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ALGEBRAIC PROPERTIES AND INVARIANTS OF POLYOMINOES

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To my family...

Contents

INTRODUCTION

In the last thirty years, a new and extremely prolific research area, called "Combinatorial Commutative Algebra", greatly developed. It was born from the intersection of two research areas, namely Commutative Algebra and Combinatorics. The starting point of this field was the proof of the "Upper bound conjecture for simplicial spheres" done by Stanley in 1975, by using Commutative Algebra methods. In the latter, the study of squarefree monomial ideals from the viewpoints of both commutative algebra and combinatorics gave rise to a very active research area on these topics. Some relevant references for this area include the monographs of Stanley [74], Bruns-Herzog [12], Miller-Sturmfels [57] and Herzog-Hibi [36]. One of the most common problems in Combinatorial commutative algebra is the one of studying how the combinatorial properties of the considered objects affect the algebraic invariants of the related structures. For example, algebraic properties such as the unmixedness or the Castelnuovo-Mumford regularity of the edge ideal of a graph are strictly related to combinatorial aspects and invariants, such as the pureness of the independence complex associated to the graph or the induced matching number of the graph.

In the early 1990s, the study of binomial ideals became of interest, due to their relevance in commutative algebra, combinatorics but also in other research areas, such as statics and symbolic computation. Binomial ideals are studied exhaustively by Eisenbud-Sturmfels [25], together with their primary decomposition. Among binomial ideals, toric ideals have a particular relevance due to their relation with algebraic geometry, in particular with the study of the toric varieties. Algebraic and combinatorial properties of toric ideals are efficiently studied in Sturmfels' book [76].

One of the most natural classes of binomial ideals arising from combinatorics is the class of join-meet ideals of finite lattices. The basic properties of finite lattices are studied in Birkhoff's book [6]. A prime join-meet ideal of a finite lattice is a toric ideal, and this is the case if and only if the lattice is distributive. By Birkhoff's theorem, any distributive lattice arises from a set of particular subsets of a given poset P, the poset ideals. Hibi

proved that some algebraic properties of toric rings of finite distributive lattices, such as the Gorensteinnes, are related to the combinatorial properties of this poset P. For this, these rings are today called Hibi rings.
Another interesting class of binomial ideals is the one of those generated by a subset of 2-minors of a $m \times n$ matrix X_{mn}. The study of the ideal of t-minors and related ideals of an $m \times n$ matrix X_{mn} of indeterminates is a classical subject of commutative algebra and algebraic geometry; see for example the lecture notes [13] and its references to original articles. Several years after the appearance of these lecture notes, a new aspect of the theory was introduced by considering Gröbner bases of determinantal ideals. This dissertation focuses on the study of the algebraic properties and invariants of a class of binomial ideals arising from 2-minors, the polyomino ideals.

Polyominoes are two-dimensional objects obtained by joining edge by edge squares of same size. Classical examples of polyominoes are Ferrer diagrams, stack and parallelogram polyominoes. They have a long history in combinatorics. Originally, polyominoes appeared in mathematical recreations, but it turned out that they have applications in various fields, for example, theoretical physics and bio-informatics. Among the most popular topics in combinatorics related to polyominoes one finds enumerating polyominoes of given size, including the asymptotic growth of the numbers of polyominoes, tiling problems, and reconstruction of polyominoes. A very nice introduction to the combinatorics of polyominoes and tilings is given in the monograph [80].

Recently in [62], Qureshi introduced a binomial ideal induced by the geometry of a given polyomino, called polyomino ideal, and its related algebra. From that moment different authors studied algebraic properties related to this ideal (see [40, 65, 73, 53]).

One of the first algebraic properties of polyomino ideals that has been studied is the primality. In particular, in [40, 65] the authors proved that if the polyomino is simple, namely the polyomino has no holes, then its coordinate ring is a Cohen-Macaulay domain. However, the classification of those polyominoes whose coordinate ring is a domain is still an open and challenging problem. One of the sections of this dissertation focuses on the study of the primality of some polyomino ideals, in particular a necessary condition for the primality could be established by looking at the geometric realization of the polyomino.

Furthermore, for some classes of polyominoes, the Gröbner basis of the polyomino ideal is a fundamental tool to determine its primality. Since the polyomino ideal is generated in degree two, the polyomino ideals whose Gröbner basis is quadratic are of interest. One of the chapters of this dissertation is devoted to the characterization of the polyominoes having a quadratic Gröbner basis with respect to $<_{\text{lex}}$ up to any rotation or/and reflection. Another property of the coordinate ring of polyomino ideals that has been studied at first is the Gorensteinnes. The classification of Cohen-Macaulay rings is a general problem in commutative algebra. Among Cohen-Macaulay rings, it is interesting to classify the Gorenstein ones, namely the ones having (Cohen-Macaulay) type equal to 1. The Gorensteinnes of stack polyominoes and convex polyominoes is studied in [62] and [1], respectively. In the latter, the author gives an upper bound for a fundamental invariant, the Castelnuovo-Mumford regularity, in the case of convex polyominoes. The argument of the Castelnuovo-Mumford regularity is also treated in [27], where, for L-convex polyominoes, it is related to a combinatorial invariant, the rook number. The well-known "rook problem" is the problem of enumerating the number of ways of placing k non-attacking rooks on a chessboard. In a similar way, one can define a rook polynomial, associated to such numbers of configurations. For Cohen-Macaulay rings, there is a strong connection between the Castelnuovo-Mumford regularity and the reduced Hilbert series. This motivated us to study the reduced Hilbert series of the coordinate ring in terms of the rook polynomial of the polyomino for the classes of simple thin polyominoes and parallelogram polyominoes. For these classes of polyominoes, we also give a characterization of the Gorenstein ones. For the Gorensteinnes of simple thin polyominoes, we use fundamental characterization due to Stanley [78, Theorem 4.4] about the Hilbert series of Cohen-Macaulay domains. For the Gorensteinnes of parallelogram polyominoes, we use a characterization of Hibi on distributive lattices.

In the following a more detailed description of each chapter is given.

In Chapter 1, we recall some notions and definitions from Commutative Algebra, Combinatorics and Lattice theory. In particular, we are interested in some algebraic invariants, such as Krull dimension and depth (Section 1.1), Betti numbers and Castelnuovo-Mumford regularity (Section 1.4), in some algebraic properties, such as the Cohen-Macaulayness (Section 1.1) and the Gorensteinnes (Section 1.4), and in some algebraic tools, such as Gröbner bases (Section 1.2), Hilbert series (Section 1.3), Hibi rings (Section 1.5) and Edge rings associated to graphs (Section 1.6).

In Chapter 2, we widely discuss polyominoes, that are the kernel of this dissertation. In particular, we firstly present their history and their usage in tiling problems and then we give some formal definitions that are helpful to their study (Section 2.1). Afterwards we focus on the combinatorial properties of two significant classes of polyominoes: L-convex polyominoes (Section 2.2) and parallelogram polyominoes (Section 2.3). Finally, we talk over the Commutative Algebra of Polyominoes, by presenting the existing results on the primality and Gorensteinnes of the polyomino ideal (Section 2.4).

In Chapter 3, we discuss questions about the primality and the Gröbner basis of the polyomino ideal. In particular, we give some combinatorial necessary condition for the primality of the polyomino ideal, that we conjecture being also sufficient, we verify it computationally for polyominoes with at most 14 cells and then we find some infinite class of prime polyominoes (Section 3.1). Furthermore, we present a necessary and sufficient condition on the geometry of polyominoes for quadratic Gröbner basis with respect to particular monomial orders and we give two infinite families of prime polyominoes satisfying such conditions (Section 3.2).

In Chapter 4, we talk over the Castelnuovo-Mumford regularity, Hilbert series, and Gorensteinnes of the polyomino ideal and its coordinate ring. In particular, we compute the Castelnuovo-Mumford regularity for the class of L-convex polyominoes in terms of the maximum number of rooks that can be placed on the polyomino and we characterize Gorenstein ones (Section 4.1). Furthermore, along the same line, we link the Hilbert series of the coordinate ring to particular rook arrangements on the polyominoes for two classes of polyominoes, simple thin polyominoes (Section 4.2) and parallelogram polyominoes (Section 4.3), and we also characterize Gorenstein ones.

Finally in Chapter 5, we present some additional results obtained during the PhD that do not involve polyominoes. In particular, we characterize the chordal circulant graphs to obtain information on the Castelnuovo-Mumford regularity of their edge ideal (Section 5.1). Furthermore, we focus on the Serre condition and Cohen-Macaulayness of binomial edge ideals of graphs, by linking them to the combinatorial conditions of accessibility and strongly unmixedness (Section 5.2).

Chapter 1

PRELIMINARIES

In this chapter we recall some definitions and notions from both Commutative Algebra and Combinatorics that are useful in the following, with the hope of keeping the dissertation as self-contained as possible. In Sections 1.1, 1.2, 1.3, 1.4 we discuss some fundamental objects and invariants that arise from Commutative Algebra, while in Section 1.5 we present some foundations of lattice theory that play a crucial role in the study of polyominoes. Finally, in Section 1.6 we talk over the edge rings associated to graphs. In Section 1.1, we discuss Cohen-Macaulay rings, namely those rings for which the Krull dimension coincides with the depth, that is the maximal length of a regular sequence. The classification of Cohen-Macaulay rings is a wide open problem in the research area of Commutative Algebra. Nevertheless, the Cohen-Macaulayness of the rings arising from combinatorics, e.g. edge ideal of graphs, binomial edge ideals, Stanley-Reisner rings, polyomino ideals, is strictly related to the combinatorial properties of the studied objects. In Section 1.2, we give some basic properties about Gröbner Bases. The latter are relevant in the study of multi-variate polynomial rings, since they are sets of generators of ideals, and they can be computed efficiently with some algorithms. In Section 1.3, we focus our attention on Hilbert function and Hilbert series. The Hilbert function, roughly speaking, counts the dimensions as K-vector space of a given graded module. Starting by the Hilbert function, one constructs the Hilbert series. The latter can be expressed as a rational function, involving also Krull dimension. In Section 1.4, we talk over the minimal free resolution of a graded module, that is a complex of modules in which any map is minimal. These numbers of generators are called Betti numbers and they play a key role in the definition of the Castelnuovo-Mumford regularity, that is an invariant that measures the “complexity” of the given module. In the Cohen-Macaulay case, Betti numbers and Castelnuovo-Mumford regularity are also related to the concept of Gorensteinnes and to the Hilbert series. In Section 1.5 we present lattice theory in a nutshell, by focusing on the properties of the ideals that can be associated to lattices, join-meet ideals. Finally, in Section 1.6, we investigate the relation between bipartite graphs and toric ideals, useful for the study of the polyomino ideals. For further information about the discussed topics one can refer to [12] (Section 1.1), [36] (Section 1.2), [82], [23] (Sections 1.3 and 1.4), [6] and [38] (Section 1.5).

1.1 KRULL DIMENSION, DEPTH AND COHEN-MACAULAY RINGS

This section is devoted to the introduction of the fundamental invariants for the discussion of Cohen-Macaulay rings. Cohen-Macaulay rings are important algebraic structures that have been studied for years (see [12]). The classification of the rings that are Cohen-Macaulay is a challenging problem in the actual research field of Commutative Algebra. The Cohen-Macaulay property plays also a crucial role in algebraic combinatorics, for example for graphs, simplicial complexes and polyominoes. Cohen-Macaulayness involves two algebraic invariants, the Krull dimension and the depth. This section also contains a little overview about graded rings and modules, because these allow us to decompose their elements in homogeneous components.

Definition 1.1.1 *Let R be a commutative ring with unit. The* Krull dimension *of R,* $\dim R$*, is the supremum of the lengths of all chains of prime ideals of R.*

In other words if

$$\mathcal{C}:\ \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$$

is a chain of prime ideals of R, we say that C has length n. Therefore, the Krull dimension is the supremum of these lengths. If the supremum of the lengths is not finite, we say that the Krull dimension is $+\infty$. As an example if $\mathbb{K}$ is a field then $\dim \mathbb{K} = 0$ since (0) is the unique prime ideal in any field. We have the following

Theorem 1.1.2. *Let $R[x]$ be a polynomial ring over a Noetherian ring R. Then* $\dim R[x] = \dim R + 1$*.*

By applying recursively Theorem 1.1.2 to $R = \mathbb{K}[x_1, \ldots, x_n]$, since for any $i \in \{2, \ldots, n\}$ it holds $\mathbb{K}[x_1, \ldots, x_i] = \mathbb{K}[x_1, \ldots, x_{i-1}][x_i]$, it follows that $\dim R = n$.

Another invariant related to Krull dimension, is the height of an ideal. We recall that $\mathrm{Spec}(R)$ is the set of all prime ideals of R. Let $\mathfrak{p}$ be a prime ideal in R, the *height* of $\mathfrak{p}$, denoted by height $(\mathfrak{p})$ is the supremum of all chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ which end at $\mathfrak{p}$. One has $\dim(R_\mathfrak{p}) = \text{height } (\mathfrak{p})$, where $R_\mathfrak{p}$ is the *localization* of R at $\mathfrak{p}$, that is

$$R_\mathfrak{p} := \left\{ \frac{a}{b} \mid a \in R \text{ and } b \in R \setminus \mathfrak{p} \right\}.$$

If I is an ideal of R, the height of I, is

$$\text{height } (I) := \min\{\text{height } (\mathfrak{p}) | \, I \subset \mathfrak{p} \text{ and } \mathfrak{p} \in \operatorname{Spec}(R)\}.$$

In general $\dim(R/I) + \text{height } (I) \leq \dim(R)$. Furthermore, given an ideal I and $f \in R$ we define the *ideal quotient* or *colon ideal*, $(I : f)$, as

$$(I : f) := \{h \in R \mid hf \in I\}.$$

As we will introduce Krull dimension on modules, we first recall a basic definition for an R-module M.

Definition 1.1.3 *Let R be a commutative ring with unit, and let M be an R-module. The* annihilator *of M on R is*

$$\operatorname{Ann}_R M := \{r \in R \mid \forall m \in M : \ rm = 0\}.$$

Hence, the *Krull dimension* of the R-module M is

$$\dim_R M := \dim(R/\operatorname{Ann_R}(M)).$$

To define the depth, we give the definition of regular sequence on M R-module. First of all we point out that $r \in R$ is a *non-zero divisor on* M if $rm = 0$ implies $m = 0$ for $m \in M$.

Definition 1.1.4 *Let M be an R-module. An* **M**-regular sequence *is a sequence of elements $r_1, \ldots, r_d \in R$ such that:*

a) r_1 is a non-zero divisor on M and for any $i \in \{2, \ldots, d\}$ the element r_i is a non-zero divisor on $M/(r_1, \ldots, r_{i-1})M$;

b) $(r_1, \ldots, r_d)M \neq M$.

Before introducing the depth, we briefly discuss graded rings and modules, and homogeneous ideals.

Definition 1.1.5 *Let $(H, +)$ be an abelian semigroup and let $(R, +, \cdot)$ be a commutative ring with unit. R is a $H-$*graded ring *if*

$$R = \bigoplus_{a \in H} R_a \quad (\text{ as a } \mathbb{Z}\text{- module})$$

such that if $x \in R_a$ and $y \in R_b$ then $xy \in R_{a+b}$, namely $R_a R_b \subseteq R_{a+b}$ for any $a, b \in H$. The direct sum is called grading *or* gradation*.*

The elements of R_a are called *homogeneous elements* of *degree a* and we refer to the degree of an element $x \in R$ as $\deg(x)$. Any element r of R may be written uniquely as a linear combination of $r_{a_i} \in R_{a_i}$

$$r = r_{a_1} + r_{a_2} + \ldots + r_{a_d}$$

and the r_{a_i} are called the *homogeneous components* of r.

We introduce gradings on modules, as well as for rings.

Definition 1.1.6 *Let R be an H-graded commutative ring with unit, and let M be an R-module. M is an H-*graded module *if*

$$M = \bigoplus_{a \in H} M_a$$

such that $M_a R_b, R_a M_b \subseteq M_{a+b}$, for any $a, b \in H$.

If we pick $a \in H$ and M graded R-module, we say that $M(a)$ is the module M *shifted* by a, in other words

$$M(a)_d = M_{a+d}.$$

In particular, the free R-module generated by an element m of degree a is $R(-a)$. We now want to grade the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$ on the field $\mathbb{K}$. Let $d_i \in \mathbb{N}_+$, for $i = 1, \ldots, n$; for $a = (a_1, \ldots, a_n)$ in $\mathbb{N}^n$ we set $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $|a| = a_1 d_1 + \cdots + a_n d_n$. We consider the *induced* $\mathbb{N}$-*grading*

$$R = \bigoplus_{i=0}^{\infty} R_i, \text{ where } R_i = \bigoplus_{|a|=i} \mathbb{K} x^a.$$

Definition 1.1.7 *The* standard grading *or* usual grading *of a polynomial ring* $\mathbb{K}[x_1, \ldots, x_n]$ *is the* $\mathbb{N}$*-grading induced by setting* $\deg(x_i) = 1$*, for all* i*.*

An ideal $I \subset R$ is called *homogeneous* if for all $r \in I$, its homogeneous components live in I. If I is a homogeneous ideal in $R = \bigoplus_{i=0}^{\infty} R_i$, then R/I is also graded with grading

$$R/I = \bigoplus_{i=0}^{\infty} (R/I)_i = \bigoplus_{i=0}^{\infty} R_i/I_i$$

where $I_i = R_i \cap I$. Let $R = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over the field $\mathbb{K}$. To state the definition of depth, we use the notation $\mathfrak{m} = (x_1, \ldots, x_n)$, namely the homogeneous maximal ideal of R.

Definition 1.1.8 *Let* M *be a finitely generated graded* R*-module. The* depth *of* M*, denoted by* depth M*, is the supremum of the lengths of all* M*-regular sequences of elements of* $\mathfrak{m}$*.*

In general it holds that depth $M \leq \dim M$. If the equality holds, then M is a *Cohen-Macaulay module.*

Definition 1.1.9 *Let* M *be a finitely generated graded* R*-module. We say that* M *is a* Cohen-Macaulay module *if*

$$\text{depth } M = \dim M.$$

Example 1.1.10 *Let* $R = \mathbb{K}[x_1, \ldots, x_n]$ *be a polynomial ring over the field* $\mathbb{K}$*. We already observed that* $\dim R = n$*.*
We observe that the sequence $x_1, x_2, \ldots, x_n$ *is an* M*-regular since it satisfies a) and b) of Definition 1.1.4 and trivially the sequence lives in* $\mathfrak{m} = (x_1, \ldots, x_n)$*. It follows that* depth $R = n$*. Therefore, we have that* R *is a Cohen-Macaulay ring.*

Example 1.1.11 *Let* $R = \mathbb{K}[x_1, x_2, x_3]$ *and* $I = (x_1x_2, x_2x_3, x_1x_3)$*. We want to find the Krull dimension of* R/I*, namely the supremum of the lengths of the chain of prime ideals containing* I*. First of all, we observe that the primary decomposition of* I *is*

$$(x_1, x_2) \cap (x_2, x_3) \cap (x_1, x_3).$$

Therefore, the primes containing I are (x_1, x_2), (x_2, x_3) and (x_1, x_3). For example, we take $\mathfrak{p}_0 = (x_1, x_2)$ that is a prime ideal. It is contained in $\mathfrak{p}_1 = (x_1, x_2, x_3)$. The chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1$$

is the longest possible. Hence $\dim R/I = 1$. We now want to compute the depth of R/I. We know that $\text{depth } R/I \leq \dim R/I = 1$ so it can be either 0 or 1. If we find a non-zero divisor of R/I we obtain $\text{depth } R/I = 1$. The elements of R/I are the powers of the variables, namely x_i^d, for $i \in \{1, 2, 3\}$ and their combinations namely

$$x_1^{d_1} + x_2^{d_2} + x_3^{d_3}$$

with $d_i \geq 0$, for $i \in \{1, 2, 3\}$. When two of the d_i are 0, e.g. d_2 and d_3, we have $x_1^{d_1}$ that multiplied by x_2 or x_3 yields $x_1^{d_1} x_2 \in I$. When $d_3 = 0$, we have $x_1^{d_1} + x_2^{d_2}$, that multiplied by x_3 returns $x_1^{d_1} x_3 + x_2^{d_2} x_3 \in I$. Hence, to obtain a non-zero divisor all the d_i have to be different from 0. In particular, the element

$$x_1 + x_2 + x_3$$

is a non-zero divisor of R/I and it represents a regular sequence of length 1. Hence R/I is a Cohen-Macaulay ring.

1.2 INITIAL IDEAL, GRÖBNER BASES AND PRIMALITY PROPERTIES

In this section we roughly discuss monomial ideals, that are fundamental for the study of multivariate polynomial rings. In fact, if we equip a multivariate polynomial ring with a monomial order, we can study many properties of ideals by looking at their initial ideal, that is the ideal containing the greatest monomial of any element of the given ideal. Buchberger described an algorithm to compute a finite set of generators whose greatest monomials generate the initial ideal for any multivariate ideal. Such a set is called *Gröbner basis*. They can be computed efficiently by using Buchberger's algorithm, and they have a crucial role in the study of systems of polynomial equations, hence in the study of algebraic varieties, but also in the computation of the radical of an ideal and in elimination theory.

Let $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ be a polynomial ring on the field $\mathbb{K}$. Any product $x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \in \mathbb{N}$ is called a *monomial*, shortened with $\mathbf{x}^{\mathbf{a}}$ where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$. We call

$\mathcal{M}$ the set of all monomials of R and we observe that it is a $\mathbb{K}$-basis for R. Therefore, for any $f \in R$ we have

$$f = \sum_{m \in \mathcal{M}} a_m m$$

with $a_m \in \mathbb{K}$ and we define the *support* of f as

$$\mathrm{supp}(f) = \{m \in \mathcal{M} \mid a_m \neq 0\}$$

for a finite number of $m \in \mathcal{M}$.

A *monomial ideal* I is an ideal that can be generated by monomials and it holds

Theorem 1.2.1. *Let $I \subseteq R$ be a monomial ideal. The following are equivalent:*

(a) I is a monomial ideal;

(b) for all $f \in R$ one has: $f \in I$ if and only if $\mathrm{supp}(f) \subseteq I$.

Now we summarize important notions about *Gröbner bases.* First of all, we introduce a partial order relation $\preceq$ on $\mathcal{M}$. Given two monomials $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$, with $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$, we have

$$\mathbf{x}^{\mathbf{a}} \preceq \mathbf{x}^{\mathbf{b}} \quad \overset{\text{def.}}{\iff} \quad a_i \leq b_i \text{ for any } i.$$

By using the relation above we define the *semigroup ideal* $\mathcal{S} \subseteq \mathcal{M}$ such that

$$t \in \mathcal{S} \Rightarrow tm \in \mathcal{S} \text{ for any } m \in \mathcal{M}.$$

We observe that a semigroup ideal $\mathcal{S}$ is generated by those elements of $\mathcal{S}$ that are minimal with respect to the relation $\preceq$, in particular it holds

Theorem 1.2.2 (Dickson's Lemma)**.** *Every semigroup ideal is finitely generated.*

Going further, a partial order on $\mathcal{M}$, $<$, is a *monomial order* if

1. $1 < m$ for any $m \in \mathcal{M}$;
2. if $m_1, m_2 \in \mathcal{M}$ and $m_1 < m_2$, then $mm_1 < mm_2$ for any $m \in \mathcal{M}$.

