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Weights for moments' geometrical localization: a canonical isomorphism

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Abstract

This paper deals with high order Whitney forms. We define a canonical isomorphism between two sets of degrees of freedom. This allows to geometrically localize the classical degrees of freedom, the moments, over the elements of a simplicial mesh. With such a localization, it is thus possible to associate, even with moments, a graph structure relating a field with its potential.

Keywords Finite element system · High order · New degrees of freedom · Exterior differential calculus

Mathematics Subject Classification (2010) 65N30

This article is dedicated to Professor Alain Bossavit on the occasion of his 80th birthday

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1 Introduction

The finite element (FE) method is a well-established technique to numerically solve partial differential equations [12]. One key aspect of FE methods is the construction of finite dimensional spaces able to provide an approximated and physically meaningful solution to the considered PDE. Suitable examples of FE spaces have been proposed to deal with differential operators such as the gradient, the curl, and the divergence. As an example, Raviart-Thomas and Brezzi-Douglas-Marini div-conforming FEs [9, 24] became popular for problems in fluid dynamics and Nédélec curl-conforming ones [19, 20] were widely adopted in electromagnetism (for further examples, see [10, 14, 18]). All these successful FEs can be unified in the finite element exterior calculus framework where physical fields are treated as instances of differential forms [4–6, 13]. In addition to the shape functions of a FE space, we must specify the degrees of freedom (DoFs) we adopt to reconstruct fields in that space. These are a unisolvent set of functionals on the shape functions. The construction of DoFs for higher order Whitney space of k-forms $\mathcal{P}_{r+1}^{-}\Lambda^k$ is classically based on moments [13] associated with a face of some dimension q with $q \geq k$. DoFs determine inter element continuity and provide interpolation operators, projections, which are defined at least for smooth fields. Moments pop up naturally from integration-by-part formulas, which thus gives a way to reconstruct differential operators and potentials.

By adopting a geometrical point of view similar to the one of Whitney in [26], new DoFs have been proposed in [22] for the interpolation of fields in the FE spaces of trimmed polynomial forms of arbitrary degree $r \geq 1$ on simplices. These new DoFs, called weights, are integrals of the field, intended as a differential k-form, on some small faces of dimension *only* $q = k$, being k the degree of the form. They have a clear physical interpretation, such as circulations along curves, fluxes across surfaces, densities in volumes, depending on the value of k. Their combinatorial and accuracy properties have been largely analyzed, see, for example, [1, 3, 11]. They have been defined also for spaces of complete polynomials (see [27] for an example in 2D) and on tensor product ones, as presented in [17].

For $k = 0$, that is, we deal with a scalar field, when $r \geq 1$, weights are evaluations of 0-forms at some points in the FEs. We can say that weights generalize to $k > 0$ the idea of r-version of Lagrangian finite elements to other (e.g., Nédélec and Raviart-Thomas) finite elements. For $r = 1$, that is, we deal with low order polynomial approximations, weights, and moments coincide, whatever is the degree k of the form. Thus, a natural question arises. What happens when the polynomial degree r of the k-form is greater than 1? In other words, is there a connection between these two sets of DoFs when $r > 1$, for any k? The answer is yes, and in this contribution, we develop this connection (see Fig. 1, where $k = 1$ and the polynomial degree is 3).

In particular, we present an isomorphism between these two sets of DoFs, the weights and the moments, for the FE spaces of trimmed polynomial k-forms of arbitrary degree $r \geq 1$ on simplices. By means of this isomorphism, we can underline the physical, geometrical, and analytical aspects hidden in the definition of moments and weights. Moreover, with an appropriate selection of the discrete space bases for $k = 0, 1$, the matrix which represents the gradient operator is the same with both sets of DoFs to reconstruct a field from its potential.

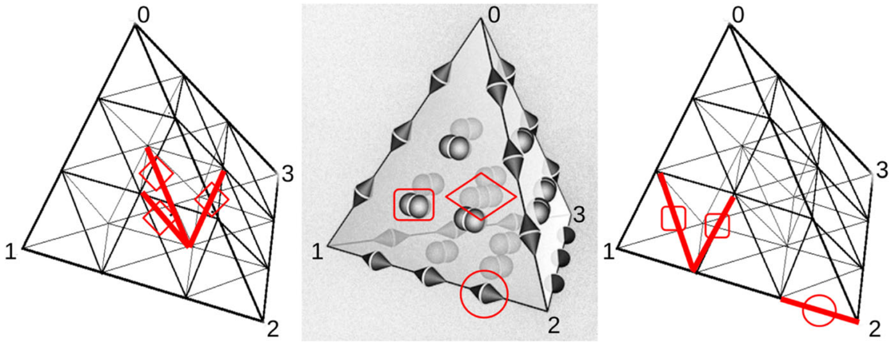

Fig. 1 Correspondence between two sets of DoFs for $\mathcal{P}_3^- \Lambda^1(T)$ in a tetrahedron T. Center, the visualization of symbols referring to the distribution of edge-type (in a circle), of face-type (in a square), and of volume-type (in a diamond) moments as given in the periodic table of FEs for $N1_3^e$ (courtesy of D. Arnold). Right and left, coded with the same symbol, the small edges supporting the corresponding weights

The paper is structured as follows. After the introduction of classical notations for the spaces of differential polynomial k-forms on a simplex T in Section 2, we explore the definition of weights and moments in Section 3. The isomorphism is detailed in Section 4. The matrix representing the exterior derivative operator working between 0- and 1-forms, in both cases of weights and moments, is analyzed in Section 5. Some concluding remarks end this contribution.

2 Notation and basic tools

The notation and theoretical results are illustrated by several examples. For the sake of clarity, we use the symbol $\square$ (resp. $\diamond$) to close a proof (resp., an example).

2.1 Increasing sequences and multi-index

Let j, l, m, and n be integers such that $0 \le l - j \le n - m$. By $\Sigma(j : l, m : n)$, we denote the set of increasing maps from $\{j, \ldots, l\}$ to $\{m, \ldots, n\}$, that is,

$$\Sigma(j : l, m : n) = \{\sigma : \{j, \ldots, l\} \longrightarrow \{m, \ldots, n\} \ : \ \sigma(j) < \sigma(j+1) < \cdots < \sigma(l)\}.$$

For a map $\sigma \in \Sigma(j : l, m : n)$, $[\![\sigma]\!]$ will indicate its range, i.e.,

$$[\![\sigma]\!] = \{\sigma(i) \ : \ i \in \{j, \ldots, l\}\} \subset \{m, \ldots, n\}.$$

We use multi-index notation and consider the sets

$$\mathcal{I}(d+1, r) := \{\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_d) \in \mathbb{N}^{d+1} \ : \ |\boldsymbol{\alpha}| = r\},$$

being $|\boldsymbol{\alpha}| = \sum_{i=0}^{d} \alpha_i$. For a multi-index $\boldsymbol{\alpha} \in \mathcal{I}(d+1, r)$, $[\![\boldsymbol{\alpha}]\!]$ will stand for the support of $\boldsymbol{\alpha}$ defined as the set

$$[\![\boldsymbol{\alpha}]\!] = \{i : \alpha_i > 0\} \subset \{0, \ldots, d\},$$

and we denote $\lfloor \boldsymbol{\alpha} \rfloor$ the minimal element of $[\![\boldsymbol{\alpha}]\!]$.

Let S be a subset of $\{0, \ldots, d\}$ and $\#S$ its cardinality. By $\mathbf{e}_S$, we denote the (unique) multi-index in $\mathcal{I}(d+1, \#S)$ such that $[\![\mathbf{e}_S]\!] = S$. The sum of multi-indexes of the same length is defined in the natural way: if $\boldsymbol{\alpha} \in \mathcal{I}(d+1, r)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r')$, then $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \mathcal{I}(d+1, r+r')$ and $(\boldsymbol{\alpha} + \boldsymbol{\beta})_i = \alpha_i + \beta_i$ for $i = 0, \ldots, d$.

Example 1 If $\sigma \in \Sigma(0:d, 0:n)$, with $d \leq n$, then $[\![\sigma]\!]$ is a subset of $\{0, 1, \ldots, n\}$ with $\#[\![\sigma]\!] = d+1$. Then, $\mathbf{e}_{[\![\sigma]\!]} \in \mathcal{I}(n+1, d+1)$ is the multi-index with entries

$$(\mathbf{e}_{[\![\sigma]\!]})_i = \begin{cases} 1 & \text{if } i \in [\![\sigma]\!] \\ 0 & \text{otherwise.} \end{cases}$$

For any $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$, the multi-index $\widetilde{\boldsymbol{\alpha}} \in \mathcal{I}(n+1, r+d+1)$ given by $\widetilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \mathbf{e}_{[\![\sigma]\!]}$ has entries

$$\widetilde{\boldsymbol{\alpha}}_i = \begin{cases} \boldsymbol{\alpha}_i + 1 & \text{if } i \in [\![\sigma]\!] \\ \boldsymbol{\alpha}_i & \text{otherwise,} \end{cases}$$

respectively. □

Given $\sigma \in \Sigma(0:d, 0:n)$, the matrix $\mathrm{E}_\sigma \in \mathbb{Z}^{(d+1)\times(n+1)}$, with entries

$$(\mathrm{E}_\sigma)_{i,j} = \begin{cases} 1 & \text{if } j-1 = \sigma(i-1) \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

allows to extend a multi-index $\boldsymbol{\beta} \in \mathcal{I}(d+1, r)$ to a multi-index $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$ by setting $\boldsymbol{\alpha} = \boldsymbol{\beta}\, \mathrm{E}_\sigma$. It is worth noting that $[\![\boldsymbol{\beta}\, \mathrm{E}_\sigma]\!] \subset [\![\sigma]\!]$, hence in particular $(\boldsymbol{\beta}\, \mathrm{E}_\sigma)_i = 0$ if $0 \leq i < \sigma(0)$.

Furthermore, the matrix $\mathrm{E}_\sigma^\top \in \mathbb{Z}^{(n+1)\times(d+1)}$ allows to restrict a multi-index $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$ to a multi-index $\boldsymbol{\beta} \in \mathcal{I}(d+1, \tilde{r})$, with $\tilde{r} \leq r$, by identifying any multi-index with a row vector, and setting $\boldsymbol{\beta} = \boldsymbol{\alpha}\mathrm{E}_\sigma^\top$. We notice that

$$\mathrm{E}_\sigma\, \mathrm{E}_\sigma^\top = I \in \mathbb{Z}^{(d+1)\times(d+1)},$$

whereas for $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$, we have

$$\boldsymbol{\alpha}\mathrm{E}_\sigma^\top\, \mathrm{E}_\sigma = \boldsymbol{\alpha} \ \text{ if and only if } \ [\![\boldsymbol{\alpha}]\!] \subset [\![\sigma]\!]. \tag{2}$$

Example 2 If $\sigma \in \Sigma(0:1, 0:3)$ has $[\![\sigma]\!] = \{1, 3\}$, the associated matrix $\mathrm{E}_\sigma \in \mathbb{Z}^{2\times 4}$ is $\mathrm{E}_\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. If $(\beta_0, \beta_1) \in \mathcal{I}(2, r)$, then we get

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\beta_0, \beta_1)\, \mathrm{E}_\sigma = (0, \beta_0, 0, \beta_1) \in \mathcal{I}(4, r).$$

It holds that $\alpha_{\sigma(j)} = \beta_j$, whereas $\alpha_i = 0$ if $i \notin [\![\sigma]\!]$. Reciprocally, if $(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathcal{I}(4, r)$, then

$$(\beta_0, \beta_1) = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)\, \mathrm{E}_\sigma^\top = (\alpha_1, \alpha_3) \in \mathcal{I}(2, \tilde{r}).$$

In this case, $\tilde{r} \leq r$. □

2.2 Simplices and barycentric coordinates

Let $T \subset \mathbb{R}^n$ be an n-simplex with vertices $x_0, x_1, \ldots, x_n$ in general position. We let $\Delta_k(T)$ be the set of subsimplices of T of dimension k, for any selected value of k between 0 and n, and $\Delta(T) = \cup_k \Delta_k(T)$.

For each $\sigma \in \Sigma(j : l, 0 : n)$, we let f_σ be the (oriented) closed convex hull of the vertices $x_{\sigma(j)}, \ldots, x_{\sigma(l)}$ which we henceforth denote by $f_\sigma = [x_{\sigma(j)}, \ldots, x_{\sigma(l)}]$. There is a one-to-one correspondence between $\Delta_k(T)$ and $\Sigma(0 : k, 0 : n)$.

Let $\mathcal{P}_r(T)$ denote the space of polynomials in n variables of degree at most r. In the following, $\lambda_{T,0}, \lambda_{T,1}, \ldots, \lambda_{T,n}$ are the barycentric coordinate functions with respect to T. Each function $\lambda_{T,i} \in \mathcal{P}_1(T)$ is determined by the equations $\lambda_{T,i}(x_j) = \delta_{i,j}$, $0 \leq i, j \leq n$, being $\delta_{\cdot,\cdot}$ the Kronecker's symbol. All together, the functions $\lambda_{T,i}$ form a basis of $\mathcal{P}_1(T)$, are non-negative on T, and sum to 1 identically on T.

To make for the higher order $r \geq 1$, we introduce the Bernstein basis of the space $\mathcal{P}_r(T)$: it consists of all monomials of degree r in the variables $\lambda_{T,i}$. We have

$$\mathcal{P}_r(T) = \operatorname{span}\{\lambda_T^{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}, \qquad \lambda_T^{\boldsymbol{\alpha}} := \lambda_{T,0}^{\alpha_0} \lambda_{T,1}^{\alpha_1} \cdots \lambda_{T,n}^{\alpha_n}.$$

Whenever a fixed simplex T is understood, we may simplify the notation by writing

$$\lambda_i \equiv \lambda_{T,i}, \quad \lambda^{\boldsymbol{\alpha}} \equiv \lambda_T^{\boldsymbol{\alpha}}.$$

2.3 Polynomial differential forms

We denote by $\Lambda^k(T)$ the space of differential k-forms over T with smooth bounded coefficients. For $k = 0$, the set $\Lambda^0(T) = C^\infty(T)$ is the space of smooth functions over T with uniformly bounded derivatives of all orders. Furthermore, $\Lambda^k(T) \neq \{0\}$ for $0 \leq k \leq n$. We recall the exterior product $\omega \wedge \eta \in \Lambda^{k+l}(T)$ for $\omega \in \Lambda^k(T)$ and $\eta \in \Lambda^l(T)$. Let $\mathrm{d} : \Lambda^k(T) \to \Lambda^{k+1}(T)$ denote the exterior derivative operator.

We write $\mathrm{d}\lambda_0, \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n \in \Lambda^1(T)$ for the exterior derivatives of the barycentric coordinate functions. Clearly,

$$\mathrm{d}\lambda_0 + \mathrm{d}\lambda_1 + \cdots + \mathrm{d}\lambda_n = 0,$$

on T since $\sum_{i=0}^n \lambda_i = 1$. If $\sigma \in \Sigma(j : l, m : n)$, we set $\mathrm{d}\lambda_\sigma := \mathrm{d}\lambda_{\sigma(j)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(l)}$ the volume $(l - j + 1)$-form.

For $k > 0$, any element ω of $\Lambda^k(T)$ can be written as

$$\omega = \sum_{\sigma \in \Sigma(0:k-1,1:n)} a_\sigma \mathrm{d}\lambda_\sigma,$$

where $a_\sigma \in C^\infty(T)$. Taking $a_\sigma \in \mathcal{P}_r(T)$, we obtain the space $\mathcal{P}_r\Lambda^k(T)$ of polynomial differential k-forms of polynomial degree at most r. Moreover, $\mathcal{P}_r\Lambda^0(T)$ coincides with $\mathcal{P}_r(T)$.

