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DOTTORATO DI RICERCA IN MATEMATICA



**PhD Thesis**

Boundary Properties for Almost-Minimizers  
of the Relative Perimeter

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# Introduction

Through the present dissertation, we aim to illustrate some outcomes related to the study of the boundary behavior for almost minimizers of the relative perimeter in an Euclidean open subset. The main results obtained are a boundary Monotonicity Formula and a Vertex Skipping Theorem, for which we refer respectively to [29] and [30].

Given an open set  $\Omega \subset \mathbb{R}^n$  and a measurable set  $E \subset \Omega$ , we define relative perimeter of  $E$  in  $\Omega$  restricted to an open set  $A \subset \mathbb{R}^n$  as

$$P_\Omega(E; A) := P(E; \Omega \cap A)$$

with the short form  $P_\Omega(E)$  when  $A \supset \Omega$ , and with  $P(E; B)$  denoting the standard perimeter of  $E$  in an open set  $B$  (see Definition 1.2.2). We say that  $E$  is an almost-minimizer of the relative perimeter in  $\Omega$  if, roughly speaking,  $E$  minimizes  $P_\Omega$  among those measurable sets  $F \subset \Omega$  that are obtained by  $E$  through a compact variation into  $\bar{\Omega}$ . In particular,  $F$  could differ from  $E$  up to the boundary of  $\Omega$  (for a more precise statement, see Definition 2.1.1).

## Boundary Monotonicity Formula

In the third chapter, we focus on the proof of the boundary Monotonicity Formula, and then we apply it to the proof of a perimeter-minimizing cone property for the limit of a blow-up sequence of an almost-minimizer. The Monotonicity Formula is a tool that for instance permits proving that if  $E$  is an almost-minimizer of the relative perimeter in an open set  $\Omega \subset \mathbb{R}^n$  and  $x_0 \in \bar{\Omega} \cap \partial E$ , the density ratios between the perimeter of  $E$  in the relative ball  $B_r(x_0) \cap \Omega$  and  $r^{n-1}$  are monotonically non-decreasing in  $r$  up to an error

term that goes to 0 as  $r \rightarrow 0^+$  in a quantified way. The Monotonicity Formula is known and classical when  $x_0$  is internal to  $\Omega$ , and a proof can be found in [19]. It is known also for  $x_0 \in \partial\Omega$ , but typically under smoothness assumptions on  $\partial\Omega$  ([2], [20]). In our main result Theorem 3.3.2 we prove a boundary Monotonicity Formula at a boundary point  $x_0 \in \partial\Omega$  provided the domain  $\Omega$  satisfies a *visibility condition* at  $x_0$ . Roughly speaking, this condition requires the existence of  $R > 0$  and a point  $V_r$ , for  $0 < r < R$ , slightly displaced with respect to  $x_0$  such that each point  $x \in \partial\Omega \cap B_r(x_0)$  is visible from  $V_r$ , i.e. the segment connecting  $V_r$  with  $x$  does not intersect  $\Omega$ . This condition leads to the construction of a quasi-conical competitor for  $E$ , allowing to prove the boundary Monotonicity Formula through a suitable adaptation of the classical argument developed by Giusti in [19], where indeed the construction of a conical competitor represents a key passage. The aforementioned visibility condition does not constitute a smoothness assumption on  $\Omega$ , since it is for instance satisfied by a generic convex set, possibly singular at 0, provided that it is suitably approximated by its tangent cone at 0 (see Example 3.2.12). Then, in the final part of Chapter 3, we show how the boundary Monotonicity Formula previously obtained can be employed to prove that any blow-up sequence of an almost-minimizer  $E$  of the relative perimeter admits a subsequence that  $L^1_{\text{loc}}$ -converges to a minimal cone in the tangent cone to  $\Omega$  at  $x_0$ , denoted by  $\Omega_{x_0}$ .

The Monotonicity Formula is an important tool in the study of the regularity of an almost-minimizer of the relative perimeter ([13], [19], [35]), and also of more general objects, like varifolds. In his seminal work [1], Allard proves an interior Monotonicity Formula for integral,  $k$ -dimensional varifolds with  $L^p$ -mean curvature,  $p > k$ , that is subsequently used in the proof of an internal regularity result for the same varifolds. In this case, the proof of the Monotonicity Formula is made using a first variation of the varifold along a suitable vector field. We also mention [2], where the same author proves a boundary Monotonicity Formula for varifolds with  $C^{1,1}$  boundary. In [20], the authors focus on the study of the boundary regularity for free-boundary integral  $k$ -varifolds. In particular, they prove that if the varifold  $V$  intersects  $\partial\Omega$  orthogonally,  $V$  has  $L^p$  mean curvature with  $p > k$  and the mass of  $V$  inside small balls centered at points of  $\partial\Omega$  is close to the volume of a half ball, then  $V$  is a  $C^{1,\alpha}$  submanifold of  $\mathbb{R}^n$  with boundary, that is diffeomorphic to a half ball. Also in this case, the Monotonicity Formula represents a key tool in the proof of the main regularity result, while its proof is based in particular

on a reflection through  $\partial\Omega$  procedure, that requires  $\Omega$  at least of class  $C^2$ . The common aspect among all the aforementioned works is the presence of a smoothness hypothesis on  $\Omega$ . Indeed, a key passage in the proof of the boundary Monotonicity Formula is to vary the varifold along a suitable vector-field tangent to  $\partial\Omega$ . The technical difficulties arising in implementing variation arguments in the presence of singularities of  $\partial\Omega$  motivate why the big majority of the works available in the literature concerning the boundary Monotonicity Formula, and consequently the study of the boundary regularity, require some smoothness assumptions on  $\Omega$ . As we remarked above, the proof of the boundary Monotonicity Formula developed in Chapter 3 is not based on a variation argument, but exploits the construction of a quasi-conical competitor, allowing to skip the strong smoothness assumptions on  $\partial\Omega$  that are typically required. It is worth mentioning that such a boundary Monotonicity Formula could be applied to the study of the boundary regularity for almost-minimizers of the relative perimeter. A regularity result in this sense is known only if  $\partial\Omega \in C^2$  [20]. An interesting development could be to exploit the boundary Monotonicity Formula proved in the present thesis in order to extend the regularity result in [20] to more general domains  $\Omega$ , for instance for those with  $\partial\Omega \in C^{1,\beta}$ .

## Behavior of the almost-minimizers near singularities

The main goal of the final two Chapters is to study the behavior of an almost-minimizer of the relative perimeter close to singular points of the domain, in particular near vertices.

## Free-boundary variations in non-smooth domains

In Chapter 4, we expose a new technique, that we developed for the study of the free-boundary area-minimality of a hypersurface  $\Sigma$  in a non-smooth domain  $\Omega \subset \mathbb{R}^n$ . We assume that  $\Omega$  is in graphical form, i.e.

$$\Omega = \{x \in \mathbb{R}^n : x_n > \omega(x_1, \dots, x_{n-1})\}$$

where  $\omega$  is a Lipschitz function. The idea is to foliate the domain  $\Omega$  through the sectional curves obtained by intersecting the boundary  $\partial\Omega$  with all the possible translations of the 2-plane generated by the last direction  $e_n$  and a fixed direction  $e_j$ , for  $j = 1, \dots, n-1$ ,

and then to vary  $\Sigma$  along these curves. This technique allows us to test the stability of  $\Sigma$  into  $\Omega$ . This approach is then applied to a specific variational problem: given a  $n$ -dimensional open circular cone  $\mathcal{C}_\alpha$  of semi-aperture  $\alpha \in (0, \pi/2)$ , and a  $(n-1)$ -plane  $\pi$  meeting  $\mathcal{C}_\alpha$  with a contact angle of 90 degrees and containing its vertex, we aim to understand whether the region  $\Sigma := \mathcal{C}_\alpha \cap \pi$  is stable, or area-minimizing, with respect to free-boundary, compact variations into  $\overline{\mathcal{C}_\alpha}$ . We will in particular show that the answer to this problem depends on the dimension  $n$ . When  $n = 3$ , in Theorem 4.2.8 we prove that  $\Sigma$  is not stable, and so in particular is not minimizing in  $\mathcal{C}_\alpha$ , for all  $\alpha$ . While, when  $n = 4$ , in Theorem 4.2.9 we show something rather surprising, namely that there exists a threshold  $\alpha^* \in (0, \pi/2)$  with the property that, if  $\alpha \geq \alpha^*$ , i.e. the width of the cone is sufficiently big, then  $\Sigma$  is stable. This interesting behaviour opens the doors to other questions. First, we do not know whether the threshold  $\alpha^*$  is optimal, and in particular, it is not clear what happens for small values of  $\alpha$ . Second, when  $\alpha \geq \alpha^*$  we just know that  $\Sigma$  is stable, but we do not know whether  $\Sigma$  is minimizing in  $\mathcal{C}_\alpha$ . Finally, there is still the question of what happens for  $n > 4$ . The case of  $n > 4$  is not covered by the present dissertation because of some difficulties in the computations of the Jacobian of the flow for  $n > 4$ . Nevertheless, we expect that the behavior for  $n > 4$  is the same as that in dimension  $n = 4$  with a dimension-dependent threshold  $\alpha^*(n)$ .

## Vertex-skipping

In the last Chapter of the thesis, we concentrate on the proof of the Vertex-skipping Theorem. The question underlying this result generalizes the question that moved us in the study of the minimality of  $\Sigma$  into  $\mathcal{C}_\alpha$ : our purpose is to understand whether the boundary of an almost-minimizer of the relative perimeter in an open, convex set  $\Omega \subset \mathbb{R}^n$  could contain vertex-type singularities of  $\partial\Omega$ . We say that a point  $x_0 \in \partial\Omega$  is a vertex for  $\Omega$  if the tangent cone  $\Omega_{x_0}$  to  $\Omega$  at  $x_0$  does not contain lines, i.e., up to isometries,  $\Omega_{x_0}$  cannot be written as  $\mathbb{R} \times C$ , for some convex cone  $C \subset \mathbb{R}^{n-1}$ . We notice that here a vertex  $x_0$  of  $\Omega$  is not the vertex of a cone, i.e. it is not required that  $x_0 + t(x - x_0) \in \Omega$ , for every  $x \in \Omega$ . With a little abuse of terminology, saying vertex we will refer both to the vertex of a cone and the vertex of an open, convex set  $\Omega$  in the sense of the definition just provided. The context will clarify which is the correct interpretation of the terminology. In our main result Theorem 5.3.1, we prove what follows.



Let  $\Omega \subset \mathbb{R}^3$  be a open, convex set, and let  $x_0 \in \partial\Omega$  be a vertex for  $\Omega$ . If  $E \subset \Omega$  is a local almost-minimizer of the relative perimeter in  $\Omega$ , then

$$x_0 \notin \overline{\partial E \cap \Omega},$$

*i.e. the closure of the boundary of  $E$  does not contain vertex-type singularities of the boundary of  $\Omega$ .*

This result generalizes Theorem 4.2.8 proved in Chapter 4. On the other hand, we notice that the validity of such a result in dimension  $n > 4$  seems improbable, owing to the stability result Theorem 4.2.9 proved in Chapter 4.

In the broader context of capillarity, it is known that if  $\Omega$  is of class  $C^{1,1}$ , a capillary surface interface in  $\Omega$  satisfies Young's law (see [40]), *i.e.*, forms a contact angle  $\beta = \arccos \gamma$  with  $\partial\Omega$ , where  $\gamma \in [-1, 1]$  is the wetting coefficient appearing in the capillary energy (without bulk terms)

$$P_\Omega(E) + \gamma P(E; \partial\Omega),$$

this resulting in the relative perimeter when  $\gamma = 0$ . The proof of Young's law relies on a boundary regularity result for local minimizers of the capillary energy, which requires  $\partial\Omega$  to be sufficiently smooth [14]. The study of the capillarity phenomena in non-smooth containers, in particular with wedge- or corner-type singularities, has been faced in various works, like in Concus-Finn [9], Lancaster [26], Chen-Finn-Miersemann [7], Lancaster-Siegel [25], Tamanini [39], Leonardi-Saracco [28].

Focusing now on the case of the relative perimeter in convex domains, we mention the contribution by Bokowski and Sperner [4] about the characterization of the minimizers of the relative isoperimetric problem in Euclidean balls. Moreover, in [34], Lions and Pacella proved the optimal isoperimetric inequality

$$P(E; \mathcal{C}) \geq P(B_E; \mathcal{C})$$

in a convex cone  $\mathcal{C}$  with a vertex at the origin, for all measurable sets  $E \subset \mathcal{C}$  having finite measure, where  $B_E$  denotes the Euclidean ball centered at the origin and such that  $|B_E \cap \mathcal{C}| = |E|$ . This inequality has been further refined to its definitive quantitative form by Figalli and Indrei in [17]. Let us spend some words about the relative isoperimetric

problem in  $\Omega$  such that  $\mathbb{R}^n \setminus \Omega$  is convex. The problem was first addressed by Choe, Ghomi and Ritoré in [8], where the authors established the optimal inequality

$$P_{\Omega}(E) \geq \frac{n\omega_n^{\frac{1}{n}}}{2} |E \cap \Omega|^{(n-1)/n},$$

including the characterization of the cases of equality, assuming that  $\partial\Omega \in C^2$ . This result has been recently generalized by Fusco and Morini [18] to the non-smooth case. With regard to the properties of minimizers, Sternberg and Zumbrun showed in [38] that the internal boundary of the local minimizers of the relative perimeter in a convex domain with boundary of class  $C^{2,\alpha}$  is either connected, or a union of parallel planes meeting  $\partial\Omega$  orthogonally. Such a property follows quite directly from another crucial result achieved in the same work, that is the concavity of the isoperimetric profile. See also [24], [36], [37], [27] for extensions to general convex sets.

The study of critical points of the relative perimeter functional deals with the classification of free-boundary minimal surfaces. We remark that the internal boundary of a local minimizer of the relative perimeter, in the absence of volume constraints, is in fact a free-boundary minimal surface. The study of free-boundary minimal surfaces started in the pioneering works of Courant [10] and Lewy [31], has been continued by many authors in the past. Recently, the topic has been tackled in various works as [23], [22], [6], [32], [33] and [11] (we refer to [5] for an overview of the recent advancements in this field). Anyway, it is important to remark that most part of the classification results available in the literature is obtained into domains, or ambient manifolds, having smooth boundary. Indeed, as we have already observed, many are the technical obstacles that arise in managing non-smooth domains. Nonetheless, it is worth citing the very recent contribution by Ede-len and Li [15], where the authors demonstrate an  $\varepsilon$ -regularity result for free-boundary minimal surfaces in domains that are close to a polyhedral-cone, i.e. are the image of a polyhedral cone through a transformation that is  $C^2$ -close to the identity. In some sense, our main result Theorem 5.3.1 is complementary to [15, Theorem 3.1], in that it excludes a specific type of boundary points from their  $\varepsilon$ -regularity result, at least in dimension  $n = 3$ .





# Chapter 1

## Bounded Variation functions

In this Chapter, we introduce the notions of variation for the gradient of a measurable function and of perimeter for a measurable set, namely the total variation of its characteristic map. Then we illustrate the main properties satisfied by the functions with bounded variation.

### 1.1 Differentiation of Radon measures

For a definition of Radon measure, real Radon measure, or vector Radon measure we refer to [3, Definition 1.40]. With a little abuse of terminology, saying Radon measure we will equivalently refer to a Radon measure, a real Radon measure, or a vector Radon measure. If  $\mu$  is a Radon measure on  $\mathbb{R}^n$  taking values in  $\mathbb{R}^p$ , we denote by  $|\mu|$  its total variation (see [3, Definition 1.4]). Let  $u = (u_1, \dots, u_p) : \mathbb{R}^n \rightarrow \mathbb{R}^p$  be summable with respect to  $|\mu|$ , then we denote by  $u \cdot \mu$  the Radon measure defined by

$$u \cdot \mu(E) := \int_E u \cdot d\mu = \sum_{q=1}^p \int_E u_q d\mu_q.$$

It can be proved (see [3, Proposition 1.23]) that

$$|u \cdot \mu| = |u| \cdot \mu. \tag{1.1.1}$$

We have the following result, whose proof can be found in [35, Corollary 5.11]. From now on, by  $B_r(x)$  we will denote a ball of center  $x$  and radius  $r$ , and denoting by  $B_r$  the ball  $B_r(0)$ .

**Lemma 1.1.1.** *Let  $\mu, \nu$  be Radon measures taking values in  $\mathbb{R}, \mathbb{R}^p$  respectively. Then, for  $\mu$ -a.e.  $x \in \mathbb{R}^n$ , the following limit exists*

$$D_\mu \nu(x) := \lim_{r \rightarrow 0^+} \frac{\nu(B_r(x))}{\mu(B_r(x))},$$

and the corresponding function  $f$  is such that  $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu; \mathbb{R}^p)$  and

$$\nu = D_\mu \nu \cdot \mu + \nu_\mu^s,$$

where  $\nu_\mu^s$  is singular with respect to  $\mu$ .

The function  $D_\mu \nu$  is called the  $\mu$ -density of  $\nu$ .

## 1.2 Variation and perimeter

We start with the following

**Definition 1.2.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open subset and  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ . The variation of  $f$  in  $\Omega$  is defined by*

$$|Df|(\Omega) = \sup \left\{ \int_\Omega f \operatorname{div} \phi \, dx : \phi \in C_c^1(\Omega; \mathbb{R}^n), \|\phi\|_{L^\infty(\Omega)} \leq 1 \right\}.$$

We say that  $f$  has bounded variation in  $\Omega$ , and we write  $f \in BV(\Omega)$ , provided  $f \in L^1(\Omega)$  and

$$|Df|(\Omega) < +\infty.$$

When  $f$  coincides with the characteristic of a measurable set, we use the following terminology.

**Definition 1.2.2.** *Let  $\Omega \subset \mathbb{R}^n$  be an open subset and  $E \subset \mathbb{R}^n$  be a measurable set. We call perimeter of  $E$  in  $\Omega$  the following quantity*

$$P(E; \Omega) = |D\mathbf{1}_E|(\Omega),$$

We say that  $E$  has finite perimeter in  $\Omega$  if  $P(E; \Omega) < +\infty$ .

In the next Chapters, we will often use the following notation: given two open sets  $\Omega$ ,  $A \subset \mathbb{R}^n$  and a measurable set  $E \subset \Omega$ , we set

$$P_\Omega(E; A) := P(E; A \cap \Omega),$$

with the short form  $P_\Omega(E; A) := P(E)$  when  $A$  contains  $\Omega$ . We call  $P_\Omega$  the relative perimeter in  $\Omega$ , and  $P_\Omega(E; A)$  the relative perimeter of  $E$  in  $\Omega$  restricted to  $A$ . Sometimes we will also use the following terminology.

**Definition 1.2.3.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set.*

(i) *We say that  $f \in L^1_{\text{loc}}(\Omega)$  has locally bounded variation in  $\Omega$  if*

$$|Df|(A) < +\infty, \quad \text{for any } A \subset\subset \Omega, A \text{ open.}$$

(ii) *We say that a measurable subset  $E \subset \Omega$  is a Caccioppoli Set (or that has locally finite perimeter) in  $\Omega$ , if*

$$P(E; A) < +\infty, \quad \text{for any } A \subset\subset \Omega, A \text{ open.}$$

We will denote by  $BV_{\text{loc}}(\Omega)$  the space of the functions  $f \in L^1_{\text{loc}}(\Omega)$  such that

$$f \in BV(\Omega \cap A), \quad \text{for all open, bounded sets } A \subset \Omega.$$

We remark that  $f \in BV_{\text{loc}}(\Omega)$  implies that  $f$  has locally bounded variation in  $\Omega$ , but the viceversa does not hold.

A bounded variation function can be characterized by its distributional gradient. The proof of the result below corresponds to the proof of Theorem 5.1 in [16].

**Theorem 1.2.4.** *Let  $\Omega \subset \mathbb{R}^n$  be open. A function  $f \in L^1(\Omega)$  has bounded variation in  $\Omega$  if and only if its distributional gradient is a Radon measure, i.e. there exists a vector-valued Radon measure  $\mu_f : \mathcal{B}(\mathbb{R}^n) \rightarrow \mathbb{R}^n$  with the following property:*

$$-\int_{\Omega} f \operatorname{div} \phi \, dx = (\phi \cdot \mu_f) \Omega, \quad \text{for all } \phi \in C_c^1(\Omega; \mathbb{R}^n).$$

Moreover, the total variation of  $\mu_f$  and the variation of  $f$  coincide, i.e.

$$|\mu_f|(\Omega) = |Df|(\Omega).$$

From now on, with a little abuse of notation, we will identify  $Df$  and  $\mu_f$ . Let now  $f \in BV(\Omega)$ . As a consequence of Lemma 1.1.1 we can consider the  $|Df|$ -density of  $Df$ , that we call  $\nu_f$ , which is defined by

$$\nu_f(x) = \lim_{r \rightarrow 0^+} \frac{Df(B_r(x))}{|Df|(B_r(x))}, \quad \text{for } |Df|\text{-a.e. } x \in \mathbb{R}^n.$$

We have  $\nu_f \in L^1(\mathbb{R}^n, |Df|; \mathbb{R}^n)$ . Since trivially  $Df$  is absolutely continuous with respect to  $|Df|$ ,  $\nu_f$  realizes

$$Df = \nu_f \cdot |Df|. \tag{1.2.1}$$

Moreover, from (1.1.1), (1.2.1), it follows that

$$|\nu_f| \cdot |Df| = |\nu_f \cdot |Df|| = |Df|,$$

and so that  $\nu_f(x) = 1$  at  $|Df|$ -a.e.  $x \in \mathbb{R}^n$ . When  $E$  has finite perimeter in  $\Omega$ , we set

$$\nu_E = \nu_{\mathbf{1}_E}. \tag{1.2.2}$$

The variation  $|Df|(\Omega)$  is lower-semicontinuous with respect to the  $L^1_{\text{loc}}$ -convergence. The proof of the following result is given in [16].

**Lemma 1.2.5.** *Let  $\Omega \subset \mathbb{R}^n$  be open, and  $\{f_j\}_{j \geq 1} \subset L^1_{\text{loc}}(\Omega)$  be a sequence of functions having locally bounded variation in  $\Omega$  that  $L^1_{\text{loc}}$  converges to a function  $f$ . Then the following inequality holds:*

$$|Df|(\Omega) \leq \liminf_j |Df_j|(\Omega).$$

## 1.3 Properties of BV functions

### 1.3.1 Approximation and compactness

BV functions can be approximated by smooth functions in a suitable sense.



**Theorem 1.3.1.** *Let  $\Omega \subset \mathbb{R}^n$  be open and  $f \in BV(\Omega)$ . Then there exists a sequence  $\{f_j\}_{j \geq 1} \in C^\infty(\Omega) \cap BV(\Omega)$  such that*

$$\|f_j - f\|_{L^1(\Omega)} \rightarrow 0, \quad |Df_j|(\Omega) \rightarrow |Df|(\Omega). \quad (1.3.1)$$

Moreover, provided (1.3.1) holds, we also have  $Df_j \rightharpoonup^* Df$ , i.e., for all  $\phi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ ,

$$\int_{\mathbb{R}^n} \phi \cdot dDf_j \rightarrow \int_{\mathbb{R}^n} \phi \cdot dDf, \quad \text{as } j \rightarrow \infty.$$

For the proof of the Theorem above, see Theorems 5.3 and 5.4 in [16].

**Remark 1.3.2.** *If condition (1.3.1) holds, we say that the sequence  $f_j$  strictly converges to  $f$  (see [3, Definition 3.14]). Thus Theorem 1.3.1 tells us in particular that  $C^\infty(\Omega) \cap BV(\Omega)$  is dense in  $BV(\Omega)$  with respect to the strict convergence.*

**Remark 1.3.3.** *The sequence  $\{f_j\}_{j \geq 1}$  built for the proof Theorem 1.3.1 turns out to have the following, further property (see [19, Remark 1.18])*

$$\lim_{\rho \rightarrow 0^+} \rho^{-N} \int_{\Omega \cap B_\rho(x_0)} |f_j - f| dx = 0, \quad \text{for all } N > 0, x_0 \in \partial\Omega, j \geq 1. \quad (1.3.2)$$

Combining Theorem 1.3.1 with the compactness result holding for Sobolev functions, one can prove the following compactness result for BV functions, provided the domain is regular enough (see [16, Theorem 5.5]).

**Theorem 1.3.4.** *Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded set with Lipschitz boundary, and  $\{f_j\}_{j \geq 1} \in BV(\Omega)$  be such that, for some constant  $C > 0$ ,*

$$\|f_j\|_{BV(\Omega)} := \|f_j\|_{L^1(\Omega)} + |Df_j|(\Omega) \leq C, \quad \text{for every } j. \quad (1.3.3)$$

Then there exists a function  $f \in BV(\Omega)$  and a subsequence  $f_{j_k}$  of  $f_j$  such that

$$\|f_{j_k} - f\|_{L^1(\Omega)} \rightarrow 0.$$

### 1.3.2 Coarea

**Theorem 1.3.5** (Coarea for  $BV$  functions). *Let  $\Omega \subset \mathbb{R}^n$  be open and  $f \in L^1(\Omega)$ . For  $t \in \mathbb{R}$ , let*

$$E_t = \{x \in \Omega : f(x) > t\}.$$

*Then the following statements hold:*

(i) *if  $f \in BV(\Omega)$ , then  $E_t$  has finite perimeter for a.e.  $t \in \mathbb{R}$ ,  $t \mapsto P(E_t; \Omega)$  is measurable, and*

$$|Df|(\Omega) = \int_{\mathbb{R}} P(E_t; \Omega) dt;$$

(ii) *conversely, if  $f \in L^1(\Omega)$ ,  $t \mapsto P(E_t; \Omega)$  is measurable, and*

$$\int_{\mathbb{R}} P(E_t; \Omega) dt < +\infty,$$

*then  $f \in BV(\Omega)$ .*

*Proof.* The proof of the measurability of  $t \mapsto P(E_t; \Omega)$  is given in [16, Lemma 5.1]. The other statements are proved in Theorem 5.9 of the same book.  $\square$

We also give the following Coarea result for Lipschitz continuous maps.

**Theorem 1.3.6** (Coarea for Lipschitz functions). *If  $\phi \in \text{Lip}(\mathbb{R}^n)$ ,  $E \subset \mathbb{R}^n$  is a Borel set, and  $f : \mathbb{R}^n \rightarrow [0, \infty)$  is a Borel function, then*

$$\int_E f |\nabla \phi| = \int_{\mathbb{R}} \int_{E \cap \{\phi=t\}} f d\mathcal{H}^{n-1} dt.$$

### 1.3.3 Traces

**Theorem 1.3.7.** *Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded set with Lipschitz boundary and denote by  $\nu_\Omega$  the outer, unit, normal vector defined  $\mathcal{H}^{n-1}$ -a.e. on  $\partial\Omega$ . Then there exists a unique linear and bounded operator*

$$\text{Tr}(f, \partial\Omega) : BV(\Omega) \rightarrow L^1(\partial\Omega, \mathcal{H}^{n-1})$$

called trace operator such that, for all  $f \in BV(\Omega)$ ,

$$\lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} |\text{Tr}(f, \partial\Omega)(x) - f(y)| dy = 0, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Omega. \quad (1.3.4)$$

Moreover, for all  $f \in BV(\Omega)$ , the following identity holds:

$$\int_{\Omega} f \operatorname{div} \phi dx = -(\phi \cdot Df)\Omega + \int_{\partial\Omega} \langle \phi, \nu_{\Omega} \rangle \text{Tr}(f, \partial\Omega) d\mathcal{H}^{n-1}, \quad \text{for any } \phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n). \quad (1.3.5)$$

*Proof.* The linearity and the uniqueness of  $\text{Tr}(f, \partial\Omega)$  immediately follow by condition (1.3.4). The existence is proven in [19, Theorem 2.10].  $\square$

The function  $\text{Tr}(f, \partial\Omega)$  is called the *trace* of  $f$  in  $\Omega$ . Identity (1.3.4) ensures in particular that

$$\text{Tr}(f, \partial\Omega)(x) = \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} f(y) dy, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega. \quad (1.3.6)$$

Hence, the trace can be interpreted as the boundary value on  $\partial\Omega$  assumed by the function  $f$ . In particular, if  $f \in BV(\Omega) \cap C^0(\overline{\Omega})$ , then (1.3.6) yields

$$\text{Tr}(f, \partial\Omega) = f|_{\partial\Omega}, \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega.$$

**Remark 1.3.8.** Under the assumptions of Theorem 1.3.7, the operator  $\text{Tr}(\cdot, \partial\Omega)$  is also continuous with respect to the strict convergence (see [19, Theorem 2.11] or [3, Theorem 3.88]). We also observe that, selecting a suitable sequence  $\{f_j\}_{j \geq 1} \subset C^\infty(\Omega) \cap BV(\Omega)$  realizing the thesis of Theorem 1.3.1, by applying (1.3.2) and (1.3.6) we have

$$\begin{aligned} |\text{Tr}(f_j, \partial\Omega)(x) - \text{Tr}(f, \partial\Omega)(x)| &= \left| \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} f_j(y) dy - \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} f(y) dy \right| \\ &\leq \lim_{r \rightarrow 0^+} \int_{B_r(x) \cap \Omega} |f_j(y) - f(y)| dy \\ &= 0, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial\Omega, \end{aligned} \quad (1.3.7)$$

and thus  $\text{Tr}(f_j, \partial\Omega) = \text{Tr}(f, \partial\Omega)$ ,  $\mathcal{H}^{n-1}$ -a.e. on  $\partial\Omega$ .

