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A not-so-expected Cartesian feature of the simple pendulum

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Abstract

This brief study suggests the pedagogical value of analysing the simple pendulum somewhat beyond the standard linear small-angle approximation. By employing video tracking techniques, we examine the Cartesian components of the pendulum's motion for relatively small amplitudes, revealing that while the horizontal displacement follows a linear harmonic motion, the vertical component exhibits a quadratic dependence on the angular displacement. This dual-frequency motion, with the vertical oscillation occurring at twice the frequency of the horizontal one, provides a tangible example for students to understand the implications of linear and quadratic approximations in physical systems. The experiment underscores the importance of considering higher-order terms in series expansions to accurately describe physical phenomena, thereby enhancing students' comprehension of approximation methods in physics.

Keywords: mechanics, pendulum, harmonic motion, Taylor series

1. Introduction

The simple pendulum has long been, and still is the subject of extensive study from multiple perspectives, particularly within educational contexts [1]. It is termed 'simple' because, being constituted by a point mass suspended from a massless and non-extensible string, it allows the direct application of Newton's second law in idealized conditions. The system is commonly introduced at the high school level because it offers a gateway to exploring approximate solutions through linearization. Otherwise, the exact equation of motion is nonlinear and solvable only through elliptic integrals or series expansions [2–4].

This short contribution highlights the pedagogical value of using the simple pendulum to emphasize the role of numerical approximations. The standard approach, adopted both in high school and introductory university mechanics courses, models the motion using a single angular coordinate θ that measures the deviation of the string from the vertical (see figure 1). Newton's second law, projected along the tangential direction of the circular trajectory, yields:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta \tag{1}$$

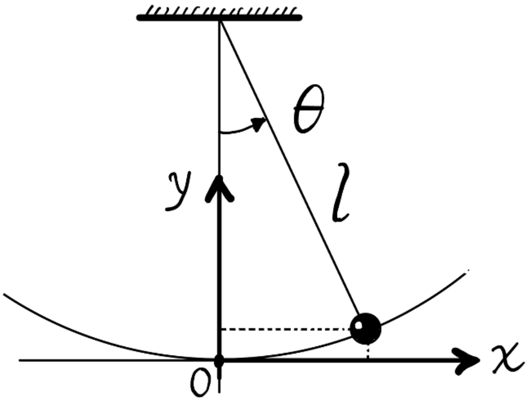

Figure 1. Angular and Cartesian pendulum's coordinates.

where g is the gravitational acceleration and l is the length of the pendulum. This equation is mathematically demanding as it has no closed-form analytical solution. It is usually linearized by substituting $\sin\theta \approx \theta$ (in radians), leading to the harmonic oscillator equation with angular frequency $\omega = \sqrt{g/l}$. Assuming the mass is released from rest at an initial angle θ_i, the solution is

$$\theta(t) = \theta_i \cos\omega t. \tag{2}$$

A straightforward experiment can be conducted by recording the motion of a real pendulum—here consisting of a (20 ± 1) mm diameter sphere suspended from a (300 ± 1) mm string—using a smartphone, possibly in slow motion to improve temporal resolution. The motion can be tracked using freely available video analysis software such as Tracker [5] to extract the angular displacement over time, allowing a direct comparison with equation (2) and as shown in figure 2. Fitting the data to equation (2) yields

$$\begin{aligned} \theta_i &= (0.242 \pm 0.002)\ \text{rad} \\ \omega &= (5.570 \pm 0.001)\ \text{rad}\,\text{s}^{-1}. \end{aligned} \tag{3}$$

The measured angular frequency agrees well with the theoretical value $\omega = \sqrt{g/l} = (5.62 \pm 0.02)\ \text{rad}\,\text{s}^{-1}$. the correction due to the finite size of the sphere is below 0.5% and thus negligible. Whether the initial angle of $\theta_i = 0.242$ rad $\cong 13.9°$ qualifies as 'small' depends on the desired precision. Since $\sin\theta_i = 0.240$ the approximation is valid to two significant digits.

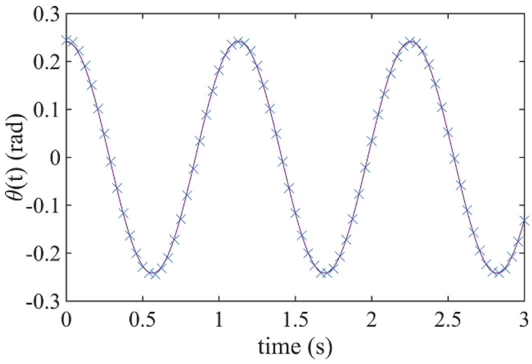

Figure 2. Angular amplitude: tracked and fitted values.

The discussion often stops here, perhaps with a brief mention of higher-order corrections for relatively large amplitude oscillations. Of course, speaking of 'small/large amplitude' has no absolute meaning and should be evaluated case by case, considering the number of significant digits involved as hinted above.