For $k > 0$,

$$\mathcal{P}_0\Lambda^k(T) = \mathrm{span}\{\mathrm{d}\lambda_\sigma \ : \ \sigma \in \Sigma(0 : k-1, 1 : n)\}.$$

Example 3 For $n = 3$, one has

$$\mathcal{P}_0\Lambda^1(T) = \mathrm{span}\{\mathrm{d}\lambda_1, \mathrm{d}\lambda_2, \mathrm{d}\lambda_3\},$$

$$\mathcal{P}_0\Lambda^2(T) = \mathrm{span}\{\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2, \mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3, \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3\},$$
$$\mathcal{P}_0\Lambda^3(T) = \mathrm{span}\{\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3\},$$

respectively. □

Furthermore, if $0 < k < n$, we can write

$$\mathcal{P}_r\Lambda^k(T) = \mathrm{span}\{\lambda^{\boldsymbol{\alpha}}\mathrm{d}\lambda_\sigma \ : \ \sigma \in \Sigma(0 : k-1, 1 : n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}.$$

The set

$$\mathcal{BP}_r\Lambda^k(T) := \{\lambda^{\boldsymbol{\alpha}}\, \mathrm{d}\lambda_\sigma \ : \ \sigma \in \Sigma(0 : k-1, 1 : n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\} \tag{3}$$

is a basis of $\mathcal{P}_r\Lambda^k(T)$.

For $k = 0$,

$$\mathcal{BP}_r\Lambda^0(T) := \{\lambda^{\boldsymbol{\alpha}} \ : \ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$$

is a basis of $\mathcal{P}_r\Lambda^0(T)$ while for $k = n$,

$$\mathcal{BP}_r\Lambda^n(T) := \{\lambda^{\boldsymbol{\alpha}}\mathrm{d}\lambda_1 \wedge \cdots \wedge \mathrm{d}\lambda_n \ : \ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$$

is a basis of $\mathcal{P}_r\Lambda^n(T)$.

A particular set of polynomial differential k-forms of polynomial degree 1 are the Whitney's differential forms. They are associated with the k-simplices f of T. If $k = n$, then $f = T$, and the Whitney's differential form w_T is the volume form of polynomial degree 0.

Definition 1 Let $k \geq 0$ and $f \in \Delta_k(T)$. The Whitney's differential form w_f associated with the subsimplex f is defined, recursively in k, as follows:

- If $k = 0$, then f is a vertex of T, namely, $f = [x_i]$ for $i = 0, \dots, n$, and $w_f = w_{[x_i]} = \lambda_i$;
- If $k > 0$, then $f = f_\sigma$ for a $\sigma \in \Sigma(0:k, 0:n)$ and

$$w_{f_\sigma} = \sum_{i=0}^{k} (-1)^i \lambda_{\sigma(i)} \mathrm{d} w_{f_\sigma \setminus [x_{\sigma(i)}]},$$

  being $f_\sigma \setminus [x_{\sigma(i)}] \in \Delta_{k-1}(T)$ the oriented $(k-1)$-face of T with the vertices of f_σ except $x_{\sigma(i)}$.

We can write $f_\sigma \setminus [x_{\sigma(i)}] = [x_{\sigma(0)}, \dots, \widehat{x_{\sigma(i)}}, \dots, x_{\sigma(k)}]$, where the wide-hat means that the underlying term is omitted from the list.

For each $\sigma \in \Sigma(0:k, 0:n)$, it holds that

$$\mathrm{d} w_{f_\sigma} = (k+1)!\, \mathrm{d}\lambda_\sigma = (k+1)!\, \mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}.$$

Then,

$$w_{f_\sigma} = \sum_{i=0}^{k} (-1)^i \lambda_{\sigma(i)} \mathrm{d} w_{f_\sigma \setminus [x_{\sigma(i)}]} = k! \sum_{i=0}^{k} (-1)^i \lambda_{\sigma(i)}\, \mathrm{d}\lambda_{\sigma(0)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\sigma(i)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\sigma(k)}.$$

Example 4 The Whitney's 1-form associated with the edge $e = [x_{\sigma(0)}, x_{\sigma(1)}]$ is

$$w_e = \lambda_{\sigma(0)} \mathrm{d}\lambda_{\sigma(1)} - \lambda_{\sigma(1)} \mathrm{d}\lambda_{\sigma(0)}.$$

The Whitney's 2-form associated with the face $f = [x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}]$ reads

$$w_f = 2(\lambda_{\sigma(0)} \mathrm{d}\lambda_{\sigma(1)} \wedge \mathrm{d}\lambda_{\sigma(2)} - \lambda_{\sigma(1)} \mathrm{d}\lambda_{\sigma(0)} \wedge \mathrm{d}\lambda_{\sigma(2)} + \lambda_{\sigma(2)} \mathrm{d}\lambda_{\sigma(0)} \wedge \mathrm{d}\lambda_{\sigma(1)}).$$

In $\mathbb{R}^3$, the Whitney's 3-form associated with $T = [x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]$ is

$$w_T = 6\, \mathrm{d}\lambda_{\sigma(1)} \wedge \mathrm{d}\lambda_{\sigma(2)} \wedge \mathrm{d}\lambda_{\sigma(3)},$$

where we have used the fact that $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1$. $\square$

In finite element exterior calculus, the space of Whitney's differential k-forms on T is denoted by

$$\mathcal{P}_1^- \Lambda^k(T) := \mathrm{span}\{w_f \ : \ f \in \Delta_k(T)\}.$$

Since there is a one-to-one correspondence between $\Delta_k(T)$ and $\Sigma(0:k, 0:n)$, we can also write

$$\mathcal{P}_1^- \Lambda^k(T) := \mathrm{span}\{w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k, 0:n)\}.$$

Definition 2 Whitney's differential k-forms of polynomial degree $r+1$ are the elements of the space

$$\mathcal{P}_{r+1}^- \Lambda^k(T) := \mathrm{span}\{\lambda^{\boldsymbol{\alpha}} w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k, 0:n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}.$$

For $k > 0$, the space $\mathcal{P}_{r+1}^{-}\Lambda^k(T) \subsetneq \mathcal{P}_{r+1}\Lambda^k(T)$.
For $k = 0$,

$$\mathcal{P}_{r+1}^{-}\Lambda^0(T) = \text{span}\{\lambda^{\boldsymbol{\alpha}}\lambda_i \ : \ i \in \{0, \ldots, n\} \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$$

$$= \text{span}\{\lambda^{\widetilde{\boldsymbol{\alpha}}} \ : \ \widetilde{\boldsymbol{\alpha}} \in \mathcal{I}(n+1, r+1)\} = \mathcal{P}_{r+1}\Lambda^0(T).$$

For $k = n$,

$$\mathcal{P}_{r+1}^{-}\Lambda^n(T) = \text{span}\{\lambda^{\boldsymbol{\alpha}}\mathrm{d}\lambda_1 \wedge \cdots \wedge \mathrm{d}\lambda_n \ : \ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\} = \mathcal{P}_r\Lambda^n(T).$$

Remark 1 It is worth noting that, in the n-simplex T with vertices $x_0, x_1, \ldots, x_n$, the elements belonging to the set

$$\{\lambda^{\boldsymbol{\alpha}} w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k, 0:n), \ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$$

are not linearly independent. As an example, for $n = 2$, if $k = 1$, and $r = 1$, it can be verified that

$$\lambda_0 w_{[x_1,x_2]} - \lambda_1 w_{[x_0,x_2]} + \lambda_2 w_{[x_0,x_1]} = 0. \tag{4}$$

Given $\sigma \in \Sigma(0:k, 0:n)$, we set

$$\mathcal{I}_\sigma(n+1, r) := \{\boldsymbol{\alpha} \in \mathcal{I}(n+1, r) \ : \ \alpha_i = 0 \ \forall i < \sigma(0)\}.$$

When $k = 0$, then f_σ is a vertex of T, namely, $f_\sigma = [x_j]$ being $\sigma(0) = j$. In this case, to be clearer, we will sometimes use the notation $\mathcal{I}_{[x_j]}(n+1, r)$ instead of $\mathcal{I}_\sigma(n+1, r)$.

A basis of $\mathcal{P}_{r+1}^{-}\Lambda^k(T)$ is

$$\mathcal{B}\mathcal{P}_{r+1}^{-}\Lambda^k(T) = \{\lambda^{\boldsymbol{\alpha}} w_{f_\sigma} \ : \ \sigma \in \Sigma(0:k, 0:n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)\}$$

(see, e.g., [16]). For $n = 2$, $k = 1$, and $r = 1$, the 8 elements of $\mathcal{B}\mathcal{P}_2^{-}\Lambda^1(T)$, with $T = [x_0, x_1, x_2]$, are

$$\begin{aligned}
\lambda_i \, w_{[x_0,x_1]} &= \lambda_i \, (\lambda_0\mathrm{d}\lambda_1 - \lambda_1\mathrm{d}\lambda_0), \quad & i &= 0, 1, 2, \\
\lambda_i \, w_{[x_0,x_2]} &= \lambda_i \, (\lambda_0\mathrm{d}\lambda_2 - \lambda_2\mathrm{d}\lambda_0), \quad & i &= 0, 1, 2, \\
\lambda_i \, w_{[x_1,x_2]} &= \lambda_i \, (\lambda_1\mathrm{d}\lambda_2 - \lambda_2\mathrm{d}\lambda_1), \quad & i &= 1, 2 \, .
\end{aligned}$$

The condition $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(3, 1)$ prevents $\lambda_0 w_{[x_1,x_2]}$ in (4) from being in the set $\mathcal{B}\mathcal{P}_2^{-}\Lambda^1(T)$.

3 Weights and moments

3.1 Small simplices and weights

The concepts of small simplices and weights for polynomial differential forms in $\mathcal{P}_{r+1}^{-}\Lambda^k(T)$ were born in [21, 22], for any order k and any polynomial degree $r \geq$

0, to solve the difficulty raised in [7]: "*The main problem with such forms is the interpretation of DoFs*" in geometrical terms. We recall these concepts here below with a notation adapted to the isomorphism we want to state between these new DoFs, the weights, and the classical ones, moments, introduced in [5, 19].

In the n-simplex T with vertices $x_0, x_1, \ldots, x_n$, the principal lattice of order $r+1$ ($r \geq 0$) is the set of points defined by their barycentric coordinates with respect to the vertices of T as follows:

$$L_{r+1}(T) := \left\{ x \in T \ : \ \lambda_i(x) \in \left\{ 0, \frac{1}{r+1}, \ldots, \frac{r}{r+1}, 1 \right\} \text{ for each } i \in \{0, \ldots, n\} \right\}.$$

To each multi-index $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$, we associate an affine function, $\tau_{\boldsymbol{\alpha}} : T \longrightarrow T$, such that $\lambda_i(\tau_{\boldsymbol{\alpha}}(x)) = \frac{\lambda_i(x)+\alpha_i}{r+1}$. If f_σ is a face of T, then

$$\tau_{\boldsymbol{\alpha}}(f_\sigma) := \{\tau_{\boldsymbol{\alpha}}(x) \ : \ x \in f_\sigma\}.$$

Definition 3 The small k-simplexes of order r in T are the elements of the set

$$\begin{aligned} S_r^k(T) &:= \{\tau_{\boldsymbol{\alpha}}(f_\sigma) \ : \ f_\sigma \in \Delta_k(T) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\} \\ &= \{\tau_{\boldsymbol{\alpha}}(f_\sigma) \ : \ \sigma \in \Sigma(0:k, 0:n) \text{ and } \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}. \end{aligned}$$

For $k > 0$, they are $1/(r+1)$-homothetic to k-faces of T, with vertices in $L_{r+1}(T)$. For $k = 0$, we have $S_r^0(T) = L_{r+1}(T)$.

Example 5 For $k = 0$, let us set $n = 2$ and $r = 1$. If $\boldsymbol{\alpha} = (1, 0, 0)$, we have that

$$\tau_{\boldsymbol{\alpha}}(x_0) = (1\,,\,0\,,\,0), \qquad \tau_{\boldsymbol{\alpha}}(x_1) = \left(\frac{1}{2}\,,\,\frac{1}{2}\,,\,0\right), \qquad \tau_{\boldsymbol{\alpha}}(x_2) = \left(\frac{1}{2}\,,\,0\,,\,\frac{1}{2}\right),$$

whereas for $\boldsymbol{\alpha} = (0, 1, 0)$, we obtain

$$\tau_{\boldsymbol{\alpha}}(x_0) = \left(\frac{1}{2}\,,\,\frac{1}{2}\,,\,0\right), \qquad \tau_{\boldsymbol{\alpha}}(x_1) = (0\,,\,1\,,\,0), \qquad \tau_{\boldsymbol{\alpha}}(x_2) = \left(0\,,\,\frac{1}{2}\,,\,\frac{1}{2}\right)$$

that are points all in $L_2(T)$. □

We recall that there is a one-to-one correspondence between the elements of $\Delta_k(T)$ and $\Sigma(0:k, 0:n)$. Moreover, for $k > 0$, there is a one-to-one correspondence between the elements of $S_r^k(T)$ and the couples $(\sigma, \boldsymbol{\alpha})$ with $\sigma \in \Sigma(0:k, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$. In fact, if $\boldsymbol{\alpha}, \boldsymbol{\alpha}' \in \mathcal{I}(n+1, r)$ and $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}'$, then $\tau_{\boldsymbol{\alpha}}(T) \cap \tau_{\boldsymbol{\alpha}'}(T)$ is either empty or an element of $S_r^0(T)$. For $k = 0$, there is not such a one-to-one correspondence. The points of the principal lattice of T that are not the vertices of T have more than one representation as small node (see Fig. 2 and Example 6).

Example 6 Let us suppose $n = 2$ and $r = 3$. The point with barycentric coordinates $(\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$ in T, has different representations, as small node. Indeed, by referring to

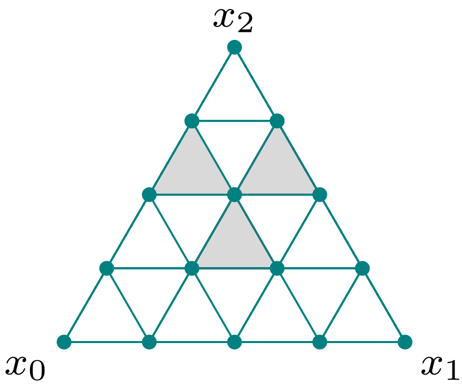

Fig. 2 Points of the principal lattice for $\mathcal{P}_4^- \Lambda^0(T)$, where T is a 2-simplex. The node with barycentric coordinates $(\frac{1}{4}, \frac{1}{4}, \frac{2}{4})$ in T is shared by the three gray small triangles

Fig. 2, this point can be $\tau_{\boldsymbol{\alpha}}(f_\sigma)$ with

$$\begin{aligned} \boldsymbol{\alpha} &= (1, 0, 2), & f_\sigma &= x_1, & &\text{in the top-left gray small triangle,} \\ \boldsymbol{\alpha} &= (0, 1, 2), & f_\sigma &= x_0, & &\text{in the top-right gray small triangle,} \\ \boldsymbol{\alpha} &= (1, 1, 1), & f_\sigma &= x_2, & &\text{in the bottom-center gray small triangle,} \end{aligned}$$

respectively. □

The weight of $\omega \in \Lambda^k(T)$ on a k-simplex s contained in T is denoted by $\int_s \omega$. If $k = 0$, for $\omega \in C^\infty(T)$ and $s \in T$, we have $\int_s \omega = \omega(s)$.

In particular, we are interested in the following set of weights.

