Elementary structure of morphism space between real algebraic varieties

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Abstract

The paper deals with the first systematic study of the spaces of regular and ratinal maps between arbitrary algebraic varieties over a real closed field R. We find conditions under which these spaces are reduced to the space of Zariski locally constant maps, we investigate the finiteness properties of the subspaces of dominating regular and rational maps and, when R is the field of real numbers, we study the topology of the space of regular maps. Our results show that the mentioned map spaces are "usually" very small. The realization of such a general study has been possible thanks to the introduction of two new classes of real algebraic invariants: the curve genera and the toric genera.

Introduction

A significant problem in Real Algebraic Geometry is to understand when the set $\mathcal{R}(M, N)$ of all regular maps between a compact affine real algebraic manifold M and an affine real algebraic manifold N is dense in the corresponding set $C^\infty(M, N)$ of all C^∞ maps, equipped with the C^∞ topology. Until now, the space $\mathcal{R}(M, N)$ has been studied only in this direction. This is due to the great importance of the approximation techniques in Geometry.

Let us recall some of the main aspects of this topic.

Differentiable, analytic and algebraic structures can coexist on a manifold. The possibility of equipping a manifold with structures of increasing richness implies the existence of more tools for the study of the manifold itself. For example, we recall the Morse theory, the Morse–Sard theorem and the Milnor–Thom inequalities. Similar considerations may be repeated when we study maps between manifolds. For example, in problems of iteration of maps from a manifold into itself, high differentiability is sometimes an important hypothesis. Let us consider the categories of C^k differentiable, real analytic and affine real Nash manifolds. The availability of the Implicit Function Theorem ensures the existence of tubular neighborhoods for any manifolds. Combining the Stone–Weierstrass theorem (or some of its variants) with the notion of tubular neighborhood, we obtain fundamental approximation results for manifolds and maps between them (see [60] and [52]). In the real algebraic setting, the Implicit Function Theorem is not true: for example, one can prove that the unique compact affine real algebraic manifold, which has an algebraic tubular neighborhood, is the single point (see [30]). For these reasons, the problem of making smooth objects algebraic turns out to be very difficult. These kinds of questions have played and continue to play a crucial role in the development of Real Algebraic Geometry. We remind the reader of Tognoli's celebrated theorem [59], which is the starting point of the systematic study of real algebraic varieties, and of subsequent efforts to characterize the topology of such varieties in the singular case (see [1], [26], [50]).

At the moment, the nature of the arduous question concerning the density of $\mathcal{R}(M,N)$ in $C^\infty(M,N)$ is well–understood essentially only when the target space N is equal to a real, complex or quaternionic grassmannian $\mathbb{G}_{n,k}(\mathbb{F})$ (where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$) equipped with its natural real algebraic structure. This follows from the deep connections existing between the notion of strongly algebraic vector bundle over M introduced in [3], the problem of approximating C^∞ maps between M and a grassmannian $\mathbb{G}_{n,k}(\mathbb{F})$ by regular maps, the algebraic and topological K–theories of M and the algebraic properties of the cohomology of M. These connections have been extensively studied in the following papers: [5, 6, 8, 9, 10, 11, 12, 13, 14, 17, 21, 24, 25, 37, 41, 42, 44, 46]. For an excellent survey of a part of results contained in these papers, we refer the reader to chapter 12 and sections 13.3 and 13.5 of [4]. Recently, interesting algebraic approximation theorems has been obtained for C^∞ maps with values in rational real algebraic surfaces or, more generally, in rational real algebraic manifolds (see [19, 20, 38, 43, 45, 48]). Further results concerning homological and homotopic properties of regular maps can be found in [7, 15, 16, 22, 23, 30, 32, 53] and in section 13.4 of [4].

The purpose of this paper is to investigate the structure of the spaces of regular and rational maps between arbitrary algebraic varieties over a real closed field R. Let X and Y be varieties of this kind. Indicate by $\mathcal{R}(X,Y)$ (resp. $\mathcal{R}atio(X,Y)$) the set of all regular (resp. rational) maps between X and Y. We study $\mathcal{R}(X,Y)$ and $\mathcal{R}atio(X,Y)$ by addressing the following three questions:

QUESTION 1: *find conditions under which* $\mathcal{R}(X,Y)$ (*resp.* $\mathcal{R}atio(X,Y)$) *is reduced to the set of Zariski locally constant maps,*

QUESTION 2: *study the finiteness properties of the subset of* $\mathcal{R}(X,Y)$ (*resp.* $\mathcal{R}atio(X,Y)$) *formed by dominating maps,*

QUESTION 3: *when* $R=\mathbb{R}$, *study the topology of* $\mathcal{R}(X,Y)$.

The algebraic varieties we consider here are understood in the sense of Serre [54] (in particular, are reduced), are not assumed to be irreducible, affine or nonsingular, but, in order to avoid trivial situations, they are assumed to have positive dimension.

The study of the spaces of regular and rational maps between such general real algebraic varieties is based on two new classes of algebraic invariants: the curve genera and the toric genera. It turns out that, from elementary relations between the curve genera of the source space and the toric genera of the target space, we are able to infer satisfactory solutions to the previous three questions. These solutions reveal that the mentioned map spaces are "usually" very small. Furthermore, if the target space is affine, then, up to a change of its algebraic structure which preserves its natural affine real Nash structure, the regular and rational map spaces are always reduced to the set of Zariski locally constant maps.

0. Summary and sketch of the main theorems

The paper is subdivided into four sections. In section 1, we introduce and study several integer–valued algebraic invariants on positive dimensional algebraic varieties over a real closed field. Among them, the most important are the curve and toric genera (biregular invariants) and the rational curve and rational toric genera (birational invariants). These genera extend the classical notion of genus of an affine irreducible real algebraic curve. In section 2, we present the main theorems. We organize these results into three subsections, each one addressing one of the preceding questions. The theorems of subsection 2.1 are followed by their proofs. The proofs of the theorems contained in subsections 2.2 and 2.3 are given in sections 3 and 4 respectively. The results of section 1 are used in such proofs.

Let us present some results of subsection 2.1 and some particular cases of results contained in subsections 2.2 and 2.3 in a way that the reader may be immediately aware of their nature.

Fix a real closed field R. Let X be an algebraic variety over R of positive dimension. Indicate by $p_c(X)$, $P_c(X)$, $p_t(X)$ and $P_t(X)$, respectively, the curve genus, the rational curve genus, the toric genus and the rational toric genus of X. It holds: $0 \leq p_t(X) \leq P_t(X) \leq p_c(X) \leq P_c(X)$. Suppose X affine. A regular map $\varphi : \widetilde{X} \longrightarrow X$ between an affine algebraic variety over R and X is said to be a weak change of the algebraic structure of X if it sends the singular locus of $\widetilde{X}$ into the singular locus of X and, equipping X and $\widetilde{X}$ with their natural affine real Nash structures, φ is a Nash isomorphism.

The following lemma is basic. It says that the new genera are Nash flexible.

Lemma. *If X is affine, then there is a weak change $\varphi : \widetilde{X} \longrightarrow X$ of its algebraic structure such that $p_t(\widetilde{X})$ is arbitrarily large.*

TOTAL ALGEBRAIC OBSTRUCTIONS. Let Y be an algebraic variety over R of positive dimension.

Theorem A. *The following assertions are verified:*

a) *If $p_c(X) < p_t(Y)$, then every regular map between X and Y is Zariski locally constant, i.e., it is constant on each irreducible component of X.*

b) *If $P_c(X) < p_t(Y)$, then every rational map between X and Y is Zariski locally constant, i.e., it is represented by a Zariski locally constant map.*

As a consequence of the previous lemma and Theorem A, we obtain:

Theorem B. *Suppose Y affine. Then, there is a weak change $\varphi : \widetilde{Y} \longrightarrow Y$ of the algebraic structure of Y such that every rational map (and hence every regular map) between X and $\widetilde{Y}$ is Zariski locally constant.*

FINITENESS OF DOMINATING MORPHISM SPACES. The following results are extensions of the classical Hurwitz–de Franchis finiteness theorem to the real algebraic setting.

Theorem C. *Suppose X and Y irreducible. If $P_t(Y) \geq 2$, then the set of all dominating rational maps between X and Y is finite and there exists an explicit upper bound for its cardinality in terms of $P_c(X)$ and $P_c(Y)$ only.*

Recall that the ordering structure on R induces a topology on X and Y called euclidean topology.

Theorem D. *Suppose X and Y equipped with the euclidean topology. If $p_t(Y) \geq 2$, then the set of all regular maps between X and Y, which are open maps also, is finite.*

Topology of regular morphism space. Consider the case $R = \mathbb{R}$. Suppose that X and Y are affine and equipped with the euclidean topology. In order to simplify the presentation, we assume that Y does not have any isolated point also. Equip the set $\mathcal{N}(X,Y)$ of all real Nash maps between X and Y with the compact–open topology and consider $\mathcal{R}(X,Y)$ as a subspace of $\mathcal{N}(X,Y)$.

Theorem E. *If $p_t(Y) \geq 1$, then $\mathcal{R}(X,Y)$ is nowhere dense in $\mathcal{N}(X,Y)$, i.e., the interior of its closure in $\mathcal{N}(X,Y)$ is void. In other words, $\mathcal{R}(X,Y)$ is topologically small in $\mathcal{N}(X,Y)$.*

Remark that the situation is quite different if $\mathcal{R}(X,Y)$ is replaced by $\mathcal{N}(X,Y)$ and $\mathcal{N}(X,Y)$ is replaced by $C^\omega(X,Y)$. In fact, in the remarkable paper [27], it is proved that, when X is compact, $\mathcal{N}(X,Y)$ is dense in $C^\omega(X,Y)$.

Theorem F. *If $p_t(Y) \geq 2$, then $\mathcal{R}(X,Y)$ can be decomposed into a finite family $\{S_i\}_i$ of locally closed subspaces such that, for each i, there is a continuous injective map between S_i and some $\mathbb{R}^{n_i}$. In particular, $\mathcal{R}(X,Y)$ is dimensionally small in $\mathcal{N}(X,Y)$.*

Suppose X and Y affine and nonsingular. Consider $\mathcal{N}(X,Y)$ as a subspace of $C^\infty(X,Y)$ equipped with the weak topology (see chapter 2 of [34]) and $\mathcal{R}(X,Y)$ as a subspace of $\mathcal{N}(X,Y)$. Theorem F can be restated without changes, while Theorem E assumes the following form.

Theorem E′. *Indicate by $\Omega(X,Y)$ the complement of the closure of $\mathcal{R}(X,Y)$ in $\mathcal{N}(X,Y)$. If $p_t(Y) \geq 1$, then the closure of $\Omega(X,Y)$ in $\mathcal{N}(X,Y)$ contains all maps in $\mathcal{N}(X,Y)$ which are not finite–to–one. In particular, $\mathcal{R}(X,Y)$ is not dense in $\mathcal{N}(X,Y)$.*

As a consequence of the latter result, we have that, if $\mathcal{R}(X,Y)$ is dense in $\mathcal{N}(X,Y)$ (or, equivalently, in $C^\infty(X,Y)$ if X is compact), then $p_t(Y) = 0$. We underline that the toric genus of each grassmannian $\mathbb{G}_{n,k}(\mathbb{F})$ is zero. Really, this is true for every rational real algebraic varieties.

In subsection 2.3, we extend the previous topological results in several ways. For example, the varieties X and Y are not necessarily affine, Y may have isolated points and $\mathcal{R}(X,Y)$ is also considered as a subspace of the sets $C^0(X,Y)$ and $C^\omega(X,Y)$ (resp. the set $C^k(X,Y)$ with $k \in \{1,2,\ldots\} \cup \{\infty,\omega\}$ when X and Y are nonsingular) equipped with topologies of a very general type. The results sketched above were presented in our announcement [31]. We remark that several nomenclatures are changed with respect to the ones used in this announcement. For further results, we refer the reader to [18].

Contents

1 Two new classes of algebraic invariants

1.1 Preliminaries and notations

Let us start recalling some classic notions concerning algebraic varieties over an arbitrary field (see sections 30–35 of [54]). Let R be a field. Equip each affine space R^n with the Zariski topology. An *algebraic variety over* R is a topological space X equipped with a subsheaf $\mathcal{R}_X$ of the sheaf of germs of functions on X with values in R such that there exist a finite open cover $\{V_i\}_{i\in I}$ of X and, for each $i \in I$, a map $\varphi_i : V_i \longrightarrow U_i$ where U_i is a locally closed subset of some R^{n_i} with the following two properties:

VA_I) For each $i \in I$, φ_i is an isomorphism between X_i equipped with the restricted sheaf $\mathcal{R}_X|_{V_i}$ and U_i equipped with its sheaf of germs of regular functions.

VA_{II}) Denote by Δ the diagonal of $X \times X$ and, for each $i, j \in I$, replace the product $R^{n_i} \times R^{n_j}$ with $R^{n_i+n_j}$ and consider $U_i \times U_j$ as a subspace of $R^{n_i+n_j}$. Then, for each $i, j \in I$, the set $(\varphi_i \times \varphi_j)(\Delta \cap (V_i \times V_j))$ is closed in $U_i \times U_j$.

We indicate such a variety by $(X, \mathcal{R}_X)$ and, when no confusion is possible, we write simply X in place of $(X, \mathcal{R}_X)$. The topology of X is called Zariski topology. Given a subset S of X, we denote by $\mathrm{Zcl}_X(S)$ the closure of S

in X. As a topological space, X is noetherian so it can be expressed uniquely as a finite union of irreducible closed subsets, no one containing another. They are called irreducible components of X. If the trivial cover $\{X\}$ of X and a map $\varphi : X \longrightarrow U$ satisfy property VA_I (and hence property VA_{II}), then X is said to be affine. A point x of X is nonsingular of dimension r if the stalk $\mathcal{R}_{X,x}$ of $\mathcal{R}_X$ at x is a regular local ring of dimension r. The dimension $\dim(X)$ of X is the largest dimension of nonsingular points of X. We denote by $\mathrm{Nonsing}(X)$ the set of all nonsingular points of X of dimension $\dim(X)$ and define $\mathrm{Sing}(X) := X \setminus \mathrm{Nonsing}(X)$. If $X = \mathrm{Nonsing}(X)$, then X is called nonsingular. A locally closed subset of X, equipped with the natural structure of algebraic variety over R induced by X, is called algebraic subvariety of X. Let $(Y, \mathcal{R}_Y)$ be an algebraic variety over R. The cartesian product $X \times Y$ equipped with its natural structure of algebraic variety over R is called product variety of X and Y. Morphisms between X and Y, viewed as locally ringed spaces, are called regular maps. Let $h : X \longrightarrow Y$ be a bijective map. If both h and h^{-1} are regular, then we say that h is a biregular isomorphism and X and Y are biregularly isomorphic. A rational map between X and Y is an equivalence class of pairs (U, f_U) where U is a dense open subset of X, f_U is a regular map between U and Y and (U, f_U) is equivalent to (V, f_V) if $f_U = f_V$ on $U \cap V$. Let α be a rational map between X and Y represented by (U, f_U). If f_U is nonconstant, then α is said to be nonconstant. Let us give the notion of dominating rational map between algebraic varieties not necessarily irreducible. This notion permits to define the composition between rational maps. We say that α is *dominating* if, for each irreducible component X' of X, there is an irreducible component Y' of Y such that $\mathrm{Zcl}_Y(f_U(U \cap X')) = Y'$. Suppose α dominating. Let β be a rational map between Y and an algebraic variety Z over R represented by (V, g_V). We define the composition map $\beta \circ \alpha$ as the rational map between X and Z represented by $(f_U^{-1}(V), g_V \circ f_U|_{f_U^{-1}(V)})$. The map α is called birational isomorphism if there exists a dominating rational map γ between Y and X such that $\gamma \circ \alpha$ is represented by the identity map on X and $\alpha \circ \gamma$ is represented by the identity map on Y. If such a map γ exists, then it is unique, is denoted by α^{-1} and X and Y are said to be birationally isomorphic.

In what follows, R is a fixed real closed field and C is its algebraic closure. We recall that the algebraic extension $R[i] = R[x]/(x^2 + 1)$ of R coincides with C and the ordering structure on R induces on every algebraic variety over R a topology finer than the Zariski one called euclidean topology. By *real algebraic variety*, we mean an algebraic variety over R of positive dimension. We use both the Zariski topology and the euclidean topology. However, unless otherwise indicated, all real algebraic varieties are considered equipped with the euclidean topology. As is usual, the notions of irreducible real algebraic variety and of irreducible components of a real algebraic variety refer to the Zariski topology. By *real algebraic curve*, we mean a 1–dimensional affine irreducible real algebraic variety. Let X be a real algebraic variety. If an algebraic subvariety D of X is a real algebraic curve, then we say that D is a real algebraic curve of X. Let Y be a real algebraic variety. We indicate by $\mathcal{R}(X,Y)$ (resp. $\mathcal{R}atio(X,Y)$) the set of all regular (resp. rational) maps between X and Y and, when X is irreducible, by $\mathcal{R}^*(X,Y)$ (resp. $\mathcal{R}atio^*(X,Y)$) the set of all nonconstant regular (resp. rational) maps between X and Y.

Let us recall some fact concerning the complexification of real algebraic curves and of nonconstant rational maps between them. Equip each projective space $\mathbb{P}^n(R)$ (resp. $\mathbb{P}^n(C)$) with its natural structure of algebraic variety over R (resp. over C). Indicate by $\sigma_n : \mathbb{P}^n(C) \longrightarrow \mathbb{P}^n(C)$ the conjugation map and identify canonically $\mathbb{P}^n(R)$ with the fixed point set of σ_n. Let S be a subset of $\mathbb{P}^n(C)$. Define the real part $S(R)$ of S by $S(R) := S \cap \mathbb{P}^n(R)$. The set S is said to be defined over R if it is σ_n–invariant, i.e, $\sigma_n(S) = S$. Remark that if S is an algebraic subvariety of $\mathbb{P}^n(C)$, then $S(R)$ is an algebraic subvariety of $\mathbb{P}^n(R)$. Let D be a real algebraic curve and let A be a 1–dimensional nonsingular irreducible Zariski closed subspace of some $\mathbb{P}^n(C)$ defined over R. If there is a birational isomorphism α between $A(R)$ and D, then the pair (A, α) (or simply A) is called a *projective nonsingular complexification of* D. By normalization, it follows that D always admits a projective nonsingular complexification. Let E be a real algebraic curve, let $\gamma \in \mathcal{R}atio^*(D,E)$ and let (D_C, α) and (E_C, β) be projective nonsingular complexifications of D and E respectively. It is known that there is a unique regular map γ_C between D_C and E_C such that $\gamma_C(D_C(R)) \subset E_C(R)$ and the rational map $\beta^{-1} \circ \gamma \circ \alpha$ is represented by the restriction of γ_C between $D_C(R)$ and $E_C(R)$. Such a map γ_C is called *complexification of* γ *between* (D_C, α) *and* (E_C, β). Let $f \in \mathcal{R}^*(D,E)$ and let γ_f be the element of $\mathcal{R}atio^*(D,E)$ represented by (D, f). The complexification of f between (D_C, α) and (E_C, β) is defined to be the complexification of γ_f between (D_C, α) and (E_C, β) and is indicated by f_C. The uniqueness of the complexification of a nonconstant rational map between real algebraic curves implies that a projective nonsingular complexification of a real algebraic curve is unique up to biregular isomorphism. In this way, it is possible to define the *genus* $g(D)$ *of* D as the (geometric or arithmetic) genus of a projective nonsingular complexification of D.

We denote by $\mathbb{N}$ the set of all non–negative integers and define $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

1.2 Curve genera

The following result is the starting point to define the curve genera of a real algebraic variety. It is a weak version of Corollary 1.3 of [33].

Lemma 1.1 ([33]) *Let V be an affine irreducible real algebraic variety. Then, there exists an integer $K \in \mathbb{N}$ with the following property: for each $x, y \in V$, there is a real algebraic curve D of V containing $\{x, y\}$ such that $g(D) \leq K$.*

Let X be a real algebraic variety, let $k \in \mathbb{N}$ and let x be a point of X. We denote by $\mathcal{C}_X(k)$ the set of all real algebraic curves of X with genus k and by $\mathcal{C}_X(k, x)$ the set of all elements of $\mathcal{C}_X(k)$ containing x. We define the curve star $\mathrm{St}_X(k, x)$ of X at x with genus k by $\mathrm{St}_X(k, x) := \mathrm{Zcl}_X(\bigcup_{D \in \mathcal{C}_X(k,x)} D)$.

Lemma 1.2 *Let X be an irreducible real algebraic variety. Then, there are an integer $K \in \mathbb{N}$ and a function $\eta : X \longrightarrow \{0, 1, \ldots, K\}$ such that $\mathrm{St}_X(\eta(x), x) = X$ for each $x \in X$.*

Proof. Since X is coverable by a finite family of affine Zariski open subsets, it suffices to consider the case in which X is affine. Fix $x \in X$. By the previous lemma, we have that $\bigcup_{k=0}^{K} \mathrm{St}_X(k, x) = X$ for some $K \in \mathbb{N}$ so, being X irreducible, there exists an integer $\eta_x \in \{0, \ldots, K\}$ such that $\mathrm{St}_X(\eta_x, x) = X$. Defining the function η by $\eta(x) := \eta_x$, we complete the proof. $\square$

Definition 1.3 *Let X be a real algebraic variety. We denote by X_* the union of all irreducible components of X of positive dimension and by X_0 the union of all irreducible components of X of dimension zero, i.e., the set $X \setminus X_*$.*

The reader notes that the set of all irreducible components of X_* coincides with the set of all irreducible components of X of positive dimension.

