A SUFFICIENT CONDITION FOR THE C^H-RECTIFIABILITY OF LIPSCHITZ CURVES

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ABSTRACT. Let $\gamma : [a,b] \to \mathbf{R}^{1+k}$ be Lipschitz and $H \geq 2$ be an integer number. Then a sufficient condition, expressed in terms of further accessory Lipschitz maps, for the C^H-rectifiability of $\gamma([a,b])$ is provided.

1. INTRODUCTION

In order to state our main theorem, we need to recall that a Borel subset S of $\mathbf{R}^{1+k}$ ($k \geq 1$, k integer) is said to be C^H-rectifiable if there exist countably many curves M_j of class C^H, embedded in $\mathbf{R}^{1+k}$ and such that

$$\mathcal{H}^1(S \setminus \cup_j M_j) = 0$$

where $\mathcal{H}^1$ denotes the usual one-dimensional Hausdorff measure in $\mathbf{R}^{1+k}$, compare [1, Definition 1.1]. Observe that for $H = 1$ this is equivalent to say that S is countably 1-rectifiable, e.g. by [11, Lemma 11.1].

The present paper is devoted to prove the following result.

Theorem 1.1. *Let be given a Lipschitz map $\gamma : [a,b] \to \mathbf{R}^{1+k}$ and an integer $H \geq 2$. Then the set $\gamma([a,b])$ is C^H-rectifiable provided the following condition is met:*

There are a family of 2^{H-1} Lipschitz maps

$$\gamma_\alpha : [a,b] \to \mathbf{R}^{1+k}, \qquad \alpha \in \{0,1\}^{H-1}$$

and a family of $H-1$ bounded functions

$$c_h : [a,b] \to \mathbf{R}, \qquad h \in \{0, \dots, H-2\}$$

such that

$$\gamma_{0^{H-1}} = \gamma$$

and

$$\dot{\gamma}_{0^{H-1-h}\beta} = c_h \, \gamma_{0^{H-2-h}1\beta} \qquad \textit{(almost everywhere)} \tag{1.1}$$

for all $h \in \{0, \dots, H-2\}$ and $\beta \in \{0,1\}^h$ (where $\{0,1\}^0 := \{\emptyset\}$ and $\gamma_{\alpha\emptyset} := \gamma_\alpha$ for all $\alpha \in \{0,1\}^{H-1}$).

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In order to clarify the meaning of the condition above, let us consider a particular case.

Example. If $H = 4$, then eight Lipschitz maps

$$\gamma_{000}, \gamma_{001}, \gamma_{010}, \gamma_{011}, \gamma_{100}, \gamma_{101}, \gamma_{110}, \gamma_{111} : [a, b] \to \mathbf{R}^{1+k}$$

and three bounded functions

$$c_0, c_1, c_2 : [a, b] \to \mathbf{R}$$

have to exist such that the following equalities hold a.e. in $[a, b]$

$$\dot\gamma_{000} = c_0\gamma_{001}$$
$$(\dot\gamma_{000}, \dot\gamma_{001}) = c_1(\gamma_{010}, \gamma_{011})$$
$$(\dot\gamma_{000}, \dot\gamma_{001}, \dot\gamma_{010}, \dot\gamma_{011}) = c_2(\gamma_{100}, \gamma_{101}, \gamma_{110}, \gamma_{111}).$$

Moreover $\gamma_{000} = \gamma$. In such a case, our result asserts that $\gamma([a, b])$ is C^4-rectifiable.

Theorem 1.1 marks a new step in the long-term program that we have been embarked on since [8]. Actually, particular cases of such a result have been considered in [8] (where the program was announced) and [9]. More precisely, if σ denotes a ±1-valued function with domain $[a, b]$, the case

$$H := 2, \quad c_0 := \sigma\|\dot\gamma_0\|$$

is indagated in [8], while

$$H := 3, \quad c_0 := \sigma\|\dot\gamma_{00}\|, \quad c_1 := \sigma\|(\dot\gamma_{00}, \dot\gamma_{01})\|$$

is considered in [9]. These particular cases arise naturally in the context of one-dimensional generalized Gauss graphs (see [3, 4], for the basic definitions and results) and of 2-storey towers of one-dimensional generalized Gauss graphs (see [9]). Now, with Theorem 1.1, the program in dimension one is completed. Hence the application to one-dimensional geometric variational problems with integral functionals depending on the curvature and its derivatives becomes a realistic option for the next move. We are confident that results in such a direction can be obtained by resorting to the notion of "h-storey tower of generalized Gauss graphs" (introduced in [9], for $h = 2$). This is a special kind of integral current whose orientation provides the "wizard hat" where the derivatives of the curvature up to the order $h - 1$ can be picked-up from, through very simple operations of multilinear algebra. For $h = 1$, namely in the context of generalized Gauss graphs, applications to geometric variational problems can be found in [5, 6, 7], where no restriction on the dimension is assumed. Another step towards the achievement of our program consists in extending Theorem 1.1 to arbitrary dimension and our future efforts will surely be devoted to pursue this goal.

2. Reduction to graphs

The proof of Theorem 1.1 can be easily reduced to the following result.

Theorem 2.1. *Let H, γ and the families $\{\gamma_\alpha\}$ and $\{c_h\}$ satisfy the same assumption as in Theorem 1.1. Given a unit vector u in $\mathbf{R}^{1+k}$, consider a map*

$$f : \mathbf{R} \to (\mathbf{R}u)^\perp$$

of class C^{H-1} and define the set

$$G_f := \{xu + f(x) \,|\, x \in \mathbf{R}\}.$$

Then the set $G_f \cap \gamma([a,b])$ is C^H-rectifiable.

In order to convince ourself of this point, observe that if $\gamma([a,b])$ is C^h-rectifiable for a given $h \in \{1, \dots, H-1\}$ then countably many unit vectors

$$u_j \in \mathbf{R}^{1+k}$$

and corresponding maps of class C^h

$$f_j : \mathbf{R} \to (\mathbf{R}u_j)^\perp$$

have to exist such that

$$\mathcal{H}^1\left(\gamma([a,b]) \setminus \cup_j G_{f_j}\right) = 0.$$

If we further assume that the condition in Theorem 1.1 is verified and Theorem 2.1 holds, then $\gamma([a,b])$ has to be C^{h+1}-rectifiable. The conclusion follows by iterating this argument for $H-1$ times and recalling that $\gamma([a,b])$ is C^1-rectifiable (e.g. by [11, Lemma 11.1]).

