analysis. The possibility to obtain a Young measure from a nonstandard control allows to study generalized solutions to nonlinear variational problems by means of nonstandard techniques: such an approach has been carried out for instance by Cutland in the aforementioned papers, and by Tuckey in [87]. For a discussion of this field of research, we refer to [65].

Structure of the chapter

In this chapter, we will discuss another theory of generalized functions of nonstandard analysis, hereafter called grid functions (see Definition 2.1.1), that provide a coherent generalization both of the space of distributions and of a space of parametrized measures that extends the space of Young measures. In Section 2.1, we will define the space of grid functions, and recall some well-established nonstandard results that will be used throughout Chapter 2 and Chapter 3. In particular, we will formulate in the setting of grid functions some known results regarding the relations between the hyperfinite sum and the Riemann integral, and the finite difference operators of an infinitesimal step and the derivative of a C^1 function.

In Section 2.2, we will study the relations between the grid functions and the distributions, with the aim of proving that every distribution can be obtained from a suitable grid function. In order to reach this result, we will introduce an algebra of nonstandard test functions that can be seen as the grid function counterpart to the space $\mathscr{D}(\Omega)$ of smooth functions with compact support over $\Omega \subseteq \mathbb{R}^k$. By duality with respect to the algebra of test functions, we will define a module of bounded grid functions, and an an equivalence relation between grid functions (see Definition 2.2.3 and Definition 2.2.5). We will then prove that the set of equivalence classes of bounded grid functions with respect to this equivalence relation is a real vector space that is isomorphic to the space of distributions. The premises of this approach are similar to those by Kinoshita in [55]: in particular, our Theorem 2.2.9 can be obtained from Theorem 1 of [55] and from the properties of the equivalence relation defined at the beginning of Section 2.2.

Afterwards, we will discuss how the finite difference operators generalize not only the usual derivative for C^1 functions, but also the distributional derivative. A similar result is also established by Kinoshita in [55] for a module of functions that is smaller than our module of bounded grid functions. On the other hand, this result provides the starting point for the work of Sousa Pinto and Hoskins on a hyperfinite representation of distributions in [51]. They define a module of pre-distributions over $\mathbb{R}$ as the module of functions obtained by applying an arbitrary number of times the finite difference operator to some S-continuous function defined on a hyperfinite grid. Then, they define a module of global distributions over $\mathbb{R}$ by piecing together pre-distributions that agree on the intersection of their support. However, the authors do not prove results comparable to our Theorem 2.2.9 or to Theorem 1 of [55].

After having shown that the finite difference operator generalizes the distributional derivative, our study of the relations between grid functions and distributions concludes with a discussion of the Schwartz impossibility theorem. In particular, we will show that the space of distributions can be embedded in the space of grid functions in a way that

1. the product over the grid functions generalizes the pointwise product between continuous functions;

2. the finite difference is coherent with the distributional derivative modulo the equivalence relation induced by duality with test functions;

3. a discrete chain rule for products holds.

This theorem supports our claim that the space of grid functions provides a nontrivial generalization of the space of distributions.

In Section 2.3, we will embed the space of grid functions in the spaces ${}^*L^p$ with $1 \leq p \leq \infty$, and we will study some properties of grid functions through this embedding. Moreover, we will discuss a generalization of the embedding of $L^2(\Omega)$ in a hyperfinite subspace of ${}^*L^2(\Omega)$ due to Robinson and Bernstein [18]. This classic result will be generalized in two directions:

1. for every $1 \leq p \leq \infty$, we will embed the spaces $L^p(\Omega)$ in the space of grid functions, which is a subspace of ${}^*L^p(\Omega)$ of a hyperfinite dimension;

2. the above embedding is actually an embedding of the bigger space $\mathscr{D}'(\Omega)$ into a hyperfinite subspace of ${}^*L^p(\Omega)$ for all $1 \leq p \leq \infty$.

Moreover, this embedding is obtained with different techniques from the original result by Robinson and Bernstein.

In the second part of Section 2.3, we will establish a correspondence between grid functions and parametrized measures, in a way that is coherent with the isomorphism between equivalence classes of bounded grid functions and distributions discussed in Section 2.2. The results discussed in Section 2.3 will be used in Section 2.4, where we will discuss the grid function formulation of partial differential equations, in Section 2.5, where we will show selected applications of grid functions from different fields of functional analysis, and in Chapter 3, where we will study in detail a grid function formulation of a class of ill-posed partial differential equations with variable parabolicity direction.

In Section 2.4, we will discuss how to formulate partial differential equations in the space of grid functions in a way that coherently generalizes the standard notions of solutions. In particular, stationary PDEs will be given a fully discrete formulation, while time-dependent PDEs will be given a continuous-in-time and discrete-in-space formulation, resulting in a hyperfinite system of ordinary differential equations, as in the nonstandard formulation of the Navier-Stokes equations by Capińsky and Cutland.

In Section 2.5, we will use the theory of grid functions developed so far to study two problems in the nonlinear theory of distributions and in the calculus of variations. These problems are classically studied within different frameworks, but we will show that each of these problems can be formulated in the space of grid functions in a way that the nonstandard solutions generalize the respective standard solutions.

2.1 Terminology and preliminary notions

In this section, we will now fix some notation and recall some results from nonstandard analysis that will be useful throughout Chapter 2 and Chapter 3.

If $A \subseteq \mathbb{R}^k$, then $\overline{A}$ is the closure of A with respect to any norm in $\mathbb{R}^k$, ∂A is the boundary of A, and χ_A is the characteristic function of A. If $x \in \mathbb{R}$, then $\chi_x = \chi_{\{x\}}$. If $f : A \to \mathbb{R}$, $\operatorname{supp} f$ is the closure of the set $\{x \in A : f(x) \neq 0\}$. These definitions are generalized as expected also to nonstandard objects.

We consider the following norms over $\mathbb{R}^k$ and ${}^*\mathbb{R}^k$: if $x \in \mathbb{R}^k$ or $x \in {}^*\mathbb{R}^k$, then $|x| = \sqrt{\sum_{i=1}^k x_i^2}$ is the euclidean norm, and $|x|_\infty = \max_{i=1,\dots,k} |x_i|$ is the maximum norm.

We will denote by ${}^*\mathbb{R}_{fin}$ the set of finite numbers in ${}^*\mathbb{R}$, i.e. ${}^*\mathbb{R}_{fin} = \{x \in {}^*\mathbb{R} : x \text{ is finite}\}$. The notion of finiteness already discussed in Chapter 1 can be extended componentwise to elements of ${}^*\mathbb{R}^k$ whenever $k \in \mathbb{N}$: we will say that $x \in {}^*\mathbb{R}^k$ is finite iff all of its components are finite, and we define ${}^\circ x = ({}^\circ x_1, {}^\circ x_2, \dots, {}^\circ x_k) \in \mathbb{R}^k$. Similarly, if $x, y \in {}^*\mathbb{R}^k$, we will write $x \approx y$ if $|x - y| \approx 0$ (notice that this is equivalent to $|x - y|_\infty \approx 0$).

We will denote by $e_1, \dots, e_k$ the canonical basis of $\mathbb{R}^k$ and of ${}^*\mathbb{R}^k$. If $f : A \subseteq {}^*\mathbb{R}^m \to {}^*\mathbb{R}^k$, we will denote by $f_1, \dots, f_k$ the hyperreal valued functions that satisfy the equality $f(x) = (f_1(x), \dots, f_k(x))$ for all $x \in {}^*\mathbb{R}$.

Let $\Omega \subseteq \mathbb{R}^k$ be an open set or the closure of an open set. We will often reference the following real vector spaces:

- $C_b^0(\Omega) = \{f \in C^0(\Omega) : f \text{ is bounded and } \lim_{|x|\to\infty} f(x) = 0\}$.
- $C_c^0(\Omega) = \{f \in C_b^0(\Omega) : \operatorname{supp} f \subset\subset \Omega\}$.
- $\mathscr{D}(\Omega) = \{f \in C^\infty(\Omega) : \operatorname{supp} f \subset\subset \Omega\}$.
- In the sequel, measurable will mean measurable with respect to μ_L, the Lebesgue measure over $\mathbb{R}^n$. Consider the equivalence relation given by equality almost everywhere: two measurable functions f and g are equivalent if $\mu_L(\{x \in \Omega : f(x) \neq g(x)\}) = 0$. We will not distinguish between the function f and its equivalence class, and we will say that $f = g$ whenever the functions f and g are equal almost everywhere.

  For all $1 \leq p < \infty$, $L^p(\Omega)$ is the set of equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ that satisfy
  $$\int_\Omega |f|^p dx < \infty.$$
  If $f \in L^p(\Omega)$, the L^p norm of f is defined by
  $$\|f\|_p^p = \int_\Omega |f|^p dx.$$

$L^\infty(\Omega)$ is the set of equivalence classes of measurable functions that are essentially bounded: we will say that $f : \Omega \to \mathbb{R}$ belongs to $L^\infty(\Omega)$ if there exists $y \in \mathbb{R}$ such that $\mu_L(\{x \in \Omega : |f(x)| > y\}) = 0$. In this case,

$$\|f\|_\infty = \inf\{y \in \mathbb{R} : \mu_L(\{x \in \Omega : f(x) > y\}) = 0\}.$$

If $1 < p < \infty$, we recall that p' is defined as the unique solution to the equation

$$\frac{1}{p} + \frac{1}{p'} = 1,$$

while $1' = \infty$ and $\infty' = 1$.

- $\mathbb{M}(\mathbb{R}) = \{\nu : \nu$ is a Radon measure over $\mathbb{R}$ satisfying $|\nu|(\mathbb{R}) < +\infty\}$.
- $\mathbb{M}_{\mathbb{P}}(\mathbb{R}) = \{\nu \in \mathbb{M}(\mathbb{R}) : \nu$ is a probability measure$\}$.

Following [2, 3, 88] and others, measurable functions $\nu : \Omega \to \mathbb{M}_{\mathbb{P}}(\mathbb{R})$ will be called Young measures. Measurable functions $\nu : \Omega \to \mathbb{M}(\mathbb{R})$ will be called parametrized measures, even though in the literature the term parametrized measure is used as a synonym for Young measure. If ν is a parametrized measure and if $x \in \Omega$, we will write ν_x instead of $\nu(x)$.

If $f \in C^1(\mathbb{R})$, we will denote the derivative of f by $\frac{df}{dx}$, f' or Df. If $f : [0,T] \times \Omega \to \mathbb{R}$, we will think of the first variable of f as the time variable, denoted by t, and we will write f_t for the derivative $\frac{\partial f}{\partial t}$. We adopt the multi-index notation for partial derivatives and, if α is a multi-index, we will denote by $D^\alpha f$ the function

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_k^{\alpha_k}}.$$

If $\alpha = (\alpha_1, \dots, \alpha_k)$ is a multi-index, then $\alpha - e_i = (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_k)$.

We recall that a real distribution over Ω is an element of $\mathscr{D}'(\Omega)$, i.e. a continuous linear functional $T : \mathscr{D}(\Omega) \to \mathbb{R}$. If T is a distribution and φ is a test function, we will denote the action of T over φ by $\langle T, \varphi \rangle_{\mathscr{D}(\Omega)}$. When T can be identified with a L^p function, we will sometimes write $\int_\Omega T\varphi dx$ instead of $\langle T, \varphi \rangle_{\mathscr{D}(\Omega)}$.

If $T \in \mathscr{D}'(\mathbb{R})$, we will denote the derivative of T by T' or DT. Recall that T' is defined by the formula

$$\langle DT, \varphi \rangle_{\mathscr{D}(\Omega)} = -\langle T, D\varphi \rangle_{\mathscr{D}(\Omega)}.$$

If $T \in \mathscr{D}'(\Omega)$ and α is a multi-index, the distribution $D^\alpha T$ is defined in a similar way:

$$\langle D^\alpha T, \varphi \rangle_{\mathscr{D}(\Omega)} = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle_{\mathscr{D}(\Omega)}.$$

It is well-established that the distributional derivative allows to define a notion of weak derivative for L^p functions (see for instance [80, 82]). L^2

functions whose weak derivatives up to order $p < \infty$ are still L^2 functions are of a particular relevance in the study of partial differential equations. We will now recall the definition of the space of such functions. For $p \in \mathbb{N}$, $p \geq 1$, the space $H^p(\Omega)$ is defined as

$$H^p(\Omega) = \{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for every multi-index } \alpha \text{ with } |\alpha| \leq p\}.$$

We also consider the following norm over the space $H^p(\Omega)$:

$$\|f\|_{H^p} = \sum_{|\alpha|\leq p} \|D^\alpha f\|_2,$$

and we will call it the H^p norm. Recall also that $H^p_0(\Omega) \subset H^p(\Omega)$ is defined as the closure of $\mathscr{D}(\Omega)$ in $H^p(\Omega)$ with respect to the H^p norm. For the properties of the space $H^p(\Omega)$ and of the space $H^p_0(\Omega)$, we refer to [80, 82].

We will now introduce the space of grid functions.

Definition 2.1.1. *Let $N_0 \in {}^*\mathbb{N}$ be an infinite hypernatural number. Set $N = N_0!$ and $\varepsilon = 1/N$, and define*

$$\mathbb{X} = \{n\varepsilon : n \in [-N^2, N^2] \cap {}^*\mathbb{Z}\}.$$

We will say that an internal function $f : \mathbb{X}^k \to {}^*\mathbb{R}$ is a grid function and, if $A \subseteq \mathbb{X}^k$ is internal, we denote by $\mathbb{G}(A)$ the space of grid functions defined over A: $\mathbb{G}(A) = \mathbf{Intl}({}^*\mathbb{R}^A) = \{f : A \to {}^*\mathbb{R}$ and f is internal$\}$.

2.1.1 Some elements of nonstandard topology

In the next definition, we will give a canonical extension of subsets of the standard euclidean space $\mathbb{R}^k$ to internal subsets of the grid $\mathbb{X}^k$.

Definition 2.1.2. *For any $A \subseteq \mathbb{R}^k$, we define $A_\mathbb{X} = {}^*A \cap \mathbb{X}^k$. Notice that $A_\mathbb{X}$ is an internal subset of $\mathbb{X}^k$, and in particular it is hyperfinite.*

In general, we expect that for a generic set $A \subseteq {}^*\mathbb{R}^k$, ${}^\circ A_\mathbb{X} \neq \overline{A}$. For instance, if $A \cap \mathbb{Q}^k = \emptyset$, then $A_\mathbb{X} = {}^\circ A_\mathbb{X} = \emptyset$. In this section, we will prove that if A is an open set, then indeed $A_\mathbb{X}$ is a faithful extension of A, in the sense that ${}^\circ A_\mathbb{X} = {}^\circ \overline{A}_\mathbb{X} = \overline{A}$. Moreover, there is a nice characterization of the boundary of $A_\mathbb{X}$ which is projected to the boundary of A via the standard part map.

In order to prove these results, we need to show that for an open set A, $\mu(x) \cap {}^*A \neq \emptyset$ is equivalent to $\mu(x) \cap A_\mathbb{X} \neq \emptyset$ for all $x \in \overline{A}$.

Lemma 2.1.3. *If $A \subseteq \mathbb{R}^k$ is an open set, then for all $x \in \overline{A}$ it holds*

$$\mu(x) \cap {}^*A \neq \emptyset \iff \mu(x) \cap A_\mathbb{X} \neq \emptyset. \tag{2.1.1}$$

Proof. Let $x \in \overline{A}$. The hypothesis $N = N_0!$ for an infinite $N_0 \in {}^*\mathbb{N}$ ensures that for all $p \in \mathbb{Q}^k$, $p \in \mathbb{X}^k$. As a consequence, for all $n \in \mathbb{N}$ there exists $p \in A_\mathbb{X}$ with $|x - p| < 1/n$. By overspill, for some infinite $M \in {}^*\mathbb{N}$ there exists $p \in A_\mathbb{X}$ that satisfies $|x - p| < 1/M$. □

We want to define a boundary for the set $A_\mathbb{X}$ that is coherent with the usual notion of boundary for A. The idea is to define the $\mathbb{X}$-boundary of $A_\mathbb{X}$ as the set of points of $A_\mathbb{X}$ that are within a step of length ε from a point of ${}^*A^c$.

Definition 2.1.4. *Let $A \subseteq \mathbb{X}^k$. We define the $\mathbb{X}$-boundary of A as*

$$\partial_\mathbb{X} A = \{x \in A : \exists y \in {}^*A^c \textit{ satisfying } |x - y|_\infty \leq \varepsilon\}.$$

This definition is coherent with the usual boundary of an open set.

Proposition 2.1.5. *Let $A \subseteq \mathbb{R}^k$ be an open set. Then ${}^\circ A_\mathbb{X} = \overline{A}$ and ${}^\circ(\partial_\mathbb{X} A_\mathbb{X}) = \partial A$.*

Proof. The equality ${}^\circ A_\mathbb{X} = \overline{A}$ is a consequence of Lemma 2.1.3.

Recall the nonstandard characterization of the boundary of A: $x \in \partial A$ if and only if there exists $y \in {}^*A$, $x \neq y$, and $z \in {}^*A^c$ with $x \approx y \approx z$. This is sufficient to conclude that $\partial A \supseteq {}^\circ(\partial_\mathbb{X} A_\mathbb{X})$.

To prove that the other inclusion holds, we only need to show that if $x \in \partial A$, then there exists $y \in \partial_\mathbb{X} A_\mathbb{X}$ with $y \approx x$. Let $x \in \partial A$: since $A_\mathbb{X}$ is a hyperfinite set, we can pick $y \in A_\mathbb{X}$ satisfying

$$|{}^*x - y|_\infty = \min_{z \in A_\mathbb{X}} \{|{}^*x - z|_\infty\}.$$

Recall that $x \in A^c$, since A is open: as a consequence, for our choice of y we have $y \neq {}^*x$ and $|{}^*x - y|_\infty > 0$. We claim that $y \in \partial_\mathbb{X} A_\mathbb{X}$. In fact, suppose towards a contradiction that $y \notin \partial_\mathbb{X} A_\mathbb{X}$: in this case, for all $z \in A^c$, $|y - z|_\infty > \varepsilon$ and, in particular, $|{}^*x - y|_\infty > \varepsilon$. Let ${}^*x - y = \sum_{i=1}^k a_i e_i$, let $I = \{i \leq k : |a_i| = |{}^*x - y|_\infty\}$, and define

$$\tilde{y} = y + \sum_{i \in I} \frac{a_i}{|a_i|} \varepsilon e_i.$$

Since $|\tilde{y} - y|_\infty = \varepsilon$ and since $y \notin \partial_\mathbb{X} A_\mathbb{X}$, then $\tilde{y} \in A_\mathbb{X}$. Moreover,

$$|{}^*x - \tilde{y}|_\infty = \max_{i \notin I} \{|{}^*x - y| - \varepsilon, |a_i|\} < |x - y|_\infty,$$

contradicting $|{}^*x - y|_\infty = \min_{z \in A_\mathbb{X}} \{|{}^*x - z|_\infty\}$. □

From now on, let $\Omega \subseteq \mathbb{R}^k$ be an open set or the closure of an open set. By Proposition 2.1.5, this hypothesis is sufficient to ensure the equalities ${}^\circ\Omega_\mathbb{X} = \overline{\Omega}$ and ${}^\circ(\partial_\mathbb{X}\Omega_\mathbb{X}) = \partial\Omega$.

2.1.2 Derivatives and integrals of grid functions

Since grid functions are defined on a discrete set, there is no notion of derivative for grid functions. However, in nonstandard analysis it is fairly usual to replace the derivative by a finite difference operator with an infinitesimal step.

Definition 2.1.6 (Grid derivative)**.** *For an internal grid function $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, we define the i-th forward finite difference of step ε as*

$$\mathbb{D}_i f(x) = \mathbb{D}_i^+ f(x) = \frac{f(x + \varepsilon e_i) - f(x)}{\varepsilon}$$

and the i-th backward finite difference of step ε as

$$\mathbb{D}_i^- f(x) = \frac{f(x) - f(x - \varepsilon e_i)}{\varepsilon}.$$

If $n \in {}^*\mathbb{N}$, $\mathbb{D}_i^n$ is recursively defined as $\mathbb{D}_i(\mathbb{D}_i^{n-1})$ and, if α is a multi-index, then $\mathbb{D}^\alpha$ is defined as expected:

$$\mathbb{D}^\alpha f = \mathbb{D}_1^{\alpha_1} \mathbb{D}_2^{\alpha_2} \ldots \mathbb{D}_n^{\alpha_n} f.$$

These definitions can be extended to $\mathbb{D}^-$ by replacing every occurrence of $\mathbb{D}$ with $\mathbb{D}^-$.

For further details about the properties of the finite difference operators, we remand to Hanqiao, St. Mary and Wattenberg [45], to Keisler [53] and to van den Berg [16, 17].

Remark 2.1.7. *Notice that if $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ and if α is a standard multi-index, then $\mathbb{D}^\alpha f$ is not defined on all of $\Omega_{\mathbb{X}}$. However, if we let*

$$\Omega_{\mathbb{X}}^\alpha = \{x \in \Omega_{\mathbb{X}} : \mathbb{D}^\alpha f \text{ is defined}\} = \{x \in \Omega_{\mathbb{X}} : x + \alpha\varepsilon \in \Omega_{\mathbb{X}}\}$$

then we have ${}^\circ\Omega_{\mathbb{X}}^\alpha = {}^\circ\Omega_{\mathbb{X}} = \overline{\Omega}$, since for every $x \in \Omega_{\mathbb{X}}^\alpha$ we have $x + \alpha\varepsilon \in \Omega_{\mathbb{X}}$ and $x + \alpha\varepsilon \approx x$ by the standardness of α.

In a similar way, if we define

$$\partial_{\mathbb{X}}^\alpha \Omega_{\mathbb{X}} = \{x \in \Omega_{\mathbb{X}} : x + \alpha\varepsilon \in \partial_{\mathbb{X}}\Omega_{\mathbb{X}}\},$$

then, from the relation $x + \alpha\varepsilon \approx x$ and from Proposition 2.1.5, we deduce that it holds also the equality ${}^\circ\partial_{\mathbb{X}}^\alpha\Omega_{\mathbb{X}} = {}^\circ\partial_{\mathbb{X}}\Omega_{\mathbb{X}} = \partial\Omega$. In section 2.4.1, we will use this result in order to show show how Dirichlet boundary conditions can be expressed in the sense of grid functions.

Since $\Omega_{\mathbb{X}}^\alpha$ is a faithful extension of Ω in the sense of proposition 2.1.5, we will often abuse notation and write $\mathbb{D}^\alpha f \in \mathbb{G}(\Omega_{\mathbb{X}})$ instead of the correct $\mathbb{D}^\alpha f \in \mathbb{G}(\Omega_{\mathbb{X}}^\alpha)$.

In the setting of grid functions, integrals are replaced by hyperfinite sums.

Definition 2.1.8 (Grid integral and inner product)**.** *Let $f, g : {}^*\Omega \to {}^*\mathbb{R}$ and let $A \subseteq \Omega_{\mathbb{X}} \subseteq \mathbb{X}^k$ be an internal set. We define*

$$\int_A f(x) d\mathbb{X}^k = \varepsilon^k \cdot \sum_{x \in A} f(x)$$

and

$$\langle f, g \rangle = \int_{\mathbb{X}^k} f(x) g(x) d\mathbb{X}^k = \varepsilon^k \cdot \sum_{x \in \mathbb{X}^k} f(x) g(x),$$

with the convention that, if $x \notin {}^*\Omega$, $f(x) = g(x) = 0$.

A simple calculation shows that the fundamental theorem of calculus holds. In particular, for all $f : \mathbb{G}(\mathbb{X}) \to {}^*\mathbb{R}$ and for all $a, b \in \mathbb{X}$, $b < N$, we have

$$\varepsilon \sum_{x=a}^{b} \mathbb{D} f(x) = f(b+\varepsilon) - f(a) \text{ and } \mathbb{D} \left(\varepsilon \sum_{x=a}^{b} f(x) \right) = f(b+\varepsilon).$$

The next Lemma is a well-known compatibility result between the grid integral and the Riemann integral of continuous functions.

Lemma 2.1.9. *Let $\Omega \subset \mathbb{R}^k$ be a compact set. If $f \in C^0(\Omega)$, then*

$$\int_{\Omega_{\mathbb{X}}} {}^*f(x) d\mathbb{X}^k \approx \int_{\Omega} f(x) dx.$$

Proof. See for instance Section 1.11 of [60]. □

2.1.3 S^α functions and C^α functions

We will now recall some well-known facts about S-continuity. This property has been widely used as a bridge between discrete functions of nonstandard analysis and standard continuous functions.

Definition 2.1.10. *We will say that $x \in \Omega_{\mathbb{X}}$ is nearstandard in Ω iff there exists $y \in \Omega$ such that $x \approx y$.*

Definition 2.1.11. *We say that a function $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ is S-continuous on $\Omega_{\mathbb{X}}$ iff $f(x)$ is finite for some nearstandard $x \in \Omega_{\mathbb{X}}$ and for every nearstandard $x, y \in \Omega_{\mathbb{X}}$, $x \approx y$ implies $f(x) \approx f(y)$.*

We also define functions of class S^α for every multi-index α:

- *f is of class $S^0(\Omega_{\mathbb{X}})$ if f is S-continuous on $\Omega_{\mathbb{X}}$;*
- *f is of class $S^\alpha(\Omega_{\mathbb{X}})$ if $\mathbb{D}^\alpha f \in S^0(\Omega_{\mathbb{X}})$.*

- *f is of class $S^\infty(\Omega_\mathbb{X})$ if $\mathbb{D}^\alpha f \in S^0(\Omega_\mathbb{X})$ for any standard multi-index α.*

Notice that if $f \in S^\alpha(\Omega_\mathbb{X})$ for some standard multi-index α, then $f(x)$ is finite at all nearstandard $x \in \Omega_\mathbb{X}$.

In the study of S-continuous functions, we find it useful to introduce the following equivalence relation.

Definition 2.1.12. *Let $f, g \in \mathbb{G}(\Omega_\mathbb{X})$. We say that $f \equiv_S g$ iff $(f-g)(x) \approx 0$ for all nearstandard $x \in \Omega_\mathbb{X}$. From the properties of $\approx$, it can be proved that $\equiv_S$ is an equivalence relation. We will denote by π_S the projection from $\mathbb{G}(\Omega_\mathbb{X})$ to the quotient space $\mathbb{G}(\Omega_\mathbb{X})/\equiv_S$, and will denote by $[f]_S$ the equivalence class of f with respect to $\equiv_S$.*

The rest of this section is devoted to the proof that the quotient $S^\alpha(\Omega_\mathbb{X})/\equiv_S$ is real algebra isomorphic to the algebra of C^α functions over Ω. This result is a reformulation in the language of grid functions of some results by van den Berg [16] and by Wattenberg, Hanqiao, and St. Mary [45].

Lemma 2.1.13. *For every standard multi-index α, $S^\alpha(\Omega_\mathbb{X})$ with pointwise sum and product is an algebra over ${}^*\mathbb{R}_{fin}$, and $S^\alpha(\Omega_\mathbb{X})/\equiv_S$ inherits a structure of real algebra from $S^\alpha(\Omega_\mathbb{X})$.*

Proof. The only non-trivial assertion that needs to be verified is closure of $S^\alpha(\Omega_\mathbb{X})$ with respect to pointwise product. This property is a consequence of Proposition 2.6 of [16]. □

Theorem 2.1.14. *$S^0(\Omega_\mathbb{X})/\equiv_S$ is a real algebra isomorphic to $C^0(\Omega)$. The isomorphism is given by $i[f]_S = {}^\circ f$. The inverse of i is the function $i^{-1}(f) = [{}^*f_{|\mathbb{X}}]_S$.*

Proof. If $f \in S^0(\Omega_\mathbb{X})$, then it is well-known that ${}^\circ f$ is a well-defined function and that ${}^\circ f \in C^0(\Omega)$. Surjectivity of ${}^\circ$ is a consequence of Lemma II.6 of [45]. Since

$$\ker({}^\circ) = \{f \in S^\alpha(\Omega_\mathbb{X}) : f(x) \approx 0 \text{ for all finite } x \in \Omega_\mathbb{X}\} = [0]_S,$$

we deduce that i is injective and surjective. Since ${}^\circ(x+y) = {}^\circ x + {}^\circ y$ and ${}^\circ(xy) = {}^\circ x {}^\circ y$ for all $x, y \in {}^*\mathbb{R}_{fin}$, i_α is an isomorphism of real algebras. □

We will now show that, for grid functions of class S^α, the finite difference operators $\mathbb{D}_i^+$ and $\mathbb{D}_i^-$ assume the role of the usual partial derivative for C^α functions. In particular, these finite difference operators can be seen as generalized derivatives.

Theorem 2.1.15. *For all $1 \leq i \leq k$ and for all standard multi-indices α with $\alpha_i \geq 1$, the diagram*

$$\begin{array}{ccc} S^\alpha(\Omega_{\mathbb{X}}) & \stackrel{\mathbb{D}_i^+}{\longrightarrow} & S^{\alpha-e_i}(\Omega_{\mathbb{X}}) \\ i \circ \pi_S \downarrow & & \downarrow i \circ \pi_S \\ C^\alpha(\Omega) & \stackrel{D_i}{\longrightarrow} & C^{\alpha-e_i}(\Omega) \end{array}$$

and the diagram

$$\begin{array}{ccc} S^\alpha(\Omega_{\mathbb{X}}) & \stackrel{\mathbb{D}_i^-}{\longrightarrow} & S^{\alpha-e_i}(\Omega_{\mathbb{X}}) \\ i \circ \pi_S \downarrow & & \downarrow i \circ \pi_S \\ C^\alpha(\Omega) & \stackrel{D_i}{\longrightarrow} & C^{\alpha-e_i}(\Omega) \end{array}$$

commute.

Proof. By Theorem 2.1.14, if $f \in S^\alpha(\Omega_{\mathbb{X}}) \subseteq S^0(\Omega_{\mathbb{X}})$ then $(i_\alpha \circ \pi_S)(f) = {}^\circ f$ and, by Lemma II.7 of [45], ${}^\circ(\mathbb{D}_i^\pm f) = D_i {}^\circ f$. $\square$

By Theorem 2.1.15, the isomorphism i defined in Theorem 2.1.14 induces an isomorphism between $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ and $C^\alpha(\Omega)$ as real algebras.