Lemma 1.2 ensures that the following definition is consistent.

Definition 1.4 *Let X be a real algebraic variety.*

First, suppose X irreducible. We define the curve genus $p_c(X, x)$ of X at a point x of X by $p_c(X, x) := \min\{k \in \mathbb{N} \,|\, \mathrm{St}_X(k, x) = X\}$, the curve genus $p_c(X)$ of X by $p_c(X) := \min_{x \in X} p_c(X, x)$ and, setting $\mathcal{Z}(X)$ equal to the set of all non–void Zariski open subsets of X, define the rational curve genus $P_c(X)$ of X by $P_c(X) := \max_{Z \in \mathcal{Z}(X)} \min_{x \in Z} p_c(X, x)$.

If X is reducible and $X_1, \ldots, X_s$ are the irreducible components of X_*, then, using the same nomenclature, we define $p_c(X,x) := \max_{\{i \mid X_i \ni x\}} p_c(X_i, x)$ for each $x \in X_*$, $p_c(X) := \max_{i \in \{1,\ldots,s\}} p_c(X_i)$ and $P_c(X) := \max_{i \in \{1,\ldots,s\}} P_c(X_i)$.

The function $p_{c,X} : X_* \longrightarrow \mathbb{N}$ which sends x into $p_c(X,x)$ is called curve genus function of X.

Remark 1.5 *If D is a real algebraic curve, then $p_c(D) = P_c(D) = g(D)$ and $p_{c,D}$ is constantly equal to $g(D)$.*

We describe the basic properties of curve genera. First, we need a lemma.

Lemma 1.6 *Let D and E be real algebraic curves. If $g(D) < g(E)$, then every rational map (and hence every regular map) between D and E is constant.*

Proof. Suppose $\mathcal{R}atio^*(D,E) \neq \emptyset$. Let γ be an element of $\mathcal{R}atio^*(D,E)$ and let $\gamma_C : D_C \longrightarrow E_C$ be a complexification of γ. Since $\gamma_C \in \mathcal{R}^*(D_C, E_C)$, by Hurwitz's formula, it would follow that $g(D) = g(D_C) \geq g(E_C) = g(E)$ which contradicts our assumption. □

Lemma 1.7 *Let X be a real algebraic variety and let U be a Zariski open subset of X such that $U \cap X_* \neq \emptyset$. It holds: $p_c(X) \leq P_c(X)$ and $p_{c,U} = p_{c,X}|_{U \cap X_*}$. Moreover, if X is irreducible, $P_c(X)$ coincides with the minimum integer $k \in \mathbb{N}$ such that $p_{c,X}^{-1}(k)$ is Zariski dense in X.*

Proof. Easy consequence of Definition 1.4. □

Lemma 1.8 *Let X and Y be irreducible real algebraic varieties. The following assertions are verified.*

a) *Let $f \in \mathcal{R}(X,Y)$. Suppose f dominating, i.e., $\mathrm{Zcl}_Y(f(X)) = Y$. Then, it holds: $p_{c,X} \geq p_{c,Y} \circ f$ and $p_c(X) \geq p_c(Y)$.*

b) *If there is a dominating rational map between X and Y, then it holds: $P_c(X) \geq P_c(Y)$.*

Proof. Let $x \in X$, let $y := f(x)$ and let $k := p_c(X,x)$. It holds: $Y = \mathrm{Zcl}_Y(f(\mathrm{St}_X(k,x))) = \mathrm{Zcl}_Y(\bigcup_{D \in \mathcal{C}_X(k,x)} f(D)) = \mathrm{Zcl}_Y(\bigcup_{D \in \mathcal{C}_X(k,x)} \overline{f}(D))$ where $\overline{f}(D)$ is the Zariski closure of $f(D)$ in Y. For each $D \in \mathcal{C}_X(k,x)$, $\overline{f}(D) = \{y\}$ or there is an affine Zariski open neighborhood V of y in Y such that $D' := V \cap \overline{f}(D)$ is a real algebraic curve of Y. In the latter case, Lemma 1.6

ensures that $g(D') \leq k$. It follows that $\bigcup_{h=0}^{k} \mathrm{St}_Y(h,y) = Y$ so $p_c(Y,y) \leq k$ as desired. Let now $\xi : U \longrightarrow Y$ be a dominating regular map between a non–void Zariski open subset U of X and Y. Since $P_c(X) = P_c(U)$, it suffices to show that $P_c(U) \geq P_c(Y)$. By the previous part of the proof, we know that $p_{c,U} \geq p_{c,Y} \circ \xi$. Let $K := P_c(U)$ and let $U_K := p_{c,U}^{-1}(K)$. By the last part of Lemma 1.7, we have that U_K is Zariski dense in U, hence, being ξ dominating, $\xi(U_K)$ is Zariski dense in Y. Moreover, $\xi(U_K)$ is contained in $\bigcup_{h=0}^{K} p_{c,Y}^{-1}(h)$ because $K = p_{c,U}(x) \geq p_{c,Y}(\xi(x))$ for each $x \in U_K$. In this way, $p_{c,Y}^{-1}(h)$ is Zariski dense in Y for some $h \in \{0. \ldots, K\}$ so, applying the last part of Lemma 1.7 again, it follows that $P_c(Y) \leq K$. □

Corollary 1.9 a) *The curve genus at a point is a Zariski local biregular invariant, i.e., if X and Y are real algebraic varieties, x is a point of X_*, y is a point of Y_*, U is a Zariski open neighborhood of x in X, V is a Zariski open neighborhood of y in Y and $\varphi : U \longrightarrow V$ is a biregular isomorphism such that $\varphi(x) = y$, then $p_c(X,x) = p_c(U,x) = p_c(V,y) = p_c(Y,y)$.*

b) *The curve genus is a biregular invariant, i.e., if X and Y are real algebraic varieties with X_* and Y_* biregularly isomorphic, then $p_c(X) = p_c(Y)$.*

c) *The rational curve genus is a birational invariant, i.e., if X and Y are real algebraic varieties with X_* and Y_* birationally isomorphic, then $P_c(X) = P_c(Y)$.*

d) *The curve genus function is preserved by biregular isomorphisms, i.e., if X and Y are real algebraic varieties and $\varphi : X_* \longrightarrow Y_*$ is a biregular isomorphism, then $p_{c,X} = p_{c,Y} \circ \varphi$.*

1.3 Toric genera and toric dimensions

Toric genera. Let us start with a lemma.

Lemma 1.10 *Let X be a real algebraic variety. Then, there is an integer $H \in \mathbb{N}$ which satisfies the following assertion: if there are a Zariski open subset Z of X with $Z \cap X_* \neq \emptyset$, a finite family $\{D_i\}_{i \in I}$ of real algebraic curves and an injective regular map between Z and the product variety $\prod_{i \in I} D_i$, then $\min_{i \in I} g(D_i) \leq H$.*

Proof. We may suppose X irreducible. For each $h \in \mathbb{N}$, define $X_h := \bigcup_{D \in \mathcal{C}_X(h)} D$ and $\overline{X}_h := \mathrm{Zcl}_X(X_h)$. By Lemma 1.2, there is $H \in \mathbb{N}$ such that $\overline{X}_H = X$. Let Z, $\{D_i\}_{i \in I}$ and an injective regular map $\varphi : Z \longrightarrow \prod_{i \in I} D_i$ be as in the statement of the lemma. For each $j \in I$, let $\pi_j : \prod_{i \in I} D_i \longrightarrow D_j$ be the natural projection. Since X_H is Zariski dense in X, we can find a real algebraic curve D of Z with genus H. The restriction of φ to D is nonconstant so, for some $j \in I$, the regular map $\pi_j \circ \varphi|_D : D \longrightarrow D_j$ is nonconstant also. From Lemma 1.6, it follows that $g(D_j) \leq H$ and the proof is complete. $\square$

Before giving the definitions of toric genera, we need the preliminary affine notion of global toric genus. Let V be an affine real algebraic variety. Indicate by $\mathcal{T}(V)$ the set of all integers $h \in \mathbb{N}$ such that there are a finite family $\{D_i\}_{i \in I}$ of real algebraic curves with $\min_{i \in I} g(D_i) = h$ and an injective regular map between V and $\prod_{i \in I} D_i$. Remark that, being V affine, $\mathcal{T}(V)$ always contains 0. Moreover, by Lemma 1.10, we have that $\sup \mathcal{T}(V) < +\infty$. We define the *global toric genus* $gp_t(V)$ *of* V by $gp_t(V) := \max \mathcal{T}(V)$. Evidently, the global toric genus is a biregular invariant on affine real algebraic varieties.

Lemma 1.10 ensures that the following definition is consistent.

Definition 1.11 *Let X be a real algebraic variety.*

First, suppose X irreducible. For each $x \in X$, we denote by $\mathcal{Z}_x^{\mathrm{aff}}(X)$ the set of all affine Zariski open neighborhoods of x in X. We define the toric genus $p_t(X,x)$ of X at a point x of X by $p_t(X,x) := \max_{V \in \mathcal{Z}_x^{\mathrm{aff}}(X)} gp_t(V)$, the toric genus $p_t(X)$ of X by $p_t(X) := \min_{x \in X} p_t(X,x)$ and the rational toric genus $P_t(X)$ by $P_t(X) := \max_{x \in X} p_t(X,x)$.

If X is reducible and $X_1, \ldots, X_s$ are the irreducible components of X_*, then, using the same nomenclature, we define $p_t(X,x) := \min_{\{i \,|\, X_i \ni x\}} p_t(X_i,x)$ for each $x \in X_*$, $p_t(X) := \min_{i \in \{1,\ldots,s\}} p_t(X_i)$ and $P_t(X) := \min_{i \in \{1,\ldots,s\}} P_t(X_i)$.

The function $p_{t,X} : X_* \longrightarrow \mathbb{N}$ which sends $x \in X_*$ into $p_t(X,x)$ is called toric genus function of X.

Remark 1.12 *If D is a real algebraic curve, then $p_t(D) = P_t(D) = g(D)$ and $p_{t,D}$ is constantly equal to $g(D)$.*

In the following results, we present the basic properties of toric genera. We omit the simple proofs.

Lemma 1.13 *Let X be a real algebraic variety and let U be a Zariski open subset of X such that $U \cap X_* \neq \emptyset$. It holds: $p_t(X) \leq P_t(X)$ and $p_{t,U} = p_{t,X}|_{U \cap X_*}$. Moreover, if X is affine, $gp_t(X) \leq p_t(X)$.*

Lemma 1.14 *Let X and Y be real algebraic varieties. The following assertions are verified.*

a) *Let $f : X_* \longrightarrow Y_*$ be an injective regular map. Then, it holds: $p_{t,X} \geq p_{t,Y} \circ f$ and $p_t(X) \geq p_t(Y)$. In particular, if X' be an algebraic subvariety of X (of positive dimension), then, $p_{t,X'} \geq p_{t,X}|_{X'_*}$ and $p_t(X') \geq p_t(X)$.*

b) *If there is a rational map between X_* and Y_* represented by (U, f_U) with f_U dominating and injective, then $P_t(X) \geq P_t(Y)$.*

Corollary 1.15 a) *The toric genus at a point is a Zariski local biregular invariant.*

b) *The toric genus is a biregular invariant.*

c) *The rational toric genus is a birational invariant.*

d) *The toric genus function is preserved by biregular isomorphisms.*

Example 1.16 (toric genera of a n–torus) *Let $\{D_i\}_{i\in I}$ be a finite family of real algebraic curves and let T be the product variety $\prod_{i\in I} D_i$. Define $a := \min_{i\in I} g(D_i)$ and let $b := \max_{i\in I} g(D_i)$. From Lemma 1.8 and the proof of Lemma 1.10, it follows that $p_{c,X} \geq b$ and $p_{t,T}$ is constantly equal to a. In particular, $P_c(T) \geq p_c(T) \geq b$ and $p_t(T) = P_t(T) = a$.*

Let us present a variant of the global toric genus. Let V be an affine real algebraic variety. Indicate by $\mathcal{T}_{emb}(V)$ the set of all integers $h \in \mathbb{N}$ such that there are a finite family $\{D_i\}_{i\in I}$ of real algebraic curves with $\min_{i\in I} g(D_i) = h$ and a biregular embedding of V into $\prod_{i\in I} D_i$, i.e., an injective regular map $\varphi : V \longrightarrow \prod_{i\in I} D_i$ such that $\varphi(V)$ is an algebraic subvariety of $\prod_{i\in I} D_i$ and $\varphi^{-1} : \varphi(V) \longrightarrow V$ is a regular map also. We define the *biregular global toric genus $bgp_t(V)$ of V* by $bgp_t(V) := \max \mathcal{T}_{emb}(V)$. Evidently, $0 \in \mathcal{T}_{emb}(V)$ and $\mathcal{T}_{emb}(V) \subset \mathcal{T}(V)$, hence $bgp_t(V)$ is well-defined and $bgp_t(V) \leq gp_t(V)$. The biregular global toric genus is a biregular invariant on affine real algebraic varieties.

TORIC DIMENSIONS. Let us introduce the notions of toric dimensions.

Definition 1.17 *Let X be a real algebraic variety.*

First, suppose X is irreducible. The toric dimension $\mathrm{tdim}(X)$ *of X is the minimum integer $n \in \mathbb{N} \setminus \{0\}$ such that there are a non–void affine Zariski open subset V of X, real algebraic curves $D_1, \ldots, D_n$ with $\min_{i\in\{1,\ldots,n\}} g(D_i) \geq p_t(X)$ and an injective regular map between V and $\prod_{i=1}^n D_i$. Replacing $p_t(X)$*

with $P_t(X)$, we obtain the definition of rational toric dimension $\mathrm{Tdim}(X)$ *of X.*

If X is reducible and $X_1, \ldots, X_s$ are the irreducible components of X_*, then, using the same nomenclature, we define $\mathrm{tdim}(X) := \max_{i \in \{1,\ldots,s\}} \mathrm{tdim}(X_i)$ *and* $\mathrm{Tdim}(X) := \max_{i \in \{1,\ldots,s\}} \mathrm{Tdim}(X_i)$.

Remark 1.18 *If D is a real algebraic curve, then both* $\mathrm{tdim}(D)$ *and* $\mathrm{Tdim}(D)$ *are equal to* 1.

The toric dimension is a biregular invariant and the rational toric dimension is a birational invariant on real algebraic varieties.

1.4 Comparison, variation and bounds

COMPARISON. We define a new biregular invariant on real algebraic varieties.

Definition 1.19 *Let X be a real algebraic variety. We indicate by $e_X : X_* \longrightarrow \mathbb{N}$ the function which sends x into the minimum genus of a real algebraic curve of X containing x and define $e(X) := \min_{x \in X_*} e_X(x)$.*

The consistency of the previous definition follows from Lemma 1.1 (or, simply, from the Dubois–Efroymson dimension theorem [28]).

Lemma 1.20 *Let X be a real algebraic variety. Then, it holds:*

$$p_{t,X} \leq e_X \leq p_{c,X}, \;\; p_t(X) \leq e(X) \leq p_c(X),$$

$$p_t(X) \leq P_t(X) \leq p_c(X) \leq P_c(X)$$

and

$$\dim(X) \leq \mathrm{tdim}(X) \leq \mathrm{Tdim}(X).$$

Moreover, if X is affine, $bgp_t(X) \leq gp_t(X) \leq p_t(X)$.

Proof. We may suppose X irreducible. Following the argument used in the proof of Lemma 1.10, it is easy to see that $P_t(X) \leq p_c(X)$ and, for each $x \in X$, $p_{t,X}(x) \leq e_X(x)$. In particular, it follows that $p_t(X) \leq e(X)$. The remaining inequalities are evident or just known. $\square$

Variation. A *real Nash set* (*over* R) is a topological space X equipped with a subsheaf $\mathcal{N}_X$ of the sheaf of germs of functions on X with values in R isomorphic to a Nash subset of some R^n equipped with its sheaf of germs of Nash functions (see chapter 8 of [4]). Evidently, an affine real algebraic variety has a natural structure of real Nash set. The notions of Nash map and Nash isomorphism between real Nash sets can be defined similarly to the algebraic case.

Definition 1.21 *Let X and $\widetilde{X}$ be affine real algebraic varieties. A regular map $\varphi : \widetilde{X} \longrightarrow X$ is said to be a weak change of the algebraic structure of X if $\varphi(\mathrm{Sing}(\widetilde{X})) \subset \mathrm{Sing}(X)$ and, equipping X and $\widetilde{X}$ with their natural structure of real Nash sets, φ is a Nash isomorphism.*

Remark 1.22 *If X is nonsingular, then $\widetilde{X}$ is nonsingular also.*

The following result is elementary, but fundamental, because it reveals two important aspects of our genera in the affine case:

i) The real Nash set structure of an affine real algebraic variety does not determinate any of our genera. In this sense, we can say that the nature of such invariants is completely algebraic.

ii) Our genera are not trivial.

Lemma 1.23 *Let X be an affine real algebraic variety and let $H \in \mathbb{N}$. Then, there exists a weak change $\varphi : \widetilde{X} \longrightarrow X$ of the algebraic structure of X such that $bgp_t(\widetilde{X}) \geq H$. In particular, $gp_t(\widetilde{X})$, $p_t(\widetilde{X})$, $P_t(\widetilde{X})$, $p_c(\widetilde{X})$ and $P_c(\widetilde{X})$ satisfy the same inequality.*

Proof. By using the real algebraic Alexandrov compactification of X, we may suppose that X is a bounded algebraic subvariety of some R^n. For each $i \in \{1, \ldots, n\}$, let H_i be the hyperplane of R^n defined by the equation $x_i = 1$. Using a translation of R^n if needed, we may suppose that $X \cap \bigcup_{i=1}^n H_i = \emptyset$ also. Fix an odd integer d such that $\frac{1}{2}(d-1)(d-2) \geq H$. Define the real algebraic curve D_d by $D_d := \{(x, y) \in R^2 \mid x^d + y^d = 1\}$ and the regular map $\psi : D_d \longrightarrow R$ by $\psi(x, y) := x$. Let $\psi^n : D_d^n \longrightarrow R^n$ be the n^{th}–power of ψ, let $\widetilde{X} := (\psi^n)^{-1}(X)$ and let $\varphi : \widetilde{X} \longrightarrow X$ be the restriction of ψ^n between $\widetilde{X}$ and X. The map φ is a weak change of the algebraic structure of X and $bgp_t(\widetilde{X}) \geq g(D_d) = \frac{1}{2}(d-1)(d-2) \geq H$. $\square$

Remark 1.24 *Making use of the previous lemma and the real algebraic blowing down, it is possible to prove the following assertion: "Let X be an affine real algebraic variety with* $\dim(X) \geq 2$ *and let $H, K \in \mathbb{N}$. Then, there exists a weak change $\varphi : \widetilde{X} \longrightarrow X$ of the algebraic structure of X such that $p_t(\widetilde{X}) \geq H$ and $p_c(\widetilde{X}) \geq p_t(\widetilde{X}) + K$".*

Bounds for curve and toric genera. We present an explicit upper bound for the rational curve genus and a criterion to obtain a lower bound for the toric genus of a real algebraic variety. For simplicity, we consider the affine irreducible case only.

Let X be an irreducible algebraic subvariety of R^n of dimension r. First, suppose $r < n$. We define the *complete intersection degree* $\mathrm{cideg}(X, R^n)$ *of X in R^n* as the minimum integer c such that there are a point $p \in \mathrm{Nonsing}(X)$ and polynomials $P_1, \ldots, P_{n-r}$ vanishing on X with independent gradients at p and $c = \prod_{i=1}^{n-r} \deg(P_i)$. If $r = n$, then we consider $\mathrm{cideg}(X, R^n)$ equal to 1.

We call *Castelnuovo function* the function $\mathrm{Castel} : \mathbb{N}^* \times \mathbb{N}^* \longrightarrow \mathbb{N}$ defined as follows: for each (d, n) with d or n equal to 1, $\mathrm{Castel}(d, n) := 0$ and, for each (d, n) with $d \geq 2$ and $n \geq 2$, $\mathrm{Castel}(d, n) := \frac{1}{2}a(a-1)(n-1) + ab$ where a and b are the unique non–negative integers such that $d - 1 = a(n - 1) + b$ and $b \in \{0, 1, \ldots, n-2\}$. It is easy to see that $\mathrm{Castel}(d, n) \leq \frac{1}{2}(d-1)(d-2)$.

