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Volterra integro-differential equations with accretive operators and non-autonomous perturbations

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1 Introduction

The type of Volterra equations studied in this paper is the non-linear evolution equation

$$\begin{cases} \dfrac{d}{dt}\left(k_0(u(t)-x) + \displaystyle\int_0^t k_1(t-s)(u(s)-x)\,ds\right) \\ \qquad\qquad\qquad\qquad\qquad +G(u(t)) = F(t,u(t)), \\ t \in (0,\infty), \qquad u(0+) = x, \end{cases} \tag{1.1}$$

in a real Banach space X. Here, k_0 is a constant and k_1 is a real, non-negative function, that satisfy Hypothesis 1a below, G is an accretive operator in X, see Hypothesis 1b, and we shall consider the operator $F(t,u)$ as a non-linear, non-autonomous perturbation of the operator G, see Hypothesis 1c for details.

Since the early 1970s, the case where $F(t,u) = f(t)$ have been under consideration; this problem has an interest also in our setting, and it shall be further discussed in Section 2.1. The next step in the literature were to consider functional perturbations of such problem, compare Crandall & Nohel (1978) or Gripenberg (1985).

In this paper, on the contrary, we consider perturbation operators acting on X, but we can allow such operators to be non-autonomous. The study of (1.1) with the operator $F(t,u)$ is based on the results for the inhomogeneous problem $F = f(t)$ and a fixed point argument; this should justify the appellative of "perturbation term" given to $F(t,u)$.

In order to state the main result of the paper, we shall introduce the main assumptions on the coefficients of (1.1). A comprehensive explanation of the notation employed in the paper will be given in Section 3.

Hypothesis 1. *The kernel $k(t) = k_0 + \int_0^t k_1(s)\,ds$ is a Bernstein function associated to a kernel $a(t)$,*

1a *the kernel $a : (0,\infty) \to \mathbb{R}$ is completely monotone, $a \in L^1_{loc}(0,\infty)$, and the relation between $a(t)$ and $k(t)$ is given by*

$$k_0 a(t) + \int_0^t k_1(t-s) a(s)\,ds = 1, \qquad t \in (0,\infty);$$

1b *$G(x)$ is an operator in X, with domain $D(G) \subset X$, of type ω: G belongs to $\widetilde{\Lambda}_{mc}(X)$,* i.e., *for some $\omega \geq 0$, $G + \omega I$ is m-accretive in X,*

1c *the perturbation term F maps $C(\mathbb{R}_+; X)$ into $C(\mathbb{R}_+; X)$, it is uniformly continuous on bounded sets of $\mathbb{R}_+ \times X$ and for each $t \geq 0$ $F(t,\cdot)$ is m-dissipative on X.*

Our main result provides the existence of a generalized solution for problem (1.1).

Theorem 1.1. *Assume X is a real Banach space and let the coefficients in* (1.1) *satisfy Hypothesis 1. Then, for any $x \in \widehat{D}(G)$, there exists a unique generalized solution to the abstract non-linear Volterra equation* (1.1).

The paper is organized as follows. In Section 2, we shall discuss how the results provided here are related with those already known in literature. Our notation, and some preliminary result about the coefficients of (1.1), are given in Section 3. In particular, in Section 3.4 we discuss some properties of the linear Volterra operator

$$Lu(t) = \frac{d}{dt}\left(k_0 u(t) + \int_0^t k_1(t-s) u(s)\,ds\right).$$

Although there exists a large literature about this subject, we obtain a representation of the Yosida approximations $L_\mu = L\left(I + \frac{1}{\mu}L\right)^{-1}$ which seems to be new and may deepen the understanding of the relation with the associated completely monotone kernel. Finally, the remaining sections are devoted to study (1.1), first in the case $F(t,x) = f(t)$, then in the case of a Lipschitz non-linearity, and the last section provides the proof of Theorem 1.1.

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2 Nonlinear equations with accretive operators

The equation that we consider in this paper is a non-autonomous perturbation of the inhomogeneous problem

$$\begin{cases} \dfrac{d}{dt}\left(k_0(u(t)-x)+\displaystyle\int_0^t k_1(t-s)(u(s)-x)\,ds\right)+G(u(t))=f(t), \\ t\in(0,\infty), \qquad u(0+)=x. \end{cases} \tag{2.1}$$

There is a wide literature concerning such equations, also due to their relevance in applications. Actually, Volterra integro-differential equations of convolution type with completely monotone kernel arise naturally in several fields, as heat conduction in materials with memory and in the theory of thermo-viscoelasticity: see for instance the monograph of Prüss (1993) and the references therein.

In this section, we shall discuss further the results obtained in this paper and compare them with the existing literature.

2.1 The case of a perturbation independent of u

We start by considering the simpler case where the perturbation in the right-hand side of (1.1) is independent of u. This case shall provide us with the estimates that we need in order to study the general case of equation (1.1), compare also Gripenberg *et al.* (2000). Therefore, in this section we are concerned with the equation (2.1)

In order to define a generalized solution to (2.1), we shall consider an approximate equation, where the operator L is replaced by its Yosida approximation $L_\mu = L(I+\frac{1}{\mu}L)^{-1}$, $\mu>0$. Let u_μ be the solution of the following equation

$$L_\mu[u_\mu(\cdot)-x](t)+G(u_\mu(t))=f(t), \qquad t\in(0,\infty). \tag{2.2}$$

In the next theorem, we establish the existence of a generalized solution of (2.1).

Theorem 2.1. *Assume that the coefficients in (2.1) satisfy Hypotheses 1a-1b and let $x\in\overline{D(G)}$ and $f\in C(\mathbb{R}_+;X)$. Then, for every $\mu>0$ equation (2.2) has a unique solution $u_\mu(\cdot)\in C(\mathbb{R}_+;X)$.*

As $\mu\to\infty$, there exists a function $u=U(x,f)$ with $u\in L^1_{loc}(\mathbb{R}_+;X)$ such that $u_\mu\to u$ in $L^1_{loc}(\mathbb{R}_+;X)$.

If $x\in\widehat{D}(G)$ then the convergence takes place also in $L^\infty_{loc}(\mathbb{R}_+;X)$ and the limit function u belongs to $C(\mathbb{R}_+;X)$.

The function $u=U(x,f)$, that exists according to Theorem 2.1, is said the generalized solution for problem (2.1). The main ideas in the proof of the theorem, see also the estimates in Theorem 4.7, are almost the same to those introduced in Gripenberg (1985), Theorem 1. For the reader's convenience, we exploit in Section 4 all the details of the proofs.

Let us discuss briefly our setting as compared to that of Gripenberg (1985). The results in that paper distinguish the cases $k_0 = 0$ and $k_0 \neq 0$. In the latter case, the quoted result fully describes the case $\omega = 0$ (ω is the type of the operator G). In general, however, we may write $G(u) = \widetilde{G}(u) - \omega u$, $\widetilde{G}$ is a m-accretive operator of negative type, and ωu is a linear perturbation, so that this case may be as well treated by means of Theorem 3 of that paper.

In section 4, for reader convenience, we shall discuss the case $k_0 = 0$ in full details. Here, actually, the results in Gripenberg (1985) does not suffice and a refinement of the estimates for the solution is necessary. We collect in Theorem 4.7 the relevant estimates that we obtain in our setting. In case $k_0 = 0$ and G an m-accretive operator on X, similar results were already proved in Cockburn *et al.* (1996), see also formula (4.16) here.

Remark 2.1. Using the estimates in Cockburn *et al.* (1996), Gripenberg *et al.* (2000) solved the problem of existence of a strong solution for (2.1). In our setting, the extension of this result does not seem straighforward, since one of the relevant estimates failed to be proved with our techniques: see Remark 4.2 for more details. We hope to return on this problem in a subsequent paper.

2.2 The case of a Lipschitz perturbation

Now we return to the original equation (1.1). Before we discuss the case of dissipative non-linearities, that is the object of Theorem 1.1, we shall consider the case of a Lipschitz non-linearity. We shall say that $u(t)$ is a generalized solution of (1.1) if $u = U(x, F(\cdot, u))$.

Theorem 2.2. *Let the assumptions of Theorem 2.1 be fulfilled and assume that the non linear term $F(t, \cdot)$ satisfies*

$$F : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X) \tag{2.3}$$

and there exists a function $\eta(t) \in L^\infty_{loc}(\mathbb{R}_+)$ such that, for any $t \in \mathbb{R}_+$

$$\|F(t, u) - F(t, v)\| \leq \eta(t) \|u - v\|. \tag{2.4}$$

Then there exists a unique generalized solution to equation (1.1)

$$\begin{cases} L[u(\cdot) - x](t) + G(u(t)) = F(t, u(t)), \\ t \in (0, \infty), \qquad u(0+) = x. \end{cases}$$

As we mentioned before, Theorem 3 in Gripenberg (1985) is concerned with the existence of a generalized solution to (1.1). As in previous section, the cases $k_0 = 0$ and $k_0 \neq 0$ are treated separately and, again, the second case, $k_0 > 0$, is fully described there. Instead, in case $k_0 = 0$, the Lipschitz perturbation term in Theorem 2.2 is not contained in the assumption of Theorem 3 in Gripenberg (1985), that is

$$\|F(v_1) - F(v_2)\|_{L^1(0,t;X)} \leq \int_0^t \eta(s) \|v_1 - v_2\|_{L^1(0,s;X)}\, ds, \quad t \in \mathbb{R}_+.$$

2.3 The case of a non-autonomous dissipative perturbation

In the last section we finish the proof of the main result stated in Theorem 1.1. We are concerned here with the case of a continuous and m-dissipative operator $F(t,u)$, see Hypothesis 1c. Since this term is non-autonomous it is non possible to include it into G and to apply the previous theorems, also if we suppose that $G + F$ is m-dissipative.

The techniques applied in this part, although very different from those employed in the previous sections, are usually applied in the theory of dissipative systems; in particular, we refer to the proof of Theorem 7.13 of Da Prato & Zabczyk (1992).

3 Notation and preliminary results

We shall denote the norm in the Banach space X by $\|\cdot\|$.