Definition 4 Let $\omega \in \Lambda^k(T)$, $\sigma \in \Sigma(0:k, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}(n+1, r)$.

$$W_{\sigma,\boldsymbol{\alpha}}(\omega) := \int_{\tau_{\boldsymbol{\alpha}}(f_\sigma)} \omega. \tag{5}$$

The weights of Definition 4 are determinant in $\mathcal{P}_{r+1}^- \Lambda^k(T)$, namely, if $\omega \in \mathcal{P}_{r+1}^- \Lambda^k(T)$ and $\int_s \omega = 0$ for all $s \in S_r^k(T)$, then $\omega = 0$ (see [11] for a proof). However, for $0 < k < n$, the cardinality of the set of weights $\{W_{\sigma,\boldsymbol{\alpha}}(\omega) : \sigma \in \Sigma(0 : k, 0 : n),\ \boldsymbol{\alpha} \in \mathcal{I}(n+1, r)\}$ is greater than the dimension of $\mathcal{P}_{r+1}^- \Lambda^k(T)$. Hence, in the sequel, we often consider the following set of weights:

$$W^k := \{W_{\sigma,\boldsymbol{\alpha}}(\omega) : \sigma \in \Sigma(0 : k, 0 : n),\ \boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)\}. \tag{6}$$

It is worth noting that W^k is determinant (see [1]) and its cardinality coincides with the dimension of $\mathcal{P}_{r+1}^- \Lambda^k(T)$.

Remark 2 Only the second one of the three representations in Example 6 verifies the condition $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$ required to support a weight of the set defined in (6). In the first representation $\sigma(0) = 1$, hence $\mathcal{I}_\sigma(3, 3)$ is the set of multi-indices $\boldsymbol{\alpha} \in \mathcal{I}(3, 3)$ with $\alpha_0 = 0$ and $\boldsymbol{\alpha} = (1, 0, 2) \notin \mathcal{I}_\sigma(3, 3)$. In the second representation $\sigma(0) = 0$, hence $\mathcal{I}_\sigma(3, 3) = \mathcal{I}(3, 3)$ and $\boldsymbol{\alpha} = (0, 1, 2) \in \mathcal{I}(3, 3)$. In the third representation

$\sigma(0) = 2$, hence $\mathcal{I}_\sigma(3,3)$ is the set of multi-indices $\boldsymbol{\alpha} \in \mathcal{I}(3,3)$ with $\alpha_0 = \alpha_1 = 0$, and $\boldsymbol{\alpha} = (1,1,1) \notin \mathcal{I}_\sigma(3,3)$.

3.2 Moments associated with a particular basis of polynomial differential forms

Let ω be a smooth differential k-form defined on $T \subset \mathbb{R}^n$. For each d-face f_ζ of T, with $\zeta \in \Sigma(0:d, 0:n)$ and $k \le d \le n$, the moments of ω in f_ζ of degree $r-(d-k)$ are

$$M_{\zeta,\eta}(\omega) := \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\omega \wedge \eta, \qquad \forall\, \eta \in \mathcal{P}_{r-(d-k)}\Lambda^{d-k}(f_\zeta)\,, \tag{7}$$

where Tr_{f_ζ} is the trace operator on f_ζ.

It is well known that these moments are determinant in $\mathcal{P}^-_{r+1}\Lambda^k(T)$. Taking η in a basis of each space $\mathcal{P}_{r-(d-k)}\Lambda^{d-k}(f_\zeta)$, one obtains a determinant set of moments with cardinality equal to the dimension of $\mathcal{P}^-_{r+1}\Lambda^k(T)$ (see [5] and [11], for two different proofs).

The goal of the present work is to point out an isomorphism between moments and weights which, in a sense specified in the next sections, is consistent with the exterior derivative operator. To do that, we will consider a particular basis of the space $\mathcal{P}_{r-(d-k)}\Lambda^{d-k}(f_\zeta)$ in (7).

– If $d = k$, we adopt the Bernstein's basis of the space $\mathcal{P}_r(f_\zeta)$, namely

$$\mathcal{B}\mathcal{P}_r\Lambda^0(f_\zeta) = \{\lambda^{\boldsymbol{\beta}}_{f_\zeta} \;:\; \boldsymbol{\beta} \in \mathcal{I}(d+1, r)\},$$

where $\lambda^{\boldsymbol{\beta}}_{f_\zeta} = \lambda^{\beta_0}_{f_\zeta,0}\cdots\lambda^{\beta_d}_{f_\zeta,d} = \lambda^{\beta_0}_{T,\zeta(0)}\cdots\lambda^{\beta_d}_{T,\zeta(d)}$.

– If $d > k$, we rely on the basis indicated in (3), namely,

$$\mathcal{B}\mathcal{P}_{r-(d-k)}\Lambda^{d-k}(f_\zeta) = \{\lambda^{\boldsymbol{\beta}}_{f_\zeta}(\mathrm{d}\lambda_{f_\zeta})_\rho \;:\; \rho \in \Sigma(0:d-(k+1), 1:d),\\ \boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))\}.$$

Here

$$\begin{aligned}(\mathrm{d}\lambda_{f_\zeta})_\rho &= \mathrm{d}\lambda_{f_\zeta,\rho(0)} \wedge \cdots \wedge \mathrm{d}\lambda_{f_\zeta,\rho(d-(k+1))}\\ &= \mathrm{d}\lambda_{T,\zeta(\rho(0))} \wedge \cdots \wedge \mathrm{d}\lambda_{T,\zeta(\rho(d-(k+1)))}.\end{aligned}$$

Example 7 For $k = 1$,

– If $f_\zeta = [x_0, x_2, x_3] \in \Delta_2(T)$, then $d = 2$, $d-k = 1$, and

$$\mathcal{B}\mathcal{P}_{r-1}\Lambda^1(f_\zeta) = \{\lambda^{\boldsymbol{\beta}}_{f_\zeta}\mathrm{d}\lambda_2 \;:\; \boldsymbol{\beta} \in \mathcal{I}(3, r-1)\} \cup \{\lambda^{\boldsymbol{\beta}}_{f_\zeta}\mathrm{d}\lambda_3 \;:\; \boldsymbol{\beta} \in \mathcal{I}(3, r-1)\};$$

– If $f_\zeta = [x_0, x_1, x_2, x_3] \in \Delta_3(T)$ (for $n = 3$, it means $f_\zeta = T$), then $d-k = 2$ and

$$\begin{aligned}\mathcal{B}\mathcal{P}_{r-2}\Lambda^2(f_\zeta) = \{&\lambda^{\boldsymbol{\beta}}_{f_\zeta}\,\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2 \;:\; \boldsymbol{\beta} \in \mathcal{I}(4, r-2)\}\\ &\cup\{\lambda^{\boldsymbol{\beta}}_{f_\zeta}\,\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3 \;:\; \boldsymbol{\beta} \in \mathcal{I}(4, r-2)\} \cup \{\lambda^{\boldsymbol{\beta}}_{f_\zeta}\,\mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3 \;:\; \boldsymbol{\beta} \in \mathcal{I}(4, r-2)\},\end{aligned}$$

respectively. □

With these choices of basis, we obtain the following moments for $\omega \in \Lambda^k(T)$: for each $\zeta \in \Sigma(0:k,0:n)$, and $\boldsymbol{\beta} \in \mathcal{I}(k+1,r)$

$$M_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega) := \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\omega \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}; \tag{8}$$

for each $d > k$, $\zeta \in \Sigma(0:d,0:n)$, $\rho \in \Sigma(0:d-(k+1),1:d)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1,r-(d-k))$

$$M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) := \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\omega \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho. \tag{9}$$

We use the notation "$\rho = \emptyset$" when $d = k$ since $\Sigma(0:d-(k+1),1:d)$ has not been defined for $d = k$. We thus have the following set of moments for $\omega \in \mathcal{P}_{r+1}^-\Lambda^k(T)$:

$$\begin{aligned} M^k := \{M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) \ : \ \zeta \in \Sigma(0:d,0:n),\ \rho \in \Sigma(0:d-(k+1),1:d),\\ \text{and } \boldsymbol{\beta} \in \mathcal{I}(d+1,r-(d-k)) \text{ with } k \le d \le n\}. \end{aligned} \tag{10}$$

Remark 3 If $\omega \in \Lambda^0(T)$,

- when $d = k = 0$, then $\zeta \in \Sigma(0:0,0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(1,r)$, so $f_\zeta = [x_{\zeta(0)}]$ and $\boldsymbol{\beta} = (r)$ (the "multi-index" $\boldsymbol{\beta}$ has only one component that takes the value r). We have
$$M_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega) = (\lambda_{\zeta(0)}^r\omega)(x_{\zeta(0)}) = \omega(x_{\zeta(0)}) =: \widehat{M}_{\zeta,\boldsymbol{\beta}}(\omega)$$
- when $d > 0$, then $\zeta \in \Sigma(0:d,0:n)$, $\rho \in \Sigma(0:d-1,1:d)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1,r-d)$. It is worth noting that $\Sigma(0:d-1,1:d)$ has a unique element and $(\mathrm{d}\lambda_{f_\zeta})_\rho = \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}$, namely

$$M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) = \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\omega \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) =: \widehat{M}_{\zeta,\boldsymbol{\beta}}(\omega).$$

This means that in $\Lambda^0(T)$ moments depend on two parameters, $\zeta \in \Sigma(0:d,0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1,r-d)$. Hence, in $\Lambda^0(T)$ to denote the moments, we will often prefer the notation $\widehat{M}_{\zeta,\boldsymbol{\beta}}$ with $\zeta \in \Sigma(0:d,0:n)$, $\boldsymbol{\beta} \in \mathcal{I}(d+1,r-d)$, and $d \in \{0,\ldots,n\}$.

4 Isomorphism

We establish an isomorphism, one for each value of $k \in \{0,\ldots,n\}$, between the set of moments M^k defined in (10) and the set of weights W^k defined in (6).

We distinguish two cases, when the support f_ζ of the moment has dimension d either equal to the order k of the differential form or higher.

The moment $M_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega)$, with $\zeta \in \Sigma(0:k, 0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(k+1,r)$ is linked to the weight $W_{\sigma,\boldsymbol{\alpha}}(\omega)$, with $\sigma = \zeta$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}\,\mathrm{E}_\zeta$, the extension of $\boldsymbol{\beta}$ to a multi-index in $\mathcal{I}(n+1,r)$ by the matrix E_ζ defined in (1). In this case, the small simplex $s = \tau_{\boldsymbol{\alpha}}(f_\sigma)$ is not only parallel to $f_\sigma = f_\zeta$, but it is in fact contained in f_ζ.

In order to associate a weight with the moment $M_{\zeta,\rho,\boldsymbol{\beta}}(\omega)$ when $d > k$, we first prove the following lemma.

Lemma 1 *If $\zeta \in \Sigma(0:d, 0:n)$ and $\rho \in \Sigma(0:d-(k+1), 1:d)$ for some d with $k < d \leq n$, then $\zeta \circ \rho \in \Sigma(0:d-(k+1), 1:n)$ and the face of $T = [x_0, \ldots, x_n]$ with vertices $\{x_j : j \in [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]\}$ is a k-face of f_ζ that contains $x_{\zeta(0)}$, the first vertex of f_ζ.*

Proof If $\zeta \in \Sigma(0:d, 0:n)$ and $\rho \in \Sigma(0:d-(k+1), 1:d)$, then $\rho(0) > 0$ and $\zeta(\rho(0)) > \zeta(0) \geq 0$, hence $\zeta \circ \rho \in \Sigma(0:d-(k+1), 1:n)$.

We notice that $\#[\![\zeta]\!] = d+1$, $\#[\![\zeta \circ \rho]\!] = d-k$, and $[\![\zeta \circ \rho]\!] \subset [\![\zeta]\!]$, hence $\#([\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]) = k+1$ and the face of $T = [x_0, \ldots, x_n]$ with vertices $\{x_j : j \in [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]\}$ is a k-face of $f_\zeta = [x_{\zeta(0)}, \ldots, x_{\zeta(d)}]$.

Using again that $\rho \in \Sigma(0:d-(k+1), 1:d)$ one has $\zeta(0) < \zeta(1) \leq \zeta(\rho(j))$ for all $j \in \{0, \ldots, d-(k+1)\}$. Hence, $\zeta(0) \notin [\![\zeta \circ \rho]\!]$ and the vertex $x_{\zeta(0)}$ belongs to the set $\{x_j : j \in [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]\}$. □

We identify moments $M_{\zeta,\rho,\boldsymbol{\beta}}(\omega)$ with weights $W_{\sigma,\boldsymbol{\alpha}}(\omega)$ in small simplices that are parallel to the k-face of f_ζ with vertices $\{x_j : j \in [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]\}$ and that are not completely contained in the boundary of f_ζ.

The map $\sigma \in \Sigma(0:k, 0:n)$ is such that f_σ is the element of $\Delta_k(T)$ with vertices $[\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]$.[1]

The multi-index $\boldsymbol{\alpha}$ is constructed in the following way. Since $[\![\rho]\!]$ is a subset of $\{0, \ldots, d\}$ with $d-k$ elements and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))$, the multi-index $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta} + \mathbf{e}_{[\![\rho]\!]}$ belongs to $\mathcal{I}(d+1,r)$. We set $\boldsymbol{\alpha} = \widetilde{\boldsymbol{\beta}}\,\mathrm{E}_\zeta$, the extension of $\widetilde{\boldsymbol{\beta}}$ to a multi-index in $\mathcal{I}(n+1,r)$ by the matrix E_ζ.

Example 8 For $n = 3$ and $k = 1$, we explain which weights are associated with some selected elements of the set of moments unisolvent in $\mathcal{P}_5^- \Lambda^1(T)$ $(r = 4)$.

- Let us consider the moment $\int_{[x_2,x_3]} \omega\, \lambda_2 \lambda_3^3$, thus $d = 1 = k$ and $\rho = \emptyset$.

  Here, we have $\sigma = \zeta$, and the multi-index $\boldsymbol{\alpha}$ is the extension of $\boldsymbol{\beta}$ (in this case, $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta}$). More precisely, $\boldsymbol{\alpha} = (1,3)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (0,0,1,3)$.
  The associated weight is $\int_s \omega$ with $s = \tau_{(0,0,1,3)}([x_2, x_3]) = \tau_{\boldsymbol{\alpha}}(f_\sigma)$.

- Let us consider the moment $\int_{[x_0,x_2,x_3]} \omega \wedge \lambda_0^2 \lambda_3 (\mathrm{d}\lambda_2)$, thus $d = 2 > 1 = k$.

  Here, we have $[\![\zeta]\!] = \{0,2,3\}$, $[\![\rho]\!] = \{1\}$, and $[\![\zeta \circ \rho]\!] = \{2\}$.
  Then $\sigma \in \Sigma(0:1, 0:3)$ with $[\![\sigma]\!] = \{0,2,3\} \setminus \{2\} = \{0,3\}$ so $\sigma(0) = 0$, $\sigma(1) = 3$.

[1] If $\rho^* \in \Sigma(0:k, 0:d)$ is the complementary map of ρ, namely, $[\![\rho]\!] \cup [\![\rho^*]\!] = \{0,1,\ldots,d\}$, then $\sigma = \zeta \circ \rho^*$.

Concerning the multi-index, first, we compute $\widetilde{\boldsymbol{\beta}} = (2, 0, 1)+(0, 1, 0) = (2, 1, 1)$.
Then, we extend it as $\boldsymbol{\alpha} = (2, 1, 1)\begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix} = (2, 0, 1, 1)$.
The associated weight is $\int_s \omega$ with $s = \tau_{(2,0,1,1)}([x_0, x_3]) = \tau_{\boldsymbol{\alpha}}(f_\sigma)$.