3. The proof of Theorem 2.1
Preliminaries I: The derivatives of f in terms of $\{\gamma_\alpha\}$

For $h \geq 1$ let $B_h^1 := \{1^h\}$, while, for $h \geq 2$ and $i \in \{2, \dots, h\}$, let B_h^i denote the set of all i-tuples $(\beta_1, \dots, \beta_i)$ whose elements β_j belong to $\{0,1\}^h$ and are such that

- $\beta_1 > \ldots > \beta_i > 0$ as binary numbers;
- for each $l \in \{1, \dots, h\}$, there exists one and only one β_j with a 1 occupying the l-th position.

Examples. One has

$$\begin{aligned}
B_3^3 &= \{(100, 010, 001)\}, \\
B_4^4 &= \{(1000, 0100, 0010, 0001)\}, \\
B_3^2 &= \{(100, 011); (101, 010); (110, 001)\}, \\
B_4^3 &= \{(1000, 0100, 0011); (1000, 0101, 0010); (1000, 0110, 0001); \\
&\qquad (1001, 0100, 0010); (1010, 0100, 0001); (1100, 0010, 0001)\}.
\end{aligned}$$

Now let $\{\gamma_\alpha\}$, $\{c_h\}$, u and f be as in the statement of Theorem 2.1 and introduce some further notation. First of all, if $H \geq 3$, in order to simplify the formulas below, we set the shortened notation

$$\gamma_\beta := \gamma_{0^j\beta}$$

for all $j = 1, \dots, H-2$ and $\beta \in \{0,1\}^{H-1-j}$. For example, when $H = 4$, one has

$$\gamma_0 = \gamma_{00} = \gamma_{000},\ \gamma_1 = \gamma_{01} = \gamma_{001},\ \gamma_{10} = \gamma_{010}, \dots$$

For $h \in \{1, \dots, H-1\}$ define

$$\Gamma_h^0(t) := 0, \qquad t \in [a,b], \tag{3.1}$$

and

$$\Gamma_h^i := \sum_{(\beta_1,\dots,\beta_i)\in B_h^i} \prod_{j=1}^{i} (\gamma_{\beta_j} \cdot u) \qquad (i = 1, \dots, h) \tag{3.2}$$

Remark 3.1. In particular we have $(h = 1, \dots, H-1)$

$$\Gamma_h^1 = \gamma_{1^h} \cdot u \tag{3.3}$$

and

$$\Gamma_h^h = \prod_{j=0}^{h-1} (\gamma_{10^j} \cdot u). \tag{3.4}$$

Proposition 3.1. *Let $s \in [a,b]$ be such that $\dot\gamma_\alpha(s)$ exists for all $\alpha \in \{0,1\}^{H-1}$. Then Γ_h^i is differentiable at s, for all $h \in \{1, \dots, H-1\}$ and $i \in \{0, \dots, h\}$. Moreover, if $H \geq 3$, the following formula*

$$\dot\Gamma_h^i(s) = c_h(s) \left(\Gamma_{h+1}^i(s) - [\gamma_{10^h}(s) \cdot u] \Gamma_h^{i-1}(s) \right). \tag{3.5}$$

holds for all $h \in \{1, \dots, H-2\}$ and $i \in \{1, \dots, h\}$.

Proof. The differentiability of Γ_h^i at s is obvious. As for the second assertion, observe, first of all, that

$$\dot\Gamma_h^1(s) = \dot\gamma_{1^h}(s) \cdot u = c_h(s)\, \gamma_{1^{h+1}}(s) \cdot u = c_h(s) \Gamma_{h+1}^1(s)$$

for all $h \in \{1, \dots, H-2\}$, by (3.3) and (1.1). Hence the equality (3.5) with $i = 1$ follows, by also recalling that $\Gamma_h^0(s) = 0$. For $i \geq 2$ (note: this case occurs only when $H \geq 4$), one has

$$\begin{aligned}
\dot\Gamma_h^i(s) &= \sum_{(\beta_1,\dots,\beta_i)\in B_h^i} \left(\sum_{j=1}^{i} [\dot\gamma_{\beta_j}(s) \cdot u] \prod_{\substack{l=1 \\ l\neq j}}^{i} [\gamma_{\beta_l}(s) \cdot u] \right) \\
&= c_h(s) \sum_{(\beta_1,\dots,\beta_i)\in B_h^i} \left(\sum_{j=1}^{i} [\gamma_{1\beta_j}(s) \cdot u] \prod_{\substack{l=1 \\ l\neq j}}^{i} [\gamma_{\beta_l}(s) \cdot u] \right)
\end{aligned}$$

for all $h \in \{i, \dots, H-2\}$, by (3.2) and (1.1). Now, the formula (3.5) follows observing that

$$\begin{aligned}
B_{h+1}^i &= \{(\beta_1, \dots, \beta_i) \in B_{h+1}^i \,|\, \beta_1 = 10^h\} \cup \{(\beta_1, \dots, \beta_i) \in B_{h+1}^i \,|\, \beta_1 \neq 10^h\} \\
&= \left(10^h \times B_h^{i-1} \right) \cup \left(\cup_{(\beta_1,\dots,\beta_i)\in B_h^i} \cup_{j=1}^{i} \{(1\beta_j, \beta_1, \dots, \widehat{\beta_j}, \dots, \beta_i)\} \right)
\end{aligned}$$

hence

$$\Gamma_{h+1}^i = (\gamma_{10^h} \cdot u) \Gamma_h^{i-1} + \sum_{(\beta_1,\dots,\beta_i)\in B_h^i} \left(\sum_{j=1}^{i} (\gamma_{1\beta_j} \cdot u) \prod_{\substack{l=1 \\ l\neq j}}^{i} (\gamma_{\beta_l} \cdot u) \right).$$