Let  $\Omega, \Omega' \subset \mathbb{R}^n$  open, bounded sets with Lipschitz boundary such that  $\Omega \subset\subset \Omega'$ .

We observe that  $\partial\Omega$  can be oriented by  $\nu_\Omega$ , the unit, normal vector that points out with respect to  $\Omega$ , or by  $-\nu_\Omega$ , the unit, normal vector that points out with respect to  $\Omega' \setminus \bar{\Omega}$ . For any  $f \in BV(\Omega')$ , we can then consider two traces for  $f$  on  $\partial\Omega$ , the first one with respect to  $\Omega$ , the second one with respect to  $\Omega' \setminus \bar{\Omega}$ :

$$\mathrm{Tr}^+(f, \partial\Omega) = \mathrm{Tr}(f, \partial\Omega), \quad \mathrm{Tr}^-(f, \partial\Omega) = \mathrm{Tr}(f, \partial(\Omega' \setminus \bar{\Omega})).$$

From (1.3.6), it follows that the definition of  $\mathrm{Tr}^-(f, \partial\Omega)$  does not depend on  $\Omega'$ . The functions  $\mathrm{Tr}^\pm(f, \partial\Omega) \in L^1(\partial\Omega; \mathcal{H}^{n-1})$  are called the *inner trace* and the *outer trace* of  $f$  in  $\Omega$  respectively. The following extension of *BV* functions result holds.

**Lemma 1.3.9.** *Let  $\Omega, \Omega' \subset \mathbb{R}^n$  be open subsets. In addition, let  $\Omega$  be bounded and have Lipschitz boundary, and assume that  $\Omega \subset\subset \Omega'$ . For all  $f \in BV(\Omega')$ , we have*

$$|Df|(\Omega') = |Df|(\Omega) + |Df|(\Omega' \setminus \bar{\Omega}) + \|\mathrm{Tr}^+(f, \partial\Omega) - \mathrm{Tr}^-(f, \partial\Omega)\|_{L^1(\partial\Omega; \mathcal{H}^{n-1})}. \quad (1.3.8)$$

Conversely, for any  $f_1 \in BV(\Omega)$ ,  $f_2 \in BV(\Omega' \setminus \bar{\Omega})$ , if we set  $f = f_1 \mathbf{1}_\Omega + f_2 \mathbf{1}_{\Omega' \setminus \bar{\Omega}}$ , then

$$f \in BV(\Omega').$$

*Proof.* The argument is very similar to that exploited for the proof of Theorem 5.8 in [16].  $\square$

**Remark 1.3.10.** *We observe in particular that, by (1.3.8),*

$$|Df|(\partial\Omega) = \|\mathrm{Tr}^+(f_1, \partial\Omega) - \mathrm{Tr}^-(f_2, \partial\Omega)\|_{L^1(\partial\Omega; \mathcal{H}^{n-1})}. \quad (1.3.9)$$

**Lemma 1.3.11.** *Fix an open set  $A \subset \mathbb{R}^n$  and a Lipschitz function  $\phi : A \rightarrow \mathbb{R}$  of class  $C^1$ , such that  $|\nabla\phi(x)| > 0$  for all  $x \in A$ . Set  $A_r = \phi^{-1}(-\infty, r)$ . Then for all  $f \in BV(A)$ , for  $\mathcal{L}^1$ -a.e.  $r \in \mathbb{R}$ , and  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial A_r \cap A$ , we have*

$$f(x) = \mathrm{Tr}^+(f, \partial A_r)(x) = \mathrm{Tr}^-(f, \partial A_r)(x). \quad (1.3.10)$$

*Proof.* We observe that, for a.e.  $r \in \mathbb{R}$ ,

$$\mathrm{Tr}^+(f, \partial A_r) = \mathrm{Tr}^-(f, \partial A_r), \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial A_r \cap A. \quad (1.3.11)$$

Indeed, for the proof of (1.3.11) we can combine (1.3.8) with  $|Df|(\partial A_r \cap A) = 0$  for a.e.  $r \in \mathbb{R}$ , which in turn comes from the fact that  $|Df|$  is a finite measure and  $\partial A_r \cap \partial A_s \cap A = \emptyset$  whenever  $r \neq s$ . Let  $x \in A$  be a Lebesgue point for  $f$ , then  $x \in \partial A_r$  if and only if  $r = \phi(x)$ . Thanks to the smoothness of  $\partial A_r \cap A$  (a consequence of the Implicit Function Theorem) we have

$$|A_r \cap B_\rho(x)| = \frac{1}{2} \rho^n + o(\rho^n), \quad \text{as } \rho \rightarrow 0^+,$$

and consequently

$$\begin{aligned} f(x) &= \lim_{\rho \rightarrow 0^+} \frac{1}{\omega_n \rho^n} \int_{B_\rho(x)} f(y) dy \\ &= \lim_{\rho \rightarrow 0^+} \frac{1}{\omega_n \rho^n} \left( \int_{A_r \cap B_\rho(x)} f(y) dy + \int_{B_\rho(x) \setminus A_r} f(y) dy \right) \\ &= \lim_{\rho \rightarrow 0^+} \frac{1}{2} \frac{1}{|A_r \cap B_\rho(x)|} \int_{A_r \cap B_\rho(x)} f(y) dy + \frac{1}{2} \frac{1}{|B_\rho(x) \setminus A_r|} \int_{B_\rho(x) \setminus A_r} f(y) dy \\ &= \frac{1}{2} \text{Tr}^+(f, \partial A_r)(x) + \frac{1}{2} \text{Tr}^-(f, \partial A_r)(x) \\ &= \text{Tr}^\pm(f, \partial A_r)(x). \end{aligned}$$

Since the set of Lebesgue points for  $f$  coincides with  $A$  up to a  $\mathcal{L}^n$ -negligible set, by Theorem 1.3.6 we obtain that the first equality in (1.3.10) is verified for  $\mathcal{L}^1$ -a.e.  $r \in \mathbb{R}$  and  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial A_r \cap A$ , which together with (1.3.11) concludes the proof.  $\square$

**Proposition 1.3.12.** *Under the assumptions of Lemma 1.3.11, we take  $f, f_j \in BV(A)$  for  $j \in \mathbb{N}$ , such that*

$$\|f_j - f\|_{L^1(A)} \longrightarrow 0, \quad |Df_j|(A) \longrightarrow |Df|(A).$$

*Then, for a.e.  $0 < r < 1$ , we have*

$$\text{Tr}(f, \partial A_r) := \text{Tr}^+(f, \partial A_r) = \text{Tr}^-(f, \partial A_r) \tag{1.3.12}$$

$$\text{Tr}(f_j, \partial A_r) := \text{Tr}^+(f_j, \partial A_r) = \text{Tr}^-(f_j, \partial A_r), \quad \text{for all } j \geq 1, \tag{1.3.13}$$

and

$$|Df_j|(A_r) \longrightarrow |Df|(A_r), \quad \|\mathrm{Tr}(f_j, \partial A_r) - \mathrm{Tr}(f, \partial A_r)\|_{L^1(\partial A_r)} \longrightarrow 0, \quad (1.3.14)$$

hence in particular  $f_j$  strictly converges to  $f$  on  $A_r$ .

*Proof.* Thanks to Lemma 1.3.11, the two identities (1.3.12) and (1.3.13) hold for a.e.  $0 < r < 1$ . In particular, for such  $r$ , we deduce that  $|Df|(\partial A_r) = 0$ , hence  $|Df|(\overline{A_r}) = |Df|(A_r)$ . Moreover, by Lemma 1.2.5, we have

$$\begin{aligned} \liminf_{j \rightarrow \infty} |Df_j|(A_r) &\geq |Df|(A_r) \\ &= |Df|(\overline{A_r}) = |Df|(A) - |Df|(A \setminus \overline{A_r}) \\ &= \lim_{j \rightarrow \infty} |Df_j|(A) - |Df|(A \setminus \overline{A_r}) \\ &\geq \limsup_{j \rightarrow \infty} |Df_j|(A) - |Df_j|(A \setminus \overline{A_r}) \\ &= \limsup_{j \rightarrow \infty} |Df_j|(\overline{A_r}) \\ &\geq \limsup_{j \rightarrow \infty} |Df_j|(A_r), \end{aligned}$$

which proves that

$$|Df|(A_r) = \lim_{j \rightarrow \infty} |Df_j|(A_r).$$

Since  $\|f_j - f\|_{L^1(A)} \rightarrow 0$ , we have in particular  $\|f_j - f\|_{L^1(A_r)} \rightarrow 0$ , and thus  $\{f_j\}_{j \geq 1}$  strictly converges to  $f$  in  $A_r$ . Finally, (1.3.14) holds because, as observed in Remark 1.3.8, the inner trace operator is continuous with respect to the strict convergence.  $\square$

## 1.4 Structure properties of Caccioppoli Sets

The perimeter of a set can be roughly described as a generalization of the measure of the boundary of the set itself. In this section, we give a very important structure Theorem for sets having (locally) finite perimeter. The proof of this capital result is due to Ennio De Giorgi. We start stating the following Lemma, whose proof is exactly that of Proposition 12.19 contained in [35]. The other parts of the present section are mostly inspired by Chapter 5 of [35].

**Lemma 1.4.1.** *Let  $E \subset \mathbb{R}^n$  be a set of locally finite perimeter. Then*

$$\text{spt } D\mathbf{1}_E = \{x \in \mathbb{R}^n : 0 < |E \cap B_r(x)| < \omega_n r^n, \text{ for all } r > 0\} \subset \partial E.$$

Moreover, there exists a measurable set  $E' \subset \mathbb{R}^n$  such that

$$|E' \Delta E| = 0, \quad \text{spt } |D\mathbf{1}_{E'}| = \partial E'. \quad (1.4.1)$$

**Remark 1.4.2.** *In other words, the support of  $D\mathbf{1}_E$  is contained in the topological boundary of  $E$ . In addition, up to choosing suitably a representative for  $E$ , the support of  $D\mathbf{1}_E$  coincides with  $\partial E$ .*

**Definition 1.4.3.** *Let  $E \subset \mathbb{R}^n$  be a set of locally finite perimeter. We define the reduced boundary of  $E$ , and we denote it by  $\partial^* E$ , as the set of those points  $x \in \mathbb{R}^n$  such that  $|D\mathbf{1}_E(B_r(x))| > 0$ , for all  $r > 0$ , and the following limit exists*

$$\nu_E(x) = \lim_{r \rightarrow 0^+} \frac{D\mathbf{1}_E(B_r(x))}{|D\mathbf{1}_E(B_r(x))|},$$

with  $|\nu_E(x)| = 1$ .

**Remark 1.4.4.** *We observe that  $\nu_E(x)$  is precisely the  $|D\mathbf{1}_E|$ -density of  $D\mathbf{1}_E$  that we introduced in (1.2.2). Then*

$$\begin{aligned} |\nu_E(x)| &= 1, \quad \text{for } |D\mathbf{1}_E|\text{-almost all points } x \in \mathbb{R}^n, \\ D\mathbf{1}_E &= \nu_E \cdot |D\mathbf{1}_E| \llcorner \partial^* E. \end{aligned} \quad (1.4.2)$$

**Remark 1.4.5.** *By definition,  $\partial^* E \subset \text{spt } D\mathbf{1}_E$ . Moreover, we observe that (1.4.2) implies*

$$\text{spt } D\mathbf{1}_E \subset \overline{\partial^* E}.$$

Hence  $\partial^* E \subset \text{spt } D\mathbf{1}_E \subset \overline{\partial^* E}$ . By Lemma 1.4.1, we infer that up to a suitable choice of a representative of  $E$  realizing (1.4.1), we also have

$$\partial^* E \subset \partial E = \overline{\partial^* E}.$$

We finally state the following structure Theorem due to De Giorgi. The proof of this result can be found in [35].

**Theorem 1.4.6.** *Let  $E \subset \mathbb{R}^n$  be a set of locally finite perimeter. Then*

$$|D\mathbf{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \quad D\mathbf{1}_E = \nu_E \cdot \mathcal{H}^{n-1} \llcorner \partial^* E,$$

and the following generalization of the Gauss-Green Formula holds:

$$\int_E \nabla \phi \, dx = \int_{\partial^* E} \phi \nu_E \, d\mathcal{H}^{n-1}.$$

Moreover  $\partial^* E$  is countably  $(n-1)$ -rectifiable, i.e. there exist countably many  $C^1$  hypersurfaces  $M_j \subset \mathbb{R}^n$ , compact sets  $K_j \subset M_j$  and a Borel set  $F$  with  $\mathcal{H}^{n-1}(F) = 0$  realizing

$$\partial^* E = F \cup \bigcup_{j \geq 1} K_j.$$

Finally, for every  $x \in K_j$ , we have  $\nu_E(x)^\perp = T_x M_j$ .

### 1.4.1 A local extension result

For this part, we assume that  $\Omega \subset \mathbb{R}^n$  is an open set with Lipschitz boundary. We fix  $\rho > 0$  such that, up to an isometry, there exist a Lipschitz function  $\omega : B'_\rho \rightarrow \mathbb{R}$  and a constant  $m > 0$ , with  $\omega(0) = 0$  and  $m > \|\omega\|_{L^\infty(B'_\rho)}$ , satisfying the following property: if we set  $\mathcal{C}_{\rho,m} = B'_\rho \times (-m, m)$ , we have

$$\Omega \cap \mathcal{C}_{\rho,3m} = \{x = (x', x_n) \in \mathbb{R}^n : x' \in B'_\rho, \omega(x') < x_n < 3m\}. \quad (1.4.3)$$

We aim to prove that, under this assumption, any measurable set  $E \subset \Omega$  with  $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$ <sup>1</sup> can be extended to a locally finite perimeter set  $\tilde{E}$  in  $\Omega \cup \mathcal{C}_{\rho,m}$ , in such a way that  $\tilde{E} \cap \Omega = E \cap \Omega$ ,  $P(\tilde{E}; \partial\Omega \cap \mathcal{C}_{\rho,m}) = 0$ , and  $P(\tilde{E}; S(B)) \leq C P(E; B)$ , for all Borel sets  $B \subset \mathcal{C}_{\rho,m} \setminus \Omega$  and for some constant  $C > 0$  depending on the dimension  $n$  and the function  $\omega$ . In what follows, we will denote by  $T_x E$  the approximate tangent space to  $\partial^* E$  at  $x$ <sup>2</sup>.

<sup>1</sup>We recall that in this paper  $f \in BV_{\text{loc}}(\Omega)$  means  $f \in BV(A)$  for all  $A \subset \Omega$  open and bounded.

<sup>2</sup>The approximate tangent space  $T_x E$  is given by the orthogonal complement of  $\nu_E(x)$ .



Set  $\mathcal{C}_\rho = B'_\rho \times \mathbb{R}$  and define the map  $S : \mathcal{C}_\rho \rightarrow \mathcal{C}_\rho$  as

$$S(x) := (x', 2\omega(x') - x_n). \quad (1.4.4)$$

Note that  $S$  satisfies  $S^2(x) = x$  for all  $x$ . Moreover, elementary computations show that

$$\text{Lip}(S) \leq \sqrt{3 + 6\text{Lip}(\omega)^2}. \quad (1.4.5)$$

Given  $E \subset \Omega$  measurable with  $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$ , we define  $\tilde{E} \subset \Omega \cup \mathcal{C}_\rho$  as

$$\tilde{E} = E \cup (S_\rho(E) \setminus \Omega), \quad (1.4.6)$$

where  $S_\rho(E) = S(E \cap \mathcal{C}_\rho)$ . Clearly, we have  $\tilde{E} \cap \Omega = E \cap \Omega$ . Further properties of  $\tilde{E}$  are stated in the next lemma.

**Lemma 1.4.7.** *Let  $E \subset \Omega$  be a measurable set with  $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$ . Then, for almost all  $x \in \partial^* E$ ,  $S$  restricted to  $x + T_x E$  is differentiable at  $x$ , and we denote by  $d^E S_x : T_x E \rightarrow \mathbb{R}^n$  its differential. Moreover, if  $\tilde{E}$  is the set defined in (1.4.6), we have*

$$P(\tilde{E}; \partial\Omega \cap \mathcal{C}_{\rho,m}) = 0 \quad (1.4.7)$$

and, for all Borel sets  $B \subset S(\mathcal{C}_{\rho,m} \cap \Omega)$ ,

$$P(\tilde{E}; B) = P(S_\rho(E); B) = \int_{\partial^* E \cap S(B)} J^E S(x) d\mathcal{H}^{n-1}(x) \leq C P(E; S(B)), \quad (1.4.8)$$

where  $J^E S(x) := \sqrt{\det(d^E S_x^* \circ d^E S_x)}$ ,  $d^E S_x^*$  is the adjoint of  $d^E S_x$ , and  $C = \text{Lip}(S)^{n-1}$ .

*Proof.* Owing to Theorem 1.4.6, we know that  $\partial^* E$  is countably  $(n-1)$ -rectifiable. Then the fact that  $S|_{T_x E}$  is differentiable at  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial^* E$  follows immediately from [35, Theorem 11.4].

Now, we prove (1.4.7) in the following way. Thanks to Lemma 1.3.9 we only need to check that, for  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial\Omega \cap \mathcal{C}_{\rho,m}$ , we have

$$\text{Tr}^+(\mathbf{1}_{\tilde{E}}, \partial\Omega)(x) = \text{Tr}^-(\mathbf{1}_{\tilde{E}}, \partial\Omega)(x),$$

that is,

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x). \quad (1.4.9)$$

The proof of (1.4.9) goes as follows. We employ the characterization of the trace as a limit of averages: for  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial\Omega$  we have

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = \lim_{r \rightarrow 0^+} \frac{|E \cap B_r(x) \cap \Omega|}{|B_r(x) \cap \Omega|}$$

and

$$\mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = \lim_{r \rightarrow 0^+} \frac{|S_\rho(E) \cap B_r(x) \setminus \Omega|}{|B_r(x) \setminus \Omega|}.$$

Then we combine this characterization with a consequence of (1.3.4), i.e. that the trace of a  $BV$  characteristic function coincides with a characteristic function  $\mathcal{H}^{n-1}$ -almost everywhere on  $\partial\Omega$ , to infer that we only need to show the equivalence

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = 0 \quad \Leftrightarrow \quad \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = 0.$$

One of the two required implications (the other can be discussed similarly) is

$$\mathrm{Tr}^+(\mathbf{1}_E, \partial\Omega)(x) = 0 \quad \Rightarrow \quad \mathrm{Tr}^+(\mathbf{1}_{S_\rho(E)}, \partial(\mathbb{R}^n \setminus \Omega))(x) = 0.$$

This implication can be restated as

$$|E \cap B_r(x) \cap \Omega| = o(r^n) \quad \Rightarrow \quad |S_\rho(E) \cap B_r(x) \setminus \Omega| = o(r^n) \quad \text{as } r \rightarrow 0^+. \quad (1.4.10)$$

Up to taking  $r > 0$  small enough, we have  $B_r(x) \subset \mathcal{C}_{\rho, m}$ , hence setting  $L = \mathrm{Lip}(S)$  we get

$$\begin{aligned} |S_\rho(E) \cap B_r(x) \setminus \Omega| &\leq |S(E \cap S(B_r(x)) \cap \Omega)| \\ &\leq L^n |E \cap B_{Lr}(x) \cap \Omega| \\ &= L^n o(L^n r^n) = o(r^n) \quad \text{as } r \rightarrow 0^+, \end{aligned}$$

which proves the implication (1.4.10) and concludes the proof of (1.4.7).

Finally, for the proof of (1.4.8), it is enough to show that

$$\mathcal{H}^{n-1}(\partial^* S(E) \Delta S(\partial^* E)) = 0. \quad (1.4.11)$$

Indeed, if (1.4.11) holds, Theorem 1.4.6 ensures that

$$P(S(E); B) = \mathcal{H}^{n-1}(\partial^* S(E) \cap B) = \mathcal{H}^{n-1}(S(\partial^* E) \cap B) = \mathcal{H}^{n-1}(S(\partial^* E \cap S(B))),$$

thus (1.4.8) is an immediate consequence of the Area Formula for rectifiable sets (see [35, Theorem 11.6]). Let us demonstrate (1.4.11). Again by Theorem 1.4.6, it suffices to prove that

$$S(E)^{(0)} = S(E^{(0)}), \quad S(E)^{(1)} = S(E^{(1)}). \quad (1.4.12)$$

Let us prove the first of the previous identities, as the proof of the other one is obtained by observing that  $(\mathbb{R}^n \setminus E)^{(0)} = E^{(1)}$ . Let  $0 < \delta < \rho$ , and  $x \in \Omega \cap (B'_{\rho-\delta} \times \mathbb{R})$ . Set  $L = \text{Lip}(S)$  as before, then by construction, for any  $0 < r < \delta$ ,  $S(B_r(S(x))) \subset B_{Lr}(x)$ , and thus the Area Formula yields

$$|S(E) \cap B_r(S(x))| = \int_{E \cap B_r(S(x))} JS \, dx \leq L^n |B_{Lr}(x)|. \quad (1.4.13)$$

Since  $r$  is arbitrary and  $S^{-1} = S$ , it is easy to check that, thanks to (1.4.13), if  $x \in E^{(0)}$  then  $S(x) \in S(E)^{(0)}$ , which proves the inclusion  $S(E^{(0)}) \subset S(E)^{(0)}$ . The reverse inclusion is proved in a completely analogous way. The proof of the lemma is then achieved thanks to (1.4.5).  $\square$



# Chapter 2

## Almost minimality

In this Chapter, we introduce the definition of (local) almost-minimizer of the relative perimeter in an open subset. This notion will play a key role in the thesis, since the most part of our results will apply to almost minimizers.

### 2.1 Almost minimizers

**Definition 2.1.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set and  $E \subset \mathbb{R}^n$  be measurable. We say that  $E$  is a local almost-minimizer of  $P_\Omega$  if, for any  $x \in \overline{\Omega}$  there exists  $r_x > 0$  such that, for any  $0 < r < r_x$  and any measurable subset  $F$  of  $\Omega$  with  $F \Delta E \subset \subset B_r(x)$ , one has*

$$P_\Omega(E; B_r(x)) \leq P_\Omega(F; B_r(x)) + |F \Delta E|^{\frac{n-1}{n}} \psi_\Omega(E; x, r), \quad (2.1.1)$$

for a suitable function  $\psi_\Omega(E; x, r)$  such that  $\lim_{r \rightarrow 0^+} \psi_\Omega(E; x, r) = 0$ .

If  $\psi_\Omega(E; x, r) = 0$ , for all  $x \in \overline{\Omega}$ , we say that  $E$  is a minimizer of  $P_\Omega$ .

If there exists a function  $\psi_\Omega(E; r) : (0, r_0) \rightarrow \mathbb{R}$  such that  $\lim_{r \rightarrow 0^+} \psi_\Omega(E; r) = 0$ , and

$$\psi_\Omega(E; x, r) \leq \psi_\Omega(E; r), \quad \text{for all } x \in \overline{\Omega},$$

we say that  $E$  is a almost-minimizer of  $P_\Omega$ .

Various notions of almost minimality can be found in literature. The notion chosen in

Definition 2.1.1 turns out to be rather weak, in the sense that the other, most common definitions of almost-minimality require conditions that are stronger than (2.1.1). This is the case of the so-called  $\Lambda$ -minimizers. We say that  $E$  is a  $\Lambda$ -*minimizer* of  $P_\Omega$  if, for any  $x \in \bar{\Omega}$  there exists  $r_x > 0$  such that, for any  $0 < r < r_x$  and any measurable subset  $F$  of  $\Omega$  with  $F\Delta E \subset\subset B_r(x)$ , one has

$$P_\Omega(E; B_r(x)) \leq P_\Omega(F; B_r(x)) + \Lambda|F\Delta E|. \quad (2.1.2)$$

It is then clear that condition (2.1.2) is stronger than (2.1.1): in particular,  $\Lambda$ -minimizers satisfy (2.1.1) with

$$\psi_\Omega(E; x, r) = \Lambda r \omega_n^{1/n}.$$

### 2.1.1 Minimality gap

**Definition 2.1.2.** *Let  $\Omega, A \subset \mathbb{R}^n$  be open sets and  $f \in BV_{\text{loc}}(\Omega)$ . The minimality gap of  $f$  in  $A$  relative to  $\Omega$  is*

$$\Psi_\Omega(f; A) = |Df|(\Omega \cap A) - \inf\{|Dg|(\Omega \cap A) : g \in BV_{\text{loc}}(\Omega), g - f \text{ has compact support in } A\}.$$

If  $f = \mathbf{1}_E$  for some measurable set  $E$  with  $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$ , the minimality gap measures how far is a measurable subset  $E$  from minimizing  $P_\Omega$ . When  $f = \mathbf{1}_E$ , for some measurable subset  $E \subset \mathbb{R}^n$ , we denote  $\Psi_\Omega(\mathbf{1}_E; A)$  by  $\Psi_\Omega(E; A)$ .