2. Cartesian harmonic motions

More insightful, however, can be the analysis of the Cartesian components of the motion. Rather than tracking the angular displacement, one can analyse the horizontal (x) and vertical (y) positions of the mass with respect to a reference frame centered at the equilibrium position (the plumb line). From the geometry:

$$x = l\sin\theta, \quad y = l(1 - \cos\theta) \tag{4}$$

and a linear approximation (within first-power of θ) yields

$$x \cong l\theta = l\theta_i \cos\omega t, \quad y \cong 0 \tag{5}$$

indicating purely horizontal motion. However, this neglects any vertical displacement, which is questionable for $\theta_i = 0.242$ rad, since $\cos\theta_i = 0.971$ implies $y = 0.03\ l$, not negligible beyond one significant digit. Students may readily notice the vertical motion and question the approximation in equation (5).

To improve the model, we consider the next order term in the Taylor expansion (θ^2) which affects only the y component (the sine function remains linear within this order):

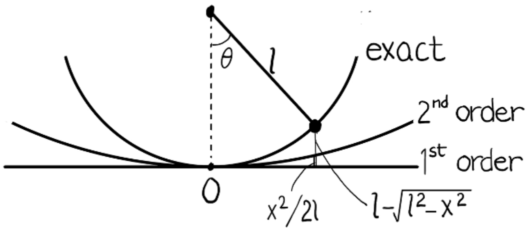

Figure 3. Comparison between orders of approximation and trajectories. The parabolic path corresponds to the second-order Taylor expansion.

$$\cos\theta \approx 1 - \frac{\theta^2}{2} \Rightarrow y \cong \frac{l\theta^2}{2} = \frac{l\theta_i{}^2}{2}\cos^2\omega t = \frac{l\theta_i^2}{4}\left[1 + \cos\left(2\omega t\right)\right]. \tag{6}$$

This reveals a vertical harmonic motion with double the frequency of the horizontal component, as expected since the pendulum completes two vertical oscillations per full horizontal cycle. The resulting trajectory, in this approximation, is not linear (as in the first-order model) nor circular (as requested by the physical constraint), but parabolic:

$$y = l\theta^2/2 = x^2/(2l). \tag{7}$$

This geometric feature can be used to foster students' understanding of approximation in physical systems. Figure 3 illustrates these paths and figure 4 shows experimental tracking data of $x(t)$ and $y(t)$, fitted to the expected functional forms

$$x(t) = A\cos(\omega_x t), \quad y(t) = B\left[1 + \cos(\omega_y t)\right] \tag{8}$$

leading to

$$\begin{array}{l} A = (73.5 \pm 0.2)\ \text{mm},\ \omega_x = (5.570 \pm 0.002)\ \text{rad}\,\text{s}^{-1} \\ B = (4.69 \pm 0.04)\ \text{mm},\ \omega_y = (11.13 \pm 0.01)\ \text{rad}\,\text{s}^{-1} \end{array}. \tag{9}$$

These values align well with theory: $A = l\theta_i$, $B = l\theta_i{}^2/4$, with $l = (300 \pm 1)$ mm gives $\theta_i = (0.25 \pm 0.003)$ rad and $\omega_y = 2\omega_x$. Notably, the ratio $\frac{B}{A} = \theta_i/4$ offers a practical diagnostic of the validity of the linear approximation.

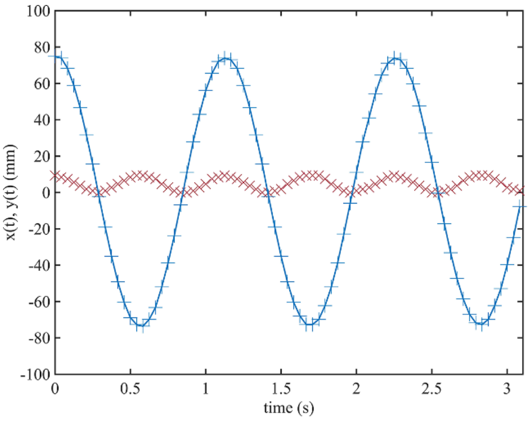

Figure 4. Cartesian coordinates: tracked (x, plus, y, cross) and fitted values.

3. Conclusions

This analysis demonstrates that a simple video-based analysis of a pendulum allows students to explore the basic implications of linear and quadratic approximations in the motion of a pendulum. This provides a didactic entry point into understanding how different components of motion respond to the order of approximation, and why harmonic solutions of frequency ratio 1:2 naturally emerge when series expansions are truncated at the second order. Further development could include higher-order terms, nonlinear frequency corrections, the emergence of odd/even harmonics, the tension in the string as a function of the angle or energy conservation, opening a rich landscape for deeper investigation.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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[5] Open Source Physics Tracker Video Analysis and Modeling Tool (available at: https://opensourcephysics.github.io/tracker-website/)

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