Lemma 1.25 *Let X be a r–dimensional irreducible algebraic subvariety of R^n (with $r \geq 1$) and let $c := \mathrm{cideg}(X, R^n)$. Suppose X is nondegenerate in R^n, i.e, is not contained in any affine hyperplane of R^n. Then, it holds:*

$$P_c(X) \leq \max p_{c,X} \leq \mathrm{Castel}(c, n - r + 1).$$

Proof. This is an immediate consequence of Theorem 1.4 of [33]. □

Let X be an affine irreducible real algebraic variety and let (x, y) be a pair of distinct points of X. Denote by $\mathrm{Sep}_X(x, y)$ the set of all integers $h \in \mathbb{N}$ such that there are a real algebraic curve D of genus h and $f \in \mathcal{R}(X, D)$ which distinguishes x and y. Remark that, being X affine, $0 \in \mathrm{Sep}_X(x, y)$. Moreover, from Lemma 1.1 and Lemma 1.6, it follows that $\sup \mathrm{Sep}_X(x, y) < +\infty$. We define the *separation genus* $\mathrm{sep}_X(x, y)$ *of x and y in X* by $\mathrm{sep}_X(x, y) := \max \mathrm{Sep}_X(x, y)$.

Lemma 1.26 *Let X be an affine irreducible real algebraic variety and let Δ be the diagonal of $X^2 = X \times X$. Then, it holds:*

$$gp_t(X) = \min{}_{(x,y) \in X^2 \setminus \Delta} \mathrm{sep}_X(x, y)$$

Proof. Define $k := gp_t(X)$ and $h := \min_{(x,y)\in X^2\setminus\Delta} \mathrm{sep}_X(x,y)$. By definition of $gp_t(X)$, there are a finite family $\{D_i\}_{i\in I}$ of real algerbaic curves with $\min_{i\in I} g(D_i) = k$ and an injective regular map $\varphi : X \longrightarrow \prod_{i\in I} D_i$. For each $j \in I$, let $\pi_j : \prod_{i\in I} D_i \longrightarrow D_j$ be the natural projection. Fix $(x,y) \in X^2 \setminus \Delta$ and choose $j \in I$ in such a way that the composition map $\pi_j \circ \varphi$ distinguishes x and y. It follows that $\mathrm{sep}_X(x,y) \geq g(D_j) \geq k$. In particular, it holds: $h \geq k$. Let us show the converse inequality. For each $(x,y) \in X^2 \setminus \Delta$, choose a real algebraic curve $D_{x,y}$ with $g(D_{x,y}) = \mathrm{sep}_X(x,y)$ and $f_{x,y} \in \mathcal{R}(X, D_{x,y})$ which distinguishes x and y. Remark that $\Delta = \bigcap_{(x,y)\in X^2\setminus\Delta}(f_{x,y} \times f_{x,y})^{-1}(\Delta_{x,y})$ where $\Delta_{x,y}$ is the diagonal of $D_{x,y} \times D_{x,y}$. By noetherianity, it is possible to extract a finite family $\{f_{x_i,y_i}\}_{i\in I}$ from $\{f_{x,y}\}_{(x,y)\in X^2\setminus\Delta}$ in such a way that $\Delta = \bigcap_{i\in I}(f_{x_i,y_i} \times f_{x_i,y_i})^{-1}(\Delta_{x_i,y_i})$. Let $f_i := f_{x_i,y_i}$ and $D_i := D_{x_i,y_i}$ for each $i \in I$. It follows that the regular map $\prod_{i\in I} f_i : X \longrightarrow \prod_{i\in I} D_i$ is injective so $k \geq \min_{i\in I} g(D_i) \geq h$. This completes the proof. □

As an immediate consequence, we have the following corollary.

Corollary 1.27 *Let X be an affine irreducible real algebraic variety. Suppose there is an integer $h \in \mathbb{N}$ with the following property: for each pairs (x,y) of distinct points of X, there are a real algebraic curve $D_{x,y}$ with $g(D_{x,y}) \geq h$ and $f_{x,y} \in \mathcal{R}(X, D_{x,y})$ which distinguishes x and y. Then, it holds:*

$$p_t(X) \geq gp_t(X) \geq h.$$

Upper bound for toric dimensions. We present an upper bound for the rational toric dimension of a real algebraic variety having rational toric genus greater than or equal to 2. We will use such a bound in subsection 2.2.

Recall that C indicates the algebraic closure of the fixed real closed field R. In what follows, by algebraic curve over C, we mean a 1–dimensional nonsingular irreducible Zariski closed subspace of some projective space $\mathbb{P}^n(C)$. Let A be an algebraic curve over C and let $Im_2(A)$ be the set of all nonconstant regular maps between A and some algebraic curve B over C with genus greater than or equal to 2. Define an equivalent relation F_C on $Im_2(A)$ as follows: $f \in \mathcal{R}^*(A,B)$ is F_C–equivalent to $f' \in \mathcal{R}^*(A,B')$ in $Im_2(A)$ if and only if there is a biregular isomorphism $h : B \longrightarrow B'$ such that $f' = h \circ f$. Indicate by $\mathfrak{Im}_2(A)$ the set of all F_C–equivalence classes of $Im_2(X)$.

Theorem (de Franchis [29], Harris, Howard–Sommese [35]). *For each algebraic curve A over C, the set $\mathfrak{Im}_2(A)$ is finite and there exists an upper bound for its cardinality $\sharp\mathfrak{Im}_2(A)$ in terms of the genus $g(A)$ of A only.*

This result ensures the consistency of the following definition.

Definition 1.28 *We call de Franchis function the function* $\mathcal{F} : \mathbb{N} \longrightarrow \mathbb{N}$ *which sends* $a \in \mathbb{N}$ *into the maximum integer* $k \in \mathbb{N}$ *such that there is an algebraic curve* A *over* C *with* $g(A) = a$ *and* $\sharp\mathfrak{Im}_2(A) = k$. *We indicate by* $\mathcal{F}_*$ *the smallest non–decreasing function* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *such that* $\mathcal{F} \leq f$.

Evidently, it holds: $\mathcal{F}(0) = \mathcal{F}(1) = 0$ and $\mathcal{F}(2) = 1$.

Remark 1.29 *In [35], Howard and Sommese gave the first explicit upper bound for* $\mathcal{F}$. *This bound has been improved successively by Kani [39] and Alzati and Pirola [2]. Really, all these bounds are upper bounds for* $\mathcal{F}_*$ *also. For completeness, we recall the Alzati–Pirola bound (which is the better): for each* $a \geq 3$, $\mathcal{F}_*(a) \leq \exp\left\{\frac{4}{3}(\ln 3)(a^2-1) + [\log_2(a)]\ln(84a) + \ln(12\sqrt{2})\right\}$ *where* $[x]$ *indicates the integer part of a real number* x. *The previous bound implies the following one:* $\mathcal{F}_*(a) \leq 3^{(4/3)(a+1)^2}$ *for each* $a \geq 3$.

The next lemma contains the mentioned upper bound.

Lemma 1.30 *Let* X *be a real algebraic variety with* $P_t(X) \geq 2$. *Then, it holds:*

$$\mathrm{Tdim}(X) \leq \mathcal{F}_*(P_c(X)).$$

Moreover, when X *is irreducible, it holds:*

$$\mathrm{Tdim}(X) \leq \mathcal{F}(P_c(X)).$$

Before proving this result, we need a real version of the previous theorem. Let X be an irreducible real algebraic variety and let $G_2(X)$ be the set of all nonconstant rational maps between X and some real algebraic curve D with $g(D) \geq 2$. Define an equivalence relation F on $G_2(X)$ as follows: $\alpha \in \mathcal{R}atio^*(X, D)$ is F–equivalent to $\beta \in \mathcal{R}atio^*(X, D')$ in $G_2(X)$ if and only if there is birational isomorphism γ between D and D' such that $\beta = \gamma \circ \alpha$. Indicate by $\mathcal{G}_2(X)$ the set of all F–equivalence classes of $G_2(X)$.

The following lemma will be proved in section 3.

Lemma 1.31 *Let* X *be an irreducible real algebraic variety. Then, the set* $\mathcal{G}_2(X)$ *is finite and it holds:* $\sharp\mathcal{G}_2(X) \leq \mathcal{F}(P_c(X))$.

Proof of Lemma 1.30. It suffices to prove the lemma in the case X is irreducible. Let $n := \mathrm{Tdim}(X)$ and $m := \sharp\mathcal{G}_2(X)$. We will show that $n \leq m$

so, by Lemma 1.31, the proof will be complete. Suppose on the contrary that $m < n$. By definition of rational toric dimension, there are a non–void affine Zariski open subset Z of X, real algebraic curves $D_1, \ldots, D_n$ with $\min_{i \in \{1,\ldots,n\}} g(D_i) = P_t(X)$ and an injective regular map φ between Z and $T := \prod_{i=1}^n D_i$. For each $i \in \{1, \ldots, n\}$, let $\pi_i : T \longrightarrow D_i$ be the natural projection, let $\varphi_i : Z \longrightarrow D_i$ be the composition $\pi_i \circ \varphi$ and let Φ_i be the rational map between X and D_i represented by φ_i. Up to rearrange the indices, we may suppose that there are a function $\eta : \{m+1, \ldots, n\} \longrightarrow \{1, \ldots, m\}$ and, for each $i \in \{m+1, \ldots, n\}$, a birational isomorphism α_i between D_i and $D_{\eta(i)}$ such that $\Phi_{\eta(i)} = \alpha_i \circ \Phi_i$. For each $i \in \{m+1, \ldots, n\}$, choose a non–void Zariski open subset Ω_i of D_i and a biregular embedding $\psi_i : \Omega_i \longrightarrow D_{\eta(i)}$ which represents α_i. Let Ω be the Zariski open subset of T defined by $\Omega := \prod_{i=1}^m D_i \times \prod_{i>m}^n \Omega_i$. Let us prove that $\varphi(Z) \cap \Omega \neq \emptyset$. If this would not be true, then $\varphi(Z) \subset T \setminus \Omega = \bigcup_{j>m}^n \Big(\prod_{i=1}^{j-1} D_i \times (D_j \setminus \Omega_j) \times \prod_{i>j}^n D_i\Big)$. Since each $D_j \setminus \Omega_j$ is finite and $\mathrm{Zcl}_T(\varphi(Z))$ is irreducible, there would exist $j \in \{m+1, \ldots, n\}$ and $p \in D_j \setminus \Omega_j$ such that $\varphi(Z) \subset \prod_{i=1}^{j-1} D_i \times \{p\} \times \prod_{i>j}^n D_i$. This contradicts the definition of n. In this way, we have that $\varphi(Z) \cap \Omega \neq \emptyset$ so $Z' := \varphi^{-1}(\Omega)$ is a non–void affine Zariski open subset of X. Let $\varphi' : Z' \longrightarrow \Omega$ be the restriction of φ between Z' and Ω, let $\psi : \Omega \longrightarrow \prod_{i=1}^m D_i \times \prod_{i>m}^n D_{\eta(i)}$ be the regular map defined by $\psi := \prod_{i=1}^m id_{D_i} \times \prod_{i>m}^n \psi_i$ where id_{D_i} indicates the identity map on D_i and let $\pi : \prod_{i=1}^m D_i \times \prod_{i>m}^n D_{\eta(i)} \longrightarrow \prod_{i=1}^m D_i$ be the natural projection. At this point, since $\varphi_{\eta(i)} = \psi_i \circ \varphi_i$ on Z' for each $i \in \{m+1, \ldots, n\}$, it is immediate to see that the composition $\pi \circ \psi \circ \varphi' : Z' \longrightarrow \prod_{i=1}^m D_i$ is an injective regular map. This contradicts the definition of n again. □

Lemma 1.30 has the following corollary.

Corollary 1.32 *Let X be a real algebraic variety with $P_t(X) \geq 2$ and let $r := \dim(X)$. Then, it holds:*

$$P_c(X) \geq -1 + \tfrac{1}{2}\sqrt{3 \log_3(r)}.$$

Proof. Since $r \leq \mathrm{Tdim}(X)$, by Lemma 1.30 and Remark 1.29, we have that $r \leq 3^{(4/3)(P_c(X)+1)^2}$. This inequality is equivalent to the one stated above. □

2 The main theorems

2.1 Total algebraic obstructions

The new algebraic invariants generate some elementary, but deep, obstructions to the existence of non–elementary rational and regular maps between real algebraic varieties. Let us present such obstructions. First, we recall the notion of Zariski locally constant map.

Definition 2.1 *Let X and Y be real algebraic varieties. A map $f : X \longrightarrow Y$ is said to be Zariski locally constant if, for each $x \in X$, there is a Zariski neighborhood of x in X on which f is constant or, equivalently, if f is constant on each irreducible component of X.*

Remark that every constant map is Zariski locally constant and every Zariski locally constant map is regular.

Theorem 2.2 *Let X and Y be real algebraic varieties. The following assertions are verified.*

a) *If $p_c(X) < p_t(Y)$ (or, better, $p_c(X) < e(Y)$), then every regular map between X and Y is Zariski locally constant.*

b) *If $P_c(X) < p_t(Y)$ (or, better, $P_c(X) < e(Y)$), then every rational map between X and Y is Zariski locally constant, i.e., it is represented by a Zariski locally constant map.*

Proof. a) Since $p_t(Y) \leq e(Y)$, it suffices to prove that, if there exists a regular map between X and Y which is not Zariski locally constant, then $p_c(X) \geq e(Y)$. Let f be such a map. There is an irreducible component X' of X_* such that $f|_{X'}$ is nonconstant. Let $Y' := \mathrm{Zcl}_Y(f(X'))$ and let $f' : X' \longrightarrow Y'$ be the restriction of f between X' and Y'. Remark that Y' is irreducible, $\dim(Y') \geq 1$, $Y' \subset Y_*$ and f' is dominating. By Lemma 1.8 and Lemma 1.20, we have that $p_{c,X'} \geq p_{c,Y'} \circ f' \geq e_{Y'} \circ f' \geq e_Y \circ f|_{X'}$ so $p_c(X) \geq p_c(X') \geq e(Y)$ as desired.

b) Suppose there is a rational map between X and Y represented by (U, f_U) with f_U not Zariski locally constant. From Corollary 1.15, Lemma 1.20 and part a) of this proof, we obtain that $P_c(X) = P_c(U) \geq p_c(U) \geq e(Y)$. □

Corollary 2.3 *Let X be a r–dimensional nondegenerate irreducible algebraic subset of R^n (with $r \geq 1$), let $c := \mathrm{cideg}(X, R^n)$ and let Y be a real algebraic variety. Then, if $\mathrm{Castel}(c, n-r+1) < p_t(Y)$ (or, better, $\mathrm{Castel}(c, n-r+1) < e(Y)$), every rational map (and hence every regular map) between X and Y is constant.*

Proof. Immediate from the previous result and Lemma 1.25. □

Let X and Y be real algebraic varieties and let $\alpha \in \mathcal{R}atio(X, Y)$. Suppose α is represented by the pairs (U, f_U) and $(U', f_{U'})$. Since $f_U = f_{U'}$ on $U \cap U'$, there is a regular map $f : U \cup U' \longrightarrow Y$ such that $f|_U = f_U$ and $f|_{U'} = f_{U'}$. This means that there exists a largest Zariski dense open subset U_α of X on which α is represented by a regular map $f_\alpha : U_\alpha \longrightarrow Y$. The points of $X \setminus U_\alpha$ are called fundamental point of α.

Theorem 2.4 *Let X and Y be irreducible real algebraic varieties, let $x \in X$ and let $y \in Y$ such that $p_{c,X}(x) < p_{t,Y}(y)$ (or, better, $p_{c,X}(x) < e_Y(y)$). The following assertions are verified.*

a) *Let $f \in \mathcal{R}(X, Y)$. If f sends x into y, then it is constant.*

b) *Let $\alpha \in \mathcal{R}atio^*(X, Y)$ represented by (U, f_U). Suppose x is an accumulation point of $X \setminus U$ in X and y is a limit point of f_U at x. Then, x is a fundamental point of α.*

Proof. Repeat the argument used in the proof of Theorem 2.2, a). □

In the next result, we will see that, when the target space is affine, up to a weak change of its algebraic structure, every rational (resp. regular) map is Zariski locally constant.

Theorem 2.5 *Let X and Y be real algebraic varieties. Suppose Y affine. Then, there is a weak change $\varphi : \widetilde{Y} \longrightarrow Y$ of the algebraic structure of Y such that every rational map (and hence every regular map) between X and $\widetilde{Y}$ is Zariski locally constant.*

Proof. By Lemma 1.23, we can find a weak change $\varphi : \widetilde{Y} \longrightarrow Y$ of the algebraic structure of Y such that $p_t(\widetilde{Y}) > P_c(X)$ so, from Theorem 2.2, it follows that every rational (resp. regular) map between X and $\widetilde{Y}$ is Zariski locally constant. □

In the remainder of this subsection, we consider only the case $R = \mathbb{R}$.

Let X be an affine real algebraic variety. A $\mathbb{Z}/2$–homology class of X is called algebraic if it is represented by a Zariski closed subset of X. The homology of X is said to be *totally algebraic* if each $\mathbb{Z}/2$–homology class of X is algebraic. This notion plays a crucial role in the study of the classical problem of making smooth objects algebraic (see chapters 11–14 of [4]). We say that the homotopy of X is *algebraically trivial* if, for each integer $n \in \mathbb{N}^*$, every regular map between the standard n–sphere and X is homotopic to a constant.

Theorem 2.6 *Let X be an affine real algebraic variety. Then, there exists a weak change $\varphi : \widetilde{X} \longrightarrow X$ of the algebraic structure of X such that the homotopy of $\widetilde{X}$ is algebraically trivial. Moreover, if the homology of X is totally algebraic, then the homology of $\widetilde{X}$ is totally algebraic also.*

Proof. By Lemma 1.23, there is a weak change $\varphi : \widetilde{X} \longrightarrow X$ of the algebraic structure of X such that $p_t(\widetilde{X}) \geq 1$. Since the rational curve genus of each standard sphere is zero, from Theorem 2.2, it follows that the homotopy of X is algebraically trivial. The last part of the statement follows easily from the properties of φ. □

As an immediate consequence of this theorem and Theorem 11.3.12 of [4], we have the following corollary.

Corollary 2.7 *For every compact C^∞ manifold M of dimension less than or equal to 5, there is an affine nonsingular real algebraic variety X diffeomorphic to M such that the homology of X is totally algebraic, while its homotopy is algebraically trivial.*

2.2 Finiteness of dominating morphism spaces

Let us recall a classical finiteness theorem.

Theorem (Hurwitz [36], de Franchis [29], Martens [49]). *Let A and B be nonsingular irreducible projective complex algebraic curves. Suppose the genus $g(B)$ of B is greater than or equal to 2. Then, the set $\mathcal{R}^*_{\mathbb{C}}(A, B)$ of all nonconstant regular maps between A and B is finite and there exists an explicit upper bound for its cardinality in terms of $g(A)$ and $g(B)$ only.*

This subsection deals with some extensions of the previous result to the real algebraic setting. We study the finiteness properties of three kinds of dominating maps between real algebraic varieties: dominating regular maps, dominating rational maps and weakly open regular maps.

Dominating regular and rational maps. Let X and Y be real algebraic varieties and let $f \in \mathcal{R}(X,Y)$. We say that a map f is *dominating* if, for each irreducible component X' of X, there exists an irreducible component Y' of Y such that $\mathrm{Zcl}_Y(f(X')) = Y'$. Remark that a rational map between X and Y is dominating (see the definition at page 7) if and only if it is represented by a dominating regular map. We denote by $rDom(X,Y)$ (resp. $RDom(X,Y)$) the set of all dominating regular (resp. rational) maps between X and Y and by $rAut(X)$ (resp. $RAut(X)$) the set of all biregular (resp. birational) automorphisms of X.

Definition 2.8 *Denote by $\mathcal{M} : \mathbb{N} \times (\mathbb{N} \setminus \{0,1\}) \longrightarrow \mathbb{N}$ the function which sends (a,b) into the maximum integer $k \in \mathbb{N}$ such that there are nonsingular irreducible projective complex algebraic curves A and B with $g(A) = a$, $g(B) = b$ and $\sharp\mathcal{R}^*_{\mathbb{C}}(A,B) = k$. Since $\mathcal{M}(a,b) = 0$ if $a < b$, we can define the function $\mathcal{M}_*$ as the smallest function $f : \mathbb{N} \times (\mathbb{N} \setminus \{0,1\}) \longrightarrow \mathbb{N}$ such that $\mathcal{M} \leq f$ and $f(a,b+1) \leq f(a,b) \leq f(a+1,b)$ for each $(a,b) \in \mathbb{N}\times(\mathbb{N}\setminus\{0,1\})$.*

It is well–known that $\mathcal{M}(a,a) \leq 84(a-1)$ if $a \geq 2$. This bound, due to Hurwitz, is sharp for infinitely many values of a (see [47]). In [49], Martens gave the first explicit upper bound for $\mathcal{M}$. Remark that, from the de Franchis–Harris–Howard–Sommese theorem (stated at page 17), it follows that $\mathcal{M}(a,b) \leq 84(b-1)\mathcal{F}(a)$ where $\mathcal{F}$ is the de Franchis function. In this way, the explicit upper bounds for $\mathcal{F}$ mentioned in Remark 1.29 induces explicit upper bounds for $\mathcal{M}$. In 1999, Tanabe [58] proved that $\mathcal{M}(a,b) \leq 2(a-1)(2b-1)\left(4\frac{a-1}{b-1}+1\right)^{2a}$ if $a > b \geq 2$. At the moment, to the best of our knowledge, this is the better upper bound for $\mathcal{M}(a,b)$ if $a > b \geq 2$.