- Let us consider the moment $\int_{[x_0,x_1,x_2,x_3]} \omega \wedge \lambda_0\lambda_2 \,(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2)$, thus $d = 3 > 1 = k$.

  Here, we have $[\![\zeta]\!] = \{0, 1, 2, 3\}$, $[\![\rho]\!] = \{1, 2\} = [\![\zeta \circ \rho]\!]$.
  Then, $\sigma \in \Sigma(0 : 1, 0 : 3)$ with $[\![\sigma]\!] = \{0, 1, 2, 3\} \setminus \{1, 2\} = \{0, 3\}$, so $\sigma(0) = 0, \sigma(1) = 3$.
  Concerning the multi-index $\boldsymbol{\beta} = (1, 0, 1, 0) \in \mathcal{I}(4, 4 - (3 - 1)) = \mathcal{I}(4, 2)$.
  Finally, $\boldsymbol{\alpha} = \widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta} + \mathbf{e}_{\{1,2\}} = (1, 0, 1, 0) + (0, 1, 1, 0) = (1, 1, 2, 0)$. Note that, in this case, it is not necessary to extend $\widetilde{\boldsymbol{\beta}}$.
  The associated weight is $\int_s \omega$ with $s = \tau_{(1,1,2,0)}([x_0, x_3]) = \tau_{\boldsymbol{\alpha}}(f_\sigma)$. □

Example 9 We consider the three types of moments for $\mathcal{P}_3^- \Lambda^1(T)$ indicated by different symbols in Fig. 1, center. They can be geometrically localized in T by resorting to the small edges s (shown in Fig. 1, right and left) supporting the corresponding weights. Indeed, we have as follows:

$$\begin{aligned}
\int_{[1,2]} \omega\, \lambda_2^2 \quad &(\Longleftrightarrow \quad \int_s \omega \text{ with } s = \tau_{(0,0,2,0)}([1, 2])\,) \quad \Longleftrightarrow \quad s = \bigcirc \\
\int_{[0,1,2]} \omega \wedge \lambda_1(\mathrm{d}\lambda_1) \quad &(\Longleftrightarrow \quad \int_s \omega \text{ with } s = \tau_{(0,2,0,0)}([0, 2])\,) \quad \Longleftrightarrow \quad s = \square_1 \\
\int_{[0,1,2]} \omega \wedge \lambda_1(\mathrm{d}\lambda_2) \quad &(\Longleftrightarrow \quad \int_s \omega \text{ with } s = \tau_{(0,1,1,0)}([0, 1])\,) \quad \Longleftrightarrow \quad s = \square_2 \\
\int_{[0,1,2,3]} \omega \wedge (\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2) \quad &(\Longleftrightarrow \quad \int_s \omega \text{ with } s = \tau_{(0,1,1,0)}([0, 3])\,) \quad \Longleftrightarrow \quad s = \Diamond_1 \\
\int_{[0,1,2,3]} \omega \wedge (\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3) \quad &(\Longleftrightarrow \quad \int_s \omega \text{ with } s = \tau_{(0,1,0,1)}([0, 2])\,) \quad \Longleftrightarrow \quad s = \Diamond_2 \\
\int_{[0,1,2,3]} \omega \wedge (\mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3) \quad &(\Longleftrightarrow \quad \int_s \omega \text{ with } s = \tau_{(0,0,1,1)}([0, 1])\,) \quad \Longleftrightarrow \quad s = \Diamond_3.
\end{aligned}$$

The moments (on the left) are in correspondence ($\Longleftrightarrow$) with the weights (in the center) as it is established by the isomorphism described in the present section. Weights have a precise geometrical localization in T, namely, they are supported on precise small edges s (indicated in the center). As a result, moments can be geometrically localized in T by associating with each of them the small simplex s (on the right) supporting the weight they correspond with. ◇

The set of moments defined in (10) and the set of weights defined in (6) are subsets of $(\mathcal{P}_{r+1}^- \Lambda^k(T))^*$, the dual space of $\mathcal{P}_{r+1}^- \Lambda^k(T)$. We are interested in the maps $\mathcal{W}^k$ defined from this set of moments to the set of weights in the following way:

Definition 5 For each $k, d, n \in \mathbb{N}$, $0 \le k \le d \le n$, $\zeta \in \Sigma(0 : d, 0 : n)$, $\rho \in \Sigma(0 : d - (k + 1), 1 : d)$ ($\rho = \emptyset$ if $d = k$) and $\boldsymbol{\beta} \in \mathcal{I}(d + 1, r - (d - k))$, we set

- If $d = k$

$$\mathcal{W}^k(M_{\zeta,\emptyset,\boldsymbol{\beta}}) = W_{\zeta,\boldsymbol{\beta}\,\mathrm{E}_\zeta};$$

– If $d > k$

$$\mathcal{W}^k(M_{\zeta,\rho,\boldsymbol{\beta}}) = W_{\zeta\circ\rho^*,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\,\mathrm{E}_\zeta},$$

where $\zeta \circ \rho^* \in \Sigma(0:k, 0:n)$ is such that $[\![\zeta \circ \rho^*]\!] = [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]$.

The following proposition shows that the image of this map is, in fact, the set of weights considered in (6).

Proposition 1 *If $W_{\sigma,\boldsymbol{\alpha}} = \mathcal{W}^k(M_{\zeta,\emptyset,\boldsymbol{\beta}})$, with $\zeta \in \Sigma(0:d, 0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r)$, or $W_{\sigma,\boldsymbol{\alpha}} = \mathcal{W}^k(M_{\zeta,\rho,\boldsymbol{\beta}})$ for a triplet*

$$(\zeta, \rho, \boldsymbol{\beta}) \in \Sigma(0:d, 0:n) \times \Sigma(0:d-(k+1), 1:d) \times \mathcal{I}(d+1, r-(d-k))$$

with $k < d \leq n$, then $(\sigma, \boldsymbol{\alpha}) \in \Sigma(0:k, 0:n) \times \mathcal{I}_\sigma(n+1, r)$.

Proof Note that $(\sigma, \boldsymbol{\alpha}) \in \Sigma(0:k, 0:n) \times \mathcal{I}(n+1, r)$ by construction. We have thus to prove that $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$, namely, that $\alpha_i = 0$ for all $i < \sigma(0)$.

We recall that if $\zeta \in \Sigma(0:d, 0:n)$ and $\widetilde{\boldsymbol{\beta}} \in \mathcal{I}(d+1, \tilde{r})$, then $(\widetilde{\boldsymbol{\beta}}\,\mathrm{E}_\zeta)_i = 0$ if $0 \leq i < \zeta(0)$, hence $\widetilde{\boldsymbol{\beta}}\,\mathrm{E}_\zeta \in \mathcal{I}_\zeta(n+1, \tilde{r})$.

If $W_{\sigma,\boldsymbol{\alpha}} = \mathcal{W}^k(M_{\zeta,\emptyset,\boldsymbol{\beta}})$, then $\sigma = \zeta$ and $\boldsymbol{\alpha} = \boldsymbol{\beta}\,\mathrm{E}_\zeta \in \mathcal{I}_\zeta(n+1, r) = \mathcal{I}_\sigma(n+1, r)$.

If $d > k$ and $W_{\sigma,\boldsymbol{\alpha}} = \mathcal{W}^k(M_{\zeta,\rho,\boldsymbol{\beta}})$, then $\boldsymbol{\alpha} = (\boldsymbol{\beta} + \mathbf{e}_{[\![\rho]\!]})\,\mathrm{E}_\zeta \in \mathcal{I}_\zeta(n+1, r)$. We notice that $\sigma(0) = \zeta(0)$ because $[\![\sigma]\!] = [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!]$ and $\zeta(0) \notin [\![\zeta \circ \rho]\!] \subset \{1, \ldots, n\}$. Hence, $\mathcal{I}_\zeta(n+1, r) = \mathcal{I}_\sigma(n+1, r)$. □

Similarly, we can define a map $\mathcal{M}^k$ from the set of weights in (6) to the set of moments in (10).

Definition 6 Given a couple $(\sigma, \boldsymbol{\alpha}) \in \Sigma(0:k, 0:n) \times \mathcal{I}_\sigma(n+1, r)$, we denote

$$d = \#([\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!]) - 1.$$

- If $[\![\boldsymbol{\alpha}]\!] \subset [\![\sigma]\!]$, then $[\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!] = [\![\sigma]\!]$ and $d = k$. We set $\zeta = \sigma$, $\rho = \emptyset$ and $\boldsymbol{\beta} = \boldsymbol{\alpha}\mathrm{E}_\zeta^\top (= \boldsymbol{\alpha}\mathrm{E}_\sigma^\top)$.
- If $[\![\boldsymbol{\alpha}]\!] \not\subset [\![\sigma]\!]$, then $[\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!] \supsetneq [\![\sigma]\!]$ and $d > k$. We set $\zeta \in \Sigma(0:d, 0:n)$ such that $[\![\zeta]\!] = [\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!]$, $\rho \in \Sigma(0:d-(k+1), 1:d)$ such that $[\![\zeta \circ \rho]\!] = [\![\boldsymbol{\alpha}]\!] \setminus [\![\sigma]\!]$, and $\boldsymbol{\beta} = \boldsymbol{\alpha}\mathrm{E}_\zeta^\top - \mathbf{e}_{[\![\rho]\!]}$.

Then, we set $\mathcal{M}^k(W_{\sigma,\boldsymbol{\alpha}}) := M_{\zeta,\rho,\boldsymbol{\beta}}$.

Proposition 2 *For each $(\sigma, \boldsymbol{\alpha}) \in \Sigma(0:k, 0:n) \times \mathcal{I}_\sigma(n+1, r)$, the element $\mathcal{M}^k(W_{\sigma,\boldsymbol{\alpha}})$ belongs to the set of moments defined in (10).*

Proof In fact, $k \leq d \leq n$, $\zeta \in \Sigma(0:d, 0:n)$ and $\rho = \emptyset$ if $d = k$ or $\rho \in \Sigma(0:d-(k+1), 1:d)$, if $d > k$ by construction. Furthermore, $\boldsymbol{\beta}$ is a multi-index with $d+1$ components. Since $[\![\boldsymbol{\alpha}]\!] \subset [\![\zeta]\!]$, then $|\boldsymbol{\alpha}\mathrm{E}_\zeta^\top| = |\boldsymbol{\alpha}|$ and $|\boldsymbol{\beta}| = |\boldsymbol{\alpha}| - (d-k) = r - (d-k)$. Hence, $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))$. □

Remark 4 From a geometric point of view, we associate with the multi-index $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$ a subsimplex $f_{a(\boldsymbol{\alpha})}$ of T with vertices those in $[\![\boldsymbol{\alpha}]\!]$, namely, x_i is a vertex of $f_{a(\boldsymbol{\alpha})}$ if and only if $\alpha_i \neq 0$. Given f_σ and $f_{\widetilde{\sigma}}$ two subsimplices of T, we denote by $f_\sigma \vee f_{\widetilde{\sigma}}$ the subsimplex of T with vertices those of f_σ and $f_{\widetilde{\sigma}}$. The moment associated with the weight $W_{\sigma,\boldsymbol{\alpha}}$ is an integral on the face $f_\zeta = f_\sigma \vee f_{a(\boldsymbol{\alpha})}$.

- If $[\![\boldsymbol{\alpha}]\!] \subset [\![\sigma]\!]$, then $f_\sigma \vee f_{a(\boldsymbol{\alpha})} = f_\sigma$ and $\rho = \emptyset$.
- If $[\![\boldsymbol{\alpha}]\!] \not\subset [\![\sigma]\!]$, then $f_\sigma \vee f_{a(\boldsymbol{\alpha})} \neq f_\sigma$ and ρ involves all the vertices of $f_\sigma \vee f_{a(\boldsymbol{\alpha})}$ that are not in f_σ.

Example 10 Let us consider $T \subset \mathbb{R}^3$, $\omega \in \mathcal{P}_4^- \Lambda^1(T)$ $(r = 3, k = 1)$ and $f_\sigma = [x_1, x_3]$ (namely, $\sigma \in \Sigma(0:1, 0:3)$, $\sigma(0) = 1$ and $\sigma(1) = 3$), we have

$$\begin{aligned}
\mathcal{M}^1 W_{\sigma,(0,3,0,0)}(\omega) &= \textstyle\int_{[x_1,x_3]} \omega\, \lambda_1^3 && (f_{a(\boldsymbol{\alpha})} = [x_1]) \\
\mathcal{M}^1 W_{\sigma,(0,2,0,1)}(\omega) &= \textstyle\int_{[x_1,x_3]} \omega\, \lambda_1^2 \lambda_3 && (f_{a(\boldsymbol{\alpha})} = [x_1, x_3]) \\
\mathcal{M}^1 W_{\sigma,(0,2,1,0)}(\omega) &= \textstyle\int_{[x_1,x_2,x_3]} \omega \wedge \lambda_1^2 (\mathrm{d}\lambda_2) && (f_{a(\boldsymbol{\alpha})} = [x_1, x_2]).
\end{aligned}$$

For $f_\sigma = [x_0, x_3]$ (namely, $\sigma \in \Sigma(0:1, 0:3)$, $\sigma(0) = 0$ and $\sigma(1) = 3$), we have

$$\mathcal{M}^1 W_{\sigma,(0,2,1,0)}(\omega) = \textstyle\int_{[x_0,x_1,x_2,x_3]} \omega \wedge \lambda_1 (\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_2) \quad (f_{a(\boldsymbol{\alpha})} = [x_1, x_2]).$$

For $\omega \in \mathcal{P}_4^- \Lambda^2(T)$ $(r = 3, k = 2)$ and $f_\sigma = [x_0, x_1, x_3]$ (namely, $\sigma \in \Sigma(0:2, 0:3)$, $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(2) = 3$), we have

$$\begin{aligned}
\mathcal{M}^2 W_{\sigma,(0,3,0,0)}(\omega) &= \textstyle\int_{[x_0,x_1,x_3]} \omega\, \lambda_1^3 && (f_{a(\boldsymbol{\alpha})} = [x_1]) \\
\mathcal{M}^2 W_{\sigma,(1,1,0,1)}(\omega) &= \textstyle\int_{[x_0,x_1,x_3]} \omega\, \lambda_0 \lambda_1 \lambda_3 && (f_{a(\boldsymbol{\alpha})} = [x_0, x_1, x_3]) \\
\mathcal{M}^2 W_{\sigma,(1,1,1,0)}(\omega) &= \textstyle\int_{[x_0,x_1,x_2,x_3]} \omega \wedge \lambda_0 \lambda_1 (\mathrm{d}\lambda_2) && (f_{a(\boldsymbol{\alpha})} = [x_0, x_1, x_2]) \\
\mathcal{M}^2 W_{\sigma,(0,0,3,0)}(\omega) &= \textstyle\int_{[x_0,x_1,x_2,x_3]} \omega \wedge \lambda_2^2 (\mathrm{d}\lambda_2) && (f_{a(\boldsymbol{\alpha})} = [x_2]).
\end{aligned}$$

It is worth noting that, with this geometric rule, we associate to a couple $(\sigma, \boldsymbol{\alpha}) \in \Sigma(0:k, 0:n) \times \mathcal{I}(n+1, r)$, with $\boldsymbol{\alpha} \notin \mathcal{I}_\sigma(n+1, r)$, a weight that is not in (6) and a moment that is not in (10). For instance, if $f_\sigma = [x_1, x_3]$ and $\boldsymbol{\alpha} = (1, 1, 0, 1)$, we have $f_{a(\boldsymbol{\alpha})} = [x_0, x_1, x_3]$ and then the moment $\int_{[x_0,x_1,x_3]} \omega \wedge \lambda_1 \lambda_3 (\mathrm{d}\lambda_0)$. If $\boldsymbol{\alpha} = (1, 1, 1, 0)$, then $f_{a(\boldsymbol{\alpha})} = [x_0, x_1, x_2]$ and the corresponding moment should be $\int_{[x_0,x_1,x_2,x_3]} \omega \wedge \lambda_1 (\mathrm{d}\lambda_0 \wedge \mathrm{d}\lambda_2)$. □

Proposition 3 *For any* $(\sigma, \boldsymbol{\alpha}) \in \Sigma(0:k, 0:n) \times \mathcal{I}_\sigma(n+1, r)$, *it holds that*

$$\mathcal{W}^k \mathcal{M}^k (W_{\sigma,\boldsymbol{\alpha}}) = W_{\sigma,\boldsymbol{\alpha}}.$$

Proof If $[\![\boldsymbol{\alpha}]\!] \subset [\![\sigma]\!]$, then $d = k$, and using (2), we have

$$\mathcal{W}^k \mathcal{M}^k (W_{\sigma,\boldsymbol{\alpha}}) = \mathcal{W}^k (M_{\sigma,\emptyset,\boldsymbol{\alpha}\mathrm{E}_\sigma^\top}) = W_{\sigma,\boldsymbol{\alpha}\mathrm{E}_\sigma^\top \mathrm{E}_\sigma} = W_{\sigma,\boldsymbol{\alpha}}\,,$$

where the last equality holds because $[\![\boldsymbol{\alpha}]\!] \subset [\![\sigma]\!]$.