□

Remark 3.2. Define L as the set of $t \in \gamma_0^{-1}(G_f)$ such that:

$$\dot\gamma_\beta(t) \text{ exists and (1.1) holds at } t$$

for all $\beta \in \{0,1\}^h$ with $h = 0, \dots, H-2$, and

$$\dot\gamma_0(t) \neq 0.$$

From the Lusin Theorem it follows that, for any given real number $\varepsilon > 0$, there exists

$$L_\varepsilon \subset L, \qquad L_\varepsilon \text{ closed}$$

such that

$$\mathcal{L}^1(L \backslash L_\varepsilon) \leq \varepsilon$$

and

$$\text{the map } \dot{\gamma}_\alpha | L_\varepsilon \text{ is continuous}$$

for all $\alpha \in \{0,1\}^h$ with $h = 0, \ldots, H-2$. If L_ε^* denotes the set of density points of L_ε, then

$$L_\varepsilon^* \subset L_\varepsilon$$

in that L_ε is closed. Moreover one has

$$\mathcal{L}^1(L_\varepsilon \backslash L_\varepsilon^*) = 0$$

by a well-known Lebesgue's result. In the special case when $\mathcal{L}^1(L) = 0$ we define $L_\varepsilon := \emptyset$, hence $L_\varepsilon^* = \emptyset$.

Now, by the same argument as in [9, §2], one can prove that

$$\mathcal{H}^1\left(G_f \cap \gamma_0([a,b]) \backslash \cup_{j=1}^\infty \gamma_0(L_{1/j}^*)\right) = 0.$$

Then, in order to prove Theorem 2.1, it will be enough to verify that

$$\gamma_0(L_\varepsilon^*) \text{ is } C^H\text{-rectifiable} \tag{3.6}$$

for all $\varepsilon > 0$.

Remark 3.3. Setting

$$\beta := \begin{cases} \emptyset & \text{if } h = 0 \\ 0^h & \text{if } h = 1, \ldots, H-2 \text{ (provided } H \geq 3) \end{cases}$$

in the equality (1.1), we find

$$c_h(t) \neq 0 \tag{3.7}$$

for all $t \in L$ and for $h = 0, \ldots, H-2$.

Proposition 3.2. *Define*

$$x(t) := \gamma_0(t) \cdot u, \qquad t \in [a,b],$$

let $\varepsilon > 0$ *and consider*

$$s \in L_\varepsilon^*.$$

Then one has

$$x'(s) \neq 0 \tag{3.8}$$

and

$$\Gamma_h^h(s) \neq 0, \qquad h \in \{1, \ldots, H-1\}. \tag{3.9}$$

Moreover the following formula holds

$$\sum_{i=1}^h f^{(i)}(x(s))\Gamma_h^i(s) = \gamma_{1^h}(s) - [\gamma_{1^h}(s) \cdot u]u, \qquad h \in \{1, \ldots, H-1\}. \tag{3.10}$$

Proof. Observe that

$$f(x(t)) = \gamma_0(t) - [\gamma_0(t) \cdot u]u = \gamma_0(t) - x(t)u$$

for all $t \in \gamma_0^{-1}(G_f)$. Moreover the members of this equality are both differentiable at s, in that $L_\varepsilon^* \subset L$. Since s is a limit point of $L_\varepsilon \subset \gamma_0^{-1}(G_f)$, it follows that

$$x'(s)f'(x(s)) = \gamma_0'(s) - x'(s)u \tag{3.11}$$

which implies (3.8).

Recalling (1.1), we find

$$c_i(s)[\gamma_{10^i}(s) \cdot u] = \dot{\gamma}_0(s) \cdot u = x'(s) \neq 0, \qquad i \in \{0, \dots, H-2\}.$$

Then

$$\gamma_{10^i}(s) \cdot u \neq 0, \qquad i \in \{0, \dots, H-2\}$$

by (3.7). Now (3.9) follows at once from (3.4).

We will prove (3.10) by induction on h. As for $h = 1$, the formula follows from (3.11) recalling that

$$\dot{\gamma}_0(s) = c_1(s)\gamma_1(s), \qquad c_1(s) \neq 0$$

by (1.1) and (3.7), respectively, while

$$\Gamma_1^1 = \gamma_1 \cdot u$$

by (3.3). The argument proceed now under the hypothesis $H \geq 3$ (for $H = 2$ the proof is completed). Let us assume that (3.10) holds for a generic $h \leq H - 2$ and at all $s \in L_\varepsilon^*$. We shall prove that

$$\sum_{i=1}^{h+1} f^{(i)}(x(s))\Gamma_{h+1}^i(s) = \gamma_{1^{h+1}}(s) - [\gamma_{1^{h+1}}(s) \cdot u]u \tag{3.12}$$

for all $s \in L_\varepsilon^*$. Actually, by the same argument as above, we can differentiate (3.10) and get (let us omit, for simplicity, the argument s)

$$c_h\left[\gamma_{1^{h+1}} - (\gamma_{1^{h+1}} \cdot u)u\right] = \sum_{i=1}^{h} c_h f^{(i+1)}(x)(\gamma_{10^h} \cdot u)\Gamma_h^i + f^{(i)}(x)\dot{\Gamma}_h^i$$

by (1.1), namely

$$\begin{aligned} c_h\left[\gamma_{1^{h+1}} - (\gamma_{1^{h+1}} \cdot u)u\right] &= c_h \sum_{i=2}^{h+1} f^{(i)}(x)(\gamma_{10^h} \cdot u)\Gamma_h^{i-1} + c_h \sum_{i=1}^{h} f^{(i)}(x)\left[\Gamma_{h+1}^i - (\gamma_{10^h} \cdot u)\Gamma_h^{i-1}\right] \\ &= c_h\Bigg(f'(x)\Gamma_{h+1}^1 + \sum_{i=2}^{h} f^{(i)}(x)\left[\Gamma_{h+1}^i - (\gamma_{10^h} \cdot u)\Gamma_h^{i-1} + (\gamma_{10^h} \cdot u)\Gamma_h^{i-1}\right] \\ &\qquad + f^{(h+1)}(x)\Gamma_{h+1}^{h+1}\Bigg) \end{aligned}$$

by (3.1), (3.4), (3.5) and (1.1). Hence (3.12) follows recalling (3.7). $\square$

Now consider the $(H-1)$-order lower triangular matrix field

$$\Gamma := \begin{pmatrix} \Gamma_1^1 & 0 & \cdots & 0 \\ \Gamma_2^1 & \Gamma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{H-1}^1 & \Gamma_{H-1}^2 & \cdots & \Gamma_{H-1}^{H-1} \end{pmatrix}$$

and the orthogonal projection

$$P : \mathbf{R}^{1+k} \to (\mathbf{R}u)^\perp$$

that is

$$Pv = v - (v \cdot u)u = (u \wedge v) \llcorner u.$$

One has the following result.