Definition 2.9 *We call Hurwitz–Tanabe function the function $HT : \mathbb{N} \times (\mathbb{N} \setminus \{0,1\}) \longrightarrow \mathbb{N}$ defined as follows: $H(a,b) := 84(a-1)$ if $a = b \geq 2$, $H(a,b) := 2(a-1)(2b-1)\left(4\frac{a-1}{b-1}+1\right)^{2a}$ if $a > b \geq 2$ and $H(a,b) := 0$ otherwise.*

Remark 2.10 *It is easy to see that $\mathcal{M}_* \leq HT$ and, for each $a, b \in \mathbb{N}\setminus\{0,1\}$, $H(a,b) \leq 6(a-1)(4a-3)^{2a}$.*

We are now in position to present our finiteness results.

Theorem 2.11 *Let X and Y be irreducible real algebraic varieties. If $P_t(Y) \geq 2$, then $RDom(X,Y)$ and $RAut(Y)$ (and hence $rDom(X,Y)$ and $rAut(Y)$) are finite.*

More precisely, the following assertions are verified.

a) *If $p_t(Y) \geq 2$, then*

$$\sharp rDom(X,Y) \leq \mathcal{M}_*(P_c(X), p_t(Y))^{\mathrm{tdim}(Y)} \leq HT(P_c(X), p_t(Y))^{\mathcal{F}(P_c(Y))}$$

and

$$\sharp rAut(Y) \leq \mathcal{M}_*(P_c(Y), p_t(Y))^{\mathrm{tdim}(Y)} \leq HT(P_c(Y), p_t(Y))^{\mathcal{F}(P_c(Y))}.$$

b) *If $P_t(Y) \geq 2$, then the inequalities obtained by the previous ones replacing $rDom(X,Y)$ with $RDom(X,Y)$, $rAut(Y)$ with $RAut(Y)$, $p_t(Y)$ with $P_t(Y)$ and* $\mathrm{tdim}(Y)$ *with* $\mathrm{Tdim}(Y)$ *are verified also.*

Furthermore, in the mentioned inequalities, when Y is a real algebraic curve, we can replace the function $\mathcal{M}_*$ with $\mathcal{M}$ and $\mathcal{F}(P_c(Y))$ with 1.

Let X and Y be real algebraic varieties. Define:
u as the number of irreducible components of X_*,
w as the cardinality of X_0,
U as the number of irreducible components of Y_*,
V as the number of Zariski connected components of Y_*,
W as the cardinality of Y_0
and
Z as the maximum number of irreducible components of a Zariski connected components of Y_*.

Theorem 2.11$'$ (general case)**.** *If $P_t(Y) \geq 2$, then $RDom(X,Y)$ and $RAut(Y)$ (and hence $rDom(X,Y)$ and $rAut(Y)$) are finite.*

More precisely, the following assertions are verified.

a) *If $p_t(Y) \geq 2$, then*

$$\sharp rDom(X,Y) \leq W^w \cdot \left(W + V \cdot Z \cdot \mathcal{M}_*(P_c(X), p_t(Y))^{\mathrm{tdim}(Y)}\right)^u$$

and

$$\sharp rAut(Y) \leq V! \cdot W! \cdot Z^U \cdot \mathcal{M}_*(P_c(Y), p_t(Y))^{U \cdot \mathrm{tdim(Y)}}$$

where 0^0 is considered equal 1.

b) *If $P_t(Y) \geq 2$, then the inequalities obtained by the previous ones replacing $rDom(X,Y)$ with $RDom(X,Y)$, $rAut(Y)$ with $RAut(Y)$, $p_t(Y)$ with $P_t(Y)$ and* $\mathrm{tdim}(Y)$ *with* $\mathrm{Tdim}(Y)$ *are also verified.*

Furthermore, in all mentioned inequalities, we can replace the function $\mathcal{M}_*$ with the Hurwitz–Tanabe function HT and $\mathrm{tdim}(Y)$ *and* $\mathrm{Tdim}(Y)$ *with* $\mathcal{F}_*(P_c(Y))$.

The following corollary is a real version of the Hurwitz–de Franchis–Martens theorem.

Corollary 2.12 *If $P_t(Y) \geq 2$, then $RDom(X,Y)$ is finite and there exists an explicit upper bound for its cardinality in terms of u, w, V, W, Z, $P_c(X)$ and $P_c(Y)$ only.*

Remark 2.13 *Suppose that X is an algebraic subvariety of R^n and Y is an algebraic subvariety of R^m with $P_t(Y) \geq 2$. Let $\{X_i\}_{i=1}^u$ be the irreducible components of X_* and let $\{Y_j\}_{j=1}^U$ be the irreducible components of Y_*. For each $i \in \{1,\ldots,u\}$ and for each $j \in \{1,\ldots,U\}$, define: $c_i := \mathrm{cideg}(X_i, R^n)$, $r_i := \dim(X_i)$, n_i as the minimum dimension of an affine subspace of R^n containing X_i, $d_j := \mathrm{cideg}(Y_j, R^m)$, $s_j := \dim(Y_j)$ and m_j as the minimum dimension of an affine subspace of R^m containing Y_j. Then, from Lemma 1.25 and Corollary 2.12 (see Theorem 1.4 and Lemma 2.4 of [33]), it follows that $RDom(X,Y)$ is finite and there are an explicit upper bound for its cardinality in terms of u, w, V, W, Z, $\{c_i, r_i, n_i\}_{i=1}^u$ and $\{d_j, s_j, m_j\}_{j=1}^U$ only and a less accurate upper bound for its cardinality in terms of u, w, V, W, Z, $\{c_i\}_{i=1}^u$ and $\{d_j\}_{j=1}^U$ only.*

Weakly open regular maps. Let X and Y be real algebraic varieties and let $f \in \mathcal{R}(X,Y)$. We say that f is *weakly open* if, for each irreducible component X' of X, there exists an irreducible component Y' of Y such that $f(X') \subset Y'$ and the interior of $f(\mathrm{Nonsing}(X'))$ in Y' is non–void. We denote by $wo\mathcal{R}(X,Y)$ the set of all weakly open regular maps between X and Y. It is easy to see that if a regular map is dominating or is open in the usual sense, then it is weakly open also. In other words, indicating by $o\mathcal{R}(X,Y)$ the set of all maps in $\mathcal{R}(X,Y)$ which are open in the usual sense, it holds: $rDom(X,Y) \cup o\mathcal{R}(X,Y) \subset wo\mathcal{R}(X,Y)$. However, there are real algebraic varieties X and Y for which the converse inclusion is false. For example, if X is the line R and Y is the Whitney umbrella $\{(x,y,z) \in R^3 \,|\, y^2 - zx^2 = 0\}$, then the regular map $f : X \longrightarrow Y$ which sends x into $(0,0,x)$ is weakly open, but, it is neither dominating nor open.

The following is one of the deeper result of this paper.

Theorem 2.14 *Let X and Y be real algebraic varities. If $p_t(Y) \geq 2$, then $wo\mathcal{R}(X, Y)$ is finite. In particular, there are only finitely many open regular maps between X and Y.*

APPLICATIONS AND CONJECTURES. Let us apply the previous results.

Theorem 2.15 *Let Y be an affine real algebraic variety. Then, there is a weak change $\varphi : \widetilde{Y} \longrightarrow Y$ of the algebraic structure of Y with the following properties:*

a) *For each real algebraic variety X, the sets $RDom(X, \widetilde{Y})$ and $wo\mathcal{R}(X, \widetilde{Y})$ are finite.*

b) *The group $RAut(\widetilde{Y})$ (and hence the group $rAut(\widetilde{Y})$) is finite.*

Remark 2.16 *In a future paper, we will prove that, if X is irreducible, in point a) of Theorem 2.15, it is possible to replace $wo\mathcal{R}(X, \widetilde{Y})$ with $\mathcal{R}^*(X, \widetilde{Y})$.*

We have a conjecture.

Conjecture 2.17 (1^{st} version) *For each affine irreducible real algebraic variety Y, there is a weak change $\varphi : \widetilde{Y} \longrightarrow Y$ of the algebraic structure of Y such that the unique birational automorphism of $\widetilde{Y}$ is the identity, i.e., the rational map represented by the identity map on $\widetilde{Y}$.*

(2^{nd} version) *For each affine real algebraic variety Y, there are an affine real algebraic variety $\widetilde{Y}$ and a semi–algebraic homeomorphism $\psi : \widetilde{Y} \longrightarrow Y$ such that the unique birational automorphism of $\widetilde{Y}$ is the identity and ψ restricts to a Nash isomorphism between* $\mathrm{Nonsing}(\widetilde{X})$ *and* $\mathrm{Nonsing}(X)$.

2.3 Topology of regular morphism space

Throughout this subsection, we consider only the case $R = \mathbb{R}$.

TOPOLOGICAL PRELIMINARIES. We recall some classical notions and give some new definitions concerning function spaces and topological dimension.

Let X and Y be fixed topological spaces (for basic definitions in General Topology, see chapter 1 of [40]). Indicate by $C^0(X, Y)$ the set of all continuous maps between X and Y and fix a subset F of $C^0(X, Y)$. For each compact subset K of X and for each open subset U of Y, define $F(K, U)$ as the set of all maps in F which send K into U. The family of finite intersections of sets of the form $F(K, U)$ is a base for the compact–open (or weak) topology for F.

Example 2.18 *The notions of real analytic set and of real analytic maps between them can be defined similarly to the real Nash case (see page 15). Suppose X and Y are affine real algebraic varieties. These varieties have natural structures of real Nash set and of real analytic set. Therefore, it makes sense to speak of real Nash and real analytic maps between X and Y. Indicate by $\mathcal{N}(X, Y)$ (resp. $C^\omega(X, Y)$) the set of all real Nash (resp. real analytic) maps between X and Y. Since $\mathcal{N}(X, Y) \subset C^\omega(X, Y) \subset C^0(X, Y)$, we can equip $\mathcal{N}(X, Y)$ and $C^\omega(X, Y)$ with the compact–open topology.*

Let τ and ξ be two topologies for F and let $f \in F$. We say that τ *and* ξ *coincides at* f if and only if the neighborhood system of f in F with respect to τ is equal to the neighborhood system of f in F with respect to ξ.

Definition 2.19 *We say that a topology τ for F is locally compact–open if it satisfies the following two conditions:*

a) τ is finer than the compact–open topology for F.

b) For each $x \in X$ and for each $f \in F$, there is a compact neighborhood K of x in X (depending on x and f) such that, defining $F(f, K) := \{g \in F \mid g = f \text{ on } X \setminus K\}$, the compact–open topology for $F(f, K)$ and the relativization of τ to $F(f, K)$ coincide at f.

Remark that if K satisfies previous condition b), then every compact neighborhood K' of x contained in K satisfies the same condition.

When Y is metrizable, the locally compact–open topologies for F can be easily described. Let d be a metric for Y. For each $f \in F$, for each non–void subset K of X and for each $\varepsilon \in \mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$, define $F_\varepsilon^{(d)}(f, K) := \{g \in F(f, K) \mid \sup_{x \in K} d(g(x), f(x)) < \varepsilon\}$. A topology τ for F finer than the compact–open topology is locally compact–open if and only if, for each $x \in X$ and for each $f \in F$, there is a compact neighborhood K of x in X such that the set $\{F_{1/n}^{(d)}(f, K)\}_{n \in \mathbb{N}^*}$ is a base of the neighborhood system of f in $F(f, K)$ equipped with the relative topology induced by τ.

Example 2.20 *The compact–open topology, the Whitney (or strong) topology and, when Y is metrizable by a metric d, the topology of uniform convergence with respect to d for F are always locally compact–open.*

Recall that, given a subset A of the topological product space Y^X, the topology for A induced by Y^X is called pointwise topology (or topology of the pointwise convergence). Remark that F is a subset of Y^X.

Definition 2.21 *Let F_{const} be the set of all maps in F which are locally constant, i.e., constant on each connected component of X. A topology τ for F is nice if the relativization of τ to F_{const} coincides with the pointwise topology for F_{const}.*

Example 2.22 *Suppose X and Y are real algebraic varieties and let d be a metric for Y. Since X has a finite family of connected components, the compact–open topology and the topology of uniform convergence with respect to d for F are nice.*

Let $k \in \mathbb{N}^* \cup \{\infty\}$ and let M and N be C^k manifolds, i.e., second countable, Hausdorff, locally Euclidean spaces equipped with differentiable structures of class C^k. Indicate by $C^k(M,N)$ the set of all C^k maps between M and N, by $J^k(M,N)$ the topological space of k–jets between M and N and by $j^k : C^k(M,N) \longrightarrow C^0(M, J^k(M,N))$ the map which sends f into its k–jets (see section 4, chapter 2 of [34]). Fix a subset G of $C^k(M,N)$ and denote by $i : G \hookrightarrow C^k(M,N)$ the inclusion map. We define the C^k *compact–open* (or *weak*) *topology for* G as the topology induced by the composition $j^k \circ i$ from the compact–open topology of $C^0(M, J^k(M,N))$, i.e., the smallest topology for G such that, equipping $C^0(M, J^k(M,N))$ with the compact–open topology, $j^k \circ i$ is continuous.

Remark 2.23 *Suppose M and N are nonsingular real algebraic varieties. These varieties have natural differentiable structures of class C^k for each $k \in \mathbb{N}^* \cup \{\infty\}$. In this way, it makes sense to speak of C^k maps between M and N. Remark that $\mathcal{N}(M,N) \subset C^\omega(M,N) \subset C^\infty(M,N)$ so we can equip $\mathcal{N}(M,N)$ and $C^\omega(M,N)$ with the C^∞ compact–open topology.*

Definition 2.24 *We say that a topology for G is locally C^k compact–open if it is induced by $j^k \circ i$ from a locally compact–open topology of $C^0(M, J^k(M,N))$.*

Evidently, the C^k compact–open (or weak) topology and the Whitney C^k (or strong) topology for G are locally C^k compact–open. Moreover, the former is nice. A class of locally C^k compact–open topologies for G can be obtained following the construction of "C^r topologies" given in section II.1 of [56].

Let us introduce the notions of Brouwer dimension and of quasi–euclidean (resp. euclidean) set and stratification. Fix a non–void topological space S. For each $n \in \mathbb{N}$, let D^n be the closed ball of $\mathbb{R}^n$ centered at the origin with radious 1. Indicate by $\mathfrak{B}(S)$ the subset of all integer $n \in \mathbb{N}$ such that there is an injective continuous map between D^n and S. Remark that $\mathfrak{B}(S)$ always contains 0 because D^0 is a single point. We define the *Brouwer dimension* $\dim_{\mathfrak{B}} S$ *of* S by $\dim_{\mathfrak{B}} S := \sup \mathfrak{B}(S)$.

Remark 2.25 *The Brouwer dimension is a topological invariant and extends the notion of vector dimension of* $\mathbb{R}^n$. *In fact, by Brouwer's theorem on invariance of domain, it follows immediately that* $\dim_{\mathfrak{B}} \mathbb{R}^n = n$. *Really, the Brouwer dimension is closely related with the classical covering dimension used in Dimension Theory (see Definition I.4 of [51]). Denote by* $\dim_{\mathfrak{C}} S$ *the covering dimension of* S. *It is possible to prove that, if* S *is metrizable,* $\dim_{\mathfrak{B}} S \leq \dim_{\mathfrak{C}} S$ *and, if* S *is a locally compact polyhedron with simplicial dimension* r, $\dim_{\mathfrak{B}} S = \dim_{\mathfrak{C}} S = r$ *(see Theorems II.1, II.3, II.7, III.6, III.7 and IV.5 of [51]).*

We say that S is a *quasi–euclidean* (resp. *euclidean*) *set* if there is an injective continuous map (resp. a homeomorphism into its image) between S and some $\mathbb{R}^n$. Let I be a finite set ordered by a reflexive and transitive relation $\preceq$ such that $i \preceq j$ and $j \preceq i$ imply $i = j$. A partition $\mathcal{S} = \{S_i\}_{i \in I}$ of S is called a *quasi–euclidean* (resp. *euclidean*) *stratification of* S if each subspace S_i of S (called stratum of $\mathcal{S}$) is a quasi–euclidean (resp. euclidean) set and, for each $i \in I$, $\bigsqcup_{\{j \in I \,|\, i \preceq j\}} S_j$ is closed in S. Furthermore, such a stratification is called *singular* if I is totally ordered by $\preceq$ and, for each $i, j \in I$ with $i \preceq j$ and $i \neq j$, $\dim_{\mathfrak{B}} S_i > \dim_{\mathfrak{B}} S_j$.

Suppose S has a quasi–euclidean stratification $\{S_i\}_{i \in I}$. Then, the following assertions are verified.

a) Each stratum S_i of S is locally closed in S.

b) $\dim_{\mathfrak{B}} S$ is finite and it holds: $\dim_{\mathfrak{B}} S = \max_{i \in I} \dim_{\mathfrak{B}} S_i$.

For each $i \in I$, the stratum S_i is the difference between the following two closed subsets of S: $\bigsqcup_{\{j \in I \,|\, i \preceq j\}} S_j$ and $\bigcup_{\{j \in I \,|\, i \preceq j, i \neq j\}} \bigsqcup_{\{k \in I \,|\, j \preceq k\}} S_k$. In this way, assertion a) is proved. Let us prove assertion b). Since each S_i has finite Brouwer dimension and the inequality $\dim_{\mathfrak{B}} S \geq \max_{i \in I} \dim_{\mathfrak{B}} S_i$ is evident, it suffices to prove that $\mathfrak{B}(S) \subset \bigcup_{i \in I} \mathfrak{B}(S_i)$. Let $n \in \mathfrak{B}(S)$, let $\varphi : D^n \longrightarrow S$ be an injective continuous map, let $I_\varphi := \{i \in I \,|\, S_i \cap \varphi(D^n) \neq \emptyset\}$ and let m be a minimal element of I_φ. Define $S_\varphi := \bigsqcup_{i \in I_\varphi} S_i$

and $S_\varphi^* := \bigcup_{i\in I_\varphi\setminus\{m\}} \bigsqcup_{\{j\in I\,|\,i\preceq j\}} S_j$. Remark that $\varphi(D^n)\subset S_\varphi$, S_φ^* is closed in S and $S_\varphi\setminus S_\varphi^* = S_m$. In particular, it holds: $\varphi^{-1}(S_m) = D^n\setminus\varphi^{-1}(S_\varphi^*)$. It follows that $\varphi^{-1}(S_m)$ is a non–void open subset of D^n so $n\in\mathfrak{B}(S_m)$.

The theorems: general case. Let X and Y be real algebraic varieties. Denote by $\mathrm{Isol}(X)$ (resp. $\mathrm{Isol}(Y)$) the set of all isolated points of X (resp. Y) and define:

a as the number of Zariski connected components of X,
b as the number of irreducible components of X,
w as the cardinality of X_0,
δ as the cardinality of $\mathrm{Isol}(X)$,
η as the number of connected components of $X\setminus\mathrm{Isol}(X)$,
s as the dimension of Y

and

σ as the cardinality of $\mathrm{Isol}(Y)$.

Evidently, it holds: $b\geq a>w$, $\delta\geq w$ and $\eta\geq a-w$. A map in $C^0(X,Y)$ is said to be *trivial* if it sends $X\setminus\mathrm{Isol}(X)$ into $\mathrm{Isol}(Y)$. We denote by $\mathrm{Triv}(X,Y)$ the set of all trivial maps between X and Y. Let L be a subset of $C^0(X,Y)$. Equip $C^0(X,Y)$ with some topological structure ν. We say that L *is nowhere dense up to trivial maps in* $C^0(X,Y)$ if $L\setminus\mathrm{Triv}(X,Y)$ is nowhere dense in $C^0(X,Y)$, i.e., the interior of the closure of $L\setminus\mathrm{Triv}(X,Y)$ in $C^0(X,Y)$ is void. Suppose ν is locally compact–open. It is easy to see that $\mathrm{Triv}(X,Y)$ is open and closed in $C^0(X,Y)$ and contains the isolated points of $C^0(X,Y)$. Moreover, if ν is nice, $\mathrm{Triv}(X,Y)$ is homeomorphic to the disjoint topological sum of σ^η copies of Y^δ where $Y^\emptyset$ is considered equal to a point.

By the symbol $\mathrm{Map}(X,Y)$, we indicate one of the topological spaces described below:

i) $C^0(X,Y)$ equipped with a locally compact–open topology,

ii) $\mathcal{N}(X,Y)$, or $C^\omega(X,Y)$, equipped with the compact–open topology, where X and Y are assumed to be affine.

We denote by τ the topology of $\mathrm{Map}(X,Y)$. If, in case i), τ is the compact–open topology, then we say that $\mathrm{Map}(X,Y)$ is equipped with the compact–open topology. Unless otherwise indicated, $\mathcal{R}(X,Y)$ is considered as a subspace of $\mathrm{Map}(X,Y)$.

Before stating our main theorem concerning the topology of $\mathcal{R}(X,Y)$, we present a lemma which describes the elementary structure of the space of trivial regular maps and a dimensional property of $\mathrm{Map}(X,Y)$.