3.1 Properties of accretive operators

For the sake of completeness, we recall the main properties of accretive operators from the book of Da Prato (1976).

We denote $\Lambda_{mc}(X)$ the space of accretive operators[1] on X such that their resolvent contains $\mathbb{R}_+$; such operator are said m-accretive.

We also denote $\widetilde{\Lambda}_{mc}(X)$ the space of operators G on X such that $G + \omega I$ belongs to $\Lambda_{mc}(X)$ for a suitable real number ω. If $G \in \widetilde{\Lambda}_{mc}(X)$ we set $\omega_G = \inf\{\omega \in \mathbb{R} : G + \omega I \in \Lambda_{mc}(X)\}$; then we denote ω_G the type of G; if $\omega_G < 0$ we say that G is of negative type.

As stated in the introduction, we assume that the operator G belongs to $\widetilde{\Lambda}_{mc}(X)$ and we denote $\omega = \omega_G \geq 0$ the type of G. If G is of negative type then we choose $\omega = 0$.

The resolvent operator J_μ, associated with $\widetilde{G} = G + \omega I$, is defined by

$$J_\mu = \left(I + \tfrac{1}{\mu}\widetilde{G}(\cdot)\right)^{-1}.$$

[1]An operator G on X is said to be accretive if for any $x, y \in D(G)$ then $\|x - y\| \leq \|x - y + \lambda(G(x) - G(y))\|$, for all $\lambda > 0$; on the other hand, an operator F on X is said to be dissipative if $-F$ is accretive.

We have that J_μ satifies the following properties:

$$\|J_\mu(x) - J_\mu(y)\| \le \|x - y\|, \quad \forall x, y \in X,$$

and

$$\lim_{\mu \to \infty} J_\mu(x) = x, \quad \forall x \in \overline{D(G)}.$$

We also introduce the Yosida approssimations G_α, $\alpha > 0$, of $\widetilde{G}$ by setting

$$G_\alpha(x) = \widetilde{G}(J_\alpha(x)) = \alpha(x - J_\alpha(x)), \quad x \in X.$$

We remark that $G_\alpha(x)$ is a Lipschitz continuous mapping and it holds that $\|G_\alpha(x)\| \le \|\widetilde{G}(x)\|$ for any $x \in D(G)$.

We denote $\widehat{D}(G)$ the set $\{x \in X \ : \ \sup_{\alpha>0} \|G_\alpha(x)\| < +\infty\}$; we have $D(G) \subseteq \widehat{D}(G) \subseteq \overline{D(G)}$. If X is not reflexive, then it is possible that $D(G) \subsetneq \widehat{D}(G)$.

3.2 Properties of the scalar kernel

A function $f : (0,\infty) \to \mathbb{R}$ is called *completely monotonic* if f belongs to $C^\infty(0,\infty)$ and

$$(-1)^n \frac{d^n}{dx^n} f(x) \ge 0, \qquad x > 0, \quad n = 1, 2, \dots.$$

Below we list some properties of completely monotonic functions.

Remark 3.1. Assume that $f : (0,\infty) \to \mathbb{R}$ is completely monotonic; then

i. if $f(x_0) = 0$ for some $x_0 > 0$ then $f(x)$ is identically zero;

ii. the function f has an analytic extension to $\{z \in \mathbb{C} : \Re(z) > 0\}$;

iii. if $f(0+) = +\infty$ then $(-1)^n \frac{d^n}{dx^n} f(0+) = +\infty$ for $n = 1, 2, \dots$;

iv. $(-1)^n \frac{d^n}{dx^n} f(+\infty) = 0$ for $n = 1, 2, \dots$.

For an exhaustive introduction to completely monotonic functions, as well as a proof of these properties, we refer to Gripenberg *et al.* (1990) and Prüss (1993), compare also the Introduction in Homan (2003).

Definition 3.1. *A* Bernstein function $k(t)$ *is a* C^∞ *function* $k : (0,\infty) \to \mathbb{R}$ *such that* $k(t) \ge 0$ *for* $t > 0$ *and* $k'(t)$ *is completely monotonic.*

Proposition 3.2. *(Prüss (1993)) If* $k(t)$ *is a Bernstein function, then there exists a unique completely monotonic function* $a(t)$ *such that*

$$k_0 a(t) + \int_0^t k_1(t-s) a(s)\, ds = 1, \qquad t \in (0,\infty). \tag{3.1}$$

If $k_0 = 0$ *then* $a(0+) = +\infty$.

We consider in the following table some examples of Bernstein functions $k(t)$ and corresponding completely monotonic functions.

$k(t)$	$a(t)$
1	1
$1+t$	e^{-t}
$\int_0^t E_1(s)\,ds$	$\int_0^\infty e^{-t} t^{\rho-1} \frac{d\rho}{\Gamma(\rho)}$
$\int_0^t \frac{s^{-\alpha}}{\Gamma(1-\alpha)},\ ds$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$

Here, $\alpha \in (0,1)$ and $E_1(x) = \int_x^\infty e^{-t} \frac{dt}{t}$.

Let us denote the family of functions $s_\mu(t)$, $t \geq 0$, $\mu \in \mathbb{R}$, where $s_\mu(t)$ is the solution of the scalar Volterra equation

$$s_\mu(t) + \mu \int_0^t s_\mu(t-\vartheta) a(\vartheta)\, d\vartheta = 1, \qquad t > 0. \tag{3.2}$$

Under Hypothesis 1a, it follows that $s_\mu(t)$ is positive and nonincreasing with respect to $t > 0$, for every $\mu > 0$.

The following table contains examples of scalar resolvent functions for various completely monotonic functions.

$a(t)$	$s(t;\mu)$
1	$e^{-\mu t}$
e^{-t}	$(1+\mu)^{-1}[1+\mu e^{-(1+\mu)t}]$
$\int_0^\infty e^{-t} t^{\rho-1} \frac{d\rho}{\Gamma(\rho)}$	$1 - \int_0^\infty \mu e^{-\mu\rho} \left[\int_0^t e^{-\tau} \tau^{\rho-1}\, d\tau\right] \frac{d\rho}{\Gamma(\rho)}$
$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$	$\mathcal{E}_\alpha(\mu t^\alpha)$

Here, $\mathcal{E}_\alpha(x) = \sum_{k=0}^\infty \frac{(-x)^k}{\Gamma(\alpha k+1)}$ is known as Mittag-Leffler's function; as before, $\alpha \in (0,1)$.

Proposition 3.3. *For any* $\mu \in \mathbb{R}$*:*

$$\frac{d}{d\mu} s_\mu(t) \leq 0 \qquad \text{for all } t > 0.$$

Proof. For a proof we refer to Prüss (1993), page 98, noticing that the case $\mu < 0$ can be treated similarly to the case $\mu > 0$. □

Notice that the above proposition not only implies that $s_\mu(t) \leq 1$ for any $\mu > 0$, but also that $s_\mu(t) \geq 1$ for any $\mu < 0$.

Let us denote $r_\mu(\cdot)$ the solution to the integral equation

$$r_\mu(t) + \mu \int_0^t r_\mu(t-s) a(s)\, ds = \mu a(t). \tag{3.3}$$

By Lemma 4.1 in Prüss (1993), since $a(t)$ is completely monotonic, we know that for any $\mu > 0$, $r_\mu(t)$ belongs to $L^1(\mathbb{R}_+) \cap C(0,\infty)$, it is completely monotonic, $0 \le r_\mu(t) \le \mu a(t)$ and

$$\int_0^\infty r_\mu(s)\,ds = \widehat{r}_\mu(0) = \frac{\mu \widehat{a}(0)}{1 + \mu \widehat{a}(0)} \le 1.$$

Moreover, if $\mu < 0$, then $r_\mu(t)$ belongs to $L^1_{loc}(\mathbb{R}_+) \cap C(0,\infty)$ and $r_\mu(t) \le \mu a(t) < 0$, compare also Friedman (1963).

The relation between $s_\mu(t)$ and $r_\mu(t)$ is clarified in the following statement.

Proposition 3.4. *It holds that*

$$s_\mu(t) = \left(1 - \int_0^t r_\mu(\tau)\,d\tau\right), \qquad t > 0. \tag{3.4}$$

We shall resume, in the next proposition, some results about the limit behaviour of $r_\mu(\cdot)$ and $s_\mu(\cdot)$ as $\mu \to \infty$.

Proposition 3.5. *The following relation holds between $s_\mu(t)$ and the function $k(t)$:*

$$\mu s_\mu(t) = (r_\mu * k_1)(t) + k_0 r_\mu(t). \tag{3.5}$$

Moreover,

$$\mu \int_0^t s_\mu(\tau)\,d\tau \to k(t)$$

for a.e. $t > 0$.

Proof. Let us briefly sketch the idea of the proof. Taking convolution in (3.3) with $k_1(t)$, recalling that from (3.1) it follows that $k_0 a(t) + (k_1 * a)(t) = 1$, we obtain

$$(r_\mu * k_1)(t) + \mu\big((r_\mu * k_1) * a\big)(t) = \mu(a * k_1)(t) = \mu(1 - k_0 a(t)).$$

On the other hand, again from (3.3) and (3.2) it follows that

$$(\mu s_\mu - k_0 r_\mu)(t) + \mu\big((\mu s_\mu - k_0 r_\mu) * a\big)(t) = \mu(1 - k_0 a(t)),$$

and comparing this expression with the previous one we prove (3.5).

The Laplace transform of $\int_0^t r_\mu(s)\,ds$ verifies

$$\left[\widehat{\int r_\mu(s)\,ds}\right](\lambda) = \frac{1}{\lambda}\,\frac{\mu \widehat{a}(\lambda)}{1 + \mu \widehat{a}(\lambda)} \to \frac{1}{\lambda}$$

as $\mu \to \infty$, hence

$$\int_0^t r_\infty(s)\,ds = 1 \qquad \text{for a.e. } t > 0. \tag{3.6}$$

Integrating both sides of (3.5) in $(0,t)$, using the limit in (3.6), yields the thesis. □

3.3 A Gronwall-type lemma

In this section, we shall discuss a Gronwall-type lemma, that will allow us to prove estimates for the solution of a Volterra equation. We shall state two results, the first more general, while the second is best adapted to our needs.