Corollary 3.1. *Let $\varepsilon > 0$ and $s \in L_\varepsilon^*$. The following facts hold:*

(1) *The matrix $\Gamma(s)$ is invertible;*
(2) *If $N_{ij}(s)$ denote the elements of $\Gamma(s)^{-1}$ and define*

$$\Theta_i(s) := \sum_{j=1}^{i} N_{ij}(s)\, \gamma_{1^j}(s), \qquad i \in \{1, \dots, H-1\} \tag{3.13}$$

then

$$f^{(i)}(x(s)) = P\Theta_i(s) = [u \wedge \Theta_i(s)] \llcorner u, \qquad i \in \{1, \dots, H-1\}. \tag{3.14}$$

Proof. (1) is an immediate consequence of (3.9). As for (2), let $\{e_1, \dots, e_k\}$ be an orthonormal basis of $(\mathbf{R}u)^\perp$ and set

$$f_m := f \cdot e_m, \qquad m \in \{1, \dots, k\}.$$

Then the equality (3.10) can be written as follows

$$\Gamma(s) \left(f_m^{(1)}(x(s)), \dots, f_m^{(H-1)}(x(s)) \right)^t = ([\gamma_1(s) \cdot e_m], \dots, [\gamma_{1^{H-1}}(s) \cdot e_m])^t$$

i.e.

$$\left(f_m^{(1)}(x(s)), \dots, f_m^{(H-1)}(x(s)) \right)^t = \Gamma(s)^{-1} ([\gamma_1(s) \cdot e_m], \dots, [\gamma_{1^{H-1}}(s) \cdot e_m])^t$$

for all $m \in \{1, \dots, k\}$. Hence we get

$$f^{(i)}(x(s)) = \sum_{m=1}^{k} f_m^{(i)}(x(s)) e_m = \sum_{m=1}^{k} \sum_{j=1}^{i} N_{ij}(s) [\gamma_{1^j}(s) \cdot e_m] e_m$$

for all $i \in \{1, \dots, H-1\}$, that is just (3.14). □

4. The proof of Theorem 2.1
Preliminaries II: Taylor-type residues formulae

Let us continue to consider $\{\gamma_\alpha\}$, $\{c_h\}$, u and f as in the statement of Theorem 2.1. The following result provides the highest order Taylor-type residue formula we are interested to, in order to apply the Whitney extension theory.

Theorem 4.1. *Let $\varepsilon > 0$ and $s \in L^*_\varepsilon$. Define*

$$\Delta_s := \gamma_0 - \gamma_0(s),$$

$$\Phi_{1,s} := \gamma_1 - \sum_{i=1}^{H-1} \frac{(\Delta_s \cdot u)^{i-1}\Gamma_1^1}{(i-1)!}\Theta_i(s)$$

and, for $h \in \{2, \dots, H-1\}$ (provided $H \geq 3$)

$$\Phi_{h,s} := \gamma_{1^h} - \sum_{i=1}^{h-1} \Theta_i(s) \sum_{j=0}^{i-1} \frac{(\Delta_s \cdot u)^j \Gamma_h^{i-j}}{j!} - \sum_{i=h}^{H-1} \Theta_i(s) \sum_{j=i-h}^{i-1} \frac{(\Delta_s \cdot u)^j \Gamma_h^{i-j}}{j!}.$$

Then the maps $\Phi_{h,s}$ are Lipschitz, for all $h \in \{1, \dots, H-1\}$, and the following equalities hold:

(1) *For all $t \in \gamma_0^{-1}(G_f)$,*

$$f(x(t)) - \sum_{i=0}^{H-1} \frac{f^{(i)}(x(s))}{i!}[x(t) - x(s)]^i = P\left(\int_s^t c_0 \Phi_{1,s}\right) = \left(u \wedge \int_s^t c_0 \Phi_{1,s}\right) \llcorner u;$$

(2) *For $h \in \{1, \dots, H-1\}$,*

$$\Phi_{h,s}(s) = 0;$$

(3) *For $\rho \in [a, b]$, $H \geq 3$ and $h \in \{1, \dots, H-2\}$,*

$$\Phi_{h,s}(\rho) = \int_s^\rho c_h \Phi_{h+1,s}.$$

Proof. Since the maps γ_α are Lipschitz, the functions Γ_h^j have to be Lipschitz too. Hence the lipschitzianity of the $\Phi_{h,s}$ follows.

(1) Let $t \in \gamma_0^{-1}(G_f)$. Then, invoking (3.14) and observing that

$$f(x(t)) - f(x(s)) = \gamma_0(t) - x(t)u - [\gamma_0(s) - x(s)u] = \Delta_s(t) - [\Delta_s(t) \cdot u]u = P(\Delta_s(t)),$$

we obtain

$$\begin{aligned} f(x(t)) - \sum_{i=0}^{H-1} \frac{f^{(i)}(x(s))}{i!}[x(t) - x(s)]^i &= P\left(\Delta_s(t) - \sum_{i=1}^{H-1} \frac{\Theta_i(s)}{i!}[\Delta_s(t) \cdot u]^i\right) \\ &= P\left(\int_s^t \left[\dot{\gamma}_0 - \sum_{i=1}^{H-1} \frac{\Theta_i(s)}{i!} i(\Delta_s \cdot u)^{i-1}(\dot{\gamma}_0 \cdot u)\right]\right). \end{aligned}$$

The first claim follows now, recalling that

$$\dot{\gamma}_0 = c_0 \gamma_1, \qquad \text{a.e. in } [a, b]$$

by (1.1), while

$$\gamma_1 \cdot u = \Gamma_1^1$$

by (3.3).