**Lemma 2.1.3.** *Let  $\Omega, A \subset \mathbb{R}^n$  be open sets and  $E \subset \Omega$  be such that  $\mathbf{1}_E \in BV_{\text{loc}}(\Omega)$ . Then*

$$\Psi_\Omega(E; A) = P_\Omega(E; A) - \inf\{P_\Omega(F; A) : \mathbf{1}_F \in BV_{\text{loc}}(\Omega), F\Delta E \subset\subset A \cap \bar{\Omega}\}.$$

*Proof.* Let us define

$$\begin{aligned} \mathcal{I}_1 &= \inf\{|Dg|(\Omega \cap A) : g \in BV_{\text{loc}}(\Omega), \text{spt}(g - \mathbf{1}_E) \subset\subset A \cap \bar{\Omega}\} \\ \mathcal{I}_2 &= \inf\{P_\Omega(F; A) : \mathbf{1}_F \in BV_{\text{loc}}(\Omega), F\Delta E \subset\subset A \cap \bar{\Omega}\}. \end{aligned}$$

It suffices to show that  $\mathcal{I}_1 = \mathcal{I}_2$ . For sure,  $\mathcal{I}_1 \leq \mathcal{I}_2$  because we can take  $g = \mathbf{1}_F$  in the definition of  $\mathcal{I}_1$ . Fix now  $\varepsilon > 0$ , and let  $g \in BV_{\text{loc}}(\Omega)$  be such that  $\text{spt}(g - \mathbf{1}_E) \subset\subset A \cap \bar{A}$

and

$$|Dg|(\Omega \cap A) \leq \mathcal{I}_1 + \varepsilon. \quad (2.1.3)$$

For  $t \in \mathbb{R}$ , let us set

$$G_t = \{x \in \Omega : g(x) > t\}.$$

We observe that, for each  $0 < t < 1$ ,  $G_t \setminus \text{spt}(g - \mathbf{1}_E) = E \setminus \text{spt}(g - \mathbf{1}_E)$ , and so

$$G_t \Delta E \subset \text{spt}(g - \mathbf{1}_E) \subset\subset A \cap \bar{\Omega}. \quad (2.1.4)$$

We can now exploit Theorem 1.3.5 and (2.1.4) to infer that

$$|Dg|(\Omega \cap A) = \int_{\mathbb{R}} P_{\Omega}(G_t; A) dt \geq \int_0^1 P_{\Omega}(G_t; A) dt \geq \mathcal{I}_2. \quad (2.1.5)$$

By (2.1.3) and (2.1.5), we deduce that

$$\mathcal{I}_1 \leq \mathcal{I}_2 \leq \mathcal{I}_1 + \varepsilon.$$

Owing to the arbitrariness of  $\varepsilon$ , we finally conclude that  $\mathcal{I}_1 = \mathcal{I}_2$ . □

**Lemma 2.1.4.** *Under the assumptions of Definition 2.1.2, if  $A \subset\subset A'$ , then*

$$\Psi_{\Omega}(f; A) \leq \Psi_{\Omega}(f; A').$$

*Proof.* Let us fix  $\varepsilon > 0$ , and let  $g$  be such that

$$|Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \geq \Psi_{\Omega}(f; A) - \varepsilon.$$

We can take

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \in \Omega \cap A \\ f(x) & \text{if } x \in \Omega \setminus \bar{A}. \end{cases}$$

Since  $A \subset\subset A'$  and the fact that  $f$  and  $\tilde{g}$  coincide out of  $A$ , by (1.3.8), we get

$$\begin{aligned} \Psi_\Omega(f; A') &\geq |Df|(\Omega \cap A') - |D\tilde{g}|(\Omega \cap A') \\ &= |Df|(\Omega \cap A) + |Df|(\Omega(A' \setminus \bar{A})) + |Df|(\Omega \cap \partial A) - \\ &\quad - |D\tilde{g}|(\Omega \cap A) - |D\tilde{g}|(\Omega \cap (A' \setminus \bar{A})) - |D\tilde{g}|(\Omega \cap \partial A) \\ &= |Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \\ &\geq \Psi_\Omega(f; A) - \varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary, we conclude. □

**Remark 2.1.5.** *The minimality gap of a local almost-minimizer satisfies a suitable decay on balls. Indeed, under the assumptions of Definition 2.1.2, if  $E$  is a local almost-minimizer (resp. a minimizer) of  $P_\Omega$ , then, owing to Lemma 2.1.3 and the trivial estimate  $|F\Delta E| \leq \omega_n r^n$ , we have*

$$\Psi_\Omega(E; B_r(x)) \leq \omega_n^{\frac{n-1}{n}} r^{n-1} \psi_\Omega(E; x, r) \quad (\text{resp. } \Psi_\Omega(E; B_r(x)) = 0), \quad (2.1.6)$$

for all  $x \in \bar{\Omega}$ ,  $0 < r < r_x$ , where  $\psi_\Omega(E; x, r)$  is the function appearing in (2.1.1). Conversely, if for any  $x \in \bar{\Omega}$  there exists  $r_x > 0$  such that for all  $0 < r < r_x$ ,

$$\Psi_\Omega(E; B_r(x)) = 0,$$

then  $E$  is a minimizer of  $P_\Omega$ .

## 2.1.2 Density estimates

In this subsection we establish perimeter and volume density estimates for almost-minimizers at a point either in  $\Omega$  or on  $\partial\Omega$ .

**Lemma 2.1.6.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set with a Lipschitz boundary. Let  $E$  be an almost-minimizer in  $\Omega$ , and let  $x \in \bar{\Omega}$ . Assume that  $P_\Omega(E; B_r(x)) > 0$  for all  $r > 0$ .*



Then, there exist constants  $C \geq 1$ ,  $\bar{r} > 0$ , depending on  $\Omega$ ,  $E$  and  $x$ , such that

$$C^{-1}r^{n-1} \leq P_{\Omega}(E; B_r(x)) \leq Cr^{n-1} \quad (2.1.7)$$

$$\min\left(|E \cap B_r(x) \cap \Omega|, |(B_r(x) \cap \Omega) \setminus E|\right) \geq C^{-1}r^n, \quad (2.1.8)$$

for all  $0 < r < \bar{r}$ .

*Proof.* Up to a translation, we assume that  $x = 0$ . We start proving (2.1.8). Given  $r > 0$  we set

$$m(r) := |B_r \cap \Omega \cap E|, \quad \mu(r) := |B_r \cap \Omega \setminus E|.$$

Both  $m$  and  $\mu$  are non-decreasing, thus differentiable for almost all  $r > 0$ . By [35, Example 13.3], for almost all  $r > 0$ , we have

$$m'(r) = \mathcal{H}^{n-1}(E \cap \partial B_r \cap \Omega), \quad \mu'(r) = \mathcal{H}^{n-1}(\partial B_r \cap \Omega \setminus E).$$

Since the one-parameter family of rescaled domains  $D_r := r^{-1}(\Omega \cap B_r)$ ,  $0 < r \leq 1$  is precompact with respect to the  $L^1$ -convergence in the class of Lipschitz and connected domains, there exists a constant  $\bar{C} > 0$  such that the following, relative isoperimetric inequality holds

$$P_{\Omega}(E; B_r) \geq \bar{C} \min\{m(r), \mu(r)\}^{\frac{n-1}{n}}, \quad (2.1.9)$$

for all  $0 < r < 1$ . Set  $0 < t < r$  and define the competitor

$$F_t = \begin{cases} E \cup B_t \cap \Omega & \text{if } m(t) > \mu(t), \\ E \setminus B_t \cap \Omega & \text{otherwise.} \end{cases}$$

We note that in the first case  $F_t \Delta E = B_t \cap \Omega \setminus E$ , while in the second case  $F_t \Delta E = B_t \cap \Omega \cap E$ . In any case, we have  $F_t \Delta E \subset\subset B_r \cap \Omega$ . Thus, by the almost-minimality of  $E$  in  $\Omega$ , and for almost all  $0 < t < r$ , we infer that either

$$\begin{aligned} P_{\Omega}(E; B_r) &\leq P_{\Omega}(F_t; B_r) + \mu(t)^{\frac{n-1}{n}} \psi(r) \\ &\leq P_{\Omega}(E; B_r \setminus \bar{B}_t) + \mathcal{H}^{n-1}(\partial B_t \cap \Omega \setminus E) + \mu(r)^{\frac{n-1}{n}} \psi(r), \end{aligned} \quad (2.1.10)$$

or

$$\begin{aligned} P_\Omega(E; B_r) &\leq P_\Omega(F_t; B_r) + m(t)^{\frac{n-1}{n}} \psi(r) \\ &\leq P_\Omega(E; B_r \setminus \overline{B_t}) + \mathcal{H}^{n-1}(\partial B_t \cap \Omega \cap E) + m(r)^{\frac{n-1}{n}} \psi(r). \end{aligned} \quad (2.1.11)$$

where  $\psi(r) := \psi_\Omega(E; 0, r)$ . Taking the limit as  $t \nearrow r$  in (2.1.10) and (2.1.11), and using (2.1.9), we deduce that, if  $m(r) > \mu(r)$ , then for almost all  $0 < r < r_0$  we have

$$\mu'(r) + \mu(r)^{\frac{n-1}{n}} \psi(r) \geq \overline{C} \mu(r)^{\frac{n-1}{n}},$$

while otherwise, we have

$$m'(r) + m(r)^{\frac{n-1}{n}} \psi(r) \geq \overline{C} m(r)^{\frac{n-1}{n}}.$$

Therefore, calling  $s(r) := \min\{m(r), \mu(r)\}$  and owing to the infinitesimality of  $\psi(r)$  as  $r \rightarrow 0$ , we obtain

$$\frac{s'(r)}{s(r)^{\frac{n-1}{n}}} \geq C,$$

for  $0 < r < \bar{r}$ , for some  $C, \bar{r} > 0$ . Integrating this inequality on the interval  $(0, r)$  we obtain (2.1.8). Then, the first inequality in (2.1.7) follows from (2.1.8) and (2.1.9). Finally, the second inequality in (2.1.7) follows from the observation that, taking the limit as  $t \nearrow r$  in (2.1.10) and possibly redefining  $\bar{r}$  and  $C$ , we have

$$P_\Omega(E; B_r) \leq \mathcal{H}(\partial B_r \cap \Omega \setminus E) + \mu(r)^{\frac{n-1}{n}} \psi(r) \leq Cr^{n-1},$$

for every  $0 < r < \bar{r}$ . □

### 2.1.3 Some estimates for the minimality gap

Here we prove two key properties of the minimality gap. The first one is the lower semicontinuity property of the minimality gap for uniform sequences of local almost-minimizers. The second is an upper bound for the difference between the minimality gaps of two  $BV_{\text{loc}}$  functions.

In what follows, we will say that  $\Omega_j \rightarrow \Omega$  locally in Hausdorff distance if there exist

$r_0, m, L > 0$ , with  $r_0 < m$ , and  $L$ -Lipschitz functions  $\omega_j, \omega : B'_{r_0} \rightarrow (-m, m)$  providing local graphical representations of  $\partial\Omega_j, \partial\Omega$ , respectively, as in (1.4.3), such that  $\omega_j \rightarrow \omega$  uniformly in  $B'_{r_0}$ .

**Lemma 2.1.7.** *For  $j \in \mathbb{N}$  we let  $\Omega_j, \Omega \subset \mathbb{R}^n$  be open sets with uniformly Lipschitz boundary, such that  $0 \in \partial\Omega$  and  $\Omega_j \rightarrow \Omega$  locally in Hausdorff distance. Let  $E_j, E$  be sets of locally finite perimeter, such that  $E_j \subset \Omega_j$ ,  $E \subset \Omega$ , and  $E_j \rightarrow E$  in  $L^1(B_{r_0})$ . Finally, we assume that  $E_j$  satisfies (2.1.1) for all  $0 < r < r_0$  and  $x = 0$ , and that moreover we have*

$$\lim_{r \rightarrow 0^+} \sup_j \psi_{\Omega_j}(E_j; 0, r) = 0.$$

Then,  $E$  satisfies (2.1.1) for all  $0 < r < r_0$ , with  $\psi_{\Omega}(E; 0, r) = \sup_j \psi_{\Omega_j}(E_j; 0, r)$ , and

$$\liminf_j \Psi_{\Omega_j}(E_j; B_{r_0}) \geq \Psi_{\Omega}(E; B_{r_0}). \quad (2.1.12)$$

Moreover, if  $\Psi_{\Omega_j}(E_j; B_{r_0}) \rightarrow 0$  as  $j \rightarrow \infty$ , then for almost all  $0 < r < r_0$  we have

$$\lim_{j \rightarrow \infty} P_{\Omega_j}(E_j; B_r) = P_{\Omega}(E; B_r), \quad \text{for almost all } r > 0. \quad (2.1.13)$$

*Proof.* Let us fix  $\varepsilon > 0$ . Let  $F \subset \Omega$  be such that  $F \Delta E \subset\subset B_{r_0}$  and

$$\Psi_{\Omega}(E; B_{r_0}) \leq P_{\Omega}(E; B_{r_0}) - P_{\Omega}(F; B_{r_0}) + \varepsilon. \quad (2.1.14)$$

By Lemma 1.3.11, for all  $j \geq 1$ , for a.e.  $0 < r < r_0$  and  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial B_r$ , we have

$$\mathbf{1}_{E_j}(x) = \text{Tr}^{\pm}(E_j, \partial B_r)(x) \quad \text{and} \quad \mathbf{1}_E(x) = \text{Tr}^{\pm}(E, \partial B_r)(x), \quad (2.1.15)$$

where  $\text{Tr}^{\pm}(A, \partial B_r) := \text{Tr}^{\pm}(\mathbf{1}_A, \partial B_r)$ . By the  $L^1_{\text{loc}}$ -convergence of  $E_j$  to  $E$ , we can choose  $r < r_0$  with the above property and the additional

$$P_{\Omega}(E; B_r) \geq P_{\Omega}(E; B_{r_0}) - \varepsilon, \quad (2.1.16)$$

then take  $j_\varepsilon$  large enough, such that

$$F\Delta E \subset\subset B_r, \quad (2.1.17)$$

$$\int_{\partial B_r} |\mathbf{1}_{E_j} - \mathbf{1}_E| d\mathcal{H}^{n-1} < \varepsilon \quad \text{for } j \geq j_\varepsilon. \quad (2.1.18)$$

Let us fix  $\delta > 0$ , define

$$U_\delta := \left\{ x = (x', x_n) \in \mathcal{C}_{r,m} : \omega(x') - \delta < x_n \leq \omega(x') \right\},$$

and assume  $\delta$  so small that

$$\mathcal{H}^{n-1}(\partial B_r \cap U_\delta) < \varepsilon \quad (2.1.19)$$

and

$$P_\Omega(E; B_r) \leq P(E; B_r \cap (\Omega + \delta e_n)) + \varepsilon. \quad (2.1.20)$$

Owing to the uniform convergence of  $\omega_j$  to  $\omega$ , we can select  $j_\delta \geq 1$  such that  $\|\omega_j - \omega\|_\infty < \delta$  for all  $j \geq j_\delta$ , then for those  $j$  we define

$$A_j := \Omega_j \cap \Omega \cap B_r, \quad B_j := (\Omega_j \setminus \Omega) \cap B_r, \quad (2.1.21)$$

and observe that  $B_j \subset U_\delta$ . Now we set

$$F_j := (\tilde{F} \cap \Omega_j \cap B_r) \cup (E_j \cap (B_{r_0} \setminus B_r)), \quad (2.1.22)$$

where  $\tilde{F} = F \cup (S(F) \setminus \Omega)$  and  $S$  is the symmetry through  $\partial\Omega$  defined in (1.4.4) (with  $\rho = r_0$ ). Thanks to (2.1.17), we have  $F_j \subset \Omega_j$  and  $F_j\Delta E_j \subset\subset B_{r_0}$ , which means that  $F_j$  is a competitor for  $E_j$  in the definition of  $\Psi_{\Omega_j}(E_j, B_{r_0})$ . Moreover, by (2.1.22), (2.1.18), and (2.1.19), we have

$$\begin{aligned} P_{\Omega_j}(F_j; B_{r_0}) &\leq P_{\Omega_j}(F_j; B_r) + P_{\Omega_j}(E_j; B_{r_0} \setminus \overline{B_r}) \\ &\quad + \int_{\partial B_r \cap \Omega_j \cap \Omega} |\mathbf{1}_{E_j} - \mathbf{1}_E| d\mathcal{H}^{n-1} + \mathcal{H}^{n-1}(\partial B_r \cap U_\delta) \\ &\leq P_{\Omega_j}(F_j; B_r) + P_{\Omega_j}(E_j; B_{r_0} \setminus \overline{B_r}) + 2\varepsilon. \end{aligned} \quad (2.1.23)$$

Let us compute  $P(F_j; B_r \cap \Omega_j)$ . By Lemma 1.4.7,  $P(\tilde{F}; \partial\Omega \cap \mathcal{C}_{r_0, m}) = 0$ , hence

$$P_{\Omega_j}(F_j; B_r) = P(F; A_j) + P(S(F); B_j). \quad (2.1.24)$$

Again by Lemma 1.4.7, up to possibly taking a smaller  $\delta$ , we have

$$P(S(F); B_j) = P(\tilde{F}; B_j) \leq P(\tilde{F}; U_\delta) \leq \varepsilon. \quad (2.1.25)$$

Putting together (2.1.23), (2.1.24), and (2.1.25), and taking into account (2.1.15), we obtain

$$\begin{aligned} \Psi_{\Omega_j}(E_j; B_{r_0}) &\geq P_{\Omega_j}(E_j; B_{r_0}) - P_{\Omega_j}(F_j; B_{r_0}) \\ &\geq P_{\Omega_j}(E_j; B_r) - P(F; A_j) - 3\varepsilon \\ &\geq P_{\Omega_j}(E_j; B_r) - P_\Omega(F; B_{r_0}) - 3\varepsilon. \end{aligned} \quad (2.1.26)$$

Now, since for all  $j \geq j_\varepsilon$  we have  $\|\omega_j - \omega\|_\infty < \delta$ , we infer that

$$B_r \cap (\Omega + \delta e_n) \subset B_r \cap \Omega_j.$$

This inclusion combined with (2.1.26) and (2.1.20) gives

$$\Psi_{\Omega_j}(E_j; B_{r_0}) \geq P(E_j; B_r \cap (\Omega + \delta e_n)) - P(F; B_{r_0} \cap \Omega) - 3\varepsilon, \quad (2.1.27)$$

so that taking the liminf as  $j \rightarrow \infty$  in (2.1.27), and using the lower semicontinuity of the perimeter, (2.1.20), (2.1.16), and (2.1.14), we find

$$\begin{aligned} \liminf_j \Psi_{\Omega_j}(E_j; B_{r_0}) &\geq \liminf_j P(E_j; B_r \cap (\Omega + \delta e_n)) - P_\Omega(F; B_{r_0}) - 3\varepsilon \\ &\geq P(E; B_r \cap (\Omega + \delta e_n)) - P_\Omega(F; B_{r_0}) - 3\varepsilon \\ &\geq P_\Omega(E; B_r) - P_\Omega(F; B_{r_0}) - 4\varepsilon \\ &\geq P_\Omega(E; B_{r_0}) - P_\Omega(F; B_{r_0}) - 5\varepsilon \\ &\geq \Psi_\Omega(E; B_{r_0}) - 6\varepsilon. \end{aligned}$$

Then, the arbitrary choice of  $\varepsilon$  implies (2.1.12). Now, the fact that  $E$  satisfies (2.1.1)

with  $\psi_\Omega(E; 0, r)$  as in the statement can be proved with the same argument used to show (2.1.12), also taking into account that  $|E_j \Delta F_j| \rightarrow |E \Delta F|$  as  $j \rightarrow \infty$ . Finally, to prove (2.1.13) we consider the sequence  $F_j$  defined as before, but now with  $F = E$ . Choosing  $\varepsilon > 0$  arbitrarily, for  $j$  large enough we obtain as before

$$\begin{aligned} \Psi_{\Omega_j}(E_j; B_{r_0}) &\geq P_{\Omega_j}(E_j; B_{r_0}) - P_{\Omega_j}(F_j; B_{r_0}) \\ &\geq P_{\Omega_j}(E_j; B_r) - P_\Omega(E; B_r) - 3\varepsilon, \end{aligned}$$

which gives the desired conclusion.  $\square$

**Lemma 2.1.8.** *Let  $\Omega, A \subset \mathbb{R}^n$  be open sets, with  $\partial\Omega$  Lipschitz and  $A$  bounded, of class  $C^2$ , and such that  $\mathcal{H}^{n-1}(\partial A \cap \partial\Omega) = 0$ . Let  $f, g \in BV_{\text{loc}}(\Omega)$ , then*

$$\begin{aligned} |\Psi_\Omega(f, A) - \Psi_\Omega(g, A)| &\leq \left| |Df|(\Omega \cap A) - |Dg|(\Omega \cap A) \right| \\ &\quad + \|\text{Tr}^+(f_0, \partial A) - \text{Tr}^+(g_0, \partial A)\|_{L^1(\partial A)}, \end{aligned} \quad (2.1.28)$$

where  $f_0, g_0$  denote the zero-extensions of, respectively,  $f$  and  $g$  on  $A \setminus \Omega$ .

*Proof.* Given  $\varepsilon > 0$ , there exists  $h \in BV_{\text{loc}}(\Omega)$  with  $\text{spt}(h - f) \subset\subset A$ , such that

$$\begin{aligned} \Psi_\Omega(f, A) &\leq |Df|(A \cap \Omega) - |Dh|(A \cap \Omega) + \varepsilon \\ &\leq \left| |Df|(A \cap \Omega) - |Dg|(A \cap \Omega) \right| + \Psi_\Omega(g, A) + |D\tilde{h}|(A \cap \Omega) - |Dh|(A \cap \Omega) + \varepsilon, \end{aligned} \quad (2.1.29)$$

where  $\tilde{h} \in BV_{\text{loc}}(\Omega)$  will be suitably chosen, so that in particular  $\text{spt}(\tilde{h} - g) \subset\subset A$ . For the definition of  $\tilde{h}$ , we claim that it is possible to construct a sequence  $A^{(k)}$  of inner parallel sets of  $A$  that converge to  $A$ , for which  $|Df_0|(\partial A^{(k)}) = |Dg_0|(\partial A^{(k)}) = 0$  and, moreover,

$$\lim_k \int_{\partial A^{(k)}} |\text{Tr}^+(f_0 - g_0, \partial A^{(k)})| d\mathcal{H}^{n-1} = \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1}. \quad (2.1.30)$$

For the proof of (2.1.30) we argue as follows. Since  $A$  is of class  $C^2$ , there exists  $\delta > 0$  such that, for all  $0 < t < \delta$ , the map  $\zeta_t(x) = x + t\nu_A(x)$  is a diffeomorphism of class  $C^1$  between  $\partial A$  and the boundary  $\partial A_t$  of the inner parallel set  $A_t = \{x \in A : \text{dist}(x, \partial A) > t\}$ . Now, we consider two sequences  $f_{0,j}, g_{0,j}$  of smooth approximations of  $f_0, g_0$  on  $A$ , with traces  $\text{Tr}^+(f_{0,j}, \partial A) = \text{Tr}^+(f_0, \partial A)$  and  $\text{Tr}^+(g_{0,j}, \partial A) = \text{Tr}^+(g_0, \partial A)$ , respectively (see Remark

1.3.8). By inspecting the proof of Anzellotti-Giaquinta's approximation theorem, it is not restrictive to ask that the sequences  $f_{0,j}, g_{0,j}$  also satisfy

$$\int_{A \setminus A_{1/k}} (|f_{0,j} - f_0| + |g_{0,j} - g_0|) dx \leq \frac{1}{k^2}, \quad (2.1.31)$$

for all  $j$  and for  $k > \delta^{-1}$ . We note that the tangential Jacobian of  $\zeta_t$  satisfies  $J\zeta_t(x) = 1 + O(t)$ , hence the area formula gives

$$\int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)| d\mathcal{H}^{n-1}(y) = (1 + O(t)) \int_{\partial A} |f_{0,j}(\zeta_t(x)) - g_{0,j}(\zeta_t(x))| d\mathcal{H}^{n-1}(x). \quad (2.1.32)$$

As  $t \rightarrow 0^+$  we have  $\zeta_t(x) \rightarrow x$  uniformly. Therefore, by following the same Cauchy-sequence argument as in the classical construction of the trace (see, e.g., [16]), the compositions  $f_{0,j}(\zeta_t(x))$  and  $g_{0,j}(\zeta_t(x))$  can be shown to converge in  $L^1(\partial A)$  to some limits  $\hat{f}_{0,j}$  and  $\hat{g}_{0,j}$ , respectively. Hence (2.1.32) implies

$$\lim_{t \rightarrow 0^+} \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)| d\mathcal{H}^{n-1}(y) = \int_{\partial A} |\hat{f}_{0,j}(x) - \hat{g}_{0,j}(x)| d\mathcal{H}^{n-1}(x). \quad (2.1.33)$$

At the same time, if we choose a vector field  $\xi \in C^1(\mathbb{R}^n; \mathbb{R}^n)$  and set either  $u_j = f_{0,j}$  or  $u_j = g_{0,j}$ , by Gauss-Green Theorem we obtain

$$\begin{aligned} \int_{A_t} (u_j \operatorname{div} \xi + \nabla u_j \cdot \xi) dx &= - \int_{\partial A_t} u_j \xi \cdot \nu_{A_t} d\mathcal{H}^{n-1} \\ &= -(1 + O(t)) \int_{\partial A} u_j(\zeta_t(x)) \xi(\zeta_t(x)) \cdot \nu_A(x) d\mathcal{H}^{n-1}(x), \end{aligned}$$

hence taking the limit as  $t \rightarrow 0^+$  gives

$$\int_A (u_j \operatorname{div} \xi + \nabla u_j \cdot \xi) dx = - \int_{\partial A} \hat{u}_j \xi \cdot \nu_A d\mathcal{H}^{n-1},$$

which means that  $\hat{f}_{0,j}$  and  $\hat{g}_{0,j}$  coincide, respectively, with  $\operatorname{Tr}^+(f_0, \partial A)$  and  $\operatorname{Tr}^+(g_0, \partial A)$  up to  $\mathcal{H}^{n-1}$ -null sets and for all  $j$ , by the uniqueness of the trace. We can thus rewrite (2.1.33) as

$$\lim_{t \rightarrow 0^+} \int_{\partial A_t} |f_{0,j}(y) - g_{0,j}(y)| d\mathcal{H}^{n-1}(y) = \int_{\partial A} |\operatorname{Tr}^+(f_0 - g_0, \partial A)(x)| d\mathcal{H}^{n-1}(x). \quad (2.1.34)$$

To get (2.1.30) from (2.1.34), we must choose  $A^{(k)}$  appropriately. To this aim, we apply the coarea formula to the integral in (2.1.31) and average the resulting inequality, deducing the existence of  $0 < t_k < 1/k$  such that for all  $j$

$$\int_{\partial A^{(k)}} (|f_{0,j} - \text{Tr}(f_0, \partial A^{(k)})| + |g_{0,j} - \text{Tr}(g_0, \partial A^{(k)})|) d\mathcal{H}^{n-1} \leq \frac{1}{k}, \quad (2.1.35)$$

where we have set  $A^{(k)} = A_{t_k}$ . By the triangle inequality and (2.1.35) we obtain

$$\int_{\partial A^{(k)}} |\text{Tr}^+(f_0 - g_0, \partial A^{(k)})| d\mathcal{H}^{n-1} \leq \int_{\partial A^{(k)}} |f_{0,j} - g_{0,j}| d\mathcal{H}^{n-1} + \frac{1}{k},$$

which gives (2.1.30) at once from (2.1.34).

Now we observe that  $|Df_0|(\partial A^{(k)}) = |Dg_0|(\partial A^{(k)}) = 0$  because the inner and outer traces of  $f_0$  and  $g_0$  on  $\partial A^{(k)}$  coincide, hence we can define

$$\tilde{h} = h\mathbf{1}_{A^{(k)}} + g\mathbf{1}_{A \setminus A^{(k)}}.$$

Note that  $\text{spt}(\tilde{h} - g) \subset\subset A$  and, if  $k$  is large enough,  $\text{spt}(h - f) \subset\subset A^{(k)}$ , so that

$$|D\tilde{h}|(A \cap \Omega) \leq |Dh|(A^{(k)} \cap \Omega) + |Dg_0|(A \setminus \overline{A^{(k)}}) + \int_{\partial A^{(k)}} |\text{Tr}(f_0 - g_0, \partial A^{(k)})| d\mathcal{H}^{n-1}.$$

By choosing  $k$  large enough we obtain  $|Dg_0|(A \setminus A^{(k)}) < \varepsilon$  and thus

$$|D\tilde{h}|(A \cap \Omega) - |Dh|(A \cap \Omega) \leq \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1} + \varepsilon. \quad (2.1.36)$$

By combining (2.1.29) and (2.1.36), we get

$$\Psi_\Omega(f, A) \leq \left| |Df|(A \cap \Omega) - |Dg|(A \cap \Omega) \right| + \Psi_\Omega(g, A) + \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1} + 2\varepsilon.$$

Since  $\varepsilon$  is arbitrary, we conclude

$$\Psi_\Omega(f, A) - \Psi_\Omega(g, A) \leq \left| |Df|(A \cap \Omega) - |Dg|(A \cap \Omega) \right| + \int_{\partial A} |\text{Tr}^+(f_0 - g_0, \partial A)| d\mathcal{H}^{n-1}$$

and, by exchanging the role of  $f$  and  $g$  in the argument above, we obtain (2.1.28).  $\square$



# Chapter 3

## Boundary Monotonicity Formula

The aim of the present Chapter is to prove a boundary Monotonicity Formula holding for local almost-minimizers of the relative perimeter at a boundary point  $x_0$  of an open set  $\Omega \subset \mathbb{R}^n$  with Lipschitz continuous boundary, under the assumption that  $\Omega$  satisfies a *visibility condition* at  $x_0$ . The main results proved in this chapter are contained in [29]. From now on, given  $x \in \mathbb{R}^n$ , we write  $x = (x', x_n)$ , where  $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^n$ . We denote by  $B'_r(x')$  the  $(n-1)$ -dimensional ball of radius  $r$  and center  $x'$ . Moreover, given a Lipschitz function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ , we define  $\text{Lip}(f, A)$  as the Lipschitz constant of  $f|_A$ , for all sets  $A \subset \mathbb{R}^n$ , and we set  $\text{Lip}(f) := \text{Lip}(f, \mathbb{R}^n)$ .