Lemma 3.6. *Let $v(t)$ be a continuous, nonnegative function, which satisfies the estimate*

$$v(t) \le g(t) + \gamma(a * v)(t) \tag{3.7}$$

where $\gamma > 0$, $g(t) \in L^1_{loc}(\mathbb{R}_+)$ and $a(t)$ satisfies Hypothesis 1a. Then

$$v(t) \le \frac{d}{dt}(g * s_{-\gamma})(t) = g(t) - (g * r_{-\gamma})(t), \tag{3.8}$$

where $s_{-\gamma}(t)$ and $r_{-\gamma}(t)$ are solutions to (3.2) and (3.3), respectively.

The proof of this and the other results in this section are postponed to the Appendix. Let us now get to a more particular case.

Lemma 3.7. *Let $v(t)$ be a continuous, non negative function which satisfies the estimate*

$$v(t) \le s_\lambda(t)x + \tfrac{1}{\lambda}f(t) + \tfrac{\omega}{\lambda}v(t) + r_\lambda * v(t), \tag{3.9}$$

where $s_\lambda(t)$ and $r_\lambda(t)$ are defined in (3.2) and (3.3) respectively. Then

$$v(t) \le \frac{d}{dt}\left(\tfrac{\omega_\lambda}{\omega}(x + \tfrac{1}{\lambda}f + a * f) * s_{-\omega_\lambda}\right)(t), \tag{3.10}$$

where $s_{-\omega_\lambda}(t)$ is defined as in (3.2) with $\omega_\lambda = \frac{\lambda\omega}{\lambda-\omega}$.

Remark 3.2. In case $f \equiv 0$ we obtain from the above lemma the following estimate:

$$v(t) \le \tfrac{\omega_\lambda}{\omega} x\, s_{-\omega_\lambda}(t). \tag{3.11}$$

If we consider, instead, the case $\omega = 0$, then estimate (3.10) becomes

$$v(t) \le x + \tfrac{1}{\lambda}f(t) + (a * f)(t). \tag{3.12}$$

3.4 Volterra operators

In this section we shall discuss some properties of the linear Volterra operator

$$Lu(t) = \frac{d}{dt}\left[k_0 u(t) + \int_0^t k_1(t-s)u(s)\,ds\right], \qquad t > 0. \tag{3.13}$$

The operator L is m-accretive in $L^p(\mathbb{R}_+; X)$, for any $p \ge 1$, see Clément & Nohel (1981). There is a natural representation of its inverse operator L^{-1} in terms of the kernel $a(t)$.

Lemma 3.8. *Given the operator L defined in (3.13), the operator L^{-1} is defined by*

$$L^{-1}v(t) = \int_0^t a(t-s)v(s)\,ds. \tag{3.14}$$

Proof. Let us prove one implication, say that $L(a*v)(t) = v(t)$, the other being similar. We start from (3.1), taking the convolution of both sides with $v(t)$, to get

$$k_0(a*v)(t) + (k_1 * a * v)(t) = (1*v)(t). \tag{3.15}$$

Next, observe that the definition of L implies

$$L(a*v)(t) = \frac{d}{dt}\left[k_0(a*v)(t) + (k_1*(a*v))(t)\right];$$

if we substitute what we have found in (3.15), and use the identity $\frac{d}{dt}(1*f)(t) = f(t)$, we obtain the thesis. □

We now proceed to analyze the operator $L_\mu = L(I + \frac{1}{\mu}L)^{-1}$.

Lemma 3.9. *The operator $L_\mu = L(I+\frac{1}{\mu}L)^{-1}$ is given by*

$$L_\mu v(t) = \mu\left(v(t) - \int_0^t v(t-s)r_\mu(s)\,ds\right), \tag{3.16}$$

where $r_\mu(t)$ is a solution to (3.3).

Proof. Let $y = L_\mu v$; then

$$(I+\tfrac{1}{\mu}L)L^{-1}y = v \quad\Longrightarrow\quad L^{-1}y + \tfrac{1}{\mu}y = v \quad\Longrightarrow\quad a*y + \tfrac{1}{\mu}y = v.$$

If we take convolution with r_μ, recalling (3.3), we get

$$a*y = r_\mu * v \qquad\Longrightarrow\qquad \mu(r_\mu * v) + y = \mu v.$$

□

Remark 3.3. We shall use (3.16) in this equivalent form:

$$L_\mu v(t) = \mu\frac{d}{dt}(v(\cdot) * s_\mu(\cdot))(t). \tag{3.17}$$

3.5 Some estimates on convolution operators

Let α be a positive real number, $\alpha \in (0,1)$, and $a(t)$ be a completely monotonic function on $\mathbb{R}_+$ and $a(t) \in L^1_{loc}(0,\infty)$. We define a measure $\rho([0,s]) = \alpha + \int_0^s a(\sigma)\,d\sigma$. The following lemmas treat the estimates on the convolution powers of $a(t)$ and ρ, respectively.

Lemma 3.10. *Let $a(t)$ satisfies Hypothesis 1a; then, for each $T > 0$ and for any constant $C > 0$,*

$$C^n \|a^{*n}\|_{L^1(0,T)} \to 0.$$

More precisely, we have

$$\sum_{n=0}^{\infty} C^n \|a^{*n}\|_{L^1(0,T)} < \infty. \tag{3.18}$$

Proof. Let $C > 0$ be fixed, and define the operator $\mathcal{A} : L^1(0,T) \to L^1(0,T)$ as

$$\mathcal{A}v(t) = C(a * v)(t), \qquad t \in (0,T).$$

$\mathcal{A}$ is a linear bounded operator from $L^1(0,T)$ into itself. We claim that the spectral radius $\sigma(\mathcal{A})$ is 0. Then it will follow, from the formula

$$\sigma(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|^{\frac{1}{n}}_{\mathcal{L}(L^1(0,T))}$$

(here $\|\cdot\|_{\mathcal{L}(L^1(0,T))}$ is the norm of operators on $L^1(0,T)$), that

$$C\|a^{*n}\|^{\frac{1}{n}}_{L^1(0,T)} \le \|\mathcal{A}^n\|^{\frac{1}{n}}_{\mathcal{L}(L^1(0,T))} \to 0.$$

In particular, from the *root test* for the convergence of a series we have

$$\sum_{n=0}^{\infty} C^n \|a^{*n}\|_{L^1(0,T)} < \infty,$$

and so it holds that

$$C^n \|a^{*n}\|_{L^1(0,T)} \to 0.$$

It remains to show that $\sigma(\mathcal{A}) = 0$. From the definition of spectral radius it is sufficient to show that for any $\alpha > 0$ and any function $u \in L^1(0,T)$ the following problem has a solution $v \in L^1(0,T)$:

$$u(t) = C(a * v)(t) + C\alpha v(t).$$

But, since $a(t)$ is a completely monotonic kernel, we have

$$v(t) = \tfrac{1}{\alpha C}\big(u(t) - (r_{1/\alpha} * u)(t)\big),$$

and this shows the lemma. □

Next, we state a useful generalization of the previous lemma.

Lemma 3.11. *Let ρ a completely positive measure on $\mathbb{R}$, defined by*

$$\rho([0,t]) = \alpha + \int_0^t a(s)\,ds,$$

where $\alpha \in (0,1)$ and a satisfies Hypothesis 1a. Let us define

$$\rho^{*i}([0,t]) = \int_0^t \rho([0,t-\sigma])\,\rho^{*(i-1)}(d\sigma).$$

Then we have that

$$\sum_{n=0}^{\infty} \rho^{*i}([0,t]) < +\infty.$$

Proof. By direct calculations it follows that

$$\rho^{*n}([0,t]) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \|a^{*k}\|_{L^1(0,t)},$$

so we have

$$\begin{aligned}\sum_{n=0}^{\infty} \rho^{*n}([0,t]) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \|a^{*k}\|_{L^1(0,t)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \|a^{*k}\|_{L^1(0,t)} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\alpha^{n-k}.\end{aligned}$$

Now

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\alpha^{n-k} = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)\alpha^n.$$

But

$$(n+k)(n+k-1)\cdots(n+1)\alpha^n = \frac{d^k}{d\alpha^k}\alpha^{n+k},$$

then

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\alpha^{n-k} = \frac{d^k}{d\alpha^k}\sum_{n=0}^{\infty}\alpha^{n+k} = \frac{d^k}{d\alpha^k}\frac{\alpha^k}{1-\alpha}.$$

Since

$$\frac{\alpha^k}{1-\alpha} = \frac{1}{1-\alpha} - (1+\alpha+\cdots+\alpha^{k-1}),$$

we have

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\alpha^{n-k} = \frac{d^k}{d\alpha^k}\frac{1}{1-\alpha} = \frac{k!}{(1-\alpha)^{k+1}}.$$

Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty} \rho^{*n}([0,t]) &= \sum_{k=0}^{\infty} \frac{1}{k!}\|a^{*k}\|_{L^1(0,t)} \frac{k!}{(1-\alpha)^{k+1}} \\ &= \frac{1}{1-\alpha}\sum_{k=0}^{\infty} \frac{\|a^{*k}\|_{L^1(0,t)}}{(1-\alpha)^k} < \infty,\end{aligned}$$

where the last series converges thanks to estimate (3.18) in Lemma 3.10. □

4 Construction of the approximate solution

In this section, we shall prove the results stated in Theorem 2.1. As explained in section 2.1, we shall only be concerned with the case $k_0 = 0$. We first consider the approximate equation:

$$L_\mu(u_\mu(\cdot) - x)(t) + G(u_\mu(t)) = f(t), \qquad t > 0. \tag{4.1}$$

Applying J_μ to both sides of (4.1), we get that this is equivalent to the following

$$u_\mu(t) = J_\mu\left(\tfrac{\omega}{\mu}u_\mu(t) + \tfrac{1}{\mu}f(t) + s_\mu(t)x + \int_0^t u_\mu(t-s)r_\mu(s)\,ds\right). \tag{4.2}$$