If $[\![\boldsymbol{\alpha}]\!] \not\subset [\![\sigma]\!]$, then $d > k$ and

$$\mathcal{W}^k \mathcal{M}^k(W_{\sigma,\boldsymbol{\alpha}}) = \mathcal{W}^k(M_{\zeta,\rho,\boldsymbol{\beta}}),$$

with $\zeta \in \Sigma(0:d, 0:n)$ such that $[\![\zeta]\!] = [\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!]$, $\rho \in \Sigma(0:d-(k+1), 1:d)$ such that $[\![\zeta \circ \rho]\!] = [\![\boldsymbol{\alpha}]\!] \setminus [\![\sigma]\!]$, and $\boldsymbol{\beta} = \boldsymbol{\alpha}\mathrm{E}_\zeta^\top - \mathbf{e}_{[\![\rho]\!]}$. Then, $\mathcal{W}^k(M_{\zeta,\rho,\boldsymbol{\beta}}) = W_{\widetilde{\sigma},\widetilde{\boldsymbol{\alpha}}}$, with

$$\widetilde{\sigma} \in \Sigma(0:k, 0:n) \text{ such that } [\![\widetilde{\sigma}]\!] = [\![\zeta]\!] \setminus [\![\zeta \circ \rho]\!] = ([\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!]) \setminus ([\![\boldsymbol{\alpha}]\!] \setminus [\![\sigma]\!]) = [\![\sigma]\!]$$

and, using again (2), we obtain

$$\widetilde{\boldsymbol{\alpha}} = (\boldsymbol{\beta} + \mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta = \boldsymbol{\alpha}\mathrm{E}_\zeta^\top \mathrm{E}_\zeta = \boldsymbol{\alpha}.$$

Also, in this case, the last equality holds because $[\![\boldsymbol{\alpha}]\!] \subset [\![\zeta]\!] = [\![\boldsymbol{\alpha}]\!] \cup [\![\sigma]\!]$. □

Remark 5 Since the cardinality of both sets M^k and W^k defined in (10) and (6), respectively, is equal to the dimension of $\mathcal{P}_{r+1}^- \Lambda^k(T)$, from Proposition 3, it follows that $\mathcal{M}^k \mathcal{W}^k$ is also equal to the identity.

5 The matrix of the gradient operator

Let us fix a set R^0 of unisolvent degrees of freedom for $\mathcal{P}_{r+1}^- \Lambda^0(T)$ and a set R^1 of unisolvent degrees of freedom for $\mathcal{P}_{r+1}^- \Lambda^1(T)$. We denote by $\mathbf{r}^0(\varphi)$ the vector collecting the degrees of freedom of the set R^0 evaluated on the 0-form φ and by $\mathbf{r}^1(\omega)$ the vector collecting the degrees of freedom of the set R^1 evaluated on the 1-form ω. Then, there exists a unique matrix G_R (that depends on the sets R^0 and R^1) such that

$$\mathbf{r}^1(\mathrm{d}\varphi) = G_R\, \mathbf{r}^0(\varphi), \qquad \forall \varphi \in \mathcal{P}_{r+1}^- \Lambda^0(T).$$

When the two sets, R^0 and R^1, of degrees of freedom contain the weights defined in Definition 4 (namely when $R^k = W^k$, for $k = 0, 1$), the matrix G_R, denoted by G_W, has a clear geometrical meaning. By Stokes' theorem, G_W is the transposed of the all-nodes incidence matrix of the graph $\mathcal{M}^G$ with nodes the points of the principal lattice of T and arcs the oriented small edges corresponding to couples $(\sigma, \boldsymbol{\alpha})$ with $\sigma \in \Sigma(0:1, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$. This geometrical characterization is at the basis of the tree-cotree techniques used in electromagnetism that are well known in the low order case $r = 0$ and that have been recently extended to the high order case $r > 0$ using weights (see [25]; see also [15] for an analogous result in the framework of the isogeometric analysis).

We claim that the isomorphism defined in the previous section *preserves* the matrix G_R. This means that, if $\mathbf{m}^k$ is the vector ($\mathbf{r}^k$) collecting the moments in M^k for φ ($k = 0$) and $\mathrm{d}\varphi$ ($k = 1$), associated, through this isomorphism $\mathcal{M}^k$, with $\mathbf{w}^k$, the vector ($\mathbf{r}^k$)

collecting the weights in W^k for φ and dφ, i.e.,

$$\mathcal{M}^0 W_j^0 : \varphi \longmapsto (\mathcal{M}^0 W_j^0)(\varphi) = M_j^0(\varphi) := (\mathbf{m}^0)_j \qquad \forall\, j = 1, \ldots, \dim \mathcal{P}_{r+1}^- \Lambda^0(T),$$
$$\mathcal{M}^1 W_\ell^1 : \mathrm{d}\varphi \longmapsto (\mathcal{M}^1 W_\ell^1)(\mathrm{d}\varphi) = M_\ell^1(\mathrm{d}\varphi) := (\mathbf{m}^1)_\ell \;\; \forall\, \ell = 1, \ldots, \dim \mathcal{P}_{r+1}^- \Lambda^1(T),$$

and

$$W_j^0 : \varphi \longmapsto W_j^0(\varphi) := (\mathbf{w}^0)_j \qquad \forall\, j = 1, \ldots, \dim \mathcal{P}_{r+1}^- \Lambda^0(T),$$
$$W_\ell^1 : \mathrm{d}\varphi \longmapsto W_\ell^1(\mathrm{d}\varphi) := (\mathbf{w}^1)_\ell \;\; \forall\, \ell = 1, \ldots, \dim \mathcal{P}_{r+1}^- \Lambda^1(T),$$

then we have

$$\mathbf{m}^1 = G_M \mathbf{m}^0, \qquad \mathbf{w}^1 = G_W \mathbf{w}^0, \qquad \text{with} \qquad G_M = G_W.$$

The matrix G_R does not change, namely $G_M = G_W$ (see Fig. 3 for a visualization of this property); this means that the matrix ($G_R = G_M$) which represents the gradient operator for moments is the same as the one ($G_R = G_W$) which represents the same operator for weights. This gives a geometrical meaning to the set of moments M^k, at least for $k = 0, 1$, and allows to extend in a very natural way the tree-cotree techniques to the high order case when the two sets of degrees of freedom are the moments in M^k. To illustrate this fact, we complete the analysis previously done in [2], by involving the isomorphism, and other results that we recall here below.

– The integration by parts (IBP) formula (see, e.g., [5]) reads

$$\int_f \mathrm{Tr}_f \, \mathrm{d}u \wedge \eta = \int_{\partial f} \mathrm{Tr}_{\partial f} \, (\mathrm{Tr}_f u \wedge \eta) + (-1)^{k-1} \int_f \mathrm{Tr}_f u \wedge \mathrm{d}\eta \,, \quad u \in \Lambda^k(T).$$

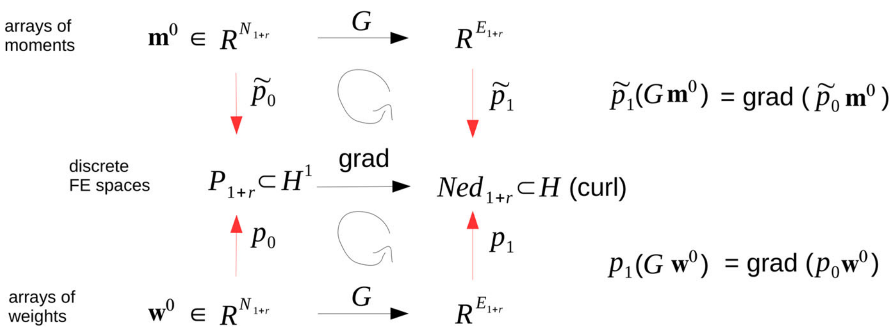

Fig. 3 Schematic graph which gives an insight, for $k = 0, 1$, on the toolkit of mathematical concepts sharpened by Alain Bossavit and that takes part in the foundation of computational methods in applied mathematics. In the scheme, G is the gradient matrix of size $E_{r+1} \times N_{r+1}$, with $E_{r+1} = \dim \mathcal{P}_{r+1}^- \Lambda^1(T)$ and $N_{r+1} = \dim \mathcal{P}_{r+1}^- \Lambda^0(T)$, $\mathbb{R}^q$ is the set of arrays with q real components, $\mathbf{w}^0$ (resp. $\mathbf{m}^0$) is the array of weights (resp. moments) for the 0-form φ, p_k and $\tilde{p}_k$ are FE reconstruction operators, and the cycling symbol stands for commutativity. If we adopt the isomorphism $\mathcal{M}^k$ linking weights to moments, then $G_M = G_W = G$

The boundary term results from the use of Stokes' theorem, stating that $\int_C \mathrm{d}u = \int_{\partial C} u$, where u is a k-form and C a $(k+1)$-chain. In [2] we explained, for any $k \in \{0, \dots, n\}$, the information an IBP formula can provide. We have shown that an IBP formula allows to identify the unknowns for fields in polynomial spaces. Moreover, it gives the way to reconstruct differential operators and potentials, once the unknowns (for fields and potentials) have been fixed. Hence, for $k = 0, 1$, the IBP formula reconstructs the gradient operator.

- The affine function $\tau_{\boldsymbol{\alpha}}$ associated with each multi-index $\boldsymbol{\alpha}$, as described in Section 3.1, is indeed a chain map (see details in [8]), namely, it commutes with the boundary operator, that is $\tau_{\boldsymbol{\alpha}}(\partial f_\sigma) = \partial(\tau_{\boldsymbol{\alpha}}(f_\sigma))$, with $f_\sigma \in \Delta_k(T)$, for any $k > 0$.
- The trace operator commutes with the exterior derivative operator (see an application of this property in [5] to prove Lemma 4.24).
- Recalling that for each $\zeta \in \Sigma(0:d, 0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, \mathrm{s})$ we denote $\lambda_{f_\zeta}^{\boldsymbol{\beta}} = \prod_{i=0}^{d} \lambda_{T,\zeta(i)}^{\beta_i} \in \Lambda^0(T)$, one has

$$\mathrm{d}\lambda_{f_\zeta}^{\boldsymbol{\beta}} = \sum_{j=0}^{d} \beta_j \left(\prod_{i=0}^{d} \lambda_{T,\zeta(i)}^{\beta_i - \delta_{i,j}} \right) \mathrm{d}\lambda_j = \sum_{j=0}^{d} \beta_j \lambda_{f_\zeta}^{\boldsymbol{\beta} - \mathbf{e}_j} \mathrm{d}\lambda_j \in \Lambda^1(T).$$

5.1 Weights: the matrix G_W

If $\varphi \in \Lambda^0(T)$, then $\mathrm{d}\varphi \in \Lambda^1(T)$. Its weights in $\mathcal{P}_{r+1}^{-}\Lambda^1(T)$ are

$$\begin{aligned} W_{\sigma,\boldsymbol{\alpha}}(\mathrm{d}\varphi) &= \int_{\tau_{\boldsymbol{\alpha}}(f_\sigma)} \mathrm{d}\varphi \\ &= \int_{\partial(\tau_{\boldsymbol{\alpha}}(f_\sigma))} \varphi \\ &= \int_{\tau_{\boldsymbol{\alpha}}(\partial f_\sigma)} \varphi \\ &= \int_{\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)})} \varphi - \int_{\tau_{\boldsymbol{\alpha}}(x_{\sigma(0)})} \varphi, \end{aligned}$$

for all $\sigma \in \Sigma(0:1, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$. Note that the weights of a 0-form in $\mathcal{P}_{r+1}^{-}\Lambda^0(T)$ are the values at the points of the principal lattice, so

$$W_{\sigma,\boldsymbol{\alpha}}(\mathrm{d}\varphi) = \varphi(\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)})) - \varphi(\tau_{\boldsymbol{\alpha}}(x_{\sigma(0)})).$$

Let us denote $\sigma_1 \in \Sigma(0:0, 0:n)$ such that $\sigma_1(0) = \sigma(1)$, and $\sigma_0 \in \Sigma(0:0, 0:n)$ such that $\sigma_0(0) = \sigma(0)$. We recall that $\lfloor \boldsymbol{\alpha} \rfloor$ denotes the minimal element of $[\![\boldsymbol{\alpha}]\!]$.

Since $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$, then $\sigma(0) \le \lfloor \boldsymbol{\alpha} \rfloor$ and $\boldsymbol{\alpha} \in \mathcal{I}_{\sigma_0}(n+1, r)$. This means that $\varphi(\tau_{\boldsymbol{\alpha}}(x_{\sigma(0)})) = W_{\sigma_0,\boldsymbol{\alpha}}(\varphi)$, being $W_{\sigma_0,\boldsymbol{\alpha}}$ an element of W^0.