Corollary 2.1.16. *For any multi-index α, the isomorphism i restricted to $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ induces an isomorphism between $S^\alpha(\Omega_{\mathbb{X}})/\equiv_S$ and $C^\alpha(\Omega)$ as real algebras.*

Thanks to this isomorphism, if $f \in S^\alpha(\Omega_{\mathbb{X}})$, we can identify the equivalence class $[f]_S$ with the standard function ${}^\circ f \in C^\alpha(\Omega)$.

2.2 Grid functions as generalized distributions

In this section, we will study the relations between the space of grid functions and the space of distributions. In particular, we will prove that the space of grid functions can be seen as generalization of the space of distributions, and the operators $\mathbb{D}^+$ and $\mathbb{D}^-$ coherently extend the distributional derivative to the space of grid functions.

In order to prove the above results, we start by defining a projection from an external ${}^*\mathbb{R}_{fin}$-submodule of $\mathbb{G}(\Omega_{\mathbb{X}})$ to the space of distributions. This projection is defined by duality with an external ${}^*\mathbb{R}_{fin}$-algebra of grid functions that is a counterpart to the space of test functions.

Definition 2.2.1 (Algebra of test functions)**.** *We define the algebra of test functions over $\Omega_{\mathbb{X}}$ as follows:*

$$\mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) = \{f \in S^\infty(\Omega_{\mathbb{X}}) : {}^\circ\mathrm{supp}\, f \subset\subset \Omega\}.$$

The above definition provides a nonstandard counterpart of the usual space of smooth functions with compact support.

Lemma 2.2.2. *The isomorphism i defined in Theorem 2.1.14 induces an isomorphism between the real algebras $\mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv_S$ and $\mathscr{D}(\Omega)$. The isomorphism preserves integrals, i.e. for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, it holds the equality*

$$^\circ\int_{\Omega_{\mathbb{X}}} \varphi d\mathbb{X}^k = \int_{\Omega} i[\varphi]_S dx. \tag{2.2.1}$$

Moreover, if $\varphi \in \mathscr{D}(\Omega)$, then ${}^*\varphi_{|\mathbb{X}} \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, so that $i^{-1}(\varphi) = \left[{}^*\varphi_{|\mathbb{X}}\right]_S \cap \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

Proof. From Theorem 2.1.14, from Theorem 2.1.15 and from the definition of $\mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, we can conclude that the hypothesis $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ ensures that $i[\varphi] \in \mathscr{D}(\Omega)$. Since $\mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) \subset S^0(\Omega_{\mathbb{X}})$, injectivity of i is a consequence of Theorem 2.1.14.

Similarly, surjectivity of i can be deduced from Theorem 2.1.14 and from Theorem 2.1.15. In fact, suppose towards a contradiction that there exists $\psi \in \mathscr{D}(\Omega)$ such that $\psi \neq i[\varphi]$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega)$. Since $\psi \in C^0(\Omega)$, Theorem 2.1.14 ensures that there exists $\phi \in S^0(\Omega_{\mathbb{X}})$ with $i[\phi] = \psi$. If $\phi \notin S^\infty(\Omega_{\mathbb{X}})$, then for some standard multi-index α, $\mathbb{D}^\alpha \phi \notin S^0(\Omega_{\mathbb{X}})$, contradicting Theorem 2.1.15. As a consequence, i is an isomorphism between $\mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv_S$ and $\mathscr{D}(\Omega)$.

Equality 2.2.1 is a consequence of the hypothesis $^\circ\mathrm{supp}\,\varphi \subset\subset \Omega$ and of Lemma 2.1.9.

Now let $\varphi \in \mathscr{D}(\Omega)$: by Theorem 2.1.14 and by Theorem 2.1.15, ${}^*\varphi_{|\mathbb{X}} \in S^\infty(\Omega_{\mathbb{X}})$. Let $A = \mathrm{supp}\,\varphi$: since A is the closure of an open set, by Proposition 2.1.5 $^\circ A_{\mathbb{X}} = A \subset\subset \Omega$, from which we deduce ${}^*\varphi_{|\mathbb{X}} \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. As a consequence, $i^{-1}(\varphi) = \left[{}^*\varphi_{|\mathbb{X}}\right]_S \cap \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, as we claimed. □

The duality with respect to the space of test functions can be used to define an equivalence relation on the space of grid functions. This equivalence relation plays the role of a weak equality.

Definition 2.2.3. *Let $f, g \in \mathbb{G}(\Omega_{\mathbb{X}})$. We say that $f \equiv g$ iff for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ it holds $\langle f, \varphi\rangle \approx \langle g, \varphi\rangle$. We will call π the projection from $\mathbb{G}(\Omega_{\mathbb{X}})$ to the quotient $\mathbb{G}(\Omega_{\mathbb{X}})/\equiv$, and we will denote by $[f]$ the equivalence class of f with respect to $\equiv$.*

The new equivalence relation $\equiv$ is coarser than $\equiv_S$.

Lemma 2.2.4. *For all $f, g \in \mathbb{G}(\Omega_{\mathbb{X}})$, $f \equiv_S g$ implies $f \equiv g$.*

Proof. We will show that $f \equiv_S g$ implies $\langle f - g, \varphi\rangle \approx 0$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$: by linearity of the hyperfinite sum, this result is equivalent to $f \equiv g$.

Let $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, and let $\eta = \max_{x\in\mathrm{supp}\,\varphi}\{|(f-g)(x)|\}$. The hypothesis that $f \equiv_S g$ and the hypothesis that $^\circ\mathrm{supp}\,\varphi$ is bounded are sufficient to ensure that $\eta \approx 0$. As a consequence, we have the following inequalities

$$|\langle f - g, \varphi\rangle| \leq \langle |f-g|, |\varphi|\rangle \leq |\eta| \int_{\Omega_{\mathbb{X}}} |\varphi(x)| d\mathbb{X}^k \approx 0,$$

that are sufficient to conclude the proof. $\square$

We can now define a duality pairing with respect to the inner product defined in 2.1.8.

Definition 2.2.5. *For any $V \subseteq \mathbb{G}(\Omega_{\mathbb{X}})$, we define*

$$V' = \{f \in \mathbb{G}(\Omega_{\mathbb{X}}) : \langle g, f\rangle \text{ is finite for all } g \in V\}.$$

The ${}^*\mathbb{R}_{fin}$-module

$$\mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) = \{f \in \mathbb{G}(\Omega_{\mathbb{X}}) \mid \langle f, \varphi\rangle \text{ is finite for all } \varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})\}$$

is called the module of bounded grid functions.

The rest of this section is devoted to the proof that the quotient $\mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv$ is real vector space isomorphic to the space of distributions $\mathscr{D}'(\Omega)$.

Lemma 2.2.6. *For any $V \subseteq \mathbb{G}(\Omega_{\mathbb{X}})$, V' with pointwise sum and product is a module over ${}^*\mathbb{R}_{fin}$. Moreover, $V'/\equiv$ inherits a structure of real vector space from V'.*

Notice that, contrary to what happened for the space $S^0(\Omega_{\mathbb{X}})$, V' is not an algebra, since in general the hypothesis $f, g \in V'$ is not sufficient to ensure that $fg \in V'$.

The following characterization of bounded generalized distributions will be used in the proof of the isomorphism between the quotient of the module of the bounded generalized distributions and the space of distributions.

Lemma 2.2.7. *The following are equivalent:*

1. $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$*;*
2. $\langle f, \varphi\rangle \approx 0$ *for all* $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ *satisfying* $\varphi(x) \approx 0$ *for all* $x \in \Omega_{\mathbb{X}}$.

Proof. (1) implies (2), by contrapositive. Suppose that $\langle f, \varphi\rangle \not\approx 0$ for some $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ with $\varphi(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$. Now take some $\psi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ with $\psi(x) \geq n\varphi(x)$ for all $x \in \Omega_{\mathbb{X}}$ and for all $n \in \mathbb{N}$. From the inequality $\langle f, \psi\rangle \geq n\langle f, \varphi\rangle$ for all $n \in \mathbb{N}$, we deduce that $\langle f, \psi\rangle$ is infinite, i.e. that $f \notin \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

(2) implies (1), by contrapositive. Suppose that $\langle f, \varphi\rangle = M$ is infinite for some $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. Since $\varphi/M \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varphi/M(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$, we deduce that (2) does not hold. $\square$

From the above Lemma, we deduce that the action of a bounded generalized distribution over the space of test functions is continuous.

Corollary 2.2.8 (Continuity)**.** *If $\varphi, \psi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varphi \equiv_S \psi$, then $\langle f, \varphi\rangle \approx \langle f, \psi\rangle$ for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.*

Proof. The hypotheses $\varphi, \psi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varphi \equiv_S \psi$ imply $\varphi - \psi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $(\varphi - \psi)(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$. Then, by Lemma 2.2.7, we have $\langle f, \varphi - \psi \rangle \approx 0$ for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, as we wanted. $\square$

We are now ready to prove that $\mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv$ is isomorphic to the space of distributions over Ω.

Theorem 2.2.9. *The function* $\Phi : (\mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv) \to \mathscr{D}'(\Omega)$ *defined by*

$$\langle \Phi([f]), \varphi \rangle_{\mathscr{D}(\Omega)} = {}^\circ\langle f, {}^*\varphi \rangle$$

is an isomorphism of real vector spaces.

Proof. At first, we will show that the definition of Φ does not depend upon the choice of the representative for $[f]$. Let $g, h \in [f]$: then, by definition of $\equiv$, ${}^\circ\langle g, \varphi \rangle = {}^\circ\langle h, \varphi \rangle$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. By Lemma 2.2.2, for all $\varphi \in \mathscr{D}'(\Omega)$, ${}^*\varphi_{|\mathbb{X}} \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, so that if $g, h \in [f]$, then ${}^\circ\langle g, {}^*\varphi \rangle = {}^\circ\langle h, {}^*\varphi \rangle$ so that the definition of Φ is independent on the choice of the representative for $[f]$.

Lemma 2.2.8 ensures that for all $[f] \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})/\equiv$, $\Phi([f]) \in \mathscr{D}'_{\mathbb{X}}(\Omega)$, and in particular that $\Phi([f])$ is continuous.

We will prove by contradiction that Φ is injective. Suppose that $\langle \Phi([f]), \varphi \rangle = 0$ for all $\varphi \in \mathscr{D}(\Omega)$ and that $[f] \neq [0]$. The latter hypothesis implies that there exists $\psi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ such that $\langle f, \psi \rangle \not\approx 0$. But, since ${}^*({}^\circ\psi)_{|\mathbb{X}} \equiv_S \psi$, by Corollary 2.2.8 we deduce

$$\langle \Phi([f]), {}^\circ\psi \rangle_{\mathscr{D}(\Omega)} = {}^\circ\langle f, {}^*({}^\circ\psi) \rangle = {}^\circ\langle f, \psi \rangle \neq 0,$$

contradicting the hypothesis $\langle \Phi([f]), \varphi \rangle = 0$ for all $\varphi \in \mathscr{D}(\Omega)$. As a consequence, Φ is injective.

Surjectivity of Φ is a consequence of Theorem 1 of [55]. $\square$

In view of the isomorphism Φ, from now on we will identify the equivalence class $[f]$ with the distribution $\Phi([f])$. Notice that if $f \in S^0(\Omega_{\mathbb{X}})$, this identification is coherent with $[f]_S$.

Corollary 2.2.10. *If* $f \in S^0(\Omega_{\mathbb{X}})$*, then* $[f] = [f]_S = {}^\circ f$.

Proof. Since f is S-continuous, by Lemma 2.1.9 and by Lemma 2.2.2 we have the equality

$$\int_\Omega {}^\circ f \varphi dx = {}^\circ\langle f, {}^*\varphi \rangle$$

for all $\varphi \in \mathscr{D}(\Omega)$, and this is sufficient to deduce the thesis. $\square$

Remark 2.2.11. *If* $k \in \mathbb{N}$*, define*

$$\mathscr{D}'_{\mathbb{X}}(\Omega, {}^*\mathbb{R}^k) = \left\{ f : \Omega_{\mathbb{X}} \to \mathbb{R}^k : f_i \in \mathscr{D}'_{\mathbb{X}}(\Omega) \textit{ for all } 1 \leq i \leq k \right\}.$$

If $f \in \mathscr{D}'_{\mathbb{X}}(\Omega, {}^*\mathbb{R}^k)$, then we can define a functional $[f]$ over the dual of the space of vector-valued test functions

$$\mathscr{D}(\Omega, \mathbb{R}^k) = \left\{\varphi : \Omega \to \mathbb{R}^k : \varphi_i \in \mathscr{D}(\Omega) \text{ for all } 1 \leq i \leq k\right\}$$

by posing $\langle [f], \varphi \rangle_{\mathscr{D}(\Omega,\mathbb{R}^k)} = \sum_{i=1}^k {}^\circ\langle f_i, {}^*\varphi_i\rangle$ for all $\varphi \in \mathscr{D}(\Omega, \mathbb{R}^k)$. From Theorem 2.2.9, we deduce that the quotient of the ${}^*\mathbb{R}_{fin}$-module $\mathscr{D}'_{\mathbb{X}}(\Omega, {}^*\mathbb{R}^k)$ with respect to $\equiv$ is isomorphic to the real vector space of linear continuous functionals over $\mathscr{D}(\Omega, \mathbb{R}^k)$.

Remark 2.2.12. *Theorem 2.2.9 can be used to define more general projections of nonstandard functions. For instance, if $f \in {}^*C^0({}^*\mathbb{R}, \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}))$, then for all $T \in \mathbb{R}$ f induces a continuous linear functional $[f]$ over the space $C^0([0,T], \mathscr{D}'(\Omega))$ defined by the formula*

$$\int_0^T \langle [f], \varphi \rangle_{\mathscr{D}(\Omega)} dt = {}^\circ\left({}^*\int_0^T \langle f(t), {}^*\varphi(t)\rangle dt\right)$$

for all $\varphi \in C^0([0,T], \mathscr{D}'(\Omega))$. Moreover, if $f \in {}^*C^1({}^*\mathbb{R}, \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}))$, then $[f]$ allows for a weak derivative with respect to time: for all $T \in \mathbb{R}$, $[f]_t$ is the distribution that satisfies

$$\int_0^T \langle [f]_t, \varphi \rangle_{\mathscr{D}(\Omega)} dt = -{}^\circ\left({}^*\int_0^T \langle f(t), {}^*\varphi(t)\rangle dt\right)$$

for all $\varphi \in C^1([0,T], \mathscr{D}'(\Omega))$.

2.2.1 Discrete derivative and distributional derivative

In this section, we will show that the finite difference operators $\mathbb{D}_i^+$ and $\mathbb{D}_i^-$ generalize the distributional derivative to the setting of grid functions, i.e. that $[\mathbb{D}_i^\pm f] = D_i[f]$ for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. For a matter of commodity, we will suppose that $\Omega_{\mathbb{X}} \subseteq \mathbb{X}$: the generalization to an arbitrary dimension can be deduced from the proof of Theorem 2.2.15 with an argument relying on Theorem 2.1.15.

Recall the discrete summation by parts formula: for all grid functions f and g and for all $a, b \in {}^*\mathbb{N}$ with $N^2 \leq a < b < N^2$ it holds the equality

$$\begin{aligned}\sum_{n=a}^b (f((n+1)\varepsilon) - f(n\varepsilon))g(n\varepsilon) &= f((b+1)\varepsilon)g((b+1)\varepsilon) - f(a\varepsilon)g(a\varepsilon) + \\ &\quad - \sum_{n=a}^b f((n+1)\varepsilon)(g((n+1)\varepsilon) - g(n\varepsilon))\end{aligned}$$

that, in particular, implies

$$\langle \mathbb{D}f, \varphi \rangle = -\langle f(x+\varepsilon), \mathbb{D}\varphi \rangle \qquad (2.2.2)$$

for all $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ and for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

Inspired by the above formula, we will now prove that if we shift a bounded generalized distribution by an infinitesimal displacement, we still obtain the same generalized distribution.

Lemma 2.2.13. *Let $f \in \mathbb{G}(\Omega_{\mathbb{X}})$. Then $f(x) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ if and only if $f(x+\varepsilon) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. If $f(x) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ then, $[f(x)] = [f(x+\varepsilon)]$.*

Proof. The hypothesis that for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ it holds $^\circ\mathrm{supp}\,\varphi \subset\subset \Omega$ ensures the equality

$$\langle f(x), \varphi(x) \rangle = \langle f(x+\varepsilon), \varphi(x+\varepsilon) \rangle$$

from which we deduce the equivalence $f(x) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ if and only if $f(x+\varepsilon) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

We will now prove that, $f(x) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, then $[f(x)] = [f(x+\varepsilon)]$. By equation 2.2.2, we have

$$\langle f(x+\varepsilon) - f(x), \varphi \rangle = -\langle f(x+\varepsilon), \varepsilon\mathbb{D}\varphi \rangle \tag{2.2.3}$$

for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. Notice that $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ implies that $\varepsilon\mathbb{D}\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $\varepsilon\mathbb{D}\varphi(x) \approx 0$ for all $x \in \Omega_{\mathbb{X}}$. Hence, by the hypothesis $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and by Lemma 2.2.7, we deduce that $\langle f(x+\varepsilon), \varepsilon\mathbb{D}\varphi \rangle \approx 0$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. By equation 2.2.3, this is sufficient to deduce the equality $[f(x)] = [f(x+\varepsilon)]$. $\square$

As a consequence of the above Lemma, we can characterize a nonstandard counterpart of the shift operator.

Corollary 2.2.14. *Let $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. For all n such that $n\varepsilon$ is finite, $[f(x \pm n\varepsilon)] = [f](x \pm {}^\circ(n\varepsilon))$.*

We are now ready to prove that the finite difference operators generalize the distributional derivative.

Theorem 2.2.15. *The diagram*

$$\begin{array}{ccc} \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}^+} & \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi\circ\pi\downarrow & & \downarrow\Phi\circ\pi \\ \mathscr{D}'(\Omega) & \xrightarrow{D} & \mathscr{D}'(\Omega) \end{array}$$

and the diagram

$$\begin{array}{ccc} \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{\mathbb{D}^-} & \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi\circ\pi\downarrow & & \downarrow\Phi\circ\pi \\ \mathscr{D}'(\Omega) & \xrightarrow{D} & \mathscr{D}'(\Omega) \end{array}$$

commute.

Proof. We will prove that the first diagram commutes, as the proof for the second is similar.

Let $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$: we have the following equality chain

$$\langle D[f], \varphi \rangle_{\mathscr{D}(\Omega)} = -\langle [f], D\varphi \rangle_{\mathscr{D}(\Omega)} = -^{\circ}\langle f, {}^*D^*\varphi \rangle.$$

By Theorem 2.1.15, ${}^*D^*\varphi \equiv_S \mathbb{D}^{\pm *}\varphi$ and, by Corollary 2.2.8,

$$\langle f, {}^*D^*\varphi \rangle \approx \langle f, \mathbb{D}^{\pm *}\varphi \rangle.$$

By the discrete summation by parts formula 2.2.2 and by Lemma 2.2.13 we have

$$\langle f, \mathbb{D}^{\pm *}\varphi \rangle \approx -\langle \mathbb{D}^{\pm} f, {}^*\varphi \rangle$$

from which we deduce

$$\langle D[f], \varphi \rangle_{\mathscr{D}(\Omega)} = {}^{\circ}\langle [\mathbb{D}^{\pm} f], {}^*\varphi \rangle$$

for all $\varphi \in \mathscr{D}(\Omega)$. □

By composing finite difference operators, we obtain the grid function counterpart of many differential operators. In Section 3.3, we will use the grid function counterpart of the gradient, the divergence and the Laplacian.

Definition 2.2.16 (Grid gradient, divergence and Laplacian)**.** *If $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, we define the forward and backward grid gradient of f as:*

$$\nabla^{\pm}_{\mathbb{X}} f = (\mathbb{D}^{\pm}_1 f, \ldots, \mathbb{D}^{\pm}_i, \ldots, \mathbb{D}^{\pm}_k f).$$

In a similar way, if $f : \Omega_{\mathbb{X}} \to {}^*\mathbb{R}^k$, we define the forward and backward grid divergence as

$$\operatorname{div}^{\pm}_{\mathbb{X}}(f(x,t)) = \sum_{i=1}^{k} \mathbb{D}^{\pm}_i f(x,t).$$

The grid Laplacian of $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ is defined as

$$\Delta_{\mathbb{X}} f = \operatorname{div}^{-}_{\mathbb{X}}(\nabla^{+}_{\mathbb{X}}(f)) = \operatorname{div}^{+}_{\mathbb{X}}(\nabla^{-}_{\mathbb{X}}(f)) = \sum_{i=1}^{k} \mathbb{D}^{+}_i \mathbb{D}^{-}_i f.$$

In the sequel, we will mostly drop the symbol + from the above definitions: for instance, we will write $\nabla_{\mathbb{X}}$ instead of $\nabla^{+}_{\mathbb{X}}$.

It is a consequence of Theorem 2.1.15 that, if $f \in S^{(1,\ldots,1)}(\Omega_{\mathbb{X}})$, then ${}^{\circ}(\nabla_{\mathbb{X}}(f))$ is the usual gradient of ${}^{\circ}f$, and similar results holds for $\nabla^{-}_{\mathbb{X}}$, $\operatorname{div}_{\mathbb{X}}$, $\operatorname{div}^{-}_{\mathbb{X}}$ and $\Delta_{\mathbb{X}}$. Moreover, by Theorem 2.2.15, the operators $\nabla_{\mathbb{X}}$ and $\nabla^{-}_{\mathbb{X}}$ satisfy the formula

$${}^{\circ}\langle \nabla_{\mathbb{X}} f, {}^*\varphi \rangle = {}^{\circ}\langle \nabla^{-}_{\mathbb{X}} f, {}^*\varphi \rangle = -\langle [f], \operatorname{div} \varphi \rangle_{\mathscr{D}(\Omega)}$$

for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all functions $\varphi \in \mathscr{D}(\Omega, \mathbb{R}^k)$, and $\mathrm{div}_{\mathbb{X}}$ and $\mathrm{div}^-_{\mathbb{X}}$ satisfy the formula

$$^\circ\langle \mathrm{div}_{\mathbb{X}} f, {}^*\varphi\rangle = {}^\circ\langle \mathrm{div}^-_{\mathbb{X}} f, {}^*\varphi\rangle = -\langle [f], \nabla\varphi\rangle_{\mathscr{D}(\Omega)}$$

for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}, {}^*\mathbb{R}^k)$ and for all $\varphi \in \mathscr{D}(\Omega)$. For the discrete Laplacian $\Delta_{\mathbb{X}}$, it holds

$$^\circ\langle \Delta_{\mathbb{X}} f, {}^*\varphi\rangle = \langle [f], \Delta\varphi\rangle_{\mathscr{D}(\Omega)}$$

for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all $\varphi \in \mathscr{D}(\Omega)$.

2.2.2 Discrete chain rule for generalized distributions and the Schwartz impossibility theorem

For the usual distributions, it is well-known that some of the derivation rules that hold for smooth functions do not hold in general. In particular, it is a consequence of the impossibility theorem by Schwartz that no extension of the distributional derivative satisfies a product rule.

However, for the grid functions there are some discrete product rules that generalize the product rule for smooth functions. Indeed, the following identities can be established by a simple calculation.

Proposition 2.2.17 (Discrete product rules)**.** *Let $f, g \in \mathbb{G}(\Omega_{\mathbb{X}})$. Then*

$$\begin{aligned} \mathbb{D}^+(f\cdot g)(x) &= \frac{f(x+\varepsilon)g(x+\varepsilon) - f(x)g(x)}{\varepsilon} \\ &= f(x+\varepsilon)\mathbb{D}^+g(x) + g(x)\mathbb{D}^+f(x) \\ &= f(x)\mathbb{D}^+g(x) + g(x+\varepsilon)\mathbb{D}^+f(x) \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}^-(f\cdot g)(x) &= \frac{f(x)g(x) - f(x-\varepsilon)g(x-\varepsilon)}{\varepsilon} \\ &= f(x)\mathbb{D}^-g(x) + g(x-\varepsilon)\mathbb{D}^-f(x) \\ &= f(x-\varepsilon)\mathbb{D}^-g(x) + g(x)\mathbb{D}^-f(x). \end{aligned}$$

Example 2.2.18 (Derivative of the sign function and the product rule)**.** *For an in-depth discussion of this example and of the limitations in the definition of a product rule for the distributional derivative, we refer to [82]. Consider the following representative of the sign function*

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

For this function u, $u^2 = 1$ and $u^3 = u$, but the distributional derivative $u_x = 2\delta_0$ is different from $(u^3)_x = 3u^2u_x = 3u_x = 6\delta_0$. So, even if u^2 is smooth, the product rule does not hold.

If we regard u as a grid function, however, it is easy to see that $u \in \mathscr{D}'_{\mathbb{X}}(\mathbb{X})$, and with a simple calculation we obtain:

$$\mathbb{D}u(x) = \begin{cases} 2\varepsilon^{-1} & \text{if } x = -\varepsilon \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.4}$$

Notice also that $[\mathbb{D}u] = 2\delta_0$, as we expected from Theorem 2.2.15. Applying one of the chain rule formulas of Lemma 2.2.17 and taking into account that $u^2(x) = 1$ for all $x \in \mathbb{X}$, we obtain

$$\begin{aligned} \mathbb{D}u^3(x) &= u(x)\mathbb{D}u^2(x) + u^2(x+\varepsilon)\mathbb{D}u(x) \\ &= u(x)(u(x)\mathbb{D}(x) + u(x+\varepsilon)\mathbb{D}u(x)) + \mathbb{D}u(x) \\ &= \mathbb{D}u(x)(2 + u(x)u(x+\varepsilon)) \end{aligned}$$

so that

$$\mathbb{D}u^3(x) = \begin{cases} \mathbb{D}u(-\varepsilon) = 2\varepsilon^{-1} & \text{if } x = -\varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

in agreement with 2.2.4.

We can summarize the results obtained so far as follows: the space of grid functions

- is a vector space over ${}^*\mathbb{R}$ that extends the space of distributions in the sense of Theorem 2.2.9;
- has a well-defined pointwise multiplication that extends the one defined for S^0 functions;
- has a derivative $\mathbb{D}$ that generalizes the distributional derivative and for which the discrete version of the chain rule established in Proposition 2.2.17 holds.

These properties are the nonstandard, discrete counterparts to the ones itemized in the impossibility theorem by Schwartz [77]. As a consequence, the space of grid functions can be seen as a non-trivial generalization of the space of distributions, as we claimed at the beginning of this section.

We will complete our discussion about the relation of the space of grid functions and the space of distributions by showing that the space of distributions can be embedded, albeit in a non-canonical way, in the space of grid functions. Notice that we cannot ask to this embedding to be fully coherent with derivatives: in fact, there is already an infinitesimal discrepancy between the usual derivative and the discrete derivative in the set of polynomials: the derivative of x^2 is $2x$, but $\mathbb{D}x^2 = 2x + \varepsilon$. However, as shown in Theorem 2.1.15, for all $f \in C^n$, $D^n f = [\mathbb{D}^n({}^*f_{|\mathbb{X}})]$. In fact, the canonical linear embedding $l : C^0(\mathbb{R}) \hookrightarrow S^0(\mathbb{X})$ given by $l(f) = {}^*f_{|\mathbb{X}}$ does not preserve derivatives, but it has the weaker property

$$l(f') \equiv \mathbb{D}(l(f)). \tag{2.2.5}$$

This will be the weaker coherence request that we will impose on the embedding from the space of distributions to the space of grid functions.

Theorem 2.2.19. *Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a partition of unity, and let H be a Hamel basis for $\mathscr{D}'(\mathbb{R})$. There is a linear embedding $l : \mathscr{D}'(\mathbb{R}) \to \mathscr{D}'_{\mathbb{X}}(\mathbb{X})$, that depends on $\{\psi_n\}_{n\in\mathbb{N}}$ and H, that satisfies the following properties:*

1. *$\Phi \circ l = id$;*
2. *the product over $\mathscr{D}'_{\mathbb{X}}(\mathbb{X}) \times \mathscr{D}'_{\mathbb{X}}(\mathbb{X})$ generalizes the pointwise product over $C^0(\mathbb{R}) \times C^0(\mathbb{R})$;*
3. *the derivative $\mathbb{D}$ over $\mathscr{D}'_{\mathbb{X}}(\mathbb{X})$ extends the distributional derivative in the sense of equation 2.2.5;*
4. *the chain rule for products holds in the form established in Lemma 2.2.17.*

Proof. We will define l over H and extend it to all of $\mathscr{D}'(\mathbb{R})$ by linearity. Let $T \in H$. From the representation theorem of distributions (see for instance [80]), we obtain

$$T = \sum_{n\in\mathbb{N}} T\psi_n = \sum_{n\in\mathbb{N}} D^{a_n} f_n \tag{2.2.6}$$

with $f_n \in C^0(\mathbb{R})$ and $\operatorname{supp}(D^{a_n} f_n) \subseteq \operatorname{supp} \psi_n$ for all $n \in \mathbb{N}$. Moreover, the sum is locally finite and for all $\varphi \in \mathscr{D}(\Omega)$ there exists a finite set $I_\varphi \subset \mathbb{N}$ such that

$$\langle T, \varphi\rangle_{\mathscr{D}(\Omega)} = \langle \sum_{i\in I_\varphi} D^{a_i} f_i, \varphi\rangle_{\mathscr{D}(\Omega)}. \tag{2.2.7}$$

Let $\{\phi_n\}_{n\in^*\mathbb{N}}$ be the nonstandard extension of the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, and let $\{b_n\}_{n\in^*\mathbb{N}}$ be the nonstandard extension of the sequence $\{a_n\}_{n\in\mathbb{N}}$. By transfer, from the representation 2.2.6 we obtain

$$^*T = \sum_{n\in^*\mathbb{N}} {}^*T\phi_n = \sum_{n\in^*\mathbb{N}} {}^*D^{b_n} g_n \tag{2.2.8}$$

with $g_i \in {}^*C^0(\mathbb{R})$ and $\operatorname{supp}(D^{b_n} g_n) \subseteq \operatorname{supp} \psi_n$ for all $n \in {}^*\mathbb{N}$. We may also assume that the representation 2.2.8 has the following properties:

1. $b_n = \min\left\{m \in {}^*\mathbb{N} : {}^*T\phi_n = {}^*D^m f \text{ with } f \in {}^*C^0(\mathbb{R})\right\}$ for all $n \in {}^*\mathbb{N}$
2. if ${}^*T\phi_n = {}^*D^{b_n} g = {}^*D^{b_n} h$ with $g, h \in {}^*C^0(\mathbb{R})$, then $g - h$ is a polynomial of a degree not greater than $b_n - 1$;
3. if n is finite and ${}^*T\phi_n = {}^*D^{b_n} g_n$, then $g_n = {}^*f_n$ and $b_n = a_n$, where f_n and a_n satisfy $T\psi_n = D^{a_n} f_n$.