Lemma 4.1. *Let $\mu > \omega$; then for each $T > 0$ there exists a unique solution u_μ to (4.1) in $C([0,T];X)$.*

Proof. For fixed $f \in C(\mathbb{R}_+;X)$ and $x \in X$, we define the mapping

$$\mathcal{K}(v)(t) = J_\mu\left(\tfrac{\omega}{\mu}v(t) + \tfrac{1}{\mu}f(t) + s_\mu(t)x + \int_0^t v(t-s)r_\mu(s)\,ds\right), \qquad t > 0.$$

It is easy to show that $\mathcal{K}$ maps $C([0,T];X)$ into itself; moreover, we can bound the norm of $\mathcal{K}$ by

$$\|\mathcal{K}(v_2) - \mathcal{K}(v_1)\|(t) \le \tfrac{\omega}{\mu}\|(v_2 - v_1)(t)\| + (r_\mu * \|v_2 - v_1\|)(t)$$

(recall that J_μ is nonexpansive). Let us introduce the measure ρ on $\mathbb{R}$ by

$$\rho([0,t]) = \tfrac{\omega}{\mu} + \int_0^t r_\mu(s)\,ds.$$

Then ρ is a completely positive measure; moreover

$$\|\mathcal{K}^i(v_2) - \mathcal{K}^i(v_1)\|_{L^\infty(0,T)} \le \|v_2 - v_1\|_{L^\infty(0,T)} \rho^{*i}([0,T]),$$

where $\rho^{*i}([0,t]) = \int_0^t \rho^{*(i-1)}([0,t-s])\,\rho(ds)$.

It holds that

$$\rho^{*n}([0,t]) = \sum_{k=0}^{n} \binom{n}{k} \left(\tfrac{\omega}{\mu}\right)^{n-k} \|r_\mu^{*k}\|_{L^1(0,T)},$$

and from Lemma 3.11 this goes to zero. □

Let us denote $U(x,f,\mu)$ the solution to (4.1) constructed in Lemma 4.1. Before we establish the convergence of $U(x,f,\mu)$, we proceed to study *a priori* estimates.

Lemma 4.2. *Let $u_1 = U(x_1,f_1,\mu)$ and $u_2 = U(x_2,f_2,\mu)$ be two solutions to (4.1); then it holds that*

$$\|u_2(t) - u_1(t)\| \le \tfrac{\omega_\mu}{\omega} \|x_2 - x_1\| s_{-\omega_\mu}(t) + \tfrac{\omega_\mu}{\omega} \frac{d}{dt} \Big(\big(\tfrac{1}{\mu} \|f_2(\cdot) - f_1(\cdot)\| + (a * \|f_2(\cdot) - f_1(\cdot)\|) \big) * s_{-\omega_\mu} \Big)(t). \quad (4.3)$$

Proof. It follows from (4.2) and the fact that J_μ is nonexpansive that

$$\|u_2(t) - u_1(t)\| \le s_\mu(t)\|x_2 - x_1\| + \tfrac{1}{\mu}\|f_2(t) - f_1(t)\| + \tfrac{\omega}{\mu}\|u_2(t) - u_1(t)\| + \big(r_\mu * \|u_2(\cdot) - u_1(\cdot)\|\big)(t),$$

so we have from Lemma 3.7 that for every $\mu > \omega$

$$\|u_2(t) - u_1(t)\| \le \tfrac{\omega_\mu}{\omega} \|x_2 - x_1\| s_{-\omega_\mu}(t) + \tfrac{\omega_\mu}{\omega} \frac{d}{dt} \Big(\big(\tfrac{1}{\mu} \|f_2(\cdot) - f_1(\cdot)\| + (a * \|f_2(\cdot) - f_1(\cdot)\|) \big) * s_{-\omega_\mu} \Big)(t)$$

where $s_{-\omega_\mu}(t)$ is defined in (3.2) with $\omega_\mu = \frac{\mu\omega}{\mu-\omega}$. □

Lemma 4.3. *Assume further that*

$$f(t) \in BV_{loc}(\mathbb{R}_+; X), \quad (4.4)$$

and let the assumptions of Theorem 2.1 be satisfied. Then the solution $u_\mu(\cdot)$ belongs to $BV_{loc}(\mathbb{R}_+)$ and it holds that

$$\begin{aligned} \operatorname{var}(\|u_\mu(\cdot) - x\|; [t_1,t_2]) &\le \big(\|x\| + \tfrac{1}{\omega}\|G_\mu(x)\| + \tfrac{1}{\omega}\|f(0+)\|\big)(s_{-\omega_\mu}(t_2) - s_{-\omega_\mu}(t_1)) \\ &\quad - \tfrac{1}{\omega} \int_0^T r_{-\omega_\mu}(s) \operatorname{var}(f; [\max\{0, t_1 - s\}, t_2 - s])\, ds \\ &\quad + \tfrac{1}{\mu} \operatorname{var}(f; [t_1,t_2]). \end{aligned} \quad (4.5)$$

Proof. For any $h > 0$, from (4.2) it holds that

$$
\begin{aligned}
u_\mu(t+h) - u_\mu(t) = J_\mu\Bigg(& s_\mu(t+h)x + \tfrac{\omega}{\mu} u_\mu(t+h) \\
& + \tfrac{1}{\mu} f(t+h) + \int_0^{t+h} u_\mu(t+h-\tau) r_\mu(\tau)\, d\tau \Bigg) \\
- J_\mu\Bigg(& s_\mu(t)x + \tfrac{\omega}{\mu} u_\mu(t) \\
& + \tfrac{1}{\mu} f(t) + \int_0^{t} u_\mu(t-\tau) r_\mu(\tau)\, d\tau \Bigg)
\end{aligned}
$$

Taking the norm, since J_μ is nonexpansive, we get

$$
\begin{aligned}
\|u_\mu(t+h) - u_\mu(t)\| \le & \tfrac{\omega}{\mu} \|u_\mu(t+h) - u_\mu(t)\| + \tfrac{1}{\mu} \|f(t+h) - f(t)\| \\
& + \int_0^t \|u_\mu(t+h-\tau) - u_\mu(t-\tau)\| r_\mu(\tau)\, d\tau \\
& + \int_t^{t+h} \|u_\mu(t+h-\tau) - x\| r_\mu(\tau)\, d\tau.
\end{aligned}
$$

Thanks to Lemma 3.7 we obtain the estimate

$$
\|u_\mu(t+h) - u_\mu(t)\| \le \tfrac{1}{\mu} q(\mu,h,t) + (q(\mu,h,\cdot) * a)(t), \tag{4.6}
$$

where we set

$$
\begin{aligned}
q(\mu,h,t) = & \mu \int_t^{t+h} \|u_\mu(t+h-\tau) - x\| r_\mu(\tau)\, d\tau + \|f(t+h) - f(t)\| \\
& - \mu \int_0^t \left(\int_s^{s+h} \|u_\mu(s+h-\tau) - x\| r_\mu(\tau)\, d\tau \right) r_{-\omega_\mu}(t-s)\, ds \\
& - \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds \\
= & \mu \int_t^{t+h} \|u_\mu(t+h-\tau) - x\| \left(r_\mu(\tau) - \int_0^t r_\mu(\tau - s) r_{-\omega_\mu}(s)\, ds \right) d\tau \\
& + \|f(t+h) - f(t)\| - \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds.
\end{aligned}
$$

Let us consider next the convolution term which appears in (4.6):

$$
\begin{aligned}
(q(\mu,h,\cdot) * a)(t) = & \int_0^t d\vartheta \Bigg[\mu a(t-\vartheta) \int_0^h \|u_\mu(h-\tau) - x\| \\
& \left(r_\mu(\vartheta + \tau) - \int_0^\vartheta r_\mu(\vartheta + \tau - s) r_{-\omega_\mu}(s)\, ds \right) d\tau \Bigg] \\
& + \int_0^t \|f(s+h) - f(s)\| a(t-s)\, ds \\
& - \int_0^t \left[\int_0^\vartheta \|f(s+h) - f(s)\| r_{-\omega_\mu}(\vartheta - s)\, ds \right] a(t-\vartheta)\, d\vartheta
\end{aligned}
$$

$$= \mu \int_t^{t+h} d\tau \Big[\|u_\mu(t+h-\tau) - x\|$$

$$\int_0^t a(\vartheta) \left(r_\mu(\tau-\vartheta) - \int_0^{t-\vartheta} r_\mu(\tau-\vartheta-s) r_{-\omega_\mu}(s)\, ds \right) d\vartheta \Big]$$

$$+ \int_0^t \|f(s+h) - f(s)\|$$

$$\left(a(t-s) - \int_0^{t-s} a(t-s-\vartheta) r_{-\omega_\mu}(\vartheta - s)\, d\vartheta \right) ds.$$

Finally, since $r_{-\omega_\mu} \le 0$, we obtain the following bound

$$\|u_\mu(t+h) - u_\mu(t)\| \le \left(\sup_{t\in(0,h)} \|u_\mu(t) - x\| \right)$$

$$\int_t^{t+h} d\tau \left[\left(r_\mu(\tau) + \mu \int_0^t r_\mu(\tau-\vartheta) a(\vartheta)\, d\vartheta \right) - \left(\int_0^t r_\mu(\tau - s) r_{-\omega_\mu}(s)\, ds \right.\right.$$

$$\left.\left. + \mu \int_0^t \int_0^{t-\vartheta} r_\mu(\tau-\vartheta-s) r_{-\omega_\mu}(s)\, ds\, a(\vartheta)\, d\vartheta \right) \right]$$

$$+ \tfrac{1}{\mu} \left(\|f(t+h) - f(t)\| - \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds \right)$$

$$- \tfrac{1}{\omega_\mu} \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds$$

$$\le \left(\sup_{t\in(0,h)} \|u_\mu(t) - x\| \right)$$

$$\int_t^{t+h} d\tau \left[\left(r_\mu(\tau) + \mu \int_0^t r_\mu(\tau-\vartheta) a(\vartheta)\, d\vartheta \right) \right.$$

$$\left. - \left(\int_0^t r_{-\omega_\mu}(s) \left(r_\mu(\tau - s) \; + \mu \int_0^{t-s} r_\mu(\tau-s-\vartheta)\, a(\vartheta)\, d\vartheta \right) ds \right) \right]$$

$$+ \tfrac{1}{\mu} \|f(t+h) - f(t)\| - \tfrac{1}{\omega} \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds \qquad (4.7)$$