On the other hand, if $\lfloor \boldsymbol{\alpha} \rfloor < \sigma(1)$, then $\boldsymbol{\alpha} \notin \mathcal{I}_{\sigma_1}(n+1, r)$, hence $W_{\sigma_1,\boldsymbol{\alpha}}$ defined as $W_{\sigma_1,\boldsymbol{\alpha}}(\varphi) = \varphi(\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)}))$ is not an element of W^0. However, being $\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)})$ a point in the principal lattice of T of order $r+1$, there exists a couple $(\sigma_*, \boldsymbol{\beta}) \in \Sigma(0:0, 0:n) \times \mathcal{I}_{\sigma_*}(n+1, r)$ such that $\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)}) = \tau_{\boldsymbol{\beta}}(x_{\sigma_*(0)})$ (see Fig. 2). In fact,

$\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)}) = \tau_{\boldsymbol{\beta}}(x_{\lfloor \boldsymbol{\alpha} \rfloor})$ with $\boldsymbol{\beta} = \boldsymbol{\alpha} + \mathbf{e}_{\sigma(1)} - \mathbf{e}_{\lfloor \boldsymbol{\alpha} \rfloor}$ since

$$\lambda_i(\tau_{\boldsymbol{\alpha}}(x_{\sigma(1)})) = \frac{\lambda_i(x_{\sigma(1)}) + \alpha_i}{r+1} = \begin{cases} \frac{1+\alpha_i}{r+1} & \text{if } i = \sigma(1) \\ \frac{\alpha_i}{r+1} & \text{otherwise} \end{cases}$$

and

$$\lambda_i(\tau_{\boldsymbol{\beta}}(x_{\lfloor \boldsymbol{\alpha} \rfloor})) = \frac{\lambda_i(x_{\lfloor \boldsymbol{\alpha} \rfloor}) + \beta_i}{r+1} = \begin{cases} \frac{1+\alpha_i - 1}{r+1} & \text{if } i = \lfloor \boldsymbol{\alpha} \rfloor \\ \frac{\alpha_i + 1}{r+1} & \text{if } i = \sigma(1) \\ \frac{\alpha_i}{r+1} & \text{otherwise.} \end{cases}$$

Moreover, if $\sigma_* \in \Sigma(0:0, 0:n)$ is such that $\sigma_*(0) = \lfloor \boldsymbol{\alpha} \rfloor$, then $\boldsymbol{\beta} \in \mathcal{I}_{\sigma_*}(n+1, r)$ since $\lfloor \boldsymbol{\beta} \rfloor \geq \lfloor \boldsymbol{\alpha} \rfloor$ if $\lfloor \boldsymbol{\alpha} \rfloor < \sigma(1)$. Hence, $W_{\sigma_*, \boldsymbol{\beta}} \in W^0$ and we can write

$$\begin{aligned} W_{\sigma, \boldsymbol{\alpha}}(\mathrm{d}\varphi) &= \varphi(\tau_{\boldsymbol{\alpha} + \mathbf{e}_{\sigma(1)} - \mathbf{e}_{\lfloor \boldsymbol{\alpha} \rfloor}}(x_{\lfloor \boldsymbol{\alpha} \rfloor})) - \varphi(\tau_{\boldsymbol{\alpha}}(x_{\sigma(0)})) \\ &= W_{\sigma_*, \boldsymbol{\beta}}(\varphi) - W_{\sigma_0, \boldsymbol{\alpha}}(\varphi). \end{aligned} \tag{11}$$

Example 11 For the sake of simplicity, in the following, in order to refer to the weight of a particular $\sigma \in \Sigma(0:k, 0:n)$, we will write $W_{\sigma, \boldsymbol{\alpha}}$ instead of $W_{f_\sigma, \boldsymbol{\alpha}}$.

In $\mathcal{P}_4^- \Lambda^k(T)$, we have

$$W_{[x_0, x_3], (0,2,1,0)}(\mathrm{d}\varphi) = \varphi(\tau_{(0,2,1,0)}(x_3)) - \varphi(\tau_{(0,2,1,0)}(x_0)).$$

The multi-index (0, 2, 1, 0) does not belong to $\mathcal{I}_{[x_3]}(4, 3)$. However,

$$\tau_{(0,2,1,0)}(x_3) = \tau_{(0,1,1,1)}(x_1),$$

and the multi-index (0, 1, 1, 1) belongs to $\mathcal{I}_{[x_1]}(4, 3)$. So we have

$$W_{[x_0, x_3], (0,2,1,0)}(\mathrm{d}\varphi) = \varphi(\tau_{(0,1,1,1)}(x_1)) - \varphi(\tau_{(0,2,1,0)}(x_0)),$$

respectively. □

Let us consider the (oriented) graph $\mathcal{M}^G$ with nodes the small vertices (namely, the points of the principal lattice) and arcs the (oriented) small edges corresponding to couples $(\sigma, \boldsymbol{\alpha})$ with $\sigma \in \Sigma(0:1, 0:n)$ and $\boldsymbol{\alpha} \in \mathcal{I}_\sigma(n+1, r)$. Relation (11) states that the all-nodes incidence matrix of $\mathcal{M}^G$ is $G_W^\top$, with G_W the matrix representing the gradient operator at the discrete level using the weights W^k for $k = 0, 1$ (thus extending to $r > 0$ the presentation done in [6], Chap.5, for $r = 0$). For the construction of a global spanning tree of this graph for any $r \geq 0$, see [25]. For the tree-cotree technique intended as a way to impose uniqueness for a vector potential problem formulation, see [23].

5.2 Moments: the matrix G_M

We aim at associating an oriented graph, $\widetilde{\mathcal{M}}^G$, with the gradient operator, when working with moments.

We note that when $\omega \in \Lambda^1(T)$, the moments defined in (9), $M_{\zeta,\rho,\boldsymbol{\beta}}(\omega)$, have $\rho \in \Sigma(0:d-2, 1:d)$. So there exists a unique element of $\{1,\ldots,d\}$ that is not in $[\![\rho]\!]$. Let us set $j_\rho := \{1,\ldots,d\} \setminus [\![\rho]\!]$. Furthermore, if $\boldsymbol{\beta} \in \mathcal{I}(d+1, \mathrm{s})$, we set $\boldsymbol{\beta}! := \beta_0! \cdots \beta_d!$. We use these notations in the following proposition:

Proposition 4 *For $k = 0, 1$, let us consider the following moments for $\mathcal{P}^-_{r+1}\Lambda^k(T)$:*

- *For $d = k$*

$$\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega) := \frac{1}{\boldsymbol{\beta}!} M_{\zeta,\emptyset,\boldsymbol{\beta}}(\omega)$$

  for each $\zeta \in \Sigma(0:d, 0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r)$;
- *For $d > k$*

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\omega) := \begin{cases} \frac{(-1)^{j_\rho - 1}}{\boldsymbol{\beta}!} M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) & \text{if } k = 1 \\ \frac{1}{\boldsymbol{\beta}!} M_{\zeta,\rho,\boldsymbol{\beta}}(\omega) & \text{if } k = 0 \end{cases}$$

  for each $\zeta \in \Sigma(0:d, 0:n)$, $\rho \in \Sigma(0:d-(k+1), 1:d)$, $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))$.

Then, if $\varphi \in \Lambda^0(T)$, each moment of $\mathrm{d}\varphi \in \Lambda^1(T)$ is the difference of two moments of φ.

Proof If $\varphi \in \Lambda^0(T)$, then $\mathrm{d}\varphi \in \Lambda^1(T)$ and its moments for $\mathcal{P}^-_{r+1}\Lambda^1(T)$ are as follows.
• Case $d = k = 1$. For each $\zeta \in \Sigma(0:1, 0:n)$ and $\boldsymbol{\beta} \in \mathcal{I}(2, r)$,

$$\begin{aligned} M_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) &= \textstyle\int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \mathrm{d}\varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}} \\ &= \textstyle\int_{f_\zeta} \mathrm{d}(\mathrm{Tr}_{f_\zeta} \varphi) \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}} \\ &= -\textstyle\int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \varphi \wedge \mathrm{d}\lambda_{f_\zeta}^{\boldsymbol{\beta}} + \int_{\partial f_\zeta} \mathrm{Tr}_{\partial f_\zeta} \varphi \wedge \mathrm{Tr}_{\partial f_\zeta} \lambda_{f_\zeta}^{\boldsymbol{\beta}}. \end{aligned}$$

Since $f_\zeta = [x_{\zeta(0)}, x_{\zeta(1)}]$, then $\partial f_\zeta = [x_{\zeta(1)}] - [x_{\zeta(0)}]$

$$M_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) = -\int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \varphi \wedge \left(\beta_0 \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0} \mathrm{d}\lambda_{\zeta(0)} + \beta_1 \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_1} \mathrm{d}\lambda_{\zeta(1)}\right)$$

$$+(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(1)}) - (\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(0)}).$$

In $[x_{\zeta(0)}, x_{\zeta(1)}]$, we have $\mathrm{d}\lambda_{\zeta(0)} = -\mathrm{d}\lambda_{\zeta(1)}$, so

$$\begin{aligned} M_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) = &\int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \varphi \wedge \beta_0 \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0} \mathrm{d}\lambda_{\zeta(1)} - \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \varphi \wedge \beta_1 \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_1} \mathrm{d}\lambda_{\zeta(1)} \\ &+(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(1)}) - (\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(0)}). \end{aligned}$$

Multiplying $M_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi)$ by $\frac{1}{\boldsymbol{\beta}!}$, we obtain

$$\begin{aligned} \frac{1}{\boldsymbol{\beta}!} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \mathrm{d}\varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}} = &\frac{\beta_0}{\boldsymbol{\beta}!} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0} \mathrm{d}\lambda_{\zeta(1)} - \frac{\beta_1}{\boldsymbol{\beta}!} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta} \varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_1} \mathrm{d}\lambda_{\zeta(1)} \\ &+\frac{1}{\boldsymbol{\beta}!}(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(1)}) - \frac{1}{\boldsymbol{\beta}!}(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(0)}). \end{aligned} \tag{12}$$

We notice that

$$\frac{\beta_j}{\boldsymbol{\beta}!} = \begin{cases} \frac{1}{(\boldsymbol{\beta}-\mathbf{e}_j)!} & \text{if } \beta_j \neq 0 \\ 0 & \text{if } \beta_j = 0\,. \end{cases}$$

Moreover, only two of these four terms are different from zero because if $\beta_0 \neq 0$, then $(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(1)}) = 0$, whereas if $\beta_1 \neq 0$, then $(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(0)}) = 0$.

In conclusion,

– If $\beta_0 \neq 0$ and $\beta_1 \neq 0$, then

$$\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{1}{(\boldsymbol{\beta}-\mathbf{e}_0)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_0}(\varphi) - \frac{1}{(\boldsymbol{\beta}-\mathbf{e}_1)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_1}(\varphi);$$

– If $\beta_0 = 0$, then $\beta_1 = r \neq 0$ and

$$\begin{aligned}\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) &= -\tfrac{1}{(\boldsymbol{\beta}-\mathbf{e}_1)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_1}(\varphi) + \tfrac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta_1,\boldsymbol{\beta}}(\varphi) \\ &= -\tfrac{1}{(\boldsymbol{\beta}-\mathbf{e}_1)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_1}(\varphi) + \tfrac{1}{r!}\varphi(x_{\zeta(1)}),\end{aligned}$$

being $\zeta_1 \in \Sigma(0:0,0:n)$ such that $\zeta_1(0) = \zeta(1)$;

– If $\beta_1 = 0$, then $\beta_0 = r \neq 0$ and

$$\begin{aligned}\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) &= \tfrac{1}{(\boldsymbol{\beta}-\mathbf{e}_0)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_0}(\varphi) - \tfrac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta_0,\boldsymbol{\beta}}(\varphi) \\ &= \tfrac{1}{(\boldsymbol{\beta}-\mathbf{e}_0)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_0}(\varphi) - \tfrac{1}{r!}\varphi(x_{\zeta(0)})\end{aligned}$$

being $\zeta_0 \in \Sigma(0:0,0:n)$ such that $\zeta_0(0) = \zeta(0)$.

• Case $d > k = 1$. For each $\zeta \in \Sigma(0:d,0:n)$, $\rho \in \Sigma(0:d-2,1:d)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-1))$,

$$\begin{aligned}M_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) &= \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\mathrm{d}\varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho = \int_{f_\zeta} \mathrm{d}(\mathrm{Tr}_{f_\zeta}\varphi) \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho \\ &= -\int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge \mathrm{d}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho) + \int_{\partial f_\zeta} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho), \\ &= -\sum_{i=0}^{d} \beta_i \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_i}\mathrm{d}\lambda_{\zeta(i)}) \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho \\ &\quad + \sum_{i=0}^{d}(-1)^i \int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(i)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho)\,.\end{aligned}$$

Taking into account the fact that if $i \in [\![\rho]\!]$, then $\mathrm{Tr}_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(i)}},\ldots,x_{\zeta(d)}]}(\mathrm{d}\lambda_{f_\zeta})_\rho = 0$ and $\mathrm{d}\lambda_{\zeta(i)} \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho = 0$, both sums reduce to $i \in \{0,\ldots,d\} \setminus [\![\rho]\!]$. Recalling that $k = 1$, $\rho \in \Sigma(0:d-2,1:d)$ so both sums reduce, in fact, to two terms $i = 0$ and

$i = \{1, \ldots, d\} \setminus [\![\rho]\!] =: j_\rho$. So we have

$$\begin{aligned} M_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = &-\beta_0 \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0} \mathrm{d}\lambda_{\zeta(0)}) \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho \\ &-\beta_{j_\rho} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_{j_\rho}} \mathrm{d}\lambda_{\zeta(j_\rho)}) \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho \\ &+ \int_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho) \\ &+(-1)^{j_\rho} \int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho). \end{aligned}$$

Taking into account that, in f_ζ, $\mathrm{d}\lambda_{\zeta(0)} = -\sum_{i=1}^{d} \mathrm{d}\lambda_{\zeta(i)}$ and by recalling that

$$(\mathrm{d}\lambda_{f_\zeta})_\rho = \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}$$

it follows that $\mathrm{d}\lambda_{\zeta(0)} \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho = -\mathrm{d}\lambda_{\zeta(j_\rho)} \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho$, so

$$\begin{aligned} M_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = &+\beta_0 \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0} \mathrm{d}\lambda_{\zeta(j_\rho)}) \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho \\ &-\beta_{j_\rho} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_{j_\rho}} \mathrm{d}\lambda_{\zeta(j_\rho)}) \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho \\ &+ \int_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho) \\ &+(-1)^{j_\rho} \int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho). \end{aligned}$$

Moreover,

$$\begin{aligned} \mathrm{d}\lambda_{\zeta(j_\rho)} \wedge (\mathrm{d}\lambda_{f_\zeta})_\rho &= \mathrm{d}\lambda_{\zeta(j_\rho)} \wedge (\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\ &= (-1)^{j_\rho - 1} \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}. \end{aligned}$$

Then,

$$\begin{aligned} M_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = &+(-1)^{j_\rho - 1}\beta_0 \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0} \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\ &+(-1)^{j_\rho}\beta_{j_\rho} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_{j_\rho}} \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\ &+ \int_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho) \\ &+(-1)^{j_\rho} \int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}} (\mathrm{d}\lambda_{f_\zeta})_\rho)\,. \end{aligned}$$

If $j_\rho = 1$, then $(\mathrm{d}\lambda_{f_\zeta})_\rho = \mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}$. If $j_\rho > 1$, we can use that in $[x_{\zeta(1)}, \ldots, x_{\zeta(d)}]$, it holds $\mathrm{d}\lambda_{\zeta(1)} = -\sum_{j=2}^d \mathrm{d}\lambda_{\zeta(j)}$ to write

$$
\begin{aligned}
(\mathrm{d}\lambda_{f_\zeta})_\rho &= \mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)} \\
&= -\mathrm{d}\lambda_{\zeta(j_\rho)} \wedge \mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)} \\
&= -(-1)^{j_\rho} \mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)} \\
&= (-1)^{j_\rho - 1} \mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
M_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = (-1)^{j_\rho-1}\beta_0 \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0}\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\
+(-1)^{j_\rho}\beta_{j_\rho} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_{j_\rho}}\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\
+(-1)^{j_\rho-1} \int_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)})) \\
+(-1)^{j_\rho} \int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)})).
\end{aligned}
$$

Multiplying $M_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi)$ by $\frac{(-1)^{j_\rho-1}}{\boldsymbol{\beta}!}$, we have

$$
\begin{aligned}
\frac{(-1)^{j_\rho-1}}{\boldsymbol{\beta}!} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\mathrm{d}\varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{f_\zeta})_\rho = \frac{\beta_0}{\boldsymbol{\beta}!} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0}\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\
-\frac{\beta_{j_\rho}}{\boldsymbol{\beta}!} \int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_{j_\rho}}\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\
+\frac{1}{\boldsymbol{\beta}!} \int_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)})) \\
-\frac{1}{\boldsymbol{\beta}!} \int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)})).
\end{aligned}
$$

Finally, we notice that if $\beta_0 \neq 0$, then $\mathrm{Tr}_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]}\lambda_{f_\zeta}^{\boldsymbol{\beta}} = 0$ while if $\beta_{j_\rho} \neq 0$, then $\mathrm{Tr}_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]}\lambda_{f_\zeta}^{\boldsymbol{\beta}} = 0$. This means that only two of these four terms are different from zero.