(2) Indeed, for $h = 1$, one has

$$\Theta_1(s) = N_{1,1}(s)\gamma_1(s) = \frac{\gamma_1(s)}{\Gamma_1^1(s)}$$

by (3.13), hence

$$\Phi_{1,s}(s) = \gamma_1(s) - \sum_{i=1}^{H-1} \frac{[\Delta_s(s) \cdot u]^{i-1}\Gamma_1^1(s)}{(i-1)!}\Theta_i(s) = \gamma_1(s) - \Gamma_1^1(s)\Theta_1(s) = 0.$$

As for $H \geq 3$ and $h \in \{2, \ldots, H-1\}$ (note: for $H = 2$ there is nothing more to prove), we find

$$\begin{aligned}
\Phi_{h,s}(s) &= \gamma_{1^h}(s) - \sum_{i=1}^{h-1} \Theta_i(s)\Gamma_h^i(s) - \Theta_h(s)\Gamma_h^h(s) \\
&= \gamma_{1^h}(s) - \sum_{i=1}^{h} \Theta_i(s)\Gamma_h^i(s) \\
&= \gamma_{1^h}(s) - \sum_{i=1}^{h}\sum_{j=1}^{i} N_{ij}(s)\Gamma_h^i(s)\gamma_{1^j}(s) \\
&= \gamma_{1^h}(s) - \sum_{i=1}^{H-1}\sum_{j=1}^{H-1} N_{ij}(s)\Gamma_h^i(s)\gamma_{1^j}(s) \\
&= \gamma_{1^h}(s) - \sum_{j=1}^{H-1} \delta_{hj}\gamma_{1^j}(s) \\
&= 0
\end{aligned}$$

again by (3.13).

(3) According to the claim, we assume $H \geq 3$. Since the $\Phi_{h,s}$ are Lipschitz and vanish at s, by (2), one has

$$\Phi_{h,s}(\rho) = \Phi_{h,s}(\rho) - \Phi_{h,s}(s) = \int_s^\rho \dot{\Phi}_{h,s} \tag{4.1}$$

for all $\rho \in [a, b]$ and $h \in \{1, \ldots, H-2\}$. Hence, by also recalling (1.1), (3.4) and (3.5), it follows that

$$\begin{aligned}
\Phi_{1,s}(\rho) &= \int_s^\rho \dot{\gamma}_1 - \dot{\Gamma}_1^1 \sum_{i=1}^{H-1} \frac{(\Delta_s \cdot u)^{i-1}}{(i-1)!}\Theta_i(s) - \Gamma_1^1 \sum_{i=2}^{H-1} \frac{(\Delta_s \cdot u)^{i-2}(\dot{\gamma}_0 \cdot u)}{(i-2)!}\Theta_i(s) \\
&= \int_s^\rho c_1 \left(\gamma_{1^2} - \Gamma_2^1 \sum_{i=1}^{H-1} \frac{(\Delta_s \cdot u)^{i-1}}{(i-1)!}\Theta_i(s) - \Gamma_2^2 \sum_{i=2}^{H-1} \frac{(\Delta_s \cdot u)^{i-2}}{(i-2)!}\Theta_i(s)\right) \\
&= \int_s^\rho c_1 \left(\gamma_{1^2} - \Gamma_2^1\Theta_1(s) - \sum_{i=2}^{H-1} \left[\frac{(\Delta_s \cdot u)^{i-2}}{(i-2)!}\Gamma_2^2 + \frac{(\Delta_s \cdot u)^{i-1}}{(i-1)!}\Gamma_2^1\right] \Theta_i(s)\right) \\
&= \int_s^\rho c_1\Phi_{2,s}
\end{aligned}$$

for all $\rho \in [a,b]$, which completes the proof in the case $H = 3$. For $H \geq 4$ and $h = 2$, by (4.1), (1.1), (3.4) and (3.5), we get

$$
\begin{aligned}
\Phi_{2,s}(\rho) &= \int_s^\rho \dot{\gamma}_{1^2} - \Theta_1(s)\dot{\Gamma}_2^1 - \Theta_2(s)\left[\dot{\Gamma}_2^2 + (\dot{\gamma}_0 \cdot u)\Gamma_2^1 + (\Delta_s \cdot u)\dot{\Gamma}_2^1\right] + \\
&\qquad - \sum_{i=3}^{H-1} \Theta_i(s)\left[\frac{(\Delta_s \cdot u)^{i-2}}{(i-2)!}\Gamma_2^2 + \frac{(\Delta_s \cdot u)^{i-1}}{(i-1)!}\Gamma_2^1\right]' \\
&= \int_s^\rho c_2\Big(\gamma_{1^3} - \Theta_1(s)\Gamma_3^1 - \Theta_2(s)\left[\Gamma_3^2 - (\gamma_{10^2} \cdot u)\Gamma_2^1 + (\gamma_{10^2} \cdot u)\Gamma_2^1 + (\Delta_s \cdot u)\Gamma_3^1\right] + \\
&\qquad - \sum_{i=3}^{H-1} \Theta_i(s)\Big[\frac{(\Delta_s \cdot u)^{i-3}(\gamma_{10^2} \cdot u)\Gamma_2^2}{(i-3)!} + \frac{(\Delta_s \cdot u)^{i-2}(\Gamma_3^2 - (\gamma_{10^2} \cdot u)\Gamma_2^1)}{(i-2)!} + \\
&\qquad + \frac{(\Delta_s \cdot u)^{i-2}(\gamma_{10^2} \cdot u)\Gamma_2^1}{(i-2)!} + \frac{(\Delta_s \cdot u)^{i-1}\Gamma_3^1}{(i-1)!}\Big]\Big) \\
&= \int_s^\rho c_2\Big(\gamma_{1^3} - \Theta_1(s)\Gamma_3^1 - \Theta_2(s)\left[\Gamma_3^2 + (\Delta_s \cdot u)\Gamma_3^1\right] - \sum_{i=3}^{H-1} \Theta_i(s) \sum_{j=i-3}^{i-1} \frac{(\Delta_s \cdot u)^j \Gamma_3^{i-j}}{j!}\Big) \\
&= \int_s^\rho c_2 \Phi_{3,s}
\end{aligned}
$$

for all $\rho \in [a,b]$. We are now reduced to the case $H \geq 5$. Under this assumption, consider $h \in \{3, \dots, H-2\}$ and observe that the following equalities hold trivially:

$$
\sum_{j=1}^{i-1} \frac{(\Delta_s \cdot u)^{j-1}}{(j-1)!}\Gamma_h^{i-j} - \frac{(\Delta_s \cdot u)^j}{j!}\Gamma_h^{i-j-1} = \Gamma_h^{i-1} \qquad (i = 2, \dots, h)
$$

and

$$
\sum_{j=i-h}^{i-1} \frac{(\Delta_s \cdot u)^{j-1}}{(j-1)!}\Gamma_h^{i-j} - \frac{(\Delta_s \cdot u)^j}{j!}\Gamma_h^{i-j-1} = \frac{(\Delta_s \cdot u)^{i-h-1}}{(i-h-1)!}\Gamma_h^h \qquad (i = h+1, \dots, H-1).
$$