Now, we divide the interval $[t_1, t_2]$ in $(N+1)$ intervals of length h, and we compute the variation of $U(x, f, \mu)$ along this partition to get

$$\sum_{k=0}^N \|u_\mu((k+1)h) - u_\mu(kh)\| \le \left(\sup_{t\in(0,h)} \|u_\mu(t) - x\| \right)$$

$$\sum_{k=0}^N \int_{kh}^{(k+1)h} d\tau \left[\left(r_\mu(\tau) + \mu \int_0^{kh} r_\mu(\tau-\vartheta) a(\vartheta)\, d\vartheta \right) \right.$$

$$\left. - \left(\int_0^{kh} r_{-\omega_\mu}(s) \left(r_\mu(\tau - s) + \mu \int_0^{kh-s} r_\mu(\tau-s-\vartheta)\, a(\vartheta)\, d\vartheta \right) ds \right) \right]$$

$$
\begin{aligned}
&+\tfrac{1}{\mu}\sum_{k=0}^{N}\|f((k+1)h)-f(kh)\| \\
&-\tfrac{1}{\omega}\sum_{k=0}^{N}\int_0^{kh}\|f((k+1)h-s)-f(kh-s)\|r_{-\omega_\mu}(s)\,ds. \qquad (4.8)
\end{aligned}
$$

Now we estimate the expression $\sup_{t\in(0,h)}\|u_\mu(t)-x\|$. Subtracting to both sides of (4.2) $J_\mu(x)$ we have:

$$
\begin{aligned}
&u_\mu(t)-J_\mu(x) \\
&\quad = J_\mu\left(s_\mu(t)x+\tfrac{\omega}{\mu}u_\mu(t)+\tfrac{1}{\mu}f(t)+\int_0^t u_\mu(t-s)r_\mu(s)\,ds\right)-J_\mu(x)
\end{aligned}
$$

then

$$
\begin{aligned}
&\|u_\mu(t)-J_\mu(x)\|\le\tfrac{\omega}{\mu}\|u_\mu(t)-x\| \\
&\qquad\qquad +\tfrac{\omega}{\mu}\|x\|+\tfrac{1}{\mu}\|f(t)\|+\int_0^t\|u_\mu(t-s)-x\|r_\mu(s)\,ds
\end{aligned}
$$

and since $J_\mu(x)-x=\frac{1}{\mu}G_\mu(x)$, we have

$$
\begin{aligned}
&\|u_\mu(t)-x\|\le\tfrac{\omega}{\mu}\|u_\mu(t)-x\| \\
&\qquad +\tfrac{\omega}{\mu}\|x\|+\tfrac{1}{\mu}\|f(t)\|+\tfrac{1}{\mu}\|G_\mu(x)\|+\int_0^t\|u_\mu(t-s)-x\|r_\mu(s)\,ds.
\end{aligned}
$$

Using Lemma 3.7, we obtain

$$
\begin{aligned}
\|u_\mu(t)-x\| &\le \tfrac{\omega_\mu}{\omega}(\omega\|x\|+\|G_\mu(x)\|)\left[\tfrac{1}{\mu}s_{-\omega_\mu}(t)+(a*s_{-\omega_\mu})(t)\right] \\
&\quad +\tfrac{\omega_\mu}{\omega}\frac{d}{dt}\Big(\big(\tfrac{1}{\mu}\|f(\cdot)\|+(a*\|f(\cdot)\|)\big)*s_{-\omega_\mu}\Big)(t) \\
&= \tfrac{\omega_\mu}{\omega}(\omega\|x\|+\|G_\mu(x)\|)\left[\tfrac{1}{\mu}+\tfrac{1}{\omega}(s_{-\omega_\mu}(t)-1)\right] \\
&\quad +\tfrac{\omega_\mu}{\omega}\frac{d}{dt}\Big(\big(\tfrac{1}{\mu}\|f(\cdot)\|+(a*\|f(\cdot)\|)\big)*s_{-\omega_\mu}\Big)(t),
\end{aligned}
$$

therefore

$$
\begin{aligned}
\sup_{t\in(0,h)}\|u_\mu(t)-x\| &\le \tfrac{\omega_\mu}{\omega}(\omega\|x\|+\|G_\mu(x)\|)\left[\tfrac{1}{\mu}+\tfrac{1}{\omega}(s_{-\omega_\mu}(h)-1)\right] \\
&\qquad +\tfrac{\omega_\mu}{\omega}\sup_{t\in(0,h)}\|f(t)\|\Bigg(\tfrac{1}{\mu}+\int_0^h a(s)\,ds \\
&\qquad\quad -\tfrac{1}{\mu}\int_0^h r_{-\omega_\mu}(s)\,ds-\int_0^h(a*r_{-\omega_\mu}(s))\,ds\Bigg)
\end{aligned}
$$

so

$$\sup_{t\in(0,h)} \|u_\mu(t)-x\| \le \tfrac{\omega_\mu}{\omega}\left(\tfrac{1}{\mu}+\tfrac{1}{\omega}(s_{-\omega_\mu}(h)-1)\right)$$
$$\left[\omega\|x\| + \|G_\mu(x)\| + \sup_{t\in(0,h)} \|f(t)\|\right]. \tag{4.9}$$

In case $\omega = 0$ the above estimate simplifies to

$$\sup_{t\in(0,h)} \|u_\mu(t)-x\| \le \left(\tfrac{1}{\mu}+\int_0^h a(s)\,ds\right)\left[\|G_\mu(x)\| + \sup_{t\in(0,h)} \|f(t)\|\right].$$

Sending $h \to 0$, the right hand side of (4.8) becomes

$$\begin{aligned}
&\tfrac{1}{\mu-\omega}(\omega\|x\| + \|G_\mu(x)\| + \|f(0+)\|)\\
&\quad\int_{t_1}^{t_2} d\tau \left[\left(r_\mu(\tau)+\mu\int_0^\tau r_\mu(\tau-\vartheta)a(\vartheta)\,d\vartheta\right)\right.\\
&\qquad\left.-\left(\int_0^\tau r_{-\omega_\mu}(s)\left(r_\mu(\tau-s)+\mu\int_0^{\tau-s} r_\mu(\tau-s-\vartheta)\,a(\vartheta)\,d\vartheta\right)\,ds\right)\right]\\
&\quad+\tfrac{1}{\mu}\operatorname{var}(f;[t_1,t_2]) - \tfrac{1}{\omega}\int_{t_1}^{t_2} r_{-\omega_\mu}(s)\operatorname{var}(f;[\max\{0,t_1-s\},t_2-s])\,ds\\
&\quad= \tfrac{\mu}{\mu-\omega}(\omega\|x\| + \|G_\mu(x)\| + \|f(0+)\|)\\
&\qquad\int_{t_1}^{t_2} d\tau\left[a(\tau) - \left(\int_0^\tau r_{-\omega_\mu}(s)a(\tau-s)\,ds\right)\right]\\
&\qquad+\tfrac{1}{\mu}\operatorname{var}(f;[t_1,t_2]) - \tfrac{1}{\omega}\int_{t_1}^{t_2} r_{-\omega_\mu}(s)\operatorname{var}(f;[\max\{0,t_1-s\},t_2-s])\,ds\\
&\quad= \left(\|x\| + \tfrac{1}{\omega}\|G_\mu(x)\| + \tfrac{1}{\omega}\|f(0+)\|\right)\left(-\int_{t_1}^{t_2} r_{-\omega_\mu}(\tau)\,d\tau\right)\\
&\qquad+\tfrac{1}{\mu}\operatorname{var}(f;[t_1,t_2]) - \tfrac{1}{\omega}\int_{t_1}^{t_2} r_{-\omega_\mu}(s)\operatorname{var}(f;[\max\{0,t_1-s\},t_2-s])\,ds\\
&\quad= (\|x\| + \tfrac{1}{\omega}\|G_\mu(x)\| + \tfrac{1}{\omega}\|f(0+)\|)(s_{-\omega_\mu}(t_2)-s_{-\omega_\mu}(t_1))\\
&\qquad+\tfrac{1}{\mu}\operatorname{var}(f;[t_1,t_2]) - \tfrac{1}{\omega}\int_{t_1}^{t_2} r_{-\omega_\mu}(s)\operatorname{var}(f;[\max\{0,t_1-s\},t_2-s])\,ds.
\end{aligned}$$

Therefore, the thesis follows:

$$\begin{aligned}
&\operatorname{var}(\|u_\mu(\cdot)-x\|;[t_1,t_2]) \le (\|x\| + \tfrac{1}{\omega}\|G_\mu(x)\|)(s_{-\omega_\mu}(t_2)-s_{-\omega_\mu}(t_1))\\
&\quad+\tfrac{1}{\mu}\operatorname{var}(f;[t_1,t_2]) - \tfrac{1}{\omega}\int_0^T r_{-\omega_\mu}(s)\operatorname{var}(f;[\max\{0,t_1-s\},t_2-s])\,ds.
\end{aligned}$$

□

Corollary 4.4. *Let the assumptions of Theorem 2.1 and (4.4) be satisfied. Then it follows from (4.9) that*

$$\|U(x,f,\mu)(0+) - x\| \le \tfrac{1}{\mu-\omega}\Big(\omega\|x\| + \|G_\mu(x)\| + \|f(0+)\|\Big). \qquad (4.10)$$

Remark 4.1. We note that for μ large enough respect to ω, and $x \in \widehat{D}(G)$, the estimates in (4.5) and (4.10) are bounded by constants independent from μ.

In order to make the paper self contained, we recall the following result, that is proved in Gripenberg (1985), Lemma 3.4, and will be needed in our proofs.