In conclusion,

– If $\beta_0 \neq 0$ and $\beta_{j_\rho} \neq 0$, then

$$
\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{1}{(\boldsymbol{\beta}-\mathbf{e}_0)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_0}(\varphi) - \frac{1}{(\boldsymbol{\beta}-\mathbf{e}_{j_\rho})!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_{j_\rho}}(\varphi);
$$

– If $\beta_0 = 0$ and $\beta_{j_\rho} \neq 0$, then

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = -\frac{1}{(\boldsymbol{\beta} - \mathbf{e}_{j_\rho})!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_{j_\rho}}(\varphi) + \frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta^0,\boldsymbol{\beta}}(\varphi),$$

being $\zeta^0 \in \Sigma(0 : d-1, 0 : n)$ such that $\zeta^0(i) = \zeta(i+1)$;

– If $\beta_{j_\rho} = 0$ and $\beta_0 \neq 0$, then

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{1}{(\boldsymbol{\beta} - \mathbf{e}_0)!}\widehat{M}_{\zeta,\boldsymbol{\beta}-\mathbf{e}_0}(\varphi) - \frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta^{j_\rho},\boldsymbol{\beta}}(\varphi),$$

being $\zeta^{j_\rho} \in \Sigma(0 : d-1, 0 : n)$ such that $\zeta^{j_\rho}(i) = \zeta(i)$ if $i < j_\rho$ and $\zeta^{j_\rho}(i) = \zeta(i+1)$ if $i \geq j_\rho$;

– If $\beta_0 = \beta_{j_\rho} = 0$, then

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta^0,\boldsymbol{\beta}}(\varphi) - \frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta^{j_\rho},\boldsymbol{\beta}}(\varphi).$$

□

Proposition 4 allows to associate an oriented graph, $\widetilde{\mathcal{M}}^G$, with the gradient operator as follows: the set of nodes is the set of moments

$$\widetilde{M}^0 := \{\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}} \in (\mathcal{P}^-_{r+1}\Lambda^0(T))^* \ : \ \zeta \in \Sigma(0 : d, 0 : n),\ \rho \in \Sigma(0 : d-1, 1 : d),$$
$$\boldsymbol{\beta} \in \mathcal{I}(d+1, r-d),\ \text{with } 0 \leq d \leq n\}$$

and the set of arcs is the set of moments

$$\widetilde{M}^1 := \{\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}} \in (\mathcal{P}^-_{r+1}\Lambda^1(T))^* \ : \ \zeta \in \Sigma(0 : d, 0 : n),\ \rho \in \Sigma(0 : d-2, 1 : d),$$
$$\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-1))\ \text{ with } 1 \leq d \leq n\}.$$

The arc corresponding with the moment $\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}} \in \widetilde{M}^1$ goes from the node $\widetilde{M}_{\zeta_I,\rho_I,\boldsymbol{\beta}_I}$ to the node $\widetilde{M}_{\zeta_F,\rho_F,\boldsymbol{\beta}_F}$, both in $\widetilde{M}^0$, if, for any $\varphi \in \Lambda^0(T)$, we have

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \widetilde{M}_{\zeta_F,\rho_F,\boldsymbol{\beta}_F}(\varphi) - \widetilde{M}_{\zeta_I,\rho_I,\boldsymbol{\beta}_I}(\varphi).$$

As a consequence, also in this case, the all-nodes incidence matrix of the graph $\widetilde{\mathcal{M}}^G$ is $G_M^\top$ with G_M the matrix representing the gradient operator at the discrete level when using these moments.

For $k = 0, 1$ let us consider the map g from the sets of moments $\widetilde{M}^k$ to the set of moments M^k defined as $g(\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}) := M_{\zeta,\rho,\boldsymbol{\beta}}$ for $d \in \{k, \ldots, n\}$, $\zeta \in \Sigma(0 : d, 0 : n)$, $\rho = \emptyset$ if $d = k$ or $\rho \in \Sigma(0 : d-(k+1), 1 : d)$ if $d \neq k$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-k))$. Then, $\widetilde{\mathcal{W}}^k = \mathcal{W}^k \circ g$ is an isomorphism from the sets of moments $\widetilde{M}^k$ to the set of weights W^k and

$$\widetilde{\mathcal{W}}^k(\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}) := \mathcal{W}^k(\, g(\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})\,) = \mathcal{W}^k(M_{\zeta,\rho,\boldsymbol{\beta}})\,,$$

for each $\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}} \in \widetilde{M}^k$. The isomorphism $\widetilde{\mathcal{W}}^k$ is illustrated here below:

$$\begin{array}{ccc} M^k & \xrightarrow{\mathcal{W}^k} & W^k \\ {\scriptstyle g}\uparrow & \nearrow{\scriptstyle \widetilde{\mathcal{W}}^k} & \\ \widetilde{M}^k & & \end{array}$$

Proposition 5 *The two graphs $\mathcal{M}^G$ and $\widetilde{\mathcal{M}}^G$ coincide, that is, by considering these moments, the map $\widetilde{\mathcal{W}}^k$ preserves the matrix of the gradient operator in the following sense: for any $\varphi \in \Lambda^0(T)$, if*

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \widetilde{M}_{\zeta_F,\rho_F,\boldsymbol{\beta}_F}(\varphi) - \widetilde{M}_{\zeta_I,\rho_I,\boldsymbol{\beta}_I}(\varphi)$$

then

$$(\widetilde{\mathcal{W}}^1 \widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})(\mathrm{d}\varphi) = (\widetilde{\mathcal{W}}^0 \widetilde{M}_{\zeta_F,\rho_F,\boldsymbol{\beta}_F})(\varphi) - (\widetilde{\mathcal{W}}^0 \widetilde{M}_{\zeta_I,\rho_I,\boldsymbol{\beta}_I})(\varphi).$$

The proof of Proposition 5 is given in the Appendix. Here, we propose to illustrate the claim of Proposition 5 by presenting three rather general examples for $n = 3$ and $r = 3$. In the three cases, we proceed as follows:

- We start from $\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi)$, and by IBP, we obtain the moments $\widetilde{M}_{\zeta_F,\rho_F,\boldsymbol{\beta}_F}$ and $\widetilde{M}_{\zeta_I,\rho_I,\boldsymbol{\beta}_I}$ such that

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \widetilde{M}_{\zeta_F,\rho_F,\boldsymbol{\beta}_F}(\varphi) - \widetilde{M}_{\zeta_I,\rho_I,\boldsymbol{\beta}_I}(\varphi);$$

- We compute $(\widetilde{\mathcal{W}}^1 \widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})(\mathrm{d}\varphi)$, and by Stokes' theorem, we write it as the difference of two weights

$$(\widetilde{\mathcal{W}}^1 \widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})(\mathrm{d}\varphi) = W_{\sigma_F,\boldsymbol{\alpha}_F}(\varphi) - W_{\sigma_I,\boldsymbol{\alpha}_I}(\varphi);$$

- Finally, we check that, for $L = I, F$, we have

$$\widetilde{\mathcal{W}}^0 \widetilde{M}_{\zeta_L,\rho_L,\boldsymbol{\beta}_L}(\varphi) = W_{\sigma_L,\boldsymbol{\alpha}_L}(\varphi).$$

Example 12 ($d = 1$) Let us consider $\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi)$ with $\zeta \in \Sigma(0:1, 0:3)$, $\zeta(0) = 1$, $\zeta(1) = 3$ and $\boldsymbol{\beta} = (1, 2)$.

- $\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{1}{1!\,2!} M_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{1}{2}\int_{[x_1,x_3]} \mathrm{d}\varphi\,(\lambda_1\lambda_3^2)$ by IBP, and using that on the edge $[x_1, x_3]$ one has $\mathrm{d}\lambda_1 = -\mathrm{d}\lambda_3$, we get

$$\begin{aligned} \frac{1}{2}\int_{[x_1,x_3]} \mathrm{d}\varphi\,(\lambda_1\lambda_3^2) &= \frac{1}{2}\Big(-\int_{[x_1,x_3]} \varphi\,\mathrm{d}(\lambda_1\lambda_3^2) + (\varphi\,\lambda_1\lambda_3^2)(x_3) - (\varphi\,\lambda_1\lambda_3^2)(x_1)\Big) \\ &= -\frac{1}{2}\int_{[x_1,x_3]} \varphi\,\mathrm{d}(\lambda_1\lambda_3^2) \\ &= -\frac{1}{2}\int_{[x_1,x_3]} \varphi\,\lambda_3^2(\mathrm{d}\lambda_1) - \int_{[x_1,x_3]} \varphi\,\lambda_1\lambda_3(\mathrm{d}\lambda_3) \\ &= \frac{1}{2}\int_{[x_1,x_3]} \varphi\,\lambda_3^2(\mathrm{d}\lambda_3) - \int_{[x_1,x_3]} \varphi\,\lambda_1\lambda_3(\mathrm{d}\lambda_3) \\ &= \frac{1}{2} M_{\zeta,\rho,(0,2)}(\varphi) - M_{\zeta,\rho,(1,1)}(\varphi) \\ &= \widetilde{M}_{\zeta,\rho,(0,2)}(\varphi) - \widetilde{M}_{\zeta,\rho,(1,1)}(\varphi), \end{aligned}$$

being ρ the unique element of $\Sigma(0:0,1:1)$.

– The corresponding weight is

$$(\widetilde{\mathcal{W}}^1 \widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}})(\mathrm{d}\varphi) = (\mathcal{W}^1 M_{\zeta,\emptyset,\boldsymbol{\beta}})(\mathrm{d}\varphi) = \int_{\tau_{(0,1,0,2)}([x_1,x_3])} \mathrm{d}\varphi.$$

Since the boundary of $\tau_{(0,1,0,2)}([x_1, x_3])$ is

$$\partial\tau_{(0,1,0,2)}([x_1, x_3]) = \tau_{(0,1,0,2)}(x_3) - \tau_{(0,1,0,2)}(x_1),$$

by Stokes' theorem, we have

$$\begin{aligned}\textstyle\int_{\tau_{(0,1,0,2)}([x_1,x_3])} \mathrm{d}\varphi &= \varphi(\tau_{(0,1,0,2)}(x_3)) - \varphi(\tau_{(0,1,0,2)}(x_1)) \\ &= \varphi(\tau_{(0,0,0,3)}(x_1)) - \varphi(\tau_{(0,1,0,2)}(x_1)) \\ &= W_{[x_1],(0,0,0,3)}(\varphi) - W_{[x_1](0,1,0,2)}(\varphi).\end{aligned}$$

– Finally, we observe that

$$\widetilde{\mathcal{W}}^0 \widetilde{M}_{\zeta,\rho,(0,2)}(\varphi) = \mathcal{W}^0 \left(\int_{[x_1,x_3]} \varphi\, \lambda_3^2(\mathrm{d}\lambda_3) \right) = W_{[x_1],(0,0,0,3)}(\varphi)$$

and

$$\widetilde{\mathcal{W}}^0 \widetilde{M}_{\zeta,\rho,(1,1)}(\varphi) = \mathcal{W}^0 \left(\int_{[x_1,x_3]} \varphi\, \lambda_1\lambda_3(\mathrm{d}\lambda_3) \right) = W_{[x_1],(0,1,0,2)}(\varphi),$$

respectively. ◇

Example 13 ($d = 2$) Let us consider the moment $\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi)$ with $\zeta \in \Sigma(0:2,0:3)$, $\zeta(0) = 0$, $\zeta(1) = 1$, $\zeta(2) = 3$, $\rho \in \Sigma(0:0,1:2)$, $\rho(0) = 2$, and $\boldsymbol{\beta} = (0,1,1)$.

– In this example, $j_\rho = 1$ and $\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{(-1)^{1-1}}{1!\,1!} \int_{[x_0,x_1,x_3]} \mathrm{d}\varphi \wedge \lambda_1\lambda_3(\mathrm{d}\lambda_3)$. By IBP, we obtain

$$\begin{aligned}&\textstyle\int_{[x_0,x_1,x_3]} \mathrm{d}\varphi \wedge \lambda_1\lambda_3(\mathrm{d}\lambda_3) \\ &\qquad = -\textstyle\int_{[x_0,x_1,x_3]} \varphi\, \mathrm{d}(\lambda_1\lambda_3\mathrm{d}\lambda_3) + \int_{\partial[x_0,x_1,x_3]} \varphi \wedge \lambda_1\lambda_3(\mathrm{d}\lambda_3) \\ &\qquad = -\textstyle\int_{[x_0,x_1,x_3]} \varphi\, \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3) + \int_{[x_1,x_3]} \varphi \wedge \lambda_1\lambda_3(\mathrm{d}\lambda_3)\end{aligned}$$

since $\mathrm{d}(\lambda_1\lambda_3\mathrm{d}\lambda_3) = \lambda_3\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3$, $\lambda_1 = 0$ in $[x_0, x_3]$ and $\lambda_3 = 0$ in $[x_0, x_1]$.

– The corresponding weight is

$$(\widetilde{\mathcal{W}}^1 \widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})(\mathrm{d}\varphi) = \int_{\tau_{(0,1,0,2)}([x_0,x_1])} \mathrm{d}\varphi.$$

Since the boundary of $\tau_{(0,1,0,2)}([x_0, x_1])$ is

$$\partial\tau_{(0,1,0,2)}([x_0, x_1]) = \tau_{(0,1,0,2)}(x_1) - \tau_{(0,1,0,2)}(x_0),$$

by Stokes' theorem, we have

$$\int_{\tau_{(0,1,0,2)}([x_0,x_1])} \mathrm{d}\varphi = \varphi(\tau_{(0,1,0,2)}(x_1)) - \varphi(\tau_{(0,1,0,2)}(x_0)).$$

– Finally, we observe that

$$\mathcal{W}^0\left(\int_{[x_0,x_1,x_3]} \varphi\,\lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3)\right) = \mathcal{W}^0(\widehat{M}_{\zeta,(0,0,1)}(\varphi)) = \varphi(\tau_{(0,1,0,2)}(x_0))$$

and, setting $\xi \in \Sigma(0:1, 0:3)$, $\xi(0) = 1$, $\xi(1) = 3$

$$\mathcal{W}^0\left(\int_{[x_1,x_3]} \varphi \wedge \lambda_1\lambda_3(\mathrm{d}\lambda_3)\right) = \mathcal{W}^0(\widehat{M}_{\xi,(1,1)}(\varphi)) = \varphi(\tau_{(0,1,0,2)}(x_1)),$$

respectively. (Note that, since $k = 0$, the moments do not depend on ρ.) □

Example 14 ($d = 3$) Let us consider the moment $\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi)$ with $\zeta \in \Sigma(0:3, 0:3)$, $\rho \in \Sigma(0:1, 1:3)$, $\rho(0) = 1$, $\rho(1) = 3$ and $\boldsymbol{\beta} = (0, 0, 0, 1)$.

