Measurement 259 (2026) 119637

Contents lists available at ScienceDirect

Measurement

journal homepage: www.elsevier.com/locate/measurement

Analysis of the Cramér-Rao Lower Bounds for unbiased parameter estimators of noisy damped sinusoids

Dario Petri [a], Daniel Belega [b,*]

[a] Department of Industrial Engineering, University of Trento, Via Sommarive 9, Povo, Trento 38123, Italy

[b] Department of Measurements and Optical Electronics, Politehnica University Timişoara, Bv. V. Pârvan, Nr. 2, 300223 Timişoara, Romania

A R T I C L E I N F O

Keywords:
Cramér-Rao Lower Bound
Damped sinusoids
Statistical analysis
Unbiased estimators

A B S T R A C T

In this paper the Cramér-Rao Lower Bounds (CRLBs) for unbiased estimators of the parameters of complex-valued noisy damped sinusoid are comprehensively analyzed assuming that enough samples are acquired in an interval duration equal to the signal time constant. At first accurate, but simple, expressions for the CRLBs are derived. Then, even simpler expressions are obtained assuming that the observation duration is either enough smaller or enough greater than the signal time constant. Leveraging on the derived expression, constraints on the number of analyzed samples that allow the optimization of the tradeoff between estimation accuracy, estimation delay, and processing effort are derived. They can be useful to optimize measurement procedures aimed at estimating unknown signal parameters. The accuracies of the derived results are verified through computer calculations and the application of a state-of-the-art almost unbiased and statistically efficient algorithm.

1. Introduction

In many applications such as nuclear magnetic resonance [1–3], optics [4,5], atomic magnetometers [6–8], power systems [9], the parameters (frequency, decay rate, amplitude, and phase) of noisy damped sinusoids need to be accurately estimated. To this aim time-domain or frequency-domain algorithms have been proposed in the literature. Time-domain algorithms, such as the Prony, the Steiglitz-McBride (iterative Prony) [3], the Matrix-Pencil [3], and the signal-fit algorithms [10,11] may provide very accurate estimates, but they require to know the number of signal components and the processing effort is high due to the involved matrix operations [3]. Frequency-domain algorithms, such as those based on the interpolated Discrete-Time Fourier Transform (IpDTFT) [12–20], require a significantly smaller processing effort and are robust with respect to the lack of knowledge about the number of signal components.

Theoretically optimum accuracy is ensured by unbiased and statistically efficient estimators, that is estimators with expectation equal to the true value of the unknown parameter and variance attaining the so called Cramér-Rao Lower Bound (CRLB) [21]. Imposing somewhat restrictive constraints on the signal parameters, analytical expressions for the CRLBs of any unbiased parameters estimators of real-valued multi-tone damped sinusoid embedded in white Gaussian noise have been derived in [22]. Moreover, leveraging these results, analytical expressions for the worst and best case CRLBs for the parameters of a low-frequency single-tone noisy damped sinusoid have been proposed in [23] under the assumption of low normalized decay rate and high number of analyzed samples. Exact analytical expressions for the CRLBs of complex-valued noisy multi-tone damped sinusoid parameters have been derived in [24]. Specifically, it has been shown that the CRLBs for real-valued and complex-valued single-tone parameters are very close to each other when both the number of analyzed samples, and the frequency separation between the image and the fundamental modes are enough high. However, the formulas derived in [24] are quite complicated so that their behavior with respect to the involved variables is not clear. In order to analyze that behavior, simple fitting expressions have been proposed in [25], but their accuracy is quite limited. This paper is aimed to provide accurate and simple expressions for the CRLB formulas derived in [24] in the case of complex-valued noisy single-tone signals under assumptions that can be often satisfied in practice. In particular, their behavior is analyzed with respect to the observation interval duration. The derived expressions allow us to derive constraints on the number of analyzed samples that ensure almost optimal accuracy with good computational and delay performances. The effectiveness of the proposed CRLB analysis is verified by applying the Matrix Pencil (MP) algorithm [26–28], which is capable to provide almost unbiased and

* Corresponding author.
E-mail addresses: dario.petri@unitn.it (D. Petri), daniel.belega@upt.ro (D. Belega).

https://doi.org/10.1016/j.measurement.2025.119637
Received 17 July 2025; Received in revised form 15 October 2025; Accepted 5 November 2025
Available online 8 November 2025

statistically efficient estimators.

The remaining of the paper is organized as follows. In Section 2 accurate, but simple, expressions for the CRLBs of any unbiased complex-valued single-tone damped sinusoid parameter estimators are derived under the constraint that the sampling rate is enough higher than the signal time constant. Assuming a fixed sampling rate, the behavior of the derived expressions versus the observation interval duration or, equivalently, versus the number of analyzed samples is analyzed in Section 3. In Section 4 the accuracies of the proposed CRLB expressions are analyzed through computer calculations and compared with the approximations given in [25]. The effectiveness of the proposed CRLB analysis is verified by applying of the MP algorithm to simulated data in Section 5. Finally, some conclusions are reported in Section 6.

2. Simple analytical expressions for the CRLBs

The considered complex-valued continuous-time noisy damped sinusoid is modeled as:

$$y(t) = x(t) + e(t) = Ae^{-\frac{t}{\tau}}e^{j(2\pi ft+\phi)} + e(t), \tag{1}$$

where A ($A > 0$), τ, f, and ϕ ($0 \le \phi < 2\pi$ rad) are the amplitude, the time constant, the frequency, and the initial phase of the damped sinusoid $x(\bullet)$, respectively, while $e(\cdot)$ is a white Gaussian noise with zero mean and variance σ_n^2. The Signal-to-Noise Ratio (SNR) of the signal (1) is defined as $\mathrm{SNR} \triangleq \frac{A^2}{\sigma_n^2}$.

Let's assume that M samples of signal (1) are acquired with a sampling rate f_s, that is:

$$y(m) = x(m) + e(m) = Ae^{-\frac{m}{\tau f_s}}e^{j\left(2\pi\frac{f}{f_s}m+\phi\right)} + e(m) = Ae^{-\beta m}e^{j(2\pi f_0 m+\phi)} + e(m)$$
$$= Ae^{-\frac{\beta_M}{M}m}e^{j\left(2\pi\frac{\nu}{M}m+\phi\right)} + e(m), m = 0,1,2,\cdots,M-1 \tag{2}$$

in which the following parameters have been defined:

– normalized frequency f_0:

$$f_0 \triangleq \frac{f}{f_s} = \frac{\nu}{M}, \quad 0 \le f_0 < 0.5, \tag{3}$$

where ν is real-valued and it represents the number of analyzed signal cycles;

– normalized decay rate β:

$$\beta \triangleq \frac{1/\tau}{f_s} = \frac{\beta_M}{M}, \tag{4}$$

which represents the inverse of the number of samples acquired in time interval of length equal to the signal time constant. In (4) $\beta_M \triangleq M/(f_s\tau) = M\beta$ represents the number of time constants that falls in the observation interval, whose duration is M/f_s.