For $T \in H$, we define

$$l(T) = \sum_{n \in {}^*\mathbb{N}:\ b_n \leq N} \mathbb{D}^{b_n}(g_{n|\mathbb{X}}),$$

and we extend l to $\mathscr{D}'(\mathbb{R})$ by linearity. Notice that l does not depend on the choice of the functions $\{g_n\}_{n \in {}^*\mathbb{N}}$. In fact, suppose that ${}^*T\phi_n = {}^*D^{b_n} g = {}^*D^{b_n} h$ with $g, h \in {}^*C^0(\mathbb{R})$. By property (2) of the representation 2.2.8, $g-h$ is a polynomial of a degree not greater than $b_n - 1$. Recall that, if $p \in \mathbb{G}(\mathbb{X})$ is a polynomial of degree at most $b_n - 1$, then $\mathbb{D}^{b_n} p = 0$. As a consequence, $\mathbb{D}^{b_n}(g_{|\mathbb{X}}) = \mathbb{D}^{b_n}(h_{|\mathbb{X}})$, as we wanted.

We will now show that, for all $T \in H$, $\langle \Phi([l(T)]), \varphi \rangle_{\mathscr{D}(\Omega)} = \langle T, \varphi \rangle_{\mathscr{D}(\Omega)}$ for all $\varphi \in \mathscr{D}'(\mathbb{R})$. This and linearity of l entail that $\Phi \circ l = id$. Let $\varphi \in \mathscr{D}(\mathbb{R})$, and let $I_\varphi \subset \mathbb{N}$ a finite set such that equality 2.2.7 holds. We claim that whenever $i \notin I_\varphi$, then $\langle \mathbb{D}^{b_i}(g_{i|\mathbb{X}}), {}^*\varphi \rangle = 0$. In fact, if $i \notin I_\varphi$ is finite, then by formula 2.2.7 and by property (3) of the representation 2.2.8 we have

$$^\circ\langle \mathbb{D}^{b_i}(g_{i|\mathbb{X}}), \varphi \rangle = {}^\circ\langle \mathbb{D}^{a_i}({}^*f_{i|\mathbb{X}}), \varphi \rangle = \langle D^{a_i} f_i, \varphi \rangle_{\mathscr{D}(\Omega)} = 0.$$

We want to show that $\langle \mathbb{D}^{b_i}(g_{i|\mathbb{X}}), {}^*\varphi \rangle = 0$ also when i is infinite. Notice that if $x \in {}^*\mathbb{R}_{fin}$, then for sufficiently large $n \in \mathbb{N}$ it holds $x \notin \operatorname{supp} \phi_n$: otherwise, we would also have ${}^\circ x \in \operatorname{supp} \psi_n$ for arbitrarily large n, against the fact that for all $x \in {}^*\mathbb{R}_{fin}$, ${}^\circ x \in \operatorname{supp} \phi_n$ only for finitely many n. As a consequence, $\operatorname{supp} \phi_i \cap {}^*\mathbb{R}_{fin} = \emptyset$, and by the inclusion $\operatorname{supp}(D^{b_i} g_i) \subseteq \operatorname{supp} \phi_i$, then also $\operatorname{supp}(D^{b_i} g_i) \cap {}^*\mathbb{R}_{fin} = \emptyset$. Taking into account property (2) of the representation 2.2.8, we deduce that the restriction of g_i to ${}^*\mathbb{R} \setminus \operatorname{supp}(D^{b_i} g_i)$ is a polynomial p of degree at most $b_n - 1$. We have already observed that $\mathbb{D}^{b_i} p = 0$ and, as a consequence, ${}^\circ\langle \mathbb{D}^{b_i}(g_{i|\mathbb{X}}), \varphi \rangle = 0$.

We then have the following equality:

$$\langle l(T), {}^*\varphi \rangle = \langle \sum_{i \in I_\varphi} \mathbb{D}^{a_i}({}^*f_{i|\mathbb{X}}), {}^*\varphi \rangle.$$

By Theorem 2.2.15, we obtain

$$\langle l(T), {}^*\varphi \rangle = \langle \sum_{i \in I_\varphi} \mathbb{D}^{a_i}({}^*f_{i|\mathbb{X}}), {}^*\varphi \rangle = \langle \sum_{i \in I_\varphi} D^{a_i} f_i, \varphi \rangle_{\mathscr{D}(\Omega)} = \langle T, \varphi \rangle_{\mathscr{D}(\Omega)},$$

that is sufficient to conclude that $\Phi([l(T)]) = T$.

Assertion (2) is a consequence of Lemma 2.1.13, assertion (3) is a consequence of Theorem 2.2.15, and assertion (4) is a consequence of Proposition 2.2.17. □

2.3 Grid functions as ${}^*L^p$ functions and as parametrized measures

The main goal of this section is to show that there is an external ${}^*\mathbb{R}_{fin}$-submodule of the space of grid functions whose elements correspond to

Young measures, and that this correspondence is coherent with the projection Φ defined in Theorem 2.2.9. Moreover, we will show how this correspondence can be generalized to arbitrary grid functions. Before we prove these results, we find it useful to discuss some properties of grid functions when they are interpreted as ${}^*L^p$ functions. These properties will be used also in Section 2.4, when we will discuss the grid function formulation of partial differential equations.

Recall that for all $1 \leq p \leq \infty$, a function $f \in L^p(\Omega)$ induces a distribution $T_f \in \mathscr{D}'(\Omega)$ defined by

$$\langle T_f, \varphi \rangle_{\mathscr{D}(\Omega)} = \int_\Omega f \varphi dx$$

for all $\varphi \in \mathscr{D}(\Omega)$. As a consequence, by identifying f with T_f we have the inclusions $L^p(\Omega) \subset \mathscr{D}'(\Omega)$ for all $1 \leq p \leq \infty$. Since Φ is surjective, we expect that for all $f \in L^p(\Omega)$ there exists $g \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ satisfying $[g] = T_f$. In this case, we will often write $[g] = f$ and $[g] \in L^p(\Omega)$. If $[g] \in L^p(\Omega)$, thanks to the Riesz representation theorem, we can think of $[g]$ either as a functional acting on $L^{p'}(\Omega)$, or as a member of an equivalence class of $L^p(\Omega)$ functions. To our purposes, we find it more convenient to treat $[g]$ as a function. With this interpretation, if $f = [g]$ and $f \in L^p(\Omega)$, then it holds the equality $f(x) = [g](x)$ for almost every $x \in \Omega$.

2.3.1 Grid functions as ${}^*L^p$ functions

If $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, then we can identify f with a piecewise constant function defined on all of ${}^*\mathbb{R}^k$. Among many different extensions, we choose to represent f by the function $\widehat{f}$ defined by

$$\widehat{f}(x) = \begin{cases} f((n_1, n_2, \dots, n_k)\varepsilon) & \text{if } n_i\varepsilon \leq x_i < (n_i + 1)\varepsilon \text{ for all } 1 \leq i \leq k \\ 0 & \text{if } |x_i| > N \text{ for some } 1 \leq i \leq k, \end{cases}$$

with the agreement that $f((n_1, n_2, \dots, n_k)\varepsilon) = 0$ if $(n_1, n_2, \dots, n_k)\varepsilon \notin \Omega_{\mathbb{X}}$.

If f is a grid function, the function $\widehat{f}$ is an internal *simple function and, as such, it belongs to ${}^*L^p(\mathbb{R}^k)$ for all $1 \leq p \leq \infty$. The integral of $\widehat{f}$ is related with the grid integral of f by the following formula:

$${}^*\!\int_{{}^*\mathbb{R}^k} \widehat{f} dx = \int_{\Omega_{\mathbb{X}}} f(x) d\mathbb{X}^k = \varepsilon^k \sum_{x \in \Omega_{\mathbb{X}}} f(x).$$

As a consequence, the ${}^*L^p$ norm of $\widehat{f}$ can be expressed by

$$\|\widehat{f}\|_p^p = \varepsilon^k \sum_{x \in \Omega_{\mathbb{X}}} |f(x)|^p \text{ if } 1 \leq p < \infty, \text{ and } \|\widehat{f}\|_\infty = \max_{x \in \Omega_{\mathbb{X}}} |f(x)|.$$

Moreover, notice that if $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, then ${}^\circ\text{supp}\, \widehat{f} \subseteq {}^\circ\text{supp}\, \widehat{\chi_{\Omega_{\mathbb{X}}}} = \overline{\Omega}$. If we define $\widehat{\Omega} = \text{supp}\, \widehat{\chi_{\Omega_{\mathbb{X}}}}$, then from the above inclusion we can write

$\widehat{f} \in {}^*L^p(\widehat{\Omega})$ for all $1 \le p \le \infty$. By identifying $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ with $\widehat{f}$, for all $1 \le p \le \infty$ the space of grid functions is identified with a subspace of ${}^*L^p(\widehat{\Omega})$ which is closed with respect to the ${}^*L^p$ norm. Since $\widehat{\Omega}$ is *bounded in ${}^*\mathbb{R}^k$, for $1 \le p \le \infty$ we have the usual relations between the ${}^*L^p$ norms of $f \in \mathbb{G}(\Omega_{\mathbb{X}})$:

$$\|\widehat{f}\|_1 \le \|\widehat{f}\|_p \le \|\widehat{f}\|_\infty.$$

From now on, when there is no risk of confusion, we will often abuse the notation and write f instead of $\widehat{f}$.

We begin our study of grid functions as ${}^*L^p$ functions by showing that if a grid function f has finite ${}^*L^p$ norm for some $1 \le p \le \infty$, then $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and, as a consequence, $[f]$ is a well-defined distribution.

Lemma 2.3.1. *If $\|f\|_p \in {}^*\mathbb{R}_{fin}$ for some $1 \le p \le \infty$, then $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.*

Proof. Notice that $\mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}) \subset {}^*L^p(\widehat{\Omega})$ for all $1 \le p \le \infty$ and, for any $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $\|\varphi\|_p \in {}^*\mathbb{R}_{fin}$ for all $1 \le p \le \infty$. By the discrete Hölder's inequality

$$|\langle f, \varphi\rangle| \le \|f\varphi\|_1 \le \|f\|_p \|\varphi\|_{p'}$$

so that if $\|f\|_p \in {}^*\mathbb{R}_{fin}$, then $\langle f, \varphi\rangle \in {}^*\mathbb{R}_{fin}$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, as desired. □

From the previous Lemma we deduce that, if the L^p norm of the difference of two grid functions f and g is infinitesimal, then $f \equiv g$.

Corollary 2.3.2. *Let $f, g \in \mathbb{G}(\Omega_{\mathbb{X}})$. If $\|f - g\|_p \approx 0$ for some $1 \le p \le \infty$, then $f \equiv g$.*

Proof. If $\|f - g\|_p \approx 0$, then by Lemma 2.3.1

$$\langle f - g, \varphi\rangle \le \|f - g\|_p \|\varphi\|_{p'} \approx 0$$

for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. As a consequence, $f \equiv g$. □

Notice that the other implication does not hold, in general. As an example, consider the grid function $f(n\varepsilon) = (-1)^n$. Since $\langle f, \varphi\rangle \approx 0$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, we deduce that $[f] = 0$, but $\|f\|_p = 1$ for all $1 \le p \le \infty$. Notice also that $\|f\|_p$ is finite, but $\|\widehat{f} - {}^*g\|_p \not\approx 0$ for all $g \in L^p(\Omega)$ and for all $1 \le p \le \infty$.

In the next section, we will show that the hypothesis $\|f\|_\infty \in {}^*\mathbb{R}_{fin}$ is sufficient to ensure that $[f] \in L^\infty(\Omega)$. If $1 \le p < \infty$, however, the hypothesis $\|f\|_p \in {}^*\mathbb{R}_{fin}$ is not sufficient to imply that $[f] \in L^p(\Omega)$. An example is given by $N\chi_0 \in \mathbb{G}(\mathbb{X})$, a representative of the Dirac distribution centred at 0. It can be calculated that

$$\|N\chi_0\|_1 = \varepsilon N = 1,$$

but $[N\chi_0] = \delta_0 \notin L^p(\mathbb{R})$ for any p. In general, whenever $[f] \in L^p(\Omega)$, it holds the inequality $\|f\|_p \ge \|[f]\|_p$.

Proposition 2.3.3. *For all $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ and for all $1 \leq p \leq \infty$, if $[f] \in L^p(\Omega)$, then*

1. *if $[|f|] \in L^p(\Omega)$, then $[|f|] \geq |[f]|$ a.e. in Ω;*

2. $^\circ\|f\|_p \geq \|[f]\|_p$.

Proof. Define $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \min\{f(x), 0\}$, so that $f = f^+ + f^-$ and $|f|^p = |f^+|^p + |f^-|^p$ for all $1 \leq p < \infty$. If $[|f|] \in L^p(\Omega)$, then $[f^+]$ and $[f^-] \in L^p(\Omega)$ and, by linearity of Φ,

$$[|f|](x) = [f^+](x) - [f^-](x) \geq [f^+](x) + [f^-](x) = [f](x)$$

for a.e. $x \in \Omega$.

Let $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ and suppose that $[f] \in L^p(\Omega)$ with $p < \infty$. If either $|f^+| \notin \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^+|^p \notin \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^-| \notin \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ or $|f^-| \notin \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ then by Lemma 2.3.1 we would have $|f|^p \notin \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and, as a consequence,

$$\|f\|_p^p = \||f|^p\|_1 \notin {}^*\mathbb{R}_{fin},$$

so that inequality (2) would hold. Suppose then that $|f^+| \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^+|^p \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $|f^-| \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $|f^-|^p \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. As a consequence, both $|f| \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and $|f|^p \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$. If $[|f|] \in L^p(\Omega)$, then (2) is a consequence of (1). The only case left is $[|f|] \notin L^p(\Omega)$.

For a matter of commodity, let $g = [f]$, and let $g^+(x) = \max\{g(x), 0\}$ and $g^-(x) = \min\{g(x), 0\}$. Since

$$[f^+] + [f^-] = [f] = g^+ + g^- \text{ in } \mathscr{D}'(\Omega),$$

we deduce that

$$[f^+] - g^+ = -([f^-] - g^-).$$

Since $[f^+] \notin L^p(\Omega)$, then also $[f^+] - g^+ \notin L^p(\Omega)$. Let $K = \operatorname{supp}([f^+] - g^+)$: then for all $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp}\varphi \subset K$ and with $\varphi(x) \geq 0$ for all $x \in \Omega$,

$$0 \leq \langle [f^+] - g^+, \varphi \rangle_{\mathscr{D}(\Omega)} = {}^\circ\langle f^+, {}^*\varphi \rangle - \int_\Omega g^+ \varphi dx.$$

Similarly,

$$0 \leq -\langle [f^-] - g^-, \varphi \rangle_{\mathscr{D}(\Omega)} = {}^\circ\langle |f^-|, {}^*\varphi \rangle - \int_\Omega |g^-| \varphi dx.$$

From the arbitrariness of φ, we deduce $\|f\chi_{K_{\mathbb{X}}}\|_p \geq \|g\chi_K\|_p$. Since $K = \operatorname{supp}([f^+] - g^+)$, we also have

$$\|([f^+] - g^+)\chi_{\Omega \setminus K}\|_p = \|([f^-] - g^-)\chi_{\Omega \setminus K}\|_p = \|0\|_p = 0,$$

from which we conclude that (2) indeed holds.

Suppose now that $[f] \in L^\infty(\Omega)$. If $\|f\|_\infty \notin {}^*\mathbb{R}_{fin}$, then inequality (2) holds. If $\|f\|_\infty \in {}^*\mathbb{R}_{fin}$, let $c_f \in \mathbb{G}(\Omega_\mathbb{X})$ satisfy $c_f(x) = \|f\|_\infty$ for all $x \in \Omega_\mathbb{X}$. Then $[c_f](x) = {}^\circ\|f\|_\infty$ for all $x \in \Omega$, so that $[c_f] \in L^\infty(\Omega)$. Since $c_f(x) \geq \max\{f^+(x), |f^-(x)|\}$ for all $x \in \Omega_\mathbb{X}$, then also $[c_f](x) \geq [f](x)$ for all $x \in \Omega_\mathbb{X}$. This is sufficient to conclude that inequality (2) holds. □

If $[f] \in L^p(\Omega_\mathbb{X})$ and ${}^\circ\|f\|_p > \|[f]\|_p$, then f features some oscillations that are compensated by the linearity of Φ. In this case, we can interpret f as the representative of a weak or (weak-$\star$ when $p = \infty$) limit of a sequence of functions whose L^p norm is uniformly bounded by ${}^\circ\|f\|_p$. In the next section, we will see how the behaviour of this weak-$\star$ limit can be described by a parametrized measure associated to f.

If $\|f\|_p \notin {}^*\mathbb{R}_{fin}$ but nevertheless $[f] \in L^p(\Omega)$, then f also features concentrations that are compensated by the linearity of Φ. An example is given by the function $f = \mathbb{D}\chi_0 = N\chi_{-\varepsilon} - N\chi_0$. The ${}^*L^p$ norm of f is $\|f\|_p = 2N^{p-1/p}$ for $p \neq \infty$ and N for $p = \infty$; however, from Theorem 2.2.15, we deduce that $[f] = D[\chi_0] = 0$. In the next section, we will discuss how these concentrations affect the parametrized measure associated to f.

We will now address the coherence between the nonstandard extension of a L^2 function and its projection in the space of grid functions. These technical results will be used in Section 2.4.

Definition 2.3.4. *Let $P : {}^*L^2(\widehat{\Omega}) \to \mathbb{G}(\Omega_\mathbb{X})$ be the ${}^*L^2$ projection over the closed subspace $\mathbb{G}(\Omega_\mathbb{X})$. Recall that $P(f)$ is the unique grid function satisfying*

$$\langle P(f), g\rangle = {}^*\int_{{}^*\Omega} f(x)\widehat{g}(x)dx$$

for all $g \in \mathbb{G}(\Omega_\mathbb{X})$.

Lemma 2.3.5. *For all $f \in C^0(\Omega)$, $P({}^*f) \in S^0(\Omega_\mathbb{X})$ and ${}^*f(x) \approx P({}^*f)(x)$ for all $x \in \Omega_\mathbb{X}$.*

Proof. Let $f \in C^0(\Omega)$. Since for all $g \in \mathbb{G}(\Omega_\mathbb{X})$ we have the equality

$$\langle P({}^*f), g\rangle = {}^*\int_{{}^*\Omega} {}^*f(x)\widehat{g}(x)dx,$$

by choosing $g = \varepsilon^{-k}\chi_y$, we obtain

$$P({}^*f)(y) = \langle P({}^*f), \widehat{\varepsilon^{-k}\chi_y}\rangle = \varepsilon^{-k*}\int_{[y,y+\varepsilon]^k} {}^*f(x)dx$$

for all $y \in \Omega_\mathbb{X}$. Since

$$\min_{x\in[y,y+\varepsilon]^k}\{{}^*f(x)\} \leq \varepsilon^{-k*}\int_{[y,y+\varepsilon]^k} {}^*f(x)dx \leq \max_{x\in[y,y+\varepsilon]^k}\{{}^*f(x)\},$$

by S-continuty of ${}^*f$, we deduce the thesis. □

Lemma 2.3.6. *For all $f \in L^2(\Omega)$, $[P({}^*f)] = f$.*

Proof. For all $\varphi \in \mathscr{D}'(\Omega)$ we have

$$\langle P({}^*f), {}^*\varphi_{|\mathbb{X}} \rangle = {}^*\int_{{}^*\Omega} {}^*f \widehat{{}^*\varphi_{|\mathbb{X}}} dx$$

and, by S-continuity of ${}^*\varphi$,

$${}^*\int_{{}^*\Omega} {}^*f \widehat{{}^*\varphi_{|\mathbb{X}}} dx \approx {}^*\int_{{}^*\Omega} {}^*f {}^*\varphi dx = \int_{\Omega} f\varphi dx.$$

This implies $[P({}^*f)] = f$. □

The above Lemma can be sharpened under the hypothesis that Ω has finite measure.

Lemma 2.3.7. *Let $\mu_L(\Omega) < +\infty$. For all $f \in L^2(\Omega)$, $\|{}^*f - P({}^*f)\|_2 \approx 0$.*

Proof. Let $f \in L^2(\Omega)$, and let $r = {}^*f - P({}^*f)$. By the properties of the ${}^*L^2$ projection, we have

$$\|{}^*f\|_2 = \|P({}^*f)\|_2 + \|r\|_2. \tag{2.3.1}$$

By the nonstandard Lusin's Theorem, there exists a *compact set $K \subseteq {}^*\Omega$ that satisfies ${}^*\mu_L({}^*\Omega \setminus K) \approx 0$ and $\|r\chi_K\|_2 \approx 0$. Since ${}^*\mu_L({}^*\Omega \setminus K) \approx 0$ and since $f \in L^2(\Omega)$, we have also $\|{}^*f\chi_K\|_2 \approx \|{}^*f\|_2$ and, as a consequence,

$$\|{}^*f\|_2 \approx \|{}^*f\chi_K\|_2 = \|P({}^*f)\chi_K\|_2 + \|r\chi_K\| \approx \|P({}^*f)\chi_K\|_2.$$

From the inequality chain

$$\|{}^*f\|_2 \approx \|P({}^*f)\chi_K\|_2 \leq \|P({}^*f)\|_2 \leq \|{}^*f\|_2$$

we deduce that $\|{}^*f\|_2 \approx \|P({}^*f)\|_2$ that, by equality 2.3.1, implies $\|{}^*f - P({}^*f)\|_2 \approx 0$, as we wanted. □

The previous Lemma suggests a definition of nearstandardness that will be useful in the sequel of the paper.

Definition 2.3.8. *Let $\mu_L(\Omega) < +\infty$. We will say that $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ is nearstandard in $L^2(\Omega)$ iff there exists $g \in L^2(\Omega)$ such that $\|f - P({}^*g)\|_2 \approx 0$.*

Notice that, thanks to Corollary 2.3.2 and to Lemma 2.3.7, f is nearstandard in $L^2(\Omega)$ if and only if $[f] \in L^2(\Omega)$ and $\|f - P({}^*[f])\|_2 \approx 0$.

We conclude the study of the properties of grid functions as ${}^*L^p$ functions by discussing the generalization of an embedding due to Robinson and Bernstein

$$L^2(\Omega) \subset V \subset {}^*L^2(\Omega),$$

where V is a vector space of a hyperfinite dimension (for the details, we refer to [18, 33]). In our case, by considering the embedding l of the space of distributions to the space of grid functions defined in Theorem 2.2.19 and by modifying the extension of f to $\widehat{f}$, we will obtain the inclusions

$$L^p(\Omega) \subset \mathscr{D}'(\Omega) \subset \mathbb{G}(\Omega_{\mathbb{X}}) \subset {}^*L^p(\Omega)$$

for all $1 \leq p \leq \infty$.

Proposition 2.3.9. *Let l be defined as in the proof of Theorem 2.2.19. There is an embedding $l' : \mathbb{G}(\Omega_{\mathbb{X}}) \to \bigcap_{1 \leq p \leq \infty} {}^*L^p(\Omega)$ such that*

$$^*\int_{^*\mathbb{R}^k} (l' \circ l)(f)^* \varphi dx \approx \int_{\mathbb{R}^k} f \varphi dx \tag{2.3.2}$$

for all $1 \leq p \leq \infty$, for all $f \in L^p(\Omega)$ and for all $\varphi \in \mathscr{D}(\Omega)$. As a consequence, if we identify $\mathscr{D}'(\Omega)$ with $l(\mathscr{D}'(\Omega)) \subseteq \mathbb{G}(\Omega_{\mathbb{X}})$ and $\mathbb{G}(\Omega_{\mathbb{X}})$ with $l'(\mathbb{G}(\Omega_{\mathbb{X}})) \subseteq {}^*L^p(\Omega)$, we have the inclusions

$$L^p(\Omega) \subset \mathscr{D}'(\Omega) \subset \mathbb{G}(\Omega_{\mathbb{X}}) \subset {}^*L^p(\Omega)$$

for all $1 \leq p \leq \infty$.

Proof. Define l' by $l'(f) = \widehat{f}\chi_{^*\Omega}$ for all $f \in \mathbb{G}(\Omega_{\mathbb{X}})$. Since $l'(f)$ is an internal *simple function, it belongs to ${}^*L^p(\Omega)$ for all $1 \leq p \leq \infty$. We will now prove that, for this choice of l', equality 2.3.2 holds.

Notice that for all $f \in \mathbb{G}(\Omega_{\mathbb{X}})$, if $l'(f)(x) \neq \widehat{f}(x)$, then $x \in {}^*\Omega \setminus \widehat{\Omega}$ or $x \in \widehat{\Omega} \setminus {}^*\Omega$. By the definition of $\widehat{\Omega}$, this entails ${}^\circ x \in \partial\Omega$. In particular, if $\varphi \in \mathscr{D}(\Omega)$, then ${}^\circ x \notin \operatorname{supp} \varphi$. As a consequence, for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and for all $\varphi \in \mathscr{D}(\Omega)$, it holds

$$^*\int_{^*\mathbb{R}^k} l'(f) \, {}^*\varphi dx = {}^*\int_{^*\mathbb{R}^k} \widehat{f} \, {}^*\varphi dx.$$

By S-continuity of ${}^*\varphi$, we have also

$$^*\int_{^*\mathbb{R}^k} \widehat{f} \, {}^*\varphi dx \approx \langle f, {}^*\varphi \rangle.$$

If we let $f = l(g)$ for some $g \in L^p(\Omega)$, from Theorem 2.2.19 we have

$$\langle l(g), {}^*\varphi \rangle \approx \langle g, \varphi \rangle_{\mathscr{D}(\Omega)} = \int_{\mathbb{R}^k} g \varphi dx.$$

By putting together the previous equalities, we conclude that equation 2.3.2 holds. $\square$

We conjecture that for $p = 2$ and under the hypothesis that Ω has finite Lebesgue measure then by an appropriate choice of the embedding l defined in Theorem 2.2.19, we could have $\|(l' \circ l)(f) - {}^*f\|_2 \approx 0$, as in the original embedding by Robinson and Bernstein.

2.3.2 Grid functions as parametrized measures

It is well known that weak limits of L^p functions behave badly with respect to composition with a nonlinear function [2, 39, 81, 88]. Consider for instance a bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ of $L^\infty(\Omega)$ functions: by the Banach–Alaoglu theorem, there is a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ that has a weak-$\star$ limit $u_\infty \in L^\infty(\Omega)$. Now let $f \in C^0_b(\mathbb{R})$: the sequence $\{f(u_n)\}_{n\in\mathbb{N}}$ is still bounded in $L^\infty(\Omega)$, so it has a weak-$\star$ limit f_∞. However, in general $f_\infty \neq f(u_\infty)$. To overcome this difficulty, the weak-$\star$ limit of the sequence $\{u_n\}_{n\in\mathbb{N}}$ can be represented by a Young measure, i.e. a measurable function $\nu : \Omega \to \mathbb{M}_\mathbb{P}(\mathbb{R})$ such that for all $f \in C^0_b(\mathbb{R})$ the weak-$\star$ limit of $\{f(u_n)\}_{n\in\mathbb{N}}$ is the function defined a.e. by $\overline{f}(x) = \int_\mathbb{R} f d\nu_x$, in the sense that the equality

$$\lim_{n\to\infty} \int_\Omega f(u_n(x))g(x)dx = \int_\Omega \left(\int_\mathbb{R} f d\nu_x \right) g(x)dx = \int_\Omega \overline{f}(x)g(x)dx \quad (2.3.3)$$

holds for all $g \in L^1(\Omega)$.

Example 2.3.10. *The following example is discussed in detail in [88]. Consider the Rademacher functions $u_n(x) = u_0(n^2x)$, with $u_0(x) = \chi_{[0,1/2)}(x) - \chi_{[1/2,1)}(x)$ extended periodically over $\mathbb{R}$. It can be calculated that the Young measure ν associated to the sequence $\{u_n\}_{n\in\mathbb{N}}$ is constant and that*

$$\nu_x = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$$

for almost every $x \in \Omega$, i.e. that for all $f \in C^0_b(\mathbb{R})$ and for all $g \in L^1(\Omega)$,

$$\lim_{n\to\infty} \int_\mathbb{R} f(u_n(x))g(x)dx = \left(\frac{1}{2}f(1) + \frac{1}{2}f(-1) \right) \int_\mathbb{R} g(x)dx.$$

In the setting of grid functions, instead of bounded sequences of L^∞ functions, we have grid functions with finite ${}^*L^\infty$ norm. These functions can be used to represent weak-$\star$ limits of L^∞ functions.

Example 2.3.11. *The function $u(n\varepsilon) = (-1)^n$ can be thought as a representative for the weak-$\star$ limit of the Rademacher functions: in fact, for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\Omega)$,*

$$^\circ\langle {}^*f(u), {}^*\varphi \rangle = \left(\frac{1}{2}f(1) + \frac{1}{2}f(-1) \right) \int_\mathbb{R} \varphi(x)dx.$$

Since $C^0_c(\Omega)$ is dense in $L^1(\Omega)$, this is sufficient to conclude that the above formula holds for all $\varphi \in L^1(\Omega)$.