Example 1.2.3 *An example of monomial ordering is the* lexicographic order $<_{\text{lex}}$*, namely the monomial order induced by $x_1 < x_2 < \ldots < x_n$. Another example is the* graded lexicographic order $<_{\text{grlex}}$*: given two monomials $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$, with $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ we say that $\mathbf{x}^{\mathbf{a}} <_{\text{grlex}} \mathbf{x}^{\mathbf{b}}$ if one of the following hold:*

(a) $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$;

(b) $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ *and* $\mathbf{x}^{\mathbf{a}} <_{\text{lex}} \mathbf{x}^{\mathbf{b}}$.

A very important and widely studied class of monomial orders is the class of "reverse" orders, namely the *reverse lexicographical* order $<_{\text{revlex}}$ and the *graded reverse lexicographical order* $<_{\text{grevlex}}$. They are the reverse orders of $<_{\text{lex}}$ and $<_{\text{grlex}}$. In fact, given two monomials $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{M}$, we say that $\mathbf{x}^{\mathbf{a}} <_{\text{revlex}} \mathbf{x}^{\mathbf{b}}$ if the last non-zero component of $b - a$ is negative, and we say that $\mathbf{x}^{\mathbf{a}} <_{\text{grevlex}} \mathbf{x}^{\mathbf{b}}$ if either $\deg(\mathbf{x}^{\mathbf{a}}) < \deg(\mathbf{x}^{\mathbf{b}})$ or $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ and $\mathbf{x}^{\mathbf{a}} <_{\text{revlex}} \mathbf{x}^{\mathbf{b}}$.

For any polynomial $f \in R$ and for any monomial ordering $<$ on $\mathcal{M}$, we call *leading monomial* of f, $\text{in}(f)$, a monomial in $\text{supp}(f)$ bigger than any other monomial in $\text{supp}(f)$, with respect to $<$. The coefficient of $\text{in}(f)$ in f is called the *leading coefficient* of f, $\text{lc}(f)$. Moreover, given $I \subseteq R$ ideal, we define the *initial ideal*, $\text{in}(I)$, as

$$\text{in}(I) := \{ \text{ in}(f) \mid f \in I\}.$$

From Dickson's Lemma (Theorem 1.2.2), the minimal system of monomials generating $\mathcal{M} \cap \text{in}(I)$ is a finite set of monomials $\text{in}(g_1), \text{in}(g_2), \dots, \text{in}(g_s)$ for some polynomials $g_1, g_2, \dots, g_s$ in I.

Definition 1.2.4 *Let I be a non-zero ideal of R. A finite set of non-zero polynomials $\{g_1, \dots, g_s\}$ with each $g_i \in I$ is said to be a* Gröbner basis *with respect to a monomial order $<$ if* $\text{in}(I)$ *is generated by the monomials* $\text{in}(g_1), \text{in}(g_2), \dots, \text{in}(g_s)$.

A Gröbner basis $\{g_1, \dots, g_s\}$ of an ideal of R with respect to a monomial ordering $\preceq$ is called *reduced* if the following conditions are satisfied:

- The coefficient of $\text{in}_{\preceq}(g_i)$ in g_i is 1 for any $1 \leq i \leq s$;
- If $i \neq j$, then none of the monomials belonging to $\text{supp}(g_j)$ is divided by $\text{in}_{\preceq}(g_i)$.

A reduced Gröbner basis always exists and is uniquely determined (see [38, Theorem 1.25]).

We now want to state some properties about Gröbner bases for an ideal. First of all, given $f, g \in R$ we introduce the *S-polynomial* between f and g, $S(f, g)$, as

$$S(f, g) = \frac{\text{lcm}(\text{in}(f), \text{in}(g))}{\text{lc}(f)\text{in}(f)} f - \frac{\text{lcm}(\text{in}(f), \text{in}(g))}{\text{lc}(g)\text{in}(g)} g.$$

The following proposition provides a criterion to determine whether a set of polynomials belonging to an ideal is a Gröbner basis for the ideal.

Theorem 1.2.5 (Buchberger's criterion)**.** *Let I be a non-zero ideal of R and $\mathcal{G} = \{g_1, \ldots, g_s\}$ a system of generators of I. Then $\mathcal{G}$ is a Gröbner basis of I if and only if for all $i \neq j$ we have $S(g_i, g_j)$ reduces to 0 with respect to $g_1, \ldots, g_s$.*

A useful result while one is checking the S-polynomials condition is

Theorem 1.2.6. *Let $f, g \in I$ such that $\operatorname{lcm}(\operatorname{in}(f), \operatorname{in}(g)) = \operatorname{in}(f)\operatorname{in}(g)$. Then $S(f, g)$ reduces to 0 modulo f, g.*

In Section 1.1 we have recalled the definition of ideal quotient; moreover if J is an ideal of R and $f = x_n$, then the following result (see [76, Lemma 12.1]) gives a way to compute the Gröbner basis of $(J : f)$.

Lemma 1.2.7. *Fix the graded reverse lexicographical order $x_1 >_{\text{grevlex}} \ldots >_{\text{grevlex}} x_n$, and let $\mathcal{G}$ be the reduced Gröbner basis of a homogeneous ideal $J \subset R$. Then the set*

$$\mathcal{G}' = \{f \in \mathcal{G} \mid x_n \textit{ does not divide } f\} \cup \{f/x_n \mid f \in \mathcal{G} \textit{ and } x_n \textit{ divides } f\}$$

is a Gröbner basis of $(J : x_n)$.

Among the various applications of Gröbner bases, e.g. in finding the solutions of systems of equations, in computing the radical of an ideal, the one that we will focus on is the elimination property, that has also a relevant role in the symbolic computation of Gröbner basis of toric ideals.

Definition 1.2.8 *Let $R = \mathbb{K}[x_1, \ldots, x_n]$, let I be an ideal of R and let $1 \leq l \leq n$. Then the ideal*

$$I_l = I \cap \mathbb{K}[x_1, \ldots, x_l] \subseteq \mathbb{K}[x_1, \ldots, x_l]$$

is called the l-th elimination *ideal of I.*

Roughly speaking, the l-th elimination ideal eliminates the last $n - l$ variables of S. The Gröbner basis of the l-th elimination ideal is strictly related to the Gröbner basis of the considered ideal by the following.

Theorem 1.2.9 (Elimination Theorem)**.** *Let $R = \mathbb{K}[x_1, \ldots, x_n]$ with $x_1 <_{\text{lex}} \ldots <_{\text{lex}} x_n$, let I be an ideal of R and let $\mathcal{G}$ be a Gröbner basis of I with respect to $<_{lex}$. Then, for any $1 \leq l \leq n$,*

$$\mathcal{G}_l = \mathcal{G} \cap \mathbb{K}[x_1, \ldots, x_l]$$

is a Gröbner basis of I_l.

1.3 HILBERT FUNCTION, HILBERT SERIES AND FREE RESOLUTIONS

Hilbert function computes the degree k-generators as a $\mathbb{K}$-vector space of a graded algebra M. In particular, one can construct a series where degree k-coefficient is the Hilbert function in the degree k. This is called *Hilbert-Poincaré series* and, from Hilbert-Serre theorem, it can be expressed as a rational function, that after a reduction involves the Krull dimension as a power of the denominator. We show some basic properties about Hilbert series, such as additivity with respect to exact sequences and it is multiplicativity with respect to tensor product. The former property can be applied if one considers a particular complex of modules that "resolve" a given module R-module M, the minimal free resolution. We discuss the relations between the length of the above resolution, the depths of the module M and of its base ring R (Auslander-Buchsbaum formula).

Let M be a graded $\mathbb{K}$-algebra. The *Hilbert function* $\mathrm{H}_M : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mathrm{H}_M(k) := \dim_{\mathbb{K}} M_k$$

where M_k is the k-degree component of the gradation of M, while the *Hilbert-Poincaré series* of M is

$$\mathrm{HP}_M(t) := \sum_{k \in \mathbb{N}} \mathrm{H}_M(k) t^k.$$

By the Hilbert-Serre theorem, the Hilbert-Poincaré series of M is a rational function. In particular, by reducing this rational function we get

$$\mathrm{HP}_M(t) = \frac{h(t)}{(1-t)^d}.$$

for some $h(t) \in \mathbb{Z}[t]$, where d is the Krull dimension of M. The degree of $\mathrm{HP}_M(t)$ as a rational function, namely $\deg h(t) - d$, is called *a-invariant* of M, denoted by $a(M)$.

We recall the following result about Hilbert series.

Proposition 1.3.1. *Let*

$$0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$$

be an exact sequence of graded algebras. Then $\mathrm{HP}_M(t) = \mathrm{HP}_N(t) + \mathrm{HP}_P(t)$.

A direct application of Proposition 1.3.1 is the following.

Proposition 1.3.2. *Let I be a homogeneous ideal of a graded ring R, let $f \in R$ be a homogeneous element of degree d and consider the following exact sequence.*

$$0 \longrightarrow R/(I:f) \xrightarrow{\cdot f} R/I \longrightarrow R/(I,f) \longrightarrow 0$$

Then

1. $\mathrm{HP}_{R/I}(t) = \mathrm{HP}_{R/(I,f)}(t) + t^d \mathrm{HP}_{R/(I:f)}(t)$
2. *If f is a regular element then*

$$\mathrm{HP}_{R/I}(t) = \frac{1}{1-t^d} \mathrm{HP}_{R/(I,f)}(t).$$

Moreover, the Hilbert series is multiplicative with respect to the tensor product, as stated in the following.

Proposition 1.3.3. *Let A and B be graded algebras over a field $\mathbb{K}$. Then*

$$\mathrm{HP}_{A \otimes_{\mathbb{K}} B}(t) = \mathrm{HP}_A(t) \cdot \mathrm{HP}_B(t)$$

Example 1.3.4 *We compute $\mathrm{HP}_R(t)$, where $R = \mathbb{K}[x_1, \ldots x_n]$. We start with $R = \mathbb{K}[x]$. We observe that $\mathrm{H}_R(k) = \dim_{\mathbb{K}} R_k = 1$ for any $k \in \mathbb{N}$. Hence*

$$\mathrm{HP}_R(t) = \sum_{k \geq 0} \mathrm{H}_R(k) t^k = 1 + t + t^2 + \ldots = \frac{1}{1-t},$$

where the last equality follows from the basic properties of the geometric series. Now let $R = \mathbb{K}[x_1, \ldots, x_n] = \bigotimes_{i=1}^{n} \mathbb{K}[x_i]$. From Proposition 1.3.3, we get that

$$\mathrm{HP}_R(t) = \mathrm{HP}_{\bigotimes_{i=1}^{n} \mathbb{K}[x_i]}(t) = \prod_{i=1}^{n} \mathrm{HP}_{\mathbb{K}[x_i]}(t) = \frac{1}{(1-t)^n}.$$

For a general module M, one can compute the Hilbert function H_M by comparing M with other free modules that form the *free resolution*. In particular, given homogeneous elements $m_i \in M$ with $\deg(m_i) = a_i$ that generate M as a free R-module, take $F_0 = \bigoplus_i R(-a_i)$ and a map from F_0 onto M, by sending the i-th generator of F_0 to m_i. Let $M_1 \subseteq F_0$ be the kernel of the map above. M_1 is a finitely generated module and the elements of M_1 are called *syzygies*. We choose $m'_i \in M_1$ of degree a'_i and we take a map from $F_1 = \bigoplus_i R(-a'_i)$ in F_0 with image M_1. Proceeding in the previous way we obtain a *graded free resolution*:

$$\ldots \longrightarrow F_i \xrightarrow{\varphi_i} F_{i-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 (\xrightarrow{\varphi_0} M \longrightarrow 0).$$

It is an exact sequence of *degree 0* so that it preserves the grading, i.e. a map $f : \bigoplus_{k\in\mathbb{N}} A_k \to \bigoplus_{k\in\mathbb{N}} B_k$ such that $f(A_k) \subseteq B_k$. We observe that $\mathrm{Coker}\varphi_1 \simeq M$. In fact

$$\mathrm{Coker}\varphi_1 = F_0/\mathrm{Im}\varphi_1 = F_0/M_1 = F_0/\ker\varphi_0 \simeq \mathrm{Im}\varphi_0 = M.$$

Free resolutions are finite due to the following.

Theorem 1.3.5 (Hilbert Syzygy Theorem)**.** *Any finitely generated graded R-module has a finite graded free resolution*

$$0 \longrightarrow F_m \xrightarrow{\varphi_m} F_{m-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0.$$

Moreover we may take $m \leq n$ the number of variables of R.

Among the free resolutions of a module, we find one of them that is minimal. One can obtain it by choosing at each step minimal systems of generators for the free modules F_i. The *minimal free resolution* is also unique up to isomorphism.

Definition 1.3.6 *A complex of R-modules*

$$\ldots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \ldots$$

is called minimal *if* $\forall i$ *we have* $\mathrm{Im}\delta_i \subseteq \mathfrak{m}F_{i-1}$.

By applying Proposition 1.3.1 to a minimal free resolution $\mathbf{F}$ of a graded module M. we obtain that

$$\mathrm{H}_M(k) = \sum_i (-1)^i \mathrm{H}_{F_i}(k). \tag{1.1}$$

Since the notions of projective modules and projective resolutions are not relevant for the discussion, we assume the following as a definition for the *projective dimension* of a module.

Definition 1.3.7 *Let M be a finitely generated graded R-module. The* projective dimension *of M,* $\mathrm{pd}M$*, is the length of the minimal free resolution of M.*

The projective dimension of a module M is strictly related to the depth of the module through the following well-known

Theorem 1.3.8 (Auslander-Buchsbaum formula)**.** *Let M be a non-zero finitely generated graded R-module. Then*

$$\mathrm{pd}M + \mathrm{depth}\ M = \mathrm{depth}\ R.$$

1.4 BETTI NUMBERS AND CASTELNUOVO-MUMFORD REGULARITY

Starting by the minimal free resolution of a given module M, one can define fundamental homological invariants that are the Betti numbers. In fact, they represent the numbers of degree-j generator of the i-th module of the minimal free resolution. In particular, in the Cohen-Macaulay case, the last Betti number is called "Cohen-Macaulay" type and it has a fundamental role in the classification of *Gorenstein* modules. We also study the greatest difference between degree j and the position i for which the corresponding Betti number is different from 0. This is another algebraic invariant that is called *Castelnuovo-Mumford regularity.* In the Cohen-Macaulay case, Castelnuovo-Mumford regularity has a strong relation with the projective dimension, and hence with Krull dimension and Hilbert series.

Let $\mathbf{F}$ be a free complex

$$\mathbf{F}:\ 0 \longrightarrow F_s \xrightarrow{\varphi_m} F_{s-1} \longrightarrow \ldots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

where $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$. It means that F_i requires β_{ij} generators of degree j. If $\mathbf{F}$ is a minimal free resolution, the β_{ij} are called *graded Betti numbers*, while

$$\beta_i = \sum_j \beta_{ij}$$

is called the *total Betti number* in the i-th degree. Among the total Betti numbers, the last one gives information on an algebraic invariant when the module is Cohen-Macaulay, the *Cohen-Macaulay type*. The Cohen-Macaulay type is important in the study of the Gorenstein rings (see [12, Theorem 3.2.10]). In fact a Cohen-Macaulay ring is said to be *Gorenstein* if the Cohen-Macaulay type is equal to 1. There is a nice characterization of the Cohen-Macaulay domains that are Gorenstein, involving the Hilbert series. The latter result, due to Stanley, is the following.

Theorem 1.4.1 (Theorem 4.4, [78]). *Let M be a graded Cohen-Macaulay domain, and let*

$$\mathrm{HP}_M(t) = \frac{\sum_{i=0}^{s} h_i t^i}{(1-t)^d}$$

be the reduced Hilbert series of M. Then M is Gorenstein if and only if for any $i = 0, \dots, s$ we have $h_i = h_{s-i}$.

We now define the *Castelnuovo-Mumford regularity*, an invariant that has particular relations with Krull dimension or depth.

Definition 1.4.2 *Let M be a finitely generated graded R-module with minimal free resolution*

$$\dots \longrightarrow F_i \xrightarrow{\delta_i} F_{i-1} \longrightarrow \dots$$

where $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$. The Castelnuovo-Mumford regularity *of M is defined as*

$$\mathrm{reg}\ M = \max\{j - i : \beta_{ij} \neq 0\}.$$

Example 1.4.3 *The polynomial ring $R = \mathbb{K}[x_0, \dots, x_{n-1}]$ has Castelnuovo-Mumford regularity equal to* 0*, since*

$$R = F_0 = R(0)$$

and the unique non-zero Betti number is β_{00} that is 1.

From the fact that for any module M and for any j it holds that $\mathrm{HP}_{M(-j)}(t) = t^j \mathrm{HP}_M(t)$ (see [82, Lemma 5.1.2]) and from Equation 1.1, it follows that

$$\mathrm{HP}_M(t) = \frac{1}{(1-t)^n} \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i \beta_{ij} t^j. \quad (1.2)$$

For Cohen-Macaulay modules, there are various relations between the Krull dimension. the Castelnuovo-Mumford regularity and the Hilbert series. If M is a Cohen-Macaulay module of projective dimension p, then the Castelnuovo-Mumford regularity can be read in the p-th degree, that is reg M is the greatest integer r such that β_{pp+r} is different from 0. Furthermore, let d be the Krull dimension of M. From Theorem 1.3.8, we get that $p + d = n$, and hence from Equation 1.2, we have

$$a(M) = p + r - n = r - d.$$

That is, if

$$\mathrm{HP}_M(t) = \frac{h(t)}{(1-t)^d}$$

is the reduced Hilbert series of M, then $\deg h(t) = r$.

1.5 LATTICES AND LATTICE IDEALS

In this section, we present an overview on lattice theory. An interesting class of lattices is the class of *distributive lattices*, because from Birkhoff's fundamental theorem, for any poset one can construct an associated distributive lattice and vice versa. Moreover, Hibi introduced an ideal associated to lattices, called *join-meet ideal*. The latter is a binomial ideal and it is prime if and only if the lattice is distributive. Join-meet ideals of planar distributive lattices are relatives of polyomino ideals, that will be discussed in next chapter.

Let P be a poset. A chain of P is a totally ordered subset of P. The length of a chain $\mathfrak{c}$, denoted by *length*$(\mathfrak{c})$, is $|\mathfrak{c}| - 1$. Given $a \in P$, the rank of a in P, denoted by rank (a), is the supremum of length of chains in P that descends from a. The *rank* of P, denoted by rank (P), is the supremum of length of chains of P. An order ideal of P is a subset I of P with the following property: if $a \in I$ then $b \in I$ for all $b \in P$ with $b < a$. Two element $a, b \in P$ are called incomparable if $a \not< b$ and $b \not< a$. A poset L in which any two elements a and b have a supremum $a \vee b$, called *join* and an infimum $a \wedge b$, called *meet*, is called a *lattice*. A lattice L is called *distributive* if for any $a, b, c \in L$ the following distributivity rules hold

- $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;
- $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Let L be a distributive lattice with unique minimal element $\min(L)$ and unique maximal element $\max(L)$. An element $a \in L$ is called *join-irreducible* if $a \neq \min(L)$ and $a \neq b \vee c$ for any $b, c \in L \setminus \{a\}$. Let P be the set consisting of all join-irreducible elements of L. Then P is a poset with partial order inherited from L. Let $I(P)$ be the set consisting of all order ideals of P, ordered by inclusion. In particular, $\emptyset, P \in I(P)$ and $I(P)$ is a distributive lattice with $\min(I(P)) = \emptyset$ and $\max(I(P)) = P$. In particular,

Theorem 1.5.1 (Birkhoff's fundamental theorem). *Let L be a finite distributive lattice and let P be the poset of of the join-irreducible elements of L. Then $L \cong I(P)$.*

It follows that, rank $(L) = |P|$.

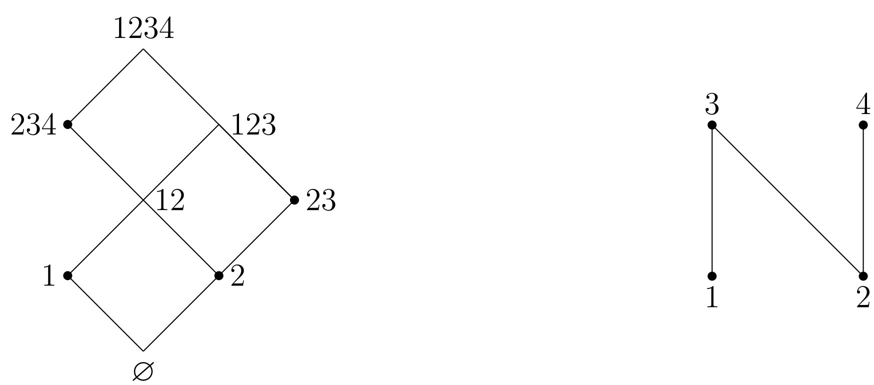


Figure 1.1: A rank 4 finite distributive lattice with the poset of join-irreducible elements

Let L be finite distributive lattice and $R = \mathbb{K}[x_a : a \in L]$. The join-meet ideal $I_L \subset S$ of L is the ideal generated by binomials $x_a x_b - x_{a \vee b} x_{a \wedge b}$, where a and b are incomparable elements in L, and it has been introduced in [42]. Due to this, the quotient ring $\mathbb{K}[L] = R/I_L$ is today called *Hibi ring*. It is known that $\mathbb{K}[L]$ is a normal, Cohen-Macaulay domain, that is I_L is a toric ideal. Moreover, if $L = I(P)$, then $\mathbb{K}[L]$ is Gorenstein if and only if the poset P is *pure*, that is all the maximal chains of P have the same length (see [42, p. 105]).

Consider the natural partial order on $\mathbb{N}^2$ defined as follows: for any $(i, j), (k, l) \in \mathbb{N}^2$, we have $(i,j) \leq (k,l)$ if and only if $i \leq k$ and $j \leq l$. With this natural partial order, $\mathbb{N}^2$ is an infinite distributive lattice. Let L be a finite sublattice of $\mathbb{N}^2$. Then L is called a *planar* distributive lattice if $(0,0) \in L$ and for any $(i,j),(k,l) \in L$ with$(i,j) < (k,l)$, there exists a chain in L of the form $(i,j) = (i_0,j_0) < (i_1,j_1) < \ldots < (i_s,j_s) = (k,l)$ such that $i_{k+1} + j_{k+1} = i_k + j_k + 1$ for all k. The condition $i_{k+1} + j_{k+1} = i_k + j_k + 1$ yields that either $(i_{k+1},j_{k+1}) = (i_k,j_k) + (0,1)$ or $(i_{k+1},j_{k+1}) = (i_k,j_k) + (1,0)$. A planar distributive lattice L is called *simple* if, for all $0 < r < \text{rank}\ (L)$, there exist at least two elements in L with rank r. Equivalently, L is simple if there is no $a \in L$ with the property that for every $b \in L$ either $a \leq b$ or $b \leq a$.

1.6 EDGE RINGS ASSOCIATED TO BIPARTITE GRAPHS

In this section, we stress the relation between bipartite graphs and toric ideals that has also a crucial role in the study of polyomino ideals (for further details see [82] and [43]). Let $R = \mathbb{K}[x_1,\ldots,x_n]$ be a polynomial ring over the field $\mathbb{K}$ with the standard grading and let G be a graph on the vertices $\{1,\ldots,n\}$ with edge set $E(G)$. The ring

$$\mathbb{K}[G] = \{x_i x_j : \{i,j\} \in E(G)\}$$

is called the *edge ring* of G. We consider a polynomial ring $T = \mathbb{K}[t_{ij} : \{i,j\} \in E(G)]$ together with a map

$$\varphi : T \to \mathbb{K}[G] \qquad t_{ij} \mapsto x_i x_j.$$

The kernel P_G of φ is a toric ideal, i.e. a prime ideal generated by (quadratic) binomials. We describe the generators of J by looking at special structures inside of the graph. A *walk* of length r between i and j in G is a sequence of edges of the form $\{i_0,i_1\},\{i_1,i_2\},\ldots,\{i_{r-1},i_r\}$ such that $i_0,\ldots,i_r$ are vertices of G with $i_0 = i$ and $i_r = j$. A *cycle* of length r in G is a closed walk where the vertices are all distinct. We say that a walk (resp. cycle) W is even or odd depending on the parity of r. Given an even closed walk W with edges $E(W) = \{\{i_0,i_1\},\{i_1,i_2\},\ldots,\{i_{2r-1},i_{2r} = i_0\}\}$ we consider the binomial of T

$$f_W = t_{i_0 i_1} t_{i_2 i_3} \ldots t_{i_{2r-2} i_{2r-1}} - t_{i_1 i_2} t_{i_3 i_4} \ldots t_{i_{2r-1} i_{2r}}$$

One has $\varphi(f_W) = 0$, that is $f_W \in P_G$. Moreover, we say that a graph G is *bipartite* if its vertex set V can be partitioned in two nonempty and disjoint sets V_1 and V_2 such that any edge of G connects a vertex of V_1 with a vertex of V_2. The following propositions describe the generators of P_G.