The analytical expressions of the CRLBs for any unbiased estimators of the parameters ν, β, A, and ϕ are [24]:

$$\left(\sigma_{\hat{\nu}}^2\right)_{CR} = M^2\left(\sigma_{\hat{f}_0}^2\right)_{CR} = \frac{M^2}{4\pi^2}\left(\sigma_{\hat{\beta}}^2\right)_{CR}$$
$$= \frac{M^2}{4\pi^2}\bullet\frac{(1-e^{-2\beta_M})(1-e^{-2\beta})^3}{-M^2e^{-2\beta_M}(1-e^{-2\beta})^2 + e^{-2\beta}(1-e^{-2\beta_M})^2}\bullet\frac{1}{2\bullet\mathrm{SNR}}, \tag{5}$$

and

$$\frac{\left(\sigma_{\hat{A}}^2\right)_{CR}}{A^2} = \left(\sigma_{\hat{\phi}}^2\right)_{CR} = \left[\frac{-M^2e^{-2\beta_M}(1-e^{-2\beta})^3 + (1-e^{-2\beta_M}-2Me^{-2\beta_M})(1-e^{-2\beta})^2e^{-2\beta}}{-M^2e^{-2\beta_M}(1-e^{-2\beta})^2 + e^{-2\beta}(1-e^{-2\beta_M})^2} + \frac{2e^{-4\beta}(1-e^{-2\beta})(1-e^{-2\beta_M})}{-M^2e^{-2\beta_M}(1-e^{-2\beta})^2 + e^{-2\beta}(1-e^{-2\beta_M})^2}\right]\bullet\frac{1}{2\bullet\mathrm{SNR}}. \tag{6}$$

In many engineering applications the sampling period is much smaller than the signal time constant, that is $\beta \ll 1$. Under that assumption, (4) implies that the number of signal time constants that falls in the observation interval is sufficiently smaller than the number of analyzed samples (i.e., $\beta_M \ll M$) and (5) and (6) can be approximated with high accuracy as follows:

Proposition 1. *When $\beta \ll 1$, the CRLBs for any unbiased estimator of noisy damped sinusoid parameters can be expressed as:*

$$\left(\sigma_{\hat{\nu}}^2\right)_{CR} = M^2\left(\sigma_{\hat{f}_0}^2\right)_{CR} = \frac{M^2}{4\pi^2}\left(\sigma_{\hat{\beta}}^2\right)_{CR}$$
$$\cong \frac{1}{\pi^2}\beta(1-\beta)\frac{\beta_M^2(1-e^{-2\beta_M})}{(1-e^{-2\beta_M})^2 - 4\beta_M^2e^{-2\beta_M}}\bullet\frac{1}{\mathrm{SNR}}, \tag{7}$$

and

$$\frac{\left(\sigma_{\hat{A}}^2\right)_{CR}}{A^2} = \left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong 2\beta(1+2\beta)$$

$$\frac{\left(\sigma_{\hat{A}}^2\right)_{CR}}{A^2} = \left(\sigma_{\hat{\phi}}^2\right)_{CR}$$
$$\cong 2\beta(1+2\beta)\frac{1-4\beta-\left[\left(2\beta_M^2(1-3\beta)+2\beta_M(1-4\beta)+1-4\beta\right)\right]e^{-2\beta_M}}{(1-e^{-2\beta_M})^2-4\beta_M^2e^{-2\beta_M}}$$
$$\bullet\frac{1}{\mathrm{SNR}}. \tag{8}$$

The proof of this Proposition is given in the Appendix A.

Moreover, when the duration of the observation interval is much smaller than the signal time constant, that is $\beta_M \ll 1$, (7) and (8) can be further simplified as specified in:

Proposition 2. *When both $\beta \ll 1$ and $\beta_M \ll 1$, the CRLBs for any unbiased estimator of noisy damped sinusoid parameters can be expressed as:*

$$\left(\sigma_{\hat{\nu}}^2\right)_{CR} = M^2\left(\sigma_{\hat{f}_0}^2\right)_{CR} = \frac{M^2}{4\pi^2}\left(\sigma_{\hat{\beta}}^2\right)_{CR} \cong \frac{3}{2\pi^2}\bullet\frac{1+(M-1)\beta}{M}\bullet\frac{1}{\text{SNR}}, \tag{9}$$

and

$$\frac{\left(\sigma_{\hat{A}}^2\right)_{CR}}{A^2} = \left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong \frac{2}{M}\left(1-\frac{3}{2M}+\frac{M-2}{2}\beta\right)\frac{1}{\text{SNR}}. \tag{10}$$

The proof of Proposition 2 is given in the Appendix B.

In particular, when $\beta = 0$ (i.e., in the case of undamped sinusoid) (9) and (10) become:

$$\left(\sigma_{\hat{\nu}}^2\right)_{CR} \cong \left(\sigma_{\hat{\nu}}^2\right)_{CR0} \cong \frac{3}{2\bullet\pi^2}\frac{1}{M\bullet\text{SNR}}, \tag{11}$$

$$\left(\sigma_{\hat{A}}^2\right)_{CR} \cong 4\frac{2M-3}{2M}\left(\sigma_{\hat{A}}^2\right)_{CR0} \cong \frac{2M-3}{M^2}\sigma^2, \tag{12}$$

and

$$\left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong \frac{2M-3}{2M}\left(\sigma_{\hat{\phi}}^2\right)_{CR0} \cong \frac{2M-3}{M}\frac{1}{M\bullet\text{SNR}}, \tag{13}$$

where $\left(\sigma_{\hat{\nu}}^2\right)_{CR0}$, $\left(\sigma_{\hat{A}}^2\right)_{CR0} = \frac{\sigma^2}{2M}$ and $\left(\sigma_{\hat{\phi}}^2\right)_{CR0} = \frac{2}{M\bullet\text{SNR}}$ represent the CRLBs for any unbiased estimators $\hat{\nu}$, $\hat{A}$ and $\hat{\phi}$ of single-tone undamped sinusoid parameters [7,29,30].

Conversely, if the observation interval length is enough greater than the signal time constant the term $e^{-2\beta_M}$ can be neglected. Thus, from (7) and (8) we have:

Proposition 3. *When both $\beta \ll 1$ and $\beta_M \gg 1$, the CRLBs for any unbiased estimator of noisy damped sinusoid parameters can be expressed as:*

$$\left(\sigma_{\hat{\nu}}^2\right)_{CR} = M^2\left(\sigma_{\hat{f}_0}^2\right)_{CR} = \frac{M^2}{4\pi^2}\left(\sigma_{\hat{\beta}}^2\right)_{CR} \cong \frac{M^2}{\pi^2}\beta^3(1-\beta)\frac{1}{\text{SNR}}, \tag{14}$$

and

$$\frac{\left(\sigma_{\hat{A}}^2\right)_{CR}}{A^2} = \left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong 2\beta(1-2\beta)\frac{1}{\text{SNR}}. \tag{15}$$

It is worth observing that, under the above constraints, (14) and (15) show that the CRLBs for the unbiased estimators of f_0, A, and ϕ are almost independent of the number of analyzed samples M.

Under the stated assumptions, Table 1 summarizes the derived approximations of the CRLBs for unbiased estimators of the parameters of a complex-valued noisy damped sinusoid.

3. Analysis of the CRLB behaviors

In this section expressions (9), (10), (14), and (15) are analyzed to determine the range of values for β and β_M in which a high approximation accuracy is ensured. To that aim it is assumed that a given signal is sampled at a constant sampling rate f_s, as often occurs in practice. In that case, the normalized decay rate β is constant and the behaviors of the CRLBs can be analyzed by assuming a variable number of analyzed samples M or, equivalently, a variable observation duration, expressed in terms of the number of time constants β_M.

At first, the behaviors of the CRLBs for the frequency and the decay rate estimators are analyzed.

3.1. Analysis of the CRLBs for the frequency and the decay rate estimators

Expression (7) for the CRLB of any unbiased estimator for the number of observed cycles estimator $\hat{\nu}$ can be rewritten as:

$$\left(\sigma_{\hat{\nu}}^2\right)_{CR} \cong \frac{1}{\pi^2}\beta(1-\beta)g_1(\beta_M)\frac{1}{\text{SNR}}, \tag{16}$$

where

Table 1
The derived approximated CRLB expressions and the assumed conditions.