Lemma 2.26 a) *Indicate by* $\mathrm{Triv}\mathcal{R}(X,Y)$ *the intersection* $\mathcal{R}(X,Y)\cap\mathrm{Triv}(X,Y)$ *equipped with the relative topology induced by* $\mathrm{Map}(X,Y)$. *Then,* $\mathrm{Triv}\mathcal{R}(X,Y)$ *coincides with the set of all Zariski locally constant maps between* X *and* Y *which send* X_* *into* $\mathrm{Isol}(Y)$ *and, if* τ *is nice, it is homeomorphic to the topological disjoint sum of* σ^{a-w} *copies of* Y^w.

b) *It holds:* $\dim_{\mathfrak{B}} \mathrm{Map}(X,Y) = +\infty$ *and, when* $\mathrm{Map}(X,Y)$ *is metrizable,* $\dim_{\mathfrak{C}} \mathrm{Map}(X,Y) = +\infty$ *also.*

The next theorem gives a description of the topology of $\mathcal{R}(X,Y)$.

Theorem 2.27 *The following assertions are verified.*

a) *If* $p_c(X) < p_t(Y)$ (*or, better,* $p_c(X) < e(Y)$), *then* $\mathcal{R}(X,Y)$ *coincides with the set of all Zariski locally constant maps between* X *and* Y, *has a quasi–euclidean stratification* $\mathcal{S}$ *and* $\dim_{\mathfrak{B}} \mathcal{R}(X,Y) \leq as$. *Moreover, if* τ *is nice,* $\mathcal{R}(X,Y)$ *is homeomorphic to* Y^a *and hence the stratification* $\mathcal{S}$ *can be choosen singular and euclidean and* $\dim_{\mathfrak{B}} \mathcal{R}(X,Y) = as$.

b) *If* $p_t(Y) \geq 1$, *then* $\mathcal{R}(X,Y)$ *is nowhere dense up to trivial maps in* $\mathrm{Map}(X,Y)$.

c) *Equip* $\mathcal{R}(X,Y)$ *with a topology* ξ *finer than the pointwise topology* (*as, for example, the one induced by* $\mathrm{Map}(X,Y)$). *If* $p_c(X) \geq p_t(Y) \geq 2$, *then* $\mathcal{R}(X,Y)$ *has a quasi–euclidean stratification* $\mathcal{S}$ *and* $\dim_{\mathfrak{B}} \mathcal{R}(X,Y) \leq bs$. *Moreover, if* ξ *is nice,* $\dim_{\mathfrak{B}} \mathcal{R}(X,Y) \geq as$ *and, if* ξ *is nice and* X *is irreducible,* $\mathcal{S}$ *can be choosen singular.*

The previous theorem asserts that, when $p_c(X) < p_t(Y)$, $\mathcal{R}(X,Y)$ is as small as possible, when $p_t(Y) \geq 1$, it is topologically small in $\mathrm{Map}(X,Y)$ and, when $p_t(Y) \geq 2$, it has a quasi–euclidean stratification and hence it is dimensionally small in $\mathrm{Map}(X,Y)$ also.

In the following result, we describe the topology of $\mathcal{R}(X,Y)$ under a hypothesis stronger than $p_t(Y) \geq 2$. The reader recalls the definition of biregular global toric genus given at page 13.

Theorem 2.28 *Suppose* Y *affine and equip* $\mathcal{R}(X,Y)$ *with the compact–open topology. If* $bgp_t(Y) \geq 2$, *then* $\mathcal{R}(X,Y)$ *is an euclidean set and it holds:* $as \leq \dim_{\mathfrak{B}} \mathcal{R}(X,Y) = \dim_{\mathfrak{C}} \mathcal{R}(X,Y) \leq bs$. *Moreover, if* X *is irreducible and* Y *is compact,* $\mathcal{R}(X,Y)$ *is compact also.*

We give an application of the previous results.

Corollary 2.29 *Let Y be an affine real algebraic variety. Then, there is a weak change $\varphi : \widetilde{Y} \longrightarrow Y$ of the algebraic structure of Y with the following property: For each real algebraic variety X, equipping* $\mathrm{Map}(X, \widetilde{Y})$ *with the compact–open topology, $\mathcal{R}(X, \widetilde{Y})$ is nowhere dense up to trivial maps in* $\mathrm{Map}(X, \widetilde{Y})$*, is an euclidean set and* $\dim_{\mathfrak{B}} \mathcal{R}(X, \widetilde{Y})$ *is finite. Moreover, if X is irreducible and Y is compact, $\mathcal{R}(X, \widetilde{Y})$ is compact also.*

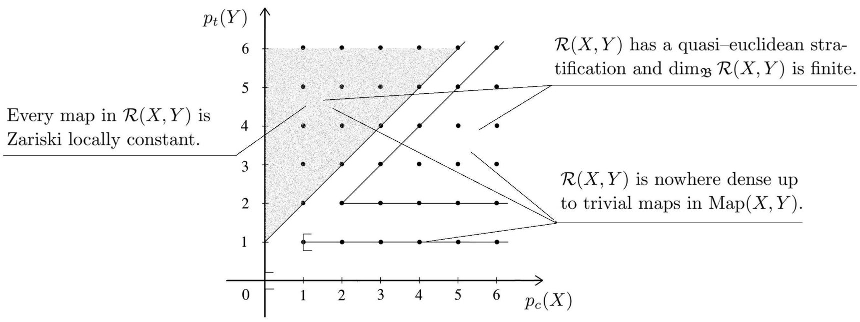

Figure 1: Topology of regular morphism space: general case

The theorems: nonsingular case. Let M and N be real algebraic manifolds, i.e., nonsingular real algebraic varieties. Define:

l as the number of irreducible components of M

and

n as the dimension of N.

By the symbol $\mathrm{Smo}(M, N)$, we indicate one of the topological spaces described below:

i′) $C^k(M, N)$ equipped with a locally C^k compact–open topology, where $k \in \mathbb{N}^* \cup \{\infty\}$,

ii′) $\mathcal{N}(M, N)$, or $C^\omega(M, N)$, equipped with the C^∞ compact–open topology, where M and N are assumed to be affine.

We indicate by λ the topology of $\mathrm{Smo}(M, N)$. If, in case i′), λ is the C^k compact–open topology, then we say that $\mathrm{Smo}(M, N)$ is equipped with the C^k compact–open topology. Unless explicitly stated otherwise, $\mathcal{R}(M, N)$ is considered as a subspace of $\mathrm{Smo}(M, N)$. Denote by $\Omega(M, N)$ the complement of the closure of $\mathcal{R}(M, N)$ in $\mathrm{Smo}(M, N)$ and by $\Sigma(M, N)$ the set of all maps $f \in \mathrm{Smo}(M, N)$ such that, for some $q \in N$, $f^{-1}(q)$ contains a

1–dimensional semi–algebraic subset of M (for basic definitions and results concerning semi–algebraic sets, see chapter 2 and section 3.2 of [4]). Remark that $\dim_{\mathfrak{B}} \mathrm{Smo}(M,N) = +\infty$ and, when $\mathrm{Smo}(M,N)$ is metrizable, $\dim_{\mathfrak{C}} \mathrm{Smo}(M,N) = +\infty$ also. Moreover, $\Sigma(M,N)$ is always non–void because it contains all Zariski locally constant maps between M and N.

The following result is the nonsingular version of Theorem 2.27. The unique substantial newness is contained in point b).

Theorem 2.30 *The following assertions are verified.*

a) *If $p_c(M) < p_t(N)$ (or, better, $p_c(M) < e(N)$), then $\mathcal{R}(M,N)$ coincides with the set of all Zariski locally constant maps between M and N, is nowhere dense in* $\mathrm{Smo}(M,N)$*, has a quasi–euclidean stratification $\mathcal{S}$ and $\dim_{\mathfrak{B}} \mathcal{R}(M,N) \leq nl$. Moreover, if λ is nice, $\mathcal{R}(M,N)$ is homeomorphic to Y^l and hence the stratification $\mathcal{S}$ can be choosen singular and euclidean and $\dim_{\mathfrak{B}} \mathcal{R}(M,N) = nl$.*

b) *Suppose $p_t(N) = 1$. Then, the closure of $\Omega(M,N)$ in* $\mathrm{Smo}(M,N)$ *contains $\Sigma(M,N)$. In particular, $\Omega(M,N)$ is non–void, i.e, $\mathcal{R}(M,N)$ is not dense in* $\mathrm{Smo}(M,N)$.

c) *Equip $\mathcal{R}(M,N)$ with a topology ξ finer than the pointwise topology (as, for example, the one induced by* $\mathrm{Smo}(M,N)$*). If $p_c(M) \geq p_t(N) \geq 2$, then $\mathcal{R}(M,N)$ is nowhere dense in* $\mathrm{Smo}(M,N)$*, has a quasi–euclidean stratification $\mathcal{S}$ and $\dim_{\mathfrak{B}} \mathcal{R}(M,N) \leq nl$. Moreover, if ξ is nice, $\dim_{\mathfrak{B}} \mathcal{R}(M,N) = nl$ and, if ξ is nice and M is irreducible, $\mathcal{S}$ can be choosen singular.*

c$'$) *Suppose N affine and equip $\mathcal{R}(M,N)$ with the C^k compact–open topology. If $bgp_t(N) \geq 2$, then $\mathcal{R}(M,N)$ is an euclidean set and $\dim_{\mathfrak{B}} \mathcal{R}(M,N) = \dim_{\mathfrak{C}} \mathcal{R}(M,N) = nl$. Moreover, if N is compact, $\mathcal{R}(M,N)$ is compact also.*

The next result is the nonsingular version of Corollary 2.29.

Corollary 2.31 *Let N be an affine real algebraic manifold. Then, there is a weak change $\varphi : \widetilde{N} \longrightarrow N$ of the algebraic structure of N with the following property: For each real algebraic manifold M, equipping* $\mathrm{Smo}(M,\widetilde{N})$ *with the C^k compact–open topology, $\mathcal{R}(M,\widetilde{N})$ is nowhere dense in* $\mathrm{Smo}(M,\widetilde{N})$*, is an euclidean set and $\dim_{\mathfrak{B}} \mathcal{R}(M,\widetilde{N})$ is finite. Moreover, if N is compact, $\mathcal{R}(M,\widetilde{N})$ is compact also.*

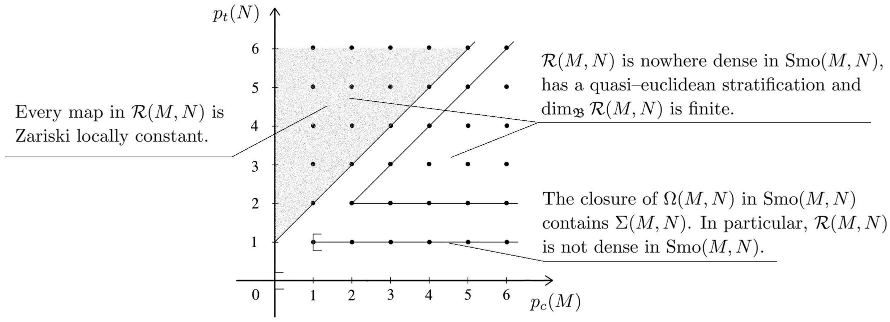

Figure 2: Topology of regular morphism space: nonsingular case

3 Proof of finiteness theorems

Lemma 3.1 *Let X be an irreducible real algebraic variety and let D be a real algebraic curve with $g(D) \geq 2$. Then, $\mathcal{R}atio^*(X,D)$ and $\mathcal{R}^*(X,D)$ are finite and it holds:*

$$\sharp\mathcal{R}atio^*(X,D) \leq \mathcal{M}(P_c(X), g(D))$$

and

$$\sharp\mathcal{R}^*(X,D) \leq \mathcal{M}(p_c(X), g(D)).$$

Proof. *Step I.* Suppose X is a real algebraic curve. Bearing in mind Definition 2.8, the lemma follows immediately from the uniqueness of the complexification of nonconstant rational (resp. regular) maps between real algebraic curves, the Hurwitz–de Franchis–Martens theorem (see page 22) and Lefschetz's principle.

Step II. Let us complete the proof. Let $h := \mathcal{M}(P_c(X), g(D))$. We must prove that $\sharp\mathcal{R}atio^*(X,D)$ is less than or equal to h. Suppose on the contrary that $\mathcal{R}atio^*(X,D)$ contains $h+1$ distinct elements $\alpha_0, \alpha_1, \ldots, \alpha_h$. Then, there are a non–void Zariski open subset Z of X and maps $f_0, f_1, \ldots, f_h$ in $\mathcal{R}^*(Z,D)$ which represent $\alpha_0, \alpha_1, \ldots, \alpha_h$ respectively. By the last part of Lemma 1.7, there is a point p of Z such that $P_c(X) = p_{c,X}(p) = p_{c,Z}(p)$. Define $\Delta := \bigcup_{i=0}^h f_i^{-1}(f_i(p)) \cup \bigcup_{i \neq j} \{x \in Z \,|\, f_i(x) = f_j(x)\}$. Since each map f_i is nonconstant and, for each $i,j \in \{0,1,\ldots,h\}$ with $i \neq j$, f_i and f_j are distinct maps, we have that Δ is a proper Zariski closed subset of Z. In this way, bearing in mind the definition of curve genus at a point, it is possible to find a real algebraic curve E of Z containing p such that

$g(E) = p_{c,Z}(p) = P_c(X)$ and $E \not\subset \Delta$. By definition of Δ, it follows that the regular maps $f_0|_E, f_1|_E, \ldots, f_h|_E$ are $h+1$ distinct elements of $\mathcal{R}^*(E, D)$. This is impossible because, thanks to *Step I*, $\sharp\mathcal{R}^*(E, D) \leq \mathcal{M}(g(E), g(D)) = h$. The proof of the inequality $\sharp\mathcal{R}^*(X, D) \leq \mathcal{M}(p_c(X), g(D))$ is similar. □

Proof of Lemma 1.31. Step I. Suppose X is a real algebraic curve. Let X_C be a projective nonsingular complexification of X. The map $\varphi : \mathcal{G}_2(X) \longrightarrow \mathfrak{Im}_2(X_C)$, which sends the F–equivalence class of an element α of $G_2(X)$ into the F_C–equivalence class of the complexification of α in $\mathfrak{Im}_2(X_C)$, is well–defined and injective. In this way, bearing in mind Definition 1.28, the de Franchis–Harris–Howard–Sommese theorem (see page 17) ensures that $\sharp\mathcal{G}_2(X) \leq \sharp\mathfrak{Im}_2(X_C) \leq \mathcal{F}(g(X)) = \mathcal{F}(P_c(X))$.

Step II. Let us complete the proof. If $P_c(X) < 2$, then, by Theorem 2.2, b), we know that $G_2(X) = \emptyset$ so the lemma is proved. Suppose $P_c(X) \geq 2$ and define $K := \mathcal{F}(P_c(X))$. We must show that $\sharp\mathcal{G}_2(X) \leq K$. Suppose on the contrary that $\mathcal{G}_2(X)$ contains $K+1$ dinstinct elements $\alpha_0 \in \mathcal{R}atio^*(X, D_0), \ldots, \alpha_K \in \mathcal{R}atio^*(X, D_K)$ for some real algebraic curves $D_0, \ldots, D_K$. Let A be the set of all pairs $(i,j) \in \{0, 1, \ldots, K\}^2$ with $i < j$ and D_i and D_j are birationally isomorphic. Fix $(i,j) \in A$ and indicate by B_{ij} the set of all birational isomorphisms between D_i and D_j. By Lemma 3.1, B_{ij} is finite. Fix a non–void Zariski open subset Ω_{ij} of D_i and choose regular maps $g_{ij1}, \ldots, g_{i,j,N_{ij}}$ between Ω_{ij} and D_j which represents the elements of B_{ij}. For each $i \in \{0, 1, \ldots, K\}$, define $A_i := \{j \,|\, (i,j) \in A\}$ and $\Omega_i := \bigcap_{j \in A_i} \Omega_{ij}$ (where $\Omega_i := D_i$ if $A_i = \emptyset$). Since each α_i is nonconstant, there exist a non–void Zariski open subset Z of X and regular maps $f_0 : Z \longrightarrow D_0, \ldots, f_K : Z \longrightarrow D_K$ which represent $\alpha_0, \ldots, \alpha_K$ respectively and satisfy the following condition: $f_i(Z) \subset \Omega_i$ for each $i \in \{0, 1, \ldots, K\}$. By the last part of Lemma 1.7, there is $p \in Z$ such that $p_c(X, p) = P_c(X)$. Let $(i,j) \in A$. Define $\Delta_{ij} := \bigcup_{h=1}^{N_{ij}} \{x \in Z \,|\, (g_{ijh} \circ f_i)(x) = f_j(x)\}$. Remark that Δ_{ij} is a proper Zariski closed subset of Z, otherwise α_i would be F–equivalent to α_j in $G_2(X)$ and our assumption would be contradicted. Let $\Delta := \bigcup_{(i,j) \in A} \Delta_{ij} \cup \bigcup_{i=0}^{K} f_i^{-1}(f_i(p))$. It follows that Δ is a proper Zariski closed subset of Z. In this way, by the definition of $p_c(X, p)$, there exists a real algebraic curve E of Z containing p such that $g(E) = P_c(X)$ and $E \not\subset \Delta$. For each $i \in \{0, 1, \ldots, K\}$, define $\xi_i : E \longrightarrow D_i$ as the restriction of f_i to E and $\beta_i \in \mathcal{R}atio^*(E, D_i)$ as the rational map represented by ξ_i. Remark that each ξ_i is nonconstant and, for each $(i,j) \in A$ and for each $h \in \{1, \ldots, N_{ij}\}$, the map ξ_j is different from the composition map $g_{ijh} \circ \xi_i$. In particular, for each $i, j \in \{0, 1, \ldots, K\}$ with $i \neq j$, β_i is not F–equivalent to β_j in $G_2(E)$ so $\sharp\mathcal{G}_2(E) \geq K+1$ which is impossible by *Step I*. □

Proof of Theorem 2.11. We will prove only part a). The proof of part b) is similar. Let $n := \mathrm{tdim}(Y)$. By definition of toric dimension, there are a non–void affine Zariski open subset V of Y, real algebraic curves $D_1, \ldots, D_n$ with $\min_{i \in \{1,\ldots,n\}} g(D_i) \geq p_t(Y) \geq 2$ and an injective regular map $\varphi : V \longrightarrow T := \prod_{i=1}^n D_i$. By Lemma 3.1, we have that, for each $i \in \{1, \ldots, n\}$, $\mathcal{R}atio^*(X, D_i)$ is finite and it holds: $\sharp\mathcal{R}atio^*(X, D_i) \leq \mathcal{M}(P_c(X), g(D_i)) \leq \mathcal{M}_*(P_c(X), p_t(Y))$. Identify $\mathcal{R}atio(X,T)$ with $\prod_{i=1}^n \mathcal{R}atio(X, D_i)$ in the natural way. For each $f \in rDom(X,Y)$, $V_f := f^{-1}(V)$ is a non–void Zariski open subset of X so it is possible to define the map $\varphi^* : rDom(X,Y) \longrightarrow \mathcal{R}atio(X,T)$ which sends f into the rational map represented by $(V_f, \varphi \circ f|_{V_f})$. Remark that φ^* is injective. Let us prove that $\varphi^*(rDom(X,Y))$ is contained in $\prod_{i=1}^n \mathcal{R}atio^*(X, D_i)$. Suppose this is not true. For each $i \in \{1, \ldots, n\}$, let $\pi_i : T \longrightarrow D_i$ be the natural projection. By the assumption just done, there exists $f \in rDom(X,Y)$, $i \in \{1, \ldots, n\}$ and $b_i \in D_i$ such that $(\pi_i \circ \varphi \circ f)(V_f) = \{b_i\}$. Without loss of generality, we may suppose $i = 1$. It follows that $\varphi(f(V_f)) \subset \{b_1\} \times \prod_{i=2}^n D_i$ so, being f dominating, $\varphi(V) \subset \{b_1\} \times \prod_{i=2}^n D_i$ also. This is impossible by definition of n. In this way, φ^* injects $rDom(X,Y)$ into $\prod_{i=1}^n \mathcal{R}atio^*(X, D_i)$ and hence $\sharp rDom(X,Y) \leq \prod_{i=1}^n \sharp\mathcal{R}atio^*(X, D_i) \leq \mathcal{M}_*(P_c(X), p_t(Y))^n$. By Lemma 1.20 and Lemma 1.30, it follows that $n \leq \mathrm{Tdim}(Y) \leq \mathcal{F}(P_c(Y))$ so $\sharp rDom(X,Y) \leq \mathcal{M}_*(P_c(X), p_t(Y))^{\mathcal{F}(P_c(Y))}$ also. □

Proof of Theorem 2.11′. We will prove only part a). The proof of part b) is similar.