Hence, invoking again (4.1), (1.1), (3.4) and (3.5), we find

$$
\begin{aligned}
\Phi_{h,s}(\rho) = \int_s^\rho \dot\gamma_{1^h} - \Theta_1(s)\dot\Gamma^1_h - \sum_{i=2}^{h-1}\Theta_i(s)&\left[\dot\Gamma^i_h + \sum_{j=1}^{i-1}\frac{(\Delta_s\cdot u)^{j-1}(\dot\gamma_0\cdot u)\Gamma^{i-j}_h}{(j-1)!} + \frac{(\Delta_s\cdot u)^j\dot\Gamma^{i-j}_h}{j!}\right] + \\
&- \Theta_h(s)\left[\dot\Gamma^h_h + \sum_{j=1}^{h-1}\frac{(\Delta_s\cdot u)^{j-1}(\dot\gamma_0\cdot u)\Gamma^{h-j}_h}{(j-1)!} + \frac{(\Delta_s\cdot u)^j\dot\Gamma^{h-j}_h}{j!}\right] + \\
&- \sum_{i=h+1}^{H-1}\Theta_i(s)\sum_{j=i-h}^{i-1}\frac{(\Delta_s\cdot u)^{j-1}(\dot\gamma_0\cdot u)\Gamma^{i-j}_h}{(j-1)!} + \frac{(\Delta_s\cdot u)^j\dot\Gamma^{i-j}_h}{j!} \\
= \int_s^\rho c_h\Bigg(\gamma_{1^{h+1}} - \Theta_1(s)\Gamma^1_{h+1} - \sum_{i=2}^{h}\Theta_i(s)\Bigg[&\Gamma^i_{h+1} - (\gamma_{10^h}\cdot u)\Gamma^{i-1}_h + \\
&+ \sum_{j=1}^{i-1}\frac{(\Delta_s\cdot u)^{j-1}(\gamma_{10^h}\cdot u)\Gamma^{i-j}_h}{(j-1)!} + \frac{(\Delta_s\cdot u)^j\left(\Gamma^{i-j}_{h+1} - (\gamma_{10^h}\cdot u)\Gamma^{i-j-1}_h\right)}{j!}\Bigg] + \\
- \sum_{i=h+1}^{H-1}\Theta_i(s)\Bigg[\sum_{j=i-h}^{i-1}&\frac{(\Delta_s\cdot u)^{j-1}(\gamma_{10^h}\cdot u)\Gamma^{i-j}_h}{(j-1)!} + \frac{(\Delta_s\cdot u)^j\left(\Gamma^{i-j}_{h+1} - (\gamma_{10^h}\cdot u)\Gamma^{i-j-1}_h\right)}{j!}\Bigg]\Bigg) \\
= \int_s^\rho c_h\Bigg(\gamma_{1^{h+1}} - \Theta_1(s)\Gamma^1_{h+1} + &\\
&- \sum_{i=2}^{h}\Theta_i(s)\left[\Gamma^i_{h+1} - (\gamma_{10^h}\cdot u)\Gamma^{i-1}_h + (\gamma_{10^h}\cdot u)\Gamma^{i-1}_h + \sum_{j=1}^{i-1}\frac{(\Delta_s\cdot u)^j}{j!}\Gamma^{i-j}_{h+1}\right] + \\
&- \sum_{i=h+1}^{H-1}\Theta_i(s)\left[\frac{(\Delta_s\cdot u)^{i-h-1}}{(i-h-1)!}(\gamma_{10^h}\cdot u)\Gamma^h_h + \sum_{j=i-h}^{i-1}\frac{(\Delta_s\cdot u)^j}{j!}\Gamma^{i-j}_{h+1}\right]\Bigg) \\
= \int_s^\rho c_h\Bigg(\gamma_{1^{h+1}} - \Theta_1(s)\Gamma^1_{h+1} - \sum_{i=2}^{h}\Theta_i(s)&\sum_{j=0}^{i-1}\frac{(\Delta_s\cdot u)^j}{j!}\Gamma^{i-j}_{h+1} + \\
&- \sum_{i=h+1}^{H-1}\Theta_i(s)\sum_{j=i-h-1}^{i-1}\frac{(\Delta_s\cdot u)^j}{j!}\Gamma^{i-j}_{h+1}\Bigg) \\
= \int_s^\rho c_h\Phi_{h+1,s}.
\end{aligned}
$$

□

In the following result, formulas for the Taylor residues, at $s\in L^*_\varepsilon$, of the $f^{(h)}(x)$ are provided in terms of the $\Phi_{i,s}$.

Theorem 4.2. *For $s,t\in[a,b]$ and $h\in\{0,1,\dots,H-1\}$, define*

$$R_{h,s}(t) := f^{(h)}(x(t)) - \sum_{i=h}^{H-1}\frac{f^{(i)}(x(s))}{(i-h)!}[x(t)-x(s)]^{i-h}.$$

Let $\varepsilon>0$. Then the following facts hold true:

(1) *For all* $s \in L_\varepsilon^*$ *and* $t \in \gamma_0^{-1}(G_f)$,

$$R_{0,s}(t) = \left(u \wedge \int_s^t c_0 \Phi_{1,s} \right) \llcorner u;$$

(2) *For all* $s, t \in L_\varepsilon^*$ *and* $h \in \{1, \dots, H-1\}$,

$$R_{h,s}(t) = \left(u \wedge \sum_{i=1}^{h} N_{hi}(t) \Phi_{i,s}(t) \right) \llcorner u.$$

Proof. The first claim just rephrases Theorem 4.1(1), so there is nothing more to prove.