### 3.1 A result on the uniform convergence.

In the next section, we will use the following technical Lemma.

**Lemma 3.1.1.** *Let  $f_j \in \text{Lip}(\mathbb{R}^n)$  be a sequence of Lipschitz continuous functions with the following properties:*

$$\sup_{j \geq 1} |f_j(0)| < \infty, \quad L := \sup_{j \geq 1} \text{Lip}(f_j) < \infty. \quad (3.1.1)$$

*Let us assume that there exists a function  $f$  such that*

$$f_j(x) \rightarrow f(x), \quad \text{for all } x \in \mathbb{R}^n.$$

Then  $f_j$  converges to  $f$  locally uniformly in  $\mathbb{R}^n$ , i.e.

$$\|f_j - f\|_{L^\infty(B_R)} \rightarrow 0, \quad \text{for all } R > 0.$$

*Proof.* We argue by contradiction. Let us assume that there exists  $R > 0$  such that

$$\|f_j - f\|_{L^\infty(B_R)} \not\rightarrow 0.$$

Then we need to have  $\ell = \limsup_{j \rightarrow \infty} \|f_j - f\|_{L^\infty(B_R)} > 0$ . Hence, in particular, for a suitable subsequence  $g_k = f_{j_k}$ , we should have

$$\|g_k - f\|_{L^\infty(B_R)} \rightarrow \ell. \tag{3.1.2}$$

Owing to (3.1.1), we are in position to apply Ascoli-Arzelà Theorem to  $g_k$  deducing that it admits a subsequence uniformly convergent to  $f$  in  $B_R$ . Thus we conclude, because this condition contradicts (3.1.2).  $\square$

## 3.2 The visibility property

In this section we introduce the visibility property and its main consequences. In what follows,  $\Omega \subset \mathbb{R}^n$  denotes an open set with Lipschitz boundary such that  $0 \in \partial\Omega$  and  $\partial\Omega$  is a graph in a neighborhood of 0, as in (1.4.3). For notational convenience, we will only consider the visibility property at 0, but of course, we could equally define the property at a generic point of  $\partial\Omega$ .

**Definition 3.2.1.** *We say that  $\Omega$  satisfies the visibility property provided there exists  $T > 0$  and a function  $u \in C^1([0, T])^1$  such that:*

(V1)  $u(0) = u'(0) = 0$  and  $0 \leq u' \leq 2^{-1}$ ;

(V2) *The function*

$$\gamma_u(t) := t^{-1} \sup_{0 < s \leq t} \sqrt{\frac{u(s)}{s} + u'(s)}$$

---

<sup>1</sup>By  $u \in C^1([0, T])$  we mean that  $u \in C^1(0, T)$  and there exist finite the limits of  $u(t)$  and  $u'(t)$  as  $t \rightarrow 0$ .

is summable on  $(0, T)$ ;

(V3) for all  $0 < t < T$ , the segment joining the point  $U_t = -u(t)e_n$  with a point  $x$  belonging to  $\partial\Omega \cap B_t$  does not intersect  $\Omega$ .

**Remark 3.2.2.** We note that (V1) and (V2) imply that  $u(t) = o(t)$  and

$$t\gamma_u(t) \rightarrow 0, \quad (3.2.1)$$

as  $t \rightarrow 0$ . Moreover, the summability of  $\gamma_u(t)$  implies that of  $t^{-2}u(t)$ , indeed  $0 \leq u(t) \leq t/2$  by (V1) and thus

$$0 \leq \frac{u(t)}{t^2} = t^{-1} \frac{u(t)}{t} \leq t^{-1} \sqrt{\frac{u(t)}{t}} \leq \gamma_u(t). \quad (3.2.2)$$

In the following proposition, we rewrite the assumption (V3) in the form of a property involving the functions  $\omega(x')$  and  $u(t)$ . This will be particularly useful when checking the visibility property for relevant classes of domains (see the examples at the end of the section).

**Proposition 3.2.3.**  $\Omega$  satisfies the property (V3) in Definition 3.2.1 if and only if, for any  $\nu \in \partial B'_1$  and for all  $0 < t < T$ , the slope  $m_t(s)$  of the line connecting  $U_t$  with  $(s, \omega(s\nu))$ , that is given by

$$m_t(s) = \frac{\omega(s\nu) + u(t)}{s},$$

is non-increasing as a function of  $s$ , for  $s > 0$  such that  $s^2 + \omega(s\nu)^2 < t^2$ .

*Proof.* Let us assume that (V3) holds, and set  $\omega_\nu(s) = \omega(s\nu)$  for more simplicity. By contradiction, let  $s_1 < s_2$  be such that  $s_i^2 + \omega_\nu(s_i)^2 < t^2$ , for  $i = 1, 2$ , and  $m_t(s_1) < m_t(s_2)$ . By definition of  $m_t$ , we have

$$\frac{\omega_\nu(s_1) + u(t)}{s_1} < \frac{\omega_\nu(s_2) + u(t)}{s_2},$$

and so equivalently

$$\omega_\nu(s_1) < \frac{s_1}{s_2} (\omega_\nu(s_2) + u(t)) - u(t).$$

This implies that the point

$$x = \left( s_1\nu, \frac{s_1}{s_2} (\omega_\nu(s_2) + u(t)) - u(t) \right)$$

is internal to  $\Omega$  and lies on the segment connecting  $(s_2\nu, \omega_\nu(s_2))$ . This contradicts (V3).

Conversely, let us suppose that  $m_t(s)$  is non-increasing in  $s$ , for  $s > 0$  such that  $s^2 + \omega_\nu(s)^2 < t^2$ . Set  $P(s) = (s\nu, \omega_\nu(s))$ , then, arguing by contradiction, assume that  $s_2 > 0$  is such that  $s_2^2 + \omega_\nu(s_2)^2 < t^2$  and there exists  $\lambda \in (0, 1)$  with the property

$$(1 - \lambda)U_t + \lambda P(s_2) = (\lambda s_2, \lambda(u(t) + \omega_\nu(s_2)) - u(t)) \in \Omega.$$

Then  $\omega_\nu(\lambda s_2) < \lambda(u(t) + \omega_\nu(s_2)) - u(t)$ . By continuity, for all  $\delta > 0$  there exists  $\lambda_\delta \in (\lambda, 1)$  such that

$$\lambda_\delta(u(t) + \omega_\nu(s_2)) - u(t) - \delta < \omega_\nu(\lambda_\delta s_2) < \lambda_\delta(u(t) + \omega_\nu(s_2)) - u(t). \quad (3.2.3)$$

Since the segment  $[U_t, P(s_2)]$  is compactly contained in  $B_t$ , by (3.2.3), we can pick  $\delta > 0$  small enough and a correspondent  $\lambda_\delta$  such that

$$P(\lambda_\delta s_2) \in B_t. \quad (3.2.4)$$

Let  $s_1 = \lambda_\delta s_2$ . We observe that (3.2.4) and (3.2.3) imply

$$m_t(s_1) < m_t(s_2), \quad s_1^2 + \omega_\nu(s_1)^2 < t^2,$$

and this contradicts our hypothesis. This completes the proof of the proposition.  $\square$

**Corollary 3.2.4.** *Assume that  $\omega$  satisfies*

$$\langle x', \nabla \omega(x') \rangle \leq \omega(x') + u(|x'|) \quad \text{for a.e. } x' \in B'_\rho, \quad (3.2.5)$$

where  $u : (0, T) \rightarrow \mathbb{R}$  is a non-decreasing function satisfying properties (V1) and (V2). Then  $\Omega$  satisfies the visibility property.

*Proof.* Since  $\omega$  is Lipschitz, the function  $m_t$  defined in the statement of Proposition 3.2.3 is a.e. differentiable, thus  $m_t$  is non-increasing if and only if  $m'_t(s) \leq 0$  at almost every  $s$ . We observe that

$$m'_t(s) = \frac{\omega'_\nu(s)}{s} - \frac{\omega_\nu(s) + u(t)}{s^2},$$

thus  $m'_t \leq 0$  if and only if

$$s \omega'_\nu(s) \leq \omega_\nu(s) + u(t). \quad (3.2.6)$$

The hypothesis (3.2.5) implies that

$$s \omega'_\nu(s) \leq \omega_\nu(s) + u(s) \quad \text{for almost all } 0 < s < T. \quad (3.2.7)$$

Hence if  $s > 0$  is such that  $s^2 + \omega_\nu(s)^2 < t^2$ , then  $s < t$ , and by (3.2.7), since  $u$  is non-decreasing, we obtain

$$s \omega'_\nu(s) \leq \omega_\nu(s) + u(t),$$

that is precisely (3.2.6). Consequently, (V3) is verified thanks to Proposition 3.2.3.  $\square$

### 3.2.1 Existence of the tangent cone

An important consequence of the visibility property is the existence of the tangent cone to  $\Omega$  at 0.

**Proposition 3.2.5.** *Let  $\Omega \subset \mathbb{R}^n$  satisfy the visibility property. Then there exists the tangent cone to  $\Omega$  at 0, denoted by  $\Omega_0$ . More precisely, if we set  $\Omega_s := s^{-1}\Omega$  for  $s > 0$ , we obtain*

$$\lim_{s \rightarrow 0^+} \text{dist}_{\mathcal{H}}(\Omega_s \cap B_R, \Omega_0 \cap B_R) = 0, \quad \text{for all } R > 0. \quad (3.2.8)$$

*Proof.* Let us fix  $\nu \in \partial B'_1$ , and let

$$\omega_\nu(s) = \omega(s\nu), \quad s \geq 0,$$

where  $\omega$  is the function realizing (1.4.3). Let  $s > 0$  be a point where  $\omega_\nu$  is differentiable and let  $t = \sqrt{s^2 + \omega_\nu(s)^2}$ . By (V3) in Definition 3.2.1, we deduce that the slope of the line connecting  $(s, \omega_\nu(s))$  with  $U_t$  needs to be bounded below by  $\omega'_\nu(s)$ , that is,

$$\omega'_\nu(s) \leq \frac{\omega_\nu(s) + u(t)}{s}.$$

We set  $L = \sqrt{1 + \text{Lip}(\omega)^2}$  and observe that  $t \leq Ls$ . Since  $u$  is non-decreasing by (V1), we infer

$$\omega'_\nu(s) \leq \frac{\omega_\nu(s) + u(Ls)}{s},$$

hence we get

$$\left(\frac{\omega_\nu(s)}{s}\right)' = \frac{\omega'_\nu(s)}{s} - \frac{\omega_\nu(s)}{s^2} \leq \frac{u(Ls)}{s^2} \leq L^2 \frac{u(Ls)}{(Ls)^2}. \quad (3.2.9)$$

We integrate (3.2.9) between  $s_1 < s_2$  thanks to (V2) (see Remark 3.2.2), and obtain

$$\frac{\omega_\nu(s_2)}{s_2} - \frac{\omega_\nu(s_1)}{s_1} \leq L^2 \int_{s_1}^{s_2} \frac{u(Ls)}{(Ls)^2} ds = L \int_{Ls_1}^{Ls_2} \frac{u(t)}{t^2} dt.$$

Thus we conclude that the function

$$s \mapsto \frac{\omega_\nu(s)}{s} - L \int_0^{Ls} \frac{u(t)}{t^2} dt$$

is monotonically non-increasing in  $s$  and bounded by  $\text{Lip}(\omega) + L \int_0^{Ls} t^{-2} u(t) dt$ , for  $0 < s < L^{-1}T$ . Therefore there exists

$$D_\nu^+ \omega(0) := \lim_{s \rightarrow 0^+} \frac{\omega_\nu(s)}{s} = \lim_{s \rightarrow 0^+} \frac{\omega_\nu(s)}{s} - L \int_0^{Ls} \frac{u(t)}{t^2} dt \in \mathbb{R}.$$

Let us define

$$\omega_0(x') = \begin{cases} |x'| D_{\frac{x'}{|x'|}}^+ \omega(0) & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0. \end{cases}$$

The function  $\omega_0$  is 1-homogeneous, therefore the set

$$\Omega_0 = \{x \in \mathbb{R}^n : x_n > \omega_0(x')\}$$

is a cone with vertex at 0. Now, for all  $s > 0$ , we set

$$\omega_s(x') = \begin{cases} \frac{\omega(sx')}{s} & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0. \end{cases} \quad (3.2.10)$$

It is immediate to observe that  $\omega_s(0) = 0$  and, setting  $t = s|x'|$ ,

$$\omega_s(x') = \frac{\omega(t(x'/|x'|))}{t} |x'| \rightarrow |x'| D_{\frac{x'}{|x'|}}^+ \omega(0) = \omega_0(x'), \quad \text{as } s \rightarrow 0^+.$$

Since  $\{\omega_s\}_{s>0}$  is a one-parameter family of locally equi-bounded and equi-Lipschitz func-

tions that pointwisely converge to  $\omega_0$  as  $s \rightarrow 0^+$ , we can apply Lemma 3.1.1 to conclude that this convergence is locally uniform. This easily implies the Hausdorff convergence stated in (3.2.8).  $\square$

**Remark 3.2.6.** *We note that for the proof of Proposition 3.2.5, the hypothesis (V2) of Definition 3.2.1 can be replaced by the weaker hypothesis of summability of  $t^{-2}u(t)$  on  $(0, T)$ . We also observe that, if  $\Omega$  is convex, then the existence of the tangent cone  $\Omega_0$  is always granted, even though (V2) may not be satisfied.*

### 3.2.2 An off-centric visibility property

The next lemma shows that the assumption (V3) in Definition 3.2.1 can be replaced by an equivalent assumption, where off-centric balls are taken instead of balls centered at 0. This off-centric visibility property will be useful later on.

**Lemma 3.2.7.** *The following properties are equivalent:*

- (i)  $\Omega$  satisfies the visibility property;
- (ii) There exist  $R > 0$  and a function  $v \in C^1(0, R)$  satisfying properties (V1), (V2) of Definition 3.2.1, and
  - (V3') for all  $0 < r < R$ , any segment joining the point  $V_r = -v(r)e_n$  with a point  $x$  belonging to  $\partial\Omega \cap B_r(V_r)$  does not intersect  $\Omega$ .

*Proof.* We prove that (i) implies (ii). Let  $z(t) = t - u(t)$  where  $u$  is as in Definition 3.2.1. We can find  $0 < T' < T$  such that  $z(t)$  is an increasing  $C^1$  diffeomorphism of the interval  $(0, T')$  with the property

$$\frac{1}{2}t \leq z(t) \leq t.$$

Let  $R = z(T')$ . Then we can consider the inverse  $z^{-1}$  of  $z$  in  $(0, R)$ , that is an increasing diffeomorphism such that

$$r \leq z^{-1}(r) \leq 2r. \quad (3.2.11)$$

Setting  $U_t = -u(t)e_n$ , it follows that  $B_{t-u(t)}(U_t) \subset B_t$ , for all  $0 < t < T$ , thus (V3) holds for all points  $x \in \partial\Omega \cap B_{t-u(t)}(U_t)$ . We then have that

$$B_r(U_{z^{-1}(r)}) \subset B_{z^{-1}(r)}, \quad \text{for any } 0 < r < R.$$

Any line segment joining  $x \in \partial\Omega \cap B_{z^{-1}(r)}$  with  $U_{z^{-1}(r)}$  does not intersect  $\Omega$ , hence the same property holds for any  $x \in \partial\Omega \cap B_r(U_{z^{-1}(r)})$ . Let

$$v(r) = u(z^{-1}(r)), \quad 0 < r < R.$$

It is clear that (V3') holds. By (3.2.11), up to possibly reducing the value of  $R$ , (V2) and  $v' \leq 2^{-1}$  follow. Since both  $u$  and  $z^{-1}$  are non-decreasing,  $v$  is non-decreasing, thus also (V1) is satisfied. A completely analogous argument shows that (ii) implies (i), and the proof is concluded.  $\square$

For all  $0 < r < R$ , we define

$$\mathcal{C}_r = \{V_r + t(z - V_r) : z \in \partial B_r(V_r) \cap \Omega, t > 0\}. \quad (3.2.12)$$

The set  $\mathcal{C}_r$  is an open cone with vertex at  $V_r$ .

**Lemma 3.2.8.** *Assume that  $\Omega$  satisfies the (off-centric) visibility property. Then, for all  $0 < r < R$ , the cone  $\mathcal{C}_r$  contains  $\Omega \cap B_r(V_r)$ .*

*Proof.* By contradiction, let  $x \in (\Omega \cap B_r(V_r)) \setminus \mathcal{C}_r$ . For all  $0 < t \leq r$ , let

$$x_t = V_r + t \frac{x - V_r}{|x - V_r|},$$

and note that  $x = x_{t_0}$  for a suitable  $0 < t_0 < r$ . It's clear that  $x_r \notin \Omega$ , otherwise  $x_{t_0}$  would belong to  $\mathcal{C}_r$ , for all  $t > 0$ , which is against our assumption. We can then select a value  $s \in (t_0, r]$  such that  $x_s \in \partial\Omega$ . This leads to a contradiction with the visibility because the segment joining  $V_r$  and  $x_s \in B_r(V_r) \cap \partial\Omega$  contains  $x_{t_0} = x \in \Omega$ .  $\square$

### 3.2.3 Foliation by off-centric spheres

Let us consider the family of off-centric balls  $B_r(V_r)$ , with  $V_r = -v(r)e_n$ , for  $0 < r < R$ . By the Implicit Function Theorem we can easily show the existence of a smooth function  $\phi$ , such that  $\partial B_r(V_r)$  is the  $r$ -level set of  $\phi$ . This means that the punctured ball  $B_R(V_R) \setminus \{0\}$  is foliated by the spheres  $\partial B_s(V_s) = \phi^{-1}(s)$  for  $0 < s < R$ .



**Lemma 3.2.9.** *There exists a function  $\phi \in C^1(B_R(V_R) \setminus \{0\})$  such that  $0 \leq \phi < R$  and*

$$\partial B_r(V_r) = \phi^{-1}(r), \quad \text{for any } 0 < r < R.$$

*In particular, for any  $x \in B_R(V_R) \setminus \{0\}$  we have*

$$\frac{\nabla \phi(x)}{|\nabla \phi(x)|} = \frac{x - V_{\phi(x)}}{|x - V_{\phi(x)}|} \quad (3.2.13)$$

*and*

$$\left| \nabla \phi(x) - \frac{x}{|x|} \right| \leq 7 \sqrt{\frac{v(\phi(x))}{\phi(x)} + v'(\phi(x))}. \quad (3.2.14)$$

*Proof.* If  $v(r)$  is identically 0, then there is nothing to prove because  $\phi(x) = |x|$  in this case. We then suppose  $v \neq 0$ . Let us start by proving the existence of the function  $\phi$ . We observe that, for any  $x \in B_R(V_R)$ , there exists a unique  $r = r_x \in [0, R)$  such that  $x \in \partial B_r(V_r)$ . Indeed, if  $x = 0$ , we can take  $r = 0$ . Otherwise, let  $F : (B_R(V_R) \setminus \{0\}) \times (0, R) \rightarrow \mathbb{R}$  be the function defined by

$$F(x, r) = |x - V_r|^2 - r^2. \quad (3.2.15)$$

It is immediate to observe that  $F$  is continuous. Moreover,

$$F(x, 0) = |x|^2 > 0 \quad \text{and} \quad F(x, R) < 0, \quad \text{for all } x \in B_R(V_R) \setminus \{0\}.$$

Hence we can find  $r \in (0, R)$  such that  $F(x, r) = 0$ , i.e. such that  $x \in \partial B_r(V_r)$ . Let us show the uniqueness. Indeed, if  $r, r' \in (0, R)$  have the property that

$$x \in \partial B_r(V_r) \cap \partial B_{r'}(V_{r'}),$$

then we get

$$|r - r'| = ||x - V_r| - |x - V_{r'}|| \leq |V_r - V_{r'}| = |v(r) - v(r')| \leq \frac{1}{2} |r - r'|,$$

thus we must have  $r = r'$ . Now we can define  $\phi(x) = r_x$ . Let us show that  $\phi \in$

$\mathcal{C}^1(B_R(R) \setminus \{0\})$ . To do so, we note that  $\phi$  is implicitly defined by

$$F(x, \phi(x)) = 0, \quad (3.2.16)$$

where  $F$  is the function defined in (3.2.15). Easy computations give

$$\partial_r F(x, r) = -2r + 2v'(r)(x_n + v(r)).$$

Therefore, if we assume  $F(x, r) = 0$  (that is,  $x \in B_r(V_r)$ ) we obtain

$$\partial_r F(x, r) = -2r + 2v'(r)(x_n + v(r)) \leq 2r + |x_n + v(r)| \leq -r,$$

where the first inequality follows from the assumption  $0 \leq v' \leq 2^{-1}$ , while the second inequality from

$$|x_n + v(r)| = |\langle x - V_r, e_n \rangle| \leq |x - V_r| = r.$$

By the Implicit Function Theorem we deduce that  $\phi \in \mathcal{C}^1(B_R(V_R) \setminus \{0\})$ . The identity (3.2.13) is a consequence of the fact that, if  $\phi(x) = r$ , then the vector  $\nabla\phi(x)$  is orthogonal to the level set  $\partial B_r(V_r) = \phi^{-1}(r)$  at  $x$ .

Let us now prove (3.2.14). We first observe that, if  $\phi(x) = r > 0$ , then

$$\nabla\phi(x) = -\frac{\partial_x F(x, r)}{\partial_r F(x, r)} = \frac{x + v(r)e_n}{r - v'(r)(x_n + v(r))} \quad (3.2.17)$$

Then (3.2.17) yields

$$\begin{aligned}
\left| \nabla \phi(x) - \frac{x}{|x|} \right|^2 &= \frac{|x + v(r)e_n|^2}{(r - v'(r)(x_n + v(r)))^2} + 1 - 2 \left\langle \frac{x + v(r)e_n}{r - v'(r)(x_n + v(r))}, \frac{x}{|x|} \right\rangle \\
&= \frac{|x + v(r)e_n|^2}{(r - v'(r)(x_n + v(r)))^2} + 1 - 2 \frac{|x + v(r)e_n|^2 - v(r)(x_n + v(r))}{(r - v'(r)(x_n + v(r)))|x|} \\
&= \frac{|x|r^2 + |x|(r - v'(r)(x_n + v(r)))^2 - 2(r - v'(r)(x_n + v(r)))[r^2 - v(r)(x_n + v(r))]}{|x|(r - v'(r)(x_n + v(r)))^2} \\
&= \frac{\frac{|x|}{r} + \frac{|x|}{r} \left(1 - \frac{v'(r)(x_n + v(r))}{r}\right)^2 - 2 \left(1 - \frac{v'(r)(x_n + v(r))}{r}\right) \left(1 - \frac{v(r)}{r} \frac{x_n + v(r)}{r}\right)}{\frac{|x|}{r} \left(1 - \frac{v'(r)(x_n + v(r))}{r}\right)^2} \\
&= \frac{1 + \left(1 - \frac{v'(r)(x_n + v(r))}{r}\right)^2 - 2 \frac{r}{|x|} \left(1 - \frac{v'(r)(x_n + v(r))}{r}\right) \left(1 - \frac{v(r)}{r} \frac{x_n + v(r)}{r}\right)}{\left(1 - \frac{v'(r)(x_n + v(r))}{r}\right)^2}.
\end{aligned} \tag{3.2.18}$$

Next we observe that  $||x| - r| = |x - |x - V_r||$ , hence

$$r - v(r) \leq |x| \leq r + v(r). \tag{3.2.19}$$

Exploiting (3.2.19), then summing and subtracting 2 in the numerator of the last term of (3.2.18), we get

$$\left| \nabla \phi(x) - \frac{x}{|x|} \right|^2 \leq \frac{2 - 2 \left(1 - \frac{v(r)}{r - v(r)}\right) \left(1 - \frac{v'(r)(x_n + v(r))}{r}\right) \left(1 - \frac{v(r)}{r} \frac{x_n + v(r)}{r}\right)}{\left(1 - \frac{v'(r)(x_n + v(r))}{r}\right)^2}.$$

We now recall that  $v' \leq 2^{-1}$ , thus in particular  $v(r) \leq 2^{-1}r$ , and we observe that  $x_n + v(r) \leq r$ , which leads to the final estimate

$$\begin{aligned}
\left| \nabla \phi(x) - \frac{x}{|x|} \right|^2 &\leq 8 \left\{ \frac{v(r)}{r - v(r)} + \frac{v'(r)(x_n + v(r))}{r} + \frac{v(r)(x_n + v(r))}{r^2} + \frac{v(r)^2 v'(r)(x_n + v(r))^2}{(r - v(r))r^2} \right\} \\
&\leq 8 \left\{ \frac{v(r)}{r - v(r)} + v'(r) + \frac{v(r)}{r} + \frac{v(r)}{2(r - v(r))} \right\} \\
&\leq 40 \left\{ \frac{v(r)}{r} + v'(r) \right\},
\end{aligned}$$

and this concludes the proof.  $\square$

### 3.2.4 Some examples

We exhibit some relevant classes of domains for which the visibility holds. We recall that

$$\omega_\nu(s) = \omega(s\nu), \quad \text{for } s \geq 0.$$

**Example 3.2.10** (Lipschitz cones and outer star-shaped sets). *Let  $\Omega$  be either a cone with respect to 0, or such that its complement  $\mathbb{R}^n \setminus \Omega$  is locally star-shaped with respect to 0. It is immediate to check that  $\Omega$  satisfies the visibility property with visibility function  $u(t) \equiv 0$ .*

**Example 3.2.11** ( $C^{1,\beta}$ -sets). *Let  $\Omega$  have  $C^{1,\beta}$  boundary and assume  $0 \in \partial\Omega$ . We show that  $\Omega$  satisfies the visibility property. Up to a rotation we can assume that  $\Omega$  admits a representation as in (1.4.3) with  $\omega \in C^{1,\beta}(B'_\rho)$ . By Corollary 3.2.4, it is enough to show that  $\omega$  satisfies (3.2.5). Since  $\nabla\omega$  is  $\beta$ -Hölder, we have*

$$\langle x', \nabla\omega(x') \rangle \leq \langle x', \nabla\omega(0) \rangle + u(|x'|). \quad (3.2.20)$$

where  $u(t) = Ct^{1+\beta}$  for some  $C > 0$ . Set  $\bar{\omega}(x') := \omega(x') - \langle \nabla\omega(0), x' \rangle$  and note that  $\bar{\omega}$  is  $C^{1,\beta}$ ,  $\bar{\omega}(0) = 0$ ,  $\nabla\bar{\omega}(0) = 0$ , and

$$\begin{aligned} |\omega(x') - \langle x', \nabla\omega(0) \rangle| &\leq \max_{|y'| \leq |x'|} |\nabla\bar{\omega}(y')| |x'| \\ &\leq (|\nabla\bar{\omega}(0)| + C|x|^\beta) |x'| \\ &= C|x'|^{1+\beta}. \end{aligned} \quad (3.2.21)$$

Putting together (3.2.20) and (3.2.21), we finally get

$$\langle x', \nabla\omega(x') \rangle \leq \omega(x') + u(|x'|), \quad \text{for a.e. } x' \in B'_\rho,$$

that is precisely (3.2.5). Since trivially  $u$  is non-decreasing and satisfies (V2), by Corollary 3.2.4 we infer that  $\Omega$  satisfies the visibility property.