Step I. Suppose that X and Y are Zariski connected. Let $X_1, \ldots, X_u$ be the irreducible components of X and let $Y_1, \ldots, Y_U$ be the irreducible components of Y. Denote by F the set of all functions between $\{1, \ldots, u\}$ and $\{1, \ldots, U\}$. For each $\gamma \in F$, let D_γ be the set of all maps $f \in rDom(X,Y)$ such that $\mathrm{Zcl}_Y(f(X_i)) = Y_{\gamma(i)}$ for each $i \in \{1, \ldots, u\}$ and let $\psi_\gamma : D_\gamma \longrightarrow \prod_{i=1}^u rDom(X_i, Y_{\gamma(i)})$ be the map which sends f into $(f_1, \ldots, f_u)$ where f_i is the restriction of f between X_i and $Y_{\gamma(i)}$. Remark that $rDom(X,Y) = \bigsqcup_{\gamma \in F} D_\gamma$ and each ψ_γ is injective. By Theorem 2.11, we have that

$$\begin{aligned} \sharp rDom(X,Y) &\leq \textstyle\sum_{\gamma \in F} \prod_{i=1}^u \mathcal{M}_*(P_c(X_i), p_t(Y_{\gamma(i)}))^{\mathrm{tdim}(Y_{\gamma(i)})} \leq \\ &\leq U^u \cdot \mathcal{M}_*(P_c(X), p_t(Y))^{u \cdot \mathrm{tdim}(Y)}. \end{aligned}$$

Step II. Let us complete the proof. Let $X_0 = \{p_1, \ldots, p_w\}$, let $Y_0 = \{q_1, \ldots, q_W\}$, let $X^{(1)}, \ldots, X^{(v)}$ be the Zariski connected components of X_* and let $Y^{(1)}, \ldots, Y^{(V)}$ be the Zariski connected components of Y_*. For each $i \in \{1, \ldots, v\}$, let u_i be the number of irreducible components of $X^{(i)}$ and, for each $j \in \{1, \ldots, V\}$, let U_j be the number of irreducible components of $Y^{(j)}$. Remark that $\sum_{i=1}^v u_i = u$ and $\max_{j \in \{1,\ldots,V\}} U_j = Z$. For each $\chi \subset \{1, \ldots, v\}$, let F_χ be the set of all functions between $\{1,\ldots,v\}\setminus\chi$ and $\{1,\ldots,V\}$ (where $F_{\{1,\ldots,v\}}$ is considered equal to the empty set). Let $f \in rDom(X,Y)$. By definition of dominating regular map, there exist (and are unique) a function $\alpha : \{1,\ldots,w\} \longrightarrow \{1,\ldots,W\}$, a subset χ of $\{1,\ldots,v\}$, a function $\beta : \chi \longrightarrow \{1,\ldots,W\}$ and a function $\gamma \in F_\chi$ such that: $f(p_i) = q_{\alpha(i)}$ for each $i \in \{1,\ldots,w\}$, $f(X^{(i)}) = \{q_{\beta(i)}\}$ for each $i \in \chi$ and $f(X^{(i)}) \subset Y^{(\gamma(i))}$ for each $i \in \{1,\ldots,v\} \setminus \chi$. In this way, it is possible to define an injective map Φ between $rDom(X,Y)$ and the following set

$$\bigsqcup_{\chi\subset\{1,\ldots,v\}} \bigsqcup_{\gamma\in F_\chi} \Big(\{1,\ldots,W\}^{(\{1,\ldots,w\}\sqcup\chi)} \times \prod_{i\in\{1,\ldots,v\}\setminus\chi} rDom(X^{(i)}, Y^{(\gamma(i))})\Big)$$

by $\Phi(f) := (\alpha\sqcup\beta, (f_i)_{i\in\{1,\ldots,v\}\setminus\chi})$ where $\alpha\sqcup\beta : \{1,\ldots,w\}\sqcup\chi \longrightarrow \{1,\ldots,W\}$ is the function which sends $i \in \{1,\ldots,w\}$ into $\alpha(i)$ and $j \in \chi$ into $\beta(j)$ and f_i is the restriction of f between $X^{(i)}$ and $Y^{(\gamma(i))}$. Define $H := Z \cdot \mathcal{M}_*(P_c(X), p_t(Y))^{\mathrm{tdim}(Y)}$ and, for each $i \in \{1,\ldots,v\} \setminus \chi$, define

$$H_i := \mathcal{M}_*(P_c(X^{(i)}), p_t(Y^{(\gamma(i))}))^{u_i\cdot \mathrm{tdim}(Y^{(\gamma(i))})}.$$

Remark that $p_t(Y^{(j)}) \geq p_t(Y) \geq 2$ for each $j \in \{1,\ldots,V\}$. By *Step I*, we have that

$$\begin{aligned}
\sharp rDom(X,Y) &\leq \textstyle\sum_{\chi\subset\{1,\ldots,v\}} \sum_{\gamma\in F_\chi} W^w \cdot W^{\sharp\chi} \cdot \prod_{i\in\{1,\ldots,v\}\setminus\chi} (U_{\gamma(i)})^{u_i} \cdot H_i \leq \\
&\leq W^w \cdot \textstyle\sum_{\chi\subset\{1,\ldots,v\}} W^{\sharp\chi} \sum_{\gamma\in F_\chi} H^{(\sum_{i\in\{1,\ldots,v\}\setminus\chi} u_i)} = \\
&= W^w \cdot \textstyle\sum_{\chi\subset\{1,\ldots,v\}} W^{\sharp\chi} \cdot V^{v-\sharp\chi} \cdot H^{(\sum_{i\in\{1,\ldots,v\}\setminus\chi} u_i)} \leq \\
&\leq W^w \cdot \textstyle\sum_{\chi\subset\{1,\ldots,v\}} W^{(\sum_{i\in\chi} u_i)} \cdot (V\cdot H)^{(\sum_{i\in\{1,\ldots,v\}\setminus\chi} u_i)} \leq \\
&\leq W^w (W + V\cdot H)^u
\end{aligned}$$

as desired. Bearing in mind that $rAut(Y)$ is contained in $rDom(Y,Y)$, the corresponding inequality for $rAut(Y)$ can be obtained by a similar argument. $\square$

Proof of Corollary 2.12. Immediate consequence of Theorem 2.11′, Remark 1.29, Lemma 1.30 and Remark 2.10. $\square$

Let X be a real algebraic variety. Recall that the accumulation points of $\mathrm{Nonsing}(X)$ in X are called central. We denote by $\mathrm{Cent}(X)$ the set of all central points of X, i.e, the closure of $\mathrm{Nonsing}(X)$ in X.

Proof of Theorem 2.14. Step I. Suppose X and Y irreducible. We need a preliminary study of two particular families of subsets of Y. Define $\{Y_i\}_{i=1}^k$ and $\{Y'_i\}_{i=1}^k$ as follows:

$$\begin{cases} Y_1 := Y, \ \ Y'_1 := \mathrm{Cent}(Y), \\ Y_{i+1} := \mathrm{Zcl}_Y\Big(\mathrm{Sing}(Y_i) \setminus \bigcup_{j=1}^i Y'_j\Big), \ \ Y'_{i+1} := \mathrm{Cent}(Y_{i+1}) \setminus \bigcup_{j=1}^i Y'_j, \\ Y_k \neq \emptyset, \ \ Y_{k+1} = \emptyset. \end{cases}$$

Since $Y_{i+1} \subset \mathrm{Sing}(Y_i)$ for each i, the previous definition is consistent.

We will prove that $\{Y'_i\}_{i=1}^k$ is a partition of Y and, for each $i \in \{1, \ldots, k\}$, $Y'_i \neq \emptyset$, $\bigsqcup_{j=1}^i Y'_j$ is closed in Y and $Y_i \setminus \bigsqcup_{j=1}^{i-1} Y'_j = Y \setminus \bigsqcup_{j=1}^{i-1} Y'_j$. Remark that, for each $i \in \{1, \ldots, k\}$, $Y_i \setminus \bigcup_{j=1}^{i-1} Y'_j = (\mathrm{Cent}(Y_i) \cup \mathrm{Sing}(Y_i)) \setminus \bigcup_{j=1}^{i-1} Y'_j = Y'_i \sqcup (Y_{i+1} \setminus \bigcup_{j=1}^i Y'_j)$ so, by induction on $i = k, k-1, \ldots, 1$, it follows that $Y_i \setminus \bigcup_{j=1}^{i-1} Y'_j = \bigcup_{j=i}^k Y'_j$. In particular, $Y = \bigcup_{j=1}^k Y'_j$. On the other hand, by definition, we know that, for each $i, j \in \{1, \ldots, k\}$ with $i \neq j$, $Y'_i \cap Y'_j = \emptyset$ so $\{Y'_i\}_{i=1}^k$ is a partition of Y. It follows that, for each $i \in \{1, \ldots, k\}$, $Y_i \setminus \bigsqcup_{j=1}^{i-1} Y'_j = \bigsqcup_{j=i}^k Y'_j = (\bigsqcup_{j=1}^k Y'_j) \setminus \bigsqcup_{j=1}^{i-1} Y'_j = Y \setminus \bigsqcup_{j=1}^{i-1} Y'_j$. Since $\bigsqcup_{j=1}^i Y'_j = \mathrm{Cent}(Y_i) \cup \bigsqcup_{j=1}^{i-1} Y'_j$ and Y_i (and hence $\mathrm{Cent}(Y_i)$) is closed in Y, proceeding by induction on i, it follows immediately that each union $\bigsqcup_{j=1}^i Y'_j$ is closed in Y. Fix $i \in \{1, \ldots, k\}$. From the definition of Y_i, we have that $\dim(Y_i) = \dim(Y_i \setminus \bigsqcup_{j=1}^{i-1} Y'_j)$. In this way, Y'_i cannot be void. Otherwise, $Y_i \setminus \bigsqcup_{j=1}^{i-1} Y'_j \subset \mathrm{Sing}(Y_i) \setminus \bigsqcup_{j=1}^{i-1} Y'_j$ and hence $\dim(Y_i) \leq \dim(\mathrm{Sing}(Y_i))$ which is impossible.

Let us proceed with the proof of this step. For each $i \in \{1, \ldots, k\}$, let $Y_{i1}, \ldots, Y_{i,m_i}$ be the irreducible components of Y_i of maximum dimension. Fix $f \in wo\mathcal{R}(X, Y)$. Let U be the interior of $f(\mathrm{Nonsing}(X))$ in Y, let $h := \min\{i \in \{1, \ldots, k\} \,|\, U \cap Y'_i \neq \emptyset\}$ and let $V := U \cap (Y_h \setminus \bigsqcup_{j=1}^{h-1} Y'_j)$. Remark that $\emptyset \neq U \cap V'_h \subset V = U \cap (Y \setminus \bigsqcup_{j=1}^{h-1} Y'_j)$ so V is a non–void open subset of Y contained in Y_h and $f^{-1}(V) \cap \mathrm{Nonsing}(X)$ is a non–void open subset of $\mathrm{Nonsing}(X)$. In particular, $f^{-1}(Y_h) = X$ or, equivalently, $f(X) \subset Y_h$. On the other hand, $\mathrm{Zcl}_{Y_h}(f(X))$ is irreducible and its interior in Y_h contains V which intersects Y'_h (and hence $\mathrm{Cent}(Y_h)$) so $\mathrm{Zcl}_{Y_h}(f(X)) = Y_{hj}$ for some (unique) $j \in \{1, \ldots, m_h\}$. In this way, we can define an injective map between $wo\mathcal{R}(X, Y)$ and the set $\bigsqcup_{i=1}^k \bigsqcup_{j=1}^{m_i} rDom(X, Y_{ij})$ which sends f into its restriction between X and Y_{hj}. By Lemma 1.14, a), it follows that, for each $i \in \{1, \ldots, k\}$ and for each $j \in \{1, \ldots, m_i\}$, $p_t(Y_{ij}) \geq p_t(Y) \geq 2$ so Theorem 2.11 ensures that $wo\mathcal{R}(X, Y)$ is finite.

Step II. Let us complete the proof. Let $X_1, \ldots, X_b$ be the irreducible components of X and let $Y_1, \ldots, Y_B$ be the irreducible components of Y. Fix $f \in wo\mathcal{R}(X, Y)$. By definition of weakly open map, there exists a map

$\psi : \{1,\ldots,b\} \longrightarrow \{1,\ldots,B\}$ such that, for each $i \in \{1,\ldots,b\}$, $f(X_i) \subset Y_{\psi(i)}$ and the restriction f_i of f between X_i and $Y_{\psi(i)}$ is weakly open. In this way, there exists an injective map between $wo\mathcal{R}(X,Y)$ and the following set

$$\bigsqcup_{\psi\in\{1,\ldots,B\}^{\{1,\ldots,b\}}} \prod_{i=1}^{b} wo\mathcal{R}(X_i, Y_{\psi(i)}).$$

It remains to prove that, for each $i \in \{1,\ldots,b\}$ and for each $j \in \{1,\ldots,B\}$, $wo\mathcal{R}(X_i,Y_j)$ is finite. If $\dim(Y_j) = 0$, then $wo\mathcal{R}(X_i,Y_j)$ contains only an element. If $\dim(Y_j)$ is positive, then $\dim(X_i)$ is positive also and, being $p_t(Y_j) \geq p_t(Y) \geq 2$, $wo\mathcal{R}(X_i,Y_j)$ is finite by *Step I.* □

4 Proof of topological theorems

Throughout this subsection, we consider only the case $R = \mathbb{R}$.

Lemma 2.26 is elementary and Theorem 2.27, a) follows easily from Theorem 2.2, a). In order to prove Theorem 2.27, b), we need some preliminary notions and results.

Let D and E be a real algebraic curves and let $f \in \mathcal{R}^*(D,E)$. Define $D_f := \mathrm{Nonsing}(D) \setminus f^{-1}(\mathrm{Sing}(E))$. Since $f^{-1}(\mathrm{Sing}(E))$ is finite, D_f is a non–void Zariski open subset of $\mathrm{Nonsing}(D)$. Let $x \in D_f$ and let $y := f(x)$. Considering D_f and $\mathrm{Nonsing}(E)$ as C^∞ manifolds, we can find a C^∞ coordinate system of D_f centered at x and a C^∞ coordinate system of $\mathrm{Nonsing}(E)$ centered at y in which the map f is given by $t \longmapsto t^a$ for some $a \in \mathbb{N}^*$. Such an integer depends only on f and x and will be indicated by $\nu_f(x)$. Remark that, being f regular and nonconstant, the set of points x of D_f with $\nu_f(x) \geq 2$ is finite. We define the *singular locus* $S(f)$ *of* f as the divisor $\sum_{x\in D_f}(\nu_f(x)-1)\cdot x$ on D and the *degree* $\deg S(f)$ *of* f by $\deg S(f) := \sum_{x\in D_f}(\nu_f(x)-1)$.

Lemma 4.1 *Let D and E be real algebraic curves and let $f \in \mathcal{R}^*(D,E)$. If $g(E) \geq 1$, then $\deg S(f) \leq 2g(D) - 2$.*

Proof. Let $f_\mathbb{C} : D_\mathbb{C} \longrightarrow E_\mathbb{C}$ be a complexification of f, let d be the degree of $f_\mathbb{C}$ and let $B(f_\mathbb{C})$ be the branch locus divisor of $f_\mathbb{C}$. Define $D_f := \mathrm{Nonsing}(D) \setminus f^{-1}(\mathrm{Sing}(E))$. Identify D_f with a Zariski open subset of $D_\mathbb{C}(\mathbb{R})$ and $\mathrm{Nonsing}(E)$ with a Zariski open subset of $E_\mathbb{C}(\mathbb{R})$. It follows immediately that $\deg S(f) \leq \deg B(f_\mathbb{C})$. On the other hand, by Hurwitz's formula, we have that $\deg B(f_\mathbb{C}) = 2g(D) - 2 - d(2g(E) - 2) \leq 2g(D) - 2$. □

Lemma 4.2 *Let X be a real algebraic variety and let $p \in X \setminus \mathrm{Isol}(X)$. Then, there are an irreducible component X' of X and an open semi–algebraic subset A of X such that $A \subset X' \setminus \mathrm{Isol}(X')$ and p is an accumulation point of A in X.*

Proof. Since the problem is local, we may suppose that X is an algebraic subset of some $\mathbb{R}^n$. For each $t \in \mathbb{R}^+$, let $B_t(p)$ be the open ball of $\mathbb{R}^n$ centered at p with radious t and let $U_t := X \cap B_t(p)$. Let $X_1, \dots, X_s$ be the irreducible components X^* of X such that p is an accumulation point of X^* in X. Choose $\varepsilon \in \mathbb{R}^+$ in such a way that $U_\varepsilon \subset \bigcup_{i=1}^s X_i$ and $U_\varepsilon \cap \bigcup_{i=1}^s \mathrm{Isol}(X_i) = \emptyset$. For each $x \in U_\varepsilon$, let I_x be the subset of $\{1, \dots, s\}$ formed by the integers i such that $x \in X_i$. Define the function $h : U_\varepsilon \longrightarrow \{1, \dots, s\}$ by $h(x) := \sharp I_x$ and the function $H : (0, \varepsilon] \longrightarrow \{1, \dots, s\}$ by $H(t) := \min h|_{U_t}$. Let $K := \max H$ and let $\eta \in (0, \varepsilon]$ such that $\min h|_{U_t} = K$ for each $t \in (0, \eta]$. For each $n \in \mathbb{N}^*$ such that $1/n < \eta$, fix $p_n \in U_{1/n} \setminus \{p\}$ in such a way that $h(p_n) = \min h|_{U_{1/n}} = K$. Since $\{1, \dots, s\}$ has a finite family of subsets with K elements, extracting a subsequence if needed, we may suppose that each I_{p_n} is equal to some non–void subset I of $\{1, \dots, s\}$. Let $A := U_\eta \setminus \bigcup_{i \notin I} X_i$. The sequence $\{p_n\}_n$ lies in A and converges to p so p is an accumulation point of A. Let $x \in A$. Since $\sharp I_x \geq K = \sharp I$ and $x \notin \bigcup_{i \notin I} X_i$, it follows that $I_x = I$. In particular, we have that $A = (B_\eta(p) \cap \bigcap_{i \in I} X_i) \setminus \bigcup_{i \notin I} X_i$. Fix $i \in I$ and define $X' := X_i$. The proof is complete. $\square$

Lemma 4.3 *Let X be an algebraic subset of $\mathbb{R}^n$, let Y be an algebraic subset of $\mathbb{R}^m$ and let $f \in \mathcal{R}(X, Y)$. Then, there are a nonsingular algebraic subset W of some $\mathbb{R}^N$, a regular map $F : X \longrightarrow W$ and a polynomial map $P : W \longrightarrow Y$ such that $f = P \circ F$.*

Proof. Let us recall the statement of the so–called general Néron desingularization (for the proof, see [55] and [57]): "Let $F \longrightarrow B$ be a regular morphism of noetherian rings where F contains a field of characteristic zero. Then, for every finitely generated F–algebra A and every F–algebra morphism $a : A \longrightarrow B$, there exist a finitely generated regular F–algebra A' and two F–algebra morphisms $p : A \longrightarrow A'$ and $a' : A' \longrightarrow B$ such that $a = a' \circ p$". When A is the $\mathbb{R}$–algebra of polynomial functions on Y, B is the $\mathbb{R}$–algebra of regular functions on X and $a : A \longrightarrow B$ is the pullback of f, this result is equivalent to the present lemma. $\square$

Indicate by $\|v\|_n$ the usual norm of a vector v of $\mathbb{R}^n$.

Lemma 4.4 *Let X be an algebraic subset of $\mathbb{R}^n$, let Y be a nonsingular Zariski locally closed subset of $\mathbb{R}^m$ and let $g \in C^0(X, Y)$ such that, for some compact subset K of X and for some $f \in \mathcal{N}(X, Y)$, $g = f$ on $X \setminus K$. Equip $C^0(X, Y)$ with the compact–open topology. Then, g is approximable by Nash maps in $C^0(X, Y)$, i.e., it is contained in the closure of $\mathcal{N}(X, Y)$ in $C^0(X, Y)$.*

Proof. Let g, K and f be as in the statement of the lemma and let $\mathcal{U}$ be a neighborhood of g in $C^0(X,Y)$. Let $\rho : U \longrightarrow Y$ be a Nash retraction of an open semi–algebraic neighborhood U of Y in $\mathbb{R}^m$ onto Y and let $\xi : X \longrightarrow \mathbb{R}$ be the function which sends x into the distance $\mathrm{dist}(g(x), \mathbb{R}^m \setminus U)$ between $g(x)$ and $\mathbb{R}^m \setminus U$ in $\mathbb{R}^m$. Since ξ is positive and coincides with a semi–algebraic function on $X \setminus K$, by Proposition 2.6.2 of [4], there are $\delta \in \mathbb{R}^+$ and $\alpha \in \mathbb{N}$ such that $\xi(x) \geq \delta(1 + \|x\|_n)^{-\alpha}$ for each $x \in X$. On the other hand, thanks to Lemma 2.8.1, b) of [1], we know that, for each $\varepsilon \in \mathbb{R}^+$ and for each $\beta \in \mathbb{N}$, there is $G_{\varepsilon,\beta} \in \mathcal{N}(X, \mathbb{R}^m)$ such that $\|G_{\varepsilon,\beta}(x) - g(x)\|_m \leq \varepsilon(1 + \|x\|_n)^{-\beta}$ for each $x \in X$. In this way, choosing ε sufficiently small and β sufficiently large, the Nash map $\rho \circ G_{\varepsilon,\beta} : X \longrightarrow Y$ is well–defined and lies in $\mathcal{U}$ (remark that, if $G_{\varepsilon,\beta}(x) \in U$ for some $x \in X$, then $\|(\varrho \circ G_{\varepsilon,\beta})(x) - g(x)\|_m \leq 2\|G_{\varepsilon,\beta}(x) - g(x)\|_m$). $\square$

Proof of Theorem 2.27, b). Step I. Suppose $\mathrm{Map}(X,Y) = C^0(X,Y)$. We must prove that, for each $f \in \mathcal{R}(X,Y) \setminus \mathrm{Triv}(X,Y)$ and each neighborhood $\mathcal{U}$ of f in $C^0(X,Y)$, there exist a map g in $\mathcal{U}$ which is not contained in the closure of $\mathcal{R}(X,Y)$ in $C^0(X,Y)$. Bearing in mind Definition 2.19 and Lemma 1.14, a), it suffices to prove the following affine version of Theorem 2.27, b).