Proposition 1.6.1. *Let G be a graph and let P_G be the toric ideal of $\mathbb{K}[G]$. Then*

- $P_G = (f_W : W$ *is an even closed walk);*
- $P_G = (f_W : W$ *is an even cycle) if G is bipartite.*

Theorem 1.6.2. *Let G be a bipartite graph. Then the following are equivalent*

(1) Every cycle in G with length greater than or equal to 6 has a chord;

(2) P_G has a Gröbner basis consisting in quadratic binomials;

(3) P_G is generated by quadratic binomials.

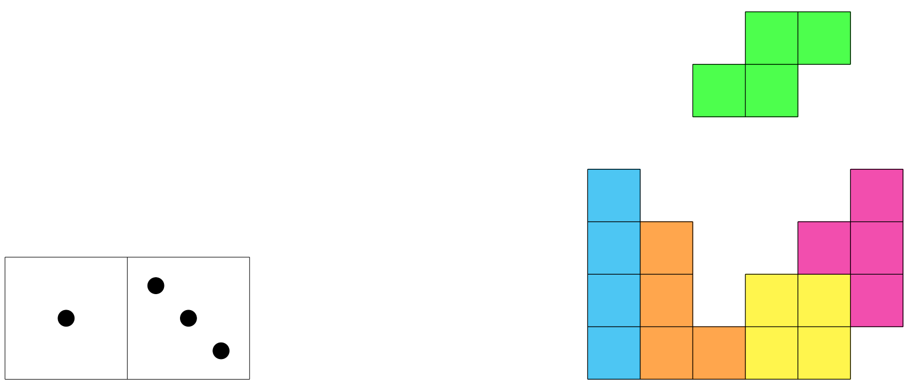
A bipartite graph satisfying property (1) of Theorem 1.6.2 is called *weakly chordal.*

Chapter 2
POLYOMINOES

This chapter is devoted to the introduction of polyominoes and to the presentation of the current state of the art in both Combinatorics (Sections 2.1, 2.2, 2.3) and Commutative Algebra (Section 2.4) about polyominoes. In Section 2.1 we give formal definitions about polyominoes, as sets of pairwise-connected unitary cells of $\mathbb{N}^2$. The latter arises from the generalization of the concept of planar distributive lattice and it is suitable to the definition of polyomino ideals. However, in the above section we also give some results about tiling and counting, described in the book of Golomb [32]. In Sections 2.2 and 2.3 we present some combinatorial properties about two classes of polyominoes, *L-convex* polyominoes and *parallelogram* polyominoes, respectively. The name "L-convex" arises from the fact that any pair of cells is connected through a path of cells having at most one change of direction, that is, any of the above paths has the shape of an L. For this class of polyominoes, it is interesting to study the number of maximal rectangles and the so-called *horizontal* and *vertical projections*, i.e. the number of cells in each row and column, respectively. In fact, given suitable horizontal and vertical projections, one can find a unique associated L-convex polyomino. Parallelogram polyominoes are the ones corresponding to a pair (α, β) of north-east paths going from the origin to a point (m, n) in the plane $\mathbb{N}^2$ with the path α staying "above" the path β. They are relevant because they can be seen as planar distributive lattices with a minimum, the origin, and a maximum, the point (m, n). Any parallelogram polyomino can be encoded by means of the so-called 2-colored *Motzkin paths*. In Section 2.4, we present the binomial ideal associated to polyominoes, the polyomino ideal and we present the known results about its algebraic invariants: primality of simple polyominoes, i.e. polyominoes without "holes" and of particular classes of nonsimple polyominoes and at the end we discuss the Gorensteinnes of stack polyominoes. For further information one can refer to [32] (Section 2.1), [16], [15] (Section 2.2), [3], [21] (Section 2.3), [62], [44], [65], [40] and [1] (Section 2.4.

2.1 HISTORY, BASIC DEFINITIONS AND PROPERTIES

Polyominoes are planar shapes made by connecting a finite number of equal-sized squares, each joined together edge by edge. The name *polyomino* was invented by Golomb in 1953, and it is the back-formation from the word *domino*, a common game piece consisting of two squares, with the first letter d- interpreted as a version of the prefix di- meaning "two" (see Figure 2.1A). The name domino for the game piece is believed to come from the spotted masquerade garment domino, from Latin *dominus*. Most of the numerical prefixes are Greek (tri-, tetro-, penta-, exa-). For example, the well-known "Tetris" is a tile-matching videogame in which the players should complete lines by moving differently shaped tetrominoes, descending in the playing field. In particular in Figure 2.1B we represented by different colors all of the different tetrominoes up to rotation and reflection. Two polyominoes are equal (resp. distinct) as *free* polyominoes if they are equal (resp. distinct) up to rotations and reflections.



(A) A typical Domino piece

(B) A typical Tetris playground

Figure 2.1: Two examples of polyominoes

Polyominoes have a crucial role in recreational mathematics, especially in tiling problems. In fact, some of the most-known challenges are the following:

- **Tiling a region with a given set of polyominoes:** a classical problem of this kind is the one of tiling a 6×10 rectangle with 12 pentominoes. A solution of the problem is given in Figure 2.2. Overall, 2339 solutions were found by using a computational approach (see [35]).

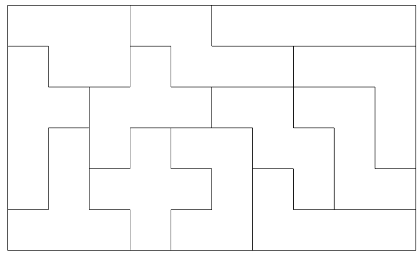


Figure 2.2: Tiling a 6×10 rectangle with 12 pentominoes

- **Tiling a region with copies of a single polyomino:** Another class of problems asks whether copies of a given polyomino can tile a region. A problem of this kind is the one of tiling a 8×8 chessboard, that has a pair of diagonally opposite corner squares removed, by using a set of dominoes (see Figure 2.3). Since a set of n dominoes covers exactly n white cells and n black cells and in our playground there are more black cells than white ones, the above problem has no solution.

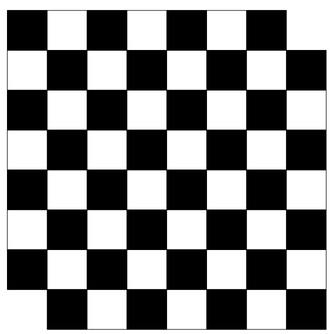


Figure 2.3: Tiling a 8×8 chessboard with a pair of diagonally opposite corner squares removed by using a set of dominoes

- **Tiling the plane with copies of a single polyomino:** The problem of which polyominoes can tile the plane has been a relevant challenge since 1965. However, the study of the above problem has been facilitated by Conway criterion: all of the polyominoes up to order 9 can be combined to tile the plane except for two nonominoes (see Figure 2.4).

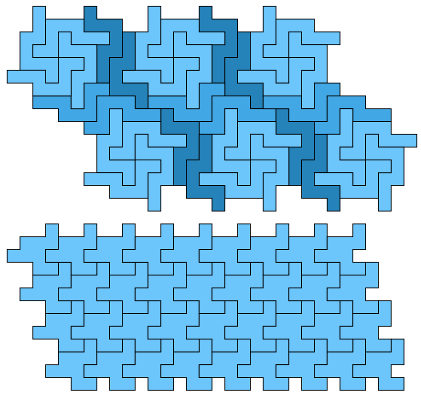


Figure 2.4: Two nonominoes that do not satisfy Conway criterion

In the following table we report the enumeration of free distinct polyominoes having n cells with $1 \leq n \leq 12$ (for further information see [32, Chapter 6] and [61]).

n	1	2	3	4	5	6	7	8	9	10	11	12
Polyominoes	1	1	2	5	12	35	108	369	1285	4655	17073	63600

Table 2.1: Enumeration of distinct polyominoes

Such enumeration has been helpful for the computation in Section 3.1.

Polyominoes find application also in statistical physics and chemistry, where they are useful in the study of the polymers.

We now give a mathematical construction of polyominoes that is similar to the construction of planar distributive lattices (see Section 1.5). Let $a = (i, j)$, $b = (k, \ell) \in \mathbb{N}^2$, with $i \leq k$ and $j \leq \ell$, the set $[a, b] = \{(r, s) \in \mathbb{N}^2 : i \leq r \leq k \text{ and } j \leq s \leq \ell\}$ is called an *interval* of $\mathbb{N}^2$. If $i < k$ and $j < \ell$, $[a, b]$ is called a *proper interval*, and the elements a, b, c, d are called corners of $[a, b]$, where $c = (i, \ell)$ and $d = (k, j)$.

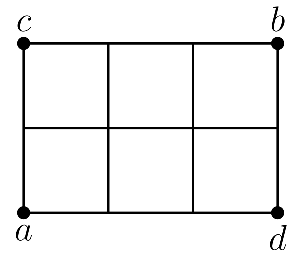


Figure 2.5: An interval of $\mathbb{N}^2$

In particular, a, b are called *diagonal corners* and c, d *anti-diagonal corners* of $[a, b]$. The corner a (resp. c) is also called the left lower (resp. upper) corner of $[a, b]$, and d (resp. b) is the right lower (resp. upper) corner of $[a, b]$. A proper interval of the form $C = [a, a + (1, 1)]$ is called a *cell*. Its vertices $V(C)$ are $a, a + (1, 0), a + (0, 1), a + (1, 1)$ and its edges $E(C)$ are

$$\{a, a + (1, 0)\}, \{a, a + (0, 1)\}, \{a + (1, 0), a + (1, 1)\}, \{a + (0, 1), a + (1, 1)\}.$$

In the following, we denote by $\ell(C)$ the left lower corner of a cell C.

Let $\mathcal{P}$ be a finite collection of cells of $\mathbb{N}^2$, and let C and D be two cells of $\mathcal{P}$. Then C and D are said to be *connected*, if there is a sequence of cells $C = C_1, \ldots, C_m = D$ of $\mathcal{P}$ such that $C_i \cap C_{i+1}$ is an edge of C_i for $i = 1, \ldots, m - 1$. In addition, if $C_i \neq C_j$ for all $i \neq j$, then $C_1, \ldots, C_m$ is called a *path* (connecting C and D). A collection of cells $\mathcal{P}$ is called a *polyomino* if any two cells of $\mathcal{P}$ are connected. We denote by $V(\mathcal{P}) = \cup_{C \in \mathcal{P}} V(C)$ the vertex set of $\mathcal{P}$. The number of cells of $\mathcal{P}$ is called the *rank* of $\mathcal{P}$, and we denote it by rank $\mathcal{P}$. A subset $\mathcal{Q} \subseteq \mathcal{P}$ is called a *subpolyomino* of $\mathcal{P}$ if Q is a polyomino itself.

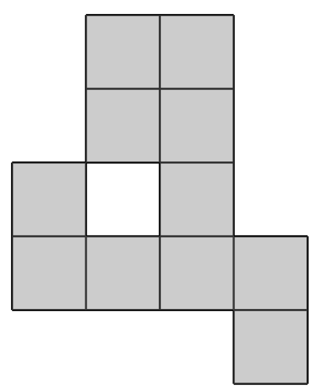
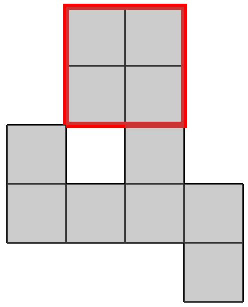
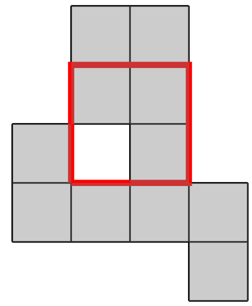


Figure 2.6: A generic polyomino

A proper interval $[a, b]$ is called an *inner interval* of $\mathcal{P}$ if all cells of $[a, b]$ belong to $\mathcal{P}$.



(A) The interval in red color is an inner interval of the polyomino



(B) The interval in red color is not an inner interval of the polyomino

We say that a polyomino $\mathcal{P}$ is *simple* if for any two cells C and D of $\mathbb{N}^2$ not belonging to $\mathcal{P}$, there exists a path $C = C_1, \dots, C_m = D$ such that $C_i \notin \mathcal{P}$ for any $i = 1, \dots, m$. If the polyomino is not simple then it is said *multiply connected* (see [32]). A finite collection $\mathcal{H}$ of cells not in $\mathcal{P}$ is called a *hole* of $\mathcal{P}$, if any two cells in $\mathcal{H}$ are connected through a path of cells in $\mathcal{H}$, and $\mathcal{H}$ is maximal with respect to the inclusion. Note that a hole $\mathcal{H}$ of a polyomino $\mathcal{P}$ is itself a simple polyomino.

An interval $[a, b]$ with $a = (i, j)$ and $b = (k, \ell)$ is called a *horizontal edge interval* of $\mathcal{P}$ if $j = \ell$ and the sets $\{(r, j), (r + 1, j)\}$ for $r = i, \dots, k - 1$ are edges of cells of $\mathcal{P}$. If a horizontal edge interval of $\mathcal{P}$ is not strictly contained in any other horizontal edge interval of $\mathcal{P}$, then we call it *maximal horizontal edge interval*. Similarly, one defines vertical edge intervals and maximal vertical edge intervals of $\mathcal{P}$.

Each proper interval $[(i, j), (k, l)]$ in $\mathbb{N}^2$ can be identified as a polyomino and it is referred to as *rectangular* polyomino, or simply as rectangle. A rectangular subpolyomino $\mathcal{P}'$ of $\mathcal{P}$ is called *maximal* if there is no rectangular subpolyomino $\mathcal{P}''$ of $\mathcal{P}$ that properly contains $\mathcal{P}'$. A rectangle has size $m \times n$ if it contains m columns and n rows of cells. Given a polyomino $\mathcal{P}$, the rectangle that contains $\mathcal{P}$ and has the smallest size with this property is called *bounding box* of $\mathcal{P}$. After a shift of coordinates, we may assume that the bounding box is $[(0, 0), (m, n)]$ for some $m, n \in \mathbb{N}$. In this case, the width of $\mathcal{P}$, denoted by $w(\mathcal{P})$ is m. Similarly, the height of $\mathcal{P}$, denoted by $h(\mathcal{P})$ is n.

A polyomino $\mathcal{P}$ is called *row convex* if for any two of its cells with lower left corners

$a = (i, j)$ and $b = (k, j)$, with $k > i$, all cells with lower left corners (l, j) with $i \leq l \leq k$ are cells of $\mathcal{P}$. Similarly, $\mathcal{P}$ is called *column convex* if for any two of its cells with lower left corners $a = (i, j)$ and $b = (i, k)$, with $k > j$, all cells with lower left corners (i, l) with $j \leq l \leq k$ are cells of $\mathcal{P}$. If a polyomino $\mathcal{P}$ is simultaneously row and column convex then $\mathcal{P}$ is called *convex*. In this dissertation, we focus on two classes of convex polyominoes, *L-convex polyominoes* (see Section 2.2) and *parallelogram polyominoes* (see Section 2.3).

2.2 L-CONVEX POLYOMINOES

In this section, we see the combinatorial properties of a class of convex polyominoes, L-convex polyominoes. There are different results on their realizations and enumerations. For further information, see [15, 16].

Let $\mathcal{C} : C_1, C_2, \ldots, C_m$ be a path of cells and (i_k, j_k) be the lower left corner of C_k for $1 \leq k \leq m$. Then $\mathcal{C}$ has a change of direction at C_k for some $2 \leq k \leq m-1$ if $i_{k-1} \neq i_{k+1}$ and $j_{k-1} \neq j_{k+1}$. A convex polyomino $\mathcal{P}$ is called k-convex if any two cells in $\mathcal{P}$ can be connected by a path of cells in $\mathcal{P}$ with at most k change of directions. The 1-convex polyominoes are simply called L-convex polyominoes. An example of L-convex polyomino is showed in Figure 2.8

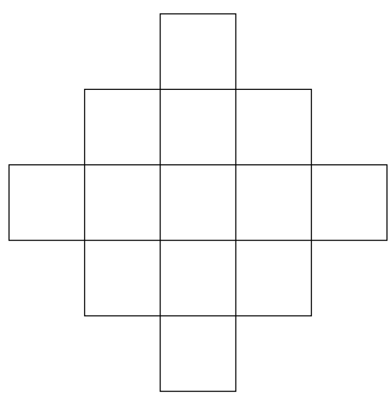


Figure 2.8: An L-convex polyomino

For the aim of giving a first characterization of L-convex polyominoes, we need the following definitions.

Definition 2.2.1 *Let $\mathcal{P}$ and $\mathcal{Q}$ be polyominoes. We say that $\mathcal{Q}$ has l occurrences in $\mathcal{P}$ if* *there exist l distinct subsets of $\mathcal{P}$ that are equal to $\mathcal{Q}$ as free polyominoes. In particular, a maximal rectangle $\mathcal{R}$ of size $m \times n$ is said to have unique occurrence in a polyomino $\mathcal{P}$, if R is the only* rectangular *subpolyomino of $\mathcal{P}$ with size $m \times n$.*

Definition 2.2.2 *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rectangles of sizes $m \times n$ and $s \times t$, respectively. We say that $\mathcal{R}$ and $\mathcal{R}'$ have a* crossing intersection *if the intersection $\mathcal{R} \cap \mathcal{R}'$ is a rectangle of size* $\min\{m, s\} \times \min\{n, t\}$. *Some examples of crossing intersections are showed in Figure 2.9.*

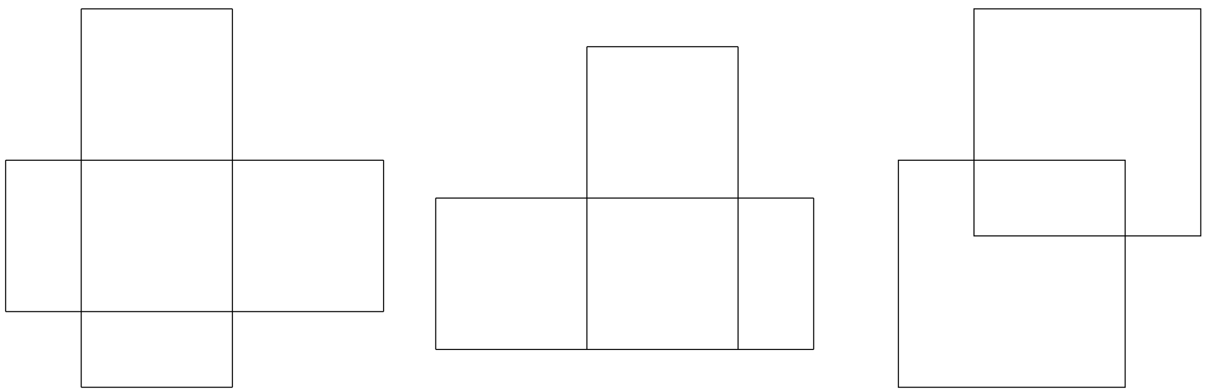


Figure 2.9: Three examples of rectangles intersecting in another rectangle. The first two are crossing intersections, while the last one is not.

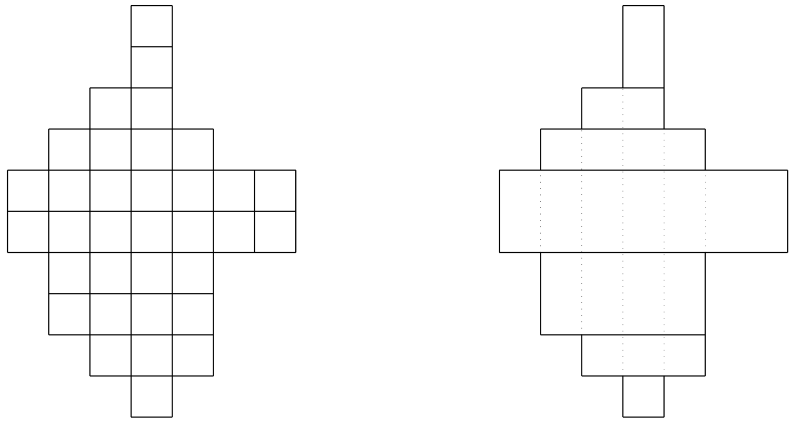
One can observe that the union of two rectangles having a crossing intersection is an L-convex polyomino. The latter is actually a characterization of the L-convex polyominoes as stated in the following Lemma.

Lemma 2.2.3. *A convex polyomino $\mathcal{P}$ is L-convex if and only if every pair of maximal rectangles occurs in $\mathcal{P}$ with a crossing intersection.*

An immediate consequence of Lemma 2.2.3 is the following

Corollary 2.2.4. *A maximal rectangle of an L-convex polyomino $\mathcal{P}$ has a unique occurrence in $\mathcal{P}$.*

The maximal rectangles of the polyomino in Figure 2.10 are of sizes 7×2, 4×5, 3×6, 2×7 and 1×10.



(A) An L-convex polyomino $\mathcal{P}$. (B) The maximal rectangles of $\mathcal{P}$.

Figure 2.10: The maximal rectangles of $\mathcal{P}$

Moreover, any convex polyomino $\mathcal{P}$ having bounding box $[(0,0),(m,n)]$ has m columns of cells, ordered increasingly from left to right, and n rows of cells, ordered increasingly from top to bottom. We attach a bipartite graph $F_{\mathcal{P}}$ to the polyomino $\mathcal{P}$ in the following way. Let $V(F_{\mathcal{P}}) = \{X_1, \ldots, X_m\} \sqcup \{Y_1, \ldots, Y_n\}$ and $\{Y_i, X_j\} \in E(F_{\mathcal{P}})$ if the i-th row of $\mathcal{P}$ intersects the j-th column of $\mathcal{P}$ non-trivially. The unique cell in the intersection of i-th row and j-th column is labelled as C_{ij}. We call *$i-th$ horizontal projection* h_i the number of cells of the $i-th$ row of $\mathcal{P}$ and *$j-th$ vertical projection* the number of cells of the $j-th$ column. Note that $h_i = \deg Y_i$ and $v_j = \deg X_j$ in the graph $F_{\mathcal{P}}$. In the sequel, we will refer to the vector $H_{\mathcal{P}} = (h_1, h_2, \ldots, h_n)$ as the horizontal projections of $\mathcal{P}$ and $V_{\mathcal{P}} = (v_1, v_2, \ldots, v_m)$ as the vertical projection of $\mathcal{P}$. In Figure 2.11, we show that in general given two vectors H and V of horizontal and vertical projections, it may happen that there are two convex polyominoes having H and V as horizontal and vertical projections.

Figure 2.11: Two convex polyominoes having $H = (1,2,2)$ and $V = (1,3,1)$

Differently, for L-convex polyominoes the horizontal and vertical projections are uniquely determined and have a nice behavior as stated in the following.