Signal parameter estimator	CRLB approximation	Assumptions
Number of analyzed cycles estimator, $\hat{\nu}$	$\left(\sigma_{\hat{\nu}}^2\right)_{CR} \cong \frac{1}{\pi^2}\beta(1-\beta)\frac{\beta_M^2\left(1-e^{-2\beta_M}\right)}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}}\frac{1}{\text{SNR}}$	$\beta \ll 1$
	$\left(\sigma_{\hat{\nu}}^2\right)_{CR} \cong \frac{3}{2\pi^2}\bullet\frac{1+(M-1)\beta}{M}\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \ll 1$
	$\left(\sigma_{\hat{\nu}}^2\right)_{CR} \cong \frac{M^2}{\pi^2}\beta^3(1-\beta)\frac{1}{\text{SNR}},$	$\beta \ll 1$ and $\beta_M \gg 1$
Normalized frequency estimator, $\hat{f}_0$	$\left(\sigma_{\hat{f}_0}^2\right)_{CR} \cong \frac{1}{\pi^2}\beta(1-\beta)\frac{\beta_M^2\left(1-e^{-2\beta_M}\right)}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}}\frac{1}{M^2\bullet\text{SNR}}$	$\beta \ll 1$
	$\left(\sigma_{\hat{f}_0}^2\right)_{CR} \cong \frac{3}{2\pi^2}\bullet\frac{1+(M-1)\beta}{M^3}\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \ll 1$
	$\left(\sigma_{\hat{f}_0}^2\right)_{CR} \cong \frac{1}{\pi^2}\beta^3(1-\beta)\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \gg 1$
Normalized decay rate estimator, $\hat{\beta}$	$\left(\sigma_{\hat{\beta}}^2\right)_{CR} \cong \frac{4}{M^2}\beta(1-\beta)\frac{\beta_M^2\left(1-e^{-2\beta_M}\right)}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}}\frac{1}{\text{SNR}}$	$\beta \ll 1$
	$\left(\sigma_{\hat{\beta}}^2\right)_{CR} \cong \frac{6}{\pi^2}\bullet\frac{1+(M-1)\beta}{M^3}\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \ll 1$
	$\left(\sigma_{\hat{\beta}}^2\right)_{CR} \cong 4\beta^3(1-\beta)\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \gg 1$
Amplitude estimator, $\hat{A}$	$\left(\sigma_{\hat{A}}^2\right)_{CR} \cong 2\beta(1+2\beta)\frac{1-4\beta-\left[\left(2\beta_M^2(1-3\beta)+2\beta_M(1-4\beta)+1-4\beta\right)\right]e^{-2\beta_M}}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}}\frac{A^2}{\text{SNR}}$	$\beta \ll 1$
	$\left(\sigma_{\hat{A}}^2\right)_{CR} \cong \frac{2}{M}\left(1-\frac{3}{2M}+\frac{M-2}{2}\beta\right)\frac{A^2}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \ll 1$
	$\left(\sigma_{\hat{A}}^2\right)_{CR} \cong 2\beta(1-2\beta)\frac{A^2}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \gg 1$
Phase estimator, $\hat{\phi}$	$\left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong 2\beta(1+2\beta)\frac{1-4\beta-\left[\left(2\beta_M^2(1-3\beta)+2\beta_M(1-4\beta)+1-4\beta\right)\right]e^{-2\beta_M}}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}}\frac{1}{\text{SNR}}$	$\beta \ll 1$
	$\left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong \frac{2}{M}\left(1-\frac{3}{2M}+\frac{M-2}{2}\beta\right)\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \ll 1$
	$\left(\sigma_{\hat{\phi}}^2\right)_{CR} \cong 2\beta(1-2\beta)\frac{1}{\text{SNR}}$	$\beta \ll 1$ and $\beta_M \gg 1$

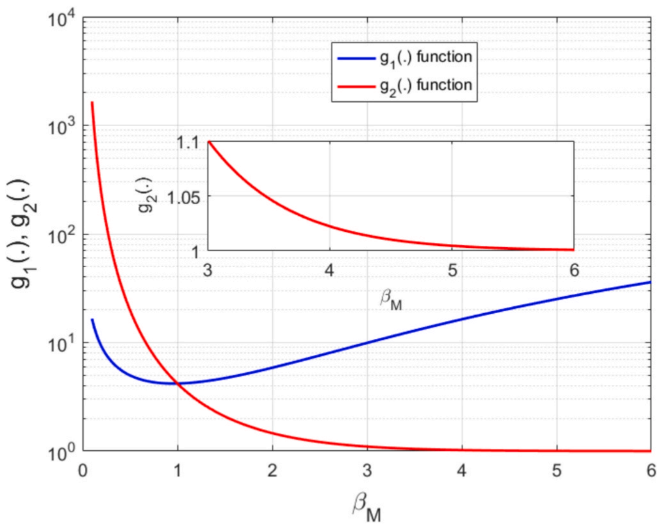


Fig. 1. Functions $g_1(\beta_M)$ and $g_2(\beta_M)$ versus the observation duration β_M expressed as a multiple of the time constant.

$$g_1(\beta_M) \triangleq \frac{\beta_M^2(1-e^{-2\beta_M})}{(1-e^{-2\beta_M})^2 - 4\beta_M^2 e^{-2\beta_M}}.$$

Conversely, from (3) and (7), the CRLBs for the normalized frequency $\widehat{f}_0$ and the decay rate $\widehat{\beta}$ unbiased estimators are given by:

$$\left(\sigma_{\widehat{f}_0}^2\right)_{\text{CR}} = \frac{\left(\sigma_{\widehat{\beta}}^2\right)_{\text{CR}}}{4\pi^2} \cong \frac{1}{\pi^2}\beta^3(1-\beta)g_2(\beta_M)\frac{1}{\text{SNR}}, \quad (17)$$

where

$$g_2(\beta_M) \triangleq \frac{g_1(\beta_M)}{\beta_M^2} = \frac{1-e^{-2\beta_M}}{(1-e^{-2\beta_M})^2 - 4\beta_M^2 e^{-2\beta_M}}.$$

Fig. 1 shows the functions $g_1(\beta_M)$ and $g_2(\beta_M)$ versus the observation duration β_M expressed as a multiple of the time constant.

As Fig. 1 shows, the function $g_1(\cdot)$ reaches a minimum when $\beta_M \cong 0.96$, where the first derivative of that function is null (as it was verified by using the MATLAB *fzero*$(\cdot)$ function). Thus, from (16) it follows that the minimum CRLB for the number of observed cycles is:

$$\left(\sigma_{\widehat{\nu}}^2\right)_{\text{CR,min}} \cong \frac{1}{\pi^2}\beta(1-\beta)g_1(0.96)\frac{1}{\text{SNR}} \cong \frac{0.42\beta(1-\beta)}{\text{SNR}}, \quad (18)$$

and it is obtained when the number of analyzed samples is:

$$M_\nu = [0.96 \bullet \tau \bullet f_s] = \left[\frac{0.96}{\beta}\right], \quad (19)$$

in which the operator $[\bullet]$ returns the integer, closest to its argument

Conversely, Fig. 1 shows that the function $g_2(\cdot)$ decreases when β_M increases and it is very close to the asymptotic value equal to 1 when β_M is greater than about 4. In that case, from (17) the CRLBs for the normalized frequency $\widehat{f}_0$ and the decay rate $\widehat{\beta}$ unbiased estimators almost reach their minimum, which is equal to:

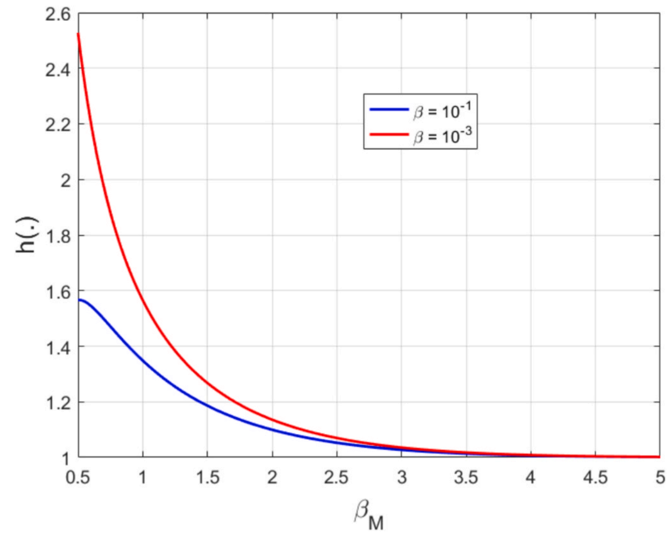


Fig. 2. Function $h(\beta, \beta_M)$ versus the observation duration β_M expressed as a multiple of the time constant. $\beta = 10^{-1}$ and $\beta = 10^{-3}$.

$$\left(\sigma_{\widehat{f}_0}^2\right)_{\text{CR,min}} = \frac{\left(\sigma_{\widehat{\beta}}^2\right)_{\text{CR,min}}}{4\pi^2} \cong \frac{0.10\beta^3(1-\beta)}{\text{SNR}}. \quad (20)$$

Values very close to (20) are achieved when the number of analyzed samples fulfills the following constraint:

$$M_{f_0}, M_\beta > [4\tau \bullet f_s] = \left[\frac{4}{\beta}\right]. \quad (21)$$

Choosing a higher number of samples increases the estimation delay and processing effort without significantly increasing the accuracy of the estimated parameters. This result is intuitive since the damped sinusoid magnitude is almost null after about four-time constants from the time origin so that the acquired samples are strongly dominated by noise and no further significant information about the signal parameters can be extracted.

3.2. Analysis of the CRLBs for the amplitude and the phase estimators

Expression (8) for the CRLB of any amplitude $\widehat{A}$ and phase $\widehat{\phi}$ unbiased estimators can be rewritten as:

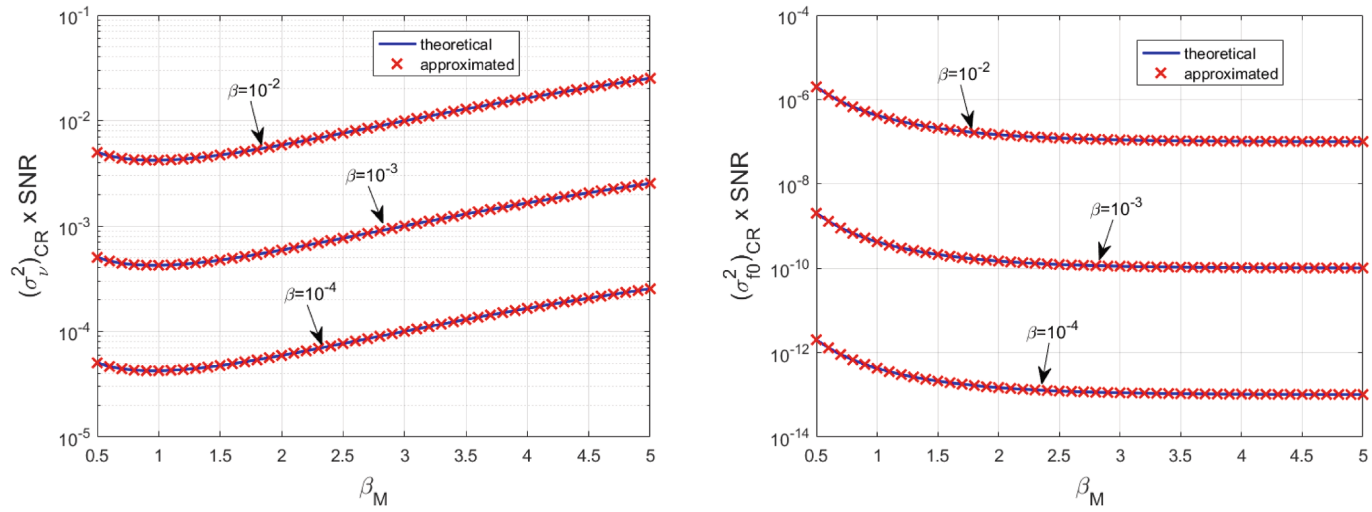
$$\frac{\left(\sigma_{\widehat{A}}^2\right)_{\text{CR}}}{A^2} = \left(\sigma_{\widehat{\phi}}^2\right)_{\text{CR}} \cong 2\beta(1-2\beta)h(\beta,\beta_M)\frac{1}{\text{SNR}}, \quad (22)$$

where

$$h(\beta,\beta_M) \triangleq \frac{1-4\beta-\left[2\beta_M^2(1-3\beta)+2\beta_M(1-4\beta)+1-4\beta\right]e^{-2\beta_M}}{(1-4\beta)\left[(1-e^{-2\beta_M})^2-4\beta_M^2e^{-2\beta_M}\right]}.$$

The function $h(\beta, \beta_M)$ is depicted in Fig. 2 as a function of β_M when $\beta = 10^{-1}$ and $\beta = 10^{-3}$.

For both considered values of β, Fig. 2 shows that $h(\beta,\beta_M)$ is very close to 1 as soon as β_M is greater than about 4, that is, when the number of analyzed samples fulfills the following constraint:



(a) (b)

Fig. 3. Exact (5) and approximated (7) expressions of $(\sigma^2_{\hat{\nu}})_{\text{CR}} \bullet \text{SNR}$ (a) and $\left(\sigma^2_{\hat{f}_0}\right)_{\text{CR}} \bullet \text{SNR}$ (b) versus β_M. β is equal to 10^{-4}, 10^{-3}, and 10^{-2}.

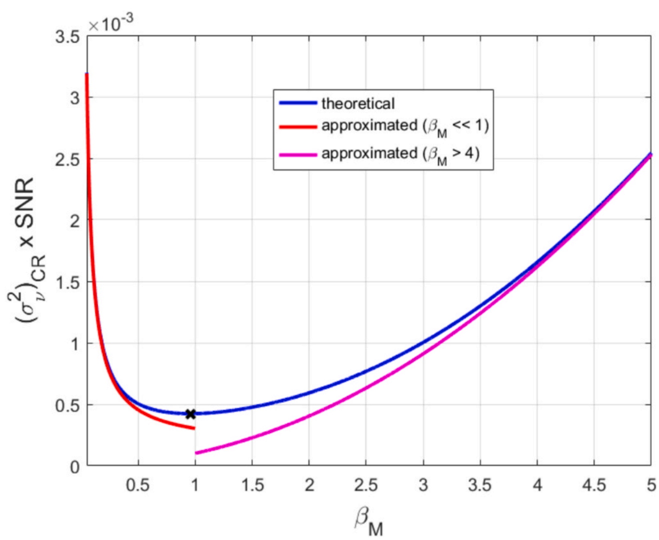


Fig. 4. Values of $(\sigma^2_{\hat{\nu}})_{\text{CR}} \bullet \text{SNR}$ returned by (5), (9), and (14) versus β_M when $\beta = 10^{-3}$. Approximated expressions (9) and (14) are obtained assuming $\beta_M \ll 1$ and $\beta_M > 4$, respectively. The minimum value of $(\sigma^2_{\hat{\nu}})_{\text{CR}} \bullet \text{SNR}$ returned by (18) is marked by a cross.

$$M_A, M_\phi > [4\tau \bullet f_s] = \left[\frac{4}{\beta}\right]. \tag{23}$$

In that situation, the CRLBs (15) almost achieve their minimum value, given by:

$$\frac{(\sigma^2_{\hat{A}})_{\text{CR,min}}}{A^2} = \left(\sigma^2_{\hat{\phi}}\right)_{\text{CR,min}} \cong 2\beta(1-2\beta)\frac{1}{\text{SNR}}. \tag{24}$$

Acquiring a higher number of samples than the one determined by (23) doesn’t significantly increase the accuracy of the estimated amplitude and phase.