**Example 3.2.12** (Convex sets satisfying (V2)). *Let  $\Omega$  be a convex set with  $0 \in \partial\Omega$ . For  $s > 0$ , let*

$$\Omega_s := \frac{1}{s} \Omega, \quad \Omega_0 := \bigcup_{s>0} \frac{1}{s} \Omega.$$

*The set  $\Omega_0$  is the tangent cone to  $\Omega$  at 0. Let*

$$u(r) := \text{dist}_{\mathcal{H}}(\Omega \cap B_r, \Omega_0 \cap B_r), \quad (3.2.22)$$

*and assume that  $u(r)$  satisfies (V2).*

*We observe that  $u$  is monotonically non-decreasing in  $r$ . Let us prove that  $\Omega$  satisfies the visibility property. As before, we assume that  $\Omega$  admits a graphical representation as in (1.4.3) with the further property that  $\omega : B'_\rho \rightarrow \mathbb{R}$  is convex. Using the notations introduced in the proof of Proposition 3.2.5, by the convexity of  $\omega$ , we can define*

$$\omega_0(x') = \begin{cases} |x'| D_{\frac{x'}{|x'|}}^+ \omega(0) & \text{if } x' \neq 0, \\ 0 & \text{if } x' = 0, \end{cases}$$

*and deduce that*

$$\Omega_0 = \{x \in \mathbb{R}^n : x_n > \omega_0(x')\}.$$

*By the definition of  $u$  given in (3.2.22), we have*

$$\|\omega - \omega_0\|_{L^\infty(B'_r)} \leq C u(r), \quad \text{for some } C > 0. \quad (3.2.23)$$

*Owing to Corollary 3.2.4, the visibility property can be proved by showing that (3.2.5) holds. Thanks to the convexity of  $\omega$ , for all  $\nu \in \partial B'_1$ , we have*

$$D_\nu^+ \omega(0) \leq \omega'_\nu \left( \frac{s^+}{2} \right) := \lim_{\sigma \rightarrow 0^+} \frac{\omega_\nu(s/2 + \sigma) - \omega_\nu(s/2)}{\sigma}, \quad \text{for all } 0 \leq s < \rho. \quad (3.2.24)$$

Moreover, for all  $0 < \sigma < s/2$ , by (3.2.23) and the convexity of  $\omega_\nu$ , we have

$$\begin{aligned}
\frac{\omega_\nu(s/2 + \sigma) - \omega_\nu(s/2)}{\sigma} &\leq \frac{\omega_\nu(s) - \omega_\nu(s/2)}{s/2} \\
&= 2 \frac{\omega_\nu(s)}{s} - \frac{\omega_\nu(s/2)}{s/2} \\
&= 2 \frac{\omega_\nu(s)}{s} - 2D_\nu^+ \omega(0) + D_\nu^+ \omega(0) - \frac{\omega_\nu(s/2)}{s/2} + D_\nu^+ \omega(0) \quad (3.2.25) \\
&= \frac{2}{s} (\omega_\nu(s) - \omega_0(s\nu) + \omega_0(s\nu/2) - \omega_\nu(s/2)) + D_\nu^+ \omega(0) \\
&\leq D_\nu^+ \omega(0) + \tilde{C} \frac{u(s)}{s}, \quad \text{for some } \tilde{C} > 0.
\end{aligned}$$

Putting together (3.2.24) and (3.2.25), we obtain

$$\left| D_\nu^+ \omega(0) - \omega'_\nu\left(\frac{s^+}{2}\right) \right| \leq \tilde{C} \frac{u(s)}{s}.$$

This suffices to conclude. In fact, if  $x' \in B'_\rho \setminus \{0\}$  is such that  $\omega$  is differentiable at  $x'$ , setting  $s := |x'|$ ,  $\nu := x'/|x'|$ , for some  $C > 0$ , we achieve

$$\begin{aligned}
\langle x', \nabla \omega(x') \rangle &= \langle s\nu, \nabla \omega(s\nu) \rangle = s \omega'_\nu(s^+) \\
&\leq s D_\nu^+ \omega(0) + C u(s) \\
&\leq \omega(s\nu) + C u(s) = \omega(x') + C u(|x'|).
\end{aligned}$$

Since  $\omega$  is convex, it is differentiable a.e. in  $B'_\rho$ , and so (3.2.5) is verified. Moreover,  $u$  is non-decreasing and satisfies (V2) by our assumption, thus we conclude.

**Example 3.2.13** (A piece-wise affine profile). For  $i \in \mathbb{N}$ , let

$$y_i = 2^{-i}, \quad \tilde{y}_i = 2^{-i} + 2^{-2i}.$$

We observe that  $y_{i+1} < \tilde{y}_{i+1} < y_i$ . For  $i \in \mathbb{N}$ , we define

$$P_i = (y_i, \tilde{y}_i), \quad Q_i = (\tilde{y}_i, y_i).$$

We can consider the polygonal curve formed by the segments  $\overline{P_i Q_{i+1}}, \overline{Q_{i+1} P_{i+1}}, i \geq 0$ . It is

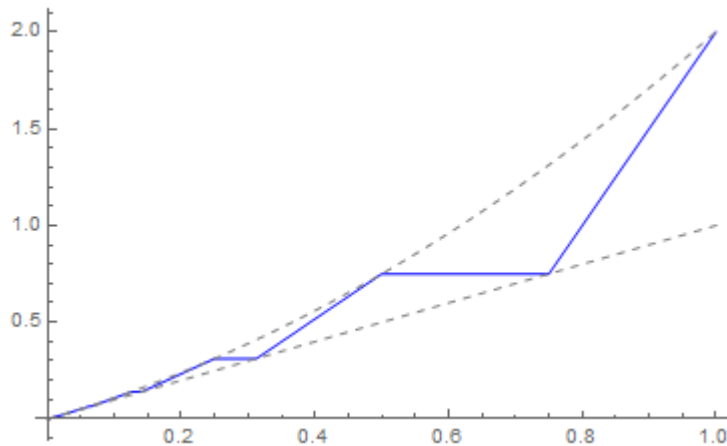


Figure 3.1: The graph of  $\omega(y)$  "bounces" between the graphs of  $y$  and  $y + y^2$ .

immediate to observe that this curve coincides with the graph of the function  $\omega : [0, 1] \rightarrow \mathbb{R}$  uniquely defined by

$$\omega(y) = \begin{cases} 2^{-(i+1)} + 2^{-2(i+1)} & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ a_i y - b_i & \text{if } y \in (\tilde{y}_{i+1}, y_i], \end{cases} \quad (3.2.26)$$

for  $i \geq 0$ , where

$$a_i = \frac{1 + 3 \cdot 2^{-(i+1)}}{1 - 2^{-(i+1)}}, \quad b_i = \frac{2^{-2i} + 2^{-(3i+1)}}{1 - 2^{-(i+1)}}.$$

Let  $\Omega \subset \mathbb{R}^2$  be an open set such that

$$\Omega \cap ((-1, 1) \times \mathbb{R}) = \{x = (y, z) \in \mathbb{R}^2 : z > \omega(|y|)\}.$$

Let us show that  $\Omega$  satisfies the visibility condition at 0. Owing to Corollary 3.2.4, it suffices to show the existence of a non-decreasing function  $u : (0, 1) \rightarrow \mathbb{R}$  satisfying (V1), (V2) and such that

$$y\omega'(y) \leq \omega(y) + u(y), \quad \text{for a.e. } y \in (-1, 1). \quad (3.2.27)$$

By (3.2.26), for every  $i \in \mathbb{N}$ , we have

$$y \omega'(y) = \begin{cases} 0 & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ a_i y & \text{if } y \in (\tilde{y}_{i+1}, y_i]. \end{cases}$$

Thus, in order to realize (3.2.27), it suffices to choose a function  $u = u(y)$  greater or equal than the function

$$\bar{u}(y) = \begin{cases} 0 & \text{if } y \in (y_{i+1}, \tilde{y}_{i+1}], \\ b_i & \text{if } y \in (\tilde{y}_{i+1}, y_i]. \end{cases}$$

We look for  $\alpha > 0$  such that

$$u_\alpha(y) = \alpha y^2 \geq \bar{u}(y). \quad (3.2.28)$$

In order to obtain the validity of (3.2.28), it suffices to impose that

$$u_\alpha(\tilde{y}_{i+1}) = \alpha \tilde{y}_{i+1}^2 \geq b_i.$$

It is immediate to observe that

$$\alpha \tilde{y}_{i+1}^2 \geq \frac{\alpha}{4} 2^{-2i}, \quad b_i \leq 4 \cdot 2^{-2i},$$

hence, if for instance we take  $\alpha = 16$ , (3.2.28) holds. Since the function  $u(y) = 16y^2$  trivially fulfills (V1) and (V2), we conclude that  $\Omega$  satisfies the visibility condition at 0.

### 3.3 Boundary Monotonicity Formula

The present section aims to prove a boundary Monotonicity Formula for local almost-minimizers of  $P_\Omega$  at a point  $x_0 \in \partial\Omega$  satisfying the visibility property up to an isometry (hence, from now on, we will directly assume  $x_0 = 0$ ). In the notation of the previous section, given  $0 < r_1 < r_2 < R$ , we recall  $V_r = -v(r)e_n$  and define

$$\mathcal{A}_{r_1, r_2} := B_{r_2}(V_{r_2}) \setminus \overline{B_{r_1}(V_{r_1})} = \phi^{-1}(r_1, r_2),$$



where  $\phi(x)$  is the function defined in Lemma 3.2.9. We also conveniently introduce some further notation. Given  $f \in BV_{\text{loc}}(\Omega)$  and  $0 < r_1 < r_2$ , we set

$$\mu_f(r) = \frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}}$$

and

$$\begin{aligned} G(f; r_1, r_2) &= \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla\phi| - 1) d|Df| d\rho \\ &\quad + \frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla\phi| - 1) d|Df| - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla\phi| - 1) d|Df|. \end{aligned} \quad (3.3.1)$$

In the next proposition, we combine the visibility property with an upper bound on  $\mu_f(r)$  and obtain the finiteness of  $\lim_{\rho \rightarrow 0} G(f; \rho, r)$ .

**Proposition 3.3.1.** *Let  $\Omega \subset \mathbb{R}^n$  satisfy the visibility property as in Definition 3.2.1, and let  $f \in BV_{\text{loc}}(\Omega)$ . Assume that  $\mu_f(r) \leq C$  for some constant  $C > 0$  and for all  $r \in (0, R)$ . Then for  $r \in (0, R)$  the limit*

$$\begin{aligned} G(f; r) &:= \lim_{\rho \rightarrow 0} G(f; \rho, r) \\ &= \int_0^r \frac{n-1}{\rho^n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla\phi| - 1) d|Df| d\rho + \frac{1}{r^{n-1}} \int_{\Omega \cap B_r(V_r)} (|\nabla\phi| - 1) d|Df| \end{aligned} \quad (3.3.2)$$

exists and is finite.

*Proof.* By (3.2.14), for all  $x \in B_\rho(V_\rho)$  we have

$$\left| |\nabla\phi(x)| - 1 \right| \leq \left| \nabla\phi(x) - \frac{x}{|x|} \right| \leq 7 \sqrt{\frac{v(\phi(x))}{\phi(x)} + v'(\phi(x))} \leq 7 \sup_{0 < r \leq \rho} \sqrt{\frac{v(r)}{r} + v'(r)} = 7\rho\gamma_v(\rho),$$

where  $\gamma_v$  is the function defined in the visibility property (V2). Then, using the upper bound on  $\mu_f$ , for  $0 < \rho < r < R$  we obtain

$$\left| \rho^{1-n} \int_{\Omega \cap B_\rho(V_\rho)} (|\nabla\phi| - 1) d|Df| \right| \leq 7C\rho\gamma_v(\rho).$$

Thanks to the summability of  $\gamma_v$  (see property (V2) of the visibility property) and to

(3.2.1), from the last inequality we easily get the proof of the proposition.  $\square$

Our monotonicity formula will then follow from the general inequality proved in the next theorem.

**Theorem 3.3.2** (Monotonicity inequality). *Let  $\Omega \subset \mathbb{R}^n$  satisfy the visibility property, and let  $f \in BV_{\text{loc}}(\Omega)$ . Then, for  $R > 0$  small enough, for almost every  $0 < r_1 < r_2 < R$ , we have*

$$\left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} |\langle \nu_f, \nabla \phi \rangle| d|Df| \right)^2 \leq 2 \left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla \phi(x)|}{\phi(x)^{n-1}} d|Df| \right) \cdot \left[ \mu_f(r_2) - \mu_f(r_1) + \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_\Omega(f; B_\rho(V_\rho)) d\rho + G(f; r_1, r_2) \right], \quad (3.3.3)$$

where  $\nu_f$  is such that  $Df = \nu_f |Df|$ .

*Proof.* We start by assuming  $f \in BV(\Omega) \cap C^1(\Omega)$ . For all  $0 < r < R$  and  $x \in \mathcal{C}_r \cap B_r(V_r)$ , where  $\mathcal{C}_r$  is defined in (3.2.12), we let

$$Y_r(x) = V_r + r \frac{x - V_r}{|x - V_r|}.$$

Standard computations yield

$$\begin{aligned} DY_r(x) &= r \left[ \frac{1}{|x - V_r|} D(x - V_r) + (x - V_r) \otimes \nabla |x - V_r|^{-1} \right] \\ &= \frac{r}{|x - V_r|} \left[ \text{Id} - \frac{x - V_r}{|x - V_r|} \otimes \frac{x - V_r}{|x - V_r|} \right]. \end{aligned} \quad (3.3.4)$$

We define

$$g_r(x) = f(Y_r(x)) \quad \text{for all } x \in \mathcal{C}_r,$$

then the “off-centric conical competitor” is

$$f_r(x) = \begin{cases} g_r(x) & \text{if } x \in \mathcal{C}_r \cap B_r(V_r) \\ f(x) & \text{if } x \in \Omega \setminus B_r(V_r). \end{cases} \quad (3.3.5)$$

By definition,  $f_r$  coincides with  $f$  in  $\Omega \setminus B_r(V_r)$ , hence we infer

$$\Psi_\Omega(f; B_r(V_r)) \geq |Df|(\Omega \cap B_r(V_r)) - |Df_r|(\Omega \cap B_r(V_r)). \quad (3.3.6)$$

Then, by (3.3.6), we deduce that

$$\begin{aligned} |Df|(\Omega \cap B_r(V_r)) - \Psi_\Omega(f; B_r(V_r)) &\leq |Df_r|(\Omega \cap B_r(V_r)) \\ &\leq |Df_r|(\mathcal{C}_r \cap B_r(V_r)) \\ &= \int_{\mathcal{C}_r \cap B_r(V_r)} |\nabla g_r(x)| dx. \end{aligned} \quad (3.3.7)$$

Let us now compute the gradient of  $g_r$ . By (3.3.4), setting

$$\nu_r(x) = \frac{x - V_r}{|x - V_r|}, \quad Y_r = Y_r(x),$$

we obtain

$$\nabla g_r(x) = DY_r \cdot \nabla f(Y_r) = \frac{r}{|x - V_r|} \nabla f(Y_r)^{\nu_r(x)^\perp},$$

where  $\nabla f(Y_r)^{\nu_r(x)^\perp}$  denotes the projection of  $\nabla f(Y_r)$  onto the hyperplane

$$\nu_r(x)^\perp := \{y \in \mathbb{R}^n : \langle \nu_r(x), y \rangle = 0\}.$$

Going on with the computations, we obtain

$$\begin{aligned} |\nabla g_r(x)| &= \frac{r}{|x - V_r|} \sqrt{|\nabla f(Y_r)|^2 - \langle \nabla f(Y_r), \nu_r(x) \rangle^2} \\ &= \frac{r}{|x - V_r|} |\nabla f(Y_r)| \sqrt{1 - \frac{\langle \nabla f(Y_r), \nu_r(x) \rangle^2}{|\nabla f(Y_r)|^2}}. \end{aligned}$$

Consequently, we get

$$\begin{aligned}
\int_{\mathcal{C}_r \cap B_r(V_r)} |\nabla g_r| dx &= \int_0^r \int_{\mathcal{C}_r \cap \partial B_\rho(V_r)} \frac{r}{\rho} |\nabla f(Y_r)| \sqrt{1 - \frac{\langle \nabla f(Y_r), \nu_r \rangle^2}{|\nabla f(Y_r)|^2}} d\mathcal{H}^{n-1} d\rho \\
&= \int_0^r \left(\frac{\rho}{r}\right)^{n-2} d\rho \int_{\mathcal{C}_r \cap \partial B_r(V_r)} |\nabla f| \sqrt{1 - \frac{\langle \nabla f, \nu_r \rangle^2}{|\nabla f|^2}} d\mathcal{H}^{n-1} \\
&\leq \frac{r}{n-1} \left\{ \int_{\Omega \cap \partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} - \frac{1}{2} \int_{\Omega \cap \partial B_r(V_r)} \frac{\langle \nabla f, \nu_r \rangle^2}{|\nabla f|} d\mathcal{H}^{n-1} \right\}.
\end{aligned} \tag{3.3.8}$$

Combining (3.3.7) and (3.3.8), we get

$$\begin{aligned}
&\frac{r}{2(n-1)} \int_{\Omega \cap \partial B_r(V_r)} \frac{\langle \nabla f, \nu_r \rangle^2}{|\nabla f|} d\mathcal{H}^{n-1} \\
&\leq \frac{r}{n-1} \int_{\Omega \cap \partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} - \int_{\Omega \cap B_r(V_r)} |\nabla f| dx + \Psi_\Omega(f; B_r(V_r)).
\end{aligned} \tag{3.3.9}$$

Multiplying both sides of (3.3.9) by  $(n-1)r^{-n}$  and observing that  $r = \phi(y)$  for any  $y \in \partial B_r(V_r)$ , we get

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega \cap \partial B_r(V_r)} \left\langle \nabla f, \frac{\nabla \phi}{|\nabla \phi|} \right\rangle^2 \frac{1}{|\nabla f| \phi^{n-1}} d\mathcal{H}^{n-1} \\
&\leq \frac{1}{r^{n-1}} \int_{\Omega \cap \partial B_r(V_r)} |\nabla f| d\mathcal{H}^{n-1} + \frac{n-1}{r^n} \int_{\Omega \cap B_r(V_r)} |\nabla f| dx + \frac{n-1}{r^n} \Psi_\Omega(f; B_r(V_r)) \\
&= \frac{d}{dr} \left( \frac{1}{r^{n-1}} \int_{\Omega \cap B_r(V_r)} |\nabla f| |\nabla \phi| dx \right) \\
&\quad + \frac{n-1}{r^n} \int_{\Omega \cap B_r(V_r)} |\nabla f| (|\nabla \phi| - 1) dx + \frac{n-1}{r^n} \Psi_\Omega(f; B_r(V_r)).
\end{aligned} \tag{3.3.10}$$

Let us now integrate (3.3.10) between  $0 < r_1 < r_2 < R$ . We then achieve

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \left\langle \nabla f(x), \frac{\nabla \phi(x)}{|\nabla \phi(x)|} \right\rangle^2 \frac{|\nabla \phi(x)|}{|\nabla f(x)| \phi(x)^{n-1}} dx \\
& \leq \frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} |\nabla f(x)| |\nabla \phi(x)| dx - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} |\nabla f(x)| |\nabla \phi(x)| dx \\
& \quad + \int_{r_1}^{r_2} \frac{n-1}{r^n} \int_{\Omega \cap B_r(V_r)} |\nabla f(x)| (|\nabla \phi(x)| - 1) dx dr + \int_{r_1}^{r_2} \frac{n-1}{r^n} \Psi_\Omega(f; B_r(V_r)) dr.
\end{aligned} \tag{3.3.11}$$

By Hölder's Inequality, we get

$$\begin{aligned}
& \left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} |\langle \nabla f(x), \nabla \phi(x) \rangle| \frac{dx}{\phi(x)^{n-1}} \right)^2 \\
& \leq \left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla \phi(x)|}{\phi(x)^{n-1}} |\nabla f(x)| dx \right) \cdot \left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \left\langle \nabla f(x), \frac{\nabla \phi(x)}{|\nabla \phi(x)|} \right\rangle^2 \frac{|\nabla \phi(x)| dx}{|\nabla f(x)| \phi(x)^{n-1}} \right).
\end{aligned} \tag{3.3.12}$$

Putting together (3.3.11) and (3.3.12), we obtain

$$\begin{aligned}
& \left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} |\langle \nabla f(x), \nabla \phi(x) \rangle| \frac{dx}{\phi(x)^{n-1}} \right)^2 \leq 2 \left( \int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \frac{|\nabla \phi(x)|}{\phi(x)^{n-1}} |\nabla f(x)| dx \right) \\
& \quad \cdot \left( \frac{1}{r_2^{n-1}} \int_{\Omega \cap B_{r_2}(V_{r_2})} |\nabla f(x)| dx - \frac{1}{r_1^{n-1}} \int_{\Omega \cap B_{r_1}(V_{r_1})} |\nabla f(x)| dx \right. \\
& \quad \left. + \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_\Omega(f; B_\rho(V_\rho)) d\rho + G(f; r_1, r_2) \right),
\end{aligned} \tag{3.3.13}$$

where  $G(f; r_1, r_2)$  is as in (3.3.1). This proves (3.3.3) for all  $f \in BV(\Omega) \cap C^1(\Omega)$ .

Let now  $f \in BV(\Omega)$ . We can select a sequence  $f_j \in BV(\Omega) \cap C^1(\Omega)$  such that

$$\|f_j - f\|_{L^1(\Omega)} \rightarrow 0, \quad |Df_j|(\Omega) \rightarrow |Df|(\Omega), \quad Df_j \xrightarrow{*} Df \quad \text{in } \Omega. \tag{3.3.14}$$

In particular, by the continuity of the trace with respect to the strict convergence, we have

$$\|\text{Tr}^+(f_j, \partial\Omega) - \text{Tr}^+(f, \partial\Omega)\|_{L^1(\partial\Omega, \mathcal{H}^{n-1})} \rightarrow 0. \tag{3.3.15}$$

Let us consider the extensions

$$f_{0,j}(x) = \begin{cases} f_j(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \quad f_0(x) = \begin{cases} f(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

We observe that, by (3.3.14), (3.3.15),

$$\|f_{0,j} - f_0\|_{L^1(\mathbb{R}^n)} \rightarrow 0, \quad |Df_{0,j}|(\mathbb{R}^n) \rightarrow |Df_0|(\mathbb{R}^n).$$

By Proposition 1.3.12, for almost all  $0 < r < R$ ,

$$|Df_{0,j}|(B_r(V_r)) \rightarrow |Df_0|(B_r(V_r)), \quad \|\mathrm{Tr}^+(f_{0,j}, \partial B_r(V_r)) - \mathrm{Tr}^+(f_0, \partial B_r(V_r))\|_{L^1(\partial B_r(V_r))} \rightarrow 0, \quad (3.3.16)$$

and in particular, owing to (3.3.15),

$$|Df_j|(\Omega \cap B_r(V_r)) = |Df_{0,j}|(\Omega \cap B_r(V_r)) \rightarrow |Df_0|(\Omega \cap B_r(V_r)) = |Df|(\Omega \cap B_r(V_r)). \quad (3.3.17)$$

Now (3.3.16), (3.3.17) allow to apply Lemma 2.1.8, deducing that

$$|\Psi_\Omega(f_j; B_r(V_r)) - \Psi_\Omega(f; B_r(V_r))| \rightarrow 0, \quad \text{as } j \rightarrow \infty,$$

for almost all  $0 < r < R$ . This implies that

$$\int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_\Omega(f_j; B_\rho(V_\rho)) d\rho \rightarrow \int_{r_1}^{r_2} \frac{n-1}{\rho^n} \Psi_\Omega(f; B_\rho(V_\rho)) d\rho.$$

Finally, to conclude that the RHS of (3.3.13) for  $f = f_j$ , passes to the limit as  $j \rightarrow \infty$ , giving precisely the RHS of (3.3.3), it suffices to show that the terms

$$\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} d|Df_j|, \quad \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla \phi| - 1) d|Df_j|, \quad \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla \phi| - 1) d|Df_j|,$$

converge as  $j \rightarrow \infty$  respectively to

$$\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi^{1-n} d|Df|, \quad \int_{\Omega \cap B_{r_1}(V_{r_1})} (|\nabla \phi| - 1) d|Df|, \quad \int_{\Omega \cap B_{r_2}(V_{r_2})} (|\nabla \phi| - 1) d|Df|.$$

To see this, it suffices to construct a suitable partition of each domain, for instance using portions of circular annuli whose boundaries are negligible for  $|Df|$  and  $|Df_j|$  for all  $j \geq 1$ , to uniformly approximate each integrand by simple functions (up to removing a small neighborhood of 0 in the case of the last two integrals). About the LHS of (3.3.13), we observe that (3.3.14) implies

$$Df_j \xrightarrow{*} Df \quad \text{in } \Omega \cap \mathcal{A}_{r_1, r_2}.$$

Now, for  $f$  smooth, the LHS of (3.3.3) and the LHS of (3.3.13) coincide. Moreover, we have

$$\int_{\Omega \cap \mathcal{A}_{r_1, r_2}} \phi(x)^{1-n} |\langle \nu_f(x), \nabla \phi(x) \rangle| d|Df|(x) = \left| \phi^{1-n} \nabla \phi \cdot Df_j \right|(\Omega \cap \mathcal{A}_{r_1, r_2}).$$

In particular, (3.3.14) implies that

$$\phi^{1-n} \nabla \phi \cdot Df_j \xrightarrow{*} \phi^{1-n} \nabla \phi \cdot Df,$$

and well-known properties of the weak-star convergence of Radon measures (see [35]) ensure that

$$\left| \phi^{1-n} \nabla \phi \cdot Df \right|(\Omega \cap \mathcal{A}_{r_1, r_2}) \leq \liminf_{j \rightarrow \infty} \left| \phi^{1-n} \nabla \phi \cdot Df_j \right|(\Omega \cap \mathcal{A}_{r_1, r_2}).$$

This implies (3.3.3) and concludes the proof of the theorem.  $\square$

The next corollary, a first important consequence of Theorem 3.3.2, states the monotonicity of a suitable function of the radius  $r$ , which is defined by three terms: the renormalized perimeter  $\mu_E(r)$ , the integral of a renormalized minimality gap  $\Psi_\Omega(E; B_r(V_r))$ , and the visibility error  $G(E, r)$ . In particular, when  $E$  is an almost-minimizer, the infinitesimality of the second and third terms implies that  $\mu_E(r)$  is “almost-increasing”, hence that it admits a finite limit as  $r \rightarrow 0$ . This limit represents the perimeter density of  $E$  at 0, see Remark 3.3.4 below.

**Corollary 3.3.3** (Boundary monotonicity for almost-minimizers). *Let  $\Omega$  be an open set satisfying the visibility property. Let  $E \subset \Omega$  be a local almost-minimizer, such that*

$P_\Omega(E, B_r) > 0$  for all  $r > 0$  and

$$\int_0^R \rho^{-n} \Psi_\Omega(E; B_\rho(V_\rho)) d\rho < +\infty.$$

Then there exists  $R'$  such that the function

$$r \mapsto \mu_E(r) + (n-1) \int_0^r \rho^{-n} \Psi_\Omega(E; B_\rho(V_\rho)) d\rho + G(E; r)$$

is non-decreasing on  $(0, R')$ . Moreover, the two terms  $\int_0^r \rho^{-n} \Psi_\Omega(E; B_\rho(V_\rho)) d\rho$  and  $G(E; r)$  are infinitesimal as  $r \rightarrow 0$ , hence in particular  $\mu_E(r)$  is “almost-monotone” and the limit

$$\theta_E(0) := \lim_{r \rightarrow 0^+} \mu_E(r)$$

exists and is finite.

*Proof of Corollary 3.3.3.* By Lemma 2.1.6 and the fact that  $B_r(V_r) \subset B_{r+v(r)}$ , we can find constants  $C, R > 0$  such that

$$\frac{|Df|(\Omega \cap B_r(V_r))}{r^{n-1}} \leq \frac{|Df|(\Omega \cap B_{r+v(r)})}{(r+v(r))^{n-1}} \left(1 + \frac{v(r)}{r}\right)^{n-1} \leq C, \quad \text{for all } 0 < r < R.$$

By combining Proposition 3.3.1 with (3.2.14) and the previous bound, up to redefining the constants  $C, R > 0$ , we obtain

$$|G(E; r)| \leq C \left( \int_0^r \gamma_v(\rho) d\rho + r\gamma_v(r) \right), \quad \text{for all } 0 < r < R. \quad (3.3.18)$$

Finally, the proof of the corollary follows directly from Theorem 3.3.2 and from the observation that the RHS of (3.3.18) is infinitesimal as  $r \rightarrow 0^+$ .  $\square$

**Remark 3.3.4.** *It is easy to check that, under the assumptions of Corollary 3.3.3, one has*

$$\exists \lim_{r \rightarrow 0^+} \frac{P_\Omega(E; B_r)}{r^{n-1}} = \theta_E(0). \quad (3.3.19)$$

*Indeed, this is an immediate consequence of the inclusions*

$$B_{r-v(r)}(V_r) \subset B_r \subset B_{r+v(r)}(V_r)$$



combined with  $v(r) = o(r)$  as  $r \rightarrow 0$ .