Proposition 4.5. *Assume that $b \in L^1_{loc}(\mathbb{R}_+)$ and $v \in BV_{loc}(\mathbb{R}_+;X)$. Then the function $t \to \int_0^t b(t-s)v(s)\,ds$ is locally absolutely continuous and differentiable almost surely on $\mathbb{R}_+$. Moreover,*

$$\int_0^T \left\|\frac{d}{dt}\int_0^t b(t-s)v(s)\,ds\right\| \le \left(\int_0^T |b(t)|\,dt\right)[\|v(0+)\| + \mathrm{var}(v;[0,T])]. \qquad (4.11)$$

Lemma 4.6. *Under the additional assumptions $f \in BV_{loc}(\mathbb{R}_+,X)$, $x \in \widehat{D}(G)$, we have*

$$\lim_{\mu\to\infty} U(x,f,\mu) \overset{def}{=} U(x,f) \qquad (4.12)$$

exists in $L^1_{loc}(\mathbb{R}_+;X)$.

Proof. If we replace λ by μ in (4.1), then, by adding and subtracting the same quantity, we get

$$L_\lambda(u_\mu - x)(t) + G(u_\mu(t)) - f(t) = L_\lambda(u_\mu - x)(t) - L_\mu(u_\mu - x)(t).$$

Setting

$$p(\lambda,\mu,t) = L_\lambda(u_\mu - x)(t) - L_\mu(u_\mu - x)(t), \qquad (4.13)$$

and using formula (3.16), we get

$$p(\lambda,\mu,t) = \lambda[u_\mu(t) - (u_\mu * r_\lambda)(t) - s_\lambda(t)x] - f(t) + G(u_\mu(t))$$

hence $u_\mu(t)$ satisfies the equation

$$u_\mu(t) = J_\lambda\left(s_\lambda(t)x + \tfrac{1}{\lambda}f(t) + \tfrac{1}{\lambda}p(\lambda,\mu,t) + \tfrac{1}{\lambda}\omega u_\mu(t) + (r_\lambda * u_\mu)(t)\right).$$

Since J_λ is nonexpansive, this equation combined with (4.2) implies

$$\|u_\lambda(t) - u_\mu(t)\| \leq \tfrac{1}{\lambda}\|p(\lambda,\mu,t)\| + \tfrac{1}{\lambda}\omega\|u_\lambda(t) - u_\mu(t)\| + \big(r_\lambda * \|u_\lambda(\cdot) - u_\mu(\cdot)\|\big)(t).$$

Using Lemma 3.7 we obtain

$$\|u_\lambda(t) - u_\mu(t)\| \leq \tfrac{\omega_\lambda}{\omega}\frac{d}{dt}\Big(\big(\tfrac{1}{\lambda}\|p(\lambda,\mu,\cdot)\| + a * \|p(\lambda,\mu,\cdot)\|\big) * s_{-\omega_\lambda}\Big)(t),$$

that in another form we can write

$$\|u_\lambda(t) - u_\mu(t)\| \leq \tfrac{1}{\lambda-\omega}\|p(\lambda,\mu,t)\| - \tfrac{\lambda}{\omega(\lambda-\omega)}(\|p(\lambda,\mu,\cdot)\| * r_{-\omega_\lambda})(t).$$

We now proceed to prove that $p(\lambda,\mu,\cdot)$ converges to 0 as $\lambda,\mu \to \infty$ in $L^1_{loc}(\mathbb{R}_+;X)$. Recall that $p(\lambda,\mu,\cdot)$ is defined by (4.13); then by (3.17)

$$p(\lambda,\mu,t) = \frac{d}{dt}\int_0^t (u_\mu(\tau) - x)(\lambda s_\lambda(t-\tau) - \mu s_\mu(t-\tau))\,d\tau.$$

By formula (4.11) we obtain

$$\int_0^T \|p(\lambda,\mu,t)\|\,dt \leq \mathrm{var}(\|u_\mu(\cdot) - x\|;[0,T])\int_0^T |\lambda s_\lambda(t) - \mu s_\mu(t)|\,dt.$$

Since the variation of $\|u_\mu(\cdot) - x\|$ is bounded by a constant for μ large enough, compare Remark 4.1, and the integral tends to 0 by Proposition 3.5, we have the thesis. □

We conclude the preparatory material for the proof of Theorem 2.1 with the following theorem, where we collect some useful estimates for the solution of problem (2.1).

Theorem 4.7. *Let $x_i \in \widehat{D}(G)$ and $f_i \in C(\mathbb{R}_+;X)$ for $i = 1,2$, and let $u_i = U(x_i,f_i)$ be the generalized solutions of equation defined in Theorem 2.1. Then we have, for each $t > 0$ and $h > 0$,*

$$\|u_2(t) - u_1(t)\| \leq \|x_2 - x_1\|s_{-\omega}(t) - \tfrac{1}{\omega}\Big(r_{-\omega} * \|f_2(\cdot) - f_1(\cdot)\|\Big)(t); \quad (4.14)$$

$$\sup_{t\in(0,h)} \|u(t) - x\| \leq (\omega(s_{-\omega}(h) - 1))\left(\omega\|x\| + \sup_{\mu>0}\|G_\mu(x)\| + \sup_{t\in(0,h)}\|f(t)\|\right). \quad (4.15)$$

Proof. Notice first that (4.15) were already proved in Lemma 4.3, see formula (4.9).

For the proof of (4.14), let for $i = 1, 2$, $u_i(\mu; t) = U(x_i, f_i, \mu)$. Now, observe that

$$\|u_2(t) - u_1(t)\| \\ \leq \|u_2(t) - u_2(\mu;t)\| + \|u_1(t) - u_1(\mu;t)\| + \|u_2(\mu;t) - u_1(\mu;t)\|.$$

Since (4.3) holds for any $\mu > 0$, while $u_i(\mu; t) \to u_i(t)$ for $i = 1, 2$ and for any $t > 0$, it follows from the previous estimate that

$$\|u_2(t) - u_1(t)\| \leq \liminf_{\mu \to \infty} \|u_2(\mu;t) - u_1(\mu;t)\|.$$

It remains to evaluate the right-hand side of the previous estimate where we get, using (4.3):

$$\liminf_{\mu \to \infty} \|u_2(\mu;t) - u_1(\mu;t)\| \leq \|x_2 - x_1\| s_{-\omega}(t) \\ + \frac{d}{dt}\Big(\big(a * \|f_2(\cdot) - f_1(\cdot)\|\big) * s_{-\omega}\Big)(t).$$

Now since

$$\frac{d}{dt}\Big(\big(a * \|f_2(\cdot) - f_1(\cdot)\|\big) * s_{-\omega}\Big)(t) = -\tfrac{1}{\omega}\Big(r_{-\omega} * \|f_2(\cdot) - f_1(\cdot)\|\Big)(t)$$

we finally obtain

$$\|u_2(t) - u_1(t)\| \leq \|x_2 - x_1\| s_{-\omega}(t) - \tfrac{1}{\omega}\Big(r_{-\omega} * \|f_2(\cdot) - f_1(\cdot)\|\Big)(t).$$

□

We are in a position to conclude the proof of Theorem 2.1. Under the additional assumptions $f \in BV_{loc}(\mathbb{R}_+, X)$, $x \in \widehat{D}(G)$, we obtain the convergence of $U(x, f, \mu)$ towards $U(x, f)$ in $L^\infty_{loc}(\mathbb{R}_+; X)$ and the continuity of the limit function via a Ascoli-Arzelà theorem, by invoking the equicontinuity of the functions $U(x, f, \mu)$ that follows from Lemma 4.3. Then it follows from Remark 4.1 and Corollary 4.4 that $U(x, f) \in BV_{loc}(\mathbb{R}_+; X)$ and

$$U(x, f)(0+) = x.$$

Now it follows from Theorem 4.7 that $U(x, f, \mu)$ converges to $U(x, f)$ in $L^1_{loc}(\mathbb{R}_+; X)$, respectively in $C(\mathbb{R}_+; X)$, also in the case that the assumptions $f \in BV_{loc}(\mathbb{R}_+, X)$, $x \in \widehat{D}(G)$ are not satisfied, but x is in $\overline{D(G)}$ and f belongs only to $L^1_{loc}(\mathbb{R}_+; X)$, resp. $C(\mathbb{R}_+; X)$.

Remark 4.2. From the proof of the Lemma 4.3, compare (4.7), we obtain, for $\omega = 0$, the estimate

$$\|u(t+h) - u(t)\| \le \int_0^t \|f(t+h-s) - f(t-s)\| a(s)\, ds + \left(\sup_{\mu>0} \|G_\mu(x)\| + \sup_{t\in(0,h)} \|f(t)\| \right) \int_t^{t+h} a(s)\, ds, \quad (4.16)$$

for each $t > 0$ and $h > 0$. This is the same formula that is proved in Cockburn *et al.* (1996). Using this result, Gripenberg *et al.* (2000) were able to prove the existence of a strong solution for (2.1).

A similar estimate, up to now, does not seem to hold for $\omega \ne 0$; we hope to return to this problem in a future work.

5 Lipschitz nonlinearity

In this section, we shall prove the results stated in Theorem 2.2.

Let us define the mapping $\mathcal{H} : C(\mathbb{R}_+; X) \to C(\mathbb{R}_+; X)$

$$\mathcal{H}(u) = U(x, F(\cdot, u)), \quad (5.1)$$

then a generalized solution to equation (1.1) is a function $u(t)$ such that

$$u = \mathcal{H}(u).$$

We can achieve the existence of the solution from a fixed point theorem if we prove that some iterate of $\mathcal{H}$ is contractive. At this purpose we need the following lemma.

Lemma 5.1. *There exists $k \ge 1$ such that the k-iterate of $\mathcal{H}$ is a contraction:*

$$\|\mathcal{H}^k(u) - \mathcal{H}^k(v)\|_{L^\infty(0,T;X)} \le \varepsilon \|u - v\|_{L^\infty(0,T;X)}$$

for some $\varepsilon < 1$.