As for the second one, it will be enough to prove that the following equality

$$\sum_{j=1}^{h} R_{j,s}(t) \Gamma_h^j(t) = [u \wedge \Phi_{h,s}(t)] \llcorner u \tag{4.2}$$

holds for all $s, t \in L_\varepsilon^*$ and $h \in \{1, \dots, H-1\}$. Indeed, the equality in (2) follows immediately from (4.2) by recalling that (N_{ij}) is the inverse matrix of Γ.

We shall prove (4.2) by induction. Since each $t \in L_\varepsilon^*$ is an accumulation point of L_ε^*, we can derive the formula in (1), thus getting

$$c_0(t)\, [u \wedge \Phi_{1,s}(t)] \llcorner u = R'_{0,s}(t) = R_{1,s}(t)\, x'(t) = c_0(t) R_{1,s}(t) \Gamma_1^1(t)$$

for all $s, t \in L_\varepsilon^*$, by (1.1) and (3.3). Hence the equality (4.2) with $h = 1$ follows at once by recalling (3.7). Now suppose $H \geq 3$ (for $H = 2$ the proof of (4.2) is completed) and assume (4.2) to be true for all $s, t \in L_\varepsilon^*$ and for a given $h \in \{1, \dots, H-2\}$. By the same argument as above, we can derive such an equality. Recalling Theorem 4.1(3), we find that

$$c_h(t)\, [u \wedge \Phi_{h+1,s}(t)] \llcorner u = \sum_{j=1}^{h} R'_{j,s}(t) \Gamma_h^j(t) + R_{j,s}(t) \dot{\Gamma}_h^j(t)$$

for all $s, t \in L_\varepsilon^*$. From (3.5) and since

$$R'_{j,s}(t) = R_{j+1,s}(t) x'(t) = R_{j+1,s}(t) c_h(t) (\gamma_{10^h}(t) \cdot u)$$

for all $s,t \in L^*_\varepsilon$, by (1.1), it follows that

$$\begin{aligned}
c_h(t)\,[u \wedge \Phi_{h+1,s}(t)] \llcorner u &= c_h(t)(\gamma_{10^h}(t)\cdot u)\sum_{j=1}^{h} R_{j+1,s}(t)\Gamma^j_h(t) + \\
&\quad + c_h(t)\sum_{j=1}^{h} R_{j,s}(t)\left[\Gamma^j_{h+1}(t) - (\gamma_{10^h}(t)\cdot u)\Gamma^{j-1}_h(t)\right] \\
&= c_h(t)(\gamma_{10^h}(t)\cdot u)\left(\sum_{j=1}^{h} R_{j+1,s}(t)\Gamma^j_h(t) - \sum_{j=1}^{h} R_{j,s}(t)\Gamma^{j-1}_h(t)\right) + \\
&\quad + c_h(t)\sum_{j=1}^{h} R_{j,s}(t)\Gamma^j_{h+1}(t) \\
&= c_h(t)(\gamma_{10^h}(t)\cdot u)\Big(R_{h+1,s}(t)\Gamma^h_h(t) - R_{1,s}(t)\Gamma^0_h(t)\Big) + \\
&\quad + c_h(t)\sum_{j=1}^{h} R_{j,s}(t)\Gamma^j_{h+1}(t)
\end{aligned}$$

for all $s,t \in L^*_\varepsilon$. Recalling (3.1), (3.4) and (3.7), we conclude that

$$[u \wedge \Phi_{h+1,s}(t)] \llcorner u = \sum_{j=1}^{h+1} R_{j,s}(t)\Gamma^j_{h+1}(t)$$

for all $s,t \in L^*_\varepsilon$. □

5. The proof of Theorem 2.1
Conclusion: Whitney-type estimates and the proof of (3.6)

In order to simplify many formulas below, for all $h \in \{0,\dots,H-2\}$, let us set

$$\|c_h\| := \sup_{[a,b]} |c_h|.$$

As a corollary of Theorem 4.1, we obtain the following estimate.

Proposition 5.1. *Let $\varepsilon > 0$ and $s \in L^*_\varepsilon$ and, for $h \in \{1,\dots,H-1\}$. Define*

$$\mu_{H-1}(s) := Lip(\Phi_{H-1,s})$$

and, in the case $H \geq 3$

$$\mu_h(s) := \frac{Lip(\Phi_{H-1,s})}{(H-h)!}\prod_{i=h}^{H-2}\|c_i\|, \qquad h \in \{1,\dots,H-2\}.$$

Then the inequality

$$\|\Phi_{h,s}(t)\| \leq \mu_h(s)|t-s|^{H-h} \tag{5.1}$$

holds for all $t \in [a,b]$ and $h \in \{1,\dots,H-1\}$.

Proof. First of all, observe that one has

$$\|\Phi_{H-1,s}(t)\| = \|\Phi_{H-1,s}(t) - \Phi_{H-1,s}(s)\| \leq \mathrm{Lip}(\Phi_{H-1,s})|t-s| = \mu_{H-1}(s)|t-s| \tag{5.2}$$

for all $t \in [a,b]$, by Theorem 4.1. In particular, for $H = 2$ the proof of (5.1) is completed.

Hence we are reduced to $H \geq 3$. In such a case, if (5.1) is verified for a certain $h \in \{2, \dots, H-1\}$ then it has to be true also for $h-1$. Indeed, by invoking again Theorem 4.1, we obtain

$$\|\Phi_{h-1,s}(t)\| = \left\| \int_s^t c_{h-1}\Phi_{h,s} \right\| \leq \frac{\|c_{h-1}\|\mu_h(s)}{H-h+1}\, |t-s|^{H-h+1} = \mu_{h-1}(s)|t-s|^{H-(h-1)}$$

for all $t \in [a,b]$. Recalling (5.2), the inequality (5.1) follows at once by induction. □

In order to proceed into proving our main theorem, we need to cover L_ε^* by sets where Whitney-type estimates hold uniformly. So (given $\varepsilon > 0$), for $j \in \{1, 2, \dots\}$, let us define $\Gamma_{\varepsilon,j}$ as the set of points $s \in L_\varepsilon^*$ such that the inequalities

$$\|R_{h,s}(t)\| \leq j\, |x(t) - x(s)|^{H-h}, \qquad h \in \{0, 1, \dots, H-1\} \tag{5.3}$$

hold for all $t \in L_\varepsilon^*$ satisfying $|t-s| \leq (b-a)/j$.