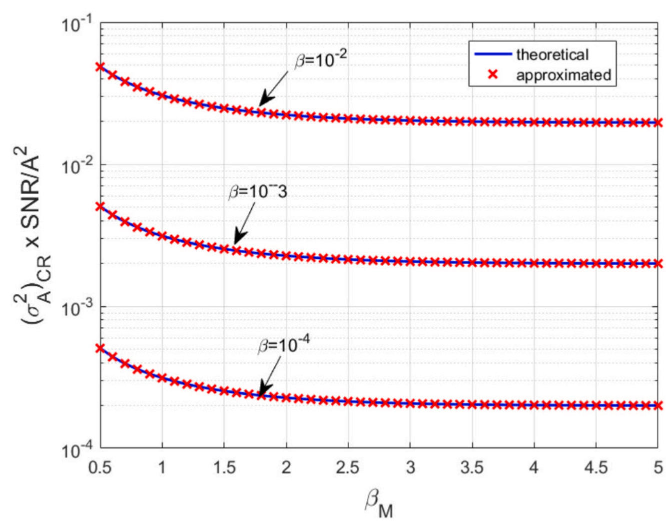


Fig. 5. Exact (6) and approximated (8) expressions of $(\sigma^2_{\hat{A}})_{\text{CR}} \bullet \text{SNR}/A^2$ versus β_M. β is equal to 10^{-4}, 10^{-3}, and 10^{-2}.

4. Accuracy of the derived expressions for the CRLBs

In this section the accuracies of the expressions derived in Section 3 are verified through computer calculations. Only the CRLBs for the estimators of the number of observed cycles $\hat{\nu}$, the normalized frequency $\hat{f}_0$, and the amplitude $\hat{A}$ are considered since the CRLBs for the estimators of the decay rate $\hat{\beta}$ and the phase $\hat{\phi}$ are related to the previous ones by a constant multiplicative factor.

Fig. 3 shows the behavior of the exact expression (5) and the

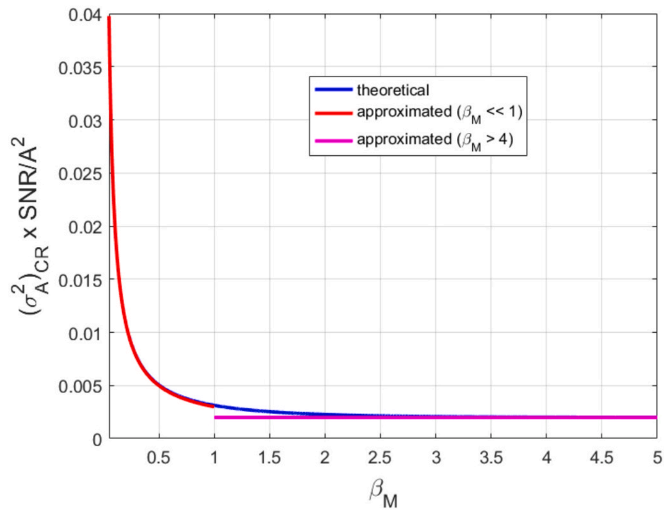


Fig. 6. Values of $(\sigma^2_{\hat{A}})_{CR} \bullet \mathrm{SNR}/A^2$ returned by (6), (10), and (15) versus β_M when $\beta = 10^{-3}$. Approximated expressions (10) and (15) are obtained assuming $\beta_M \ll 1$ and $\beta_M > 4$, respectively.

proposed expression (7) for $(\sigma^2_{\hat{\nu}})_{CR}$ (Fig. 3(a)) and $\left(\sigma^2_{\hat{f}_0}\right)_{CR}$ (Fig. 3(b)) multiplied by SNR. The observation duration β_M varies in the range [0.5, 5] time constants with step 0.1. The decay rate β is chosen equal to 10^{-4}, 10^{-3}, and 10^{-2}. As we can see, in the considered range of β_M values, the approximated and the exact expressions are in close agreement for all the considered values of the decay rate β. The maximum magnitudes of the relative errors are $4.87 \cdot 10^{-6}$%, 4. $87 \cdot 10^{-4}$%, and 4. $87 \cdot 10^{-2}$% for β equal to 10^{-4}, 10^{-3}, and 10^{-2}, respectively.

Fig. 4 shows the behaviors of the exact expression for $(\sigma^2_{\hat{\nu}})_{CR}$ multiplied by SNR and the related approximated expressions (9) and (14) as a function of β_M. The observation duration β_M varies in the range [0.05, 5] time constants with a step of 10^{-3}, while $\beta = 10^{-3}$. The corresponding number of analyzed samples M varies in the range [50,10000] with a step of 1 sample.

As we can see, the exact and the approximated values almost coincide when β_M is less than about 0.1 or greater than about 4. Indeed, the maximum magnitudes of the relative errors are about 0.49 % or 2.18 %, respectively. Observe also that the value returned by (18), marked with a cross, well agree with the exact curve minimum.

Fig. 5 shows the exact expression (6) and the approximated expression (8) for $(\sigma^2_{\hat{A}})_{CR} \bullet \mathrm{SNR}/A^2$ as a function of β_M, which varies in the ranges [0.5, 5] time constants with a step of 0.1. In that range, the maximum magnitudes of the relative errors are about $2.45 \cdot 10^{-5}$ %, $2.45 \cdot 10^{-3}$ %, and $2.53 \cdot 10^{-1}$ % when β is equal to 10^{-4}, 10^{-3}, and 10^{-2}, respectively.

Fig. 6 shows the behavior of the exact expression for $(\sigma^2_{\hat{A}})_{CR} \bullet \mathrm{SNR}/A^2$ and the related approximated expressions (10) and (15) as a function of β_M. The observation duration β_M varies in the range [0.05, 5] time constants with a step of 10^{-3}, while $\beta = 10^{-3}$. The maximum magnitudes of the relative errors in the ranges $\beta_M \leq 0.1$ or $\beta_M \geq 4$ are about 0.038 % or 0.85 %, respectively.

Fig. 7 shows a comparison between the proposed approximations and those reported in [25] as a result of curve fitting, i.e.,:

$$(\sigma^2_{\hat{\nu}})_{CR} = M^2\left(\sigma^2_{\hat{f}_0}\right)_{CR} = \frac{M^2}{4\pi^2}\left(\sigma^2_{\hat{\beta}}\right)_{CR} \cong \frac{3}{2\pi^2}\left(1+0.7\beta_M^{1.5}\right)^2\frac{1}{M \bullet \mathrm{SNR}}, \tag{25}$$

$$\frac{(\sigma^2_{\hat{A}})_{CR}}{A^2} = \left(\sigma^2_{\hat{\phi}}\right)_{CR} \cong \frac{2(1+0.7\beta_M)}{\sqrt{1+7.8\beta^2}}\frac{1}{M \bullet \mathrm{SNR}}. \tag{26}$$

A value $\beta = 10^{-3}$ is assumed, while the observation duration β_M takes values in the range [0.05, 5] time constants with a step of 10^{-3}. The exact CRLBs (5) and (6) are also shown in Fig. 7 for comparison. For visualization purposes, they are multiplied by SNR for the number of cycles (Fig. 7(a)) and the frequency estimators (Fig. 7(b)) and by SNR/A^2 for the amplitude (Fig. 7(c)) estimators. As it can be seen, the proposed approximations are much more accurate than (25) and (26). Indeed, when β is equal to 10^{-4}, 10^{-3}, and 10^{-2} the maximum relative error magnitudes associated to (25) and (26) are about 14.9 %, 13.9 %, and 13.8 % for the number of cycles and the frequency estimators and about 11.6 %, 10.0 %, and 10.2 % for the amplitude estimator.