Theorem 2.2.5. *Let $\mathcal{P}$ be an L-convex polyomino, then:*

(a) *$\mathcal{P}$ is uniquely determined by $H_{\mathcal{P}}$ and $V_{\mathcal{P}}$;*

(b) *$H_{\mathcal{P}}$ and $V_{\mathcal{P}}$ are unimodal vectors, that is there exist $r \in \{1, \ldots, n\}$ and $s \in \{1, \ldots, m\}$ such that*

$$h_1 \leq \ldots \leq h_{r-1} \leq h_r \geq h_{r+1} \geq \ldots \geq h_n \text{ and } v_1 \leq \ldots \leq v_{s-1} \leq v_s \geq v_{s+1} \geq \ldots \geq v_m;$$

(c) *Let j, j' be two different columns of $\mathcal{P}$ such that $v_j \leq v_{j'}$. Then for each row i of $\mathcal{P}$, we have $C_{ij'} \in \mathcal{P}$ if $C_{ij} \in \mathcal{P}$.*

(d) *Let i, i' be two different rows of $\mathcal{P}$ such that $h_i \leq h_{i'}$. Then for each column j of $\mathcal{P}$, we have $C_{i'j} \in \mathcal{P}$ if $C_{ij} \in \mathcal{P}$.*

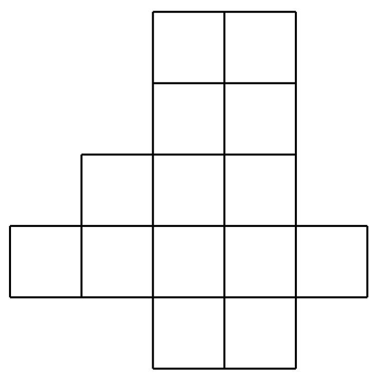


Figure 2.12: An L-convex polyomino with $H_{\mathcal{P}} = (2, 2, 3, 5, 2)$ and $V_{\mathcal{P}} = (1, 2, 5, 5, 1)$.

A special class of L-convex polyominoes is the one of *Ferrer diagrams*, namely L-convex polyominoes $\mathcal{P}$ for which both $H_{\mathcal{P}}$ and $V_{\mathcal{P}}$ are decreasing (see Figure 2.13).

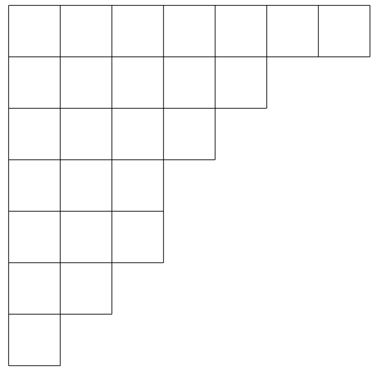


Figure 2.13: Ferrer diagram

In Section 4.1 we will prove that any L-convex polyomino can be rearranged to a Ferrer diagram, keeping fixed the sizes of the maximal rectangles.

2.3 PARALLELOGRAM POLYOMINOES

In this section, we describe parallelogram polyominoes from the point of view of combinatorics, and the ways to encode them with the so-called Motzkin paths. For further details, see [3] and [21].

Let $(a, b) \in \mathbb{N} \times \mathbb{N}$. The edge $\{(a, b), (a + 1, b)\}$ is called an *east step* and the edge $\{(a, b), (a, b + 1)\}$ is called a *north step* in $\mathbb{N} \times \mathbb{N}$. A sequence of vertices $\mathcal{S} : (a_0, b_0), (a_1, b_1) \dots (a_k, b_k)$ in $\mathbb{N} \times \mathbb{N}$ is called a *north-east* path in $\mathbb{N} \times \mathbb{N}$, if $\{(a_i, b_i), (a_{i+1}, b_{i+1})\}$ is either an east or a north step for each i. The vertices (a_0, b_0) and (a_k, b_k) are called the endpoints of $\mathcal{S}$. Let $\mathcal{S}_1 : (a_0, b_0), (a_1, b_1), \dots, (a_k, b_k)$ and $\mathcal{S}_2 : (c_0, d_0), (c_1, d_1), \dots, (c_k, d_k)$ be two north-east paths in $\mathbb{N} \times \mathbb{N}$ such that $(a_0, b_0) = (c_0, d_0)$ and $(a_k, b_k) = (c_k, d_k)$. If for all $1 \leq i, j \leq k - 1$ we have $b_i > d_j$ whenever $a_i = c_j$, then $\mathcal{S}_1$ is said to “lie above" $\mathcal{S}_2$. The *parallelogram polyomino* $\mathcal{P}$ determined by $(\mathcal{S}_1, \mathcal{S}_2)$, where $\mathcal{S}_1$ lies above $\mathcal{S}_2$, is the region bounded above by $\mathcal{S}_1$ and bounded below by $\mathcal{S}_2$. We refer to the path S_1 as the upper path of $\mathcal{P}$ and the path $\mathcal{S}_2$ as the lower path of $\mathcal{P}$. We will denote a parallelogram polyomino as $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ when we need to emphasize on its upper and lower paths. In Figure 2.14, a parallelogram polyomino is shown. The thick line in Figure 2.14 represents the upper path of $\mathcal{P}$ and the dashed line represents the lower path of $\mathcal{P}$. A parallelogram polyomino that is also L-convex is exactly a Ferrer diagram. (see 2.13).

An equivalent way to describe parallelogram polyomino is the following property: a polyomino such that its intersection with every line perpendicular to the main diagonal is a connected segment (see [22, pg. 178]).

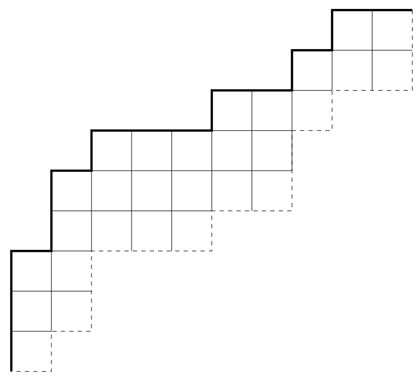


Figure 2.14: A parallelogram polyomino

We call *semiperimeter* of a parallelogram polyomino $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ the sum between the number of columns and the number of rows of $\mathcal{P}$, that also corresponds to the length of $\mathcal{S}_1$ (or $\mathcal{S}_2$). The number of parallelogram polyominoes having semiperimeter n is expressed by the *n-th Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In the following, we describe the well-known bijection between the parallelogram polyominoes and 2-colored Motzkin paths. Let $(a, b) \in \mathbb{N}$. Then

1. the edge $\{(a, b), (a + 1, b + 1)\}$ is called a *rise* step,
2. the edge $\{(a, b), (a + 1, b - 1)\}$ is called a *fall* step,
3. the edge $\{(a, b), (a + 1, b)\}$ is called a *east* step or a *horizontal* step.

A *2-colored Motzkin path*

$$\mathcal{M} : (0, 0) = (a_0, b_0), (a_1, b_1), \dots, (a_n, b_n) = (n, 0)$$

in $\mathbb{N} \times \mathbb{N}$ is a path that never passes below the x-axis and consists of rise steps, fall steps and two types of horizontal steps that are called α-colored horizontal steps and β-colored horizontal steps. Each 2-colored Motzkin path can be regarded as a 2-colored Motzkin word. Let $\mathcal{P}$ be a parallelogram polyomino determined by $(\mathcal{S}_1, \mathcal{S}_2)$ such that $\mathcal{S}_1$ and $\mathcal{S}_2$

intersect at $(0,0)$ and (m,n). Then $\mathcal{P}$ can be encoded in a unique 2-colored Motzkin path $\mathcal{M}_{\mathcal{P}}$ as described in the following algorithm given in [21].

Each north-east path in $\mathbb{N} \times \mathbb{N}$ of length n can be identified as a binary sequence with 0 representing an east step and 1 representing a north step. Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ be a parallelogram polyomino and set $u(\mathcal{P})$ be the binary tuple representing $\mathcal{S}_1$ and $\ell(\mathcal{P})$ be the binary tuple representing $\mathcal{S}_2$. Create a matrix M with $u(\mathcal{P})$ as its first row and $\ell(\mathcal{P})$ as its second row. Then M can be encoded as a Motzkin path by the coding:

$$\begin{aligned} &\begin{pmatrix}1\\0\end{pmatrix} \mapsto \text{rise step} && \begin{pmatrix}0\\1\end{pmatrix} \mapsto \text{fall step} \\ &\begin{pmatrix}1\\1\end{pmatrix} \mapsto \alpha\text{-colored horizontal step} && \begin{pmatrix}0\\0\end{pmatrix} \mapsto \beta\text{-colored horizontal step} \end{aligned} \tag{2.1}$$

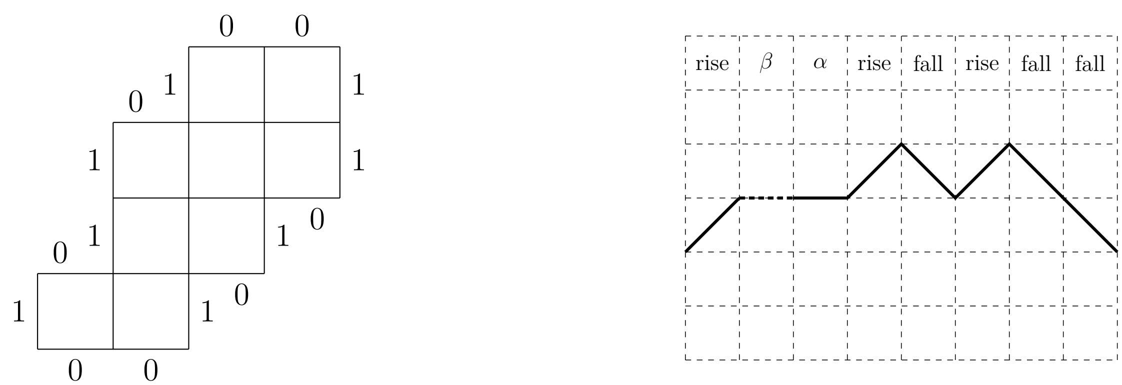


Figure 2.15: A parallelogram polyomino with its Motzkin path

For example, Figure 2.15 shows a parallelogram polyomino and the associated 2-colored Motzkin path. The β-colored horizontal steps are shown as dashed lines and the α-colored steps are shown as normal lines. We observe that

$$u(\mathcal{P}) = 10110100 \qquad \ell(\mathcal{P}) = 00101011$$

The associated matrix M of $\mathcal{P}$ described above is :

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

We also consider the reflection of $\mathcal{M}_{\mathcal{P}}$ through the x-axis. We denote this reflection by $\overline{\mathcal{M}_{\mathcal{P}}}$. The reflection $\overline{\mathcal{M}_{\mathcal{P}}}$ corresponds to the coding given in (2.1) applied to the matrix that contains $\ell(\mathcal{P})$ as first row and $u(\mathcal{P})$ as the second row.

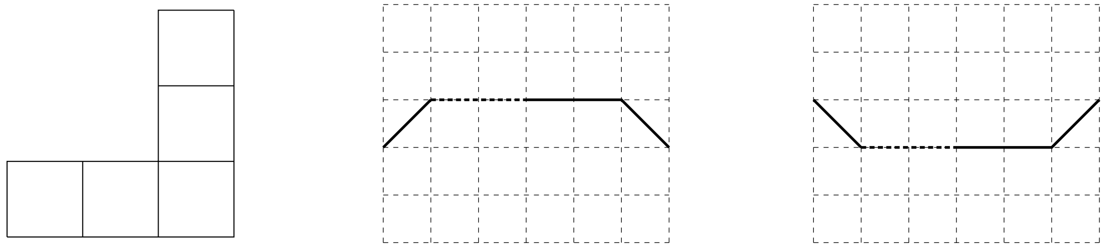


Figure 2.16: A parallelogram polyomino together with the Motzkin path $MM_{\mathcal{P}}$ and its reflection $\overline{\mathcal{M}_{\mathcal{P}}}$

2.4 POLYOMINO IDEAL AND ALGEBRAIC INVARIANTS

In this section, we start going through the Commutative Algebra of polyominoes, in particular we define the associated binomial ideals and we present some results involving Cohen-Macaulayness, Gorensteinnes, primality and Gröbner bases. The references for this section are [62] and [38].

Let $\mathcal{P}$ be a polyomino and define the polynomial ring $R = \mathbb{K}[x_v \mid v \in V(\mathcal{P})]$ over a field $\mathbb{K}$. The binomial $x_a x_b - x_c x_d \in R$ is called an *inner 2-minor* of $\mathcal{P}$ if $[a, b]$ is an inner interval of $\mathcal{P}$, where c, d are the anti-diagonal corners of $[a, b]$. The ideal $I_{\mathcal{P}} \subset R$ generated by all of the inner 2-minors of $\mathcal{P}$ is called the *polyomino ideal* of $\mathcal{P}$. The quotient ring $\mathbb{K}[\mathcal{P}] = R/I_{\mathcal{P}}$ is called the *coordinate ring* of $\mathcal{P}$. In the following, we study the algebraic properties and invariants of the polyomino ideal $I_{\mathcal{P}}$ together with its coordinate ring $I_{\mathcal{P}}$, in particular we will focus on

- **Gröbner Bases**;
- **Primality**;
- **Gorensteinnes**.

2.4.1 Gröbner bases of polyominoes

Let $a = (a_1, a_2)$ and $b = (b_1, b_2) \in V(\mathcal{P})$, we define on the vertices of $\mathcal{P}$ the following total orders:

1. $a <^1 b$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 < b_2$;

2. $a <^2 b$ if $b_1 < a_1$ or $a_1 = b_1$ and $a_2 < b_2$.

Such orders induce in a natural way the following monomial orders on R:

1. $x_a <^1_{\text{lex}} x_b$ if $a <^1 b$;
2. $x_a <^2_{\text{lex}} x_b$ if $a <^2 b$.

In [62], the author provides a necessary and sufficient condition for the set $\mathcal{M}$ of inner 2-minors to be a reduced Gröbner basis of $I_{\mathcal{P}}$, where $\mathcal{P}$ is a collection of cells of $\mathbb{N}^2$. In the following, we state the result when $\mathcal{P}$ is a polyomino.

Proposition 2.4.1. *Let $\mathcal{P}$ be a polyomino. $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^1_{\text{lex}}$ if and only if for any two intervals $[a,b]$ and $[b,e]$ of $\mathcal{P}$, at least one between $[a,f]$ and $[a,g]$ is an inner interval of $\mathcal{P}$, where f and g are the anti-diagonal corners of $[b,e]$. Similarly, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^2_{\text{lex}}$ if and only if for any two inner intervals $[a,b]$ and $[e,f]$ of $\mathcal{P}$, with d anti-diagonal corner of both the inner intervals, either a,e or b,f are anti-diagonal corners of an inner interval of $\mathcal{P}$.*

In the following example, we observe that the degree of the generators of the Gröbner basis of $\mathcal{P}$, and therefore the property of being quadratic, with respect to $<^1_{\text{lex}}$ (resp. $<^2_{\text{lex}}$) depends on the orientation of $\mathcal{P}$.

Example 2.4.2 *Let $\mathcal{P}$ be the polyomino in Figure 2.17.(A).*

Figure 2.17

The intervals $[a,b]$ and $[b,e]$ do not satisfy the condition of Proposition 2.4.1, hence the Gröbner basis of $\mathcal{P}$ with respect to $<^1_{\text{lex}}$ is not quadratic. We now consider the reflection $\bar{\mathcal{P}}$ of $\mathcal{P}$ with respect to a vertical axis. In $\bar{\mathcal{P}}$ there are no inner intervals that meet in a diagonal corner, hence the Gröbner basis of $\mathcal{P}$ with respect to $<^1_{\text{lex}}$ is quadratic.

In Section 3.2, we generalize Proposition 2.4.1 by determining similar conditions for the order $<^1_{\text{lex}}$ up to any rotation and reflection.

2.4.2 Primality of some classes of polyominoes

One of the open problems in the Commutative Algebra of polyominoes is to study whether the coordinate ring $\mathbb{K}[\mathcal{P}]$ is a domain or not.

To study such a property in the case of simple polyominoes, it is useful to consider the following construction. Let $\mathcal{P}$ be a simple polyomino. We consider the graph $G_{\mathcal{P}}$ having vertex set $\{V_i\}_{i\in I} \cup \{H_j\}_{j\in J}$, namely the set of all the maximal vertical edge intervals of $\mathcal{P}$ and all the maximal horizontal edge intervals of $\mathcal{P}$, and edge set

$$E(G_{\mathcal{P}}) = \Big\{\{H_i, V_j\} \mid H_i \cap V_j \neq \varnothing\Big\}.$$

The above graph is clearly bipartite (see Figure 2.18). Moreover, we observe that the intersection between any maximal horizontal edge interval and any maximal vertical edge interval is either empty or it is a vertex of $\mathcal{P}$. That is, $E(G_{\mathcal{P}})$ is in bijection with $V(\mathcal{P})$. Moreover, it holds $\mathbb{K}[G_{\mathcal{P}}] \cong \mathbb{K}[\mathcal{P}]$. It can be proved that such graph is weakly chordal.

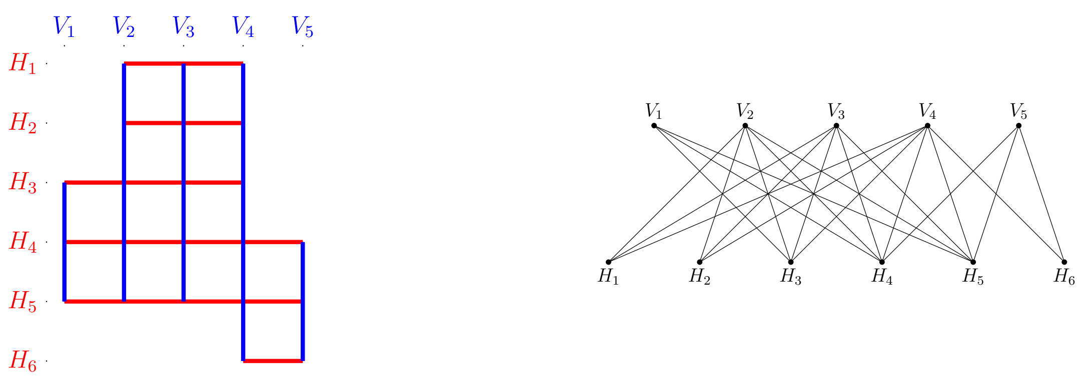


Figure 2.18

By using such a construction, it can be proved that if $\mathcal{P}$ is a simple polyomino then $\mathbb{K}[\mathcal{P}]$ is a normal Cohen-Macaulay domain ([40, Theorem 2.1] and [65, Corollary 2.3]). Combining this with [41, Corollary 3.3], one obtains the following

Lemma 2.4.3. *Let $\mathcal{P}$ be a simple polyomino. Then $\mathbb{K}[\mathcal{P}]$ is a Koszul, normal Cohen-Macaulay domain of Krull dimension $|V(\mathcal{P})| - \operatorname{rank} \mathcal{P}$.*

Moreover, in [65] and [73], the authors find a class of non simple polyominoes whose coordinate ring is a domain. Let $\mathcal{P}_{[(1,1),(m,n)]}$ be a rectangular polyomino and let $\mathcal{Q}$ be a convex polyomino. The polyomino $\mathcal{P} = \mathcal{P}_{[(1,1),(m,n)]} \setminus \mathcal{Q}$ is a multiply connected polyomino and it is proved that $I_{\mathcal{P}}$ is prime in this case. To achieve this result, in [73] the author defines a particular toric ideal $J_{\mathcal{P}}$. Let e be the lower left corner of the hole $\mathcal{Q}$, then set $\mathcal{I}_e = [(1,1), e]$ (see Figure 2.19).

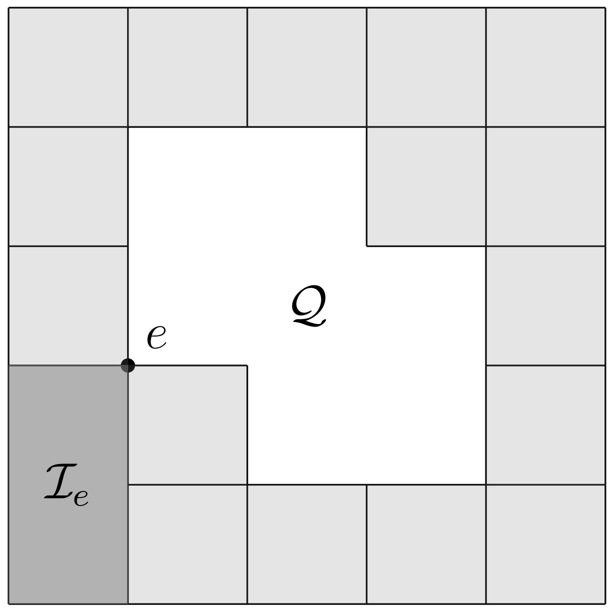


Figure 2.19

Let $\{V_i\}_{i\in I}$ be the set of all the maximal vertical edge intervals of $\mathcal{P}$, and $\{H_j\}_{j\in J}$ be the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}, \{h_j\}_{j\in J}$ be two sets of variables associated to $\{V_i\}_{i\in I}$ and $\{H_j\}_{j\in J}$, respectively, and let w be an additional variable. We consider the map

$$\alpha : V(\mathcal{P}) \longrightarrow \mathbb{K}[\{h_i, v_j \mid i \in I, j \in J\} \cup \{w\}]$$
$$a \longmapsto \prod_{a\in H_i\cap V_j} h_i v_j \prod_{a\in \mathcal{I}_e} w$$

We set $T_{\mathcal{P}} = \mathbb{K}[\alpha(a) | a \in V(\mathcal{P})] \subset \mathbb{K}[\{h_i, v_j \mid i \in I, j \in J\} \cup \{w\}]$. The ideal $J_{\mathcal{P}}$ is the kernel of the following epimorphism

$$\varphi : R \longrightarrow T_{\mathcal{P}}$$
$$x_a \longmapsto \alpha(a),$$

and the author proves that $I_{\mathcal{P}} = J_{\mathcal{P}}$. In Section 3.1, a similar construction is done for any multiply-connected polyomino.

We conclude this subsection with an example of non-prime polyomino. By computational approach, one can find that the polyomino in Figure 2.20 is not prime.

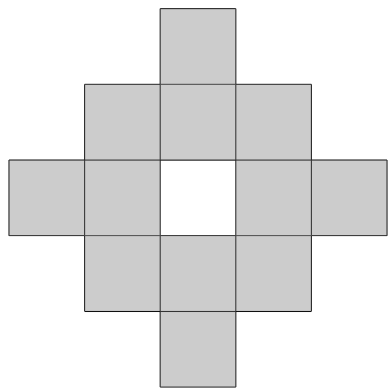


Figure 2.20: A non-prime polyomino

2.4.3 Gorensteinnes of the coordinate ring of polyominoes

Another open question is to give a complete characterization of the Gorensteinness of the algebra $\mathbb{K}[\mathcal{P}]$ when $\mathcal{P}$ is a simple polyomino. Some partial results in this direction has been obtained in [62, 1]. In the following, we present only the result of the former paper. In such a paper, the author focuses on the class of *stack* polyominoes, namely row convex bargraphs (see Figure 2.21).

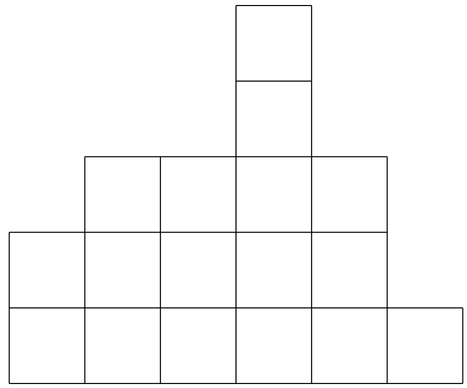


Figure 2.21: A stack polyomino

Let $\mathcal{P}$ be a stack polyomino and let $a \in V(\mathcal{P})$. We say that a is a *inside* (resp. *outside*) corner of $\mathcal{P}$ if a belongs exactly to three cells (resp. one cell) of $\mathcal{P}$.