We will now make precise the connection between grid functions and Young measure by showing that every grid function that has finite ${}^*L^\infty$ norm corresponds to a Young measure. The proof of the following theorem relies on a result by Cutland [28].

Theorem 2.3.12. *For every $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ with $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, there exists a Young measure $\nu^u : \Omega \to \mathbb{M}_{\mathbb{P}}(\mathbb{R})$ such that, for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\Omega)$,*

$$^\circ\langle {}^*f(u), {}^*\varphi\rangle = \int_\Omega \left(\int_{\mathbb{R}} f d\nu^u_x\right) \varphi(x)dx. \tag{2.3.4}$$

Proof. Since $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, there exists $n \in \mathbb{R}$ such that $|u(x)| < n$. We can identify u with a function $\tilde{u} : \widehat{\Omega} \to {}^*\mathbb{M}_{\mathbb{P}}({}^*[-n, n])$ defined by $\tilde{u}(x) = \delta_{\widehat{u}(x)}$. Notice that for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\Omega)$ it holds

$$\begin{aligned} \langle {}^*f(u), {}^*\varphi\rangle &\approx {}^*\int_{\widehat{\Omega}} {}^*f(\widehat{u}(x)){}^*\varphi(x)dx \\ &= {}^*\int_{\widehat{\Omega}} \left({}^*\int_{{}^*[-n,n]} {}^*f d\tilde{u}(x)\right) {}^*\varphi(x)dx. \end{aligned} \tag{2.3.5}$$

We define an internal measure μ over ${}^*\Omega \times {}^*[-n, n]$ by posing

$$\mu(A \times B) = {}^*\int_A \tilde{u}_x(B)dx$$

for all Borel $A \subseteq \Omega$ and for all Borel $B \subseteq {}^*[-n, n]$. Let L_μ be the Loeb measure obtained from μ (for the properties of the Loeb measure, we refer for instance to [1, 59, 60, 73]). We can define a standard measure μ_s over $\Omega \times [-n, n]$ by posing

$$\mu_s(A \times B) = L_\mu(\{x \in {}^*\Omega \times {}^*[-n, n] : {}^\circ x \in A \times B\}).$$

Since μ_s satisfies $\mu_s(A \times [-n, n]) = \mu_L(A)$ for all Borel $A \subseteq \Omega$, by Rohlin's Disintegration Theorem the measure μ_s can be disintegrated as

$$\mu_s(A \times B) = \int_A \nu^u_x(B)dx,$$

with $\nu^u : \Omega \to \mathbb{M}_{\mathbb{P}}([-n, n])$. By Lemma 2.6 of [28], ν^u satisfies

$$^\circ\left({}^*\int_{{}^*\Omega} \left({}^*\int_{{}^*[-n,n]} {}^*f d\tilde{u}(x)\right) {}^*\varphi(x)dx\right) = \int_\Omega \left(\int_{[-n,n]} f d\nu^u_x\right) \varphi(x)dx.$$

for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\Omega)$. Thanks to equality 2.3.5, we deduce that ν^u satisfies 2.3.4. We can extend ν^u_x to all of $\mathbb{M}_{\mathbb{P}}(\mathbb{R})$ by defining $\nu^u_x(A) = \nu^u_x(A \cap [-n, n])$ for all Borel sets $A \subseteq \mathbb{R}$ and for all $x \in \Omega$, thus obtaining a Young measure that satisfies equation 2.3.4. □

In [3, 88], it is shown that Young measures describe weak-$\star$ limits of bounded sequences of L^∞ functions. We will now show that grid functions with finite L^∞ norm can be similarly used to represent weak-$\star$ limits of L^∞

functions in the setting of grid functions. This is a consequence of a more general property of the correspondence between grid functions and Young measures: if $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ satisfies $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$ and ν^u is the Young measure associated to u in the sense of Theorem 2.3.12, then $[u]$ corresponds to the barycentre of ν^u.

Theorem 2.3.13. *Let $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ with $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, and let ν^u be the Young measure that satisfies equality 2.3.4. Then $[u] \in L^\infty(\Omega)$ and the following equality holds for a.e. $x \in \Omega$:*

$$[u](x) = \int_{\mathbb{R}} \tau d\nu^u_x. \tag{2.3.6}$$

Moreover,

1. *if $\{u_n\}_{n\in\mathbb{N}}$ is a sequence of L^∞ functions that converges weakly-$\star$ to ν^u in the sense of equation 2.3.3, then $u_n \stackrel{\star}{\rightharpoonup} [u]$ in L^∞;*

2. *if ν^u is Dirac, then ν^u_x is the Dirac measure centred at $[u](x)$ for a.e. $x \in \Omega$.*

Proof. Define a function f_ν by posing $f_\nu(x) = \int_{\mathbb{R}} \tau d\nu^u_x$ for all $x \in \Omega$. Since $|f_\nu(x)| \leq {}^\circ\|u\|_\infty$ for a. e. $x \in \Omega$ and since $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, $f_\nu \in L^\infty(\Omega)$. By Theorem 2.3.12, for all $\varphi \in C^0_c(\Omega)$ we have the following equalities:

$$\int_\Omega f_\nu(x)\varphi(x)dx = \int_\Omega \int_{\mathbb{R}} \tau d\nu_x \varphi(x)dx = {}^\circ\langle u, {}^*\varphi\rangle = \int_\Omega [u]\varphi dx.$$

Since $C^0_c(\Omega)$ is dense in $L^1(\Omega)$, we deduce that $f_\nu = [u]$ in $L^\infty(\Omega)$, as we wanted.

We will now prove (1). By hypothesis, from equation 2.3.3 and from equation 2.3.4, it holds

$$\lim_{n\to\infty} \int_\Omega f(u_n(x))\varphi(x)dx = \int_\Omega \left(\int_{\mathbb{R}} f d\nu_x\right) \varphi(x)dx = {}^\circ\langle {}^*f(u), {}^*\varphi\rangle$$

for all $\varphi \in C^0_c(\Omega)$. As a consequence, by considering a function $f \in C^0_b(\mathbb{R})$ with $f(x) = 1$ for all x satisfying $|x| \leq {}^\circ\|u\|_\infty$, we obtain that the weak-$\star$ limit of the sequence $\{u_n\}_{n\in\mathbb{N}}$ is equal to $[u]$.

Assertion (2) is a consequence of equality 2.3.6. □

If the sequence $\{u_n\}_{n\in\mathbb{N}}$ is not bounded in L^∞, but it is bounded in $L^p(\Omega)$ for some $1 \leq p < \infty$, then it can be proved that there exists a parametrized measure $\nu : \Omega \to \mathbb{M}(\mathbb{R})$ such that for all $f \in C^0_b(\mathbb{R})$ the weak-$\star$ limit of the sequence $\{f(u_n)\}_{n\in\mathbb{N}}$ is the function defined a.e. by $\overline{f}(x) = \int_{\mathbb{R}} f d\nu_x$ (for an in-depth discussion of this result, we refer to [3]). Notice that ν takes values in $\mathbb{M}(\mathbb{R})$ instead of $\mathbb{M}_{\mathbb{P}}(\mathbb{R})$, since the sequence $\{u_n\}_{n\in\mathbb{N}}$ could diverge in a subset of Ω with positive measure.

The grid function counterpart of this result is that for any $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ there exists a function $\nu^u : \Omega \to \mathbb{M}(\mathbb{R})$ that satisfies equation 2.3.4, even if $\|u\|_\infty \not\in {}^*\mathbb{R}_{fin}$. If $\|u\|_\infty \not\in {}^*\mathbb{R}_{fin}$, ν^u_x might not be a probability measure, but it still satisfies the inequalities $0 \le \nu^u_x(\mathbb{R}) \le 1$ for all $x \in \Omega$. In particular, the difference between $\nu^u_x(\mathbb{R})$ and 1 is due to u being unlimited at some non-negligible fraction of $\mu(x) \cap \mathbb{X}^k$.

Theorem 2.3.14. *For every $u \in \mathbb{G}(\Omega_{\mathbb{X}})$, there exists a parametrized measure $\nu^u : \Omega \to \mathbb{M}(\mathbb{R})$ such that, for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\Omega)$, equality 2.3.4 holds. Moreover, for all $x \in \Omega$ and for all Borel $A \subseteq \mathbb{R}$, $0 \le \nu^u_x(A) \le 1$.*

Proof. Let $u \in \mathbb{G}(\Omega_{\mathbb{X}})$, and for all $n \in \mathbb{N}$ define

$$u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \le n, \\ n & \text{if } u(x) > n, \\ -n & \text{if } u(x) < -n. \end{cases}$$

Since for all $n \in \mathbb{N}$ it holds $\|u_n\|_\infty \le n \in {}^*\mathbb{R}_{fin}$, by Theorem 2.3.12 there exists a Young measure ν^n that satisfies

$$^\circ\langle {}^*f(u_n), {}^*\varphi\rangle = \int_\Omega \left(\int_\mathbb{R} f d\nu^n_x \right) \varphi(x) dx. \tag{2.3.7}$$

for all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\Omega)$.

Recall that a sequence of parametrized measures $\{\mu^n\}_{n\in\mathbb{N}}$ converges weakly-$\star$ to a parametrized measure μ if for all $f \in C^0_b(\mathbb{R})$, the sequence $\{f_n\}_{n\in\mathbb{N}}$ of L^∞ functions defined by

$$f_n(x) = \int_\mathbb{R} f d\mu^n_x$$

converges weakly-$\star$ to a function $f \in L^\infty(\Omega)$ defined by

$$f(x) = \int_\mathbb{R} f d\mu_x.$$

Define ν^u as the parametrized measure satisfying $\nu^n \overset{\star}{\rightharpoonup} \nu^u$ for some subsequence (not relabelled) of $\{\nu^n\}_{n\in\mathbb{N}}$. The existence of such a weak-$\star$ limit can be obtained as a consequence of the Banach-Alaouglu theorem (for further details about the weak-$\star$ limit of measures, we refer to to [39]). We claim that ν^u satisfies equality 2.3.4 and that for all $x \in \Omega$, $0 \le \nu^u_x(\mathbb{R}) \le 1$.

Let $f \in C^0_b(\mathbb{R})$. Since $\lim_{|x|\to\infty} f(x) = 0$, there is an increasing sequence of natural numbers $\{n_i\}_{i\in\mathbb{N}}$ such that if $|x| \ge n_i$, then $|f(x)| \le 1/i$. As a consequence of this inequality, for all $i \in \mathbb{N}$ and for all $\varphi \in C^0_c(\Omega)$ it holds

$$|\langle {}^*f(u_{n_i}), {}^*\varphi\rangle - \langle {}^*f(u), {}^*\varphi\rangle| \le 2/i \|{}^*\varphi\|_1.$$

Taking into account equation 2.3.7, from the previous inequality we obtain

$$\left| \int_\Omega \left(\int_\mathbb{R} f d\nu_x^{n_i} \right) \varphi(x) dx - {}^\circ\langle {}^*f(u), {}^*\varphi \rangle \right| \leq 2/i \|\varphi\|_1.$$

As a consequence, we deduce that

$$\lim_{i\to\infty} \int_\Omega \left(\int_\mathbb{R} f d\nu_x^{n_i} \right) \varphi(x) dx = {}^\circ\langle {}^*f(u), {}^*\varphi \rangle.$$

This is sufficient to entail that $\nu^n \stackrel{\star}{\rightharpoonup} \nu^u$ and that ν^u satisfies equality 2.3.4.

The inequality $0 \leq \nu_x^u(A) \leq 1$ for all Borel $A \subseteq \mathbb{R}$ is a consequence of the lower semicontinuity of the weak-$\star$ limit of measures (see for instance theorem 3 of [39]). □

Notice that, as a consequence of Theorem 2.3.14, we deduce that the hypothesis $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$ in Theorem 2.3.12 can be relaxed. In particular, if v differs from u at some null set, then u and v induce the same parametrized measure, even if $u \not\equiv v$.

Corollary 2.3.15. *Let L_N be the Loeb measure obtained from the measure $\mu_N(A) = |A|/N^k$ for all internal $A \subseteq \mathbb{X}^k$. If for $u, v \in \mathbb{G}(\Omega_\mathbb{X})$ it holds $L_N(\{x \in \Omega_\mathbb{X} : u(x) \not\approx v(x)\}) = 0$, then $\nu^u = \nu^v$. If $\|u - v\|_p \approx 0$, then $\nu^u = \nu^v$.*

Proof. If $L_N(\{x \in \Omega_\mathbb{X} : u(x) \not\approx v(x)\}) = 0$, then also

$$L_N(\{x \in \Omega_\mathbb{X} : {}^*f(u(x)) \not\approx {}^*f(v(x))\}) = 0$$

for all $f \in C_b^0(\mathbb{R})$. This is and the hypothesis $f \in C_b^0(\mathbb{R})$ are sufficient to deduce $\langle {}^*f(u), {}^*\varphi \rangle \approx \langle {}^*f(v), {}^*\varphi \rangle$ for all $\varphi \in C_c^0(\Omega)$ that, thanks to equation 2.3.4, is equivalent to the equality $\nu^u = \nu^v$.

The hypothesis $\|u - v\|_p \approx 0$ implies $L_N(\{x \in \Omega_\mathbb{X} : u(x) \not\approx v(x)\}) = 0$, so the equality between ν^u and ν^v is a consequence of the previous part of the proof. □

The above corollary can be seen as the grid function counterpart of Corollary 3.14 of [88], that shows how Young measure ignore concentration phenomena. We find it useful to discuss this behaviour with an example, that also highlights how a grid function can describe simultaneously very different properties of a sequence of L^p functions.

Example 2.3.16. *The following example is discussed from the standard viewpoint in [88]. Consider the sequence $\{u_n\}_{n\in\mathbb{N}}$ defined by $u_n(x) = n\chi_{[1-1/n,1]}$. Notice that $\|u_n\|_\infty = n$, so that the sequence is not bounded in $L^\infty(\mathbb{R})$. For all $f \in C_b^0(\mathbb{R})$ and for all $\varphi \in C_c^0(\mathbb{R})$, it holds*

$$\lim_{n\to\infty} \int_\mathbb{R} f(u_n)\varphi dx = f(0) \int_\mathbb{R} \varphi dx$$

so that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges weakly-$\star$ to the constant Young measure $\nu_x = \delta_0$ for all $x \in \mathbb{R}$.

The sequence $\{u_n\}_{n\in\mathbb{N}}$ satisfies the L^1 uniform bound $\|u_n\|_1 = 1$ for all $n \in \mathbb{N}$. Since for all $\varphi \in \mathscr{D}(\mathbb{R})$ it holds

$$\lim_{n\to\infty} \int_{\mathbb{R}} u_n \varphi dx = \lim_{n\to\infty} n \int_{[1-1/n,1]} \varphi dx = \varphi(1)$$

the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges in the sense of distributions to δ_1, the Dirac distribution centred at 1. Indeed, it can be proved that the sequence $\{u_n\}_{n\in\mathbb{N}}$ converges weakly-$\star$ to δ_1 in the space $\mathbb{M}(\mathbb{R})$ of Radon measures.

In the setting of grid functions, a representative for the limit of the sequence $\{u_n\}_{n\in\mathbb{N}}$ is given by $u_N = N\chi_1$. For all $f \in C^0_b(\mathbb{R})$ and for all $\varphi \in C^0_c(\mathbb{R})$, it holds

$$\langle {}^*f(u_N), {}^*\varphi\rangle = \varepsilon \sum_{x\in\mathbb{X},\, x\neq 1} f(0)^*\varphi(x) + \varepsilon^* f(N)\varphi(1).$$

Since $f \in C^0_b(\mathbb{R})$, ${}^*f(N) \approx 0$ and, by Lemma 2.1.9, we deduce

$$^\circ\langle {}^*f(u_N), {}^*\varphi\rangle = f(0) \int_{\mathbb{R}} \varphi(x) dx.$$

From the above equality and from equation 2.3.4, we deduce that the Young measure associated to u_N is the constant Young measure $\nu_x = \delta_0$ for all $x \in \mathbb{R}$. Notice that the same result could have been deduced from Corollary 2.3.15 by noticing that, since $L_N(\{x \in \Omega_{\mathbb{X}} : u_N(x) \not\approx 0\}) = 0$, the Young measure associated to u_N is the same as the Young measure associated to the constant function $c(x) = 0$ for all $x \in {}^*\mathbb{R}$.

As for the distribution corresponding to $[u_N]$, since for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\mathbb{X})$ it holds $\langle N\chi_1, \varphi\rangle = \varphi(1)$, we deduce that $[u_N] = \delta_1$. In particular, the grid function u_N coherently describes the behaviour of the limit of the sequence $\{u_n\}_{n\in\mathbb{N}}$ both in the sense of Young measures and in the sense of distributions.

In the previous example we have considered a grid function u with $\|u\|_1 \in {}^*\mathbb{R}_{fin}$, and we verified that the parametrized measure associated to u was indeed a Young measure. This result holds under the more general hypothesis that $\|u\|_p \in {}^*\mathbb{R}_{fin}$.

Proposition 2.3.17. *If $\|u\|_p \in {}^*\mathbb{R}_{fin}$, then ν^u_x is a probability measure for a.e. $x \in \Omega$.*

Proof. If for some $x \in \Omega$ it holds $\nu^u_x(\mathbb{R}) < 1$, then there exists $y \in \Omega_{\mathbb{X}}$, $y \approx x$ such that $u(y) \not\in {}^*\mathbb{R}_{fin}$. The hypothesis $\|u\|_p \in {}^*\mathbb{R}_{fin}$ implies $L_N(\{y \in \Omega_{\mathbb{X}} : u(y) \not\in {}^*\mathbb{R}_{fin}\}) = 0$: this is sufficient to conclude that $\mu_L(\{x \in \Omega : \nu^u_x(\mathbb{R}) < 1\}) = 0$, as desired. □

We will conclude the discussion of the relations between grid functions and parametrized measures by determining the parametrized measure associated to a periodic grid function with an infinitesimal period. This is the grid function counterpart of the formula for the Young measure associated to the limit of a sequence of periodic functions (see Example 3.5 of [2]). We will prove this result for $k = 1$, as the generalization to an arbitrary dimension is mostly a matter of notation.

Proposition 2.3.18. *If $u \in \mathbb{G}(\mathbb{X})$ is periodic of period $M\varepsilon \approx 0$, then the parametrized measure ν associated to u is constant, and*

$$\int_{\mathbb{R}} f d\nu_x = {}^\circ\left(\frac{1}{M}\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))\right)$$

for all $x \in \Omega$ and for all $f \in C^0_b(\mathbb{R})$.

Proof. Without loss of generality, let $M \in {}^*\mathbb{N}$ and let u be periodic over $[0,(M-1)\varepsilon] \cap \mathbb{X}$, with $M\varepsilon \approx 0$.

Let $f \in C^0_b(\mathbb{R})$. At first, we will prove that $\frac{1}{M}\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))$ is finite: in fact,

$$\inf_{x\in{}^*\mathbb{R}} {}^*f(x) \leq \frac{1}{M}\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon)) \leq \sup_{x\in{}^*\mathbb{R}} {}^*f(x) \tag{2.3.8}$$

and by the boundedness of f, we deduce that $\frac{1}{M}\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))$ is finite.

Let now $\varphi \in S^0(\mathbb{X})$ with $\operatorname{supp}\varphi \subset [a,b]$, $a,b \in {}^*\mathbb{R}_{fin}$. Then there exists $h,k \in {}^*\mathbb{N}$ satisfying $a \approx Mh\varepsilon$ and $b \approx Mk\varepsilon$. We have the equalities

$$\begin{aligned}
\langle {}^*f(u),\varphi\rangle &\approx \varepsilon \sum_{x\in[Mh\varepsilon,Mk\varepsilon]_{\mathbb{X}}} f(u(x))\varphi(x) \\
&= \varepsilon \sum_{j=h}^{k}\left(\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))\varphi(jM\varepsilon+i\varepsilon)\right) \\
&= \varepsilon \sum_{j=h}^{k}\left(\left(\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))\right)(\varphi(jM\varepsilon)+e(j))\right) \\
&= \left(\frac{1}{M}\sum_{i=0}^{M-1} {}^*f(u(i\varepsilon))\right)\left(M\varepsilon\sum_{j=h}^{k}(\varphi(jM\varepsilon)+e(j))\right)
\end{aligned} \tag{2.3.9}$$

Let

$$e = \max_{0\leq i\leq M,\ k\leq j\leq h}\{|\varphi(jM\varepsilon)-\varphi(jM\varepsilon+i\varepsilon)|\}.$$

Since $\varphi \in S^0(\mathbb{X})$ and $\operatorname{supp}\varphi \subset {}^*\mathbb{R}_{fin}$, $e \approx 0$ and, as a consequence, $|e(j)| \leq e \approx 0$. We deduce

$$\left|M\varepsilon\sum_{j=k}^{h} e(j)\right| \leq M\varepsilon(k-h)e \approx (b-a)e \approx 0$$

and, by equation 2.3.8,

$$\left(\frac{1}{M}\sum_{i=0}^{M-1}{}^*f(u(i\varepsilon))\right)\left(M\varepsilon\sum_{j=h}^{k}e(j)\right)\approx 0. \tag{2.3.10}$$

Since $M\varepsilon \approx 0$,

$$M\varepsilon\sum_{j=h}^{k}(\varphi(jM\varepsilon)\approx\int_{^\circ a}^{^\circ b}{}^\circ\varphi(x)dx. \tag{2.3.11}$$

Putting together equalities 2.3.9, 2.3.10 and 2.3.11, we conclude

$$^\circ\langle{}^*f(u),\varphi(x)\rangle = {}^\circ\left(\frac{1}{M}\sum_{i=0}^{M-1}{}^*f(u(i\varepsilon))\right)\int_{^\circ a}^{^\circ b}{}^\circ\varphi(x)dx$$

as we wanted. □

2.4 The grid function formulation of partial differential equations

In this section, we will give some results that allow to coherently formulate stationary and time-dependent PDEs in the sense of grid functions in a way that, if the solutions to the grid function formulation are regular enough, they induce standard solutions to the original problem. Moreover, the existence of regular solutions to the original problem is equivalent to the existence of regular solutions to the grid function formulation. If the original problem does not have solutions in the sense of distributions, then we regard the solution to the grid function formulation as a generalized solution. It turns out that, for some nonlinear problems, the generalized solution in the sense of grid functions is related with some notions of measure-valued solutions.

2.4.1 The grid function formulation of linear PDEs

A linear PDE can be written in the most general form as

$$L(u) = f, \tag{2.4.1}$$

with $f \in \mathscr{D}'(\Omega)$, where $L : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is linear, and where the equality is meant in the sense of distributions, i.e.

$$\langle L(u),\varphi\rangle_{\mathscr{D}(\Omega)} = \langle f,\varphi\rangle_{\mathscr{D}(\Omega)}$$

for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega)$. We would like to turn problem 2.4.1 in a problem in the sense of grid functions, i.e.

$$L_{\mathbb{X}}(u) = f_{\mathbb{X}}, \tag{2.4.2}$$

with $f_{\mathbb{X}} \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ and where $L_{\mathbb{X}} : \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \to \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ is ${}^*\mathbb{R}$-linear. Moreover, we would like to determine sufficient conditions that ensure equivalence between problem 2.4.2 and problem 2.4.1, in the sense that 2.4.1 has a solution if and only if 2.4.2 has a solution.

Such a coherent formulation of linear PDEs relies upon the existence of ${}^*\mathbb{R}$-linear extensions of linear functionals over the space of distributions. Recall that every linear functional $L : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ induces an adjoint $M : \mathscr{D}_{\mathbb{X}}(\Omega) \to \mathscr{D}_{\mathbb{X}}(\Omega)$ that satisfies

$$\langle L(T), \varphi \rangle_{\mathscr{D}(\Omega)} = \langle T, M(\varphi) \rangle_{\mathscr{D}(\Omega)}$$

for all $T \in \mathscr{D}'(\Omega)$ and for all $\varphi \in \mathscr{D}(\Omega)$. If we find a ${}^*\mathbb{R}$-linear extension of M in the sense of grid functions, by taking the adjoint we are able to define a ${}^*\mathbb{R}$-linear extension of L.

Lemma 2.4.1. *For every linear $L : \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ there is a ${}^*\mathbb{R}$-linear $L_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ such that $L_{\mathbb{X}}({}^*\varphi) = {}^*(L(\varphi))_{|\Omega_{\mathbb{X}}}$ for all $\varphi \in \mathscr{D}(\Omega)$.*

Proof. For $\varphi \in \mathscr{D}'(\Omega)$ define

$$U(\varphi) = \{L_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}}) \text{ such that } L_{\mathbb{X}} \text{ is } {}^*\mathbb{R}\text{-linear and } L_{\mathbb{X}}({}^*\varphi) = {}^*L(\varphi)_{|\Omega_{\mathbb{X}}}\}$$

and let $U = \{U(\varphi) : \varphi \in \mathscr{D}(\Omega)\}$. If we prove that U has the finite intersection property, then, by saturation, $\bigcap U \neq \emptyset$, and any $L_{\mathbb{X}} \in \bigcap U$ is a ${}^*\mathbb{R}$-linear function that satisfies $L_{\mathbb{X}}({}^*\varphi) = {}^*(L(\varphi))_{\mathbb{X}}$ for all $\varphi \in \mathscr{D}(\Omega)$.

We will prove that, if $\varphi_1, \ldots, \varphi_n \in \mathscr{D}$, then $\bigcap_{i=1}^n U(\varphi_i) \neq \emptyset$ by induction over n. If $n = 1$, we need to show that $U(\varphi) \neq \emptyset$ for all $\varphi \in \mathscr{D}'(\Omega)$. If $\varphi = 0$, then the constant function $L_{\mathbb{X}} f = 0$ for all $f \in \mathbb{G}(\Omega_{\mathbb{X}})$ belongs to $U(\varphi)$. If $\varphi \neq 0$, let $f = {}^*\varphi_{|\Omega_{\mathbb{X}}}$, $g = {}^*(L(\varphi))_{|\Omega_{\mathbb{X}}}$, and let $\{f, b_2, \ldots, b_M\}$ be a *basis of $\mathbb{G}(\Omega_{\mathbb{X}})$. Define also

$$L_{\mathbb{X}} \left(a_1 f + \sum_{i=2}^{M} a_i b_i \right) = a_1 g.$$

By definition, $L_{\mathbb{X}}$ is ${}^*\mathbb{R}$-linear and $L_{\mathbb{X}} \in U(\varphi)$.

We will now show that if $\bigcap_{i=1}^{n-1} U(\varphi_i) \neq \emptyset$ for any choice of $\varphi_1, \ldots, \varphi_{n-1} \in \mathscr{D}(\Omega)$, then also $\bigcap_{i=1}^{n} U(\varphi_i) \neq \emptyset$ for any choice of $\varphi_1, \ldots, \varphi_n \in \mathscr{D}(\Omega)$. If $\{\varphi_1, \ldots, \varphi_n\}$ are linearly dependent, thanks to linearity of L, any $L_{\mathbb{X}} \in \bigcap_{i=1}^{n-1} U(\varphi_i)$ satisfies $L_{\mathbb{X}} \in \bigcap_{i=1}^{n} U(\varphi_i)$. If $\{\varphi_1, \ldots, \varphi_n\}$ are linearly independent, let $f_n = ({}^*\varphi_n)_{|\Omega_{\mathbb{X}}}$, let $g_n = {}^*(L(\varphi_n))_{|\Omega_{\mathbb{X}}}$ and let $\{f_n, b_2, \ldots, b_M\}$ be a *basis of $\mathbb{G}(\Omega_{\mathbb{X}})$. For any $L_{\mathbb{X}} \in \bigcap_{i=1}^{n-1} U(\varphi_i)$, define $\overline{L_{\mathbb{X}}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ by

$$\overline{L_{\mathbb{X}}} \left(a_1 f_n + \sum_{i=2}^{M} a_i b_i \right) = a_1 g_n + L_{\mathbb{X}} \left(\sum_{i=2}^{M} a_i b_i \right).$$

Then $\overline{L_{\mathbb{X}}} \in \bigcap_{i=1}^{n} U(\varphi_i)$. This concludes the proof. □

Theorem 2.4.2. *For every linear $L : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ there is a ${}^*\mathbb{R}$-linear $L_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ such that ${}^\circ\langle L_{\mathbb{X}} f, {}^*\varphi\rangle = \langle L[f], \varphi\rangle_{\mathscr{D}(\Omega)}$ for all $\varphi \in \mathscr{D}(\Omega)$. Moreover, if $L_{\mathbb{X}}(\mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})) \subseteq \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) & \xrightarrow{L_{\mathbb{X}}} & \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}}) \\ \Phi \circ \pi \downarrow & & \downarrow \Phi \circ \pi \\ \mathscr{D}'(\Omega) & \xrightarrow{L} & \mathscr{D}'(\Omega). \end{array} \tag{2.4.3}$$

Proof. Let M be the adjoint of L, and let $M_{\mathbb{X}}$ be the ${}^*\mathbb{R}$-linear operator coherent with M in the sense of Lemma 2.4.1. Define $\langle L_{\mathbb{X}}(f), \varphi\rangle = \langle f, M_{\mathbb{X}}(\varphi)\rangle$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$. From this definition, ${}^*\mathbb{R}$-linearity of $L_{\mathbb{X}}$ can be deduced from the ${}^*\mathbb{R}$-linearity of $M_{\mathbb{X}}$.

We will now prove that $L_{\mathbb{X}}$ satisfies ${}^\circ\langle L_{\mathbb{X}} f, {}^*\varphi\rangle = \langle L[f], \varphi\rangle_{\mathscr{D}(\Omega)}$ for all $\varphi \in \mathscr{D}(\Omega)$. For any $\varphi \in \mathscr{D}(\Omega)$, thanks to Lemma 2.4.1 we have the equalities

$$\langle L_{\mathbb{X}}(f), {}^*\varphi\rangle = \langle f, M_{\mathbb{X}}({}^*\varphi_{|\Omega_{\mathbb{X}}})\rangle = \langle f, {}^*M(\varphi)\rangle \approx \langle [f], M(\varphi)\rangle_{\mathscr{D}(\Omega)} = \langle L[f], \varphi\rangle_{\mathscr{D}(\Omega)},$$

as we wanted.