5. Verification of the CRLB behavior

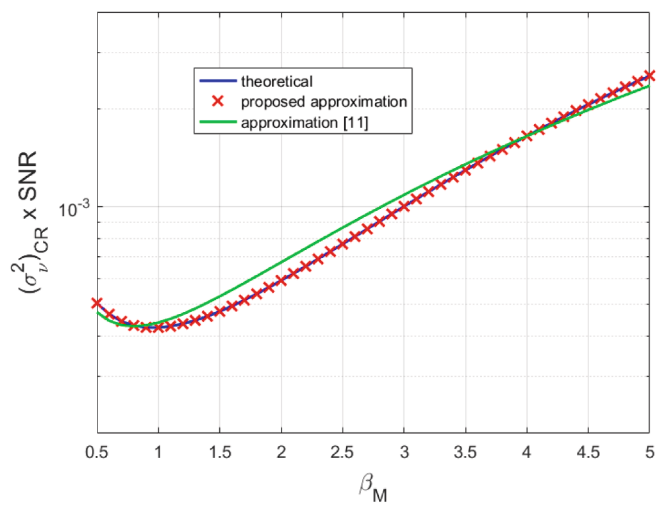
In this section, the behavior of the exact CRLBs as a function of the observation duration is verified by comparing it with the variances of the estimators returned by the MP algorithm [26–28], which are almost unbiased and statistically efficient when the pencil parameter is properly selected [28].

Fig. 8 shows the variances of the number of observed cycles, the frequency, and the amplitude estimators $\hat{\nu}, \hat{f}_0$, and $\hat{A}$ returned by the MP algorithm and the related exact CRLBs (5) and (6) as a function of the observation duration β_M expressed as a multiple of the time constant. The value of β_M varies in the range [0.5, 5] with a step of 0.1 and the decay rate is $\beta = 10^{-2}$. The values of β_M returned by (19), (21), and (23) and the definition (4) are also shown in Fig. 8. The signal parameters are: $A = 1$, $\phi = \pi/3$ rad, $\nu = 3.3$ cycles, and SNR $= 40$ dB. The number of analyzed sample M is determined from (4) using the values of β_M and β, and the pencil parameter used in the algorithm is equal to $M/2$ [28], For each value of the β_M 10,000 runs are performed.

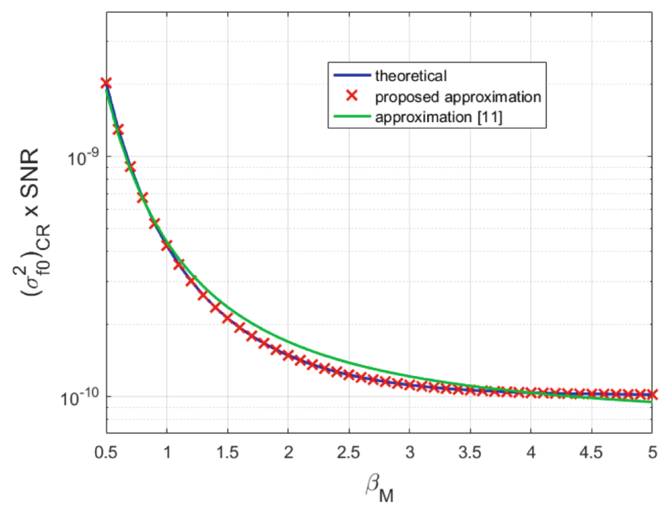
As Fig. 8 shows, the variances of the MP algorithm estimators are very close to the related CRLBs, as expected. Moreover, the MP estimator variances almost reach their minimum when the observation duration β_M assumes the values returned by (19), (21), and (23).

6. Conclusions

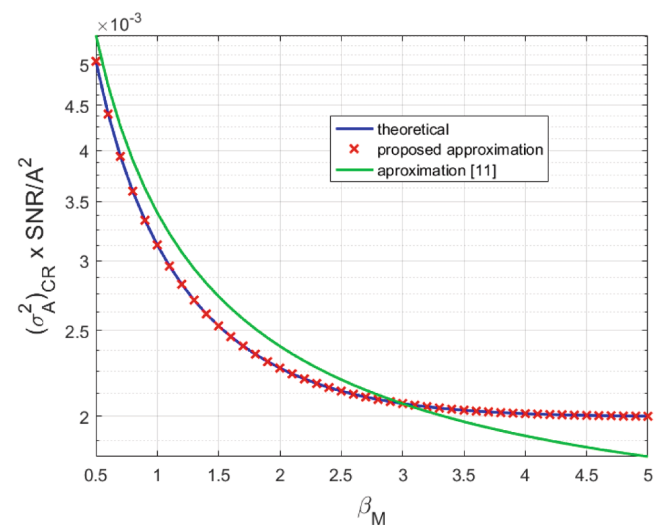
In this paper simple and very accurate expressions for the CRLBs of the unbiased estimators of a complex-valued noisy damped sinusoid parameters (frequency, decay rate, amplitude, and phase) are derived in the case of a Gaussian noise distribution assuming that the sampling period is sufficiently smaller than the signal time constant (i.e., $\beta \ll 1$). Even simpler CRLB expressions are derived under the further assumption that at least 4 signal time constants are observed (i.e., $\beta_M > 4$). Leveraging on the proposed expressions, constraints on the number of



(a)



(b)



(c)

Fig. 7. Approximated CRLBs (7), (8) and (25), (26): (a) $(\sigma_{\hat{\nu}}^2)_{CR} \bullet \mathrm{SNR}$, (b) $\left(\sigma_{\hat{f}_0}^2\right)_{CR} \bullet \mathrm{SNR}$, (c) $(\sigma_{\hat{A}}^2)_{CR} \bullet \mathrm{SNR}/A^2$ versus β_M. $\beta = 10^{-3}$. Normalized exact CRLBs (5) and (6) are shown for comparison.