Let $[c, d]$ be a vertical interval of $\mathcal{P}$ of maximal length and let $e_1, \ldots, e_s$ be the vertices of $[c, d]$ such that the maximal horizontal edge interval $[g_i, h_i]$ with $e_i \in [g_i, h_i]$ contains an inside corner of $\mathcal{P}$. Let $e_0 = c$ and $e_{s+1} = d$ and for $i \in \{0, s+1\}$ let $[g_i, h_i]$ be the maximal horizontal edge interval of $\mathcal{P}$ with $e_i \in [g_i, h_i]$. We set

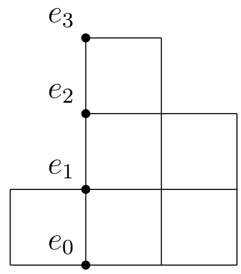
$$m_j = \Big|[g_j, h_j]\Big| - 1, \quad n_j = \Big|[e_j, e_{j+1}]\Big| - 1$$

for $j = 0, \ldots, s$ and $m_{s+1} = 0$.

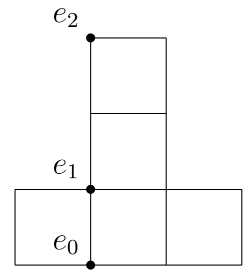
With the construction above we can state the characterization of Gorenstein stack polyominoes.

Theorem 2.4.4. *Let $\mathcal{P}$ be a stack polyomino. Then $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if $m_i = \sum_{j=i}^{s} n_j$ for $i = 0, \ldots, s$.*

Example 2.4.5 *Let $\mathcal{P}$ be the stack polyomino in Figure 2.22.(A).*



(A) A Gorenstein stack polyomino



(B) A non-Gorenstein stack polyomino

Figure 2.22

One can observe that $s = 2$, $n_0 = n_1 = n_2 = 1$ *and* $m_0 = 3$, $m_1 = 2$, $m_2 = 1$, *namely* $\mathcal{P}$ *is Gorenstein according to Theorem 2.4.4. Instead, the stack polyomino in Figure 2.22.(B) is not Gorenstein because* $s = 1$, $m_0 = m_1 = 3$, *while* $n_0 = 1$ *and* $n_1 = 2$, *that is* $m_1 \neq n_1$.

We underline that stack polyominoes are a particular subclass of L-convex polyominoes. In Section 4.1, we characterize the Gorenstein L-convex polyominoes, retrieving similar conditions to the ones of Theorem 2.4.4.

Chapter 3

PRIMALITY AND GROÖBNER BASES OF POLYOMINOES

This chapter is devoted to the study of the primality of the polyomino ideal in the case of non-simple polyominoes, the so-called multiply connected polyominoes. In Section 2.4, we have seen that simple polyominoes are prime. Among multiply connected polyominoes, one can find both prime and non-prime polyominoes. For example, in Section 2.4, we described a family of prime multiply-connected polyominoes obtained by removing a convex polyomino by a given rectangle. However, giving a complete characterization of the primality of multiply connected polyomino ideals seems to be not so easy. In order to make a further step in this direction, in Section 3.1 we observe that the non-existence of a certain sequence of inner intervals of the polyomino, called *zig-zag walk* (see Definition 3.1.2), gives a necessary condition for the primality of the polyomino ideal. Furthermore, we present a toric ideal associated to a polyomino, generalizing the construction of Section 2.4. Moreover, by computational approach, we prove that for all polyominoes with rank less than or equal to 14 the condition on zig-zag walks is also sufficient. We also conjecture that such condition gives a characterization of the primality of polyominoes. At the end of the Section, we define a new infinite family of polyominoes that we call *grid polyominoes*, that are obtained by removing rectangular holes by a given rectangle in a way that avoids the existence of zig-zag walks. We prove that grid polyominoes are prime. Beside the primality, another interesting question concerns the Gröbner basis of ideals generated by a subset of t-minors, see [59], [77] and [14]. As regards polyomino ideals, Proposition 3.2.3 provides a necessary and sufficient condition for the set of inner 2-minors to be a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to two fixed lexicographic monomial orders. In Section 3.2, we generalize such a result by giving a necessary and sufficient condition on a polyomino ideal for having the set of inner 2-minors as graded reverse lexicographic Gröbner basis, due to combinatorial properties of the polyomino itself. Moreover, we prove that when the latter holds the polyomino ideal coincides with the lattice ideal associated to the polyomino, that is the polyomino ideal is prime. In particular, we introduce the family of *thin* polyominoes, namely polyominoes that do not contain a square tetromino as subpolyomino. As an application of the results on the quadratic Gröbner basis we give two classes of prime thin polyominoes. The references for this chapter are the papers [53] and [54].

3.1 MULTIPLY CONNECTED POLYOMINOES

In this section, we investigate the primality of the multiply connected polyominoes under different point of views. In Subsection 3.1.1, we give a necessary condition for the primality of the polyomino ideal with respect to the geometric representation of the polyomino. This condition is related to a sequence of inner intervals contained in the polyomino, called a zig-zag walk, whose existence determines the non-primality of the polyomino ideal. In the same subsection,we present a toric ideal associated to a polyomino, generalizing Shikama's construction in Section 2.4. This toric ideal contains the polyomino ideal (see Proposition 3.1.1). Moreover, if the polyomino contains a zig-zag walk, the binomial associated to the zig-zag walk belongs to the toric ideal and the above inclusion is strict. The condition on zig-zag walks gives us a good filtration of primality. As an application, by implementing the algorithm described in [55], we compute all the polyominoes with rank less than or equal to 14 that are 123851 (for a complete description of the algorithm see [55]). In Subsection 3.1.2, we observe that removing 5 squares in a particular position from a given rectangle, we obtain a polyomino with a zig-zag walk (see Figure 3.6 (B)). On the other hand, by removing squares in a nice way, we construct an infinite family of prime multiply-connected polyominoes, called grid polyominoes.

3.1.1 The toric ring of generic polyominoes and zig-zag walks

In this subsection, we generalize the construction of Section 2.4 to find toric ideal associated to any generic polyomino, depending on its holes. We also notice that the existence of particular sequences of inner intervals in a multiply connected polyomino $\mathcal{P}$ gives rise to a binomial zero-divisor in $\mathbb{K}[\mathcal{P}]$, namely we have a necessary condition for the primality of $\mathcal{P}$. By computational approach, we find that such condition is also sufficient for any polyomino with rank ≤ 14, that leads us to conjecture that the latter is a characterization for the primality of any polyomino.

Let $\mathcal{P}$ be a polyomino. Let $\mathcal{H}$ be a hole of $\mathcal{P}$. We call *lower left corner* e of $\mathcal{H}$ the minimum, with respect to $<$, of the vertices of $\mathcal{H}$.

Let $\mathcal{H}_1, \dots, \mathcal{H}_r$ be holes of $\mathcal{P}$. For $k = 1, \dots, r$, we denote by $e_k = (i_k, j_k)$ the lower left corner of $\mathcal{H}_k$. For $k \in K = \{1, \dots, r\}$, we define the following subset of $V(\mathcal{P})$

$$\mathcal{F}_k = \{(i,j) \in V(\mathcal{P}) \mid i \leq i_k \text{ and } j \leq j_k\}.$$

Let $\{V_i\}_{i\in I}$ be the set of all the maximal vertical edge intervals of $\mathcal{P}$, and $\{H_j\}_{j\in J}$ be the set of all the maximal horizontal edge intervals of $\mathcal{P}$. Let $\{v_i\}_{i\in I}, \{h_j\}_{j\in J}$, and $\{w_k\}_{w\in K}$ be three sets of variables associated to $\{V_i\}_{i\in I}, \{H_j\}_{j\in J}$, and $\{\mathcal{F}_k\}_{k\in K}$, respectively. We consider the map

$$\begin{aligned} \alpha : V(\mathcal{P}) &\longrightarrow \mathbb{K}[\{h_i, v_j, w_k\} \mid i \in I, j \in J, k \in K] \\ a &\longmapsto \prod_{a \in H_i \cap V_j} h_i v_j \prod_{a \in \mathcal{F}_k} w_k \end{aligned}$$

We consider the toric ring $T_{\mathcal{P}}$ associated to $\mathcal{P}$,

$$T_{\mathcal{P}} = \mathbb{K}[\alpha(a) | a \in V(\mathcal{P})] \subset \mathbb{K}[\{h_i, v_j, w_k\} \mid i \in I, j \in J, k \in K].$$

The homomorphism

$$\begin{aligned} \varphi : R &\longrightarrow T_{\mathcal{P}} \\ x_a &\longmapsto \alpha(a) \end{aligned}$$

is surjective and the toric ideal $J_{\mathcal{P}}$ is the kernel of φ (as already seen in Section 2.4). The toric ring $T_{\mathcal{P}}$ is viewed as a standard graded $\mathbb{K}$-algebra and, therefore, the corresponding toric ideal $J_{\mathcal{P}}$ is standard graded.

By definition, $J_{\mathcal{P}}$ is a prime ideal containing $I_{\mathcal{P}}$. Moreover, the next result shows that for any polyomino $\mathcal{P}$, $(J_{\mathcal{P}})_2$, the homogeneous part of degree 2 of $J_{\mathcal{P}}$, is equal to $I_{\mathcal{P}}$, that means that the minimal generators of $I_{\mathcal{P}}$ are all and only the minimal generators of degree 2 of $J_{\mathcal{P}}$.

Lemma 3.1.1. *Let $\mathcal{P}$ be a polyomino. Then $I_{\mathcal{P}} = (J_{\mathcal{P}})_2$.*

Proof. First of all we show that $I_{\mathcal{P}} \subseteq (J_{\mathcal{P}})_2$. Let $f \in \mathcal{M}$, with $f = x_a x_b - x_c x_d$. Since $[a,b]$ is an inner interval of $\mathcal{P}$, the corners a and d (resp. b and c) lie on the same horizontal edge interval H_i (resp. H_j). In the same way, it holds that a and c (resp. b and d) lie on the same vertical edge interval V_l (resp. V_m). Therefore,

$$\varphi(x_a x_b) = h_i h_j v_l v_m \prod_{k=1,\dots,r} w_k^{p_k} \tag{3.1}$$

and

$$\varphi(x_c x_d) = h_i h_j v_l v_m \prod_{k=1,\dots,r} w_k^{n_k} \tag{3.2}$$

for some $p_k, n_k \in \{0,1,2\}$. We have to show that for any $k \in \{1,\dots,r\}$ $p_k = n_k$. If $\mathcal{P}$ has not holes then $\varphi(x_a x_b) = \varphi(x_c x_d)$, and $f \in J_{\mathcal{P}}$. Suppose that $\mathcal{H}_1, \dots, \mathcal{H}_r$ are holes of $\mathcal{P}$ and consider $\mathcal{H}_k$ for $k = 1, \dots, r$. Observe that the left lower corner e_k of $\mathcal{H}_k$ satisfies one of the following

1. $e_k < a$;
2. $a \leq e_k \leq d$;
3. $d < e_k$.

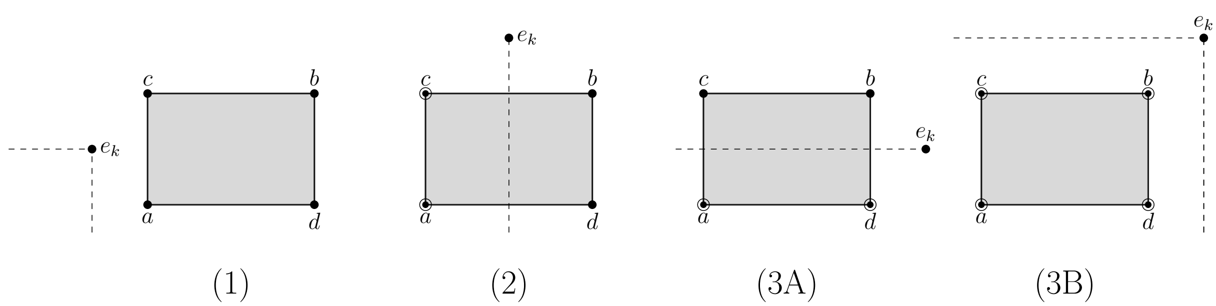


Figure 3.1: Some positions of e_k and induced flagging on $[a,b]$

Case (1). w_k does not divide $\varphi(f)$ (see Figure 3.1 (1)). Case (2). w_k divides either both $\varphi(x_a)$ and $\varphi(x_c)$ (see Figure 3.1 (2)) or it does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$. Case (3). w_k divides either $\varphi(x_a)$ and $\varphi(x_d)$ (see Figure 3.1 (3A)) or all $\varphi(x_a), \varphi(x_b), \varphi(x_c)$ and $\varphi(x_d)$ (see Figure 3.1 (3B)) or w_k does not divide neither $\varphi(x_a x_b)$ nor $\varphi(x_c x_d)$. Therefore $n_k = p_k$, and it holds for any $k = 1, \dots, r$. It follows $\varphi(x_a x_b) = \varphi(x_c x_d)$, and $f \in \ker \varphi = J_{\mathcal{P}}$. Since all generators of $I_{\mathcal{P}}$ belong to $J_{\mathcal{P}}$, the inclusion $I_{\mathcal{P}} \subseteq (J_{\mathcal{P}})_2$ is proved.

We are going to prove the other inclusion, namely $(J_{\mathcal{P}})_2 \subseteq I_{\mathcal{P}}$. Let $f \in J_{\mathcal{P}}$, $f =$

$x_ax_b - x_cx_d$. We start observing that if $a = b$ or $a \in \{c, d\}$ we obtain that f is null. Hence we assume without loss of generality $a < b$ and $c < d$. Since $\varphi(x_ax_b) = \varphi(x_cx_d)$, by (3.1) and (3.2) the vertices a and d (resp. b and c) lie on the same horizontal edge interval of $\mathcal{P}$, and a and c (resp. b and d) lie on the same vertical edge interval of $\mathcal{P}$, and all the vertices of these edge intervals belong to $\mathcal{P}$. Therefore, the vertices a, b, c, and d are the corners of the interval $[a, b]$. By contradiction, we assume that $[a, b]$ is not an inner interval of $\mathcal{P}$, namely exists a set of cells $\mathcal{C}$ that does not belong to $\mathcal{P}$ such that $[a, b] \cap \mathcal{C} \neq \emptyset$. We observe that the set $[a, b] \cap \mathcal{C}$ is a set of holes of $\mathcal{P}$ properly contained in $[a, b]$ because $[a, d]$, $[a, c]$, $[b, c]$ and $[b, d]$ are edge intervals in $\mathcal{P}$. Let $\mathcal{H}_1$ be a hole in $[a, b] \cap \mathcal{C}$ with lower left corner $e = (i, j)$. Let $\mathcal{F}_1 = \{(m, n) \in V(\mathcal{P}) \mid m \leq i \text{ and } n \leq j\}$, then a is the unique vertex in $\{a, b, c, d\}$ such that $a \in \mathcal{F}_1$, namely $w_1 | \varphi(x_ax_b)$ but $w_1 \nmid \varphi(x_cx_d)$, and $f \notin J_{\mathcal{P}}$. The assertion follows. □

Describing completely the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$ is not an easy task. However, if the polyomino contains a particular collection of inner intervals, then we have some partial information on the elements of $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$. The latter gives also a sufficient condition for the non-primality of $I_{\mathcal{P}}$, hence a necessary condition for the primality. In the rest of the subsection, we give such a condition.

Definition 3.1.2 *Let $\mathcal{P}$ be a polyomino. A sequence of distinct inner intervals $\mathcal{W}$: $I_1, \dots, I_\ell$ of $\mathcal{P}$ such that v_i, z_i are diagonal (resp. anti-diagonal) corners and u_i, v_{i+1} the anti-diagonal (resp. diagonal) corners of I_i, for $i = 1, \dots, \ell$, is a* zig-zag walk *of $\mathcal{P}$, if*

(Z1) $I_1 \cap I_\ell = \{v_1 = v_{\ell+1}\}$ and $I_i \cap I_{i+1} = \{v_{i+1}\}$, for $i = 1, \dots, \ell - 1$,

(Z2) v_i and v_{i+1} are on a same edge interval of $\mathcal{P}$, for $i = 1, \dots, \ell$,

(Z3) for any $i, j \in \{1, \dots, \ell\}$, with $i \neq j$, does not exist an inner interval J of $\mathcal{P}$ such that $z_i, z_j \in J$.

Remark 3.1.3 *Let $\mathcal{W} : I_1 \dots, I_\ell$ be a zig-zag walk of $\mathcal{P}$. Then*

(i) if v_i is a diagonal vertex of I_i, then v_{i+1} is an anti-diagonal vertex of I_{i+1};

(ii) ℓ is even.

Proof. (1) Assume that v_k, with $k \in \{1, \dots, \ell - 1\}$ is a diagonal corner of I_k. From condition

(Z2), v_{k+1} lies on the same edge interval of v_k, say E, and is an anti-diagonal corner of I_k. The line containing E divides $\mathbb{N}^2$ in two semi-planes. From condition (Z1), we have $I_k \cap I_{k+1} = \{v_{k+1}\}$, hence I_k and I_{k+1} do not lie on the same semi-plane. Therefore, v_{k+1} is anti-diagonal corner of I_{k+1}, as well. Observe that the latter justifies the name "zig-zag".
(2) Assume that the starting point v_1 is a diagonal corner of I_1. From (1) it follows that the vertex v_k is a diagonal corner of I_k if and only if k is even (resp. anti-diagonal corner if and only if k is odd). Since $v_{\ell+1} = v_1$, $\ell + 1$ is odd. □

Remark 3.1.4 *Let $\mathcal{P}$ be a polyomino and let $I_{\mathcal{P}} \subset R$ the polyomino ideal associated to $\mathcal{P}$. If $f \in I_{\mathcal{P}}$, then*

$$f = \sum f_{I_i} f_i = \sum x_{a_i} x_{b_i} f_i - \sum x_{c_i} x_{d_i} f_i,$$

where $f_{I_i} = x_{a_i} x_{b_i} - x_{c_i} x_{d_i} \in \mathcal{M}$, hence for every m, monomial of f, there are two variables in m that are (anti-)diagonal corners of an inner interval of $\mathcal{P}$.

The following proposition gives a necessary condition on $\mathcal{P}$ to have a non-prime polyomino ideal $I_{\mathcal{P}}$.

Proposition 3.1.5. *Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If there exists a zig-zag walk $\mathcal{W} : I_1, \dots, I_\ell$ in $\mathcal{P}$ then*

$$x_{v_1}, \dots, x_{v_\ell} \text{ and } f_{\mathcal{W}} = \prod_{k=1,\dots,\ell} x_{z_k} - \prod_{j=1,\dots,\ell} x_{u_j}$$

are zerodivisors of $\mathbb{K}[\mathcal{P}]$ with $x_{v_i} f_{\mathcal{W}} \in I_{\mathcal{P}}$ for $i = 1, \dots, \ell$.

Proof. For any vertex v_j in $v_1, \dots, v_\ell$, after relabelling, we may assume $j = 1$. Let $f_{I_i} \in \mathcal{M}$ be associated to the inner interval I_i.

We define the following polynomial

$$\tilde{f} = \prod_{k>1} x_{z_k} f_{I_1} + \cdots + (-1)^{i+1} \prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k} f_{I_i} + \cdots + (-1)^{\ell+1} \prod_{j<\ell} x_{u_j} f_{I_\ell}$$

Let $i = 1, \dots, \ell - 1$. Suppose that v_i is a diagonal corner of I_i, hence v_{i+1} is an anti-diagonal corner of I_{i+1}. It holds

$$\begin{aligned} &\prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k} f_{I_i} - \prod_{j<i+1} x_{u_j} \prod_{k>i+1} x_{z_k} f_{I_{i+1}} \\ =& \prod_{j<i} x_{u_j} \prod_{k>i} x_{z_k} (x_{v_i} x_{z_i} - x_{v_{i+1}} x_{u_i}) - \prod_{j<i+1} x_{u_j} \prod_{k>i+1} x_{z_k} (x_{v_{i+2}} x_{u_{i+1}} - x_{v_{i+1}} x_{z_{i+1}}) \\ =& \prod_{j<i} x_{u_j} \prod_{k\geq i} x_{z_k} x_{v_i} - \prod_{j\leq i+1} x_{u_j} \prod_{k>i+1} x_{v_{i+2}}. \end{aligned}$$

Due to the alternation of the signs in $\tilde{f}$ and by Remark 3.1.3, it follows that

$$\tilde{f} = \pm \left(\prod_{k=1,\dots,\ell} x_{z_k} x_{v_1} - \prod_{j=1,\dots,\ell} x_{u_j} x_{v_1} \right) = \pm x_{v_1} f_{\mathcal{W}},$$

and the sign of $\tilde{f}$ depends if v_1 is a diagonal corner in I_1.

Since $\tilde{f}$ is sum of polynomials in $I_{\mathcal{P}}$, then $\tilde{f} \in I_{\mathcal{P}}$. Observe that, by hypothesis, for $i \neq j$, z_i, z_j do not belong to the same inner interval of $\mathcal{P}$, and the same fact holds for u_i and u_j, with $i \neq j$. Due to this fact and by Remark 3.1.4, $f \notin I_{\mathcal{P}}$. Therefore, x_{v_1} and $f_{\mathcal{W}}$ are zerodivisors of $\mathbb{K}[\mathcal{P}]$. □

Corollary 3.1.6. *Let $\mathcal{P}$ be a polyomino and $I_{\mathcal{P}}$ the polyomino ideal associated to $\mathcal{P}$. If there exists a zig-zag walk in $\mathcal{P}$, then $I_{\mathcal{P}}$ is not prime.*

Remark 3.1.7 *The ideal $J_{\mathcal{P}}$ contains the binomials associated to zig-zag walks. Indeed, let $\mathcal{W}$ be a zig-zag walk and let $f_{\mathcal{W}}$ be its associated binomial. From the proof of Proposition 3.1.5, it arises that*

$$x_{v_1} f_{\mathcal{W}} \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$$

and, due to primality of $J_{\mathcal{P}}$, it follows $f_{\mathcal{W}} \in J_{\mathcal{P}}$.

We give an example to better understand the structure of $J_{\mathcal{P}}$.