### 3.4 Blow-up limits of almost-minimizers are cones

We now apply Theorem 3.3.2 and prove that any blow-up limit of a local almost-minimizer  $E$  of  $P_\Omega$  is a perimeter-minimizing cone.

**Theorem 3.4.1.** *Let  $\Omega \subset \mathbb{R}^n$  be an open set satisfying the visibility property. Let  $E \subset \Omega$  be a local almost-minimizer in  $\Omega$  such that*

$$\int_0^R \frac{\Psi_\Omega(E; B_r)}{r^n} dr < \infty. \quad (3.4.1)$$

*Fix a decreasing sequence  $t_j \rightarrow 0$  and set  $E_{t_j} = t_j^{-1}E$ . Then, up to subsequences,  $E_{t_j}$  converges to  $E_0 \subset \Omega_0$  in  $L^1_{\text{loc}}(\mathbb{R}^n)$ . Moreover,  $E_0$  is a nontrivial cone minimizing the relative perimeter in  $\Omega_0$ .*

*Proof.* Set  $E_j = E_{t_j}$  and  $\Omega_j = \Omega_{t_j}$  for more simplicity. Then by the upper density estimate on the relative perimeter of  $E$  (Lemma 2.1.6) coupled with analogous estimates satisfied by  $\Omega$  Lipschitz, we can find a constant  $C > 0$  such that, for every fixed  $R > 0$ ,

$$\begin{aligned} P(E_j; B_R) &\leq P(\Omega_j; B_R) + P_{\Omega_j}(E_j; B_R) \\ &= t_j^{1-n} \left( P(\Omega; B_{Rt_j}) + P_\Omega(E; B_{Rt_j}) \right) \\ &\leq CR^{n-1}. \end{aligned}$$

By the compactness property of sequences of sets with uniformly bounded relative perimeter the ball  $B_R$ , we conclude that there exists a not relabeled subsequence  $E_j$  and a set  $E_0$  of finite perimeter in  $B_R$ , such that  $E_j \rightarrow E_0$  in  $L^1(B_R)$  as  $j \rightarrow \infty$ . The fact that  $E_0 \subset \Omega_0$  up to null sets is immediate, since  $E_j \subset \Omega_j$ , for all  $j$ , and the sequence  $\Omega_j$  converges to the tangent cone  $\Omega_0$  locally in Hausdorff distance (hence, in  $L^1_{\text{loc}}(\mathbb{R}^n)$ ) thanks to Proposition 3.2.5. Up to a standard diagonal argument we can assume that the subsequence  $E_j$  converges to  $E_0$  in  $L^1_{\text{loc}}(\mathbb{R}^n)$ . Moreover by the lower-density estimates on the volume of  $E$  we also deduce that  $E_0$  can be neither the empty set, nor the whole  $\Omega_0$  up to null sets (that is,  $E_0$  is nontrivial).

By the scaling properties of the perimeter, for any fixed  $R > 0$  we have

$$\Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = \frac{1}{t_j^{n-1}} \Psi_{\Omega}(E; B_{t_j R}) \leq \omega_n^{1-\frac{1}{n}} R^{n-1} \psi_{\Omega}(E; 0, t_j R) \longrightarrow 0,$$

therefore we can apply Lemma 2.1.7 and deduce that

$$\Psi_{\Omega_0}(E_0; B_R) \leq \liminf_j \Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = 0$$

for all  $R > 0$  and, also owing to Corollary 3.3.3,

$$P_{\Omega_0}(E_0; B_R) = \lim_j P_{\Omega_{t_j}}(E_{t_j}; B_R) = \lim_j t_j^{1-n} P_{\Omega}(E; B_{t_j R} \cap \Omega) = R^{n-1} \theta_E(0).$$

Thus,  $E_0$  is a minimizer for the relative perimeter in the cone  $\Omega_0$ , such that

$$\frac{P(E_0; B_R \cap \Omega_0)}{R^{n-1}} = \theta_E(0) \quad \text{for all } R > 0.$$

Now, the monotonicity inequality (3.3.3) written for  $f = \mathbf{1}_{E_0}$  and  $\Omega = \Omega_0$  takes the form

$$\begin{aligned} & \left( \int_{\Omega_0 \cap (B_{r_2} \setminus B_{r_1})} \frac{|\langle \nu_{E_0}(x), x \rangle|}{|x|^n} d|D\mathbf{1}_E|(x) \right)^2 \\ & \leq \left( \int_{\Omega_0 \cap (B_{r_2} \setminus B_{r_1})} |x|^{1-n} d|D\mathbf{1}_{E_0}|(x) \right) \cdot \left( \frac{P_{\Omega_0}(E_0; B_{r_2})}{r_2^{n-1}} - \frac{P_{\Omega_0}(E_0; B_{r_1})}{r_1^{n-1}} \right) = 0, \end{aligned}$$

for almost all  $0 < r_1 < r_2$ . The only possibility is then that  $\langle \nu_{E_0}(x), x \rangle = 0$  at  $\mathcal{H}^{n-1}$ -a.e.  $x \in \partial^* E_0$ . By [35, Proposition 28.8] we infer that  $E_0$  is a cone with vertex at the origin, up to negligible sets, and the proof is concluded.  $\square$

# Chapter 4

## Free-boundary variations in non-smooth domains

In this Chapter, we develop a technique for constructing "free-boundary variations" in domains with non-smooth boundary. We then apply this technique to the study of the stability of a  $n$ -plane containing the vertex of a circular cone in  $\mathbb{R}^{n+1}$ .

For  $n \geq 2$ , given  $x \in \mathbb{R}^{n+1}$ , we will use the following notations:

$$\tilde{x} = (x_2, \dots, x_n) \in \mathbb{R}^{n-1}, \quad x' = (x_1, \tilde{x}) \in \mathbb{R}^n, \quad x'' = (\tilde{x}, x_{n+1}) \in \mathbb{R}^n.$$

Moreover, given a differentiable function  $f$ , for all  $1 \leq i \leq n$ , we will set

$$f_i := \frac{\partial f}{\partial x_i}.$$

### 4.1 Construction of the flow

From now on, we will deal with Lipschitz open subsets  $\Omega \subset \mathbb{R}^{n+1}$  that can be globally represented in graphical form, i.e. there exists a Lipschitz function  $\omega : \mathbb{R}^n \rightarrow \mathbb{R}$  with  $\omega(0) = 0$  such that

$$\Omega = \{x \in \mathbb{R}^{n+1} : x_{n+1} > \omega(x')\}. \quad (4.1.1)$$

This choice is suggested by the application we will do of the flow, which, as mentioned in the introduction, is represented by a variational problem involving a circular cone.

Nevertheless, we remark that for the construction of the flow, it would suffice a local representability of  $\Omega$  in graphical form (as in (1.4.3)). We will denote by  $\Sigma$  the set

$$\Sigma := \{x \in \Omega : x_1 = 0\}.$$

For  $x \in \bar{\Sigma}$ , we can define the following curve

$$\Gamma_x(t) := x + te_1 + (\omega(t, \tilde{x}) - \omega(0, \tilde{x}))e_{n+1}. \quad (4.1.2)$$

Roughly speaking,  $\Gamma$  is a foliation of  $\Omega$  with curves leaving from points of  $\Sigma$ , as the following Lemma suggests.

**Lemma 4.1.1.** *The following properties hold:*

(i)  $\Gamma_x(0) = x$ , for all  $x \in \bar{\Sigma}$ ;

(ii) if  $x, y \in \bar{\Sigma}$ , and  $x \neq y$ , then the curves  $t \mapsto \Gamma_x(t)$ ,  $u \mapsto \Gamma_y(u)$  do not cross each other, i.e.

$$\Gamma_x(t) \neq \Gamma_y(u), \quad \text{for all } t, u \in \mathbb{R};$$

(iii) for all  $x \in \bar{\Sigma}$ , the parametrization  $t \mapsto \Gamma_x(t)$  is monotone, i.e.

$$t_1 \neq t_2 \implies \Gamma_x(t_1) \neq \Gamma_x(t_2);$$

(iv) for all  $x \in \bar{\Sigma}$ , there exists a time  $\bar{t} > 0$  with  $\Gamma_x(\bar{t}) \in \partial\Omega$  if and only if  $\Gamma_x(t) \in \partial\Omega$ , for all  $t \in \mathbb{R}$ .

*Proof.* The property (i) is an immediate consequence of the definition. Let us prove (ii). If  $\Gamma_x(t) = \Gamma_y(u)$ , then in particular

$$t = \langle \Gamma_x(t), e_1 \rangle = \langle \Gamma_y(u), e_1 \rangle = u. \quad (4.1.3)$$

Thus  $\Gamma_x(t) = \Gamma_y(u)$  implies

$$x + (\omega(t, \tilde{x}) - \omega(0, \tilde{x}))e_{n+1} = y + (\omega(t, \tilde{y}) - \omega(0, \tilde{y}))e_{n+1},$$

that is equivalent to

$$\tilde{x} = \tilde{y} \quad \text{and} \quad \omega(t, \tilde{x}) + x_{n+1} - \omega(0, \tilde{x}) = \omega(t, \tilde{y}) + y_{n+1} - \omega(0, \tilde{y}).$$

But then the only possibility is that  $x = y$ . Let us prove (iii): we observe that, if  $\Gamma_x(t_1) = \Gamma_x(t_2)$ , then (4.1.3) implies  $t_1 = t_2$ , and this proves (iii). Let us finally show (iv). If  $\Gamma_x(\bar{t})$ , then we necessarily have

$$\omega(\bar{t}, \tilde{x}) + x_{n+1} - \omega(0, \tilde{x}) = \langle \Gamma_x(\bar{t}), e_{n+1} \rangle = \omega(\bar{t}, \tilde{x}), \quad (4.1.4)$$

i.e.  $x_{n+1} = \omega(0, \tilde{x})$ . But then (4.1.4) applied to a generic  $t \in \mathbb{R}$  yields  $\Gamma_x(t) \in \partial\Omega$ , and this concludes the proof.  $\square$

We can now give the following

**Definition 4.1.2.** Let  $f \in C_c^{0,1}(\bar{\Sigma})$  be a Lipschitz continuous function with compact support in  $\bar{\Sigma}$ . We define the flow associated to  $f$  and  $\Gamma$  as

$$\Phi_{f,t}(x) = \Gamma_x(tf(x)), \quad \text{for all } x \in \Sigma, t \in \mathbb{R}. \quad (4.1.5)$$

**Remark 4.1.3.** We observe that, for any fixed  $t \in \mathbb{R}$ ,  $\Phi_{f,t}$  is Lipschitz continuous and one-to-one. Indeed

$$\Phi_{f,t}(x) = \Gamma_x(tf(x)) = x + tf(x)e_1 + (\omega(tf(x), \tilde{x}) - \omega(0, \tilde{x}))e_{n+1}, \quad (4.1.6)$$

and both  $\omega$  and  $f$  are Lipschitz continuous. Moreover, if  $x \neq y$ , then from (ii) of Lemma 4.1.1 it follows that  $\Phi_{f,t}(x) \neq \Phi_{f,t}(y)$ . By (i) of Lemma 4.1.1, we also have

$$\Phi_{f,t}(x) = x, \quad \text{for all } x \notin \text{spt } f, \quad \Phi_{f,0}(x) = x, \quad \text{for all } x \in \bar{\Sigma}. \quad (4.1.7)$$

From now on, we will identify a point  $x = (0, x'') \in \bar{\Sigma}$  with  $x'' = (x_2, \dots, x_n, x_{n+1})$ . In such a way,  $\Phi_{f,t}$  is a function of  $x''$ . Since  $\Phi_{f,t}$  is Lipschitz, for a.e.  $x'' \in \Sigma$ , we can consider its Jacobian matrix  $D\Phi_{f,t}(x'')$ , that is a rectangular matrix of order  $(n+1) \times n$ . Denoting by  $D\Phi_{f,t}(x'')^T$  the transpose of  $D\Phi_{f,t}(x'')$ , we will denote by  $S[f](t)(x'')$  the matrix

$$S[f](t)(x'') := D\Phi_{f,t}(x'')^T \cdot D\Phi_{f,t}(x'').$$

We observe that  $S[f](t)$  is a symmetric matrix of order  $n$  whose entries are the scalar products

$$S[f](t)_{i,j} := \left\langle \frac{\partial \Phi_{f,t}}{\partial x_i}(x''), \frac{\partial \Phi_{f,t}}{\partial x_j}(x'') \right\rangle, \quad \text{for } i, j = 2, \dots, n+1.$$

**Lemma 4.1.4.** *Given  $x'' \in \Sigma$ , let us assume that  $\Phi_{f,t}$  is differentiable at  $x''$ . Then, for  $i = 2, \dots, n$ ,*

$$\frac{\partial \Phi_{f,t}}{\partial x_i}(x'') = tf_i(x'')e_1 + e_i + [tf_i(x'')\omega_1(tf(x''), \tilde{x}) + \omega_i(tf(x''), \tilde{x}) - \omega_i(0, \tilde{x})]e_{n+1}, \quad (4.1.8)$$

while, if  $i = n+1$ ,

$$\frac{\partial \Phi_{f,t}}{\partial x_{n+1}}(x'') = tf_{n+1}(x'')e_1 + [tf_{n+1}(x'')\omega_1(tf(x''), \tilde{x}) + 1]e_{n+1}. \quad (4.1.9)$$

Moreover, denoting by  $Q_i = Q_i[f](t)$  the quantity

$$Q_i[f](t)(x'') := \begin{cases} tf_i(x'')\omega_1(tf(x''), \tilde{x}) + \omega_i(tf(x''), \tilde{x}) - \omega_i(0, \tilde{x}) & \text{if } i = 2, \dots, n \\ tf_{n+1}(x'')\omega_1(tf(x''), \tilde{x}) & \text{if } i = n+1, \end{cases}$$

we have  $\text{spt } Q_i[f](t) \subset \text{spt } f$ , for all  $i = 2, \dots, n+1$ , and

$$S[f](t)_{i,j} = \delta_{i,j} + t^2 f_i f_j + Q_i Q_j + Q_i \delta_{j,n+1} + Q_{n+1} \delta_{i,n+1} \delta_{j,n+1}, \quad (4.1.10)$$

for any  $i, j = 2, \dots, n+1$ .

*Proof.* It suffices to observe that, in coordinates,

$$\Phi_{f,t}(x'') = (tf(x''), \tilde{x}, \omega(tf(x''), \tilde{x}) + x_{n+1} - \omega(0, \tilde{x})).$$

Then (4.1.8), (4.1.9), and (4.1.10) immediately follow by a mere computation.  $\square$

**Definition 4.1.5.** *For all  $t \in \mathbb{R}$ ,  $f \in C_c^{0,1}(\overline{\Sigma})$ , we define*

$$A[f](t) := \mathcal{H}^{n-1}(\Phi_{f,t}(\Sigma \cap \text{spt } f)).$$

By the Area Formula (see [3, Theorem 2.91]),  $A[f](t)$  can be explicitly computed in terms of  $\Phi_{f,t}$ :

$$A[f](t) = \int_{\text{spt } f} J[f](t)(x'') \, dx'', \quad (4.1.11)$$

where

$$J[f](t)^2 = \det(S[f](t)).$$

By (4.1.7), it is immediate to observe that

$$J[f](0) \equiv 1, \quad \text{a.e. in } \Sigma, \quad A[f](0) = \mathcal{H}^{n-1}(\text{spt } f). \quad (4.1.12)$$

## 4.2 The case of the $n$ -dimensional circular cone

Let us now restrict to the case in which  $\Omega$  is a  $n$ -dimensional circular cone with vertex at 0, i.e., for a fixed  $\lambda > 0$ ,

$$\omega(x') = \lambda|x'|, \quad \text{for } x' \in \mathbb{R}^n.$$

In this case, we have

$$\Phi_{f,t}(x'') = \left( tf(x''), \tilde{x}, \lambda\sqrt{t^2 f(x'')^2 + |\tilde{x}|^2} + x_{n+1} - \lambda|\tilde{x}| \right).$$

**Lemma 4.2.1.** *We have*

$$Q_i = \lambda \left( \frac{t^2 f f_i + x_i}{\sqrt{t^2 f^2 + |\tilde{x}|^2}} - \frac{x_i}{|\tilde{x}|} \right), \quad \text{for } i = 2, \dots, n, \quad Q_{n+1} = \lambda \frac{t^2 f f_{n+1}}{\sqrt{t^2 f^2 + |\tilde{x}|^2}}. \quad (4.2.1)$$

Moreover

$$\lim_{t \rightarrow 0} Q_i = 0, \quad \lim_{t \rightarrow 0} \frac{Q_{n+1}}{t} = 0, \quad \text{a.e. in } \Sigma, \quad (4.2.2)$$

and there exists a constant  $C$  (depending on  $\text{Lip } f$ ) such that, if  $|t| \leq 1$ ,

$$|Q_i| \leq C, \quad \left| \frac{Q_{n+1}}{t} \right| \leq C. \quad (4.2.3)$$

*Proof.* It is immediate to observe that, if  $x'' \in \Sigma$  and  $\tilde{x} \neq 0$ , then, for  $i = 2, \dots, n$ ,  $Q_i \rightarrow 0$  and  $t^{-1}Q_{n+1} \rightarrow 0$ . This proves (4.2.2). The estimates in (4.2.3) immediately follow by

the trivial bounds

$$\left| \frac{x_i}{\sqrt{t^2 f^2 + |\tilde{x}|^2}} \right| \leq 1, \quad \left| \frac{t^2 f f_j}{\sqrt{t^2 f^2 + |\tilde{x}|^2}} \right| \leq |t| |f_j|,$$

that hold for all  $i = 2, \dots, n$ ,  $j = 2, \dots, n+1$ .  $\square$

We are now ready to prove the following

**Lemma 4.2.2.** *Let  $n = 2, 3$ . We have*

$$J[f](t)^2 = 1 + t^2 \left( |\nabla f|^2 + \frac{2\lambda f f_{n+1}}{\sqrt{t^2 f^2 + |\tilde{x}|^2}} \right) + R_t, \quad (4.2.4)$$

where  $R_t : \Sigma \rightarrow \mathbb{R}$  is such that, for a suitable constant  $C > 0$  (depending on  $\text{Lip} f$ ), if  $|t| \leq 1$ , then

$$\text{spt } R_t \subset \text{spt } f, \quad \left| \frac{R_t}{t^2} \right| \leq C, \quad \lim_{t \rightarrow 0} \frac{R_t}{t^2} = 0, \quad \text{a.e. in } \Sigma. \quad (4.2.5)$$

*Proof.* The fact that  $\text{spt } R_t \subset \text{spt } f$  is an immediate consequence of the fact that  $J[f](t) \equiv 1$  out of  $\text{spt } f$ . Let us now divide the proof into two cases (for this proof we will set  $S = S[f](t)$ ).

*Case 1.* If  $n = 2$ , then by (4.1.10) and (4.2.1) we obtain

$$S = \begin{pmatrix} 1 + t^2 f_2^2 + Q_2^2 & t^2 f_2 f_3 + Q_2 Q_3 + Q_2 \\ t^2 f_2 f_3 + Q_2 Q_3 + Q_2 & 1 + t^2 f_3^2 + Q_3^2 + 2Q_3 \end{pmatrix}.$$

Now  $J[f](t)^2 = \det S = S_{22} S_{33} - S_{23}^2$ , and

$$\begin{aligned} S_{22} S_{33} &= 1 + t^2 |\nabla f|^2 + 2Q_3 + Q_2^2 + 2Q_2^2 Q_3 + R_t^1 \\ S_{23}^2 &= Q_2^2 + 2Q_2^2 Q_3 + R_t^2, \end{aligned}$$

where  $R_t^1, R_t^2$  gather all the terms that, by (4.2.2), (4.2.3), go to 0 punctually faster than  $t^2$ , as  $t \rightarrow 0^+$ , and are globally bounded by  $Ct^2$ , for a suitable  $C > 0$  (depending on  $\text{Lip} f$ ).



In particular,  $R_t^1, R_t^2$  satisfy (4.2.5). Observing that

$$Q_3 = \lambda \frac{t^2 f f_3}{\sqrt{t^2 f^2 + |\tilde{x}|^2}},$$

the proof for  $n = 2$  is concluded.

*Case 2.* Let now  $n = 3$ . Again by (4.1.10) and (4.2.1), we get

$$S = \begin{pmatrix} 1 + t^2 f_2^2 + Q_2^2 & t^2 f_2 f_3 + Q_2 Q_3 & t^2 f_2 f_4 + Q_2 Q_4 + Q_2 \\ t^2 f_2 f_3 + Q_2 Q_3 & 1 + t^2 f_3^2 + Q_3^2 & t^2 f_3 f_4 + Q_3 Q_4 + Q_3 \\ t^2 f_2 f_4 + Q_2 Q_4 + Q_2 & t^2 f_3 f_4 + Q_3 Q_4 + Q_3 & 1 + t^2 f_4^2 + Q_4^2 + 2Q_4 \end{pmatrix}.$$

By the Sarrus expansion of the determinant applied to the symmetric matrix  $S$ , we infer

$$\det S = S_{22}S_{33}S_{44} + 2S_{23}S_{24}S_{34} - S_{24}^2S_{33} - S_{22}S_{34}^2 - S_{23}^2S_{44}. \quad (4.2.6)$$

Let us now study all the summands in (4.2.6). In the next lines, following the same approach used for  $n = 2$ , by  $R_t^i, \tilde{R}_i$  for  $i = 1, \dots, 5$ , we will denote a function satisfying (4.2.5), that line by line gathers all the terms that by (4.2.2), (4.2.3) go to 0 as  $R_t^1, R_t^2$  of the previous case. We have

$$\begin{aligned} S_{22}S_{33}S_{44} &= (1 + t^2 f_2^2 + Q_2^2)(1 + t^2 f_3^2 + Q_3^2)(1 + t^2 f_4^2 + Q_4^2 + 2Q_4) \\ &= [1 + t^2(f_2^2 + f_3^2) + Q_2^2 + Q_3^2 + Q_2^2 Q_3^2](1 + t^2 f_4^2 + Q_4^2 + 2Q_4) + \tilde{R}_t^1 \\ &= 1 + t^2 |\nabla f|^2 + 2Q_4 + Q_2^2 + 2Q_2^2 Q_4 + Q_3^2 + 2Q_3^2 Q_4 + Q_2^2 Q_3^2 + 2Q_2^2 Q_3^2 Q_4 + R_t^1 \end{aligned}$$

$$\begin{aligned} 2S_{23}S_{24}S_{34} &= 2(t^2 f_2 f_3 + Q_2 Q_3)(t^2 f_2 f_4 + Q_2 Q_4 + Q_2)(t^2 f_3 f_4 + Q_3 Q_4 + Q_3) \\ &= 2Q_2 Q_3 (Q_2 Q_4 + Q_2)(Q_3 Q_4 + Q_3) + \tilde{R}_t^2 \\ &= 2Q_2^2 Q_3^2 (1 + Q_4)^2 + \tilde{R}_t^2 \\ &= 2Q_2^2 Q_3^2 + 4Q_2^2 Q_3^2 Q_4 + R_t^2 \end{aligned}$$

$$\begin{aligned}
-S_{24}^2 S_{33} &= -(t^2 f_2 f_4 + Q_2 Q_4 + Q_2)^2 (1 + t^2 f_3^2 + Q_3^2) \\
&= -(1 + Q_3^2)(Q_2^2 + 2Q_2^2 Q_4) + R_t^3 \\
&= -Q_2^2 - 2Q_2^2 Q_4 - Q_2^2 Q_3^2 - 2Q_2^2 Q_3^2 Q_4 + R_t^3
\end{aligned}$$

$$\begin{aligned}
-S_{22} S_{34}^2 &= -(1 + t^2 f_2^2 + Q_2^2)(t^2 f_3 f_4 + Q_3 Q_4 + Q_3)^2 \\
&= -(1 + Q_2^2)(Q_3^2 + 2Q_3^2 Q_4) + R_t^4 \\
&= -Q_3^2 - 2Q_3^2 Q_4 - Q_2^2 Q_3^2 - 2Q_2^2 Q_3^2 Q_4 + R_t^4
\end{aligned}$$

$$\begin{aligned}
-S_{23}^2 S_{44} &= -(t^2 f_2 f_3 + Q_2 Q_3)^2 (1 + t^2 f_4^2 + Q_4^2 + 2Q_4) \\
&= -Q_2^2 Q_3^2 (1 + 2Q_4) + R_t^5 \\
&= -Q_2^2 Q_3^2 - 2Q_2^2 Q_3^2 Q_4 + R_t^5.
\end{aligned}$$

Summing up all the terms, many cancellations occur, and by (4.2.6) we get

$$\det S = 1 + t^2 |\nabla f|^2 + 2Q_4 + R_t, \quad R_t = \sum_{i=1}^5 R_t^i.$$

Observing that

$$Q_4 = \lambda \frac{t^2 f f_4}{\sqrt{t^2 f^2 + |\tilde{x}|^2}},$$

the proof is concluded.  $\square$

The identity (4.2.4) provides a Taylor expansion of order 2 in  $t = 0$  of  $J[f](t)^2$  with remainder in Lagrange form. We observe that there are no terms of order 1. This suggests to apply the following change of parameter:

$$\mathcal{J}[f](s) := J[f](\sqrt{s}), \quad \mathcal{A}[f](s) := A[f](\sqrt{s}).$$

Owing to (4.1.11), we infer

$$\mathcal{A}[f](s) = \int_{\text{spt } f} \mathcal{J}[f](s)(x'') dx'' . \quad (4.2.7)$$

**Definition 4.2.3.** We define the right first variation in 0 of  $\mathcal{A}$  with respect to  $f$  as

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] := \liminf_{s \rightarrow 0^+} \frac{\mathcal{A}[f](s) - \mathcal{A}[f](0)}{s}.$$

**Definition 4.2.4.** We say that  $\Sigma$  is stable if

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] \geq 0, \quad \text{for all } f \in C_c^{0,1}(\bar{\Sigma}).$$

Otherwise, we say that  $\Sigma$  is unstable.