If $L_{\mathbb{X}}(f) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, then for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$

$$\langle L_{\mathbb{X}}(f), \varphi\rangle = \langle L_{\mathbb{X}}(f), \varphi - {}^*({}^\circ\varphi)\rangle + \langle L_{\mathbb{X}}(f), {}^*({}^\circ\varphi)\rangle.$$

Since $\varphi - {}^*({}^\circ\varphi) \equiv 0$, by Lemma 2.2.7 the hypothesis $L_{\mathbb{X}}(f) \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ allows to conclude that $\langle L_{\mathbb{X}}(f), \varphi - {}^*({}^\circ\varphi)\rangle \approx 0$, so that ${}^\circ\langle L_{\mathbb{X}}(f), \varphi\rangle = \langle L[f], {}^\circ\varphi\rangle_{\mathscr{D}(\Omega)}$. As a consequence, the hypothesis $L_{\mathbb{X}}(\mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})) \subseteq \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ is sufficient to entail that diagram 2.4.3 commutes. □

From the previous Theorem, we obtain some sufficient conditions that ensure the equivalence between the linear problem 2.4.1 in the sense of distributions and the linear problem 2.4.2 in the sense of grid functions.

Theorem 2.4.3. *Let $L : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be linear, and let $L_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ any function such that diagram 2.4.3 commutes. Let also $f \in \mathscr{D}'(\Omega)$. Then problem 2.4.1 has a solution if and only if problem 2.4.2 has a solution $u \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$ for some $f_{\mathbb{X}}$ satisfying $[f_{\mathbb{X}}] = f$.*

Proof. By Theorem 2.4.2, if problem 2.4.2 has a solution u, then $[u]$ satisfies problem 2.4.1.

The other implication is a consequence of Theorem 2.4.2 and of surjectivity of Φ: suppose that 2.4.1 has a solution v. The commutativity of diagram 2.4.3 ensures that for any $u \in \Phi^{-1}(v)$ it holds $[L_{\mathbb{X}}(u)] = f$, hence for $f_{\mathbb{X}} = L_{\mathbb{X}}(u)$ problem 2.4.2 has a solution. □

Thanks to this equivalence result, any linear PDE that admits an extension $L_{\mathbb{X}}$ such that diagram 2.4.3 commutes can be studied in the setting of grid functions with the techniques from linear algebra.

As an example of the grid function formulation of a linear PDE, we find it useful to discuss the Dirichlet problem.

Definition 2.4.4. *Let $\Omega \subset \mathbb{R}^k$ be open and bounded, $h \in \mathbb{N}$, $a_{\alpha,\beta} \in C^\infty(\Omega)$, and let*

$$L(v) = \sum_{0 \le |\alpha|, |\beta| \le h} (-1)^{|\alpha|} D^\alpha (a_{\alpha,\beta} D^\beta v).$$

The Dirichlet problem is the problem of finding v satisfying

$$\begin{cases} L(v) = f \text{ in } \Omega \\ D^\alpha u = 0 \text{ for } |\alpha| \le h-1 \text{ in } \partial\Omega. \end{cases} \tag{2.4.4}$$

If $f \in C_b(\Omega)$, then v is a classical solution of the Dirichlet problem if

$$v \in C_b^{2h}(\Omega) \cap C_b^{2h-1}(\overline{\Omega}) \text{ and } L(v) = f. \tag{2.4.5}$$

If $f \in L^2(\Omega)$, then v is a strong solution of the Dirichlet problem if

$$v \in H^{2h}(\Omega) \cap H_0^h(\overline{\Omega}) \text{ and } L(v) = f \text{ a.e.}$$

If $f \in H^{-h}(\Omega)$, then v is a weak solution of the Dirichlet problem if

$$v \in H_0^h(\Omega) \text{ and } \sum_{0 \le |\alpha|, |\beta| \le h} \int a_{\alpha,\beta} D^\beta v D^\alpha w = f(w) \text{ for all } w \in H_0^h(\Omega). \tag{2.4.6}$$

Definition 2.4.5. *A grid function formulation of the Dirichlet problem 2.4.4 is the following: let*

$$L_{\mathbb{X}}(u) = \sum_{0 \le |\alpha|, |\beta| \le h} (-1)^{|\alpha|} \mathbb{D}^\alpha ({}^*a_{\alpha,\beta} \mathbb{D}^\beta u).$$

The Dirichlet problem is the problem of finding $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ satisfying

$$\begin{cases} L_{\mathbb{X}}(u) = P({}^*f) \text{ in } \Omega_{\mathbb{X}} \\ \mathbb{D}^\alpha u = 0 \text{ in } \partial_{\mathbb{X}}^\alpha \Omega_{\mathbb{X}} \text{ for } |\alpha| \le s-1. \end{cases} \tag{2.4.7}$$

Notice that equation 2.4.7 is satisfied in the sense of grid functions, i.e. pointwise, while equation 2.4.4 assumes the different meanings shown in Definition 2.4.4.

A priori, a solution u of problem 2.4.2 induces a solution $[u]$ of problem 2.4.1 in the sense of distributions. However, if $[u]$ is more regular, it is a solution to 2.4.1 in a stronger sense.

Theorem 2.4.6. *Let u be a solution of problem 2.4.7. Then*

1. *if $f \in C_b(\Omega)$ and $[u] \in C_b^{2h}(\Omega) \cap C_b^{2h-1}(\overline{\Omega})$, then $[u]$ is a classical solution of the Dirichlet problem;*

2. *if $f \in L^2(\Omega)$ and $[u] \in H^{2h}(\Omega) \cap H_0^h(\Omega)$, then $[u]$ is a strong solution of the Dirichlet problem;*

3. *if $f \in H^{-h}(\Omega)$ and $[u] \in H_0^h(\Omega)$, then $[u]$ is a weak solution of the Dirichlet problem, i.e. $[u]$ satisfies 2.4.6.*

Proof. A solution u of problem 2.4.7 satisfies the equality

$$\begin{aligned}\langle P(^*f), \phi\rangle &= \sum_{0\le|\alpha|,|\beta|\le h} (-1)^{|\alpha|}\langle \mathbb{D}^\alpha(^*a_{\alpha,\beta}\mathbb{D}^\beta u), \varphi\rangle \\ &= \sum_{0\le|\alpha|,|\beta|\le h} \langle ^*a_{\alpha,\beta}\mathbb{D}^\beta u, \mathbb{D}^\alpha \varphi\rangle.\end{aligned}$$

for all $\varphi \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$.

We will now prove (1). If $f \in C_b(\Omega)$, then by Lemma 2.3.5, $[P(^*f)] = f$. By Theorem 2.2.15, $[\mathbb{D}^\beta u] = D^\beta[u]$, $[^*a_{\alpha,\beta}\mathbb{D}^\beta u] = a_{\alpha,\beta}D^\beta[u]$, so that

$$[(-1)^{|\alpha|}\mathbb{D}^\alpha(^*a_{\alpha,\beta}\mathbb{D}^\beta u)] = (-1)^{|\alpha|}D^\alpha(a_{\alpha,\beta}D^\beta[u]).$$

We deduce that $[u]$ satisfies equation 2.4.5 in the classical sense, as desired.

The proof of parts (2) and (3) is similar to that of part (1). The only difference is that it relies on Lemma 2.3.2 instead of Lemma 2.3.5. □

Remark 2.4.7. *While Theorem 2.4.2 and Theorem 2.4.3 do not explicitly determine an extension $L_{\mathbb{X}}$ for a given linear PDE, they determine a sufficient condition for problem 2.4.2 to be a coherent representation of problem 2.4.1 in the sense of grid function. In the practice, an explicit extension $L_{\mathbb{X}}$ of a linear $L : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ can be determined from L by taking into account that*

- *thanks to Theorem 2.2.15, derivatives can be replaced by finite difference operators;*
- *shifts can be represented in accord to Corollary 2.2.14;*
- *if $a \in C^\infty(\Omega)$, then $[^*af] = a[f]$ for all $f \in \mathscr{D}'_{\mathbb{X}}(\Omega_{\mathbb{X}})$, since for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, $^*a\varphi \in \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}})$, and we have the equalities*

$$^\circ\langle ^*af, \varphi\rangle = {}^\circ\langle f, {}^*a\varphi\rangle = \langle [f], a\varphi\rangle_{\mathscr{D}(\Omega)} = \langle a[f], \varphi\rangle_{\mathscr{D}(\Omega)}.$$

Similarly, we have not established a canonical representative $f_{\mathbb{X}}$ for f. However, observe that for all $g \in \mathbb{G}(\Omega_{\mathbb{X}})$ and for all $x \in \Omega_{\mathbb{X}}$ it holds

$$g(x) = \sum_{y\in\Omega_{\mathbb{X}}} g(y)N^k\chi_y(x)$$

Moreover, $\chi_y(x) = \chi_0(x-y)$, so that once a solution u_0 for the problem $L_{\mathbb{X}}u = N^k\chi_0$ is determined, a solution for $L_{\mathbb{X}}u = g$ can be determined from the above equality by posing

$$u_g(x) = \sum_{y\in\Omega_{\mathbb{X}}} g(y)u_0(x-y). \tag{2.4.8}$$

In fact, by linearity of $L_{\mathbb{X}}$ we have that, for all $x \in \Omega_{\mathbb{X}}$,

$$\begin{aligned} L_{\mathbb{X}}(u_g(x)) &= L_{\mathbb{X}}\left(\sum_{y\in\Omega_{\mathbb{X}}} g(y)u_0(x-y)\right) \\ &= \sum_{y\in\Omega_{\mathbb{X}}} g(y)L_{\mathbb{X}}(u_0(x-y)) \\ &= \sum_{y\in\Omega_{\mathbb{X}}} g(y)N^k\chi_0(x-y) \\ &= g(x). \end{aligned}$$

In particular, u_0 plays the role of a fundamental solution for problem 2.4.2, while equality 2.4.8 can be interpreted as the discrete convolution between g and u_0. As a consequence, the study of a linear problem 2.4.2 can be carried out by determining the solutions to the problem $L_{\mathbb{X}}(u) = N^k\chi_0$.

2.4.2 The grid function formulation of nonlinear PDEs

A nonlinear PDE can be written in the most general form as

$$F(u) = f,$$

usually with $u \in V \subseteq L^2(\Omega)$ and $F : V \to W \subseteq L^2(\Omega)$. As in the linear case, the grid function formulation of nonlinear problems is based upon the possibility to coherently extend every continuous $F : L^2(\Omega) \to L^2(\Omega)$ to all of $\mathbb{G}(\Omega_{\mathbb{X}})$. Since the proofs of the following theorems are based upon Lemma 2.3.7, we will impose the additional hypothesis that the Lebesgue measure of Ω is finite. Notice that, in contrast to what happened for Theorem 2.4.2, in the proof of Theorem 2.4.8, we will be able to explicitly determine a particular extension $F_{\mathbb{X}}$ for a given continuous $F : L^2(\Omega) \to L^2(\Omega)$.

Theorem 2.4.8. *Let $\mu_L(\Omega) < +\infty$ and let $F : L^2(\Omega) \to L^2(\Omega)$ be continuous. Then there is a function $F_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ that satisfies*

1. *whenever $u, v \in \mathbb{G}(\Omega_{\mathbb{X}})$ are nearstandard in $L^2(\Omega)$, $\|u - v\|_2 \approx 0$ implies $\|F_{\mathbb{X}}(u) - F_{\mathbb{X}}(v)\|_2 \approx 0$;*
2. *for all $f \in L^2(\Omega)$, $[F_{\mathbb{X}}(P(^*f))] = F(f)$.*

Proof. We will show that the function defined by $F_{\mathbb{X}}(u) = P(^*F(\widehat{u}))$ for all $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ satisfies the thesis. By continuity of F, whenever u and v are nearstandard in $L^2(\Omega)$ we have

$$\|u - v\|_2 \approx 0 \text{ implies } \|^*F(u) - {}^*F(v)\|_2 \approx 0,$$

and, by Lemma 2.3.7,

$$\|^*F(u) - {}^*F(v)\|_2 \approx 0 \text{ implies } \|F_{\mathbb{X}}(u) - F_{\mathbb{X}}(v)\|_2 \approx 0,$$

hence (1) is proved.

We will now prove that $[F_{\mathbb{X}}(P({}^*f))] = F(f)$. By Lemma 2.3.7, we have $\|{}^*f - P({}^*f)\|_2 \approx 0$ and, by continuity of ${}^*F$, $\|{}^*F({}^*f) - {}^*F(P({}^*f))\|_2 \approx 0$. From Lemma 2.3.6 we have $[F_{\mathbb{X}}({}^*f)] = [P({}^*F({}^*f))] = F(f)$, as desired. □

Remark 2.4.9. *In the same spirit, if $F : V \to W$ is continuous and the space of grid functions can be continuously embedded in ${}^*V$ and ${}^*W$, then one can prove similar theorems by varying condition (1) in order to properly represent the topologies on the domain and the range of F. For instance, if $F : H^1(\Omega) \to L^2(\Omega)$, then (1) would be replaced by*

$$\|u - v\|_{H^1} \approx 0 \text{ implies } \|F_{\mathbb{X}}(u) - F_{\mathbb{X}}(v)\|_2 \approx 0,$$

where $\|u - v\|_{H^1}$ is defined in the expected way as

$$\|u - v\|_{H^1} = \|u - v\|_2 + \|\nabla_{\mathbb{X}}(u - v)\|_2.$$

Condition (1) of Theorem 2.4.8 is a continuity requirement for $F_{\mathbb{X}}$, and condition (2) implies coherence of $F_{\mathbb{X}}$ with the original function F, so that theorem 2.4.8 ensures that for all continuous $F : L^2 \to L^2$ there is a function $F_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ which is continuous and coherent with F. This result allows to formulate nonlinear PDEs in the setting of grid functions.

Theorem 2.4.10. *Let $\mu_L(\Omega) < +\infty$, let $F : L^2(\Omega) \to L^2(\Omega)$ and let $F_{\mathbb{X}} : \mathbb{G}(\Omega_{\mathbb{X}}) \to \mathbb{G}(\Omega_{\mathbb{X}})$ satisfy conditions (1) and (2) of Theorem 2.4.8. Let also $f \in L^2(\Omega)$. Then the problem of finding $v \in L^2(\Omega)$ satisfying*

$$F(v) = f \tag{2.4.9}$$

has a solution if and only if there exists a solution $u \in \mathbb{G}(\Omega_{\mathbb{X}})$, u nearstandard in $L^2(\Omega)$, that satisfy

$$F_{\mathbb{X}}(u) = f_{\mathbb{X}} \tag{2.4.10}$$

for some $f_{\mathbb{X}} \in \mathbb{G}(\Omega_{\mathbb{X}})$ with $[f_{\mathbb{X}}] = f$, and in particular for $f_{\mathbb{X}} = P({}^*f)$.

Proof. Suppose that 2.4.10 with $f_{\mathbb{X}} = P({}^*f)$ has a solution u. Since $[P({}^*f)] = f$ by Corollary 2.3.6, u satisfies the equality $[F_{\mathbb{X}}(u)] = f$ in the sense of distributions. At this point, if u is nearstandard in $L^2(\Omega)$, by Lemma 2.3.7 we have $\|{}^*[u] - u\|_2 \approx 0$, so that $[u] \in L^2(\Omega)$, and condition (2) of Theorem 2.4.8 ensures that $[F_{\mathbb{X}}(u)] = F([u])$, so that $[u]$ is a solution of 2.4.9.

For the other implication, suppose that v is a solution to 2.4.9. Then, by condition (2) of Theorem 2.4.8, $[F_{\mathbb{X}}(P({}^*v))] = F(v) = f$, so that problem 2.4.10 has a solution. □

If u is a solution to 2.4.10 but it is not nearstandard in $L^2(\Omega)$, i.e. if $\|{}^*[u] - u\|_2 \not\approx 0$, $[F_{\mathbb{X}}(u)]$ needs not be equal to $F([u])$. In fact, if $[u] \in L^2(\Omega)$

and $\|{}^*[u] - u\|_2 \not\approx 0$, we have argued in Section 2.3.1 that we expect u to feature either strong oscillations or concentrations. Due to these irregularities, we have no reasons to expect that $[F_{\mathbb{X}}(u)](x)$, that represents the mean of the values assumed by $F_{\mathbb{X}}(u)$ at points infinitely close to x, is related to $F([u])(x)$, that represents the function F applied to the mean of the values assumed by u at points infinitely close to x. However, as we have seen in Section 2.3.12, if $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$, then u can be interpreted as a Young measure ν^u. If the composition $F(\nu^u)$ is defined in the sense of Equation 2.3.4, then ν^u satisfies

$$\int_\Omega \int_\mathbb{R} F(\tau) d\nu^u(x) \varphi(x) dx = {}^\circ\langle F_{\mathbb{X}}(u), \varphi \rangle = {}^\circ\langle P({}^*f), \varphi \rangle = \int_\Omega f \varphi dx$$

for all $\phi \in \mathscr{D}'(\Omega)$, and can be regarded as a Young measure solution to Equation 2.4.9. In particular, since Young measures describe weak-$\star$ limits of sequences of L^∞ functions, the relation between $F(\nu^u)$ and problem 2.4.9 is the following: there exists a family of regularized problems

$$F_\eta(u) = f_\eta$$

and a family $\{u_\eta\}_{\eta>0}$ of $L^2(\Omega) \cap L^\infty(\Omega)$ solutions of these problems such that ν^u represents the weak-$\star$ limit of a subsequence of $\{u_\eta\}_{\eta>0}$, and $F(\nu^u)$ is the corresponding weak limit of the sequence $\{F(u_\eta)\}_{\eta>0}$.

In the case that $\|u\|_\infty$ is infinite or that $[u] \notin L^2(\Omega)$, we consider u as a generalized solution of problem 2.4.9 in the sense of grid functions. Moreover, we expect u to capture both the oscillations and the concentrations we would expect from a sequence of solutions of some family of regularized problems of 2.4.9. We will see such an example while discussing the generalized solution to an ill-posed PDE in Chapter 3.

Remark 2.4.11. *Notice that if $F_{\mathbb{X}}$ satisfies the stronger continuity hypothesis*

$$u \equiv v \text{ implies } F_{\mathbb{X}}(u) \equiv F_{\mathbb{X}}(v), \tag{2.4.11}$$

then $F_{\mathbb{X}}$ has a standard part $\tilde{F}$ defined by

$$\tilde{F}(g) = [F_{\mathbb{X}}(P({}^*g))]$$

for any $g \in L^2(\Omega)$. Moreover, from Lemma 2.3.7 and from Theorem 2.4.8, we deduce that $\tilde{F} = F$. As a consequence, any grid function u that satisfies $F_{\mathbb{X}}(u) = P({}^*f)$ induces a solution to problem 2.4.9.

However, the continuity condition 2.4.11 holds only for very regular functions, and it fails for many of the functions that still satisfy the hypotheses of Theorem 2.4.8.

Remark 2.4.12. *If the function F appearing in equation 2.4.9 can be expressed as*

$$F = L \circ G,$$

where G is nonlinear and L is linear, the equivalence between the standard notions of solutions for the PDE 2.4.9 and one of its formulations in the sense of grid functions can be obtained by a suitable combination of the results of Theorem 2.4.3 and of Theorem 2.4.10.

2.4.3 Time dependent PDEs

Time dependent PDEs have been studied in the setting of nonstandard analysis by a variety of means. A possibility is to give a nonstandard representation of a given time dependent PDE by discretizing in time as well as in space, and by defining a standard solution to the original problem by the technique of stroboscopy. In [17], van den Berg showed how the stroboscopy technique can be extended to the study of a class of partial differential equations of the first and the second order by imposing additional regularity hypotheses on the time-step of the discretization. For an in-depth discussion on the stroboscopy technique and its applications to partial differential equations, we remand to [17, 75, 76].

A delicate point in the time discretization of PDEs is that the discrete time step cannot be chosen arbitrarily. In fact, it is often the case that the time-step of the discretization must be chosen in accord to some bounds that depend upon the specific problem. As an example, consider the nonstandard model for the heat equation discussed in [45], where the time-step is dependent upon the diameter of the grid and upon the diffusion coefficients. In general, if the discrete timeline $\mathbb{T}$ is a deformation of the grid $\mathbb{X}$, then the finite difference in time does not generalize faithfully the partial difference in time, and Theorem 2.1.15 fails. However, it is possible to determine sufficient conditions over $\mathbb{T}$ that imply the existence of $k \in \mathbb{N}$ such that Theorem 2.1.15 holds for derivatives up to order k. This study has been carried out in depth by van den Berg in [16].

In order to provide a general theory that is not dependent upon the specific problem, we have chosen to follow the idea of Capińsky and Cutland in [21, 24] and subsequent works: we will not discretize in time, but instead we will work with functions defined on ${}^*\mathbb{R} \times \mathbb{X}^k$, where the first variable represents time, and the other k variables represent space. In particular, we want to describe the problem

$$u_t - Fu = f \tag{2.4.12}$$

with $u : \mathbb{R} \to V \subseteq L^2(\Omega)$, $F : V \to W \subseteq L^2(\Omega)$ by the nonstandard problem

$$u_t - F_{\mathbb{X}}u = f_{\mathbb{X}} \tag{2.4.13}$$

with $u : {}^*\mathbb{R} \to \mathbb{G}(\Omega_{\mathbb{X}})$, with $[f_{\mathbb{X}}] = f$, and where $F_{\mathbb{X}}$ is a suitable extension of F in the sense of Theorems 2.4.2 and 2.4.8. Notice that, by Theorem 2.2.15 and by Theorem 2.4.2, the grid function formulation of a time dependent

PDE is formally a hyperfinite system of ordinary differential equations, and it can be solved by exploiting the standard theory of dynamical systems.

Once we have a grid function formulation for a time dependent PDE, we would like to study the relation between its solutions and the solutions to the original problem. As expected, if for a suitable choices of $F_{\mathbb{X}}$ problem 2.4.13 has a solution u and u is regular enough, then u induces a solution to problem 2.4.12.

Theorem 2.4.13. *Let $F_{\mathbb{X}}$ be coherent with F in the sense of Theorems 2.4.2 and 2.4.8. If 2.4.13 has a solution $u(t) \in {}^*C^1({}^*[0,T], \mathbb{G}(\Omega_{\mathbb{X}}))$ that satisfies the continuity hypothesis*

$$u(t) \equiv u(t') \text{ and } F(u(t)) \equiv F(u(t')) \text{ whenever } t \approx t', \tag{2.4.14}$$

then u induces two functions $[u], [F_{\mathbb{X}}(u)] \in C^0([0,T], \mathscr{D}'(\Omega))$ that satisfy

$$\int_{[0,T]\times\Omega} [u]\varphi_t + [F_{\mathbb{X}}(u)]\varphi d(t,x) + \int_{\Omega} [u(0,x)]\varphi(0,x)dx = -\int_{[0,T]\times\Omega} [f_{\mathbb{X}}]\varphi d(t,x)$$

for all $\varphi \in C^1([0,T], \mathscr{D}(\Omega))$ with $\varphi(T,x) = 0$.

Moreover, if F is linear or if $u(t)$ is nearstandard in the domain of F for all $t \in {}^*[0,T]$, then we can replace $[F_{\mathbb{X}}(u)]$ with $F([u])$ in the above equality.

Proof. By condition 2.4.14, the functions $t \mapsto [u(t)]$ and $t \mapsto [F(u(t))]$ are well-defined and continuous with respect to the weak-$\star$ topology on $\mathscr{D}'(\Omega)$. Moreover, since $u \in {}^*C^1$,

$$\int_{{}^*[0,T]} \langle u_t, {}^*\varphi\rangle dt = -\int_{{}^*[0,T]} \langle u, {}^*\varphi_t\rangle dt - \langle {}^*u(0,x), {}^*\varphi(0,x)\rangle$$

for all $\varphi \in C^1([0,T], \mathscr{D}(\Omega))$ with $\varphi(T,x) = 0$. As a consequence, u satisfies the equality

$$\int_{[0,T]} \langle u, {}^*\varphi_t\rangle + \langle F_{\mathbb{X}}(u), {}^*\varphi\rangle dt + \langle {}^*u(0,x), {}^*\varphi(0,x)\rangle = -\int_{[0,T]} \langle f_{\mathbb{X}}, {}^*\varphi\rangle dt$$

for all $\varphi \in C^1([0,T], \mathscr{D}(\Omega))$ with $\varphi(T,x) = 0$, and this is equivalent to the first part of the thesis.

The second part of the thesis is a consequence of Theorem 2.4.2 and of Theorem 2.4.8. □

If u does not satisfy 2.4.14 but $\|u(t)\|_\infty$ is finite and uniformly bounded in t, by the same argument of Theorem 2.3.12 u corresponds to a Young measure $\nu^u : [0,T] \times \Omega \to \mathbb{M}_{\mathbb{P}}(\mathbb{R})$. If the composition $F(\nu^u)$ is defined in the sense of Equation 2.3.4, then ν^u satisfies the equality

$$\begin{aligned}\int_{[0,T]\times\Omega} \int_{\mathbb{R}} \tau d\nu^u(t,x)\varphi_t + \int_{\mathbb{R}} F(\tau)d\nu^u(t,x)\varphi d(t,x) + \\ + \int_{\Omega}\int_{\mathbb{R}} \tau d\nu^u(0,x)\varphi(0,x)dx &= \int_{[0,T]\times\Omega} [f_{\mathbb{X}}]\varphi d(t,x)\end{aligned}$$

for all $\varphi \in C^1([0,T], \mathscr{D}(\Omega))$ with $\varphi(T,x) = 0$.

If $\|u(t)\|_p$ is finite for some $1 \leq p < +\infty$, the sense in which $[u]$ is a solution to problem 2.4.12 has to be addressed on a case-by-case basis. In Chapter 3, we will discuss an example where $\|u(t)\|_1$ is finite and uniformly bounded, and $[u]$ can be interpreted as a Radon measure solution to problem 2.4.12.

2.5 Selected applications

In this section, we will discuss two classic problems: the first concerns the nonlinear theory of distributions, and the second is a minimization problem from the calculus of variations. The discussion of these examples is meant to show how grid functions can be applied to a variety of problems while retaining coherence with the various standard approaches.

2.5.1 The product HH'

The following example is discussed in the setting of Colombeau algebras in [26], and it can also be formalized in the framework of algebras of asymptotic functions [67].

Let H be the Heaviside function

$$H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

and let H' be the derivative of the Heaviside function in the sense of distributions, i.e. the Dirac distribution centered at 0. It is well-known that the product HH' is not well-defined in the sense of distributions. However, this product arises quite naturally in the description of some physical phenomena. For instance, in the study of shock waves discussed in [26], it is convenient to treat H and H' as a smooth functions and performing calculations such as

$$\int_{\mathbb{R}} (H^m - H^n)H'dx = \left[\frac{H^{m+1}}{m+1}\right]_{-\infty}^{+\infty} - \left[\frac{H^{n+1}}{n+1}\right]_{-\infty}^{+\infty} = \frac{1}{m+1} - \frac{1}{n+1}. \quad (2.5.1)$$

This calculation is not justified in the theory of distributions: on the one hand, $H^m = H^n$ for all $m, n \in \mathbb{N}$, so that we intuitively expect that the integral should equal 0; on the other hand, since the products $H^m H'$ and $H^n H'$ are not defined, the integrand is not well-defined.

We will now show how in the setting of grid functions one can rigorously formulate the integral 2.5.1 and compute the product HH'. Let $M \in {}^*\mathbb{N} \setminus \mathbb{N}$ satisfy $M\varepsilon \approx 0$, and consider the grid function $h \in \mathscr{D}'_{\mathbb{X}}(\mathbb{X})$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x/(M\varepsilon) & \text{if } 0 < x < M\varepsilon \\ 1 & \text{if } x \geq M\varepsilon \end{cases}$$

The function $\mathbb{D}h$ is given by

$$\mathbb{D}h(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ and } x \geq M\varepsilon \\ 1/(M\varepsilon) & \text{if } 0 < x < M\varepsilon \end{cases}$$

In the next Lemma, we will prove that h is a representative of the Heaviside function for which the calculation 2.5.1 makes sense.

Lemma 2.5.1. *The function h has the following properties:*

1. $[h^m] = H$ *and* $[\mathbb{D}h^m] = \delta_0$ *for all* $m \in {}^*\mathbb{N}$*;*
2. $h^m \neq h^n$ *whenever* $m \neq n$*;*
3. $\langle h^m - h^n, \mathbb{D}h\rangle \approx \frac{1}{m+1} - \frac{1}{n+1}$.

Proof. (1). Let $\varphi \in \mathscr{D}_{\mathbb{X}}(\mathbb{X})$ and, without loss of generality, suppose that $\varphi(x) \geq 0$ for all $x \in \mathbb{X}$. Then for all $m \in {}^*\mathbb{N}$ we have the inequalities

$$\varepsilon \sum_{x \geq M\varepsilon} \varphi(x) \leq \langle h^m, \varphi\rangle \leq \varepsilon \sum_{x \geq 0} \varphi(x),$$

and, by taking the standard part of all the sides of the inequalities, we deduce

$$\int_0^{+\infty} {}^\circ\varphi(x)dx \leq {}^\circ\langle h^m, \varphi\rangle \leq \int_0^{+\infty} {}^\circ\varphi(x)dx.$$

This is sufficient to conclude that $[h^m] = H$ for all $m \in {}^*\mathbb{N}$. By Theorem 2.2.15, $[\mathbb{D}h^m] = H' = \delta_0$.

(2). Let $m \neq n$. Then,

$$(h^m - h^n)(x) = \begin{cases} 0 & \text{if } x \leq 0 \text{ and } x \geq M\varepsilon \\ (x/(M\varepsilon))^m - (x/(M\varepsilon))^n & \text{if } 0 < x < M\varepsilon. \end{cases}$$

In particular, $h^m - h^n \neq 0$, even if $[h^m] - [h^n] = 0$.