Example 3.1.8 *We consider the polyomino in Figure 3.2. By using* `Macaulay2`*, we computed the ideal $J_{\mathcal{P}}$ associated to $\mathcal{P}$. $J_{\mathcal{P}}$ has 50 generators, 46 having degree 2, corresponding to the inner 2-minors of $\mathcal{P}$, and 4 having degree 4 that do not belong to $I_{\mathcal{P}}$. The latter are:*

$$f_1 = x_{(1,3)} x_{(3,1)} x_{(7,4)} x_{(8,2)} - x_{(1,2)} x_{(3,4)} x_{(7,1)} x_{(8,3)},$$

$$f_2 = x_{(1,3)} x_{(2,1)} x_{(7,4)} x_{(8,2)} - x_{(1,2)} x_{(2,4)} x_{(7,1)} x_{(8,3)},$$

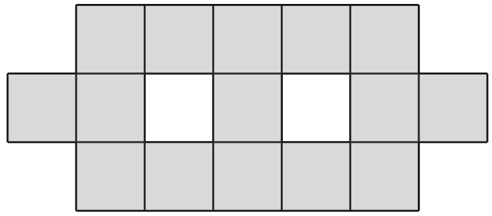


Figure 3.2

$$f_3 = x_{(1,3)}x_{(3,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(3,4)}x_{(6,1)}x_{(8,3)},$$

$$f_4 = x_{(13)}x_{(2,1)}x_{(6,4)}x_{(8,2)} - x_{(1,2)}x_{(2,4)}x_{(6,1)}x_{(8,3)}.$$

The four binomials above correspond to the four zig-zag walks drawn in Figure 3.3. In

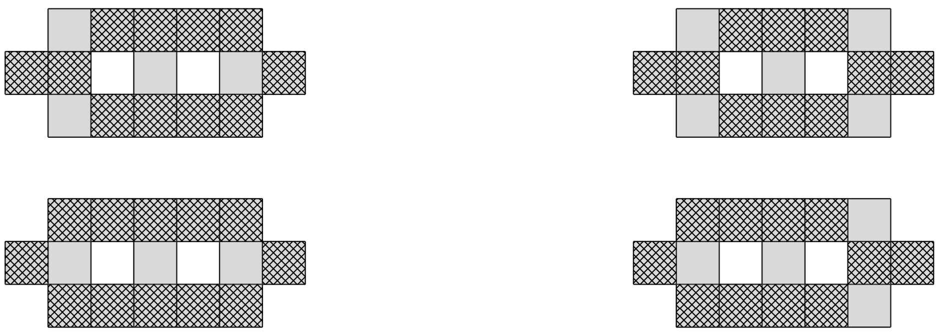


Figure 3.3: The zig-zag walks related to $f_1, \ldots, f_4$.

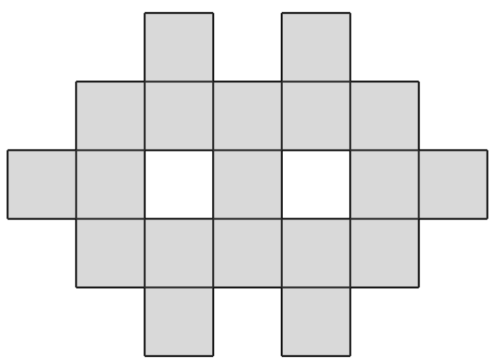
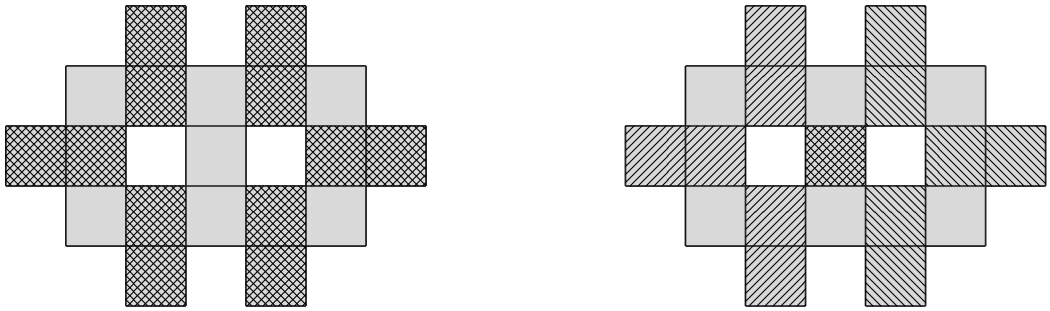


Figure 3.4

this case, the generators of $J_{\mathcal{P}}$ in $J_{\mathcal{P}} \setminus I_{\mathcal{P}}$ are all related to zig-zag walks. However, we computed $J_{\mathcal{P}}$ for the polyomino in Figure 3.4, and we found that there are generators of degree 6 that are not related to zig-zag walks, for example

$$g = x_{(1,4)}x_{(3,1)}x_{(4,6)}x_{(5,1)}x_{(6,6)}x_{(8,3)} - x_{(1,3)}x_{(3,6)}x_{(4,1)}x_{(5,6)}x_{(6,1)}x_{(8,4)}.$$

In Figure 3.5 (A), we highlight the intervals related to g. On the other hand, there are two zig-zag walks that arises from g, as in Figure 3.5 (B).



(A) g is not related to a zig-zag walk...   (B) ...but there are two zig-zag walks

Figure 3.5

The condition on zig-zag walks provides a good method to find immediately if a polyomino in non-prime. Moreover, we computationally investigate whether such condition is also sufficient. To verify that the non-existence of zig-zag walk is a sufficient condition for the primality of $I_{\mathcal{P}}$, for any multiply connected polyomino $\mathcal{P}$ of rank ≤ 14, is not an easy task. In fact, the set of polyominoes grows exponentially with respect to the rank as the following table, obtained by the implementation in [55], shows (see also Table 2.1).

Rank	7	8	9	10	11	12	13	14
Multiply connected polyominoes	1	6	37	195	979	4663	21474	96496

We now present the theorem obtained by the implementation in [55].

Theorem 3.1.9. *Let $\mathcal{P}$ be a polyomino with* rank $(\mathcal{P}) \leq 14$. *The following conditions are equivalent:*

1. *the polyomino ideal $I_{\mathcal{P}}$ is prime;*
2. *$\mathcal{P}$ contains no zig-zag walks.*

Proof. (1) $\Rightarrow$ (2) It is an immediate consequence of Corollary 3.1.6.

(2) $\Rightarrow$ (1) To prove the claim we have implemented a computer program that performs the following 3 steps:

(S1) Compute the set of all multiply connected polyominoes with rank ≤ 14, namely P.

(S2) Compute the set of polyominoes NP $\subset P$ whose associated ideals are not primes. We used a routine developed in `Macaulay2` (see [33]).

(S3) Verify that all polyominoes in NP have at least one zig-zag walk.

We refer to [55] for a complete description of the algorithm that we used.

□

From Theorem 3.1.9, a natural conjecture arises.

Conjecture 3.1.10 *Let $\mathcal{P}$ be a polyomino. The following conditions are equivalent:*

(i) the polyomino ideal $I_{\mathcal{P}}$ is prime;

(ii) $\mathcal{P}$ contains no zig-zag walks.

3.1.2 Grid Polyominoes

From a view point of finding a new class of prime polyomino ideals, due to Corollary 3.1.6, it is reasonable to consider multiply connected polyominoes with no zig-zag walks. In this subsection, we consider polyominoes obtained subtracting some inner intervals by a given interval of $\mathbb{N}^2$, similarly as done in [44] and [73]. But, if the cells are removed without a specific pattern, one can easily obtain a zig-zag walk in this case, too (see Figure 3.6(B)). Hence, we define an infinite family of polyominoes with no zig-zag walks by their intrinsic shape: the grid polyominoes.

Definition 3.1.11 *Let $\mathcal{P} \subseteq I := [(1,1),(m,n)]$ be a polyomino such that*

$$\mathcal{P} = I \setminus \{\mathcal{H}_{ij} : i \in [r],\ j \in [s]\},$$

where $\mathcal{H}_{ij} = [a_{ij}, b_{ij}]$, with $a_{ij} = ((a_{ij})_1, (a_{ij})_2)$, $b_{ij} = ((b_{ij})_1, (b_{ij})_2)$, $1 < (a_{ij})_1 < (b_{ij})_1 < m$, $1 < (a_{ij})_2 < (b_{ij})_2 < n$, and

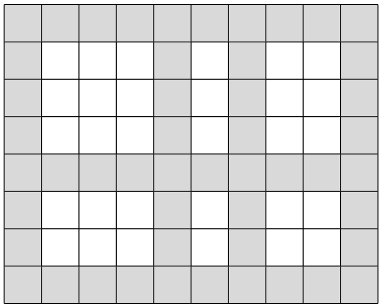
(i) for any $i \in [r]$ and $\ell, k \in [s]$ we have $(a_{i\ell})_1 = (a_{ik})_1$ and $(b_{i\ell})_1 = (b_{ik})_1$;

(ii) for any $j \in [s]$ and $\ell, k \in [r]$ we have $(a_{\ell j})_2 = (a_{kj})_2$ and $(b_{\ell j})_2 = (b_{kj})_2$;

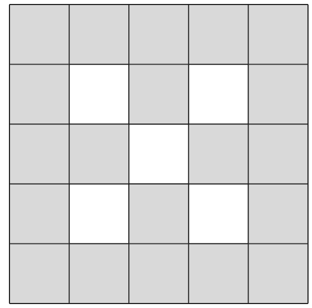
(iii) for any $i \in [r-1]$ and $j \in [s-1]$, we have $(a_{i+1j})_1 = (b_{ij})_1+1$ and $(a_{ij+1})_2 = (b_{ij})_2+1$.

We call $\mathcal{P}$ a grid polyomino.

Let $\mathcal{P}$ be a grid polyomino and let $T_{\mathcal{P}}$ and $J_{\mathcal{P}}$ be the toric ring and the toric ideal associated to $\mathcal{P}$, respectively, as defined in Subsection 3.1.1, where the hole $\mathcal{H}_{ij}$ induces the



(A) A grid polyomino



(B) A non-grid polyomino, with a zig-zag walk.

Figure 3.6

subset $\mathcal{F}_{i,j}$ and the variable $\omega_{i,j}$. We claim that the grid polyominoes are primes. In order to prove this, we are going to show that $I_{\mathcal{P}} = J_{\mathcal{P}}$.

Let $f = f^+ - f^- \in J_{\mathcal{P}}$, we define $V_+ = \{v \in V(\mathcal{P}) \mid x_v \text{ divides } f^+\}$, and, similarly, $V_- = \{v \in V(\mathcal{P}) \mid x_v \text{ divides } f^-\}$. A binomial f in a binomial ideal J is said to be *redundant* if it can be expressed as a linear combination of binomials in J of lower degree. A binomial is said to be *irredundant* if it is not redundant. The following lemma, that has been stated in [73] but only for a family of polyominoes, holds also for any $J_{\mathcal{P}}$, as defined in Subsection 3.1.1. Even if the proof is essentially the same of [73, Lemma 2.2], we report it for the sake of completeness.

Lemma 3.1.12. *Let $f = f^+ - f^- \in J_{\mathcal{P}}$ be a binomial of degree ≥ 3. If there exist three vertices $p, q \in V_+$ and $r \in V_-$ such that p, q are diagonal (resp. anti-diagonal) corners of an inner interval of $\mathcal{P}$ and r is one of the anti-diagonal (resp. diagonal) corners of the inner interval, then f is redundant in $J_{\mathcal{P}}$.*

Proof. Let s be the other corner of the inner interval determined by p, q and r. Then

$$\begin{aligned} f = f^+ - f^- &= x_p x_q \frac{f^+}{x_p x_q} - f^- \\ &= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r x_s \frac{f^+}{x_p x_q} - f^- \\ &= (x_p x_q - x_r x_s)\frac{f^+}{x_p x_q} + x_r \left(x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r} \right). \end{aligned}$$

By Lemma 3.1.1, it holds $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. Since $x_p x_q - x_r x_s \in I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$, and $J_{\mathcal{P}}$ is a prime ideal, then $x_s \frac{f^+}{x_p x_q} - \frac{f^-}{x_r} \in J_{\mathcal{P}}$, and the statement is proved. □

Let $\mathcal{P}$ be a grid polyomino, and let $\mathcal{H}_{ij}$, for $i \in [r]$ and $j \in [s]$, be its holes, enumerated as in Definition 3.1.11. Fix $i \in [r]$ and $j \in [s]$, we denote by $\mathcal{L}_{i,j}$ the set

$$\mathcal{L}_{i,j} = \mathcal{F}_{i,j} \setminus \bigcup_{\substack{k \leq i \\ h \leq j \\ (h,k) \neq (i,j)}} \mathcal{F}_{h,k}.$$

In Figure 3.7, it is displayed an example of a set $\mathcal{L}_{i,j}$. In particular, for the grid polyomino $\mathcal{P}$ in figure, $\mathcal{L}_{2,2}$ consists of all vertices of $\mathcal{P}$ in the dark grey region.

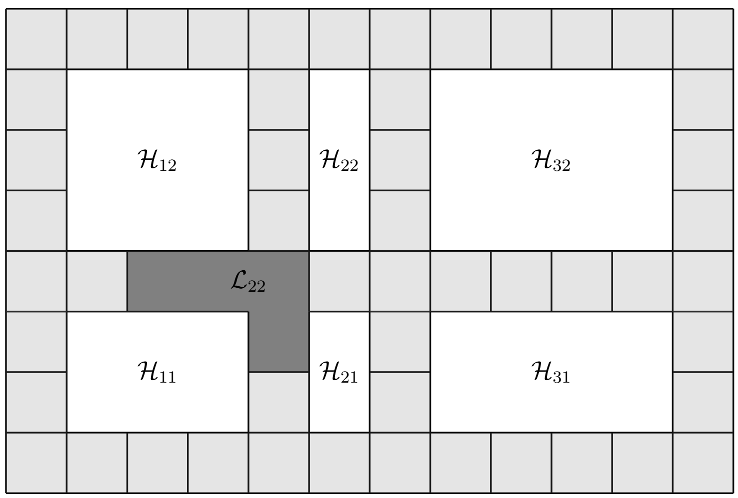


Figure 3.7: An example of $\mathcal{L}_{i,j}$.

Lemma 3.1.13. *Let $\mathcal{P}$ be a grid polyomino. Let $f = f^+ - f^- \in J_{\mathcal{P}}$. If $v \in V_+ \cap \mathcal{L}_{i,j}$, for some $i \in [r]$ and $j \in [s]$, then there exists $v' \in V_- \cap \mathcal{L}_{i,j}$.*

Proof. We prove the assertion showing that for all (i,j) and any $v \in \mathcal{L}_{i,j}$ with $v \in V_+$, there exists $v' \in V_-$ such that $v' \in \mathcal{L}_{i,j}$. Let

$$(i_1, j_1) = \min\{(k,h) \mid V_+ \cap \mathcal{F}_{k,h} \neq \emptyset\}.$$

If such a pair does not exist, there is nothing to prove. Otherwise, let $v_1 \in V_+ \cap \mathcal{L}_{i_1,j_1}$. Since $\omega_{i_1,j_1} \mid \varphi(f^+)$, then $\omega_{i_1,j_1} \mid \varphi(f^-)$. It follows there exists $v_1' \in V_- \cap \mathcal{F}_{i_1,j_1}$. By the minimality of the pair (i_1, j_1) and since $\varphi(f^+) = \varphi(f^-)$, $v_1' \in \mathcal{L}_{i_1,j_1}$. Let

$$(i_2, j_2) = \min\{(k,h) \mid (V_+ \setminus \{v_1\}) \cap \mathcal{F}_{k,h} \neq \emptyset\}.$$

If such a pair does not exist, we have done. Otherwise, let $v_2 \in (V_+ \setminus \{v_1\}) \cap \mathcal{L}_{i_2,j_2}$. We observe that because of the existence of v_1 and v_1' we have the following equation

$$f = (\prod_{\substack{k \geq i_1 \\ h \geq j_1}} \omega_{k,h})g,$$

where we have collected all $\omega_{k,h}$'s induced by v_1 and v_1'. Because of the existence of v_2, we have that

$$\omega_{i_2,j_2} \mid \varphi(g^+) = \varphi(g^-).$$

It follows there exists $v_2' \in (V_- \setminus \{v_1'\}) \cap \mathcal{F}_{i_2,j_2}$. By the minimality of the pair (i_2, j_2), $v_2' \in \mathcal{L}_{i_2,j_2}$. Iterating this procedure, the assertion follows.

□

Theorem 3.1.14. *Let $\mathcal{P}$ be a grid polyomino. Then $I_{\mathcal{P}} = J_{\mathcal{P}}$.*

Proof. By Lemma 3.1.1, $I_{\mathcal{P}} \subseteq J_{\mathcal{P}}$. We have to prove the opposite inclusion, that is $J_{\mathcal{P}} \subseteq I_{\mathcal{P}}$. Since $(J_{\mathcal{P}})_2 = I_{\mathcal{P}}$, it suffices to prove that any irredundant binomial of $J_{\mathcal{P}}$ is of degree 2. Let $f = f^+ - f^- \in J_{\mathcal{P}}$, with $\deg(f) \geq 3$. Assume by contradiction that f is irredundant. First, we show that there is no $v \in (V_+ \cup V_-) \cap \mathcal{F}$, where $\mathcal{F} = \bigcup_{i \in [r], j \in [s]} \mathcal{F}_{i,j}$. Assume by contradiction that there exists $v_1 \in (V_+ \cup V_-) \cap \mathcal{F}$. In particular, $v_1 \in \mathcal{L}_{i_1,j_1}$, for some $i_1 \in [r], j_1 \in [s]$. Without loss of generality, we may assume $v_1 \in V_+$. By Lemma 3.1.13, there exists $v_1' \in V_- \cap \mathcal{L}_{i_1,j_1}$. Note that, by the condition (3) in Definition 3.1.11, v_1 belongs to $V(\mathcal{P}) \cap V(\mathcal{H}_{ij})$, for some i, j. The same holds for v_1'. Assume $v_1 < v_1'$. We have the following 3 cases:

(1) v_1 and v_1' belong to the same maximal vertical (resp. horizontal) edge interval;

(2A) at least one between v_1 and v_1' is not a corner of an hole of $\mathcal{P}$ (e.g., see Figure 3.8 (A));

(2B) v_1 and v_1' are both diagonal (or anti-diagonal) corners of some holes of $\mathcal{P}$ (e.g., see Figure 3.8 (B)).

(1) If v_1 and v_1' belong to the same maximal vertical edge interval, there exists $v_2' \in V_-$ that lies on the same maximal horizontal edge interval of v_1. The vertices v_1, v_1' and v_2' are corners of an inner interval of $\mathcal{P}$, and by Lemma 3.1.12, f is redundant, which is a contradiction. Similarly, one shows that v_1 and v_1' do not belong to the same maximal horizontal edge interval.

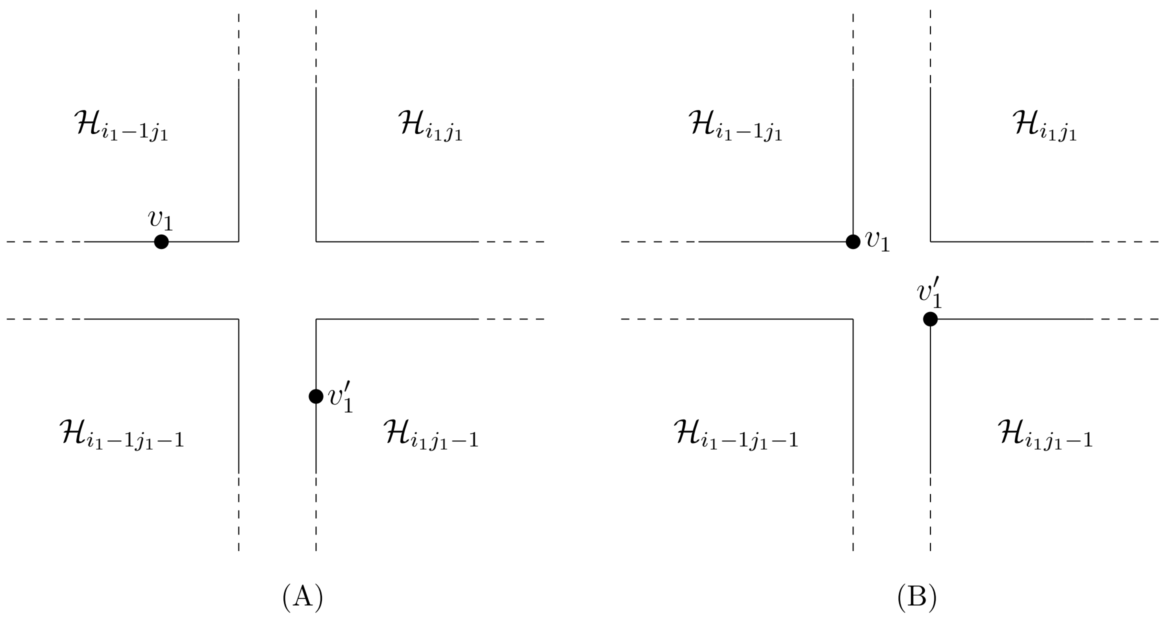


Figure 3.8

(2A) We assume that at least one between v_1 and v_1' is not a corner of an hole of $\mathcal{P}$, we say v_1. Denote by v_2' and v_3' the vertices in V_- that belong to the same horizontal and vertical edge interval of v_1, respectively. The vertices v_1, v_2', v_3' are corners of an inner interval of $\mathcal{P}$, hence by applying Lemma 3.1.12 to v_1, v_2', v_3' we obtain that f is redundant, which is a contradiction.

(2B) We denote by v_2' the vertex in V_- that belongs to the same vertical edge interval of v_1. The vertices v_1' and v_2' are diagonal (or anti-diagonal) corners of an inner interval of $\mathcal{P}$. Denote by g, h the other two corners, where g is the one on the same horizontal edge interval of v'_1. Then the binomial $x_{v'_1}x_{v'_2} - x_g x_h \in J_{\mathcal{P}}$, and

$$\begin{aligned} f = f^+ - f^- &= f^+ - x_{v'_1}x_{v'_2}\frac{f^-}{x_{v'_1}x_{v'_2}} \\ &= f^+ - x_h x_g \left(\frac{f^-}{x_{v'_1}x_{v'_2}}\right) - (x_{v'_1}x_{v'_2} - x_g x_h)\frac{f^-}{x_{v'_1}x_{v'_2}} \\ &= f' - (x_{v'_1}x_{v'_2} - x_g x_h)\frac{f^-}{x_{v'_1}x_{v'_2}}. \end{aligned}$$

Let v'_3 be the vertex in V_- that belongs to the same horizontal edge interval of v_1. The vertices v_1, v'_3, and g are corners of an inner interval of $\mathcal{P}$. Since $f' \in J_{\mathcal{P}}$, by applying Lemma 3.1.12 to v_1, v'_3 and g, we obtain that f' is redundant, and then also f is redundant, which is a contradiction.

It follows that the vertices appearing in $V_+ \cup V_-$ do not belong to $\mathcal{F}$. This means $f \in J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. Let $\mathcal{P}'$ be the subpolyomino of $\mathcal{P}$ which consists of all cells of $\mathcal{P}$ having no vertices belonging to $\mathcal{F}$. $\mathcal{P}'$ is a simple polyomino and $I_{\mathcal{P}'} = I_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. Note that $\alpha(v)$, for every $v \in V(\mathcal{P}) \setminus \mathcal{F}$, is a monomial of degree 2 determined by the maximal horizontal and vertical edge intervals to which v belongs. Then, by [65, Theorem 2.2], $I_{\mathcal{P}'} = J_{\mathcal{P}'} = J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. Hence, if f is irredundant in $J_{\mathcal{P}}$, then it is also irredundant in $J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$. But $I_{\mathcal{P}'}$ is generated by binomials of degree 2, then f is redundant in $I_{\mathcal{P}'}$, and then in $J_{\mathcal{P}} \cap \mathbb{K}[x_v \mid v \in V(\mathcal{P}) \setminus \mathcal{F}]$, which is a contradiction. □

As a corollary, we obtain the following.