Assertion. *Let X be an algebraic subset of $\mathbb{R}^n$ (of positive dimension), let Y be an algebraic subset of $\mathbb{R}^m$ (of positive dimension) with $p_t(Y) \geq 1$ and let $f \in \mathcal{R}(X,Y) \setminus \mathrm{Triv}(X,Y)$. Equip $C^0(X,Y)$ with the compact–open topology. Then, there exist a point $p \in X \setminus \mathrm{Isol}(X)$ with $f(p) \notin \mathrm{Isol}(Y)$ and a compact neighborhood K of p in X with the following properties: for each $\varepsilon \in \mathbb{R}^+$, there is a map g of $C^0(X,Y)$ such that $g = f$ on $X \setminus K$, $\sup_{x \in K} \|g(x) - f(x)\|_m < \varepsilon$ and g is not contained in the closure of $\mathcal{R}(X,Y)$ in $C^0(X,Y)$.*

Let X, Y and f be as in the statement of the previous assertion and fix a point p in $X \setminus \mathrm{Isol}(X)$ such that $q := f(p) \notin \mathrm{Isol}(Y)$. We subdivide the remainder of the proof of the previous assertion into three parts.

Part I.1. By Lemma 4.2, we can choose an irreducible component Y' of Y and an open semi–algebraic subset A of Y such that $A \subset Y' \setminus \mathrm{Isol}(Y')$ and q is an accumulation point of A in Y. Since $p_{t,Y'}(q) \geq 1$, there are a Zariski open neighborhood Z of q in Y', a finite family $\{D_i\}_{i \in I}$ of real algebraic curves with $\min_{i \in I} g(D_i) = p_{t,Y'}(q)$ and an injective regular map $\varphi : Z \longrightarrow \prod_{i \in I} D_i$. Let $A' := A \cap Z$. It is immediate to see that A' is an open semi–algebraic subset of Y contained in $Y' \setminus \mathrm{Isol}(Y')$ such that q is an accumulation point of A' in Y. Let V be the Zariski closed subset of X defined by $V := \mathrm{Isol}(X) \cup f^{-1}(\mathrm{Isol}(Y)) \cup f^{-1}(Y' \setminus Z)$. Since $p \notin V$, we can find a compact neighborhood K of p in X which does not intersect V. Indicate by Ω the interior of K in X and, for each $t \in \mathbb{R}^+$, denote by $B_t(q)$ the open ball of $\mathbb{R}^m$ centered at q with radious t. Fix $\varepsilon \in \mathbb{R}^+$. Let $\varrho : U \longrightarrow Y$ be a continuous retraction of an open neighborhood U of Y in $\mathbb{R}^m$ onto Y (recall that an affine real algebraic variety is an absolute neighborhood retract), let $U_\varepsilon := \varrho^{-1}(Y \cap B_{\varepsilon/2}(q))$ and let $\eta \in (0, \varepsilon/2)$ such that $B_{3\eta}(q) \subset U_\varepsilon$. By using the Nash Curve Selection Lemma (see Proposition 8.1.13 of [4]), it is easy to find a real algebraic curve G of X containing p such that $p \notin \mathrm{Isol}(G)$ and a real algebraic curve L of Z such that $\mathrm{Nonsing}(L) \cap A' \cap B_\eta(q) \neq \emptyset$. Since $\varphi|_L$ is nonconstant, there is $j \in I$ such that, setting $\pi_j : \prod_{i \in I} D_i \longrightarrow D_j$ equal to the natural projection, the composition map $\pi_j \circ \varphi|_L : L \longrightarrow D_j$ is nonconstant also. By Sard's theorem, there exists a finite subset L' of L containing $\mathrm{Sing}(L)$ such that $(\pi_j \circ \varphi)(L \setminus L') \subset \mathrm{Nonsing}(D_j)$ and the restriction of $\pi_j \circ \varphi$ between $L \setminus L'$ and $\mathrm{Nonsing}(D_j)$ is a local diffeomorphism. Fix $p' \in \mathrm{Nonsing}(G) \cap \Omega$ so close to p that $f(p') \in B_\eta(q)$ and fix $q' \in (L \setminus L') \cap A' \cap B_\eta(q)$. Let $\nu := \max\{0, 2g(G) - 1\}$. By using adequate C^∞ coordinate systems on $\mathrm{Nonsing}(G)$ locally at p' and on $\mathrm{Nonsing}(L)$ locally at q', it is easy to find an open neighborhood H of p' in Ω and a continuous map h between H and $(L \setminus L') \cap A' \cap B_\eta(q)$ such that $f(H) \subset B_\eta(q)$, $H \cap \mathrm{Sing}(G) = \emptyset$, $h(p') = q'$ and the map $\pi_j \circ \varphi \circ h|_{H \cap \mathrm{Nonsing}(G)} : H \cap \mathrm{Nonsing}(G) \longrightarrow \mathrm{Nonsing}(D_j)$ is C^∞, nonconstant and has at least ν distinct critical points $p_1, \ldots, p_\nu$ of order 2. Let K' be a compact subset of H such that the interior of $K' \cap G$ in $\mathrm{Nonsing}(G)$ contains $\{p_1, \ldots, p_\nu\}$.

Part I.2. Let $\{\alpha, \beta\}$ be a continuous partition of unity subordinate to the open cover $\{H, X \setminus K'\}$ of X. Define the continuous map $g' : X \longrightarrow \mathbb{R}^m$ as follows: $g' := \alpha h + \beta f$ on H and $g' := f$ on $X \setminus H$. Remark that $g' = h$ on K' and, for each $x \in H$, it holds:

$$\begin{aligned} \|g'(x) - q\|_m &\leq \|g'(x) - f(x)\|_m + \|f(x) - q\|_m < \|h(x) - f(x)\|_m + \eta \leq \\ &\leq \|h(x) - q\|_m + \|q - f(x)\|_m + \eta < 3\eta. \end{aligned}$$

Since $g'(X) \subset U$, we can define the continuous map $g : X \longrightarrow Y$ by $g := \varrho \circ g'$. We have: $g = h$ on K', $g = f$ on $X \setminus H$ (and hence on $X \setminus K$), $g(H) \subset B_{\varepsilon/2}(q)$ and, for each $x \in K$, $\|g(x) - f(x)\|_m < \varepsilon$ (because $g(x) = f(x)$ if $x \in K \setminus H$ and $\|g(x) - f(x)\|_m \leq \|g(x) - q\|_m + \|q - f(x)\|_m < \varepsilon$ if $x \in H$).

Part I.3. It remains to prove that g is not contained in the closure of $\mathcal{R}(X, Y)$ in $C^0(X, Y)$. Suppose this is not true. Equip $C^0(K' \cap G, D_j)$ with the compact–open topology. Since A' is open in Y and $g(K') \subset A'$, for each neighborhood $\mathcal{V}$ of $h^* := \pi_j \circ \varphi \circ g|_{K' \cap G}$ in $C^0(K' \cap G, D_j)$, there exists a neighborhood $\mathcal{U}$ of g in $C^0(X, Y)$ with the following properties: if $R \in \mathcal{R}(X, Y) \cap \mathcal{U}$, then $R(K') \subset A'$, $(\pi_j \circ \varphi \circ R)(K' \cap G) \subset \mathrm{Nonsing}(D_j)$ and the map $\pi_j \circ \varphi \circ R|_{K' \cap G} : K' \cap G \longrightarrow D_j$ is well–defined, is nonconstant and is contained in $\mathcal{V}$. Furthermore, if $\mathcal{V}$ is sufficiently small around h^*, then Rolle's theorem ensures that the C^∞ map $\pi_j \circ \varphi \circ R|_{K' \cap G}$ has at least ν distinct critical points in the interior of $K' \cap G$ in Nonsing(G). Fix a neighborhood $\mathcal{U}$ of g in $C^0(X,Y)$ so small that each $R \in \mathcal{R}(X,Y) \cap \mathcal{U}$ has the properties described above. Since $K' \cap G$ is Zariski dense in G and $R(K' \cap G) \subset Y'$, it follows that $R(G) \subset Y'$. Define the real algebraic curve G_R of X by $G_R := G \cap R^{-1}(Z)$ (remark that $G_R \supset K' \cap G$) and the regular map $R^* : G_R \longrightarrow D_j$ by $R^* := \pi_j \circ \varphi \circ R|_{G_R}$. We will show that the existence of R^* gives a contradiction. The proof of the previous assertion will be complete. Remark that $g(D_j) \geq 1$ and $g(G) = g(G_R)$ so $\nu = \max\{0, 2g(G_R) - 1\}$. If $g(G_R) = 0$, then Lemma 1.6 implies that R^* is constant which is false. If $g(G_R) \geq 1$, then, from Lemma 4.1, it follows that $\nu \leq \deg S(R^*) \leq 2g(G_R) - 2 = \nu - 1$ which is impossible.

Step II. Suppose $\mathrm{Map}(X,Y) = C^\omega(X,Y)$ or $\mathrm{Map}(X,Y) = \mathcal{N}(X,Y)$. It suffices to prove that, for each $f \in \mathcal{R}(X,Y) \setminus \mathrm{Triv}(X,Y)$ and for each neighborhood $\mathcal{U}$ of f in $\mathcal{N}(X,Y)$, there is $g \in \mathcal{U}$ which is not approximable by regular maps in $\mathcal{N}(X,Y)$. We may assume that X is an algebraic subset of $\mathbb{R}^n$ and Y is an algebraic subset of $\mathbb{R}^m$. Let $f \in \mathcal{R}(X,Y) \setminus \mathrm{Triv}(X,Y)$. First, suppose that f is Zariski locally constant. Let X' be a Zariski connected component of X_* and let $p \in X' \setminus \mathrm{Isol}(X')$ such that $q := f(p) \notin \mathrm{Isol}(Y)$. Repeat word for word *Part I.1* of the previous step. Let $c : X \longrightarrow Y$ be the constant map which sends X into q'. By using an adequate partition of unity on X and a C^∞ coordinate system on $L \setminus L'$, it is easy to define a continuous map $g' : X \longrightarrow L \setminus L'$ with the following properties: $g' = h$ on K', $g' = c$ on $X' \setminus K$ (we can choose K contained in X'), $g'(X') \subset (L \setminus L') \cap A' \cap B_\eta(q)$ and $g' = f$ on $X \setminus X'$. Remark that, choosing ε sufficiently small, g' is arbitrarily close to f in $C^0(X,Y)$. Applying Lemma 4.4 to g', we find a map $g \in \mathcal{N}(X,Y)$ such that $g(X') \subset L \setminus L'$ and, viewing g' as a map between X and Y, g is arbitrarily close to g' in $C^0(X,Y)$. In particular, g is arbitrarily close to f in $\mathcal{N}(X,Y)$. Repeating *Part I.3*, we see that, choosing g sufficiently close to g', g is not approximable by regular maps between X and Y. Suppose now that f is not Zariski locally constant so $\dim(\mathrm{Zcl}_Y(f(X))) \geq 1$. Remark that, as a map between X and $\mathrm{Zcl}_Y(f(X))$, f is not trivial and, from Lemma 1.14, a), it follows that $p_t(\mathrm{Zcl}_Y(f(X))) \geq p_t(Y) \geq 1$. In this way, we may suppose that $f(X)$ is Zariski dense in Y. By Lemma 4.3, there are a nonsingular algebraic subset W of some $\mathbb{R}^N$, a regular map $F : X \longrightarrow W$ and a polynomial map $P : W \longrightarrow Y$ such that $f := P \circ F$. Since $P(W) \supset f(X)$, $P(W)$ is Zariski dense in Y so, by Sard's theorem, there is a Zariski closed subset V of Y containing $\mathrm{Sing}(Y)$ such that $\dim(V) < \dim(Y)$ and the restriction of P between $P^{-1}(Y \setminus V)$ and $Y \setminus V$ is a submersion. Since $f(X)$ is Zariski dense in Y and $\dim(Y) \geq 1$, there are a nonsingular point p of X of some positive dimension and a real algebraic curve G of X such that $p \in \mathrm{Nonsing}(G)$, $q := f(p) \in Y \setminus V$ and $f|_G$ is nonconstant. Since $p_t(Y) \geq 1$ and q lies in an unique irreducible component of Y_*, there are a Zariski open neighborhood Z of q in $\mathrm{Nonsing}(Y)$, a finite family $\{E_j\}_{j \in J}$ of real algebraic curves with $\min_{j \in J} g(E_j) \geq 1$ and an injective regular map $\psi : Z \longrightarrow \prod_{j \in J} E_j$. Define the real algebraic curve L of Y by $L := \mathrm{Zcl}_Y(f(G))$. Proceeding as in *Part I.1*, we find a finite subset L' of L containing $\mathrm{Sing}(L)$ and $j \in J$ such that $L \setminus L' \subset Z$, $(\pi_j \circ \psi)(L \setminus L') \subset \mathrm{Nonsing}(E_j)$ and the restriction of $\pi_j \circ \psi$ between $L \setminus L'$ and $\mathrm{Nonsing}(E_j)$ is a local diffeomorphism. Choose a point $p' \in G$ close to p in G such that, setting $q' := f(p')$, q' is arbitrarily close to q in L, $q' \in Y \setminus V$ and lies in $L \setminus L'$ (such a point p' exists because $f^{-1}(L') \cap G$ is finite and $p \in \mathrm{Nonsing}(G)$). Let $z' := F(p')$. Since P is a submersion over $Y \setminus V$, it is easy to find a real algebraic curve S of W such that $z' \in \mathrm{Nonsing}(S)$, $P(S) \subset L$ and the restriction of P between $P^{-1}(L \setminus L') \cap \mathrm{Nonsing}(S)$ and $L \setminus L'$ is a local diffeomorphism at z' (remark that $P(z') = q'$)). Let B be an open neighborhood of z' in $\mathrm{Nonsing}(S)$ such that $B' := P(U) \subset L \setminus L'$, $B'' := (\pi_j \circ \psi \circ P)(U)$ is an open subset of $\mathrm{Nonsing}(E_j)$ and both the restriction maps $P|_B : B \longrightarrow B'$ and $\pi_j \circ \psi|_{B'} : B' \longrightarrow B''$ are diffeomorphisms. Let $\nu := \max\{0, 2g(G) - 1\}$. Equip $C^0(X, Y)$ with the compact–open topology. Following the argument used in *Part I.1* and *Part I.2* to define g', we obtain two compact neighborhoods K and K' of p' in X and a continuous map $g' : X \longrightarrow W$ arbitrarily close to F in $C^0(X, W)$ such that $g' = F$ on $X \setminus K$, K' is contained in the interior of K in X, $K' \cap G \subset \mathrm{Nonsing}(G)$, $g'(K' \cap G) \subset B$ and the restriction of g' between $K' \cap G$ and B is a nonconstant C^∞ map having ν distinct critical points of order 2 in the interior of $K' \cap G$ in $\mathrm{Nonsing}(G)$. By Lemma 4.4, there is a Nash map $g'' : X \longrightarrow W$ so close to g' in $C^0(X, W)$ that, setting $g : X \longrightarrow Y$ equal to the composition map $P \circ g''$, g is arbitrarily close to f in $\mathcal{N}(X, Y)$ and the restriction of $\pi_j \circ \psi \circ g$ between the interior of $K' \cap G$ in $\mathrm{Nonsing}(G)$ and $\mathrm{Nonsing}(E_j)$ is nonconstant and has at least ν distinct critical points. Repeating *Part I.3*, we complete the proof. $\square$

We refer the reader to [18] for further results obtained by arguments similar to the ones used in the previous proof.

Proof of Theorem 2.27, c). Step I. Let us prove the following assertion.

Assertion. *Let X and Y be real algebraic varieties (possibly zero–dimensional) such that X is irreducible and $p_t(Y) \geq 2$ if $s := \dim(Y) \geq 1$. Equip $\mathcal{R}(X, Y)$ with the pointwise topology. Then, $\mathcal{R}(X, Y)$ has a singular quasi–euclidean stratification $\mathcal{S} = \{\mathcal{R}_j\}_{j=0}^m$ with the following properties: for each*

$j \in \{0, 1, \ldots, m\}$, *there are a semi–algebraic set* S_j *with* $\dim(S_j) \leq s$ *and an injective continuous map between* $\mathcal{R}_j$ *and* S_j. *Moreover,* $\dim_{\mathfrak{B}} \mathcal{R}(X, Y) = s$.

Let us proceed by induction on s. The case $s = 0$ is trivial because $\mathcal{R}(X, Y)$ is a finite set equipped with the discrete topology so it is an euclidean set and its Brouwer dimension is zero. Let $s \geq 1$. If $\dim(X) = 0$, then X is a single point and $\mathcal{R}(X, Y)$ is homeomorphic to Y which always has a singular euclidean stratification. Suppose $\dim(X) \geq 1$. Let $Y_1, \ldots, Y_d$ be the irreducible components of Y of dimension s and let $Y_{d+1}, \ldots, Y_e$ be the remaining irreducible components. For each $j \in \{1, \ldots, d\}$, we have that $p_t(Y_j) \geq p_t(Y) \geq 2$ so there are a non–void affine Zariski open subset Z_j of Y_j, a finite family $\{D_{ji}\}_{i=1}^{n_j}$ of real algebraic curves with $\min_{i \in \{1, \ldots, n_j\}} g(D_{ji}) = p_t(Y_j)$ and an injective regular map $\varphi_j : Z_j \longrightarrow T^j := \prod_{i=1}^{n_j} D_{ji}$. Define

$$Y' := \bigcup_{j=1}^{d} (Y_j \setminus Z_j) \cup \bigcup_{j>d}^{e} Y_j \cup \bigcup_{\{(i,j) \,|\, 1 \leq i < j \leq d\}} (Y_i \cap Y_j),$$