(3). By the previous point,

$$\langle h^m - h^n, \mathbb{D}h\rangle = \frac{1}{M} \sum_{j=1}^{M} (j/M)^m - (j/M)^n.$$

Since M is infinite,

$$\frac{1}{M} \sum_{j=1}^{M} (j/M)^m - (j/M)^n \approx \int_0^1 x^m - x^n dx = \frac{1}{m+1} - \frac{1}{n+1}.$$

□

Thanks to the lemma above, we can compute the equivalence class in $\mathscr{D}'(\mathbb{R})$ of the product hh'.

Corollary 2.5.2. $[h\mathbb{D}h] = \frac{1}{2}H'$.

Proof. For any $\varphi \in \mathscr{D}_{\mathbb{X}}(\mathbb{X})$, we have

$$\langle h\mathbb{D}h, \varphi\rangle = \frac{1}{M^2}\sum_{j=1}^{M} j\psi(j\varepsilon).$$

Let $\underline{m} = \min_{1\leq j\leq M}\{\varphi(j\varepsilon)\}$ and $\overline{m} = \max_{1\leq j\leq M}\{\varphi(j\varepsilon)\}$. We have the following inequalities:

$$\frac{\underline{m}}{M}\sum_{j=1}^{M} j/M \leq \frac{1}{M^2}\sum_{j=1}^{M} j\varphi(j\varepsilon) \leq \frac{\overline{m}}{M}\sum_{j=1}^{M} j/M.$$

Since M is infinite,

$$\frac{1}{M}\sum_{j=1}^{M} j/M \approx \int_0^1 x dx = \frac{1}{2},$$

so that

$$^\circ\left(\frac{\underline{m}}{2}\right) \leq {}^\circ\langle h\mathbb{D}h, \varphi\rangle \leq {}^\circ\left(\frac{\overline{m}}{2}\right).$$

By S-continuity of φ, $\underline{m} \approx \overline{m} \approx \varphi(0)$, so that $^\circ\langle h\mathbb{D}h, \varphi\rangle = \frac{1}{2}{}^\circ\varphi(0)$ for all $\varphi \in \mathscr{D}_{\mathbb{X}}(\mathbb{X})$, which is equivalent to $[h\mathbb{D}h] = \frac{1}{2}H'$. □

Notice that h is not the only function satisfying Lemma 2.5.1 and Corollary 2.5.2. In fact, we conjecture that Lemma 2.5.1 and Corollary 2.5.2 hold for a class of grid functions that satisfy some regularity conditions yet to be determined.

2.5.2 A variational problem without a minimum

We will now discuss a grid function formulation of a classic example of a variational problem without a minimum. For an in-depth analysis of the Young measure solutions to this problem we refer to [81], and for a discussion of a similar problem in the setting of ultrafunctions, we refer to [15]. The grid function formulation consists in a hyperfinite discretization, as in Cutland [29].

Consider the problem of minimizing the functional

$$J(u) = \int_0^1 \left(\int_0^x u(t)dt\right)^2 + (u(x)^2-1)^2 dx \tag{2.5.2}$$

with $u \in L^2([0,1])$. Intuitively, a minimizer for J should have a small mean, but nevertheless it should assume values in the set $\{-1,+1\}$. Let us make precise this idea: define

$$u_0 = \chi_{[k,k+1/2)} - \chi_{[k+1/2,k+1)},\ k \in \mathbb{Z}$$

and let $u_n : [0,1] \to \mathbb{R}$ be defined by $u_n(x) = u_0(nx)$. It can be verified that $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for J, but J has no minimum. However, the sequence $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty([0,1])$, hence it admits a weak* limit in the sense of Young measures. The limit is given by the constant Young measure

$$\nu_x = \frac{1}{2}(\delta_1 + \delta_{-1}).$$

We can now evaluate $J(\nu)$:

$$\int_0^x \nu(t)dt = \int_0^x \left(\int_{\mathbb{R}} \tau d\nu_x\right) dt = 0,$$

meaning that the barycentre of ν is 0, and

$$(\nu(x)^2 - 1)^2 = \int_{\mathbb{R}} (\tau^2 - 1)^2 d\nu_x = 0$$

since the support of ν is the set $\{-1, +1\}$. As a consequence, ν can be interpreted as a minimum of J in the sense of Young measures.

In the setting of grid functions, the functional 2.5.2 can be represented by

$$J_{\mathbb{X}}(u) = \varepsilon \sum_{n=0}^{N} \left[\left(\varepsilon \sum_{i=0}^{n} u(i\varepsilon)\right)^2 + (u(n\varepsilon)^2 - 1)^2\right].$$

Observe that this representation is coherent with the informal description of J, and that the only difference between J and $J_{\mathbb{X}}$ is the replacement of the integrals with the hyperfinite sums. Let us now minimize $J_{\mathbb{X}}$ in the sense of grid functions. The minimizing sequence found in the classical case suggests us that a minimizer of $J_{\mathbb{X}}$ should assume values ± 1, and that it should be piecewise constant in an interval of an infinitesimal length. For $M \in {}^*\mathbb{N}$, let $u_M = {}^*u_0(Mx)$. If $M < M' \leq N/2$, then

$$\varepsilon \sum_{i=0}^{n} u_M(i\varepsilon) > \varepsilon \sum_{i=0}^{n} u_{M'}(i\varepsilon).$$

We deduce that a minimizer for $J_{\mathbb{X}}$ is the grid function $u_{N/2}$, that is explicitly defined by $u_{N/2}(n\varepsilon) = (-1)^n$.

We will now show that this solution is coherent with the one obtained with the classic approach, i.e. that the Young measure associated to $u_{N/2}$ corresponds to $\frac{1}{2}(\delta_1 + \delta_{-1})$. Since $\|u_{N/2}\|_\infty = 1$, Theorem 2.3.12 guarantees the existence of a Young measure ν that corresponds to $u_{N/2}$. Moreover, by Proposition 2.3.18, ν is constant, and

$$\int_{\mathbb{R}} f d\nu_x = \frac{1}{2} \sum_{i=0}^{1} f(u_{N/2}(i\varepsilon)) = \frac{1}{2}(f(1) + f(-1))$$

for all $f \in C_b^0(\mathbb{R})$. We deduce that the Young measure associated to $u_{N/2}$ is constant and equal to $\frac{1}{2}(\delta_1 + \delta_{-1})$, the minimizer of J in the sense of Young measures.

Chapter 3

A grid function formulation of a class of ill-posed parabolic equations

One of the applications of grid functions is the study of generalized solutions to partial differential equations. In this chapter, we will discuss the grid function formulation of a class of parabolic equations that depend upon a parameter u^+. This class of parabolic equations is ill-posed forward in time, in the sense that they do not have solutions in the sense of distributions. Under the hypothesis that $0 < u^+ < +\infty$, the ill-posed problem has been studied by Plotnikov in [70], where he introduced a notion of solution in the sense of Young measures. If $u^+ = +\infty$, then the problem has no solution also in the class of Young measures; however, a notion of measure-valued solution has been given by Smarrazzo [79] and can be characterized as the sum of a Young measure and a Radon measure. In Section 3.1 we will introduce the class of ill-posed parabolic equations 3.1.1 and recall the definition of the measure-valued solutions for these problems.

Despite the different notions of measure-valued solutions, that depend upon the value of u^+, we will be able to give a unique grid function formulation for the class of ill-posed PDEs. In section 3.2 we will derive the grid function formulation of problem 3.1.1 from a discrete model of diffusion, then we will discuss its solutions. In section 3.3, we will show the relations between the solutions to the grid function formulation and the solutions to the original problem 3.1.1. In particular, if the solutions to the nonstandard problem are regular enough, they induce solutions to problem 3.1.1 that are coherent with the approaches of Plotnikov and Smarrazzo. In section 3.4 we will discuss the asymptotic behaviour of the grid solutions to problem 3.1.1 by studying the asymptotic behaviour of the solution to the grid function formulation. We will also give a positive answer to a conjecture by Smarrazzo on the coarsening of the solutions to problem 3.1.1

when $u^+ = +\infty$. The chapter concludes with a brief discussion of some properties of the grid solution to problem 3.1.1 with Riemann initial data. In particular, by studying this initial value problem, we will show that the grid solution to problem 3.1.1 features a hysteresis loop, in agreement with the behaviour of the Young measure solution.

3.1 The ill-posed PDE

Consider the Neumann initial value problem

$$\begin{cases} \partial_t u = \Delta\phi(u) \text{ in } \Omega \\ \frac{\partial\phi(u)}{\partial\nu} = 0 \text{ in } [0,T]\times\partial\Omega \\ u(0,x) = u_0(x) \end{cases} \quad (3.1.1)$$

where $\Omega \subseteq \mathbb{R}^k$ is an open, bounded, and connected set, $u : \mathbb{R}\times\Omega \to \mathbb{R}$ and where the following hypotheses over ϕ are assumed:

Hypothesis 3.1.1. ϕ *satisfies:*

- $\phi(x) \geq 0$ *for all* $x \geq 0$ *and* $\phi(0) = 0$*;*
- $\phi \in C^1(\mathbb{R})$*;*
- ϕ *is non-monotone, i.e. there are* u^-, u^+ *with* $0 < u^- < u^+ \leq +\infty$ *such that* $\phi'(u) > 0$ *if* $u \in (0, u^-) \cup (u^+, +\infty)$ *and* $\phi'(u) < 0$ *for* $u \in (u^-, u^+)$*;*
- *if* $u^+ = +\infty$*, then* $\lim_{x\to+\infty}\phi(x) = 0$.

It is well-known that, if $u_0 \in L^\infty(\Omega)$ and $\operatorname{ess\,sup} v \leq u^-$ or $\operatorname{ess\,inf} v \geq u^+$, then the dynamics described by equation 3.1.1 amount to a parabolic smoothing. The main feature of problem 3.1.1 is that it is ill–posed forward in time for u in the interval (u^-, u^+): there are no weak solutions to problem 3.1.1 whenever $\|u_0\|_\infty > \phi(u^-)$ and, if we allow for measure-valued solutions, then solutions to problem 3.1.1 exist for any initial data but are not unique.

Forward-backward parabolic equations like 3.1.1 or the closely related

$$u_t = \mathrm{div}\phi(\nabla u) \quad (3.1.2)$$

with non-monotone ϕ have been used to describe various physical phenomena. Cubic-like functions with $u^+ < +\infty$ arise for instance in models of phase transitions: in this context, the function u represents the enthalpy and $\phi(u)$ the temperature distribution. Equation 3.1.1 can be deduced as a consequence of the Fourier law.

If $u^+ = +\infty$, then equation 3.1.1 has been used in models of the dynamics of aggregating populations both in a discrete approximation (see for instance Horstmann, Othmer and Painter [49] and Lizana and Padron [58]) and as a continuous diffusion approximation [68]. Equation 3.1.2 has been used to describe also shearing of granular media (see Witelski, Shaeffer and Shearer [89]). It is also noteworthy to mention that the Perona-Malik edge-enhancement algorithm via backward diffusion [69] is based on equation 3.1.2.

The hypothesis that ϕ is non-monotone is crucial both for the applications and for the description of the physical phenomena. For this reason, suitable approximations of initial value problems for equations 3.1.1 and 3.1.2 has been studied in a variety of ways. For a discussion of these approaches, we refer to [62] and to [79].

3.1.1 The Young measure solution in the case $u^+ < +\infty$

The most common approach to problem 3.1.1 and to problem 3.1.2 is to treat them as the limit of some sequence of approximating problems: the notions of solutions will accordingly depend on the chosen regularization.

In [70], Plotnikov studied problem 3.1.1 by means of the following Sobolev regularization:

$$\begin{cases} u_t = \Delta v \text{ in } \Omega \\ v = \phi(u) + \eta u_t \\ \frac{\partial v}{\partial \nu} = 0 \text{ in } [0,T] \times \partial\Omega \\ u(x) = u_0 \text{ in } \Omega. \end{cases} \tag{3.1.3}$$

with $\eta > 0$. The Neumann initial-boundary value problem for this regularized problem under the hypothesis that $u^+ < +\infty$ has been studied by Novick-Cohen and Pego [66]. In particular, Novick-Cohen and Pego proved that if ϕ is locally Lipschitz continuous and the initial data is $L^\infty(\Omega)$, then there exists a unique classical solution (u_η, v_η) to problem 3.1.3, with $u \in C^1([0,T], L^\infty(\Omega))$ and $v_\eta = \phi(u) + \eta(u_\eta)_t$, and the functions (u_η, v_η) satisfy the inequality

$$\int_0^T \int_\Omega G(u_\eta)\varphi_t - g(v_\eta)\nabla\varphi - g'(v_\eta)|\nabla v_\eta|^2 \varphi dx dt \geq 0 \tag{3.1.4}$$

for all non-decreasing $g \in C^1(\mathbb{R})$ and with $G' = g$, and for all $\psi \in \mathscr{D}([0,T] \times \Omega)$ with $\psi(t,x) \geq 0$ for all $(t,x) \in [0,T] \times \Omega$. This inequality has the role of an entropy condition for the weak solutions of problem 3.1.1, in a sense made precise by Evans [38] and by Mascia, Terracina and Tesei [62].

Since the sequences $\{u_\eta\}_{\eta>0}$ and $\{v_\eta\}_{\eta>0}$ are uniformly bounded in $L^\infty([0,T]\times\Omega)$, they have a weak-$\star$ limit $(u,v) \in L^\infty([0,T]\times\Omega)$ that satisfies equation

$$\begin{cases} u_t = \Delta v \text{ in } \Omega \\ \frac{\partial v}{\partial \nu} = 0 \text{ in } [0,T] \times \partial\Omega \\ u(x) = u_0 \text{ in } \Omega \end{cases} \tag{3.1.5}$$

in the weak sense, i.e. $u \in L^\infty([0,T]\times\Omega)$, $v \in L^\infty([0,T]\times\Omega)\cap L^2([0,T],H^1(\Omega))$ such that

$$\int_0^T \int_\Omega u\psi_t - \nabla v \cdot \nabla\psi dxdt + \int_\Omega u_0(x)\psi(0,x)dx = 0 \tag{3.1.6}$$

for all $\psi \in C^1([0,T]\times\overline{\Omega})$ with $\psi(T,x) = 0$ for all $x \in \Omega$. However, since in general weak-$\star$ convergence is not preserved by composition with a nonlinear function, we have no reason to expect that $v = \phi(u)$, so that the weak solution of 3.1.5 is not a weak solution of 3.1.1.

Thanks to the uniform bound on the L^∞ norm of $\{u_\eta\}_{\eta>0}$, Plotnikov in [70] showed that the sequence $\{u_\eta\}_{\eta>0}$ has a limit point ν in the space of Young measures, and ν can be interpreted as a weak solution to equation 3.1.1. Moreover, ν can be characterized as a superposition of at most three Dirac measures concentrated at the three branches of ϕ, and the Young measure ν and the function v defined by $v = \int_\mathbb{R} \phi(\tau)d\nu$ still satisfy the entropy inequality 3.1.4. This analysis suggests the following definition of weak solution to problem 3.1.1 in the sense of Young measures.

Definition 3.1.2. *An entropy Young measure solution of problem 3.1.1 consists of funtions $u, v, \lambda_i \in L^\infty([0,T]\times\Omega)$, $1 \le i \le 3$, satisfying the conditions:*

1. $\lambda_i \ge 0$, $\sum_{i=1}^3 \lambda_i = 1$, *and* $\lambda_1(x) = 1$ *if* $v(x) < \phi(u^+)$, $\lambda_3(x) = 1$ *if* $v(x) > \phi(u^-)$;
2. $v \in L^2([0,T],H^1(\Omega))$ *and* $u = \sum_{i=1}^3 \lambda_i S_i(v)$, *where* $S_i(v)$ *are defined as follows:*
$$S_1 : (-\infty, \phi(u^-)] \to (-\infty, u^-],$$
$$S_2 : (\phi(u^+), \phi(u^-)) \to (u^-, u^+),$$
$$S_3 : [\phi(u^+), +\infty) \to [u^+, +\infty),$$
and, for all i, $u = S_i(v)$ *iff* $v = \phi(u)$;
3. $u_t = \Delta v$ *in the weak sense, i.e.*
$$\int_0^T \int_\Omega u\psi_t - \nabla v \cdot \nabla\psi dxdt + \int_\Omega u_0(x)\varphi(0,x)dx = 0$$
for all $\varphi \in C^1([0,T]\times\overline{\Omega})$ *with* $\varphi(T,x) = 0$ *for all* $x \in \Omega$.
4. *for all* $g \in C^1(\mathbb{R})$ *with* $g' \ge 0$, *define*
$$G(x) = \int_0^x g(\phi(\tau))d\tau \text{ and } G^\star(u) = \sum_{i=1}^3 \lambda_i G(S_i(v)).$$
Then the following entropy inequality holds:
$$\int_0^T \int_\Omega G^\star(u)\varphi_t - g(v)\nabla v\nabla\varphi - g'(v)|\nabla v|^2\varphi dxdt \ge 0 \tag{3.1.7}$$

for all $\varphi \in \mathscr{D}([0,T]\times\Omega)$ with $\varphi(t,x) \geq 0$ for all $(t,x) \in [0,T]\times\Omega$.

It has been proved [62, 70] that problem 3.1.1 allows for an entropy Young measure solution, but in general these solutions are not unique, as shown for instance in [83]. Uniqueness of Young measure solutions has been proved by Mascia, Terracina and Tesei [63] under the additional constraint that the initial data and the solution do not take value in the interval (u^-, u^+). For a detailed discussion of the Young measure solutions to problem 3.1.1, we refer to [40, 62, 63, 70, 83].

3.1.2 The Radon measure solution in the case $u^+ = +\infty$

The Neumann initial-boundary value problem for 3.1.3 under the hypothesis that $u^+ = +\infty$ has been studied by Padron [68]. In analogy to the case where $u^+ < +\infty$, if ϕ is Lipschitz continuous and the initial data is $L^\infty(\Omega)$, then there exists a unique classical solution (u_η, v_η) to 3.1.3, with $u \in C^1([0,T], L^\infty(\Omega))$ and $v_\eta = \phi(u) + \eta u_t$. However, while the sequence $\{v_\eta\}_{\eta>0}$ is still uniformly bounded in the L^∞ norm, the sequence $\{u_\eta\}_{\eta>0}$ is not, so we cannot take the weak-$\star$ limit of $\{u_\eta\}_{\eta>0}$, even in the sense of Young measures. Nevertheless, thanks to the Neumann boundary conditions, $\|u_\eta(t)\|_1 = \|u_0\|_1$ for all $t \geq 0$. As a consequence, the sequence $\{u_\eta\}_{\eta>0}$ has a limit point u in the space of positive Radon measures over $[0,T]\times\Omega$. In [79], it is proved that u can be represented as the sum $u = \overline{u}+\mu$, where $\overline{u}$ is the baricenter of the Young measure $\nu(t,x)$ associated to an equi-integrable subsequence of $\{u_\eta\}_{\eta>0}$, and μ is a Radon measure over $[0,T]\times\Omega$. As a consequence, the Radon measure u and the L^∞ function v are a weak solution to problem 3.1.1 in the sense that

$$\int_0^T \langle \mu, \varphi_t\rangle_{\mathscr{D}(\Omega)} dt + \int_0^T \int_\Omega \overline{u}\varphi_t - \nabla v \nabla \varphi dx dt + \int_\Omega u_0(x)\varphi(0,x)dx = 0, \quad (3.1.8)$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$ with $\varphi(T,x) = 0$ for all $x \in \Omega$. In particular, a notion of entropy Radon measure solution can be defined for equation 3.1.1 even in the case $u^+ = +\infty$.

Definition 3.1.3. *An entropy Radon measure solution of problem 3.1.1 consists of funtions $\overline{u}, v, \lambda_i \in L^\infty([0,+\infty)\times\Omega)$, $i = 1,2$ and of a positive Radon measure $\mu \in \mathbb{M}([0,T]\times\Omega)$, satisfying the conditions:*

1. *$\lambda_i \geq 0$, $\sum_{i=1}^2 \lambda_i = 1$;*
2. *$v \in L^2([0,+\infty), H^1(\Omega))$ and*

$$\overline{u} = \begin{cases} \sum_{i=1}^2 \lambda_i S_i(v) & \text{if } v(x) > 0 \\ 0 & \text{if } v(x) = 0; \end{cases}$$

3. *$(\overline{u}+\mu)_t = \Delta v$ in the the sense of equation 3.1.8;*

4. the entropy inequality 3.1.7 holds for $\overline{u}$ and v.

Smarrazzo proved in [79] that problem 3.1.1 allows for a global entropy Radon measure solution, and we refer to her paper for an in-depth analysis of the properties of such solutions.

We conclude the discussion of the entropy Radon measure solution to problem 3.1.1 by recalling two features of the singular part of the solution. In [79], Smarrazzo showed that the singular part μ of the entropy Radon measure solution satisfies the following equality for all $t \geq 0$:

$$\mu(t) = \left(\int_{\Omega} u_0(x)dx - \int_{\Omega} \overline{u}(t,x)dx \right) \tilde{\mu}(t), \tag{3.1.9}$$

where $\tilde{\mu}(t)$ is a positive probability measure over Ω. Moreover, she conjectured that this singular term prevails over the regular term for large times. In section 3.3, we will show that the solution to problem 3.1.1 obtained from the grid function formulation satisfies an equality analogous to 3.1.9, and in section 3.4 we will show that the conjecture by Smarrazzo holds for such solutions.

3.1.3 Further remarks on problems 3.1.1 and 3.1.2

In [78], Slemrod showed that an approach similar to the one in [70] can be used to obtain Young measure solutions to problem 3.1.2. Notice however that, depending on the choice of the regularized problem, there are different notions of Young measure solutions for problems 3.1.1 and 3.1.2. For instance, Demoulini [35] has given a notion of Young measure solution to problem 3.1.2 based on a discrete-in-time energy minimization approach that sacrifices some of the physical meaning of the Sobolev approximation in favour of stability of the solution. For a comparison between Demoulini's solutions and the solutions obtained via the Sobolev approximation, see Horstmann and Schweizer [50].

Due to the wide range of their applications to the description of physical phenomena, numerical approximations of equations 3.1.1 and 3.1.2 have also been widely studied, especially focusing on particular choices of function ϕ (see for instance [48, 49, 58, 56, 69, 89]). The somewhat surprising feature of problems 3.1.1 and 3.1.2 is that, despite their ill-posedness and the absence of solutions in classical functional spaces, they nevertheless allow for stable numerical schemes that lead to successful applications. In particular, in addition to the original Perona-Malik algorithm for edge enhancing [69], the discrete-in-space counterparts of equations 3.1.1 and 3.1.2 have been shown to have a well-defined unique solution and their equilibria have been studied in depth, for instance by Lizana and Padron [58] and by Witelski, Shaeffer and Shearer [89].

3.2 The grid function formulation for the ill-posed PDE

In this section, we will derive the grid function formulation for the ill-posed problem 3.1.1 from very simple basic principles by using an elementary description that generalizes the nonstandard model for the diffusion equation by Hanqiao, St. Mary and Wattenberg [45]. This approach will allow us to choose a suitable grid function counterpart to the operator $\Delta\phi(u)$. Under the hypotheses 3.1.1 over ϕ, we will prove that the grid function formulation always has a unique well-defined solution. At the end of this section, we will discuss the coherence of the solution to the grid function formulation with the notions of solutions discussed in section 3.1.

3.2.1 Derivation of the grid function formulation

For a matter of commodity, in the derivation of the model we will use the image of a population that moves around the grid $\mathbb{X}^k$ according to some basic rules. The initial distribution of the population around the grid is described by an internal function $u_0 : \mathbb{X}^k \to {}^*[0,1]$ satisfying $\int_{\mathbb{X}^k} u_0(x) d\mathbb{X}^k = c \in {}^*\mathbb{R}$. The value $u_0(x)$ determines the number of individuals of the population inhabiting point x at time $t = 0$.

Let $\varepsilon_t = \varepsilon^2$. The population moves around the grid according to the following rules:

- the n-th move occurs between time $(n-1)\varepsilon_t$ and $n\varepsilon_t$;
- at each jump the population at each grid point breaks into $(2k+1)$ smaller groups:
  - for $i = 1, \ldots, k$, a fraction $p_i(u((n-1)\varepsilon_t, x))$ of the population at x jumps to $x + \varepsilon_x \vec{e_i}$;
  - for $i = 1, \ldots, k$, a fraction $p_i(u((n-1)\varepsilon_t, x))$ of the population at x jumps to $x - \varepsilon_x \vec{e_i}$;
  - the remaining fraction $1 - 2\sum_{i=1}^{k} p_i(u((n-1)\varepsilon_t, x))$ of the population at x remains at x.

In the above description, the functions p_i are internal functions $p_i : {}^*\mathbb{R} \to {}^*\mathbb{R}$ satisfying

- $0 \leq p_i(r)$ for all $r \in {}^*\mathbb{R}$;
- $\sum_{i=1}^{k} p_i(r) \leq 1/2$ for all $r \in {}^*\mathbb{R}$

for all $i = 1, \ldots, k$. The properties of the functions p_i determine the criteria used by the population to choose whether and how to jump to a nearby grid point. In particular, in the model outlined above an individual chooses its next movement to move according only to local informations. If the functions p_i are constant and do not depend on i, then the above model coincides with the nonstandard model of diffusion discussed in [45]. More complex behaviour can be described by different choices of functions p_i and by introducing a spatial bias.

If we denote by $u(t,x)$ the population present at time t at point x, then by arguing as in section III of [45] we deduce that $u(t,x)$ evolves according to the finite difference initial value problem

$$\begin{aligned} u(0,x) &= u_0(x) \\ u((n+1)\varepsilon_t, x) &= \left(1 - 2\sum_{i=1}^{k} p_i(u(n\varepsilon_t, x))\right) u(n\varepsilon_t, x) \\ &\quad + \sum_{i=1}^{k} p_i(u(n\varepsilon_t, x+\varepsilon_x e_i))u(n\varepsilon_t, x+\varepsilon_x e_i) \\ &\quad + \sum_{i=1}^{k} p_i(u(n\varepsilon_t, x-\varepsilon_x e_i))u(n\varepsilon_t, x-\varepsilon_x e_i)) \end{aligned}$$

From the above equation, if we define $\phi_i(u(n\varepsilon_t, x)) = p_i(n\varepsilon_t, x)u(n\varepsilon_t, x)$, we obtain

$$u((n+1)\varepsilon_t, x) - u(n\varepsilon_t, x) = \sum_{i=1}^{k} \Big[\phi_i(u(n\varepsilon_t, x+\varepsilon_x e_i)) - 2\phi_i(u(n\varepsilon_t, x)) + \phi_i(u(n\varepsilon_t, x-\varepsilon_x e_i))\Big].$$

At this point, we divide both sides of the above equation by $\varepsilon_t = \varepsilon_x^2$ and obtain

$$\frac{u((n+1)\varepsilon_t, x) - u(n\varepsilon_t, x)}{\varepsilon_t} = \sum_{i=1}^{k} \mathbb{D}_i^+ \mathbb{D}_i^- \phi_i(u(x,t)).$$

If $\phi_i = \phi$ for all $i = 1, \ldots, k$, i.e. if the population moves without spatial bias, from the above equality we deduce

$$\frac{u((n+1)\varepsilon_t, x) - u(n\varepsilon_t, x)}{\varepsilon_t} = \Delta_{\mathbb{X}} \phi(u). \tag{3.2.1}$$

Neumann boundary conditions are imposed in the discrete formulation of the Laplacian in the following way: if $x \in \partial_{\mathbb{X}} \Omega_{\mathbb{X}}$, let

$$I_x^+ = \{i : x + \varepsilon e_i \notin \Omega_{\mathbb{X}}\} \text{ and } I_x^- = \{i : x - \varepsilon e_i \notin \Omega_{\mathbb{X}}\}.$$

The Neumann boundary conditions are equivalent to

$$\sum_{i \in I_x^+} \mathbb{D}_i^{+*} \phi(u(x)) = 0 \text{ and } \sum_{i \in I_x^-} \mathbb{D}_i^{-*} \phi(u(x)), \tag{3.2.2}$$

for all $x \in \partial_{\mathbb{X}}\Omega_{\mathbb{X}}$, so that the first-order discrete approximation of the Laplacian with Neumann boundary conditions is defined by:

$$\begin{aligned}\Delta_{\mathbb{X}}{}^*\phi(u(x)) &= -\sum_{i \in I_x^+} \mathbb{D}_i^{-*}\phi(u(x)) + \sum_{i \in I_x^-} \mathbb{D}_i^{+*}\phi(u(x)) + \\ &\quad + \sum_{i \notin I_x^+ \cup I_x^-} \mathbb{D}_i^+ \mathbb{D}_i^- \phi(u(x)).\end{aligned}$$

The above argument suggests that the functional $F : L^\infty(\Omega) \cap H^1(\Omega) \to (C^1(\overline{\Omega}))'$ defined by

$$\langle F(u), \varphi \rangle_{C^1(\Omega)} = -\int_\Omega \nabla\phi(u)\nabla\varphi dx \tag{3.2.3}$$

for all $\varphi \in C^1(\overline{\Omega})$ can be represented in the sense of grid functions by $\Delta_{\mathbb{X}}{}^*\phi$. We will now prove that $\Delta_{\mathbb{X}}{}^*\phi$ is indeed coherent with F in the sense of Theorem 2.4.8. Notice how condition (1) of Theorem 2.4.8 is replaced by a different coherence condition that depends upon the definition of F.