Proof. From (4.14) and (2.4) we have

$$\begin{aligned} \|\mathcal{H}(u)(t) - \mathcal{H}(v)(t)\| &\le \tfrac{1}{\omega} \int_0^t (-r_{-\omega}(t-s)) \|F(s, u(s)) - F(s, v(s))\|\, ds \\ &\le \tfrac{1}{\omega} \int_0^t (-r_{-\omega}(t-s)) \eta(s) \|u(s) - v(s)\|\, ds. \end{aligned}$$

Iterating this procedure we have

$$
\begin{aligned}
&\|\mathcal{H}^k(u)(t) - \mathcal{H}^k(v)(t)\| \\
&\quad \le \tfrac{1}{\omega^k} \int_0^t (-r_{-\omega}(t-x_1))\eta(x_1) \int_0^{x_1} (-r_{-\omega}(x_1-x_2))\eta(x_2) \cdots \\
&\quad \cdots \int_0^{x_{k-1}} (-r_{-\omega}(x_{k-1}-x_k))\eta(x_k)\|u(x_k)-v(x_k)\|\,dx_k \dots\, dx_2\, dx_1.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|\mathcal{H}^k(u) - \mathcal{H}^k(v)\|_{L^\infty(0,T;X)} \le \|u-v\|_{L^\infty(0,T;X)}\omega^{-k}\|\eta\|^k_{L^\infty(0,T)} \\
&\quad \int_0^T \int_0^{x_1} \cdots \int_0^{x_{k-1}} (-r_{-\omega}(T-x_1)) \cdots (-r_{-\omega}(x_{k-1}-x_k))\,dx_k \dots\, dx_1,
\end{aligned}
$$

but, by a repeated use of Fubini theorem, we have

$$
\begin{aligned}
&\|\mathcal{H}^k(u) - \mathcal{H}^k(v)\|_{L^\infty(0,T;X)} \\
&\qquad \le \|u-v\|_{L^\infty(0,T;X)}\omega^{-k}\|\eta\|^k_{L^\infty(0,T)}\|(-r_{-\omega})^{*k}\|_{L^1(0,T)}.
\end{aligned}
$$

Finally by Lemma 3.10 we have that the right hand side converges to zero, so for sufficiently large k we have the lemma. □

As stated before, this lemma provides the proof of Theorem 2.2. We insist on the following explanation.

Remark 5.1. Let $u = U(x, F(\cdot, u))$ be a generalized solution to (1.1): then, by definition, this means that there exists a sequence $u_\mu(t)$ such that

$$
L_\mu(u_\mu - x)(t) + G(u_\mu(t)) = F(t, u(t))
$$

and $u_\mu \to u$ in $L^1_{loc}(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; X)$.

6 Dissipative nonlinearity

In this section, we shall prove the results stated in Theorem 1.1. Here, we follow the ideas in the proof of Theorem 7.13 of Da Prato & Zabczyk (1992). Let us introduce, for any $\alpha > 0$, the approximating equation

$$
L(u_\alpha - x)(t) + G(u_\alpha(t)) = F_\alpha(t, u_\alpha(t)), \tag{6.1}
$$

where $F_\alpha(t, \cdot)$ are the Yosida approximations of $F(t, \cdot)$. We denote with $J_\alpha^{F,t}(\cdot)$ the resolvent operators associated to $F(t, \cdot)$.

Let us recall that F_α is Lipschitz continuous; moreover, for any $x, y \in X$ and $x^* \in \partial\|x\|$,

$$
\begin{aligned}
\langle F_\alpha(t, x+y), x^* \rangle &= \langle F_\alpha(t, x+y) - F_\alpha(t, y), x^* \rangle + \langle F_\alpha(t, y), x^* \rangle \\
&\le \langle F_\alpha(t, y), x^* \rangle \le \|F(t, y)\|.
\end{aligned}
$$

From Theorem 2.2 we know that there exists a generalized solution $u_\alpha(t)$ to equation (6.1). Then, there exist sequences $u_{\alpha,\mu}$ and $\delta_{\alpha,\mu}$ such that

$$\begin{aligned} u_{\alpha,\mu} \to u_\alpha \in L^1_{loc}(\mathbb{R}_+;X) \cap C(\mathbb{R}_+;X), \\ L_\mu(u_{\alpha,\mu} - x)(t) + G(u_{\alpha,\mu}(t)) - F_\alpha(t, u_{\alpha,\mu}(t)) = \delta_{\alpha,\mu} \to 0. \end{aligned} \tag{6.2}$$

Now, let $y \in D(G)$, then for some $y^* \in \partial\|u_{\alpha,\mu}(t) - y\|$ we get, from

$$\begin{aligned} \langle L_\mu(u_{\alpha,\mu}(t) - y), y^*\rangle - \langle L_\mu(x - y), y^*\rangle + \langle G(u_{\alpha,\mu}(t)) - G(y), y^*\rangle \\ + \langle G(y), y^*\rangle - \langle F_\alpha(t, u_{\alpha,\mu}(t)), y^*\rangle = \langle \delta_{\alpha,\mu}, y^*\rangle \end{aligned}$$

the estimate

$$\begin{aligned} &\mu\Big(\|u_{\alpha,\mu}(t) - y\| - \big(\|u_{\alpha,\mu}(\cdot) - y\| * r_\mu\big)(t)\Big) \\ &\qquad \le \omega\|u_{\alpha,\mu}(t) - y\| + s_\mu(t)\|x - y\| + \|G(y)\| + \|F(t,y)\| + \|\delta_{\alpha,\mu}\|. \end{aligned}$$

Lemma 3.7 now implies

$$\begin{aligned} \|u_{\alpha,\mu}(t) - y\| \le \tfrac{\omega_\mu}{\omega}\frac{d}{dt}\Bigg(&\|x - y\| + \Big(\tfrac{1}{\mu}[\|G(y)\| + \|F(\cdot,y)\| + \|\delta_{\alpha,\mu}\|] \\ &+ \big(a * [\|G(y)\| + \|F(\cdot,y)\| + \|\delta_{\alpha,\mu}\|]\big)\Big) * s_{-\omega_\mu}\Bigg)(t) \end{aligned}$$

and passing to the limit as $\mu \to \infty$ we get

$$\|u_\alpha(t) - y\| \le \frac{d}{dt}\Big(\big(\|x - y\| + a * [\|G(y)\| + \|F(\cdot,y)\|]\big) * s_{-\omega}\Big)(t). \tag{6.3}$$

We can simplify this expression. If we consider separately the case $\omega = 0$, then the estimate (6.3) has the simpler form

$$\|u_\alpha(t) - y\| \le \|x - y\| + (a * [\|G(y)\| + \|F(\cdot,y)\|])\,(t).$$

In the general case $\omega \neq 0$ we get

$$\|u_\alpha(t) - y\| \le s_{-\omega}(t)\,\|x - y\| - \tfrac{1}{\omega}\,(r_{-\omega} * [\|G(y)\| + \|F(\cdot,y)\|])\,(t).$$

This shows that the sequence $\{u_\alpha(\cdot)\}$ is bounded uniformly on bounded sets.

To show the convergence of the sequence, we set, for any $\alpha, \beta > 0$,

$$g^{\alpha,\beta}(t) = u_\alpha(t) - u_\beta(t).$$

Let us consider the functions $g^{\alpha,\beta}_\mu(t) = u_{\alpha,\mu}(t) - u_{\beta,\mu}(t)$, where $u_{\alpha,\mu}(t)$ and $u_{\beta,\mu}(t)$ are the approximating functions solving

$$\begin{aligned} L_\mu(u_{\alpha,\mu} - x)(t) + G(u_{\alpha,\mu}(t)) &= F_\alpha(t, u_\alpha(t)) \\ L_\mu(u_{\beta,\mu} - x)(t) + G(u_{\beta,\mu}(t)) &= F_\beta(t, u_\beta(t)) \end{aligned}$$

respectively; moreover we have that

$$u_{\alpha,\mu} \to u_\alpha \quad \text{and} \quad u_{\beta,\mu} \to u_\beta$$

in $L^1_{loc}(\mathbb{R}_+; X) \cap C(\mathbb{R}^+; X)$. Then $g^{\alpha,\beta}(t)$ shall be a generalized solution to the problem

$$Lg^{\alpha,\beta}(t) + G(u_\alpha(t)) - G(u_\beta(t)) = F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)).$$

Now we have, for $y^* \in \partial\|g^{\alpha,\beta}_\mu(t)\|$,

$$\left\langle L_\mu g^{\alpha,\beta}(t), y^* \right\rangle + \langle G(u_{\alpha,\mu}(t)) - G(u_{\beta,\mu}(t)), y^*\rangle$$
$$= \langle F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)), y^*\rangle\,,$$

which becomes, thanks to (3.17):

$$\mu\left(\|g^{\alpha,\beta}_\mu(t)\| - (\|g^{\alpha,\beta}_\mu\| * r_\mu)(t)\right) - \omega\|g^{\alpha,\beta}_\mu(t)\|$$
$$\leq \langle F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)), y^*\rangle\,. \quad (6.4)$$

Let us notice that

$$\begin{aligned}
&\langle F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)), y^*\rangle \\
&\leq \langle F(t, u_{\alpha,\mu}(t)) - F(t, u_{\beta,\mu}(t)), y^*\rangle \\
&\quad + \langle F(t, J^{F,t}_\alpha(u_\alpha(t))) - F(t, u_{\alpha,\mu}(t)) + F(t, u_{\beta,\mu}(t)) - F(t, J^{F,t}_\beta(u_\beta(t))), y^*\rangle \\
&\leq \|F(t, J^{F,t}_\alpha(u_\alpha(t))) - F(t, u_{\alpha,\mu}(t))\| + \|F(t, J^{F,t}_\beta(u_\beta(t))) - F(t, u_{\beta,\mu}(t))\|
\end{aligned}$$

Now, by (6.3) and recalling that F is uniformly bounded on bounded subsets of $\mathbb{R}_+ \times X$, for a fixed $T > 0$ there exists $R > 0$ such that

$$\|u_\alpha(t)\| \leq R \quad \text{and} \quad \|F(t, u_\alpha(t))\| \leq 2R \qquad \forall\, t \in [0, T],$$

for all α. Then we have

$$\|u_\alpha(t) - J^{F,t}_\alpha(u_\alpha(t))\| \leq \tfrac{1}{\alpha}\|F_\alpha(t, u_\alpha(t))\| \leq \tfrac{2}{\alpha}R;$$

and, for μ sufficiently large

$$\|u_{\alpha,\mu}(t)\| \leq R \qquad \forall\, t \in [0.T];$$

so we have

$$\|u_\alpha(t) - u_{\alpha,\mu}(t)\| \leq 2R.$$

Therefore, it follows

$$\begin{aligned}
&\|F(t,J_\alpha^{F,t}(u_\alpha(t))) - F(t,u_{\alpha,\mu}(t))\| \\
&\qquad \le \|F(t,J_\alpha^{F,t}(u_\alpha(t))) - F(t,u_\alpha(t))\| + \|F(t,u_\alpha(t)) - F(t,u_{\alpha,\mu}(t))\| \\
&\qquad \le \rho_F(\tfrac{2}{\alpha}R) + \rho_F(\|u_\alpha(t) - u_{\alpha,\mu}(t)\|),
\end{aligned}$$

where ρ_F is the modulus of continuity[2] of $F(t,\cdot)$ restricted to $[0,T] \times B(0,2R)$.