Corollary 3.1.15. *Let $\mathcal{P}$ be a grid polyomino. Then $I_{\mathcal{P}}$ is prime.*

3.2 PRIMALITY OF POLYOMINOES BY QUADRATIC GRÖBNER BASES

In this Section, we go further in the study the primality of multiply connected polyominoes, by using Gröbner basis techniques. In Subsection 3.2.2 we define different graded reverse lexicographic monomial orders and, as in [62], we give a necessary and sufficient condition on $\mathcal{P}$ for having the set of inner 2-minors as reduced Gröbner basis of $I_{\mathcal{P}}$ (see Proposition 3.2.3). Starting from these monomial orders, for any corner v of the polyomino, we define new monomial orders $<_v$ such that the variable x_v is the smallest one with respect to $<_v$. We determine when $I_{\mathcal{P}}$ admits quadratic Gröbner basis with respect to $<_v$ (see Proposition 3.2.5). In this case, we prove that the ideal is prime (see Theorem 3.2.6). In Subsection 3.2.3 we apply all the previous results on a class of polyominoes: the *thin polyominoes* (see Definition 3.2.7). We exhibit necessary and sufficient conditions in terms of the geometry of the thin polyomino so that its ideal has a quadratic Gröbner basis with respect to some graded reverse lexicographic monomial orders (see Theorem 3.2.10). As an application we find two subclasses of thin polyominoes that are prime (see Corollary 3.2.12 and 3.2.15): one is that of *thin cycles* (see Definition 3.2.11) with inner intervals of length at least 3, and the other consists of polyominoes obtained from grid polyominoes by the deletion of some cells, that we call *subgrid polyominoes* (see Definition 3.2.14). Before going through the results on Gröbner bases, in Subsection 3.2.1, we recall the definition given in [62] of the lattice ideal associated to a polyomino $\mathcal{P}$, and we show that it is the ideal quotient of the polyomino ideal $I_{\mathcal{P}}$ and a monomial.

3.2.1 Lattice ideals of polyominoes

In this Subsection, we recall some basic definitions on lattices and lattice ideals and their relation with polyomino ideals. In addition, we prove Lemma 3.2.1, that is fundamental in the computation of the lattice ideal of any polyomino.

Given a lattice $\Lambda \subseteq \mathbb{Z}^{m\times n}$, we attach a binomial ideal I_Λ called the *lattice ideal* of Λ such that

$$x^{\mathbf{a}} - x^{\mathbf{b}} \in I_\Lambda \Leftrightarrow \mathbf{a} - \mathbf{b} \in \Lambda.$$

We say that a lattice Λ is *saturated* if for any $\mathbf{a} \in \mathbb{Z}^{m\times n}$, $c \in \mathbb{Z}$ such that $c\mathbf{a} \in \Lambda$, we have $\mathbf{a} \in \Lambda$. It is known that Λ is saturated if and only if I_Λ is prime. Let $\mathcal{P} \subseteq [(1,1),(m,n)]$ be a polyomino. Let

$$\mathcal{B} = \{\mathbf{e}_{ij} \ : i \in \{1,\dots,m\},\ j \in \{1,\dots,n\}\}$$

be the canonical basis of $\mathbb{Z}^{m\times n}$ and let $\mathcal{C} = \{C_1,\dots,C_r\}$ be the set of cells of $\mathcal{P}$. Let $\alpha : \mathcal{C} \longrightarrow \mathbb{Z}^{m\times n}$ be such that $\alpha(C_k) = \mathbf{c}_k = \mathbf{e}_{ij} + \mathbf{e}_{i+1j+1} - \mathbf{e}_{i+1j} - \mathbf{e}_{ij+1}$, where (i,j) is the lower left corner of the cell C_k.

It is known from [25] that an ideal generated by any set of adjacent 2-minors of a $m \times n$ matrix is a lattice ideal and that its corresponding lattice is saturated. Hence, the lattice $\Lambda = \langle \{\mathbf{c}_k\}_{k=1,\dots,r} \rangle$ is a saturated lattice, and I_Λ is a prime ideal. In addition, it is known from [62] that for a collection $\mathcal{P}$ of cells of $\mathbb{N}^2$, $I_\mathcal{P}$ is prime if and only if $I_\mathcal{P} = I_\Lambda$. Moreover,

Lemma 3.2.1. *Let $\mathcal{P}$ be a collection of cells of $\mathbb{N}^2$, let R be the polynomial ring associated to $\mathcal{P}$. Then, there exists a monomial $u \in R$ such that*

$$I_\Lambda = (I_\mathcal{P} : u).$$

Proof. $\supseteq$). Let $u \in R$ be a monomial and let $f \in (I_\mathcal{P} : u)$. We have that $uf \in I_\mathcal{P} \subseteq I_\Lambda$. Since I_Λ is a prime ideal and $u \notin I_\Lambda$, then $f \in I_\Lambda$.
$\subseteq$). Let $f_\mathbf{e} = x^{\mathbf{e}^+} - x^{\mathbf{e}^-}$ be a generator of I_Λ, with

$$\mathbf{e} = \mathbf{e}^+ - \mathbf{e}^- = \sum_{k=1}^{r} \lambda_k \mathbf{c}_k = \sum_{k=1}^{r} \Big((\lambda_k \mathbf{c}_k)^+ - (\lambda_k \mathbf{c}_k)^- \Big) \in \Lambda,$$

where $\lambda_k \in \mathbb{Z}$, $\mathbf{v}^+$ denotes the vector obtained from $\mathbf{v} \in \mathbb{Z}^{m \times n}$ by replacing all negative components of $\mathbf{v}$ by zero, and $\mathbf{v}^- = -(\mathbf{v} - \mathbf{v}^+)$.

Let $\mathbf{v} = \sum_{k=1}^{r} (\lambda_k \mathbf{c}_k)^+ - \mathbf{e}^+ = \sum_{k=1}^{r} (\lambda_k \mathbf{c}_k)^- - \mathbf{e}^-$. We have that all the components of $\mathbf{v}$ are non-negative, as for any $k \in \{1, \dots, r\}$ one has $(\mathbf{c}_k^+)_{ij} \geq (\mathbf{c}_k)_{ij}$, for all $1 \leq i \leq m$ and $1 \leq j \leq n$. This implies that the monomial $x^\mathbf{v} \in R$ is such that

$$x^\mathbf{v}(x^{\mathbf{e}^+} - x^{\mathbf{e}^-}) = \prod_{k=1}^{r} x^{(\lambda_k \mathbf{c}_k)^+} - \prod_{k=1}^{r} x^{(\lambda_k \mathbf{c}_k)^-} = \sum_{k=1}^{r} \mu_k (x^{\mathbf{c}_k^+} - x^{\mathbf{c}_k^-}) \in I_\mathcal{P},$$

for some $\mu_k \in R$. If we set u as the least common multiple of the elements $x^\mathbf{v}$ induced by all the generators $f_\mathbf{e}$ of I_Λ the assertion follows. □

3.2.2 Quadratic graded reverse lexicographic Gröbner basis

Consider the total orders $<^i$, with $i \in \{1, \dots, 8\}$, on $\mathbb{N}^2$ induced by the pairs of arrows displayed in Table 3.1.

$<^1$	$<^2$	$<^3$	$<^4$	$<^5$	$<^6$	$<^7$	$<^8$
$(\downarrow, \rightarrow)$	$(\downarrow, \leftarrow)$	$(\uparrow, \leftarrow)$	$(\uparrow, \rightarrow)$	$(\leftarrow, \uparrow)$	$(\rightarrow, \uparrow)$	$(\rightarrow, \downarrow)$	$(\leftarrow, \downarrow)$

Table 3.1: Pairs of arrows that induce the total orders.

Given $a = (a_1, a_2)$ and $b = (b_1, b_2)$, the horizontal arrows refer to the first coordinates, a_1 and b_1, while the vertical ones to the second coordinates, a_2 and b_2. Each arrow goes from the minimum to the maximum. For any pair of arrows, that is for any total order, we first compare the coordinate given by the second arrow, and, if they are equal, then we compare the coordinates given by the first arrow. For instance, $a <^1 b$ if $a_1 < b_1$ or $a_1 = b_1$ and $a_2 > b_2$. That is, let $a, b, c, d \in V(\mathcal{P})$ be as in Figure 3.9.

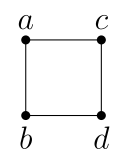


Figure 3.9

Then it holds $a <^1 b <^1 c <^1 d$. The latter explains the order of the arrows, that is, we can order a set of vertices from the minimum to the maximum by firstly following the direction given by the first arrow and then the direction given by the second one. Similarly $(a_1, a_2) <^5 (b_1, b_2)$ if $a_2 < b_2$ or $a_2 = b_2$ and $a_1 > b_1$ and then one can recover all of the other orders. In the next remark, we show the relations between the orders $<^i$.

Remark 3.2.2 *Let $\mathcal{P}$ be the polyomino in Figure 3.10.*

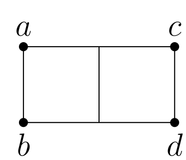


Figure 3.10: A rectangular polyomino $\mathcal{P}$

Then with respect to the orders $<^1$ induced by $(\downarrow, \rightarrow)$, $<^2$ induced by $(\downarrow, \leftarrow)$, $<^3$ induced by $(\uparrow, \leftarrow)$ we have

$$a <^1 b <^1 c <^1 d, \quad c <^2 d <^2 a <^2 b, \quad d <^3 c <^3 b <^3 a.$$

Let $\mathcal{P}'$ and $\mathcal{P}''$ be respectively the reflection of $\mathcal{P}$ with respect to the line containing the edge $\{c, d\}$ (Figure 3.11) and the 180 degree rotation of $\mathcal{P}$ (Figure 3.12).

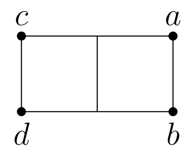


Figure 3.11: The polyomino $\mathcal{P}'$: the reflection of $\mathcal{P}$ with respect to $\{c, d\}$

We observe that in $\mathcal{P}'$ we have $c <^1 d <^1 a <^1 b$.

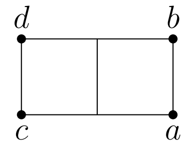


Figure 3.12: The polyomino $\mathcal{P}''$: the 180 degree rotation of $\mathcal{P}$

We observe that in $\mathcal{P}''$ we have $d <^1 c <^1 b <^1 a$. We conclude that the order $<^2$ it is equal to the order $<^1$ up to a reflection of the polyomino, while the order $<^3$ is equal to the order $<^1$ up to a 180 degree rotation of the polyomino. Similarly the other relations follow.

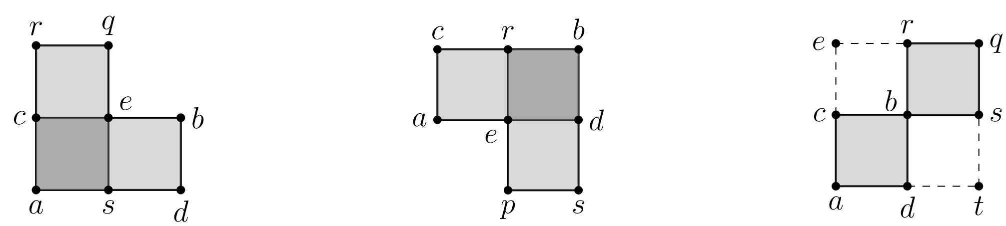
The total orders $<^i$, with $i \in \{1, \dots, 8\}$, on the vertices of $\mathcal{P}$ induce in a natural way the graded reverse lexicographic monomial orders $<^i_{\text{grevlex}}$, with $i \in \{1, \dots, 8\}$, on the polynomial ring R, respectively.

As in Proposition 2.4.1, the next proposition gives a necessary and sufficient condition on $\mathcal{P}$ for having $\mathcal{M}$ as quadratic reduced Gröbner basis of $I_{\mathcal{P}}$.

From now on, we set $\mathcal{O} = \{1, 3, 5, 7\}$ and $\mathcal{E} = \{2, 4, 6, 8\}$.

Proposition 3.2.3. *Let $\mathcal{P}$ be a polyomino. $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, for $i \in \mathcal{O}$, if and only if for any two intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$, at least one interval between $[a, f]$ and $[a, g]$ is an inner interval of $\mathcal{P}$, where f and g are the anti-diagonal corners of $[b, e]$. Similarly, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, for $i \in \mathcal{E}$, if and only if for any two inner intervals $[a, b]$ and $[e, f]$ of $\mathcal{P}$, with d anti-diagonal corner of both the inner intervals, either a, e or b, f are anti-diagonal corners of an inner interval of $\mathcal{P}$.*

Proof. We are going to prove the statement only for $<^1_{\text{grevlex}}$,then, by similar arguments and by Remark 3.2.2, the other cases follow. The others follow in a similar way. The set $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^1_{\text{grevlex}}$ if and only if all S-polynomials of inner 2-minors of $I_{\mathcal{P}}$ reduce to 0. Let $f, g \in \mathcal{M}$, where $f = x_a x_b - x_c x_d$ is associated to the inner interval $[a, b]$ of $\mathcal{P}$ and $g = x_p x_q - x_r x_s$ is associated to the inner interval $[p, q]$ of $\mathcal{P}$. In the following, we denote by S the S-polynomial between f and g and by $\text{in}(h)$ the leading monomial of a polynomial h. We consider the non-trivial cases when $\gcd(\text{in}(f), \text{in}(g)) \neq 1$. Moreover, if one of the inner intervals, namely $[a, b]$, is contained in the second one, namely $[p, q]$, S reduces to 0 since the polyomino ideal is generated by all inner 2-minors. In the following, denote by $<$ the total order $<^1$ on the vertices of $\mathcal{P}$. Without loss of generality, let $a \leq p$. Therefore, we have to consider the following cases: $a = p$, $b = q$, and $b = p$.



(A) Case $a = p$. (B) Case $b = q$. (C) Case $b = p$.

Figure 3.13

Let $a = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_a x_q - x_r x_s$, and assume $r < c < a < q < s < b < d$ as in Figure 3.13A. We have $S = x_q x_c x_d - x_b x_r x_s$ and $\text{in}(S) = x_q x_c x_d$. Since $\text{in}(f_{c,q}) = x_c x_q$, we get

$$S = x_d(x_c x_q - x_r x_e) - x_r(x_s x_b - x_e x_d),$$

that is S reduces to 0 with respect to $\mathcal{M}$.

Let $b = q$, and assume $c < a < r < p < b < d < s$ as in Figure 3.13B. We have $S = x_a x_r x_s - x_c x_d x_p$ and $\text{in}(S) = x_a x_r x_s$. Since $\text{in}(f_{a,r}) = x_a x_r$, we get

$$S = x_s(x_a x_r - x_c x_e) - x_c(x_p x_d - x_e x_s),$$

that is S reduces to 0 with respect to $\mathcal{M}$.

Let $b = p$, and assume $c < a < r < b < d < q < s$ as in Figure 3.13C. We have $S = x_a x_r x_s - x_q x_c x_d$ and $\text{in}(S) = x_a x_r x_s$. If neither $[a, s]$ nor $[a, r]$ is an inner interval of $\mathcal{P}$, then S does not reduce to 0 with respect to $\mathcal{M}$ and the Gröbner basis is not quadratic. Furthermore, if $[a, s]$ is an inner interval of $\mathcal{P}$, since $\text{in}(f_{a,s}) = x_a x_s$, we get

$$S = x_r(x_a x_s - x_c x_t) - x_c(x_d x_q - x_r x_t).$$

If $[a, r]$ is an inner interval of $\mathcal{P}$, since $\text{in}(f_{a,r}) = x_a x_r$, we get

$$S = x_s(x_a x_r - x_e x_d) - x_d(x_c x_q - x_e x_s).$$

It shows that in both situations S reduces to 0 with respect to $\mathcal{M}$. The latter shows that S reduces to 0 with respect to $\mathcal{M}$ if and only if either $[a, s]$ or $[a, r]$ is an inner interval of $\mathcal{P}$ and the thesis follows. □

Let $V(\mathcal{P}) = \{v_1, \dots, v_n\}$. Given a monomial order $<$ such that we have

$$x_{v_1} < x_{v_2} < \cdots < x_{v_n},$$

we define by $<_v$, with $v = v_k \in V(\mathcal{P})$, the following monomial order:

$$x_{v_k} < x_{v_{k+1}} < \cdots < x_{v_n} < x_{v_1} < x_{v_2} < \cdots < x_{v_{k-1}}.$$

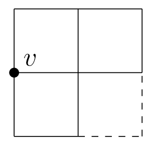
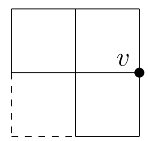
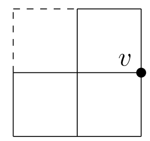
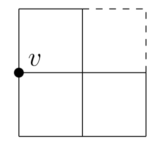
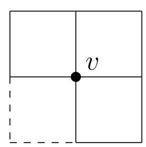
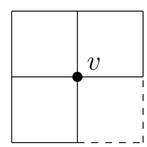
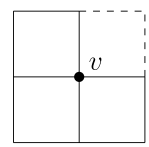
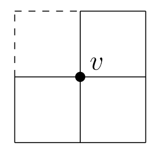
From now on, we will denote $(<^i_{\text{grevlex}})_v$ by $<^i_v$, for any $i \in \{1, \dots, 8\}$.

Definition 3.2.4 *Let $\mathcal{P}$ be a polyomino and let $v \in V(\mathcal{P})$. We say that v satisfies the condition π_1 if it fulfils at least one of the following conditions:*

(I) There exist two inner intervals $I = [a, b]$ and $J = [b, q]$ of $\mathcal{P}$, with v upper left corner of I, and s the lower right corner of J, such that $[v, q]$ is inner interval of $\mathcal{P}$, whereas the interval $[a, s]$ is not (see Table 3.2, Case π_1 (I)).

(II) There exist two inner intervals $K = [a, b]$ and $L = [p, q]$, with v lower right corner of K and upper left corner of L, such that the interval having b and q as anti-diagonal corners is inner interval of $\mathcal{P}$, whereas the interval having a and p as anti-diagonal corners is not (see Table 3.2, Case π_1 (II)).

In a similar way, by Remark 3.2.2 and by using suitable rotations and/or reflections, one can define v satisfying the condition π_i, for $i \in \{2, \dots, 8\}$, if it fulfils at least one of the cases (I) and (II) displayed in Table 3.2.

	π_1	π_2	π_3	π_4
(I)				
(II)				

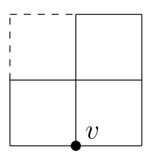
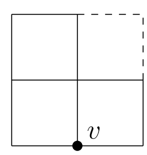
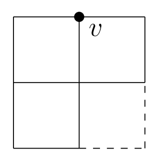
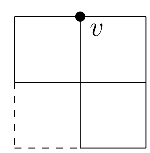
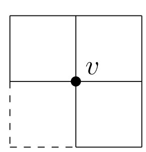
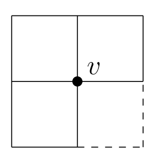
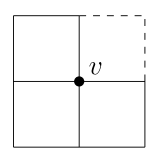
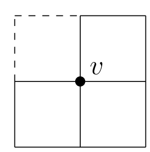
	π_5	π_6	π_7	π_8
(I)				
(II)				

Table 3.2: Conditions π_i, for $i = 1, \dots, 8$.

Proposition 3.2.5. *Let $\mathcal{P}$ be a polyomino such that $I_{\mathcal{P}}$ has $\mathcal{M}$ as reduced Gröbner basis with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$ ($i \in \mathcal{E}$, respectively). If $v \in V(\mathcal{P})$ does not satisfy π_k for some $k \in \mathcal{O}$ ($k \in \mathcal{E}$, respectively), then $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^k_v$.*

Proof. Assume that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$. Let $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_r x_s$ be associated to the inner interval $[a, b]$ and $[p, q]$ of $\mathcal{P}$, respectively. Let $v \in V(\mathcal{P})$. We have to show that for each pair of inner 2-minors, f and g, the corresponding S-polynomial reduces to 0 with respect to a fixed monomial order $<_v^i$, with $i \in \mathcal{O}$. In the following, we denote by S the S-polynomial between f and g, by $\mathrm{in}(h)$ the leading monomial of a polynomial h, and by $f_{m,n}$ the inner 2-minor associated to the inner interval $[m, n]$ of $\mathcal{P}$.

We leave to the reader the trivial cases $\{a, b, c, d\} \cap \{p, q, r, s\} = \varnothing$, and $|\{a, b, c, d\} \cap \{p, q, r, s\}| = 2$ where S reduces to 0 since the polyomino ideal is generated by all inner 2-minors.

Note that if, for all vertices $w \in \{a, b, c, d, p, q, r, s\}$ and a monomial order $<_{\text{grevlex}}^i$, for some $i \in \mathcal{O}$, it holds $x_w <_v^i x_v$ or $x_v <_v^i x_w$, then S reduces to 0 with respect to $<_v^i$, since it reduces to 0 with respect to $<_{\text{grevlex}}^i$.

If one of the inner intervals, namely $[a, b]$, is contained in the second one, namely $[p, q]$, S reduces to 0 since the polyomino ideal is generated by all inner 2-minors. In the following, denote by $<$ the total order $<^1$ on the vertices of $\mathcal{P}$. Without loss of generality, let $a \leq p$. Therefore, we have to consider the following cases:

$$a = p, \qquad b, d \in \{p, q, r, s\}, \qquad c \in \{p, r\}.$$

If v does not satisfy the condition π_k, for some $k \in \mathcal{O}$, we fix the monomial order $<_v^k$. Without loss of generality, assume $k = 1$. Then, by similar arguments and by Remark 3.2.2, the cases $k > 1$ can be done by applying suitable rotations and/or reflections.

Let $a = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_a x_q - x_r x_s$, and $r < c < a < q < s < b < d$ as in Figure 3.14.

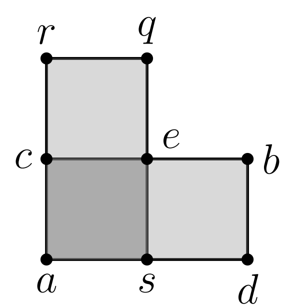


Figure 3.14: Case $a = p$.

We start by observing that if $r < v \leq b$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, we have $S = x_r x_s x_b - x_c x_d x_q$. If $b < v \leq d$, then $\text{in}(S) = x_r x_s x_b$. Since $\text{in}(f_{s,b}) = x_s x_b$, then

$$S = x_r(x_s x_b - x_e x_d) - x_d(x_c x_q - x_r x_e),$$

that is S reduces to 0 with respect to the inner 2-minors $f_{s,b}$ and $f_{c,q}$. If $v = r$, then $\text{in}(S) = x_c x_d x_q$. Since $\text{in}(f_{c,q}) = x_c x_q$, then

$$S = -x_d(x_c x_q - x_r x_e) + x_r(x_s x_b - x_e x_d),$$

that is S reduces to 0 with respect to the inner 2-minors $f_{c,q}$ and $f_{s,b}$.

Let $b = p$, that is $f = x_a x_b - x_c x_d$ and $g = x_b x_q - x_r x_s$, and$c < a < r < b < d < q < s$, as in Figure 3.15.

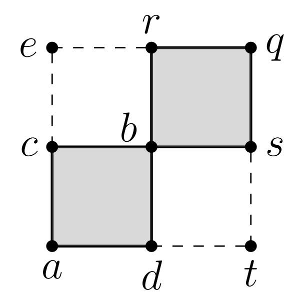


Figure 3.15: Case $b = p$.