Proposition 3.2.1. *Let ϕ be a standard function satisfying hypotheses 3.1.1, and let F be defined by equation 3.2.3. Then*

1. *if $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ satisfies $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$ and if u and $\mathbb{D}_i^\pm u$ are nearstandard in $L^2(\Omega)$, then $[\Delta_{\mathbb{X}}{}^*\phi(u)] = F([u]) \in (C^1(\overline{\Omega}))'$;*
2. *whenever $u, v \in \mathbb{G}(\Omega_{\mathbb{X}})$ are nearstandard in $L^\infty(\Omega) \cap H^1(\Omega)$, if $\|u - v\|_\infty \approx 0$ and $\|u - v\|_{H^1} \approx 0$, then $[\Delta_{\mathbb{X}}{}^*\phi(u)] = [\Delta_{\mathbb{X}}{}^*\phi(v)]$;*
3. *for all $u \in L^\infty(\Omega) \cap H^1(\Omega)$, $[\Delta_{\mathbb{X}}{}^*\phi(P({}^*u))] = F(u)$.*

Proof. By the discrete summation by parts formula and by taking into account the Neumann boundary conditions 3.2.2, for all $\varphi \in S^1(\overline{\Omega}_{\mathbb{X}})$ we have the equality

$$\langle \Delta_{\mathbb{X}}{}^*\phi(u), \varphi \rangle = -\langle \nabla_{\mathbb{X}}^- \phi(u(x+\varepsilon)), \nabla_{\mathbb{X}}^+ \varphi \rangle. \tag{3.2.4}$$

We will now show that $[\nabla_{\mathbb{X}}^{-*}\phi(u)] = [\phi'(u)]\nabla[u]$. For all $1 \le i \le k$, we have the equality

$$\begin{aligned}\mathbb{D}_i^{-*}\phi(u(x)) &= \frac{{}^*\phi(u(x)) - {}^*\phi(u(x - \varepsilon_x e_i))}{\varepsilon} \\ &= \frac{{}^*\phi(u(x)) - \varphi(u(x) - \varepsilon\mathbb{D}_i^+ u(x - \varepsilon_x e_i))}{\varepsilon\mathbb{D}_i^+ u(x - \varepsilon_x e_i)} \cdot \mathbb{D}_i^+ u(x - \varepsilon_x e_i).\end{aligned}$$

The hypothesis that $\mathbb{D}_i^+ u$ is nearstandard in $L^2(\Omega)$ ensures that there is a L_N-nullset $\Omega_0 \subset \Omega_{\mathbb{X}}$ such that $\mathbb{D}_i^+ u(x - \varepsilon_x e_i)$ is finite for all $x \in \Omega_{\mathbb{X}} \setminus \Omega_0$. Moreover, if $x \in \Omega_0$, $\varepsilon\mathbb{D}_i^+ u(x - \varepsilon_x e_i) \approx 0$, otherwise $\mathbb{D}_i^+ u$ would not be nearstandard in $L^2(\Omega)$. As a consequence, $[\mathbb{D}_i^+ u(x-\varepsilon_x e_i)] = D_i[u](^\circ x)$ and, since $\|u\|_\infty$ is finite, $[\mathbb{D}_i^{-*}\phi(u)] = [\phi'(u)]D_i^+[u]$. Taking into account these equalities, we deduce

$$\begin{aligned} {}^\circ\langle \Delta_{\mathbb{X}}{}^*\phi(u), \varphi\rangle &= \langle [\Delta_{\mathbb{X}}{}^*\phi(u)], {}^\circ\varphi\rangle_{C^1(\Omega)} \\ &= -\int_\Omega [\nabla^{-*}\phi(u)]\nabla^\circ\varphi dx \\ &= -\int_\Omega [\phi'(u)]\nabla[u]\nabla^\circ\varphi dx \end{aligned} \tag{3.2.5}$$

and, by the hypothesis that $\mathbb{D}_i^+ u$ are nearstandard in $H^1(\Omega)$ for $1 \le i \le k$, $[u] \in H^1(\Omega)$. As a consequence, the integral 3.2.5 is finite, and $[\Delta_{\mathbb{X}}{}^*\phi(u)] \in (C^1(\overline{\Omega}))'$.

In order to prove that $[\Delta_{\mathbb{X}}{}^*\phi(u)] = F([u])$, notice that the hypothesis that $\|u\|_\infty \in {}^*\mathbb{R}_{fin}$ and that u is nearstandard in $L^2(\Omega)$ entail that the Young measure associated to u is Dirac, hence a.e. equal to $[u]$ by Theorem 2.3.13. As a consequence, $[{}^*\varphi(u)] = \varphi([u])$, so that from equality 3.2.5 we deduce that $[\Delta_{\mathbb{X}}{}^*\phi(u)] = F([u])$.

We will now prove that $\|u-v\|_\infty \approx 0$ and $\|u-v\|_{H^1} \approx 0$ imply $[\Delta_{\mathbb{X}}{}^*\phi(u)] = [\Delta_{\mathbb{X}}{}^*\phi(v)]$. From the hypothesis $\|u-v\|_\infty \approx 0$ and by S-continuity of ${}^*\phi$ and of ${}^*\phi'$, we have

$$\|{}^*\phi(u) - {}^*\phi(v)\|_\infty \approx \|{}^*\phi'(u) - {}^*\phi'(v)\|_\infty \approx 0.$$

The assumption $\|u-v\|_{H^1} \approx 0$ entails also $[\nabla_{\mathbb{X}}^+ u] = [\nabla_{\mathbb{X}}^+ v]$, so that

$$[\nabla_{\mathbb{X}}^{+*}\phi(u)] = [\phi'(u)]\nabla[u] = [\phi'(v)]\nabla[v] = [\nabla_{\mathbb{X}}^{+*}\phi(v)].$$

As a consequence, from equality 3.2.5 we obtain

$$\langle [\Delta_{\mathbb{X}}{}^*\phi(u)] - [\Delta_{\mathbb{X}}{}^*\phi(v)], {}^\circ\varphi\rangle_{C^1(\Omega)} = \int_\Omega ([\phi'(u)]\nabla^-[u] - [\phi'(v)]\nabla^-[v])\nabla^\circ\varphi dx = 0,$$

so that the proof of (2) is concluded.

Part (3) of the assertion is a consequence of part (1), since if $u \in L^\infty(\Omega) \cap H^1(\Omega)$, then Lemma 2.3.7 ensures that $P({}^*u)$ satisfies the hypotheses of part (1) of the assertion and that $[P({}^*u)] = u$. □

3.2.2 The grid function formulation for the ill-posed PDE

We now have all of the elements to formulate problem 3.1.1 in the sense of grid functions.

Definition 3.2.2. *The functions $[u], [{}^*\phi(u)] \in \mathbb{G}(\Omega_{\mathbb{X}})/\equiv$ are called a grid solution of 3.1.1 if u satisfies the following system of ODEs:*

$$\begin{cases} u_t = \Delta_{\mathbb{X}}{}^*\phi(u); \\ u(0,x) = {}^*u_0(x). \end{cases} \tag{3.2.6}$$

Remark 3.2.3. *A standard version of the formulation 3.2.6 with* $\Omega_{\mathbb{X}} = [0, 1]$ *and with standard* N *has been used by Lizana and Padron [58] to describe the dynamics of a population inhabiting a finite collection of* $N+1$ *equally spaced points* $\{0, \ldots, i/N, \ldots, 1\}$ *on the interval* $[0, 1]$*. By the transfer principle, many properties of the finite model discussed in section 3 of [58] hold also for the hyperfinite system 3.2.6. Conversely, many of the results discussed in Sections 3.4 and 3.5 of this chapter can be applied to this finite model by omitting the stars and by taking* $N \in \mathbb{N}$*.*

Remark 3.2.4. *Notice that problem 3.2.6 makes sense for an arbitrary* $f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ *instead of* ${}^*\phi$ *and for arbitrary initial data. However, since we are interested not only in the solutions to problem 3.2.6, but also in the coherence with the measure-valued solutions to problem 3.1.1, we will restrict our attention to the case where* $f = {}^*\phi$*, and* ϕ *satisfies hypotheses 3.1.1, and where the initial data is the nonstandard extension of a function* $u_0 \in L^\infty(\Omega)$ *that satisfies* $u_0(x) \geq 0$ *for all* $x \in \Omega$*.*

Problem 3.2.6 can be interpreted as a hyperfinite system of ordinary differential equations: as such, the existence of solutions and their properties can be studied by the theory of ordinary differential equations. These results, in turn, apply to the grid solution for problem 3.1.1.

Theorem 3.2.5. *There exists a maximal interval* $I \subseteq {}^*\mathbb{R}$ *such that Problem 3.2.6 has a unique solution* $u \in {}^*C^1(I, \mathbb{G}(\Omega_{\mathbb{X}}))$*; Moreover,* $\|u(t)\|_1 = \|u_0\|_1$ *for all* $t \in I$*.*

Proof. By transfer, existence and uniqueness can be deduced from the theory of ordinary differential equations.

In order to prove that $\|u(t)\|_1 = \|u_0\|_1$ for all $t \in I$, notice that it holds

$$\frac{d}{dt} \int_{\Omega_{\mathbb{X}}} u(t, x) d\mathbb{X}^k = \int_{\Omega_{\mathbb{X}}} u_t(t, x) d\mathbb{X}^k = \int_{\Omega_{\mathbb{X}}} \Delta_{\mathbb{X}} \phi(u(t, x)) d\mathbb{X}^k.$$

Thanks to the Neumann boundary conditions, $\int_{\Omega_{\mathbb{X}}} \Delta_{\mathbb{X}} \phi(u(t, x)) d\mathbb{X}^k = 0$, so that the mass of the solution is preserved. □

Proposition 3.2.6 (Invariant set)**.** *For all* $t \in I$ *and for all* $x \in \Omega_{\mathbb{X}}$*,*

1. *if* $u^+ = +\infty$*, then* $u(t, x) \geq 0$*.*
2. *if* $u^+ < +\infty$*, then* $u(t, x) \in \{0, \max\{\|{}^*u_0\|_\infty, S_3(\phi(u^-))\}\}$*. In particular,* $\|u(t)\|_\infty \in {}^*\mathbb{R}_{fin}$ *is homogeneously bounded for all* $t \geq 0$*.*

Proof. If $u^+ = +\infty$, let

$$\bar{t} = \sup\{t \geq 0 : u(t, x) \geq 0 \text{ for all } t \in I \text{ and } x \in \Omega_{\mathbb{X}}\}.$$

The hypotheses over ϕ and the definition of $\bar{t}$ ensure that if $u(\bar{t}, x) = 0$, then $u_t(\bar{t}, x) = \Delta_{\mathbb{X}} \phi(u(\bar{t}, x)) \geq 0$. As a consequence, $\bar{t} = \sup I$.

Similarly, if $u^+ < +\infty$, let

$$\bar{t} = \sup\{t \geq 0 : u(t,x) \in \{0, \max\{\|^*u_0\|_\infty, S_3(\phi(u^-))\}\} \text{ for all } t \in I \text{ and } x \in \Omega_\mathbb{X}\}.$$

In this case, if $u(\bar{t}, x) = 0$, the equality $u_t(\bar{t}, x) = \Delta_\mathbb{X}\phi(u(\bar{t}, x)) \geq 0$ holds as in the previous part of the proof. If $u(\bar{t}, x) = \max\{\|^*u_0\|_\infty, \phi(u^-)\}$, then a similar calculation allows to conclude $u_t(\bar{t}, x) \leq 0$. We deduce that it holds $\bar{t} = \sup I$ also for this case. □

Since for any initial data $u_0 \in L^\infty(\Omega)$ the invariant set for the dynamical system 3.2.6 is bounded, we deduce global existence in time.

Corollary 3.2.7 (Global existence in time)**.** *The solution u of system 3.2.6 satisfies $u \in {}^*C^1({}^*[0,\infty), \mathbb{G}(\Omega_\mathbb{X}))$.*

Proof. Let u be the solution of Problem 3.2.6, and let I be the interval over which u is defined. Define also

$$\mathbb{S}^+({}^*u_0) = \{f \in \mathbb{G}(\Omega_\mathbb{X}) : f(x) \geq 0 \text{ for all } x \in \Omega_\mathbb{X} \text{ and } \|f\|_1 = \|^*u_0\|_1\}.$$

By Theorem 3.2.5 and by Proposition 3.2.6, $u(t) \in \mathbb{S}^+({}^*u_0)$ for all $t \in I$.

Let $\Omega_\mathbb{X} = \{x_1, \ldots, x_M\}$, with $M = |\Omega_\mathbb{X}|$. We identify u with a vector-valued function that, by abuse of notation, we will still denote by $u : I \to {}^*\mathbb{R}^M$, with the convention that the k-th component of $u(t)$ is $u(t, x_k)$. Since the set $\mathbb{S}^+({}^*u_0)$ is *compact in ${}^*\mathbb{R}^M$, the theory of ODEs allows to conclude that u has global existence in time. □

As a consequence of Theorem 3.2.5 and of Corollary 3.2.7, we deduce that problem 3.1.1 always has a unique global grid solution.

3.3 Coherence of the grid solution with the measure-valued solutions to the ill-posed PDE

This section is devoted to the study of the coherence of the grid solution with the notions of measure-valued solutions for problem 3.1.1 discussed in section 3.1. In particular, we will show that, if u is regular enough, then the grid solution of problem 3.1.1 coincides with an entropy Young measure solution in the case where $u^+ < +\infty$, and with the entropy Radon measure solution in the case where $u^+ = +\infty$.

Our argument relies on an equality that will be used to establish an entropy condition for the pair $[u], [{}^*\phi(u)]$.

Lemma 3.3.1. *For all internal $f, g : \mathbb{X}^k \to {}^*\mathbb{R}$, it holds*

$$\text{div}^-_\mathbb{X}\big(g(f(x)) \cdot \nabla_\mathbb{X} f(x)\big) = g(f(x))\Delta_\mathbb{X}(f(x)) + \nabla^-_\mathbb{X} f(x) \cdot \nabla^-_\mathbb{X} g(f(x)).$$

Notice that the above result is independent of the regularity of f and g.

Lemma 3.3.2 (Entropy condition)**.** *For any $g \in C^1(\mathbb{R})$ with $g' \geq 0$, define $G(u(t,x)) = \int_0^{u(t,x)} g(\phi(s))ds$. Then, if u is the solution to problem 3.2.6, it holds*

$$^*G(u)_t = \mathrm{div}^-_{\mathbb{X}}((^*g(\phi(u))\nabla^+_{\mathbb{X}}(\phi(u))) - \nabla^-_{\mathbb{X}}{}^*g'(\phi(u)) \cdot \nabla^-_{\mathbb{X}}\phi(u).$$

and, if $\nabla^-_{\mathbb{X}}\phi(u)$ is finite,

$$^*G(u)_t \approx \mathrm{div}^-_{\mathbb{X}}((^*g(\phi(u))\nabla^+_{\mathbb{X}}(\phi(u))) - {}^*g'(\phi(u))|\nabla^-_{\mathbb{X}}\phi(u)|^2. \tag{3.3.1}$$

Proof. For G, g and u it holds

$$G(u)_t = g(\phi(u))u_t = g(\phi(u))\Delta_{\mathbb{X}}\phi(u).$$

By Lemma 3.3.1,

$$\mathrm{div}^-_{\mathbb{X}}((g(\phi(u))\nabla^+_{\mathbb{X}}(\phi(u))) = g(\phi(u))\Delta_{\mathbb{X}}\phi(u) + \nabla^-_{\mathbb{X}}{}^*g(\phi(u)) \cdot \nabla^-_{\mathbb{X}}\phi(u),$$

so that the first equality is proved.

For the second equality, we have already shown in the proof of Proposition 3.2.1 that if $\nabla^-_{\mathbb{X}}\phi(u)$ is finite, then $\nabla^-_{\mathbb{X}}g(\phi(x)) \approx g'(\phi(x))\nabla^-_{\mathbb{X}}\phi(x)$. As a consequence,

$$\nabla^-_{\mathbb{X}}g(\phi(x)) \cdot \nabla^-_{\mathbb{X}}\phi(x) \approx g'(\phi(x))|\nabla^-_{\mathbb{X}}\phi(x)|^2,$$

as desired. □

Formula 3.3.1 can be regarded as an entropy condition for system 3.2.6. In particular, this equality allows us to prove that the solution obtained by the nonstandard model 3.2.6 retains the physical meaning of an entropy solution.

Now, we will prove that the grid solution of problem 3.1.1 is always a very weak solution in the sense of distributions.

Lemma 3.3.3. *Let u be the solution of problem 3.2.6. Then $[u], [^*\phi(u)] \in \mathscr{D}'(\mathbb{R} \times \Omega)$ is a very weak solution of problem 3.1.1 in the sense of distributions, i.e. $[u]$ and $[^*\phi(u)]$ satisfy*

$$\int_0^T \langle [u], \varphi_t \rangle + \langle [^*\phi(u)], \Delta\varphi \rangle dt + \int_\Omega u_0(x)\varphi(0,x)dx = 0 \tag{3.3.2}$$

for all $\varphi \in C^1([0,T], \mathscr{D}'(\Omega))$ with $\varphi(T,x) = 0$ for all $x \in \Omega$.

Proof. By Proposition 3.2.6, $\|u(t)\|_1 = c \in {}^*\mathbb{R}_{fin}$ for all $t \in {}^*\mathbb{R}_{\geq 0}$ and, by Proposition 3.2.6 if $u^+ < +\infty$ or by the boundedness of ϕ if $u^+ = +\infty$, also $\|^*\phi(u)\|_\infty \in {}^*\mathbb{R}_{fin}$ for all $t \in {}^*\mathbb{R}_{\geq 0}$.

Now let $\varphi \in C^1([0,T], \mathscr{D}'(\Omega))$ with $\varphi(T,x) = 0$ for all $x \in \Omega$, and define $\varphi_{\mathbb{X}}(t) = {}^*\varphi(t)_{|\mathbb{X}}$. Since $u \in {}^*C^1({}^*\mathbb{R}_{\geq 0}, \mathbb{G}(\Omega_{\mathbb{X}}))$ and $\varphi_{\mathbb{X}} \in {}^*C^1([0,T], \mathscr{D}_{\mathbb{X}}(\Omega_{\mathbb{X}}))$, we have

$$\int_0^T \langle u_t(t), \varphi_{\mathbb{X}}(t)\rangle dt = -\int_0^T \langle u(t), (\varphi_{\mathbb{X}})_t(t)\rangle dt - \langle {}^*u_0, \varphi_{\mathbb{X}}(0,x)\rangle. \quad (3.3.3)$$

By the discrete summation by parts formula, for all $t \in {}^*\mathbb{R}_{\geq 0}$ we have

$$\langle \Delta_{\mathbb{X}}\phi(u(t)), \varphi_{\mathbb{X}}(t)\rangle = \langle \phi(u(t)), \Delta_{\mathbb{X}}\varphi_{\mathbb{X}}(t)\rangle. \quad (3.3.4)$$

Taking into account that u satisfies 3.2.6, by equations 3.3.3 and 3.3.4, we obtain

$$\int_0^T \langle u(t), (\varphi_{\mathbb{X}})_t(t)\rangle + \langle \phi(u(t)), \Delta_{\mathbb{X}}\varphi_{\mathbb{X}}(t)\rangle dt + \langle {}^*u_0, \varphi_{\mathbb{X}}(0,x)\rangle = 0.$$

By Lemma 2.3.6,

$$^\circ\langle {}^*u_0, \varphi_{\mathbb{X}}(0,x)\rangle = \int_\Omega u_0\varphi(0,x)dx.$$

As a consequence, $[u]$ and $[{}^*\phi(u)]$ satisfy

$$^\circ\left(\int_0^T \langle u(t), {}^*\varphi(t)\rangle dt\right) = \int_{[0,T]} \langle [u], \varphi_t\rangle_{\mathscr{D}(\Omega)} dt + \int_\Omega u_0(x)\varphi(0,x)dx$$

and

$$^\circ\left(\int_0^T \langle \Delta_{\mathbb{X}}\phi(u(t)), {}^*\varphi(t)\rangle dt\right) = \int_0^T \langle [{}^*\phi(u)], \Delta\varphi\rangle_{\mathscr{D}(\Omega)} dt.$$

As a consequence, we deduce that equality 3.3.2 holds. □

3.3.1 The case $u^+ < +\infty$

We will now discuss coherence of the grid solution with the solutions of problem 3.1.1 in the case where $u^+ < \infty$. As expected, if u is regular enough, then the grid solution to problem 3.1.1 is a solution of problem 3.1.1 in a classical sense. The degree of regularity of the standard solution depends upon the regularity of u.

Theorem 3.3.4. *Let $[u], [{}^*\phi(u)]$ be the grid solution of Problem 3.2.6, and let $\nu(t,x)$ the Young measure associated to u.*

1. *If $[{}^*\phi(u)] \in L^2([0,T], H^1(\Omega))$, then $[u], [{}^*\phi(u)]$ is an entropy Young measure solution of Problem 3.1.1 in the sense of equation 3.1.6.*

2. *Under the hypotheses*

    - *$\nu(t,x)$ is Dirac a.e.,*

- $[{}^*\phi(u)] \in L^\infty([0,T] \times \Omega)$,

then $[u] \in L^\infty([0,T], L^\infty(\Omega))$ *and* $[u], [{}^*\phi(u)]$ *is a very weak solution of Problem 3.1.1.*

3. *Under the hypotheses*

- $\nu(t,x)$ *is Dirac a.e.,*
- $[{}^*\phi(u)] \in L^\infty([0,T] \times \Omega) \cap L^2([0,T], H^1(\Omega))$,

then $[u], [{}^*\phi(u)]$ *is a weak solution of Problem 3.1.1.*

4. *If* $u \in S^1({}^*[0,+\infty), S^2(\Omega_{\mathbb{X}}))$, *then* $[u] = {}^\circ u$ *is a classical global solution of Problem 3.1.1.*

Proof. (1). Since $[{}^*\phi(u(t))] \in H^1(\Omega)$ for a.e. $t \geq 0$, we deduce that $\int_\Omega \phi(\tau) d\nu(t,x)$ is single-valued for a.e. $t \geq 0$. In particular, $\nu(t,x)$ is a.e. a superposition of at most three Dirac measures centred at $S_i\left(\int_\Omega \phi(\tau) d\nu(t,x)\right)$, and $[u]$ is the barycentre of ν in the sense that

$$[u](t,x) = \int_{\mathbb{R}} \tau d\nu(t,x).$$

From these properties, we recover conditions (1)–(3) of the definition of entropy Young measure solution.

By taking into account that $[{}^*\phi(u)] \in H^1(\Omega)$, from Proposition 3.2.1 and from equation 3.3.2 we deduce that $[u]$ and $[{}^*\phi(u)]$ satisfy

$$\int_0^T \int_\Omega [u]\varphi_t - \nabla[{}^*\phi(u)]\nabla\varphi dxdt + \int_\Omega u_0(x)\varphi(0,x)dx = 0$$

for all $\varphi \in C^1([0,T] \times \overline{\Omega})$ with $\varphi(T,x) = 0$ for all $x \in \Omega$.

We will now derive the entropy condition 3.1.7 for $[u]$ and $[{}^*\phi(u)]$. Let $g \in C^1(\mathbb{R})$, $g' \geq 0$, $G(x) = \int_0^x g(\phi(\tau))d\tau$, and let $\varphi \in \mathscr{D}([0,T] \times \Omega)$ with $\varphi \geq 0$. Define also

$$G^\star([u]) = \sum_{i=1}^3 \int_0^{S_i([{}^*\phi(u)])} g(\tau)d\tau.$$

By Theorem 2.3.12, we have the following equalities

$$\begin{aligned} -\int_0^T \int_\Omega G^\star([u])\varphi_t dxdt &= -{}^\circ\int_0^T \langle {}^*G(u), {}^*\varphi_t \rangle dt \\ &= {}^\circ\int_0^T \langle {}^*G(u)_t, {}^*\varphi \rangle dt \\ &= {}^\circ\int_0^T \langle {}^*g({}^*\phi(u))u_t, {}^*\varphi \rangle dt \\ &= {}^\circ\int_0^T \langle {}^*g({}^*\phi(u))\Delta_{\mathbb{X}}\varphi(u), {}^*\psi \rangle dt. \end{aligned}$$

By Lemma 3.3.2 and by $[^*\phi(u)] \in H^1(\Omega)$, we deduce

$$\int_0^T \langle {}^*g({}^*\phi(u))\Delta_{\mathbb{X}} v, {}^*\varphi\rangle dt \approx \int_0^T \langle \mathrm{div}^-_{\mathbb{X}}({}^*g({}^*\phi(u)\nabla^+_{\mathbb{X}}\phi(u)) - {}^*g'({}^*\phi(u))|\nabla^-_{\mathbb{X}}\phi(u)|^2, {}^*\varphi\rangle dt$$

By the discrete summation by parts formula and by Theorem 2.2.15,

$$\begin{aligned} -\int_0^T \langle \mathrm{div}^-_{\mathbb{X}}({}^*g({}^*\phi(u)\nabla^+_{\mathbb{X}}\phi(u)), {}^*\varphi\rangle dt &\approx \int_0^T \langle {}^*g({}^*\phi(u))\nabla^+_{\mathbb{X}}{}^*\phi(u), \nabla^-_{\mathbb{X}}{}^*\varphi\rangle dt \\ &\approx \int_0^T\int_\Omega g([^*\phi(u)])\nabla[^*\phi(u)]\nabla\varphi dx dt \end{aligned}$$

and, by Proposition 2.3.3,

$$^\circ\int_0^T \langle {}^*g'({}^*\phi(u))|\nabla^-_{\mathbb{X}}{}^*\phi(u)|^2, {}^*\varphi\rangle dt \geq \int_0^T\int_\Omega g'([^*\phi(u)])|\nabla[^*\phi(u)]|^2\varphi dx dt.$$

Putting together the above inequalities, we deduce

$$\int_0^T\int_\Omega G^\star([u])\varphi_t - g([^*\phi(u)])\nabla([^*\phi(u)])\nabla\varphi - g'([^*\phi(u)])|\nabla[^*\phi(u)]|^2\varphi dx dt \geq 0,$$

so that $[u]$ and $[^*\phi(u)]$ satisfy the entropy condition 3.1.7.

(2). Since $\nu(t,x)$ is Dirac, by Lemma 2.3.13, it coincides with $[u]$ and, as a consequence, we also have $[^*\phi(u(t))] = \phi([u])$. By substituting $[u]$ and $\phi([u])$ in equation 3.3.2, we obtain

$$\int_0^T\int_\Omega [u](t,x)\varphi_t + \phi([u])(t,x)\Delta\varphi d(t,x) + \int_\Omega u_0(x)\varphi(0,x)dx = 0,$$

that is, $[u]$ and $[^*\phi(u)]$ are a very weak solution of Problem 3.1.1.

(3). In addition to the conclusions of point (2), we also have $\phi([u]) \in L^\infty((0,T), H^1(\Omega))$, so that by Proposition 3.2.1 the following equality

$$\langle \Delta_{\mathbb{X}}\phi(u(t))(x), {}^*\varphi(t)\rangle = -\int_\Omega \nabla\phi([u])(t)\nabla\varphi(t)dx$$

holds for a.e. $t \geq 0$. Hence, by substituting $[u]$ and $[^*\phi(u)]$ in equation 3.3.2, we deduce that $[u]$ and $[^*\phi(u)]$ are a weak solution of Problem 3.1.1.

We will now prove (4). By Theorem 2.1.14, $[u] = {}^\circ u$ and, by Corollary 2.1.16, ${}^\circ u \in C^1(\mathbb{R}_{\geq 0}, C^2(\Omega))$. Moreover, ${}^{\circ *}\phi(u) = \phi({}^\circ u) \in C^1([0,+\infty), C^2(\Omega))$ and, by Theorem 2.1.15, $[\Delta_{\mathbb{X}}{}^*\phi(u(t))] = \Delta\phi({}^\circ u(t))$ for all $t \geq 0$. The boundary conditions 3.2.2 ensure that

$$\frac{\partial\phi({}^\circ u)}{\partial\nu} = 0 \text{ in } [0,+\infty)\times\partial\Omega.$$

This is sufficient to conclude that ${}^\circ u$ is a classic global solution of Problem 3.1.1. □

3.3.2 The case $u^+ = +\infty$

We will now discuss coherence of the grid solution to problem 3.2.6 with the measure-valued solution to equation 3.1.1 under the hypothesis that $u^+ = +\infty$.

Theorem 3.3.5. *Let $[u], [^*\phi(u)]$ be the grid solution of Problem 3.2.6, and let $\nu(t,x)$ the Young measure associated to u. If $[^*\phi(u)] \in L^2([0,+\infty), H^1(\Omega))$, then $[u], [^*\phi(u)]$ is an entropy Radon measure solution of problem 3.1.1 in the sense of equation 3.1.8.*

Proof. Let

$$u_r(t,x) = \int_{\mathbb{R}} \tau d\nu(t,x)$$

be the barycentre of ν, and let $\mu(t) = [u](t) - \overline{u}(t)$. The Young measure $\nu(t,x)$ corresponds to the regular term of the solution to problem 3.1.1, and the Radon measure μ corresponds to the singular term.

The hypothesis $[^*\phi(u)](t) \in H^1(\Omega)$ ensures that $[^*\phi(u)](t,x)$ is single-valued for a.e. $t \geq 0$ and $x \in \Omega$. If $[^*\phi(u)](t,x) = c \neq 0$, this implies that $\nu(t,x)$ is a superposition of at most two Dirac measures centred at $S_1(c)$ and at $S_2(c)$. If $[^*\phi(u)](t,x) = 0$, then $\nu(t,x)$ is a Dirac Young measure centred at 0.

Notice that, for any $\varphi \in C^1([0,T] \times \overline{\Omega})$ with $\varphi(T,x) = 0$ for all $x \in \Omega$, we have the equality

$$\int_0^T \langle u, {}^*\varphi \rangle dt = \int_0^T \int_\Omega [u]\varphi dx dt = \int_0^T \int_\Omega u_r \varphi dx dt + \int_0^T \langle \mu, \varphi \rangle_{\mathscr{D}(\Omega)} dt,$$

for any arbitrary $T > 0$. By Proposition 3.2.1, by equality 3.3.2 and by the hypothesis that $[^*\phi(u)](t) \in H^1(\Omega)$, we deduce that u_r, μ and $[^*\phi(u)]$ satisfy the equality

$$\int_0^T \langle \mu, \varphi_t \rangle_{\mathscr{D}(\Omega)} dt + \int_0^T \int_\Omega u_r \varphi_t - \nabla[^*\phi(u)] \nabla \varphi dx dt + \int_\Omega u_0(x) \varphi(0,x) dx = 0,$$

so that $[u]$ and $[^*\phi(u)]$ induce a Radon entropy solution to problem 3.1.1 in the sense of equation 3.1.8.