The above construction, starting from (6.4), leads to

$$\begin{aligned}
\mu\Big(\|g_\mu^{\alpha,\beta}(t)\| - (\|g_\mu^{\alpha,\beta}\| * r_\mu)(t)\Big) - \omega\|g_\mu^{\alpha,\beta}(t)\| \\
\le \rho_F(\tfrac{2}{\alpha}R) + \rho_F(\tfrac{2}{\beta}R) + \rho_F(\varepsilon_{\alpha,\mu}) + \rho_F(\varepsilon_{\beta,\mu}),
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_{\alpha,\mu} &= \sup_{t\in[0,T]} \|u_\alpha(t) - u_{\alpha,\mu}(t)\| \le 2R, \\
\varepsilon_{\beta,\mu} &= \sup_{t\in[0,T]} \|u_\beta(t) - u_{\beta,\mu}(t)\| \le 2R.
\end{aligned}$$

Lemma 3.7 now implies

$$\begin{aligned}
\|g_\mu^{\alpha,\beta}(t)\| \le \tfrac{\omega_\mu}{\omega}\Big[\rho_F(\tfrac{2}{\alpha}R) + \rho_F(\tfrac{2}{\beta}R) + \rho_F(\varepsilon_{\alpha,\mu}) + \rho_F(\varepsilon_{\beta,\mu})\Big] \\
\Big(\tfrac{1}{\mu}s_{-\omega_\mu}(t) + \big(a * s_{-\omega_\mu}\big)(t)\Big).
\end{aligned}$$

From the above inequality, as we pass to the limit for $\mu \to \infty$, we have

$$\|g^{\alpha,\beta}(t)\| \le [\rho_F(\tfrac{2}{\alpha}R) + \rho_F(\tfrac{2}{\beta}R)]\big(a * s_{-\omega}\big)(t).$$

This yields the convergence of the sequence $u_\alpha(t)$ in $C([0,T];X)$ to a function u, which is easily seen to be a generalized solution to (1.1). The remaining of the proof now follows as in Da Prato & Zabczyk (1992).

A Proof of auxiliary results

In this section we shall give the proofs of the results introduced in section 3.3.

Proof of Lemma 3.6. Since $s_{-\gamma}(t)$ is positive, taking the convolution in both sides of (3.7) with $s_{-\gamma}(t)$ we have

$$\begin{aligned}
\int_{t_1}^{t_2} v(x)s_{-\gamma}(t_2-x)\,dx \le \int_{t_1}^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx \\
+ \int_{t_1}^{t_2}\int_0^x \gamma a(x-y)v(y)\, s_{-\gamma}(t_2-x)\,dy\,dx.
\end{aligned}$$

[2] A function $\rho = \rho_{f,T}$ is called the modulus of continuity of a function f if $\rho(s) = \sup\{\|f(t_1) - f(t_2)\| \,:\, t_1,t_2 \in [0,T], |t_1 - t_2| < s\}$.

Applying Fubini's theorem to the last integral we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^x \gamma a(x-y)v(y)s_{-\gamma}(t_2-x)\,dy\,dx \\ &= \int_{t_1}^{t_2}\left(\int_y^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx\right)v(y)\,dy \\ &\quad + \int_0^{t_1}\left(\int_{t_1}^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx\right)v(y)\,dy \\ &= \int_{t_1}^{t_2} s_{-\gamma}(t_2-y)v(y)\,dy - \int_{t_1}^{t_2} v(y)\,dy \\ &\quad + \int_0^{t_1}\left(\int_{t_1}^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx\right)v(y)\,dy.\end{aligned}$$

Then we have

$$\begin{aligned}\int_{t_1}^{t_2} v(x)s_{-\gamma}(t_2-x)\,dx \le \int_{t_1}^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx \\ + \int_{t_1}^{t_2} s_{-\gamma}(t_2-y)v(y)\,dy - \int_{t_1}^{t_2} v(y)\,dy \\ + \int_0^{t_1}\left(\int_{t_1}^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx\right)v(y)\,dy,\end{aligned}$$

and we read

$$\begin{aligned}\int_{t_1}^{t_2} v(y)\,dy \le \int_{t_1}^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx \\ + \int_0^{t_1}\left(\int_{t_1}^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx\right)v(y)\,dy. \qquad \text{(A.1)}\end{aligned}$$

We analize now the last term of previous inequality: since $y < t_1$,

$$\begin{aligned}\int_{t_1}^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx \\ &= \int_y^{t_2}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx - \int_y^{t_1}\gamma a(x-y)s_{-\gamma}(t_2-x)\,dx \\ &= [s_{-\gamma}(t_2-y)-1] - \int_y^{t_1}\gamma a(x-y)s_{-\gamma}(t_1-x)\,dx \\ &\quad - \int_y^{t_1}\gamma a(x-y)[s_{-\gamma}(t_2-x)-s_{-\gamma}(t_1-x)]\,dx \\ &= [s_{-\gamma}(t_2-y)-1-(s_{-\gamma}(t_1-y)-1)] \\ &\quad - \int_y^{t_1}\gamma a(x-y)[s_{-\gamma}(t_2-x)-s_{-\gamma}(t_1-x)]\,dx.\end{aligned}$$

If we return to (A.1), we get

$$\begin{aligned}\int_{t_1}^{t_2} v(y)\,dy \le & \int_{t_1}^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx \\ & + \int_0^{t_1} [s_{-\gamma}(t_2-y) - s_{-\gamma}(t_1-y)]v(y)\,dy \\ & - \int_0^{t_1} \left(\int_y^{t_1} \gamma a(x-y)[s_{-\gamma}(t_2-x) - s_{-\gamma}(t_1-x)]\,dx \right) v(y)\,dy \\ = & \int_{t_1}^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx \\ & + \int_0^{t_1} [s_{-\gamma}(t_2-y) - s_{-\gamma}(t_1-y)]v(y)\,dy \\ & - \int_0^{t_1} \left(\int_0^{x} \gamma a(x-y)v(y)\,dy \right) [s_{-\gamma}(t_2-x) - s_{-\gamma}(t_1-x)]\,dx\end{aligned}$$

and, since $s_{-\gamma}(t)$ is an increasing function, we can use here (3.7)

$$\begin{aligned}\le & \int_{t_1}^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx \\ & + \int_0^{t_1} [s_{-\gamma}(t_2-y) - s_{-\gamma}(t_1-y)]v(y)\,dy \\ & - \int_0^{t_1} \big(v(x) - g(x)\big)[s_{-\gamma}(t_2-x) - s_{-\gamma}(t_1-x)]\,dx\end{aligned}$$

hence

$$\int_{t_1}^{t_2} v(y)\,dy \le \int_0^{t_2} g(x)s_{-\gamma}(t_2-x)\,dx - \int_0^{t_1} g(x)s_{-\gamma}(t_1-x)\,dx. \qquad \text{(A.2)}$$

Now dividing (A.2) by t_2-t_1 and letting $t_2-t_1 \to 0$ we have the thesis. □

Proof of Lemma 3.7. If we take convolution with $a(\cdot)$ of both sides of (3.9), we have

$$(a*v)(t) \le (a*s_\lambda)(t)x + \tfrac{1}{\lambda}(a*f)(t) + \tfrac{\omega}{\lambda}(a*v)(t) + (a*r_\lambda*v)(t).$$

Using the very definition of $r_\lambda(\cdot)$ in the above expression we get

$$(a*v)(t) \le (a*s_\lambda)(t)x + \tfrac{1}{\lambda}(a*f)(t) + \tfrac{\omega}{\lambda}(a*v)(t) + (a*v)(t) - \tfrac{1}{\lambda}(r_\lambda*v)(t),$$

that we read

$$(r_\lambda*v)(t) \le \lambda(a*s_\lambda)(t)x + (a*f)(t) + \omega(a*v)(t). \qquad \text{(A.3)}$$

Now we substitute what we have found in (3.9) to get

$$v(t) \leq s_\lambda(t)x + \tfrac{1}{\lambda}f(t) + \tfrac{\omega}{\lambda}v(t) + \lambda(a * s_\lambda)(t)x + (a * f)(t) + \omega(a * v)(t),$$

and the definition of s_λ implies

$$\begin{aligned} \tfrac{\lambda-\omega}{\lambda}v(t) &\leq x + \tfrac{1}{\lambda}f(t) + (a * f)(t) + \omega(a * v)(t) \\ v(t) &\leq \tfrac{\lambda}{\lambda-\omega}\big(x + \tfrac{1}{\lambda}f(t) + (a * f)(t)\big) + \omega_\lambda(a * v)(t). \end{aligned}$$

Now we conclude since we can apply Lemma 3.6 with $g(t) = \frac{\lambda}{\lambda-\omega}\big(x+\frac{1}{\lambda}f(t)+ (a * f)(t)\big)$. □

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