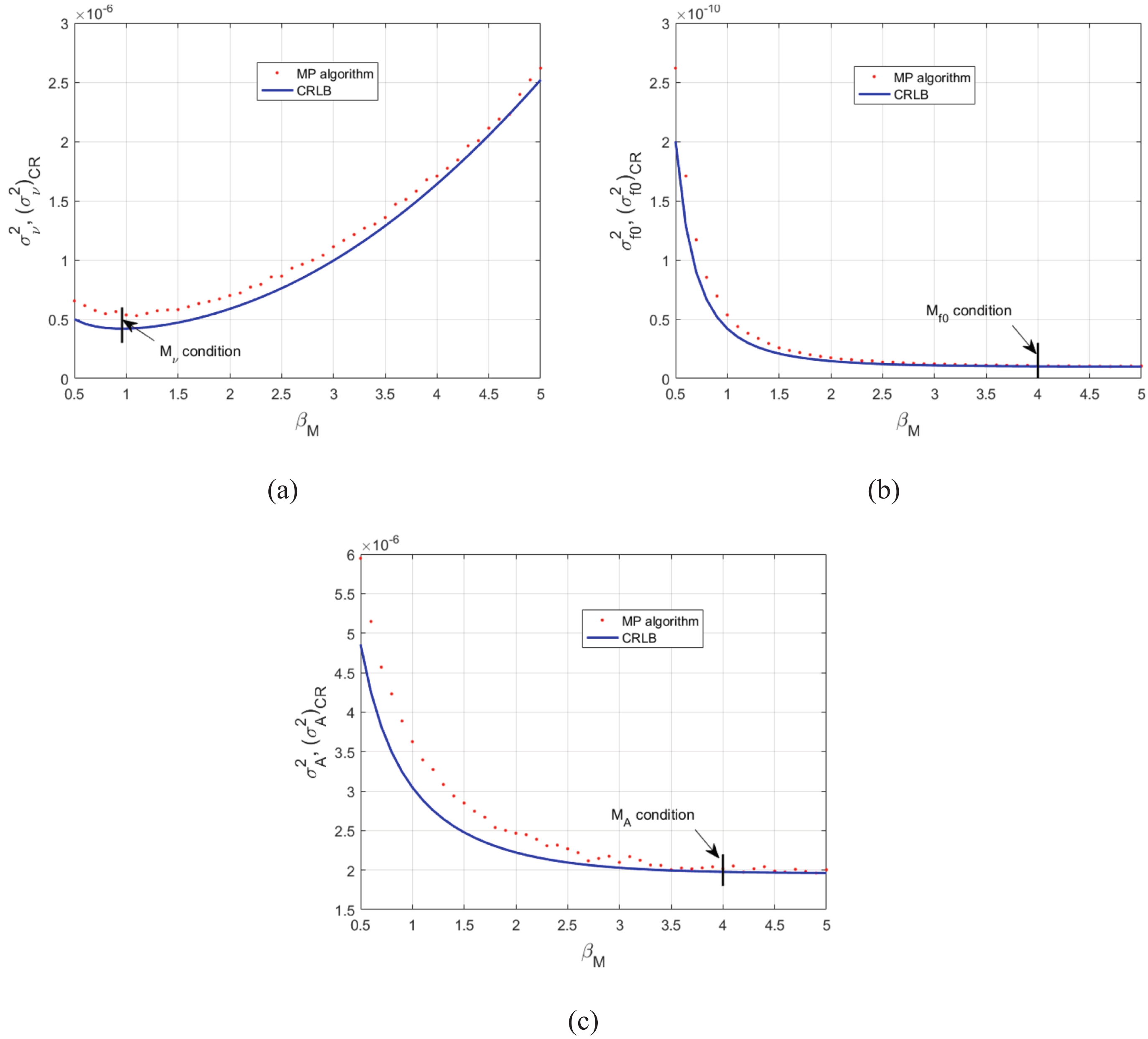


Fig. 8. Variances of (a) number of observed cycles $\hat{\nu}$, (b) frequency $\hat{f}_0$ and (c) amplitude estimator $\hat{A}$ returned by the MP algorithm and the exact CRLBs (5) and (6) versus the observation duration β_M. $\beta = 10^{-2}$ and SNR = 40 dB. 10,000 runs are performed for each value of β_M. The values of β_M determined through (19), (21), and (23) are shown.

analyzed samples that allow the optimization of both estimation accuracy and processing effort are derived. They can be exploited to optimize estimation accuracy, response time, and processing effort required by a measurement procedure aimed at estimating unknown signal parameters. The accuracies of the derived results have been verified through computer calculations and the application of the MP algorithm.

CRediT authorship contribution statement

Dario Petri: Investigation, Methodology, Supervision, Validation, Writing – review & editing. **Daniel Belega:** Writing – original draft, Software, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

Proof of Proposition 1.

Since $\beta \ll 1$ the following approximation holds:

$$e^{-2k\beta} \cong 1-k\beta+2(k\beta)^2-\frac{4(k\beta)^3}{3}+\frac{2(k\beta)^4}{3}, k=1,2,3 \tag{A.1}$$

From (A.1) after some algebra, we obtain:

$$\left(1-e^{-2\beta}\right)^2 = 1-2e^{-2\beta}+e^{-4\beta} \cong 4\beta^2(1-2\beta), \tag{A.2}$$

$$\left(1-e^{-2\beta}\right)^3 = 1-3e^{-2\beta}+3e^{-4\beta}-e^{-6\beta} \cong 8\beta^3(1-3\beta). \tag{A.3}$$

where the higher order terms of β within the brackets have been neglected.

From (A.1) with $k=1$, (A.2) and (A.3), still neglecting higher order terms of β, after some algebra the following expressions contained in (5) and (6) become:

$$\left(1-e^{-2\beta}\right)^3\left(1-e^{-2\beta_M}\right) \cong 8\beta^3(1-3\beta)\left(1-e^{-2\beta_M}\right), \tag{A.4}$$

$$-M^2e^{-2\beta_M}\left(1-e^{-2\beta}\right)^2+e^{-2\beta}\left(1-e^{-2\beta_M}\right)^2 \cong (1-2\beta)\left[\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}\right] \tag{A.5}$$

and

$$\begin{aligned}&-M^2e^{-2\beta_M}\left(1-e^{-2\beta}\right)^3+\left(1-(2M+1)e^{-2\beta_M}\right)\left(1-e^{-2\beta}\right)^2e^{-2\beta}+2e^{-4\beta}\left(1-e^{-2\beta}\right)\left(1-e^{-2\beta_M}\right)\\&\cong 2\beta\left\{2-8\beta-\left[\left(4\beta_M^2(1-3\beta)+4\beta_M(1-4\beta)+2-8\beta\right)\right]e^{-2\beta_M}\right\}.\end{aligned} \tag{A.6}$$

By replacing (A.4) and (A.5) into (5) it follows:

$$\left(\sigma_{\hat{\nu}}^2\right)_{\text{CR}} = \frac{M^2}{4\pi^2}\left(\sigma_{\hat{\beta}}^2\right)_{\text{CR}} = \frac{2}{\pi^2}\bullet\frac{1-3\beta}{1-2\beta}\bullet\frac{\beta_M^2\left(1-e^{-2\beta_M}\right)}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}}\bullet\frac{1}{\text{SNR}}. \tag{A.7}$$

Finally, using the approximations $(1-2\beta)^{-1} \cong 1+2\beta$ and $\frac{1-3\beta}{1-2\beta} \cong 1-\beta$, since $\beta << 1$, (A.7) provides (7).

Similarly, by replacing (A.5) and (A.6) into (6) and using the relationship $(1-2\beta)^{-1} \cong 1+2\beta$, expression (8) is obtained.

Appendix B

Proof of Proposition 2.

When $\beta_M \ll 1$ the following approximation holds:

$$e^{-2k\beta_M} \cong 1-2(k\beta_M)+2(k\beta_M)^2-\frac{4(k\beta_M)^3}{3}+\frac{2(k\beta_M)^4}{3}-\frac{4(k\beta_M)^5}{15}, k=1,2 \tag{B.1}$$

From (B.1) with $k=1$ and neglecting higher order terms of β within the brackets, it follows:

$$\beta_M^2\left(1-e^{-2\beta_M}\right) \cong 2\beta_M^3(1-\beta_M) \tag{B.2}$$

Using (B1) still neglecting higher order terms of β, after some algebra the following expressions contained in (7) and (8) become:

$$\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M} = 1-2e^{-2\beta_M}+e^{-4\beta_M}-4\beta_M^2e^{-2\beta_M} \cong \frac{4\beta_M^4}{3}(1-2\beta_M) \tag{B.2}$$

and

$$2\beta\left\{2-8\beta-\left[\left(4\beta_M^2(1-3\beta)+4\beta_M(1-4\beta)+2-8\beta\right)\right]e^{-2\beta_M}\right\} \cong \frac{16\beta_M^4}{3M}\left[1-\frac{3}{2M}-2\beta_M\left(\frac{3}{4}+\frac{1}{2M}\right)\right] \tag{B.4}$$

From (B.2) and (B.3) it results:

$$\frac{\beta_M^2\left(1-e^{-2\beta_M}\right)}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}} \cong \frac{3}{2\beta_M}\bullet\frac{1-\beta_M}{1-2\beta_M} \tag{B.5}$$

Since $\beta_M \ll 1$, neglecting the higher order terms of β the following approximation occurs $(1-\beta)\frac{1-\beta_M}{1-2\beta_M} \cong 1+(M-1)\beta$. Using that approximation and (B.5) expression (9) easily follows.