Lemma 4.2.2 allows to prove the following

**Theorem 4.2.5.** For  $n = 2, 3$ , the following identity holds:

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] = \frac{1}{2} \left( \int_{\Sigma} |\nabla f|^2 dx'' - \lambda \int_{\mathbb{R}^{n-1}} \frac{f(\tilde{x}, \lambda|\tilde{x}|)^2}{|\tilde{x}|} d\tilde{x} \right). \quad (4.2.8)$$

*Proof.* By (4.2.7), we have

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] = \liminf_{s \rightarrow 0^+} \frac{1}{s} \int_{\text{spt } f} \mathcal{J}[f](s) - \mathcal{J}[f](0) dx''.$$

Owing to (4.1.12), we have  $\mathcal{J}[f](0) \equiv 1$ , hence

$$\frac{1}{s} \int_{\text{spt } f} \mathcal{J}[f](s) - \mathcal{J}[f](0) dx'' = \frac{1}{s} \int_{\text{spt } f} \mathcal{J}[f](s) - 1 dx''. \quad (4.2.9)$$

Now (4.2.4) guarantees that

$$\mathcal{J}[f](s)^2 = 1 + sG, \quad G := \left( |\nabla f|^2 + \frac{2\lambda f f_{n+1}}{\sqrt{s f^2 + |\tilde{x}|^2}} + \frac{R_{\sqrt{s}}}{s} \right).$$

Thus by (4.2.9) we infer

$$\frac{1}{s} \int_{\text{spt } f} \mathcal{J}[f](s) - \mathcal{J}[f](0) dx'' = \mathcal{I}_s^1 + \mathcal{I}_s^2.$$

where

$$\mathcal{I}_s^1 := \frac{1}{s} \int_{\text{spt } f} \sqrt{1 + sG} - \left(1 + s \frac{G}{2}\right) dx'', \quad \mathcal{I}_s^2 := \int_{\text{spt } f} \frac{G}{2} dx''.$$

A mere computation yields

$$\sqrt{1 + sG} - \left(1 + s \frac{G}{2}\right) = \frac{-s^2 G^2 / 4}{\sqrt{1 + sG} + \left(1 + s \frac{G}{2}\right)},$$

consequently

$$|\mathcal{I}_s^1| \leq \frac{1}{4} \int_{\text{spt } f} \frac{sG^2}{\left|\sqrt{1 + sG} + \left(1 + s \frac{G}{2}\right)\right|} dx''.$$

Now it is immediate to observe that  $G$  is 0 out of the support of  $f$ , and

$$\sqrt{s}G \longrightarrow 0, \text{ a.e. in } \Sigma \quad |\sqrt{s}G| \leq C, \text{ for some } C > 0 \text{ depending on } f.$$

By Dominated convergence, we then get

$$\lim_{s \rightarrow 0^+} \mathcal{I}_s^1 = 0.$$

Let us now study the limit as  $s \rightarrow 0^+$  of

$$\begin{aligned} \mathcal{I}_s^2 &= \int_{\text{spt } f} \frac{1}{2} |\nabla f|^2 + \frac{\lambda f f_{n+1}}{\sqrt{s f^2 + |\tilde{x}|^2}} + \frac{1}{2} \frac{R_{\sqrt{s}}}{s} dx'' \\ &= \frac{1}{2} \int_{\Sigma} |\nabla f|^2 dx'' + \int_{\Sigma} \frac{\lambda f f_{n+1}}{\sqrt{s f^2 + |\tilde{x}|^2}} + \frac{1}{2} \frac{R_{\sqrt{s}}}{s} dx''. \end{aligned}$$

By (4.2.5) and Dominated Convergence, we have

$$\int_{\Sigma} \frac{R_{\sqrt{s}}}{s} dx'' \longrightarrow 0, \quad \text{as } s \rightarrow 0^+.$$

On the other hand, denoting by  $g(\tilde{x}) := f(\tilde{x}, \lambda|\tilde{x}|)^2$ , an integration by parts yields

$$\begin{aligned}
\int_{\Sigma} \frac{\lambda f f_{n+1}}{\sqrt{s f^2 + |\tilde{x}|^2}} dx'' &= \frac{\lambda}{s} \int_{\Sigma} \frac{s f f_{n+1}}{\sqrt{s f^2 + |\tilde{x}|^2}} dx'' \\
&= \frac{\lambda}{s} \int_{\Sigma} \frac{\partial}{\partial x_{n+1}} \sqrt{s f^2 + |\tilde{x}|^2} dx'' \\
&= \frac{\lambda}{s} \int_{\mathbb{R}^{n-1}} \left( \int_{\lambda|\tilde{x}|}^{+\infty} \frac{\partial}{\partial x_{n+1}} \sqrt{s f^2 + |\tilde{x}|^2} dx_{n+1} \right) d\tilde{x} \\
&= \frac{\lambda}{s} \int_{\mathbb{R}^{n-1}} |\tilde{x}| - \sqrt{|\tilde{x}|^2 + s g(\tilde{x})} d\tilde{x} \\
&= -\lambda \int_{\mathbb{R}^{n-1}} \frac{g(\tilde{x})}{|\tilde{x}| + \sqrt{|\tilde{x}|^2 + s g(\tilde{x})}} d\tilde{x}
\end{aligned}$$

Now the integrand

$$\frac{g(\tilde{x})}{|\tilde{x}| + \sqrt{|\tilde{x}|^2 + s g(\tilde{x})}}$$

is monotonically non-increasing in  $s$ , at any  $\tilde{x} \neq 0$ . By Beppo Levi's Theorem we then obtain

$$\lim_{s \rightarrow 0^+} \int_{\mathbb{R}^{n-1}} \frac{g(\tilde{x})}{|\tilde{x}| + \sqrt{|\tilde{x}|^2 + s g(\tilde{x})}} d\tilde{x} = \frac{1}{2} \int_{\mathbb{R}^{n-1}} \frac{g(\tilde{x})}{|\tilde{x}|} d\tilde{x}.$$

This implies that

$$\lim_{s \rightarrow 0^+} \mathcal{I}_s^2 = \frac{1}{2} \int_{\Sigma} |\nabla f|^2 dx'' - \frac{\lambda}{2} \int_{\mathbb{R}^{n-1}} \frac{g(\tilde{x})}{|\tilde{x}|} d\tilde{x},$$

and the proof is concluded.  $\square$

**Remark 4.2.6.** *Theorem 4.2.5 allows us to test in which cases  $\Sigma$  is stable. Indeed, by (4.2.8), the condition*

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] \geq 0$$

*is equivalent to the validity of the following functional inequality*

$$\int_{\Sigma} |\nabla f|^2 dx'' \geq \lambda \int_{\mathbb{R}^{n-1}} \frac{f(\tilde{x}, \lambda|\tilde{x}|)^2}{|\tilde{x}|} d\tilde{x}.$$

### 4.2.1 The case of $n = 2$

**Corollary 4.2.7.** *Let  $n = 2$ . Then, for all  $f \in C_c^{0,1}(\overline{\Sigma})$  with the property that  $f(0) \neq 0$ ,*

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] < 0.$$

*In particular,  $\Sigma$  is unstable.*

*Proof.* As observed in Remark 4.2.6, we know that

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] \geq 0$$

if and only if the following inequality holds:

$$\int_{\Sigma} |\nabla f|^2 dx'' \geq \lambda \int_{\mathbb{R}} \frac{f(x_2, \lambda|x_2|)^2}{|x_2|} dx_2. \quad (4.2.10)$$

Now the energy term on the LHS is finite, for all  $f \in C_c^{0,1}(\overline{\Sigma})$ , but the integral on the RHS is finite if and only if  $f(0) = 0$ . Thus (4.2.10) fails for all  $f \in C_c^{0,1}(\overline{\Sigma})$  such that  $f(0) \neq 0$ , and the statement is proven.  $\square$

We can then prove the following

**Theorem 4.2.8.** *Let  $n = 2$ , and let us define*

$$E := \Omega \cap \{x \in \mathbb{R}^3 : x_1 < 0\}.$$

*Then  $E$  is not a minimizer of the relative perimeter in  $\Omega$ .*

*Proof.* It suffices to demonstrate that, for all  $r > 0$ , there exists a measurable set  $F_r \subset \Omega$  such that  $F_r \Delta E \subset\subset B_r$ , and

$$P_{\Omega}(F_r; B_r) < P_{\Omega}(E; B_r). \quad (4.2.11)$$

Up to blowing up, it suffices to prove (4.2.11) for  $r = 1$ . Let  $f \in C_c^{0,1}(\overline{\Sigma})$  be such that

$$f \geq 0, \quad \text{spt } f \subset\subset \overline{\Sigma} \cap B_1, \quad f(0) > 0. \quad (4.2.12)$$

By Corollary 4.2.7, we know that

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] < 0,$$

and thus by Definition 4.2.3 we can select  $\sigma > 0$  small enough to ensure

$$\frac{\mathcal{A}[f](\sigma) - \mathcal{A}[f](0)}{\sigma} \leq \max \left\{ -1, \left. \frac{1}{2} \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] \right\} < 0. \quad (4.2.13)$$

Since  $\Phi_{f,0}$  is the identity on  $\bar{\Sigma}$ , and  $\text{spt } f$  is compact in  $\bar{\Sigma}$ , up to possibly reducing  $\sigma$ , we can also suppose that

$$\Phi_{f,\sqrt{\sigma}}(\text{spt } f) \subset B_1.$$

Let us define

$$F := \{\Gamma_{x''}(t) : x'' \in \bar{\Sigma}, t < \sqrt{\sigma}f(x)\}.$$

By construction,  $F \Delta E \subset\subset B_1$ . Owing to (iv) of Lemma 4.1.1 and (4.2.13), we can deduce that

$$\begin{aligned} P_\Omega(F; B_1) - P_\Omega(E; B_1) &= \mathcal{H}^{n-1}(\Phi_{f,\sqrt{\sigma}}(\Sigma) \cap B_1) - \mathcal{H}^{n-1}(\Sigma \cap B_1) \\ &= \mathcal{H}^{n-1}(\Phi_{f,\sqrt{\sigma}}(\text{spt } f)) - \mathcal{H}^{n-1}(\text{spt } f) \\ &= \mathcal{A}[f](\sigma) - \mathcal{A}[f](0) \\ &< 0, \end{aligned}$$

this concludes the proof. □

### 4.2.2 The case of $n = 3$

When  $n = 3$ , i.e. the ambient dimension is 4, the situation deeply changes.

**Theorem 4.2.9.** *Let  $n = 3$ . Then there exists a threshold  $\lambda^* > 0$  such that, if  $\lambda \leq \lambda^*$ , we have*

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] \geq 0, \quad \text{for all } f \in C_c^{0,1}(\bar{\Sigma}), \quad (4.2.14)$$

namely  $\Sigma$  is stable.

In order to show the validity of Theorem 4.2.9, we need the following result.

**Lemma 4.2.10.** *Let  $n \geq 3$ . Then*

$$\int_{\Sigma} |\nabla f|^2 dx'' \geq \frac{H}{(1+\lambda)^2} \int_{\mathbb{R}^{n-1}} \frac{f(\tilde{x}, \lambda|\tilde{x}|)^2}{|\tilde{x}|} d\tilde{x}, \quad \text{for all } f \in H^1(\Sigma), f \not\equiv 0, \quad (4.2.15)$$

where, denoting by  $\Gamma$  the Gamma Function, we have

$$H = H(n) = 2 \frac{\Gamma\left(\frac{n}{4}\right)^2}{\Gamma\left(\frac{n-2}{4}\right)^2}, \quad \text{for all } n \geq 3. \quad (4.2.16)$$

Inequality (4.2.15) is known as *Kato's Inequality*. A proof of a stronger version of (4.2.15), but holding just when  $\Sigma$  is a half-space, is given in [12, Theorem 1.4] (while, up to the domain  $\Sigma$ , our version coincides precisely with inequality (9) in the same work). We notice that the original proof of the Kato's Inequality was given by Herbst in [21].

*Proof of Lemma 4.2.10.* Let us denote by  $\mathbb{R}_+^n$  the half-space of the points  $x'' \in \mathbb{R}^n$  with  $x_{n+1} > 0$ . By inequality (9) in [12], we know that

$$\int_{\mathbb{R}_+^n} |\nabla g|^2 dx'' \geq H \int_{\mathbb{R}^{n-1}} \frac{g(\tilde{x}, 0)^2}{|\tilde{x}|} d\tilde{x}, \quad \text{for all } g \in H^1(\mathbb{R}_+^n), g \not\equiv 0, \quad (4.2.17)$$

where  $H$  is the constant in (4.2.16). Given  $f \in H^1(\Sigma)$ , let us extend  $f$  to  $\mathbb{R}_+^n$  setting

$$g(x'') := f(T(x'')), \quad T(x'') := (\tilde{x}, x_{n+1} + \lambda|\tilde{x}|).$$

We have  $g \in H^1(\mathbb{R}_+^n)$ . Moreover, trivially  $g(\tilde{x}, 0) \equiv f(\tilde{x}, \lambda|\tilde{x}|)$ , thus

$$\int_{\mathbb{R}^{n-1}} \frac{g(\tilde{x}, 0)^2}{|\tilde{x}|} d\tilde{x} = \int_{\mathbb{R}^{n-1}} \frac{f(\tilde{x}, \lambda|\tilde{x}|)^2}{|\tilde{x}|} d\tilde{x}. \quad (4.2.18)$$

We observe that

$$\nabla g(x'') = \nabla f(T(x'')) + \lambda f_{n+1}(T(x'')) \left( \frac{\tilde{x}}{|\tilde{x}|}, 0 \right),$$



and thus

$$|\nabla g(x'')|^2 \leq (1 + \lambda)^2 |\nabla f(T(x''))|^2. \quad (4.2.19)$$

Combining (4.2.18) and (4.2.19), owing to (4.2.17) and the fact that  $\det DT \equiv 1$ , we finally get

$$\begin{aligned} (1 + \lambda)^2 \int_{\Sigma} |\nabla f(x'')|^2 dx'' &= (1 + \lambda)^2 \int_{\mathbb{R}_+^n} |\nabla f(T(x''))|^2 dx'' \\ &\geq \int_{\mathbb{R}_+^n} |\nabla g(x'')|^2 dx'' \\ &\geq H \int_{\mathbb{R}^{n-1}} \frac{f(\tilde{x}, \lambda|\tilde{x}|)^2}{|\tilde{x}|} d\tilde{x}. \end{aligned}$$

□

We are now in position to prove Theorem 4.2.9.

*Proof of Theorem 4.2.9.* As observed in Remark 4.2.6, we know that

$$\left. \frac{d}{ds} \right|_{s=0}^+ \mathcal{A}[f] \geq 0$$

if and only if the following inequality holds:

$$\int_{\Sigma} |\nabla f|^2 dx'' \geq \lambda \int_{\mathbb{R}^2} \frac{f(\tilde{x}, \lambda|\tilde{x}|)^2}{\tilde{x}} d\tilde{x}. \quad (4.2.20)$$

By (4.2.15), the validity of (4.2.20), for all  $f \in C_c^{0,1}(\overline{\Sigma})$ , is guaranteed by the condition

$$\lambda \leq \frac{H}{(1 + \lambda)^2},$$

where  $H = H(3)$  is the constant defined in (4.2.16). Since the function  $\lambda(1 + \lambda)^2$  is monotonically increasing, there exists exactly one  $\lambda^* > 0$  such that

$$\lambda^*(1 + \lambda^*)^2 = H.$$

The positive number  $\lambda^*$  is precisely the claimed threshold.

□

**Remark 4.2.11.** *Let  $\alpha \in (0, \pi/2)$  be the half-aperture of the circular cone  $\Omega$ . It is immediate to see that  $\lambda$  and  $\alpha$  are related each other by*

$$\lambda = \cot \alpha .$$

*Hence, setting  $\cot \alpha^* := \lambda^*$ , Theorem 4.2.9 tells us what follows: if*

$$\alpha \geq \alpha^* ,$$

*i.e. the aperture of  $\Omega$  is big enough, then  $\Sigma$  is stable.*

**Remark 4.2.12.** *Theorem 4.2.9 leaves a lot of questions open. Indeed, we do not know whether the threshold  $\alpha^* > 0$  is optimal, and it is not clear what happens if  $\alpha$  is small. Moreover, Theorem 4.2.9 provides just a stability result, but it remains to understand whether  $\Sigma$  is really area-minimizing in  $\Omega$  when  $\alpha \geq \alpha^*$ . We expect that the quantitative expansion of  $J[f](t)$  provided in (4.2.4) could be employed to show that  $\Sigma$  is also minimizing. Finally, we did not treat the case of  $n > 3$ , i.e. the case in which the ambient dimension is bigger than 4. The motivation of this is related to the difficulties arising in computing the determinant  $J[f](t)$ . On the other hand, we notice that Kato's Inequality holds also as  $n > 3$ , hence if we knew that  $J[f](t)$  expands as in (4.2.4) in some dimension  $n > 3$ , we would deduce that  $\Sigma$  is stable in  $\Omega$  for  $\alpha \geq \alpha^*$ , where  $\alpha^*$  is expected to depend on  $n$ .*

# Chapter 5

## The Vertex-skipping Theorem

In this Chapter, we expose the argument for the proof of Theorem 5.3.1, as developed in [30]. The argument of the proof is by contradiction, and requires a characterization of the blow-up limit of a local almost-minimizer of the relative perimeter. This characterization will turn out to be a consequence of the boundary Monotonicity Formula. In what follows,  $\Omega$  will always denote an open, convex subset of  $\mathbb{R}^n$ .

### 5.1 Preliminary results

#### 5.1.1 Tangent cone and vertices

**Definition 5.1.1.** *Let  $x_0 \in \bar{\Omega}$ . We define the tangent cone to  $\Omega$  at  $x_0$  as*

$$\Omega_{x_0} := \lim_{t \rightarrow +\infty} t(\Omega - x_0) = \bigcup_{t > 0} t(\Omega - x_0).$$

We observe immediately that  $\Omega_{x_0}$  is a cone with vertex at 0. Indeed, if  $y \in \Omega_{x_0}$ , by definition, there exists  $t > 0$  such that

$$y \in t(\Omega - x_0).$$

Then, for all  $s > 0$ ,  $sv \in st(\Omega - x_0) \subset \Omega_{x_0}$ .

We also provide the definition of vertex for an Euclidean convex set.

**Definition 5.1.2.** Let  $x_0 \in \partial\Omega$ . We say that  $x_0$  is a vertex for  $\Omega$  if  $\Omega_{x_0}$  does not contain lines.

### 5.1.2 Minimality in the tangent cone

We start showing that a sequence of dilations of an almost-minimizer  $E$  in  $\Omega$  converges, up to subsequences, to a minimizer of the relative perimeter in the tangent cone  $\Omega_0$ . This fact is an immediate consequence of Lemma 2.1.7.

**Lemma 5.1.3.** Let  $E$  be an almost-minimizer in  $\Omega$ , and assume  $0 \in \partial\Omega$  and  $P_\Omega(E; B_r) > 0$  for all  $r > 0$ . Let  $t_j \searrow 0$  be a sequence and let

$$\Omega_{t_j} := \frac{1}{t_j} \Omega, \quad E_{t_j} := \frac{1}{t_j} E.$$

Then there exist a subsequence  $s_{j_k}$  and a measurable set  $E_0 \subset \Omega_0$  such that  $E_{s_{j_k}} \rightarrow E_0$  in  $L^1_{loc}$  and  $E_0$  is a perimeter-minimizer in  $\Omega_0$ , namely

$$\Psi_{\Omega_0}(E_0; B_R) = 0 \quad \text{for any } R > 0.$$

*Proof (of Lemma 5.1.3).* First, we fix  $R > 0$  and prove that there exist  $t_0, C > 0$  such that

$$P(E_t; B_R) \leq CR^{n-1} \quad \forall 0 < t < t_0. \quad (5.1.1)$$

In what follows, for more simplicity, we will write  $C$  to denote a constant that might change from one line to another. To prove (5.1.1), we note that

$$P(\Omega_t; B_R) = t^{1-n} P(\Omega; B_{Rt}) \leq CR^{n-1} \quad (5.1.2)$$

since  $\partial\Omega$  is Lipschitz. Then owing to (2.1.7) and (5.1.2), and assuming  $t < t_0 := \min(1, \bar{r}/R)$ , we obtain

$$P(E_t; B_R) \leq P_{\Omega_t}(E_t; B_R) + P(\Omega_t; B_R) = t^{1-n} \left( P_\Omega(E; B_{tR}) + P(\Omega; B_{Rt}) \right) \leq CR^{n-1},$$

which proves (5.1.1). Consequently, we obtain the global perimeter bound

$$P(E_t) \leq P(E_t; B_R) + P(\Omega_t; B_R) \leq CR^{n-1},$$

hence any blow-up sequence  $E_{t_j}$  admits a (not relabeled) subsequence converging in  $L^1(B_R)$  to a limit set  $E_0$ . By a standard diagonal argument, one can prove the existence of a subsequence and a limit set, still denoted respectively as  $E_{t_j}$  and  $E_0$ , such that  $E_{t_j} \rightarrow E_0$  in  $L^1_{loc}(\mathbb{R}^n)$ . Since the corresponding sequence of rescaled domains  $\Omega_{t_j}$  converges to the tangent cone  $\Omega_0$  in  $L^1_{loc}(\mathbb{R}^n)$ , we infer that  $E_0 \subset \Omega_0$ .

Now we have to show that  $\Psi_{\Omega_0}(E_0; B_R) = 0$ , for all  $R > 0$ . To do so, we can apply Lemma 2.1.7. Indeed, well-known properties of convex sets ensure that the sequence of rescaled domains  $\Omega_{t_j}$  converges to the tangent cone  $\Omega_0$  locally in Hausdorff distance. Moreover, by the rescaling properties of the perimeter, since  $E$  is an almost-minimizer of the relative perimeter in  $\Omega$ , we infer that  $E_{t_j}$  satisfies (2.1.1) for all  $0 < r < r_0$  and  $x = 0$  with the function

$$\psi_{\Omega_{t_j}}(E_{t_j}; 0, r) = \psi_{\Omega}(E; 0, t_j r), \quad 0 < r < r_0/t_j.$$

Now  $t_j \searrow 0$ , hence there exists  $M > 0$  such that  $|t_j| \leq M$ , for all  $j \in \mathbb{N}$ . Moreover, since  $\psi_{\Omega}(E; 0, r)$  is infinitesimal as  $r \rightarrow 0^+$ , we infer that, for any  $\varepsilon > 0$ , there exists  $r_\varepsilon > 0$  such that  $0 \leq \psi_{\Omega}(E; 0, r) \leq \varepsilon$ , for all  $0 < r < r_\varepsilon$ . If we take  $r < r_\varepsilon/M$ , we then have

$$\sup_j \psi_{\Omega_{t_j}}(E_{t_j}; 0, r) = \sup_j \psi_{\Omega}(E; 0, t_j r) \leq \varepsilon,$$

and this suffices to show that

$$\lim_{r \rightarrow 0^+} \sup_j \psi_{\Omega_{t_j}}(E_{t_j}; 0, r) = 0.$$

We are then in position to apply Lemma 2.1.7 to deduce that

$$\liminf_j \Psi_{\Omega_{t_j}}(E_{t_j}; B_R) \geq \Psi_{\Omega_0}(E_0; B_R), \quad \text{for all } R > 0. \quad (5.1.3)$$

Finally, by (2.1.6) we obtain

$$\Psi_{\Omega_{t_j}}(E_{t_j}; B_R) = \frac{1}{t_j^{n-1}} \Psi_{\Omega}(E; B_{t_j R}) \leq \omega_n^{1-\frac{1}{n}} R^{n-1} \psi_{\Omega}(E; 0, t_j R) \longrightarrow 0,$$

which concludes the proof.  $\square$

### 5.1.3 Flatness of the blow-up of an almost-minimizer

Thanks to Lemma 5.1.3, we know in particular that for a suitable choice of a sequence  $t_j \searrow 0$  the rescalings  $E_{t_j}$  with respect to  $0 \in \partial\Omega$  converge to a minimizer  $E_0$  in the tangent cone  $\Omega_0$ . Exploiting Theorem 3.4.1, we can prove the following result.

**Proposition 5.1.4.** *There exist a sequence  $s_k \searrow 0$  and a subset  $E_{00} \subset \Omega_0$  such that*

$$(E_0)_{s_k} \longrightarrow E_{00}, \quad \text{in } L_{\text{loc}}^1,$$

and  $E_{00}$  is a minimal cone in  $\Omega_0$ , i.e. is a cone with vertex at 0 with the property that

$$\Psi_{\Omega_0}(E_{00}; B_R) = 0 \quad \text{for any } R > 0. \quad (5.1.4)$$

*Proof.* We note that  $\Omega_0$  is a Lipschitz cone with vertex at 0, thus  $\Omega_0$  satisfies the visibility condition at 0 with the trivial choice  $u(t) \equiv 0$  (see Example 3.2.10). Moreover,  $E_0$  is a minimizer of the relative perimeter in  $\Omega$ , i.e.  $\Psi_{\Omega_0}(E_0; B_R) = 0$ , for all  $R > 0$ , and this trivially implies (3.4.1). We can then apply Theorem 3.4.1 in this special setting deducing that, choosing a suitable sequence  $s_k \searrow 0$ , there exists  $E_{00} \subset \Omega_0$  such that

$$(E_0)_{s_k} \longrightarrow E_0, \quad \text{in } L_{\text{loc}}^1,$$

and  $E_{00}$  is a minimal cone in  $\Omega_0$ , i.e. is a cone with vertex at 0 such that (5.1.4) holds true, and this concludes the proof.  $\square$

**Remark 5.1.5.** *We observe that, if  $\Omega$  satisfies the assumptions of Theorem 3.4.1, then the same result allows to prove that  $E_0$  itself is a cone with vertex at 0. In this case, the argument performed in the proof of Proposition 5.1.4 becomes unuseful, since  $E_0 = E_{00}$ .*

### 5.1.4 A Federer's Reduction Lemma

Next, we state a slight variant of the classical Federer's Reduction Lemma, which will be used in the proof of Theorem 5.2.1. In what follows, by “cone” we shall always mean a cone with respect to the origin.

**Lemma 5.1.6.** *Let  $K \subset \mathbb{R}^n$  be a convex cone,  $C \subset K$  be a minimizing cone for the relative perimeter, both with vertex at 0. Let  $x_0 \in \partial C \cap \partial K \setminus \{0\}$ . For  $t > 0$ , we set*

$$C_t := x_0 + \frac{C - x_0}{t}, \quad K_t := x_0 + \frac{K - x_0}{t}.$$

Then there exist a sequence  $t_j \searrow 0$  and two sets  $C_{x_0}, K_{x_0}$  such that

$$C_j := C_{t_j} \longrightarrow C_{x_0}, \quad K_j := K_{t_j} \longrightarrow K_{x_0} \quad (5.1.5)$$

in  $L^1_{\text{loc}}$ -topology,  $C_{x_0} \subset K_{x_0}$  and  $C_{x_0}, K_{x_0}$  are cylinders with axis coinciding with the line joining 0 to  $x_0$ . Moreover, the sets  $C'_{x_0} := C_{x_0} \cap x_0^\perp$  and  $K'_{x_0} := K_{x_0} \cap x_0^\perp$  are cones with respect to 0 in the hyperplane  $x_0^\perp$ , and  $C'_{x_0}$  is perimeter-minimizing in  $K'_{x_0}$ .

To prove this result we need the following two technical Lemmas. The proof of Lemma 5.1.7 is part of the proof of Lemma 2.4 in [19], while the proof of Lemma 5.1.8 is a slight variation of that of Lemma 9.8 in [19].

**Lemma 5.1.7.** *Let  $f$  have locally bounded variation in  $\mathbb{R}^n$ . For  $t \in \mathbb{R}$ , let*

$$f_t : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$$

be the function defined by  $f_t(y) := f(y, t)$ . Then, for almost all  $s < t$ ,  $r > 0$ , one has

$$\int_{B_r^{n-1}} |f_t - f_s| d\mathcal{H}^{n-1} \leq |D_n f|(B_r^{n-1} \times (s, t)), \quad (5.1.6)$$

where  $B_r^{n-1} := B_r \cap \{x_n = 0\}$ .

**Lemma 5.1.8.** *Let  $f$  have locally-bounded variation in  $\mathbb{R}^n$ , and  $\Omega$  be an open, bounded set in  $\mathbb{R}^{n-1}$ . Then, for any  $T > 0$ ,*

$$|Df|(\Omega \times (-T, T)) \geq \int_{-T}^T |Df_t|(\Omega) dt,$$

with equality holding if  $f$  is independent of the variable  $x_n$ .