$\mathcal{R}^* := \{f \in \mathcal{R}(X, Y) \,|\, f(X) \not\subset Y'\}$ and $\mathcal{R}' := \{f \in \mathcal{R}(X, Y) \,|\, f(X) \subset Y'\}$. Remark that Y' is Zariski closed in Y, $\dim(Y') < s$ and, by Lemma 1.14, a), $p_t(Y') \geq 2$. Moreover, $\{\mathcal{R}^*, \mathcal{R}'\}$ is a partition of $\mathcal{R}(X, Y)$, $\mathcal{R}^*$ is open in $\mathcal{R}(X, Y)$ and hence $\mathcal{R}'$ is closed in $\mathcal{R}(X, Y)$. By induction, it suffices to show that $\dim_{\mathfrak{B}} \mathcal{R}^* = s$ and there are a semi–algebraic set S with $\dim(S) \leq s$ and an injective continuous map between $\mathcal{R}^*$ and S. For each $j \in \{1, \ldots, d\}$, define $\mathcal{R}_j^* := \{f \in \mathcal{R}^* \,|\, f(X) \subset Y_j\}$. Fix $f \in \mathcal{R}^*$. Since $\mathrm{Zcl}_Y(f(X))$ is irreducible, there exists at least one integer j in $\{1, \ldots, d\}$ such that $f(X) \subset Y_j$. Such an integer must be unique, otherwise $f(X) \subset Y'$. In this way, $\mathcal{R}^*$ is the disjoint union of $\mathcal{R}_j^*$'s. Fix $j \in \{1, \ldots, d\}$. Let $f \in \mathcal{R}_j^*$ and let $p \in \mathrm{Nonsing}(X)$ such that $f(p) \in Y_j \setminus Y'$. Since $Y_j \setminus Y'$ is open in Y, there is a neighborhood $\mathcal{U}$ of f in $\mathcal{R}(X, Y)$ with respect to the pointwise topology such that, for each $g \in \mathcal{U}$, $g(p) \in Y_j \setminus Y'$. On the other hand, X is irreducible so each g in $\mathcal{U}$ sends X into Y_j, i.e., $\mathcal{U} \subset \mathcal{R}_j^*$. It follows that $\{\mathcal{R}_j^*\}_{j=1}^{d}$ is an open partition of $\mathcal{R}^*$ so it suffices to prove that, for some $j \in \{1, \ldots, d\}$, $\dim_{\mathfrak{B}} \mathcal{R}_j^* = s$ and, for each $j \in \{1, \ldots, d\}$, there are semi–algebraic sets S_j with $\dim(S_j) \leq s$ and an injective continuous map between $\mathcal{R}_j^*$ and S_j. Fix $j \in \{1, \ldots, d\}$. By Lemma 3.1, we have that the cardinality of each $\mathcal{R}atio^*(X, D_{ji})$ is finite. Let $\alpha_{ji1}, \ldots, \alpha_{j,i,N_{ji}}$ be the elements of $\mathcal{R}atio^*(X, D_{ji})$ for each $i \in \{1, \ldots, n_j\}$. Choose a non–void Zariski open subset W_j of X and, for each $i \in \{1, \ldots, n_j\}$, maps $g_{ji1}, \ldots, g_{j,i,N_{ji}}$ in $\mathcal{R}^*(W_j, D_{ji})$ which represent the rational maps $\alpha_{ji1}, \ldots, \alpha_{j,i,N_{ji}}$ respectively. For each $f \in \mathcal{R}_j^*$, let $W'_f := f^{-1}(Y_j \setminus Y')$. Remark that each W'_f is a non–void Zariski open subset of X and $f(W'_f) \subset Z_j$. Denote by $\varrho_j : \mathcal{R}_j^* \longrightarrow \mathcal{R}atio(X, Z_j)$ the map which sends f into the rational map represented by the restriction of f between W'_f and Z_j and indicate by $\varphi_j^* : \mathcal{R}atio(X, Z_j) \longrightarrow \mathcal{R}atio(X, T^j)$

the map which sends a rational map in $\mathcal{R}atio(X, Z_j)$ represented by (U, f_U) into the rational map in $\mathcal{R}atio(X, T^j)$ represented by $(U, \varphi_j \circ f_U)$. Since X is irreducible and φ_j is injective, both ϱ_j and φ_j^* are injective. Identify $\mathcal{R}atio(X, T^j)$ with $\prod_{i=1}^{n_j} \mathcal{R}atio(X, D_{ji})$ in the natural way. For each $i \in \{1, \ldots, n_j\}$, let $\pi_{ji} : T^j \longrightarrow D_{ji}$ be the natural projection. For each $\chi \subset \{1, \ldots, n_j\}$, let $\chi' := \{1, \ldots, n_j\} \setminus \chi$, let $T^j_{\chi'} := \prod_{i \in \chi'} D_{ji}$ (where $T^j_{\emptyset}$ is considered equal to a point), let $\pi_{\chi'} : T^j \longrightarrow T^j_{\chi'}$ be the natural projection and let $F_j(\chi)$ be the set of all functions ψ between χ and $\mathbb{N}$ such that $\psi(i) \in \{1, \ldots, N_{ji}\}$ for each $i \in \chi$. Remark that if $\chi = \emptyset$ or $N_{ji} = 0$ for some $i \in \chi$, $F_j(\chi)$ is void. For each $\chi \subset \{1, \ldots, n_j\}$ and for each $\psi \in F_j(\chi)$, let $\mathcal{P}_{j,\chi,\psi}$ be the set of all elements $(\beta_1, \ldots, \beta_{n_j})$ of $\mathcal{R}atio(X, T^j)$ such that, for each $i \in \chi$, $\beta_i = \alpha_{j,i,\psi(i)}$ and, for each $i \in \chi'$, β_i is represented by a constant map between X and D_{ji}. Moreover, define $\mathcal{R}^*_{j,\chi,\psi} := (\varphi_j^* \circ \varrho_j)^{-1}(\mathcal{P}_{j,\chi,\psi})$. Denote by $\mathcal{R}^*_{j,\emptyset}$ the set of all constant regular maps between X and $Y_j \setminus Y'$. Remark that $\{\mathcal{R}^*_{j,\chi,\psi}\}_{\chi \subset \{1,\ldots,n_j\}, \psi \in F_j(\chi)} \cup \{\mathcal{R}^*_{j,\emptyset}\}$ is a partition of $\mathcal{R}^*_j$. In order to complete the proof of the previous assertion, it suffices to show that:

a) for each $\chi \subset \{1, \ldots, n_j\}$ and for each $\psi \in F_j(\chi)$, $\mathcal{R}^*_{j,\chi,\psi}$ is open in $\mathcal{R}^*_j$ and there are a semi–algebraic set $S_{j,\chi,\psi}$ with $\dim(S_{j,\chi,\psi}) \leq s$ and an injective continuous map between $\mathcal{R}^*_{j,\chi,\psi}$ and $S_{j,\chi,\psi}$,

b) $\mathcal{R}^*_{j,\emptyset}$ is open in $\mathcal{R}^*_j$, $\dim_{\mathfrak{B}} \mathcal{R}^*_{j,\emptyset} = s$ and there is an injective continuous map between $\mathcal{R}^*_{j,\emptyset}$ and a s–dimensional semi–algebraic set.

We will prove property a). The proof of b) is easy. Remark that, for each $\chi \subset \{1, \ldots, n_j\}$ and for each $\psi \in F_j(\chi)$, we have that

$$\mathcal{R}^*_{j,\chi,\psi} := \{f \in \mathcal{R}^*_j \,|\, \pi_{ji} \circ \varphi_j \circ f = g_{j,i,\psi(i)} \text{ on } W_j \cap W'_f \text{ for each } i \in \chi \\ \text{and } \pi_{\chi'} \circ \varphi_j \circ f \text{ is constant on } W_j \cap W'_f\}.$$

Fix $\chi \subset \{1, \ldots, n_j\}$, $\psi \in F_j(\chi)$ and $f \in \mathcal{R}^*_{j,\chi,\psi}$. For each $i \in \chi'$, let $b_i(f)$ be the point of D_{ji} such that $(\pi_{ji} \circ \varphi_j \circ f)(W_j \cap W'_f) = \{b_i(f)\}$. Choose two points p and q of $W_j \cap W'_f \cap \text{Nonsing}(X)$ such that: $g_{jih}(p) \neq g_{jik}(p)$ for each $i \in \chi$ and for each $h, k \in \{1, \ldots, N_{ji}\}$ with $h \neq k$, $g_{jih}(p) \neq g_{jih}(q)$ for each $i \in \chi$ and for each $h \in \{1, \ldots, N_{ji}\}$ and $g_{jih}(p) \neq b_i(f)$ for each $i \in \chi'$ and for each $h \in \{1, \ldots, N_{ji}\}$. For each $i \in \chi$ and for each $h \in \{1, \ldots, N_{ji}\}$, let U^p_{jih} be a neighborhood of $g_{jih}(p)$ in D_{ji} and let U^q_{jih} be a neighborhood of $g_{jih}(q)$ in D_{ji} such that: $U^p_{jih} \cap U^p_{jik} = \emptyset$ for each $i \in \chi$ and for each $h, k \in \{1, \ldots, N_{ji}\}$ with $h \neq k$ and $U^p_{jih} \cap U^q_{jih} = \emptyset$ for each $i \in \chi$ and for each $h \in \{1, \ldots, N_{ji}\}$. For each $i \in \chi'$, let V_{ji} be a neighborhood of $b_i(f)$ in D_{ji} such that $V_{ji} \cap \{g_{ji1}(p), \ldots, g_{j,i,N_{ji}}(p)\} = \emptyset$. Let A be the neighborhood of $\varphi_j(f(p))$ in T^j defined by $A := \prod_{i \in \chi} U^p_{j,i,\psi(i)} \times \prod_{i \in \chi'} V_{ji}$ and let B be the neighborhood of $\varphi_j(f(q))$ in T^j defined by $B := \prod_{i\in\chi} U^q_{j,i,\psi(i)} \times \prod_{i\in\chi'} D_{ji}$. Since φ_j is continuous and $Y_j \setminus Y'$ is an open subset of Y containing $\{f(p), f(q)\}$, there exist a neighborhood A' of $f(p)$ in Y contained in $Y_j \setminus Y'$ and a neighborhood of B' of $f(q)$ in Y contained in $Y_j \setminus Y'$ such that $\varphi_j(A') \subset A$ and $\varphi_j(B') \subset B$. Define a neighborhood $\mathcal{U}$ of f in $\mathcal{R}^*_j$ by setting: $\mathcal{U} := \{\xi \in \mathcal{R}^*_j \,|\, \xi(p) \in A' \text{ and } \xi(p) \in B'\}$. From the properties of U^p_{jih}'s and U^q_{jih}'s, it follows easily that $\mathcal{U} \subset \mathcal{R}^*_{j,\chi,\psi}$. In particular, we have that each $\mathcal{R}^*_{j,\chi,\psi}$ is open in $\mathcal{R}^*_j$ as desired. The previous argument gives some more. The map $H_{j,\chi,\psi} : \mathcal{R}^*_{j,\chi,\psi} \longrightarrow \prod_{i\in\chi'} D_{ji}$ which sends f into $(b_i(f))_{i\in\chi'}$ is continuous. Moreover, it is injective because $\varphi^*_j \circ \varrho_j$ is and its image is contained in the semi–algebraic set $S_{j,\chi,\psi} := \pi_{\chi'}(\varphi_j(Z_j))$ whose dimension is less than or equal to $s = \dim(Z_j)$.

Step II. Let $X_1, \ldots, X_b$ be the irreducible components of X. For each $i \in \{1, \ldots, b\}$, equip $\mathcal{R}(X_i, Y)$ with the pointwise topology. By *Step I*, we know that each $\mathcal{R}(X_i, Y)$ has a singular quasi–euclidean stratification $\{\mathcal{R}^{(i)}_j\}_{j=0}^{m_i}$ such that, for each $i \in \{1, \ldots, b\}$ and for each $j \in \{0, 1, \ldots, m_i\}$, there are a semi–algebraic set S_{ji} with $\dim(S_{ji}) \leq s$ and an injective continuous map $\varphi_{ji} : \mathcal{R}^{(i)}_j \longrightarrow S_{ji}$. Moreover, $\dim_{\mathfrak{B}} \mathcal{R}(X, Y) = s$. Let $\Phi : \mathcal{R}(X, Y) \longrightarrow \prod_{i=1}^b \mathcal{R}(X_i, Y)$ be the map which sends f into $(f|_{X_1}, \ldots, f|_{X_b})$. Remark that Φ is injective and continuous. Let I be the set of all b–uples $(\sigma_1, \ldots, \sigma_b)$ in $\mathbb{N}^b$ such that $\sigma_i \in [0, m_i]$ for each $i \in \{1, \ldots, b\}$. Define an ordering on I as follows: $(\sigma_1, \ldots, \sigma_b) \preceq (\delta_1, \ldots, \delta_b)$ in I if and only if $\sigma_i \leq \delta_i$ for each $i \in \{1, \ldots, b\}$. For each $\sigma = (\sigma_1, \ldots, \sigma_b) \in I$, let $\mathcal{P}_\sigma$ be the subset of $\prod_{i=1}^b \mathcal{R}(X_i, Y)$ defined by $\mathcal{P}_\sigma := \prod_{i=1}^b \mathcal{R}^{(i)}_{\sigma_i}$ and let $\mathcal{R}_\sigma := \Phi^{-1}(\mathcal{P}_\sigma)$. Since $\{\mathcal{P}_\sigma\}_{\sigma\in I}$ is a quasi–euclidean stratification of $\prod_{i=1}^b \mathcal{R}(X_i, Y)$, $\{\mathcal{R}_\sigma\}_{\sigma\in I}$ is a quasi–euclidean stratification of $\mathcal{R}(X, Y)$ also. Fix $\sigma = (\sigma_1, \ldots, \sigma_b) \in I$. The map $(\prod_{i=1}^b \varphi_{\sigma_i,i}) \circ \Phi : \mathcal{R}_\sigma \longrightarrow S_\sigma := \prod_{i=1}^b S_{\sigma_i,i}$ is injective and continuous so $\dim_{\mathfrak{B}} \mathcal{R}_\sigma \leq \dim_{\mathfrak{B}}(S_\sigma) = \dim(S_\sigma) \leq bs$. It follows that $\dim_{\mathfrak{B}} \mathcal{R}(X, Y) = \max_{\sigma\in I} \dim_{\mathfrak{B}} \mathcal{R}_\sigma \leq bs$. If ξ is nice, then the subspace of $\mathcal{R}(X, Y)$ formed by Zariski locally constant maps between X and Y is homeomorphic to Y^a so $\dim_{\mathfrak{B}} \mathcal{R}(X, Y) \geq as$. $\square$

Proof of Theorem 2.28. We follow the proof of Theorem 2.27, c). First, suppose X irreducible (and, possibly, zero–dimensional). Since $bgp_t(Y) \geq 2$, there exist a finite family $\{D_i\}_{i\in I}$ of real algebraic curves with $\min_{i\in I} g(D_i) = bgp_t(Y)$ and an biregular embedding φ of Y into $T := \prod_{i\in I} D_i$. Fix $i \in I$. By Lemma 3.1, we know that $\mathcal{R}^*(X, D_i)$ is finite. Let $\{g_{i1}, \ldots, g_{i,N_i}\}$ be the elements of $\mathcal{R}^*(X, D_i)$ and let $\pi_i : T \longrightarrow D_i$ be the natural projection. For each non–void subset χ of I, define: $\chi' := I \setminus \chi$, $T_{\chi'} := \prod_{i\in\chi'} D_i$, $\pi_{\chi'} : T \longrightarrow T_{\chi'}$ as the natural projection and $F(\chi)$ as the set of all functions $\psi : \chi \longrightarrow \mathbb{N}$ such that $\psi(i) \in \{1, \ldots, N_i\}$ for each $i \in \chi$. Let $\mathcal{A}$ be the family of all non–void subsets χ of I such that $F(\chi) \neq \emptyset$, i.e, $N_i \geq 1$ (or, equivalently, $\mathcal{R}^*(X, D_i) \neq \emptyset$) for each $i \in \chi$. For each $\chi \in \mathcal{A}$ and for each $\psi \in F(\chi)$, define $\mathcal{R}_{\chi,\psi}$ by $\mathcal{R}_{\chi,\psi} := \{f \in \mathcal{R}(X,Y) \,|\, \pi_i \circ \varphi \circ f = g_{i,\psi(i)}$ for each $i \in \chi$ and $\pi_{\chi'} \circ \varphi \circ f$ is constant$\}$ and, for each $f \in \mathcal{R}_{\chi,\psi}$, indicate by $a(f)$ the point of $T_{\chi'}$ such that $(\pi_{\chi'} \circ \varphi \circ f)(X) = \{a(f)\}$. Let $\mathrm{Const}(X,Y)$ be the constant maps between X and Y. Proceeding as in *Step I* of the above–mentioned proof, we obtain that $\{\mathrm{Const}(X,Y)\} \cup \{\mathcal{R}_{\chi,\psi}\}_{\chi \in \mathcal{A}, \psi \in F(\chi)}$ is a partition of $\mathcal{R}(X,Y)$ whose elements are open and closed in $\mathcal{R}(X,Y)$. Moreover, for each $\chi \in \mathcal{A}$ and for each $\psi \in F(\chi)$, the map $H_{\chi,\psi} : \mathcal{R}_{\chi,\psi} \longrightarrow T_{\chi'}$ which sends f into $a(f)$ is continuous and injective. In order to prove that $\mathcal{R}(X,Y)$ is an euclidean set, it suffices to show that $\mathrm{Const}(X,Y)$ is an euclidean set and each $H_{\chi,\psi}$ is a topological embedding. $\mathrm{Const}(X,Y)$ is homeomorphic to Y so it is an euclidean set. Let d be a metric for Y and, for each $i \in I$, let d'_i be a metric for D_i. Define the metric d' for T by $d'(x,y) := \sum_{i \in I} d'_i(\pi_i(x), \pi_i(y))$. Fix $\chi \in \mathcal{A}$ and $\psi \in F(\chi)$. Let $f \in \mathcal{R}_{\chi,\psi}$, let K be a compact subset of X and let $\varepsilon \in \mathbb{R}^+$. In order to prove that $H_{\chi,\psi}$ is a homeomorphism onto its image, it suffices to show that there is $\delta \in \mathbb{R}^+$ with the following property: if $g \in \mathcal{R}_{\chi,\psi}$ and $\sup_{x \in K} d'(\varphi(g(x)), \varphi(f(x))) < \delta$, then $\sup_{x \in K} d(g(x), f(x)) < \varepsilon$. This fact follows immediately from the continuity of $\varphi^{-1} : \varphi(Y) \longrightarrow Y$. Remark that $H_{\chi,\psi}(\mathcal{R}_{\chi,\psi}) \subset \pi_{\chi'}(\varphi(Y))$ so, using Theorems II.3 and II.7 of [51], it follows that $\dim_{\mathfrak{C}} \mathcal{R}_{\chi,\psi} \leq s$. On the other hand, $\dim_{\mathfrak{C}} \mathrm{Const}(X,Y) = s$ so, by Theorems II.1 and II.7 of [51], we have that $\dim_{\mathfrak{C}} \mathcal{R}(X,Y) = s$. Suppose Y compact. It remains to show that $\mathcal{R}(X,Y)$ is compact. Since $\mathrm{Const}(X,Y)$ is homeomorphic to Y which is compact, it suffices to prove that each $\mathcal{R}_{\chi,\psi}$ is compact also. Fix $\chi \in \mathcal{A}$ and $\psi \in F(\chi)$. We will show that $\mathcal{R}_{\chi,\psi}$ is sequentially compact and hence compact (because $\mathcal{R}_{\chi,\psi}$ is metrizable). Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{R}_{\chi,\psi}$. The corresponding sequence $\{a(f_n)\}_{n \in \mathbb{N}}$ of $T_{\chi'}$ is contained in $\pi_{\chi'}(\varphi(Y))$ which is compact so, extracting a subsequence if needed, we may suppose that $\{a(f_n)\}_{n \in \mathbb{N}}$ converges to a point a in $\pi_{\chi'}(\varphi(Y))$. Define the regular map $\xi : X \longrightarrow T$ by setting $\pi_i \circ \xi = g_{i,\psi(i)}$ for each $i \in \chi$ and $\pi_{\chi'}(\xi(X)) = \{a\}$. Equip $\mathcal{R}(X,T)$ and $\mathcal{R}(X, \varphi(Y))$ with the compact–open topology. Evidently, the sequence $\{\varphi \circ f_n\}_{n \in \mathbb{N}}$ of $\mathcal{R}(X,T)$ converges to ξ. Since $\varphi(Y)$ is closed in T, it follows that $\xi(X) \subset \varphi(Y)$ so $\{\varphi \circ f_n\}_{n \in \mathbb{N}}$ converges to ξ in $\mathcal{R}(X, \varphi(Y))$ also. Define the element f of $\mathcal{R}_{\chi,\psi}$ by $f := \varphi^{-1} \circ \xi$. Since φ^{-1} is continuous, it turns out that $\{f_n\}_{n \in \mathbb{N}}$ converges to f in $\mathcal{R}(X,Y)$. This completes the proof when X is irreducible. In the reducible case, it suffices to repeat the argument of *Step II* of the previous proof. $\square$

Proof of Theorem 2.30. Following the proofs of Theorem 2.27 and Theorem 2.28, we obtain without difficulties points a), c) and c′). Let us prove point b).

We consider only the case $\text{Smo}(M, N) = C^\infty(M, N)$. The proof of other cases can be easily infered from the following proof of the above–mentioned case and from well–known analytic and Nash approximation results for C^∞ maps. Fix $f \in \Sigma(M, N)$. We must prove that there is $g \in C^\infty(M, N)$ arbitrarily close to f which is not approximable by regular maps. Choose $q \in N$ in such a way that $f^{-1}(q)$ contains a 1–dimensional semi–algebraic subset S of M. Making use of standard argument of Semi–algebraic Geometry, we can construct a real algebraic curve G of M such that the interior Ω of $S \cap \text{Nonsing}(G)$ in $\text{Nonsing}(G)$ is non–void. Fix $p \in \Omega$. Let N' be the irreducible component of N containing q. Since $p_{t,N'}(q) \geq 1$, there are an affine Zariski open neighborhood Z of q in N', a finite family $\{D_i\}_{i \in I}$ of real algebraic curves with $\min_{i \in I} g(D_i) \geq 1$ and an injective regular map $\varphi : Z \longrightarrow \prod_{i \in I} D_i$. We may also suppose that, setting $\pi_j : \prod_{i \in I} D_i \longrightarrow D_j$ equal to the natural projection for each $j \in I$, each composition map $\pi_j \circ \varphi : Z \longrightarrow D_j$ is nonconstant. From the latter assumption and Sard's theorem, it follows the existence of a proper Zariski closed subset W of Z with the following properties: for each $y \in Z \setminus W$, there is $j \in I$ such that $\pi_j(\varphi(y))$ is a nonsingular point of D_j and is a regular values of the restriction φ_j of $\pi_j \circ \varphi$ between $(\pi_j \circ \varphi)^{-1}(\text{Nonsing}(D_j))$ and $\text{Nonsing}(D_j)$. Up to compose f with a C^∞ automorphism of N arbitrarily close to the identity on N, we may suppose that $q \in Z \setminus W$. Let $j \in I$ such that $\pi_j(\varphi(q))$ has the properties described above. Fix a 1–dimensional C^∞ submanifold L of $(\pi_j \circ \varphi)^{-1}(\text{Nonsing}(D_j))$ containing q such that φ_j sends diffeomorphically an open neighborhood V of q in L into an open subset of $\text{Nonsing}(D_j)$. Let $\nu := \max\{0, 2g(G) - 1\}$. By using an adequate C^∞ partition of unity on M and C^∞ coordinate systems on M locally at p and on N locally at q, it is easy to find an open neighborhood U of p in Ω and a map g in $C^\infty(M, N)$ arbitrarily close to f such that $g(U) \subset V$ and $g|_U : U \longrightarrow V$ is nonconstant and has ν distinct critical points of order 2. Such a map is not approximable by regular maps as one can see repeating the argument used in *Part I.3* of the proof of Theorem 2.27, b). □

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