– In this example $j_\rho = 2$ and

$$\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{(-1)^{2-1}}{1!}\int_{[x_0,x_1,x_2,x_3]} \mathrm{d}\varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3).$$

By IBP, we have

$$\begin{aligned}
&-\textstyle\int_{[x_0,x_1,x_2,x_3]} \mathrm{d}\varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3)\\
&\quad= \textstyle\int_{[x_0,x_1,x_2,x_3]} \varphi \wedge \mathrm{d}(\lambda_3\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3) - \int_{\partial[x_0,x_1,x_2,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3)\\
&\quad= -\textstyle\int_{[x_1,x_2,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3) - \int_{[x_0,x_1,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3)\\
&\quad= \textstyle\int_{[x_1,x_2,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3) - \int_{[x_0,x_1,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3)
\end{aligned}$$

since $\mathrm{d}(\lambda_3\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3) = 0$, $\lambda_3\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3 = 0$ on $[x_0, x_2, x_3]$ and $[x_0, x_1, x_2]$, and $\mathrm{d}\lambda_1 = -\mathrm{d}\lambda_2 - \mathrm{d}\lambda_3$ on $[x_1, x_2, x_3]$.

– The corresponding weight is

$$(\widetilde{\mathcal{W}}^1\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})(\mathrm{d}\varphi) = \int_{\tau_{(0,1,0,2)}([x_0,x_2])} \mathrm{d}\varphi.$$

Since the boundary of $\tau_{(0,1,0,2)}([x_0, x_2])$ is

$$\partial\tau_{(0,1,0,2)}([x_0, x_2]) = \tau_{(0,1,0,2)}(x_2) - \tau_{(0,1,0,2)}(x_0),$$

by Stokes' theorem, we have

$$\begin{aligned}\int_{\tau_{(0,1,0,2)}([x_0,x_2])} \mathrm{d}\varphi &= \varphi(\tau_{(0,1,0,2)}(x_2)) - \varphi(\tau_{(0,1,0,2)}(x_0)) \\ &= \varphi(\tau_{(0,0,1,2)}(x_1)) - \varphi(\tau_{(0,1,0,2)}(x_0)),\end{aligned}$$

where we have used that $\tau_{(0,1,0,2)}(x_2) = \tau_{(0,0,1,2)}(x_1)$. We prefer this second form because $\boldsymbol{\alpha} = (0, 1, 0, 2) \notin \mathcal{I}_{[x_2]}(4, 3)$ since $\lfloor \boldsymbol{\alpha} \rfloor = 1 < 2$ while $\boldsymbol{\alpha}' = (0, 0, 1, 2) \in \mathcal{I}_{[x_1]}(4, 3)$ since $\lfloor \boldsymbol{\alpha}' \rfloor = 2 \geq 1$.

– Finally, we observe that

$$\mathcal{W}^0\left(\int_{[x_1,x_2,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_2 \wedge \mathrm{d}\lambda_3)\right) = \varphi(\tau_{(0,0,1,2)}(x_1))$$

and

$$\mathcal{W}^0\left(\int_{[x_0,x_1,x_3]} \varphi \wedge \lambda_3(\mathrm{d}\lambda_1 \wedge \mathrm{d}\lambda_3)\right) = \varphi(\tau_{(0,1,0,2)}(x_0)),$$

respectively. □

6 Conclusions

The spaces $\mathcal{P}_1^- \Lambda^k(T)$ present a high geometrical feature, having one degree of freedom per k-simplex in T, and thus being isomorphic to the space of simplicial k-cochains. These spaces were indeed introduced in 1957 by Whitney in his book [26]. For $r = 1$, the connection of Whitney's spaces with mixed finite elements, appeared in the late 70 s, was developed by Bossavit in the 80 s. With the time passing, in [22], we were able to generalize this connection to $r > 1$ and to introduce new DoFs for $\mathcal{P}_{r+1}^- \Lambda^k(T)$, the weights on the small k-simplices of T.

In this contribution, we have made a step forward, namely we have constructed isomorphisms $\mathcal{W}^k$, for any value of $k \in \{0, \dots, n\}$, between moments and weights for fields in the discrete spaces $\mathcal{P}_{r+1}^- \Lambda^k(T)$, for $r \geq 0$. Furthermore, we have shown that, with a suitable definition of moments, the newly introduced isomorphisms $\widetilde{\mathcal{W}}^k$ preserve, for example, the gradient matrix G (i.e., the matrix G has fixed entries $G_{i,j}$, whatever type of DoFs, weights or moments, are used in $\mathcal{P}_{r+1}^- \Lambda^k(T)$). We can thus transfer, for any $r \geq 0$, all achievements on tree-cotree construction for weights in, e.g, [6, 23, 25] (see also references therein) to the case of moments (see, e.g., [18, 19]). The construction of the isomorphism between weights and moments is compatible (see Fig. 3, for $k = 0, 1$) with the powerful and general toolkit sharpened by Alain Bossavit all along his career.

Appendix

We report here the proof of Proposition 5.

Proof Case $d = k = 1$, namely, $\zeta \in \Sigma(0 : 1, 0 : n)$, $\rho = \emptyset$ and $\boldsymbol{\beta} \in \mathcal{I}(2, r)$.

From Proposition 4,

$$\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}}(\mathrm{d}\varphi) = \frac{\beta_0}{\boldsymbol{\beta}!}\int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0}\mathrm{d}\lambda_{\zeta(1)}$$

$$-\frac{\beta_1}{\boldsymbol{\beta}!}\int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge \lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_1}\mathrm{d}\lambda_{\zeta(1)} + \frac{1}{\boldsymbol{\beta}!}(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(1)}) - \frac{1}{\boldsymbol{\beta}!}(\varphi\lambda_{f_\zeta}^{\boldsymbol{\beta}})(x_{\zeta(0)}).$$

From Definition 5,

– if $\beta_0 \neq 0$,

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{(\boldsymbol{\beta}-\mathbf{e}_0)!}\widehat{M}_{\zeta\,,\,\boldsymbol{\beta}-\mathbf{e}_0}\right) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+\mathbf{e}_1)\mathrm{E}_\zeta};$$

– if $\beta_1 \neq 0$,

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{(\boldsymbol{\beta}-\mathbf{e}_1)!}\widehat{M}_{\zeta\,,\,\boldsymbol{\beta}-\mathbf{e}_1}\right) = W_{\zeta_0,\boldsymbol{\beta}\mathrm{E}_\zeta};$$

– if $\beta_0 = 0$, then $\beta_1 = r \neq 0$ and

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta_1,\boldsymbol{\beta}}\right) = W_{\zeta_1,r\mathrm{E}_{\zeta_1}};$$

– if $\beta_1 = 0$, then $\beta_0 = r \neq 0$ and

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta_0,\boldsymbol{\beta}}\right) = W_{\zeta_0,r\mathrm{E}_{\zeta_0}}.$$

From Definition 5,

$$(\widetilde{\mathcal{W}}^1\widetilde{M}_{\zeta,\emptyset,\boldsymbol{\beta}})(\mathrm{d}\varphi) = W_{\zeta,\boldsymbol{\beta}\mathrm{E}_\zeta}(\mathrm{d}\varphi)$$

From (11),

$$W_{\zeta,\boldsymbol{\beta}\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_*,\boldsymbol{\beta}_*}(\varphi) - W_{\zeta_0,\boldsymbol{\beta}\mathrm{E}_\zeta}(\varphi)$$

being $\zeta_* \in \Sigma(0:0,0:n)$ with $\zeta_*(0) = \lfloor \boldsymbol{\beta}\mathrm{E}_\zeta \rfloor$ and $\boldsymbol{\beta}_* = \boldsymbol{\beta}\mathrm{E}_\zeta + \mathbf{e}_{\zeta(1)} - \mathbf{e}_{\lfloor \boldsymbol{\beta} text E_\zeta \rfloor}$.

In this case,

$$\lfloor \boldsymbol{\beta}\mathrm{E}_\zeta \rfloor = \begin{cases} \zeta(0) \text{ if} \beta_0 \neq 0 \\ \zeta(1) \text{ if } \beta_0 = 0. \end{cases}$$

Hence,

$$W_{\zeta,\boldsymbol{\beta}\mathrm{E}_\zeta}(\mathrm{d}\varphi) = \begin{cases} W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+\mathbf{e}_1)\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,\boldsymbol{\beta}\mathrm{E}_\zeta}(\varphi) & \text{if } \beta_0 \neq 0 \\ W_{\zeta_1,\boldsymbol{\beta}\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,\boldsymbol{\beta}\mathrm{E}_\zeta}(\varphi) & \text{if } \beta_0 = 0, \end{cases}$$

since $\boldsymbol{\beta}\mathrm{E}_\zeta - \mathbf{e}_{\zeta(0)} + \mathbf{e}_{\zeta(1)} = (\boldsymbol{\beta} - \mathbf{e}_0 + \mathbf{e}_1)\mathrm{E}_\zeta$. Notice also that if $\beta_0 = 0$, then $\boldsymbol{\beta}\mathrm{E}_\zeta = r\mathrm{E}_{\zeta_1}$. On the other hand, if $\beta_1 = 0$, then $\beta_0 = r$ and $\boldsymbol{\beta}\mathrm{E}_\zeta = r\mathrm{E}_{\zeta_0}$.

In conclusion,

– if $\beta_0 \neq 0$ and $\beta_1 \neq 0$,

$$W_{\zeta,\boldsymbol{\beta}\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+\mathbf{e}_1)\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,\boldsymbol{\beta}\mathrm{E}_\zeta}(\varphi);$$

– if $\beta_0 = 0$, then $\beta_1 = r \neq 0$ and

$$W_{\zeta,\boldsymbol{\beta}\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_1,r\mathrm{E}_{\zeta_1}}(\varphi) - W_{\zeta_0,\boldsymbol{\beta}\mathrm{E}_\zeta}(\varphi)$$

– if $\beta_1 = 0$, then $\beta_0 = r \neq 0$ and

$$W_{\zeta,\boldsymbol{\beta}\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+\mathbf{e}_1)\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,r\mathrm{E}_{\zeta_0}}(\varphi).$$

Case $d > k = 1$, namely, $\zeta \in \Sigma(0:d, 0:n)$, $\rho \in \Sigma(0:d-2, 1:d)$ and $\boldsymbol{\beta} \in \mathcal{I}(d+1, r-(d-1))$.

$$\begin{aligned}
\widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}}(\mathrm{d}\varphi) \quad &= \frac{\beta_0}{\boldsymbol{\beta}!}\int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_0}\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\
&- \frac{\beta_{j_\rho}}{\boldsymbol{\beta}!}\int_{f_\zeta} \mathrm{Tr}_{f_\zeta}\varphi \wedge (\lambda_{f_\zeta}^{\boldsymbol{\beta}-\mathbf{e}_{j_\rho}}\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)}) \\
&+ \frac{1}{\boldsymbol{\beta}!}\int_{[x_{\zeta(1)},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(2)} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)})) \\
&- \frac{1}{\boldsymbol{\beta}!}\int_{[x_{\zeta(0)},\ldots,\widehat{x_{\zeta(j_\rho)}},\ldots,x_{\zeta(d)}]} \mathrm{Tr}_{\partial f_\zeta}\varphi \wedge \mathrm{Tr}_{\partial f_\zeta}(\lambda_{f_\zeta}^{\boldsymbol{\beta}}(\mathrm{d}\lambda_{\zeta(1)} \wedge \cdots \wedge \widehat{\mathrm{d}\lambda_{\zeta(j_\rho)}} \wedge \cdots \wedge \mathrm{d}\lambda_{\zeta(d)})).
\end{aligned}$$

From Definition 5:,

– if $\beta_0 \neq 0$,

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{(\boldsymbol{\beta}-\mathbf{e}_0)!}\widehat{M}_{\zeta\,,\,\boldsymbol{\beta}-\mathbf{e}_0}\right) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+(0,1,\ldots,1))E_\zeta};$$

– if $\beta_{j_\rho} \neq 0$,

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{(\boldsymbol{\beta}-\mathbf{e}_{j_\rho})!}\widehat{M}_{\zeta\,,\,\boldsymbol{\beta}-\mathbf{e}_{j_\rho}}\right) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_{j_\rho}+(0,1,\ldots,1))E_\zeta};$$

(notice that in this case, if we denote $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta} - \mathbf{e}_{j_\rho} + (0, 1, \ldots, 1)$, one has $\widetilde{\beta}_{j_\rho} \neq 0$);

– if $\beta_0 = 0$,

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta^0\,,\,\boldsymbol{\beta}}\right) = W_{\zeta_1,(\boldsymbol{\beta}-\mathbf{e}_1+(0,1,\ldots,1))\mathrm{E}_\zeta};$$

– if $\beta_{j_\rho} = 0$,

$$\widetilde{\mathcal{W}}^0\left(\frac{1}{\boldsymbol{\beta}!}\widehat{M}_{\zeta^{j_\rho}\,,\,\boldsymbol{\beta}}\right) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_{j_\rho}+(0,1,\ldots,1))\mathrm{E}_\zeta};$$

(notice that in this case, if we denote $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta} - \mathbf{e}_{j_\rho} + (0, 1, \ldots, 1)$, one has $\widetilde{\beta}_{j_\rho} = 0$).

(ζ^0 and ζ^{j_ρ} are the elements of $\Sigma(0:d-1, 0:n)$ defined in the conclusion of the proof of Proposition 4.)

From Definition 5,

$$(\widetilde{\mathcal{W}}^1 \widetilde{M}_{\zeta,\rho,\boldsymbol{\beta}})(\mathrm{d}\varphi) = W_{\zeta\circ\rho^*,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\mathrm{d}\varphi)\,.$$

Since $k=1$, $f_{\zeta\circ\rho^*} = [x_{\zeta(0)}, x_{\zeta(j_\rho)}]$. More precisely, $\eta := \zeta\circ\rho^* \in \Sigma(0:1, 0:n)$, $\eta(0) = \zeta(0)$ and $\eta(1) = \zeta(j_\rho)$. From Stokes' theorem,

$$W_{\zeta\circ\rho^*,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_{j_\rho},(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\varphi)$$

and

$$W_{\zeta_{j_\rho},(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\varphi) = \begin{cases} W_{\zeta_0,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]}+\mathbf{e}_{j_\rho}-\mathbf{e}_0)\mathrm{E}_\zeta}(\varphi) \text{ if } \beta_0 \neq 0 \\ W_{\zeta_1,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]}+\mathbf{e}_{j_\rho}-\mathbf{e}_1)\mathrm{E}_\zeta}(\varphi) \text{ if } \beta_0 = 0. \end{cases}$$

Since $\mathbf{e}_{[\![\rho]\!]} = (0,1,\ldots,1) - \mathbf{e}_{j_\rho}$, hence

$$W_{\zeta_{j_\rho},(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\varphi) = \begin{cases} W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+(0,1,\ldots,1))\mathrm{E}_\zeta}(\varphi) \text{ if } \beta_0 \neq 0 \\ W_{\zeta_1,(\boldsymbol{\beta}-\mathbf{e}_1+(0,1,\ldots,1))\mathrm{E}_\zeta}(\varphi) \text{ if } \beta_0 = 0. \end{cases}$$

In conclusion,

- if $\beta_0 \neq 0$, then

$$W_{\zeta\circ\rho^*,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_0+(0,1,\ldots,1))\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_{j_\rho}+(0,1,\ldots,1))\mathrm{E}_\zeta}(\varphi);$$

- if $\beta_0 = 0$, then

$$W_{\zeta\circ\rho^*,(\boldsymbol{\beta}+\mathbf{e}_{[\![\rho]\!]})\mathrm{E}_\zeta}(\mathrm{d}\varphi) = W_{\zeta_1,(\boldsymbol{\beta}-\mathbf{e}_1+(0,1,\ldots,1))\mathrm{E}_\zeta}(\varphi) - W_{\zeta_0,(\boldsymbol{\beta}-\mathbf{e}_{j_\rho}+(0,1,\ldots,1))\mathrm{E}_\zeta}(\varphi);$$

and this ends the proof. □

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Declarations

Conflict of Interest The authors declare no competing interests.

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