The entropy condition under the hypothesis that G is equi-integrable can be deduced from Lemma 3.3.2 and from an argument analogous to the one in the proof of point (1) of Theorem 3.3.4. □

We can also prove that the singular part of the Radon measure solution can be disintegrated as in equation 3.1.9.

Proposition 3.3.6. *Let μ be defined as in the proof of Theorem 3.3.5. There exists a function $\tilde{\mu} : L^\infty([0,+\infty), \mathbb{M}_{\mathbb{P}}(\Omega))$ such that*

$$\mu(t) = \left(\int_\Omega u_0(x) dx - \int_\Omega u_r(t,x) dx \right) \tilde{\mu}(t).$$

Moreover, the support of $\tilde{\mu}(t)$ is a null-set with respect to the k-th dimensional Lebesgue measure for all $t \geq 0$.

Proof. By definition of μ, μ can be interpreted as a function $\mu : L^\infty([0, +\infty), \mathbb{M}(\Omega))$ that satisfies

$$\int_\Omega u_r(t, x)dx + \int_\Omega d\mu(t) = {}^\circ\|u(x, t)\|_1.$$

By Theorem 3.2.5, ${}^\circ\|u(x, t)\|_1 = \int_\Omega u_0(x)dx$, so the first part of the assertion is proved. The second part of the assertion is a consequence of $\|u(t)\|_1 \in {}^*\mathbb{R}_{fin}$. □

3.4 Asymptotic behaviour of the grid solutions to the ill-posed PDE

In this section, we will draw conclusions about the asymptotic behaviour of the grid solutions to problem 3.1.1 by studying the asymptotic behaviour of the solutions to the grid function formulation 3.2.6. In particular, we will carry out this study by determining the stability of the steady states of problem 3.2.6.

A steady state of problem 3.2.6 is a grid function $\overline{u} \in \mathbb{G}(\Omega_\mathbb{X})$ that satisfies $\Delta_\mathbb{X}{}^*\phi(\overline{u}) = 0$. By definition of $\Delta_\mathbb{X}{}^*\phi$, $\overline{u}$ is a steady state if and only if ${}^*\phi(\overline{u}(x)) = c$ for all $x \in \Omega_\mathbb{X}$. In particular, $\overline{u}$ can assume up to three values $\omega_1 \in (0, u^-)$, $\omega_2 \in (u^-, u^+)$ and, when $u^+ < +\infty$, $\omega_3 \in (u^+, +\infty)$ satisfying $\phi(\omega_1) = \phi(\omega_2) = \phi(\omega_3)$. By Proposition 3.2.1, the steady states of the grid function formulation 3.2.6 induce a steady state for problem 3.1.1.

Notice however that a steady state of problem 3.1.1 corresponds to a grid function $\widetilde{v}$ that satisfies only the weaker condition $\Delta_\mathbb{X}{}^*\phi(\widetilde{v}) \approx 0$. If $\|\widetilde{v}\|_\infty \in {}^*\mathbb{R}_{fin}$, then we expect that there exists a steady state $\overline{u}$ of problem 3.2.6 with $\|\overline{u} - \widetilde{v}\|_\infty \approx 0$. In this case, the stability of $\widetilde{v}$ can be determined by studying the stability of $\overline{u}$. On the other hand, if $u^+ = +\infty$ and $\|\widetilde{v}\|_\infty \notin {}^*\mathbb{R}_{fin}$, then $\widetilde{v}$ induces a measure-valued steady state of problem 3.1.1, but there might not exist a steady state $\overline{u}$ of the grid function formulation 3.2.6 which satisfies $\|\overline{u} - \widetilde{v}\|_\infty \approx 0$. Nevertheless, in section 3.4.4, we will show that the asymptotic behaviour of the grid solutions to problem 3.1.1 can be characterized a posteriori from the asymptotic behaviour of the solutions to problem 3.2.6.

3.4.1 Asymptotic behaviour of the solutions to problem 3.2.6

Since problem 3.2.6 corresponds to a hyperfinite dynamical system, we need to introduce an appropriate notion of stability for its steady states. Our choice is to use the nonstandard counterpart of the classical notion of stability in the L^∞ norm for discrete dynamical systems. In the following definition, it is useful to keep in mind that $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ can be identified with a vector in the euclidean space ${}^*\mathbb{R}^{|\Omega_{\mathbb{X}}|}$.

Definition 3.4.1. *Let $f : \mathbb{G}(\Omega_{\mathbb{X}}) \to {}^*\mathbb{R}$ and let $v(t) : {}^*\mathbb{R} \to \mathbb{G}(\Omega_{\mathbb{X}})$ be the solution of the nonstandard differential equation $u' = f(u)$ with initial data $v(0)$. We will say that $u \in \mathbb{G}(\Omega_{\mathbb{X}})$ is*

- **stable iff for all $\eta \in {}^*\mathbb{R},\ \eta > 0$ there exists $\delta \in {}^*\mathbb{R},\ \delta > 0$ such that $\|u - v(0)\|_\infty < \delta$ implies $\|u - v(t)\|_\infty < \eta$ for all $t \in {}^*\mathbb{R}_+$;*
- **attractive iff there exists $\rho \in {}^*\mathbb{R},\ \rho > 0$ such that $\|u - v(0)\|_\infty < \rho$ implies ${}^*\lim_{t\to+\infty} \|u - v(t)\|_\infty = 0$;*
- *asymptotically *stable iff it is *stable and *attractive;*
- *globally asymptotically *stable iff it is *stable and for all $v(0) \in \Omega_{\mathbb{X}}$ ${}^*\lim_{t\to+\infty} |u - v(t)| = 0$;*
- **unstable iff it is not *stable.*

Notice that a necessary condition for u to be *stable or *attractive is that $f(u) = 0$, i.e. u must be an equilibrium point of the differential equation.

Since the ${}^*L^\infty$ norm over $\Omega_{\mathbb{X}}$ is equivalent to the euclidean norm in ${}^*\mathbb{R}^{|\Omega_{\mathbb{X}}|}$, the stability in the ${}^*L^\infty$ norm for the grid function formulation 3.2.6 can be studied by exploiting the theory of finite dynamical systems.

For the following analysis of the asymptotic behaviour of solutions of system 3.2.6, we assume that they are isolated in $\mathbb{S}^+({}^*u_0)$, i.e. that there is only a hyperfinite number of steady states in $\mathbb{S}^+({}^*u_0)$. For a discussion of this hypothesis and for sufficient conditions that ensure the existence of a hyperfinite number of steady states in $\mathbb{S}^+({}^*u_0)$, we refer to Lizana and Padron [58]. Their hypothesis is a sharpening of the condition that S_1', S_2' and S_3' must be linearly independent on the spinoidal interval (u^-, u^+), already discussed in [66].

Proposition 3.4.2. *If the steady states of 3.2.6 are isolated in $\mathbb{S}^+({}^*u_0)$ and if M is the largest positively invariant set contained in*

$$\mathbb{S}^+({}^*u_0) \cap \{f \in \mathbb{G}(\Omega_{\mathbb{X}}) : \phi(f(x)) \text{ is constant over } \Omega_{\mathbb{X}}\},$$

then ${}^*\lim_{t\to+\infty} u(t) \in M$. In particular, system 3.2.6 has at least an asymptotically *stable steady state.

Proof. This is a consequence of Proposition 2 of [58]. □

We observe that, under the hypothesis that the steady states of system 3.2.6 are isolated in $\mathbb{S}^+({}^*u_0)$, then *stability is equivalent to asymptotic *stability.

Lemma 3.4.3. *If $\overline{u}$ is a *stable steady state of system 3.2.6 and if the steady states are isolated in $\mathbb{S}^+(^*u_0)$, then $\overline{u}$ is asymptotically *stable.*

Proof. Suppose that $\overline{u}$ is *stable: since the steady states of system 3.2.6 are isolated, we can find $\rho > 0$ such that if $|\overline{u} - v| < \rho$ then v is not a steady state of system 3.2.6. By the *stability of $\overline{u}$, we can find $\delta > 0$ such that if $|\overline{u} - v| < \delta$ then, denoting by $v(t)$ the solution of system 3.2.6 with initial data v, $|\overline{u} - v(t)| < \rho$ for all $t \in {}^*\mathbb{R}_+$. Moreover, by Proposition 3.4.2 $v(t)$ converges to a steady state of system 3.2.6. By our choice of ρ, this steady state must be $\overline{u}$, hence $\overline{u}$ is *attractive. □

3.4.2 Steady states of problem 3.2.6

For a matter of commodity, we will carry out the study of the steady states of problem 3.2.6 in the case where $k = 1$, and where the spatial domain is $[0,1]_{\mathbb{X}} = [0, \varepsilon, \dots, N\varepsilon = 1]$, but the analysis can be carried out in higher dimension and with other domains. Moreover, we identify a grid function $u \in \mathbb{G}([0,1]_{\mathbb{X}})$ with a vector $u \in {}^*\mathbb{R}^{N+1}$, with the convention that u_i, the i-th component of u, satisfies $u_i = u(i\varepsilon)$. If $u : {}^*\mathbb{R} \to \mathbb{G}([0,1]_{\mathbb{X}})$, we will identify it with a vector-valued function $u : {}^*\mathbb{R} \to {}^*\mathbb{R}^{N+1}$, with the convention that $u_i(t) = u(t, i\varepsilon)$.

We begin the study of the *stability of the steady states of system 3.2.6 by discussing its homogeneous steady state $\overline{u}_h = (||^*u_0||_1, \dots, ||^*u_0||_1)$.

Proposition 3.4.4. *The homogeneous steady state $\overline{u}_h$ of system 3.2.6 has the following properties:*

- *if $||^*u_0||_1 < u^-$ or $||^*u_0||_1 > u^+$, then $\overline{u}_h$ is *stable;*
- *if $\overline{u}_h$ is the only steady state of 3.2.6, then $\overline{u}_h$ is globally asymptotically *stable;*
- *if $u^- < ||^*u_0||_1 < u^+$, then $\overline{u}_h$ is *unstable. Moreover, if $^*u_0 \not\approx \overline{u}_h$ and if the steady states are isolated in $\mathbb{S}(^*u_0)$, then u converges to a non-homogeneous steady state.*

Proof. It is a consequence of Proposition 3 and of Corollary 4 of [58]. □

In addition to the homogeneous steady state $\overline{u}_h$, system 3.2.6 may have many non-homogeneous steady states. If we denote by n_i the number of components of $\overline{u}$ that assume the value ω_i, by Proposition 3.2.5 we obtain the relations

$$n_3 = N+1-(n_1+n_2),\ n_1\omega_1+n_2\omega_2+(N+1-(n_1+n_2))\omega_3 = (N+1)||^*u_0||_1$$

that in the case where $u^+ = +\infty$ become

$$n_2 = N+1-n_1, \qquad n_1\omega_1 + (N+1-n_1)\omega_2 = (N+1)||^*u_0||_1. \quad (3.4.1)$$

In the first step of the study of the *stability of the non-homogeneous steady states of system 3.2.6, we will prove that all the steady states with $n_2 > 1$ are *unstable.

Proposition 3.4.5. *If* $\overline{u} \in {}^*\mathbb{R}^{N+1}$ *is a steady state of 3.2.6 with* $n_2 > 1$*, then it is* **unstable.*

Proof. As in the proof of Proposition 4 of Witelski, Schaeffer and Shearer [89], we will show that $\overline{u}$ is not a stable steady state of 3.2.6 by showing that it is not a local minimum of a suitable Lyapunov function: the thesis follows from this result. In order to simplify the notation, suppose that $n_3 = 0$, as the proof for the general case can be deduced by the argument below.

Consider the perturbed steady state given by

$$\begin{cases} u_{i_1}(t) &= \omega_2 + q \\ u_{i_2}(t) &= \omega_2 - q \\ u_{i_k}(t) &= \omega_2 \qquad \text{for } k = 3, 4, \dots, n_2 \\ u_i(t) &= \omega_1 \qquad \text{otherwise} \end{cases}$$

Let now $V(u_i) = \int_0^{u_i} \phi(s)ds$, and $L(u) = \sum_{i=0}^{N+1} V(u_i)$. From Proposition 4 of [89], it can be deduced that L is a Lyapunov function for system 3.2.6. By evaluating L as a function of q, we get

$$L(q) = \frac{V(\omega_2 + q) + V(\omega_2 - q) + (n_2 - 2)V(\omega_2) + (N + 1 - n_2)V(\omega_1)}{N}$$

so we deduce

$$\left.\frac{dL}{dq}\right|_0 = 0 \quad \text{and} \quad \left.\frac{d^2L}{dq^2}\right|_0 = \frac{2}{N}\phi'(q_2) < 0,$$

where the last inequality follows from the hypothesis that $\omega_2 \in (u^-, u^+)$. We conclude that $\overline{u}$ is not a local minimum of L and, as a consequence, that $\overline{u}$ is *unstable. ☐

The characterization of the asymptotically *stable non-homogeneous steady states of system 3.2.6 is based on the following bound on $\phi'(q_2)$.

Lemma 3.4.6. *If* $\overline{u}$ *is an asymptotically* **stable non-homogeneous steady state of 3.2.6, then it holds the inequality*

$$|\phi'(\omega_2)| < \frac{\max\{\phi'(\omega_1), \phi'(\omega_3)\}^2}{N \min\{\phi'(\omega_1), \phi'(\omega_3)\}}. \qquad (3.4.2)$$

Proof. For a matter of commodity, suppose that

$$\begin{cases} \overline{u}(0) &= \omega_2 \\ \overline{u}(i) &= \omega_1 \quad \text{for } i = 1, 2, \dots, n_1 \\ \overline{u}(i) &= \omega_3 \quad \text{otherwise.} \end{cases}$$

Let $X_1(\overline{u}) = -(\phi'(\overline{u}(0)) + \phi'(\overline{u}(1)))$ and define by recursion

$$X_{i+1}(\overline{u}) = -\phi'(\overline{u}(i+1))X_i(\overline{u}) + (-1)^{i+1}\prod_{j=0}^{i}\phi'(\overline{u}(j))$$

It is a consequence of Proposition 8 of [58] that asymptotic *stability of $\overline{u}$ is equivalent to $(-1)^i X_i(\overline{u}) > 0$ for $i = 1, \ldots, N$. Notice that, as long as $i \leq n_1$,

$$X_i(\overline{u}) = (-1)^i \phi'(\omega_1)^{i-1}(\phi'(\omega_1) + i\phi'(\omega_2)),$$

so that $(-1)^i X_i(\overline{u}) > 0$ is equivalent to

$$|\phi'(\omega_2)| < \frac{\phi'(\omega_1)}{i} \leq \frac{\phi'(\omega_1)}{n_1}.$$

For $i = n_1 + 1, \ldots, N$, a similar computation shows that $(-1)^i X_i(\overline{u}) > 0$ implies

$$\begin{aligned}|\phi'(\omega_2)| &< \frac{\phi'(\omega_1)\phi'(\omega_3)}{n_1\phi'(\omega_3) + (i - n_1)\phi'(\omega_1)} \\ &\leq \frac{\phi'(\omega_1)\phi'(\omega_3)}{n_1\phi'(\omega_3) + (N - n_1)\phi'(\omega_1)}.\end{aligned}$$

From the inequality

$$\frac{\phi'(\omega_1)\phi'(\omega_3)}{n_1\phi'(\omega_3) + (N - n_1)\phi'(\omega_1)} \leq \frac{\max\{\phi'(\omega_1), \phi'(\omega_3)\}^2}{N\min\{\phi'(\omega_1), \phi'(\omega_3)\}}$$

we deduce that the desired result holds. □

3.4.3 Asymptotic behaviour of the grid solutions under the hypothesis $u^+ < +\infty$

We will now discuss the asymptotic behaviour of the grid solutions to problem 3.1.1 under the hypothesis that $u^+ < +\infty$. Under this hypothesis, the steady states of the grid function formulation with $n_2 = 0$ are all asymptotically *stable.

Proposition 3.4.7. *Let $\overline{u}$ be a steady state of system 3.2.6 with $n_2 = 0$. Then $\overline{u}$ is asymptotically *stable.*

Proof. It is a consequence of Proposition 8 of [58]. □

It turns out that, thanks to the hypotheses over ϕ, all *stable non-homogeneous steady states of system 3.2.6 for which $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$ must have $n_2 = 0$, giving a partial converse to Proposition 3.4.7.

Proposition 3.4.8. *If $\overline{u}$ is an asymptotically *stable non-homogeneous steady state of 3.2.6 with $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$, then $n_2 = 0$.*

Proof. Suppose towards a contradiction that $n_2 = 1$. The hypotheses $\omega_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$ imply $\min\{^*\phi'(\omega_1), {}^*\phi'(\omega_3)\} \not\approx 0$, otherwise either $\phi'(^\circ\omega_1) = 0$ or $\phi'(^\circ\omega_3) = 0$, against the hypotheses 3.1.1. As a consequence, $N \min\{^*\phi'(\omega_1), {}^*\phi'(\omega_3)\}$ is infinite. Thanks to inequality 3.4.2, we deduce that $|^*\phi'(\omega_2)| \approx 0$. By the hypotheses over ϕ, there exists $\omega_2 \in {}^*(u^-, u^+)$ with $|^*\phi'(\omega_2)| \approx 0$ if and only if $\omega_2 \approx u^-$ or $\omega_2 \approx u^+$. However, $\omega_2 \approx u^-$ implies $\omega_1 \approx u^-$ and $\omega_2 \approx u^+$ implies $\omega_3 \approx u^+$, in contradiction with the hypotheses $q_1 \not\approx u^-$ and $\omega_3 \not\approx u^+$. □

Putting together the results of this section, we can characterize the asymptotic behaviour of a grid solution of problem 3.1.1. In particular, for a.e. initial data, the grid solution converges to a steady state that is a superposition of at most two Dirac measures centred at the stable branches of ϕ.

Proposition 3.4.9. *Let $[u], [^*\phi(u)]$ be the grid solution of problem 3.1.1 with initial data $^*u_0$. For almost every $^*u_0 \in L^\infty(\Omega)$, $[u]$ converges to a steady state ν satisfying:*

1. *there exists $c \in \mathbb{R}$ such that $\int_{\mathbb{R}} \phi(\tau) d\nu(x) = c$ for all $x \in \Omega_{\mathbb{X}}$;*
2. *there exist $\omega_1 \in [0, u^-]$, $\omega_3 \in [u^+, +\infty)$, and $\lambda_1, \lambda_2 : \Omega \to [0, 1]$, such that*
   - *(a) $\nu(x) = \lambda_1(x)\delta_{\omega_1} + \lambda_2(x)\delta_{\omega_2}$ for a.e. $x \in \Omega$;*
   - *(b) $\phi(\omega_1) = \phi(\omega_2) = c$;*
   - *(c) $\lambda_1(x) + \lambda_2(x) = 1$ for a.e. $x \in \Omega$.*

3.4.4 Asymptotic behaviour of the grid solutions under the hypothesis $u^+ = +\infty$

If $u^+ = +\infty$, the bound of Lemma 3.4.6 becomes

$$|\phi'(\omega_2)| < \frac{\phi'(\omega_1)}{N}. \tag{3.4.3}$$

From this inequality we will deduce that a necessary condition for the asymptotic *stability of a non-homogeneous steady state p is that $\phi'(\omega_2) \approx 0$, and this is possible only when ω_2 is infinite.

Proposition 3.4.10. *Suppose that $u^+ = +\infty$ and that $\overline{u}$ is an asymptotically *stable non-homogeneous steady state of 3.2.6. Then ω_2 is infinite.*

Proof. Since the steady state is non-homogeneous, $\omega_2 > u^-$. By inequality 3.4.3, and since $\phi \in C^1(\mathbb{R})$, it must hold ${}^*\phi'(\omega_2) \approx 0$. Since hypotheses 3.1.1 entails the inequality ${}^*\phi'(x) < 0$ for all $x > u^-$, and this can only happen if ω_2 is infinite, as desired. □

This result together with Proposition 3.4.5 implies that any non-homogeneous asymptotically *stable steady states of system 3.2.6 in the case where $u^+ = +\infty$ are piecewise constant with a single spike. Proposition 3.4.10 implies that an infinite amount of the mass is concentrated in the spike. This result is in accord with both the theoretical results and the numerical experiments of [48, 49, 58, 68, 89]. However, as we observed previously, we do not expect that $[u]$ converges to a steady state which satisfies Proposition 3.4.10. Proposition 3.4.10 should be interpreted as a confirmation of a conjecture by Smarazzo that, for a grid solution $[u]$ of problem 3.1.1, the regular part of the solution eventually vanishes, and the singular part of the solution prevails. More precisely, we obtain the following result.

Proposition 3.4.11. *Let $[u], [{}^*\phi(u)]$ be the grid solution of problem 3.1.1 with initial data ${}^*u_0$. For almost every ${}^*u_0 \in L^\infty(\Omega)$, $[u]$ converges to a steady state $\nu + \mu$ satisfying:*

1. *ν is a homogeneous Dirac Young measure centred at 0, i.e. $\nu \in L^\infty(\Omega)$ and $\nu(x) = 0$ a.e.;*
2. *$\mu = \|{}^*u_0\|_1 \tilde{\mu}$, and $\tilde{\mu}$ is a probability measure over Ω.*

In particular, for almost every initial data ${}^*u_0$, $[u]$ converges to a steady state with null regular part.

3.5 The Riemann problem

In the study of problems 3.1.1 and 3.1.2 in the case when $u^+ < +\infty$, the dynamics of solutions with Riemann initial data are of particular interest both in the theoretical and in the numerical setting (see for instance [41, 56]). We will discuss the Riemann problem where the initial data ${}^*u_0$ satisfies

$$ {}^*u_0(i\varepsilon) = \begin{cases} \omega_1 \in [0, u^-] & \text{for } 0 \leq i \leq n \\ \omega_3 \in [u^+, +\infty) & \text{for } n+1 \leq i \leq N \end{cases} \tag{3.5.1} $$

for some $n \leq N$. In order to understand the evolution of system 3.2.6 with initial data 3.5.1, we need to focus on the behaviour of the solution near the discontinuity in the data. In particular, we will discuss the conditions at which $u_i(t) \in (0, u^-)$, $u_{i+1} \in (u^+, +\infty)$ and either $u_i(t+\tau) \in (u^-, u^+)$ or $u_{i+1}(t+\tau) \in (u^-, u^+)$ for some small $\tau > 0$. If $u_i(t) \in (0, u^-)$ and $u_i(t+\tau) \in (u^-, u^+)$ we will say that there is an upward phase transition at $u_i(t)$; if $u_{i+1}(t) \in (u^+, +\infty)$ and $u_{i+1}(t+\tau) \in (u^-, u^+)$ we will say that there is a downward phase transition at $u_{i+1}(t)$.

Proposition 3.5.1. *Let u be a solution of system 3.2.6 with initial data 3.5.1. Then an upward phase transition occurs at $u_i(t)$ for some $t > 0$ and for some $0 \leq i \leq N$ iff $u_i(t) = u^-$,*

$$^*\phi(u_{i-1}(t)) + {}^*\phi(u_{i+1}(t)) > 2^*\phi(u^-) \tag{3.5.2}$$

and

$$i = \max_{j \in [0,1]_{\mathbb{X}}} \{j : u_m(t) \leq u^- \text{ for all } m \leq j\}. \tag{3.5.3}$$

A downward phase transition occurs at time t at some $0 \leq i \leq N$ iff $u_i(t) = u^+$,

$$^*\phi(u_{i-1}(t)) + {}^*\phi(u_{i+1}(t)) < 2^*\phi(u^+) \tag{3.5.4}$$

and

$$i = \max_{j \in [0,1]_{\mathbb{X}}} \{j : u_m(t) \geq u^+ \text{ for all } m \leq j\}. \tag{3.5.5}$$

Proof. Suppose that $u_i(t) = u^-$ for some $t \in {}^*\mathbb{R}_+$ and for some $i \leq N$. Then there is a phase transition iff

$$\varepsilon^2 u_i'(t) = {}^*\phi(u_{i-1}(t)) - 2^*\phi(u_i(t)) + {}^*\phi(u_{i+1}(t)) > 0$$

from which 3.5.2 follows. Inequality 3.5.4 can be proved in a similar way. Notice that the two inequalities imply that if at time t $u_i(t)$, $u_{i+1}(t)$ and $u_{i-1}(t)$ are in the same stable phase, then u_i cannot have a transition at time t. This is sufficient to entail 3.5.3 and 3.5.5 for Riemann initial data. □

Proposition 3.5.2. *Let u be a solution of system 3.2.6 with initial data 3.5.1. For every $t \in {}^*\mathbb{R}_+$, there exists at most one $i \leq N$ such that $u_i(t) \in (u^-, u^+)$.*

Proof. Conditions 3.5.3 and 3.5.5 imply that if $u_i(t)$ and $u_{i+1}(t) \in (0, u^-)$ or if $u_i(t)$ and $u_{i+1}(t) \in (u^+, +\infty)$, then they cannot have a simultaneous phase transition. If both $u_i(t)$ and $u_{i+1}(t) \notin (u^-, u^+)$, there cannot be an upwards phase transition at point $u_i(t)$ and a downward phase transition at point $u_{i+1}(t)$: otherwise, from 3.5.2 and 3.5.4 we would have $^*\phi(u_{i+1}(t)) > {}^*\phi(u^-)$ or $^*\phi(u_i(t)) < {}^*\phi(u^+)$, against the necessity that $u_i(t) = u^-$ and $u_{i+1}(t) = u^+$. If $u_i(t) \in (u^-, u^+)$ and if $u_{i-1}(t)$ had an upwards phase transition, from 3.5.2 we would have $^*\phi(u_{i-2}(t)) > {}^*\phi(u^-)$, contradicting 3.5.3. If $u_i(t) \in (u^-, u^+)$ and if $u_{i+1}(t)$ had a downward phase transition, from 3.5.4 we would have $^*\phi(u_{i+2}(t)) < {}^*\phi(u^+)$, against 3.5.5. □

Notice that Propositions 3.5.1 and 3.5.2 can be generalized to any piecewise S-continuous initial data taking values in $(0, u^-) \cup (u^+, +\infty)$: in this case, if the initial data has n discontinuities, then $u_i(t) \in (u^-, u^+)$ for at most n values of $i \leq N$. In particular, if the initial data has finitely many discontinuities, then the dynamics of the system outside of the stable branches of ϕ

is negligible. In these cases, it could be argued by the above proposition that the phase transitions of $[u]$ trace a clockwise hysteresis loop, in agreement with the behaviour of two-phase solutions to 3.1.1 studied in [38, 40, 62].

We conclude our discussion of the Riemann problem with initial data 3.5.1 with a characterization of the asymptotic behaviour of the solution.

Corollary 3.5.3. *Let u be the solution of system 3.2.6 with initial data 3.5.1. If ${}^*\phi(\omega_1) > {}^*\phi(\omega_2)$ then no phase transitions occur.*

Proof. It is a consequence of 3.5.2 and 3.5.4 of Proposition 3.5.1 and of the fact that ${}^*\phi(\omega_1) > {}^*\phi(\omega_2)$ implies $u_n'(0) < 0$ and $u_{n+1}'(0) > 0$. □

Corollary 3.5.4. *Let $[u]$ be the grid solution of problem 3.1.1 with initial data 3.5.1. Then $[u]$ converges to an asymptotically stable state that is either constant or Riemann-shaped.*

Proof. If no phase transitions occur, then the thesis is a consequence of Proposition 3.4.2. If phase transitions occur, this is a consequence of Proposition 3.4.2 and of Proposition 3.4.8. □

Chapter 4

Research perspectives

The study of grid functions and their applications in functional analysis is far from concluded. We will briefly outline some promising directions for further research.

4.1 Regularity of the solutions to grid function problems

In Chapters 2 and 3 we have shown that, if the solution u of a grid function formulation of a partial differential equation is regular enough, then $[u]$ is a solution to the original PDE in some standard sense. Moreover, for the ill-posed problem 3.1.1, we have shown in Lemma 3.3.3 and in Theorems 3.3.4 and 3.3.5 which regularity hypotheses over u ensure that $[u]$ is a measure-valued, very weak, weak or classical solution to the original problem.

We believe that it would be of importance to sharpen the sufficient conditions for u that ensure regularity of $[u]$ as a standard solution to a PDE, and to study if these conditions are met for some relevant classes of PDEs. Possibly, the most natural starting point for this line of research would be to study the regularity of the solution u to the grid function formulation for problem 3.1.1.

4.2 Grid functions and the mathematical description of physical phenomena

We have seen in Section 2.5.1 that, in some cases, it is possible to coherently define the product between two distributions T and S by studying the product between some classes of grid function representatives of T and S. This, in turn, allows for the rigorous formulation of some real-world phenomena that cannot be formulated in the sense of distributions. Inspired by this example, we believe it would be interesting to determine to what extent the space of grid functions allows for the rigorous formulation of the physical phenomena that cannot be formalized coherently in the space of distributions.

In order to carry out a research in this direction, we believe it would be also relevant to study some grid function counterparts of some useful operators on the space of distributions, such as the convolution operator and the trace operator.

4.3 Relations between some notions of generalized functions beyond distributions

The examples discussed in Section 2.5 and the theoretical results from Section 2.4 suggest that some problems in functional analysis are addressed coherently in the setting of Colombeau algebras, algebras of asymptotic functions, ultrafunctions and grid functions. These evidences seem to suggest that there is a "common core" to these theories of generalized functions beyond distributions.

We believe that a study of this common core could be beneficial for the development of all of these theories. Some results in this direction are already known: in [84, 86], Todorov showed that the algebra of asymptotic functions can be seen as a generalization of the Colombeau algebras. It would be interesting to determine whether the space of grid functions can be interpreted as a subspace of some algebras of asymptotic functions, or if the opposite inclusion might hold. Similar questions arise when considering the ultrafunctions and the algebras of asymptotic functions, or the ultrafunctions and the grid functions. In particular, since both the grid functions and the spaces of ultrafunctions can be seen as subspaces of ${}^*L^2(\Omega)$, we believe it would be possible to study in depth the relation between these two spaces of generalized functions.

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