By replacing (B.3) and (B.4) into (8) it follows:

$$\frac{2\beta\left\{2-8\beta-\left[\left(4\beta_M^2(1-3\beta)+4\beta_M(1-4\beta)+2-8\beta\right)\right]e^{-2\beta_M}\right\}}{\left(1-e^{-2\beta_M}\right)^2-4\beta_M^2e^{-2\beta_M}} \cong \frac{4}{M}\frac{1-\frac{3}{2M}-2\beta_M\left(\frac{3}{4}+\frac{1}{2M}\right)}{1-2\beta_M} \tag{B.6}$$

Using the approximations $(1-2\beta_M)^{-1} \cong 1+2\beta_M$, and $\frac{(1+2\beta)\left[1-\frac{3}{2M}-2\beta_M\left(\frac{3}{4}+\frac{1}{2M}\right)\right]}{1-2\beta_M} \cong \frac{2M-3}{2M}+\frac{M-2}{2}\beta$, since $\beta_M \ll 1$, and neglecting the higher order terms of β, (B.6) returns expression (10).

Data availability

No data was used for the research described in the article.

References

[1] H. Günther, NMR Spectroscopy: Basic Principles, Concepts and Applications in Chemistry, John Wiley & Sons, 2013.
[2] K. Duda, T.P. Zielinski, L.B. Magalas, M. Majewski, DFT based estimation of damped oscillation's parameters in low frequency mechanical spectroscopy, IEEE Trans. Instrum. Meas. 60 (11) (2011) 3608–3618.
[3] K. Duda, T.P. Zielinski, Efficacy of the frequency and damping estimation of a real-value sinusoid, IEEE Instrum. Meas. Mag. 16 (2) (2013) 48–58.
[4] J.C. Visschers, O. Tretiak, D. Budker, L. Bougas, Continuous-wave cavity ring-down polarimetry, J. Chem. Phys. 152 (16) (2020).
[5] L. Bougas, D. Sofikitis, G.E. Katsoprinakis, A.K. Spiliotis, P. Tzallas, B. Loppinet, T. P. Rakitzis, Chiral cavity ring-down polarimetry: chirality and magnetometry measurements using signal reversals, J. Chem. Phys. 143 (10) (2015).
[6] D. Hunter, S. Piccolomo, J.D. Pritchard, N.L. Brockie, T.E. Dyer, E. Riis, Free-induction-decay magnetometer based on a microfabricated Cs vapor cell, Phys. Rev. Appl. 10 (1) (2018).
[7] D. Hunter, R. Jiménez-Martínez, J. Herbsommer, S. Ramaswamy, W. Li, E. Riis, Waveform reconstruction with a Cs based free-induction-decay magnetometer, Opt. Exp. 26 (23) (2018) 30523–30531.
[8] H. Dong, H. Liu, J. Ge, Z. Yuan, Z. Zhao, A high-precision frequency measurement algorithm for FID signal of proton magnetometer, IEEE Trans. Instrum. Meas. 65 (4) (2016) 898–904.
[9] J.A. de la O Serna, Synchrophasor estimation using Prony's method, IEEE Trans. Instrum. Meas. 62 (8) (2013) 2119–2128.
[10] A. Fort, E. Landi, R. Moretti, P. Carbone, A. Moschitta, Damped sine wave parameter extraction: application to QCM-D signals for accurate measurements, IEEE Trans. Instrum. Meas. 74 (2025).
[11] D. Belega, D. Petri, Algorithms for real-valued noisy damped sinusoid parameter estimation, IEEE Open Journal of Instrum. Meas. 1 (2022).
[12] M. Bertocco, C. Offelli, D. Petri, Analysis of damped sinusoidal signals via a frequency-domain interpolation algorithm, IEEE Trans. Instrum. Meas. 43 (2) (1994) 245–250.
[13] R. Diao, Q. Meng, An interpolation algorithm for discrete Fourier transforms of weighted damped sinusoidal signals, IEEE Trans. Instrum. Meas. 63 (6) (2014) 1505–1523.
[14] D. Agrež, Estimation of parameters of the weakly damped sinusoidal signals in the frequency domain, Comput. Stand. Interf. 33 (2011) 117–121.
[15] E. Aboutanios, Estimation of the frequency and decay factor of a decaying exponential in noise, IEEE Trans. Signal Process. 58 (2) (2010) 501–509.
[16] D. Belega, D. Petri, Frequency and damping factor estimation of real-valued damped sinusoids by means of an improved two-point interpolated DFT algorithm, Digital Signal Process. 153 (2024).
[17] D. Belega, D. Petri, D. Dallet, Generalized multi-point interpolated DTFT frequency and damping factor estimators of real-valued damped sinusoid, Measurement 234 (2024).
[18] J. Song, A. Mingotti, J. Zhang, L. Perotto, H. Wen, Accurate damping factor and frequency estimation for damped real-valued sinusoidal signals, IEEE Trans. Instrum. Meas. 71 (2022).
[19] H. Wu, L. Fan, X. Zhao, M. Li, L. Lu, J. Jin, Frequency and damping factor estimation of damped sinusoid by using DFT and DTFT, Measurement 242 (2025).
[20] D. Belega, D. Petri, Accurate sinusoid frequency and damping factor estimation by means of three-point interpolated DTFT algorithms, IEEE Trans. Instrum. Meas. 74 (2025).
[21] S.M. Kay, Fundamentals of Statistical Signal Processing: Estimation Theory Prentice-Hall, Upper Saddle River, NJ, 1993.
[22] T. Wigren, A. Nehorai, Asymptotic Cramér-Rao bounds for estimation of the parameters of damped sine waves in noise, IEEE Trans. Signal Process. 39 (4) (1991) 1017–1020.
[23] Y. Yilmaz, E. Dalaveroğlu, Asymptotic worst and best case Cramér-Rao bounds for estimation of parameters of low-frequency damped sinusoid, Electron. Lett. 41 (2) (2005).
[24] Y. Yao, S.M. Pandit, Cramér-Rao lower bounds for a damped sinusoidal process, IEEE Trans. Signal Process. 43 (4) (1995) 878–885.
[25] D.N. Swingler, Approximations to the Cramer-Rao lower bound for a single damped exponential signal, Signal Process. 75 (1999) 197–200.
[26] Y. Hua, T.K. Sarkar, Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise, IEEE Trans. Acoust. Speech Signal Process. 38 (5) (1990) 814–824.
[27] T.K. Sarkar, O. Pereira, Using the matrix pencil method to estimate the parameters of a sum of complex exponentials, IEEE Trans. Antennas Propag. Mag. 37 (1) (1995) 48–55.
[28] Y. Hua, T.K. Sarkar, Generalized pencil-of-function method for extracting poles of an EM system from its transient response, IEEE Trans. Antennas Propag. 37 (2) (1989) 229–234.
[29] D. Belega, D. Petri, Cramér–Rao lower bound for unbiased estimators of sampled noisy sine-wave parameters, IEEE Trans. Instrum. Meas. 70 (2021).
[30] D. Belega, D. Petri, Analysis of asymptotic and exact CRLBs on the parameters of noisy periodic signal, Digit Signal Process. 135 (2023).