*Proof of Lemma 5.1.6.* Up to a rotation, we can suppose that  $x_0 = (0, \dots, 0, a)$ , for some  $a > 0$ . We can argue as in the proof of Lemma 5.1.3 to deduce that there are a sequence  $t_j \searrow 0$  and sets  $C_{x_0}, K_{x_0}$  such that (5.1.5) hold. We can also assume that  $D\mathbf{1}_{C_{t_j}} \xrightarrow{*} D\mathbf{1}_{C_{x_0}}$ . Furthermore,  $K_{x_0}$  is the tangent cone to  $K$  at  $x_0$  and  $C_{x_0}$  is perimeter-minimizing in  $K_{x_0}$ . Let us show that  $C_{x_0}$  is a cylinder with axis coinciding with the  $x_n$ -axis. Let us denote by  $\nu_C$  the inner, unit normal to  $\partial^*C$ , and let us set  $\nu_C^n := \langle \nu_C, e_n \rangle$ . As  $C$  is a cone with respect to 0, we have

$$\langle x, \nu_C(x) \rangle = 0,$$

for  $\mathcal{H}^{n-1}$ -almost all  $x \in \partial^*C$ . Thus

$$-\langle x - x_0, \nu_C(x) \rangle = \langle x_0, \nu_C(x) \rangle = a \nu_C^n(x),$$

and consequently we obtain ( $a = |x_0|$ )

$$|\nu_C^n(x)| \leq \frac{|x - x_0|}{|x_0|}.$$

So, for some constant  $c > 0$ , for all  $\rho > 0$ , we have

$$\begin{aligned} \int_{B_\rho(x_0) \cap \partial^*C_t} |\nu_{C_t}^n(x)| d\mathcal{H}^{n-1}(x) &= t^{1-n} \int_{B_{\rho t}(x_0) \cap \partial^*C_{x_0}} |\nu_{C_{x_0}}^n(x)| d\mathcal{H}^{n-1}(x) \\ &\leq \frac{t^{2-n} \rho}{|x_0|} P(C, B_{\rho t}(x_0)) \\ &\leq c \frac{\rho^n t}{|x_0|}. \end{aligned} \tag{5.1.7}$$

Now by Reshetnyak's lower semicontinuity Theorem (see Theorem 20.11 in [35], in particular we apply it for  $\Phi(u) = |\langle u, e_n \rangle|$ ) we have

$$\int_{B_\rho(x_0) \cap \partial^*C_{x_0}} |\nu_{C_{x_0}}^n(x)| d\mathcal{H}^{n-1}(x) \leq \liminf_{j \rightarrow \infty} \int_{B_\rho(x_0) \cap \partial^*C_{t_j}} |\nu_{C_{t_j}}^n(x)| d\mathcal{H}^{n-1}(x) = 0,$$

and consequently

$$\nu_{C_{x_0}}^n = 0, \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial^*C_{x_0}.$$



Owing to Lemma 5.1.7, we deduce that  $\mathbf{1}_{C_{x_0}}$  is independent of  $x_n$ , hence there exists  $C'_{x_0} \subset \mathbb{R}^{n-1}$  such that

$$C_{x_0} = C'_{x_0} \times \mathbb{R}.$$

From the conicity of  $C_{x_0}$  with respect to 0, it follows that  $C'_{x_0}$  is a cone with respect to 0 in  $\mathbb{R}^{n-1}$ . The same argument developed up to now can be performed for  $K_{x_0}$ , deducing that there is a cone  $K'_{x_0}$  with respect to the origin in  $\mathbb{R}^{n-1}$  with the property that

$$K_{x_0} = K'_{x_0} \times \mathbb{R}.$$

To conclude, it remains to show the minimality of  $C'_{x_0}$  in  $K'_{x_0}$ . By contradiction, let  $C'_{x_0}$  be not perimeter-minimizing in  $K'_{x_0}$ . Then we can pick a competitor  $E'_0$  such that, for some  $R, \epsilon > 0$ , we have

$$E'_0 \Delta C'_{x_0} \subset\subset K'_{x_0} \cap B_R^{n-1}, \quad P_{K'_{x_0}}(E'_0, B_R^{n-1}) \leq P_{K'_{x_0}}(C'_{x_0}, B_R^{n-1}) - \epsilon.$$

Given  $T > 0$ , let

$$E_0 := E'_0 \times (-T, T) \cup (C_{x_0} \cap \{x_n \geq T\}).$$

By construction,  $E_0 = C_{x_0}$  in  $\mathbb{R}^n \setminus (B_R^{n-1} \times (-T, T))$ . In addition, by minimality of  $C_{x_0}$  in  $K_{x_0}$ , we observe that

$$P_{K_{x_0}}(C_{x_0}, B_R^{n-1} \times [-T, T]) \leq P_{K_{x_0}}(E_0, B_R^{n-1} \times [-T, T]). \quad (5.1.8)$$

On the other hand, we have

$$P_{K_{x_0}}(C_{x_0}, B_R^{n-1} \times [-T, T]) = 2T P_{K'_{x_0}}(C'_{x_0}, B_R^{n-1}),$$

and also

$$\begin{aligned} P_{K_{x_0}}(E_0, B_R^{n-1} \times [-T, T]) &\leq 2T P_{K'_{x_0}}(E'_0, B_R^{n-1}) + 2\omega_{n-1}R^{n-1} \\ &\leq 2T P_{K'_{x_0}}(C'_{x_0}, B_R^{n-1}) - 2T\epsilon + 2\omega_{n-1}R^{n-1} \\ &= P_{K_{x_0}}(C_{x_0}, B_R^{n-1} \times [-T, T]) - 2T\epsilon + 2\omega_{n-1}R^{n-1}, \end{aligned}$$

and this estimate contradicts (5.1.8) for  $T$  large enough. So  $C'_{x_0}$  is a minimal cone in  $K'_{x_0}$

and the proof is concluded.  $\square$

## 5.2 Characterization of the conical minimizer in $\mathbb{R}^3$ .

Starting from an almost-minimizer  $E$  in  $\Omega$  which satisfies volume density estimates at the origin, and applying Lemma 5.1.3 at most twice, we have obtained a conical minimizer  $E_{00}$  of the relative perimeter in the tangent cone  $\Omega_0$ . The next theorem shows that, in dimension  $n = 3$ ,  $\partial E_{00} \cap \Omega_0$  coincides with a convex angle contained in a 2-plane through the origin, that meets  $\partial\Omega_0$  orthogonally.

**Theorem 5.2.1.** *Let  $n = 3$  and  $E_{00}$  be the conical minimizer obtained in the previous subsection. Then  $\partial E_{00} \cap \Omega_0$  coincides with a 2-plane intersected with  $\Omega_0$ , that meets  $\partial\Omega_0$  orthogonally.*

*Proof.* Set  $F = E_{00}$  for brevity, then the proof is accomplished by showing that there exists exactly one geodesic arc  $\gamma \subset \partial B_1 \cap \Omega_0$  such that

$$\partial F \cap \partial B_1 \cap \Omega_0 = \gamma, \quad (5.2.1)$$

and  $\gamma$  meets  $\partial\Omega_0 \cap \partial B_1$  orthogonally. We split the proof into some steps.

*Step 1.* We claim that  $\partial F \cap \partial B_1 \cap \Omega_0$  is made of countably-many (open) geodesic arcs  $\gamma_i$ ,  $i \geq 1$  integer, such that

$$\gamma_i \cap \gamma_j = \emptyset, \text{ for } i \neq j, \quad \bigcup_i \gamma_i = \partial F \cap \partial B_1 \cap \Omega_0.$$

By interior regularity,  $\partial F \cap \Omega_0$  is smooth, and its outer normal vector  $\nu_F$  is orthogonal to the radial directions (recall that  $F$  is a cone with vertex at the origin). Hence,  $\partial F$  intersects transversally  $\partial B_1 \cap \Omega_0$  along smooth curves  $\gamma_i$  that cannot cross each other. Since  $F$  has locally-finite perimeter, the family of these curves is at most countable. Let us now show that  $\gamma_i$  is a geodesic arc, for all  $i$ . With a slight abuse of notation, we identify  $\gamma_i$  with its arc-length parametrization defined on the interval  $(0, L_i)$ , where  $L_i$  is the length of the curve. The connected component of  $\partial F$  that intersects  $\partial B_1$  along  $\gamma_i$  can

be then parametrized through

$$\sigma_i(s, t) := s \gamma_i(t), \text{ for } s > 0, t \in (0, L_i). \quad (5.2.2)$$

We can choose the parametrization  $\gamma_i(t)$  in such a way that  $\nu_F(\sigma_i(s, t)) = \gamma_i(t) \times \gamma_i'(t)$ , for all  $s > 0$ . Exploiting (5.2.2), and using  $\operatorname{div}_{\partial F} \nu_F = 0$  by the minimality of  $F$ , we infer

$$0 = \operatorname{div}_{\partial F} \nu_F(\sigma_i(s, t)) \quad (5.2.3)$$

$$\begin{aligned} &= \frac{d}{dt}(\gamma_i(t) \times \gamma_i'(t)) \cdot \gamma_i'(t) \\ &= -\gamma_i(t) \times \gamma_i'(t) \cdot \gamma_i''(t), \quad \text{for all } t. \end{aligned} \quad (5.2.4)$$

Since we also have  $\gamma_i'(t) \cdot \gamma_i''(t) = 0$  by the choice of the arc-length parametrization, and observing that  $\{\gamma_i'(t), \gamma_i(t) \times \gamma_i'(t)\}$  is an orthonormal basis for the tangent space to  $\partial B_1$  at  $\gamma(t)$ , we conclude that  $\gamma_i''(t)$  is orthogonal to the tangent space to  $\partial B_1$  at  $\gamma(t)$ , which is precisely the definition of geodesic arc.

*Step 2.* We prove that  $\gamma_i$  meets  $\partial\Omega_0$  orthogonally at its endpoints. More precisely, if  $p$  is an endpoint of  $\gamma_i$ , then  $\Omega_0$  admits a unique supporting plane at  $p$ , hence the outer unit normal vector  $\nu_0(p)$  to  $\partial\Omega_0$  at  $p$  is well-defined, and moreover if we denote by  $\nu_i$  the constant unit outer normal to the connected component of  $\partial F$  containing  $\gamma_i$ , we have

$$\nu_i \cdot \nu_0(p) = 0. \quad (5.2.5)$$

Let us first prove that  $\Omega_0$  admits a unique supporting plane at  $p$ . Up to a rotation, we may assume that  $p = (0, 0, 1)$ , hence it follows that  $\nu_i \cdot e_3 = 0$ . Owing to Lemma 5.1.6, we can find a sequence  $t_j \searrow 0$  such that  $\Omega_0^{p, t_j} := t_j^{-1}(\Omega_0 - p)$  locally converge to a cylinder of type  $C \times \mathbb{R}$ , and  $F^{p, t_j} := t_j^{-1}(F - p)$  locally converge to a cylinder that can be written as  $G \times \mathbb{R}$ , where  $G \subset C$ . Moreover, both  $C$ , and  $G$  are cones with respect to 0 in the plane  $z = 0$ , and  $G$  is perimeter-minimizing in  $C$ . By convexity of  $C$ , up to a further rotation, we can assume that

$$C = \{(x_1, x_2, 0) : x_2 > \lambda|x_1|\},$$

for some  $\lambda \geq 0$ . Clearly,  $\lambda = 0$  if and only if  $p$  admits a unique supporting plane for  $\Omega_0$ . The only possibility is then that  $\partial G \cap C$  is made of finitely many half-lines  $L_1, \dots, L_k$  of

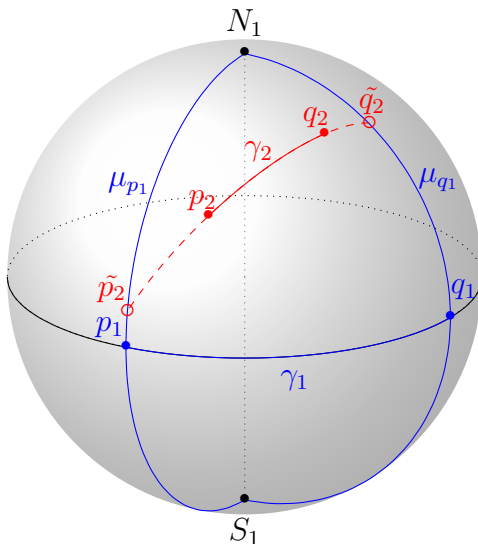


Figure 5.1

the form  $L_j = \{tv_j : t \geq 0\}$ , for some unit vectors  $v_j$ . Up to relabeling, we can assume

$$0 < v_1 \cdot e_2 \leq v_j \cdot e_2 \quad \text{for all } j.$$

If either  $\lambda > 0$  or  $k > 1$ , we could replace an initial portion of  $L_1$  with a projection segment onto the closest side of  $C$ , which strictly decreases the perimeter and thus contradicts the fact that  $G$  is perimeter-minimizing in  $C$ . Hence we necessarily have  $\lambda = 0$  and  $k = 1$ , i.e., there exists a unique supporting plane to  $\Omega_0$  at  $p$  with  $\nu_0(p) = -e_2$ , and moreover  $v_1 = e_2$ . This proves the claim and, additionally, shows that two different geodesic arcs cannot share a common endpoint.

*Step 3.* Finally, we prove that  $\partial F \cap \partial B_1 \cap \Omega_0$  is made of exactly one geodesic arc.

Suppose by contradiction that there exist two geodesic arcs  $\gamma_1 \neq \gamma_2$  contained in  $\partial F \cap \partial B_1 \cap \Omega_0$ . From the previous step we know that

$$\overline{\gamma_1} \cap \overline{\gamma_2} = \emptyset.$$

For  $i = 1, 2$  we denote by  $\Pi_i$  the plane through the origin that contains  $\gamma_i$ , and by  $p_i, q_i$  the boundary points of  $\gamma_i$ .

For  $i = 1, 2$ , we consider the point  $N_i \in \partial B_1$  such that  $\gamma_i$  is contained in the equator

with north pole  $N_i$  and south pole  $S_i = -N_i$ , and denote by  $\mu_{p_i}, \mu_{q_i}$  the corresponding meridians connecting  $N_i$  with  $S_i$  and passing through  $p_i$  and  $q_i$ , respectively. These meridians bound a region of  $\partial B_1$ , that we denote as  $\Sigma_i$ , which satisfies

$$\Omega_0 \cap \partial B_1 \subset \Sigma_i, \quad i = 1, 2.$$

Incidentally, thanks to the previous step,  $\Sigma_i$  is also obtained by intersecting the sphere  $\partial B_1$  with the wedge  $W_i$  given by the intersection of the two supporting half-spaces to  $\Omega_0$  at  $p_i$  and  $q_i$ , respectively. Since in particular  $W_i$  is a convex cone, it is immediate to check that  $\Sigma_i = W_i \cap \partial B_1$  is geodesically convex. Moreover, using the fact that the angle formed by the vectors  $p_i, q_i$  is strictly smaller than  $\pi$  (recall that the origin is an isolated vertex for  $\Omega_0$ ) we infer that the internal angle formed by the two geodesic sides of  $\Sigma_i$ , i.e. the meridians  $\mu_{p_i}$  and  $\mu_{q_i}$ , at  $N_i$  (or  $S_i$ ) is strictly smaller than  $\pi$ .

Now, observe that the closure of  $\Sigma_i$  is the union of two closed geodesic triangles  $T_{N_i}, T_{S_i}$  with vertices  $p_i, q_i, N_i$  and  $p_i, q_i, S_i$ , respectively. Since in particular  $\gamma_2 \subset \Omega_0 \cap \partial B_1 \subset \Sigma_1$ , and  $\gamma_2$  has a strictly positive distance from  $\gamma_1$ , we must have that either  $\gamma_2 \subset T_{N_1}$  or  $\gamma_2 \subset T_{S_1}$ . Without loss of generality, we assume  $\gamma_2 \subset T_{N_1}$ .

Now, we set  $\tilde{p}_2 = \Pi_2 \cap \mu_{p_1}$  and  $\tilde{q}_2 = \Pi_2 \cap \mu_{q_1}$ , and denote by  $\tilde{\gamma}_2$  the geodesic connecting  $\tilde{p}_2$  and  $\tilde{q}_2$ , and by  $\tilde{\Sigma}_2$  the associated geodesically convex region bounded by the meridians  $\mu_{\tilde{p}_2}, \mu_{\tilde{q}_2}$  meeting at poles  $\tilde{N}_2 = N_2$  and  $\tilde{S}_2 = S_2$ . Clearly we have  $\gamma_2 \subset \tilde{\gamma}_2$  and thus  $\Sigma_2 \subset \tilde{\Sigma}_2$ . Moreover, we have

$$\tilde{p}_2, \tilde{q}_2 \in T_{N_1} \setminus \{p_1, q_1, N_1\}.$$

Indeed, the geodesic  $\tilde{\gamma}_2$  cannot intersect  $\gamma_1$ , hence it is contained in  $T_{N_1}$  and its closure is disjoint from  $\gamma_1$  because  $\Pi_2 \cap \overline{\gamma_1} = \emptyset$ ; moreover, if we had  $\tilde{p}_2 = N_1$  (or  $\tilde{q}_2 = N_1$ ) we would conclude that  $\mu_{q_1} \subset \Pi_2$  (respectively,  $\mu_{p_1} \subset \Pi_2$ ), but this is impossible because  $\Pi_2$  and  $\overline{\gamma_1}$  are disjoint.

Now, consider the geodesic quadrilateral  $D$  determined by the four points  $p_1, \tilde{p}_2, \tilde{q}_2, q_1$ . By the previous argument,  $D \subset T_{N_1}$ . Denote by  $\alpha_1, \tilde{\alpha}_2, \tilde{\beta}_2, \beta_1$  the angles formed by the pairs of geodesic sides meeting at the respective vertices. Then, consider the two geodesic triangles  $R_1 = p_1 \tilde{p}_2 q_1$  and  $R_2 = \tilde{p}_2 \tilde{q}_2 q_1$ . Call  $\tilde{\alpha}_{2,1}$  the internal angle to  $R_1$  at  $\tilde{p}_2$ , and  $\tilde{\alpha}_{2,2}$  the internal angle to  $R_2$  at  $\tilde{p}_2$ . Similarly, call  $\beta_{1,1}$  the internal angle to  $R_1$  at  $q_1$ , and

$\beta_{1,2}$  the internal angle to  $R_2$  at  $q_1$ . We thus have  $\tilde{\alpha}_2 = \tilde{\alpha}_{2,1} + \tilde{\alpha}_{2,2}$ ,  $\beta_1 = \beta_{1,1} + \beta_{1,2}$ , and therefore we deduce

$$\alpha_1 = \beta_1 = \pi/2, \quad (5.2.6)$$

$$\max(\tilde{\alpha}_2, \tilde{\beta}_2) \leq \pi/2. \quad (5.2.7)$$

Indeed, (5.2.6) follows from the orthogonality of  $\gamma_1$  with the meridians  $\mu_{p_1}$  and  $\mu_{q_1}$ , while (5.2.7) follows from the fact that the quadrilateral  $D$  is contained in one of the two geodesic triangles  $\tilde{T}_{N_2} = \tilde{p}_2\tilde{q}_2N_2$ ,  $\tilde{T}_{S_2} = \tilde{p}_2\tilde{q}_2S_2$  (indeed, by a symmetric argument, we have either  $\gamma_1 \subset \tilde{T}_{N_2}$  or  $\gamma_1 \subset \tilde{T}_{S_2}$ ) and we know by construction that the internal angles to  $\tilde{T}_{N_2}$  (or  $\tilde{T}_{S_2}$ ) at  $\tilde{p}_2$  and at  $\tilde{q}_2$  are both equal to  $\pi/2$ .

Now we notice that  $R_1$  must be a non-degenerate geodesic triangle because it possesses an internal angle at  $p_1$  measuring  $\alpha_1 = \pi/2$ , and the other two vertices do not coincide. Thus we have that the sum of the internal angles of  $R_1$  satisfies

$$\alpha_1 + \tilde{\alpha}_{2,1} + \beta_{1,1} > \pi. \quad (5.2.8)$$

At the same time, the sum of the internal angles of  $R_2$  is not smaller than  $\pi$ :

$$\tilde{\alpha}_{2,2} + \tilde{\beta}_2 + \beta_{1,2} \geq \pi. \quad (5.2.9)$$

By combining (5.2.6), (5.2.7), (5.2.8) and (5.2.9), we reach the contradiction

$$2\pi < (\alpha_1 + \tilde{\alpha}_{2,1} + \beta_{1,1}) + (\tilde{\alpha}_{2,2} + \tilde{\beta}_2 + \beta_{1,2}) = \alpha_1 + \tilde{\alpha}_2 + \tilde{\beta}_2 + \beta_1 \leq 2\pi$$

and this completes the proof of the theorem.  $\square$

### 5.3 Proof of the main result

We now dispose of all the necessary tools to prove the following

**Theorem 5.3.1** (Vertex-skipping). *Let  $\Omega \subset \mathbb{R}^3$  be an open, convex set, and  $E \subset \Omega$  be a local almost-minimizer of  $P_\Omega$ . Then  $\overline{\partial E} \cap \overline{\Omega}$  does not contain vertices of  $\Omega$ .*

We argue by contradiction. By the results of the previous section, via a blow-up

argument we can restrict the proof to the case of a domain  $\Omega_0$  being a convex cone with vertex at the origin, and of a minimizer given by the intersection with  $\Omega_0$  of a half-space whose boundary plane passes through the origin and meets  $\partial\Omega_0$  orthogonally.

Before giving the proof of Theorem 5.3.1, we introduce a class of convex cones that will play a key role in the first part of the proof.

**Definition 5.3.2.** *We say that a convex cone  $C \subset \mathbb{R}^3$  is a pyramid provided there exist two wedges  $W_1, W_2$  with orthogonally incident spines such that  $C = W_1 \cap W_2$ .*

We note that a pyramid  $C \subset \mathbb{R}^3$  is always a cone with vertex at the point  $V_0$  in which the spines of the wedges intersect each other. Moreover, up to a rotation and a translation, there exist  $a, b > 0$  such that

$$C = C_{a,b} := \{x \in \mathbb{R}^3 : x_3 \geq \max\{a|x_1|, b|x_2|\}\}.$$

*Proof (of Theorem 5.3.1).* The proof is split into two steps. In the first, we show the result under the assumption that the conical container is a pyramid, i.e., a cone over a rectangle. In the second, we employ a “packing-boy” technique that allows us to reduce the case of a general convex cone to that of a suitably associated pyramid.

*Step 1.* Consider a pyramid cone  $C_{a,b}$ . We want to show that the plane

$$\pi_0 = \{x \in \mathbb{R}^3 : x_1 = 0\}$$

is not locally area-minimizing in  $C_{a,b}$ . To do so, we build a family of competitors that improve the area of  $\pi_0$  in  $C_{a,b}$ . For  $\varepsilon \geq 0$ , let  $\pi_\varepsilon = \{x \in \mathbb{R}^3 : x_1 = \varepsilon\}$  and define

$$R_\varepsilon := C_{a,b} \cap \{x \in \pi_\varepsilon : x_3 \leq 1\}, \quad A_\varepsilon := \mathcal{H}^2(R_\varepsilon).$$

We note that  $R_0$  is a triangle in the plane  $\pi_0$ , and that  $R_\varepsilon$  is a trapezium in the plane  $\pi_\varepsilon$  whenever  $0 < \varepsilon < 1$ . Moreover, up to translations,  $R_\varepsilon$  is obtained from  $R_0$  by removing a triangle of area  $\frac{a^2\varepsilon^2}{b}$ , so that we have

$$A_\varepsilon = A_0 - \frac{a^2}{b}\varepsilon^2. \tag{5.3.1}$$

The idea is now to connect the trapezium  $R_\varepsilon$  with  $\pi_0 \cap C_{a,b}$  in order to obtain a local

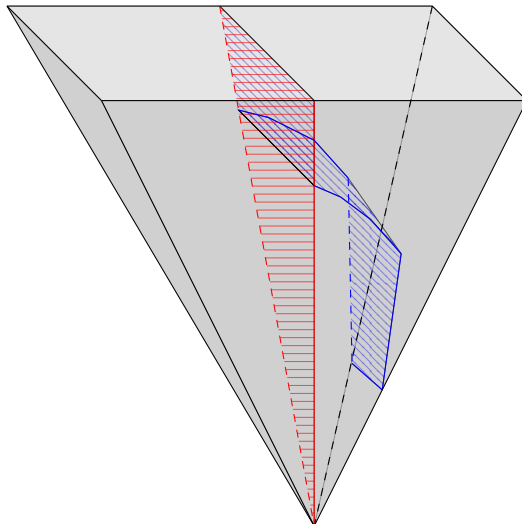


Figure 5.2

variation of  $\pi_0 \cap C_{a,b}$ . We formulate the problem in the following way. Let  $h > 0$  and let  $T_h$  be the trapezium defined as

$$T_h = C_{a,b} \cap \{x \in \pi_0 : 1 \leq x_3 \leq 1 + h\}.$$

We immediately note that

$$\mathcal{H}^2(T_h) = \frac{h(2+h)}{b}. \quad (5.3.2)$$

We look for those smooth functions  $\phi_h$  defined on the segment  $\{1 \leq x_3 \leq 1+h\}$  satisfying the following conditions:

$$\phi_h(1) = 1, \quad \phi_h(1+h) = 0. \quad (5.3.3)$$

We observe that, looking at  $\phi_h$  as a function of both variables  $x_2$  and  $x_3$  defined in  $T_h$ , the ruled surface

$$G_\varepsilon(\phi_h) = \{(\varepsilon\phi_h(x_3), x_2, x_3) : (x_2, x_3) \in T_h\}$$

connects  $R_\varepsilon$  with  $\pi_0 \cap C_{a,b}$  (see Figure 5.2, where the red region is  $\pi_0 \cap C_{a,b}$ , while the blue one coincides with  $R_\varepsilon \cup G_\varepsilon(\phi_h)$ ). By suitably choosing  $h$  and the map  $\phi_h$ , we claim that

$$A_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_h)) < A_0 + \mathcal{H}^2(T_h). \quad (5.3.4)$$



Using (5.3.1), (5.3.2), and the area formula, (5.3.4) turns out to be equivalent to

$$\iint_{T_h} \sqrt{1 + \varepsilon^2 |\phi'_h(x_3)|^2} dx_2 dx_3 < \frac{a^2}{b} \varepsilon^2 + \frac{h(2+h)}{b}. \quad (5.3.5)$$

To guarantee (5.3.5), we only need to impose that the second-order derivative at 0 of the left-hand side is strictly smaller than the same derivative of the right-hand side. We first observe that, by Dominated Convergence, one has

$$\int_1^{1+h} t \phi'_h(t)^2 dt < a^2. \quad (5.3.6)$$

Then, for  $\alpha > 0$ , we choose

$$\phi_h = \phi_{h,\alpha}(t) := \frac{h^\alpha t^{-\alpha} - 1}{h^\alpha - 1}.$$

We observe that  $\phi_{h,\alpha}$  fulfills (5.3.3). Taking  $\phi_h = \phi_{h,\alpha}$ , condition (5.3.6) becomes

$$\frac{\alpha(1+h)^\alpha + 1}{2(1+h)^\alpha - 1} = \int_1^{1+h} t \phi'_{h,\alpha}(t)^2 dt < a^2. \quad (5.3.7)$$

As  $h \rightarrow +\infty$ , the term on the left-hand side of (5.3.7) tends to  $\frac{\alpha}{2}$ , hence it is enough to choose  $\alpha < 2a^2$  and  $h$  large enough to enforce (5.3.7). This ultimately proves (5.3.4) and shows that  $\pi_0$  cannot be area-minimizing in  $C_{a,b}$ .

*Step 2.* Let now  $\Omega_0$  be a generic convex cone with vertex at the origin. Thanks to Theorem 5.2.1, and up to rotations, we may suppose that the boundary of the minimizer  $E_{00}$  is the intersection of the plane  $\pi_0$  with  $\Omega_0$ , hence there exists  $b > 0$  such that

$$\partial E_{00} \cap \Omega_0 = \{(0, x_2, x_3) : x_3 \geq b|x_2|\}.$$

Now, by Theorem 5.2.1 we have

$$\Omega_0 \subset W_1 := \{x \in \mathbb{R}^3 : x_3 \geq b|x_2|\}.$$

Since the origin is an isolated vertex for  $\Omega_0$ , it is not possible that  $\partial\Omega_0$  contains the whole

line  $\{(t, 0, 0) : t \in \mathbb{R}\}$ , hence there must exist  $a > 0$  such that the pyramid  $C_{a,b}$  verifies either

$$\Omega_0 \cap \{x_1 \geq 0\} \subset C_{a,b} \cap \{x_1 \geq 0\} \quad (5.3.8)$$

or

$$\Omega_0 \cap \{x_1 \leq 0\} \subset C_{a,b} \cap \{x_1 \leq 0\}. \quad (5.3.9)$$

We can assume for instance that (5.3.8) holds true, otherwise we simply flip the argument.

We take  $\varepsilon > 0$  and set

$$\hat{R}_\varepsilon := \Omega_0 \cap \{x \in \pi_\varepsilon : x_3 \leq 1\}, \quad \hat{A}_\varepsilon := \mathcal{H}^2(\hat{R}_\varepsilon).$$

With the choice of suitable values  $h$  and  $\alpha$ , we already know that the connection map  $\phi_{h,\alpha}$  constructed in the proof of Theorem 5.2.1 satisfies

$$A_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_{h,\alpha})) < A_0 + \mathcal{H}^2(T_h)$$

whenever  $\varepsilon$  is small enough. Finally, we observe that

$$\begin{aligned} \hat{A}_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_{h,\alpha}) \cap \Omega_0) &\leq A_\varepsilon + \mathcal{H}^2(G_\varepsilon(\phi_{h,\alpha})) \\ \hat{A}(0) &= A(0), \end{aligned}$$

which shows that  $\pi_0$  cannot be a minimizer in  $\Omega_0$ . This concludes the proof of Theorem 5.3.1.

□

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