One obviously has

$$\Gamma_{\varepsilon,j} \subset \Gamma_{\varepsilon,j+1} \subset L_\varepsilon^* \tag{5.4}$$

for all j. More difficult is to prove that the $\Gamma_{\varepsilon,j}$ actually cover L_ε^*, as the following result states.

Proposition 5.2. *Let $\varepsilon > 0$. Then $\cup_j \Gamma_{\varepsilon,j} = L_\varepsilon^*$.*

Proof. By virtue of (5.4), it is enough to prove that any fixed $s \in L_\varepsilon^*$ has to belong to some $\Gamma_{\varepsilon,j}$.

To this aim, observe that

$$\begin{aligned} \|R_{0,s}(t)\| &\leq \left\| \int_s^t c_0 \Phi_{1,s} \right\| \leq \|c_0\| \left| \int_s^t \|\Phi_{1,s}\| \right| \leq \|c_0\|\mu_1(s) \left| \int_s^t |\tau - s|^{H-1} d\tau \right| \\ &= \frac{\|c_0\|\mu_1(s)}{H}|t-s|^H \end{aligned} \tag{5.5}$$

for all $t \in \gamma_0^{-1}(G_f)$, by Theorem 4.2(1) and (5.1).

Now we can invoke (3.9) and an obvious continuity argument to find an open interval I_s, centered at s, such that

$$\nu_h(s) := \max_i \sup_{I_s} |N_{hi}| < +\infty, \qquad h \in \{1, \dots, H-1\}.$$

Then, if $H \geq 3$ and $h \in \{1, \dots, H-2\}$, we find

$$\begin{aligned}
\|R_{h,s}(t)\| &\leq \left\| \sum_{i=1}^{h} N_{hi}(t)\Phi_{i,s}(t) \right\| \leq \sum_{i=1}^{h} |N_{hi}(t)| \, \|\Phi_{i,s}(t)\| \\
&\leq \nu_h(s) \sum_{i=1}^{h} \left| \int_s^t |c_i| \, \|\Phi_{i+1,s}\| \right| \\
&\leq \nu_h(s) \sum_{i=1}^{h} \|c_i\| \mu_{i+1}(s) \left| \int_s^t |\tau - s|^{H-i-1} d\tau \right| \qquad (5.6) \\
&= \nu_h(s) \sum_{i=1}^{h} \frac{\|c_i\| \mu_{i+1}(s)}{H-i} |t-s|^{H-i} \\
&\leq \nu_h(s) \left[\sum_{i=1}^{h} \frac{\|c_i\| \mu_{i+1}(s)}{H-i} (b-a)^{h-i} \right] |t-s|^{H-h}
\end{aligned}$$

for all $t \in L_\varepsilon^* \cap I_s$, by Theorem 4.2(2), Theorem 4.1(3) and (5.1).

As for $H \geq 3$ and $h = H-1$, we have

$$\|R_{H-1}(t)\| \leq \sum_{i=1}^{H-1} |N_{H-1,i}(t)| \, \|\Phi_{i,s}(t)\| \leq \nu_{H-1}(s) \left[\sum_{i=1}^{H-1} \mathrm{Lip}(\Phi_{i,s}) \right] |t-s| \tag{5.7}$$

for all $t \in L_\varepsilon^* \cap I_s$, by Theorem 4.2(2) and Theorem 4.1(2).

From the inequalities (5.5), (5.6) and (5.7) it follows that a constant $C(H, s)$ has to exist such that

$$\|R_{h,s}(t)\| \leq C(H, s)|t-s|^{H-h}, \qquad h \in \{0, 1, \dots, H-1\}$$

for all $t \in L_\varepsilon^* \cap I_s$. Since x is differentiable at s and $x'(s) \neq 0$, by (3.8), one has

$$\left| \frac{x(t) - x(s)}{t - s} \right| \geq \frac{|x'(s)|}{2} > 0$$

provided $|t - s|$ is small enough. Then

$$\|R_{h,s}(t)\| \leq \frac{2^{H-h} C(H, s)}{|x'(s)|^{H-h}} |x(t) - x(s)|^{H-h}, \qquad h \in \{0, 1, \dots, H-1\}$$

whenever $t \in L_\varepsilon^*$ and $|t - s|$ is small enough. Hence the conclusion follows immediately. □

We can now proceed to the proof of (3.6).

Proof of (3.6). First of all observe that, as a consequence of Proposition 5.2, we are reduced to prove that

$$\gamma_0(\Gamma_{\varepsilon,j}) \text{ is } C^H\text{-rectifiable} \tag{5.8}$$

for all $\varepsilon > 0$ and $j \in \{1, 2 \dots\}$. To this aim, let us define

$$a_{ji} := a + \frac{(b-a)i}{j}$$

for $i \in \{0, 1, \dots, j\}$ and

$$\Gamma_{ji} := \Gamma_{\varepsilon,j} \cap [a_{ji}, a_{j,i+1}], \qquad F_{ji} := \overline{x(\Gamma_{ji})}$$

for $i \in \{0, \dots, j-1\}$. Then, given arbitrarily

$$\xi, \eta \in F_{ji},$$

two sequences

$$\{s_l\}, \{t_l\} \subset \Gamma_{ji}$$

have to exist such that

$$\lim_l x(s_l) = \xi, \qquad \lim_l x(t_l) = \eta.$$

Since (5.3) holds with $s = s_l$ and $t = t_l$, namely

$$\|R_{h,s_l}(t_l)\| \leq j|x(t_l) - x(s_l)|^{H-h}, \qquad h \in \{0, 1, \dots, H-1\},$$

we get (by letting $l \to \infty$)

$$\left\| f^{(h)}(\eta) - \sum_{i=h}^{H-1} \frac{f^{(i)}(\xi)}{(i-h)!}(\eta - \xi)^{i-h} \right\| \leq j\,|\eta - \xi|^{H-h}, \qquad h \in \{0, 1, \dots, H-1\}.$$

By the Whitney extension Theorem [12, Ch. VI, §2.3], it follows that each $f|F_{ji}$ can be extended to a map in $C^{H-1,1}(\mathbf{R}, (\mathbf{R}u)^\perp)$. Finally the Lusin type result [10, §3.1.15] implies that $\gamma_0(\Gamma_{ji})$ is C^H-rectifiable (compare [2, Proposition 3.2]). Hence (5.8) follows. □

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