If $c < v \leq q$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, we have $S = x_a x_r x_s - x_q x_c x_d$. If $q < v \leq s$, then $\text{in}(S) = x_q x_c x_d$. By hypothesis, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ with $i \in \mathcal{O}$, hence, from Proposition 3.2.3, either $[c, q]$ or $[d, q]$ is an inner interval of $\mathcal{P}$, with $\text{in}(f_{c,q}) = x_c x_q$ and $\text{in}(f_{d,q}) = x_d x_q$, and then

$$S = x_d(x_c x_q - x_e x_s) - x_s(x_a x_r - x_e x_d)$$

or

$$S = -x_c(x_d x_q - x_r x_t) + x_r(x_a x_s - x_c x_t),$$

that is S reduces to 0 with respect to the inner 2-minors either $f_{c,q}$ and $f_{a,r}$ or $f_{d,q}$ and $f_{a,s}$. If $v = c$, then $\text{in}(S) = x_a x_r x_s$. By hypothesis, either $[a, r]$ or $[a, s]$ is an inner interval of $\mathcal{P}$, with $\text{in}(f_{a,r}) = x_e x_d$ and $\text{in}(f_{a,s}) = x_a x_s$. If $[a, r]$ is an inner interval of $\mathcal{P}$, but $[a, s]$ is not, then v satisfies the condition π_1, so we have not to consider this case. Whereas, if $[a,s]$ is an inner interval, since $\mathrm{in}(f_{a,s}) = x_a x_s$, then

$$S = x_r(x_a x_s - x_c x_t) - x_c(x_d x_q - x_r x_t),$$

it follows that S reduces to 0.
Note that when $v = c$, if $[a,r]$ is an inner interval of $\mathcal{P}$, but $[a,s]$ is not, that is v satisfies π_1, in particular the condition π_1 (I), then S does not reduce to 0 with respect to $\mathcal{M}$ and $<_v^1$. In fact, $\mathrm{in}(S) = x_a x_r x_s$, but the monomials $x_a x_r$, $x_a x_s$, and $x_r x_s$ are not leading monomials of any inner 2-minor of $\mathcal{P}$. This situation justifies the hypothesis v not satisfying the condition π_1.

Let $b = r$, that is that is $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_b x_s$. We have to distinguish two different situations: $p < d$ (see Figure 3.16 (A)) or $p > d$ (see Figure 3.16 (B)).

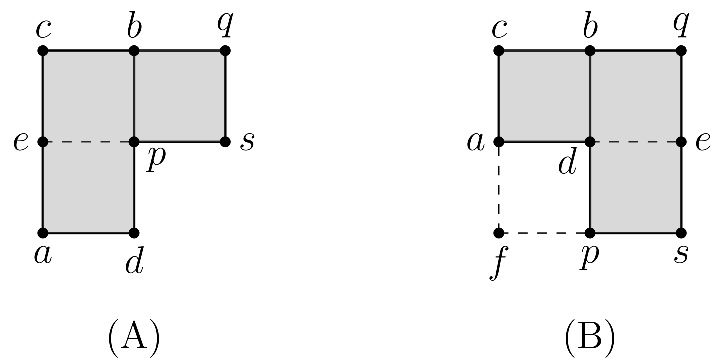


Figure 3.16: Case $b = r$.

Assume $p < d$, then $c < a < b < p < d < q < s$, as in Figure 3.16 (A). If $c \leq v \leq b$ or $q < v \leq s$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, $S = x_a x_p x_q - x_c x_d x_s$. If $b < v \leq p$ or $d < v \leq q$, then $\mathrm{in}(S) = x_c x_d x_s$ and $\mathrm{in}(f_{e,q}) = x_c x_s$. If $p < v \leq d$, then $\mathrm{in}(S) = x_a x_p x_q$ and $\mathrm{in}(f_{a,p}) = x_a x_p$. Therefore,

$$S = x_d(x_e x_q - x_c x_s) + x_q(x_a x_p - x_e x_d),$$

that is S reduces to 0 in all of these cases.
Assume $p > d$, then $c < a < b < d < p < q < s$, as in Figure 3.16 (B). If $c \leq v \leq b$ or $q < v \leq s$, then $\gcd(\mathrm{in}(f), \mathrm{in}(g)) = 1$. In the other cases, we have $S = x_a x_p x_q - x_c x_d x_s$. If

$b < v \leq d$, then $\text{in}(S) = x_a x_p x_q$. By hypothesis, v does not satisfy the condition π_1, hence $[f, d]$ is an inner interval of $\mathcal{P}$. Since $\text{in}(f_{f,d}) = x_a x_p$, then

$$S = -x_q(x_f x_d - x_a x_p) + x_d(x_f x_q - x_c x_s),$$

that is S reduces to 0. If $d < v \leq q$, then $\text{in}(S) = x_c x_d x_s$. Since $\text{in}(f_{p,e}) = x_d x_s$, it follows

$$S = x_c(x_p x_e - x_d x_s) + x_p(x_a x_q - x_c x_e),$$

that is S reduces to 0.

Note that when $b < v \leq d$, if $[f, d]$ is not an inner interval of $\mathcal{P}$, then v satisfies π_1, in particular the condition π_1 (II). In this case, S does not reduce to 0 with respect to $\mathcal{M}$ and $<^1_v$. In fact, $\text{in}(S) = x_a x_p x_q$, but the monomials $x_a x_p$, $x_a x_q$, and $x_p x_q$ are not leading monomials of any inner 2-minor of $\mathcal{P}$. This situation justifies, once again, the hypothesis v not satisfying the condition π_1.

Let $d = q$, that is $f = x_a x_b - x_c x_d$ and $g = x_p x_d - x_r x_s$, and $c < a < r < p < b < d < s$, as showed in Figure 3.17.

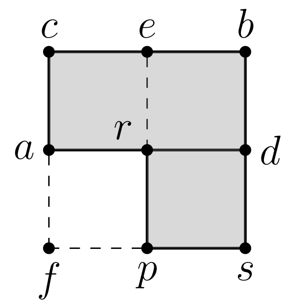


Figure 3.17: Case $d = q$.

If either $v = c$ or $r < v \leq s$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, we have $S = x_a x_b x_p - x_c x_r x_s$. If $c < v \leq a$, then $\text{in}(S) = x_c x_r x_s$. Since $\text{in}(f_{c,r}) = x_c x_r$, then

$$S = x_s(x_a x_e - x_c x_r) + x_a(x_p x_b - x_s x_e),$$

that is S reduces to 0. If $a < v \leq r$, then $\text{in}(S) = x_a x_b x_p$. By hypothesis, v does not satisfy π_1, that is $[f, r]$ is an inner interval of $\mathcal{P}$. Since $\text{in}(f_{f,r}) = x_a x_p$, then

$$S = -x_b(x_f x_r - x_a x_p) + x_r(x_f x_b - x_c x_s),$$

that is S reduces to 0.

Let $c = r$, that is $f = x_a x_b - x_c x_d$ and $g = x_p x_q - x_c x_s$, and $c < p < a < b < d < q < s$, as showed in Figure 3.18.

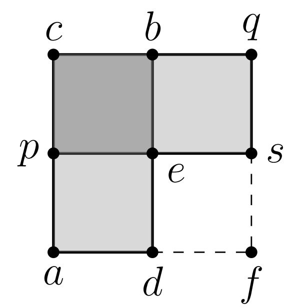


Figure 3.18: Case $c = r$.

If either $v = c$ or $b < v \leq s$, then $\gcd(\text{in}(f), \text{in}(g)) = 1$. In the other cases, we have $S = x_a x_b x_s - x_d x_p x_q$. If $c < v \leq p$, $\text{in}(S) = x_a x_b x_s$. Since v does not satisfy π_1, then $[a, s]$ is an inner interval of $\mathcal{P}$ and $\text{in}(f_{a,s}) = x_a x_s$. Therefore,

$$S = x_b(x_a x_s - x_p x_f) - x_p(x_d x_q - x_b x_f),$$

that is S reduces to 0. If $p < v \leq b$, then $\text{in}(S) = x_d x_p x_q$ and $\text{in}(f_{a,e}) = x_p x_d$. Therefore,

$$S = x_q(x_a x_e - x_p x_d) - x_a(x_e x_q - x_b x_s),$$

that is S reduces to 0. For the sake of brevity, we leave to readers to check, in a similar way, that if $b \in \{q, s\}$, $d \in \{p, r, s\}$, and $c = p$, then all the S-polynomials reduce to 0. Moreover, for no one of the corners v in these cases it needs to require the hypothesis that v does not satisfy the condition π_1.

□

We now prove the main theorem of this section.

Theorem 3.2.6. *Let $\mathcal{P}$ be a polyomino such that $I_{\mathcal{P}}$ has $\mathcal{M}$ as reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$ ($i \in \mathcal{E}$, respectively). If, for all $v \in V(\mathcal{P})$, there exists a $k_v \in \mathcal{O}$ ($k_v \in \mathcal{E}$, respectively) such that v does not satisfy π_{k_v}, then*

1. *$\mathcal{M}$ forms a reduced Gröbner basis with respect to $<^{k_v}_v$, for all $v \in V(\mathcal{P})$;*

2. $I_{\mathcal{P}}$ is prime.

Proof. (1) It is an immediate consequence of Proposition 3.2.5.
(2) Fix $v \in V(\mathcal{P})$. By (1), let $<_v$ denote the monomial order for which $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$. By Lemma 1.2.7, the reduced Gröbner basis of $(I_{\mathcal{P}} : x_v)$ with respect to $<_v$ is given by

$$\{f \in \mathcal{M} \mid x_v \text{ does not divide } f\} \cup \{f/x_v \mid f \in \mathcal{M} \text{ and } x_v \text{ divides } f\}.$$

Since all $f \in \mathcal{M}$ are not divisible by x_v, the reduced Gröbner basis of $(I_{\mathcal{P}} : x_v)$ with respect to $<_v$ is $\mathcal{M}$. Therefore $(I_{\mathcal{P}} : x_v) = I_{\mathcal{P}}$, for all $x_v \in V(\mathcal{P})$. It follows that $(I_{\mathcal{P}} : u) = I_{\mathcal{P}}$ for any monomial $u \in R$. By Lemma 3.2.1, we have that there exists a monomial $u \in R$ such that $I_\Lambda = (I_{\mathcal{P}} : u)$. Then

$$I_\Lambda = (I_{\mathcal{P}} : u) = I_{\mathcal{P}}.$$

It follows that $I_{\mathcal{P}}$ coincides with the lattice ideal I_Λ, which is prime. Therefore, $I_{\mathcal{P}}$ is a prime ideal, as well. □

3.2.3 Thin polyominoes

In this subsection, we introduce the class of thin polyominoes and we rephrase the geometric condition for the quadratic Gröbner basis of $I_{\mathcal{P}}$ in Proposition 3.2.3 in terms of some subpolyominoes of the thin polyomino $\mathcal{P}$. Thanks to the above interpretation, we find two new classes of thin polyominoes having a prime polyomino ideal: the thin cycle with no maximal inner interval of length 2 and the subgrid polyominoes.

Definition 3.2.7 *Let $\mathcal{P}$ be a polyomino. We say that $\mathcal{P}$ is* thin *if $\mathcal{P}$ does not have the polyomino $\mathcal{Q}$ in Figure 3.19 as a subpolyomino.*

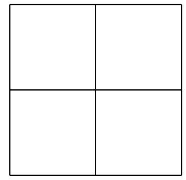


Figure 3.19: The polyomino $\mathcal{Q}$

Theorem 3.2.8. *Let $\mathcal{P}$ be a thin polyomino such that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \mathcal{O}$ (for $i \in \mathcal{E}$, respectively). Then, for any $v \in V(\mathcal{P})$, there exists $k \in \mathcal{O}$ ($k \in \mathcal{E}$, respectively) such that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^k_v$.*

Proof. Assume that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$. Let $v \in V(\mathcal{P})$. From Proposition 3.2.5 it suffices to show that there exists $k \in \mathcal{O}$ such that v does not satisfy π_k.

We claim that v can not satisfy simultaneously π_1 and π_3. In fact, if v satisfies simultaneously π_1 and π_3, then there exist four cells C, D, E, F of $\mathcal{P}$ such that $C \cap D \cap E \cap F = \{v\}$. From Table 3.2, if v satisfies π_1 then there exist two cells C, D of $\mathcal{P}$ such that v is simultaneously the lower left corner of C and the upper left corner of D, while if $\mathcal{P}$ satisfies π_3 then there exist two cells E, F of $\mathcal{P}$ such that v is simultaneously the lower right corner of E and the upper right corner of F. Since v satisfies simultaneously π_1 and π_3, the cells C, D, E, F are the ones desired. This implies that the polyomino $\mathcal{Q}$ in Figure 3.19 is a subpolyomino of $\mathcal{P}$ and then $\mathcal{P}$ is not thin, which is a contradiction. It follows that there exists at least a $k \in \mathcal{O}$ such that v does not satisfy π_k, as desired. □

Corollary 3.2.9. *Let $\mathcal{P}$ be a thin polyomino such that $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \{1, \dots, 8\}$. Then $I_{\mathcal{P}}$ is prime.*

Proof. By Theorem 3.2.8, for any $v \in V(\mathcal{P})$, $\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^k_v$, for some $k \in \{1, \dots, 8\}$. By Theorem 3.2.6, it follows that $I_{\mathcal{P}}$ is prime. □

Theorem 3.2.10. *Let $\mathcal{P}$ be a thin polyomino. The following facts are equivalent:*

1. *$\mathcal{M}$ forms a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \mathcal{O}$ ($i \in \mathcal{E}$, respectively);*
2. *there are no cells $C, D \notin \mathcal{P}$ and $E, F \in \mathcal{P}$ such that $C \cap D \cap E \cap F \neq \emptyset$ as in Figure 3.20 (a) (Figure 3.20 (b), respectively) and the polyominoes in Figure 3.21 (i) and (ii) (in Figure 3.21 (iii) and (iv), respectively) are not subpolyominoes of $\mathcal{P}$.*

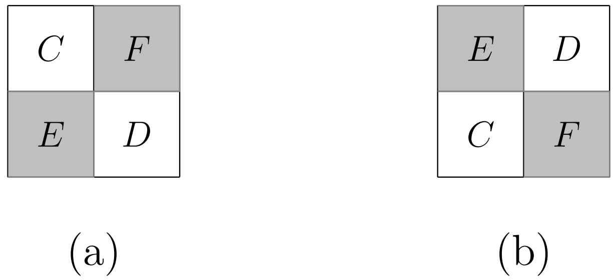


Figure 3.20

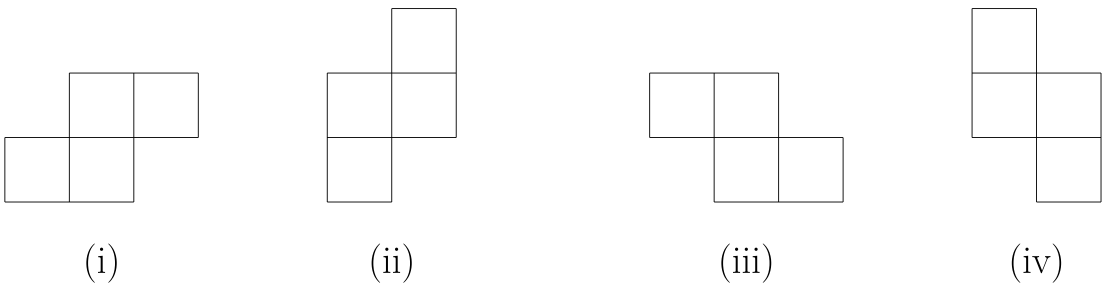


Figure 3.21

Proof. We prove the equivalent statements for $<^i_{\text{grevlex}}$, with $i \in \mathcal{O}$. The case $<^i_{\text{grevlex}}$ for $i \in \mathcal{E}$ can be done similarly.

(1) $\Rightarrow$ (2). Firstly, let E, F be two cells of $\mathcal{P}$ as in Figure 3.20 (a). Since, by hypothesis, $\mathcal{M}$ is a quadratic Gröbner basis, by Proposition 3.2.3, at least one cell between C and D must be a cell of P. That is the situation displayed in Figure 3.20 (a) is not possible. Secondly, assume, by contradiction, that the polyominoes in Figure 3.21 (i) and (ii) are subpolyominoes of $\mathcal{P}$. Then we consider the inner intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$ as in Figure 3.22.

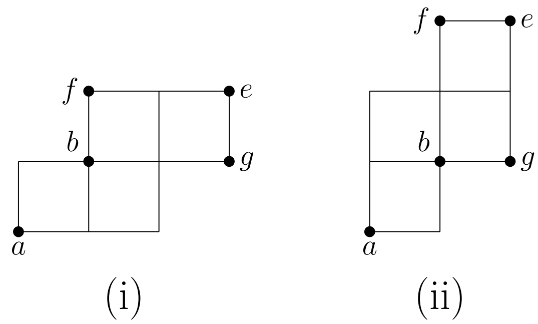


Figure 3.22

By Proposition 3.2.3, at least one between $[a, g]$ and $[a, f]$ is an inner interval of $\mathcal{P}$, where f and g are the anti-diagonal corners of $[b, e]$. In both cases, we get a polyomino that is not thin, which is a contradiction.

(2) $\Rightarrow$ (1). Assume, by contradiction, that $\mathcal{M}$ does not form a quadratic Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \mathcal{O}$. According to Proposition 2.1, there exist two inner intervals $[a, b]$ and $[b, e]$ of $\mathcal{P}$, where $[a, b]$ has anti-diagonal corners c and d, and $[b, e]$ has anti-diagonal corners f and g, such that neither $[a, f]$ nor $[a, g]$ is an inner interval of $\mathcal{P}$. Let E and F be respectively cells of $[a, b]$ and $[b, e]$ such that $E \cap F = \{b\}$. Let C and D be respectively cells of $[a, f]$ and $[a, g]$ such that $E \cap C \cap D \cap F = \{b\}$. Since $\mathcal{P}$ is thin, the cells C and D can not simultaneously be cells of $\mathcal{P}$. If neither C nor D is a cell of $\mathcal{P}$, then C, D, E, and F are cells as in Figure 3.20 (a) and this is a contradiction. Assume, without loss of generality, that $C \notin \mathcal{P}$, but $D \in \mathcal{P}$. Since $[a, g]$ is not an inner interval of $\mathcal{P}$, then d and g are not both corners of D.

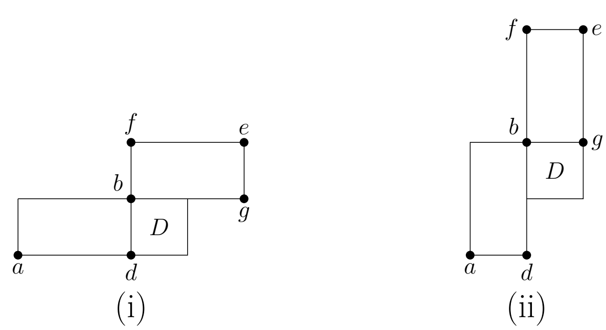


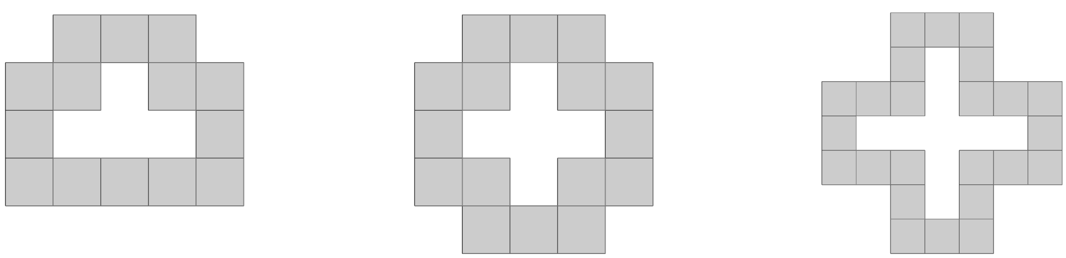
Figure 3.23

Let $\mathcal{P}'$ be the subpolyomino of $\mathcal{P}$ given by the union of the cells of $[a,b]$, $[b,e]$ and D, as in Figure 3.23. Then, one of the two subpolyominoes displayed in Figure 3.21 (i) and (ii) is a subpolyomino of $\mathcal{P}'$, and then of $\mathcal{P}$, which is a contradiction.

□

Definition 3.2.11 *Let $\mathcal{P} = \{C_1, \dots, C_n\}$ be a thin polyomino. If there exists a relabelling of the cells of $\mathcal{P}$ such that $C_1, C_2, \dots C_n$ is a path of cells, C_1 and C_n have an edge in common, and $C_i \cap C_j = \emptyset$ for all $j > i+2$, then $\mathcal{P}$ is called thin cycle.*

Note that a thin cycle is a polyomino with exactly one hole. In Figure 3.24 three thin cycles are displayed. In particular, the polyominoes in (A) and (B) have the polyominoes in Figure 3.21 (i)–(iv) as subpolyominoes. This implies that in both cases $\mathcal{M}$ is not a reduced Gröbner basis of $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$ for $i \in \{1, \dots, 8\}$. However, the polyomino in (A) is prime, whereas the polyomino in (B) is not. Surprisingly, in the next result we exhibit a class of thin cycles having a prime ideal. The polyomino in Figure 3.24 (C) belongs to such a class.



(A) (B) (C)

Figure 3.24: Examples of thin cycle polyominoes.

Corollary 3.2.12. *Let $\mathcal{P}$ be a thin cycle polyomino whose all maximal inner intervals have length at least* 3*. Then $I_{\mathcal{P}}$ is prime.*

Proof. First of all, we observe that such a $\mathcal{P}$ satisfies the condition (2) of Theorem 3.2.10. In fact, by definition of thin cycle, there are no cells C, D, E and F such that $E, F \in \mathcal{P}$ intersect in one vertex, $C, D \notin \mathcal{P}$ and $C \cap D \cap E \cap F \neq \emptyset$, as in Figure 3.20. Moreover, by hypothesis, there is no maximal inner intervals of length 2 as in Figure 3.21. By Theorem

3.2.10, $\mathcal{M}$ is a quadratic Gröebner basis for $I_{\mathcal{P}}$ with respect to $<^i_{\text{grevlex}}$, for all $i \in \{1,\dots,8\}$. By Corollary 3.2.9, the thesis follows. □

As another application of the results obtained for thin polyominoes, we consider the grid polyominoes, that we introduced in Subsection 3.1.2. They are prime and, by definition, thin. One can see, by applying Proposition 3.2.3, that grid polyominoes have quadratic Gröbner basis with respect to $<^i_{\text{grevlex}}$, for all $i \in \{1,\dots,8\}$. In the following, we will define a new infinite family of prime polyominoes, obtained by the deletion of certain cells from grid polyominoes.

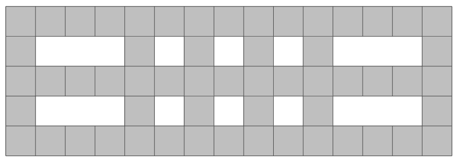
Remark 3.2.13 *We observe that a grid polyomino $\mathcal{P}$ can be regarded as the disjoint union of two collections of cells, namely $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$, where $\mathcal{P}_1 = \{C \in \mathcal{P} \mid C$ is properly contained in exactly one maximal inner interval of $\mathcal{P}\}$ and $\mathcal{P}_2 = \{C \in \mathcal{P} \mid C$ is properly contained in 2 maximal inner intervals of $\mathcal{P}\}$.*

Definition 3.2.14 *Let $\mathcal{P}$ be a grid polyomino with $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2$ and $\mathcal{P}_1$ and $\mathcal{P}_2$ as in Remark 3.2.13. Let $\mathcal{P}'_1$ be a subset of $\mathcal{P}_1$ such that $\mathcal{P}' = \mathcal{P} \setminus \mathcal{P}'_1$ is a polyomino. We call $\mathcal{P}'$ a subgrid polyomino of $\mathcal{P}$.*

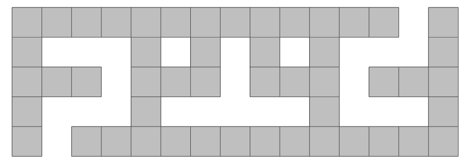
Corollary 3.2.15. *Let $\mathcal{P}'$ be a subgrid polyomino of a grid polyomino $\mathcal{P}$. Then $I_{\mathcal{P}'}$ is prime.*

Proof. First of all, we claim that $\mathcal{P}'$ satisfies the condition (2) of Theorem 3.2.10. By contradiction, assume that there exist E and F cells of $\mathcal{P}'$ as in Figure 3.20, but neither C nor D is a cell of $\mathcal{P}'$. By definition of grid polyomino, either C or D is a cell of $\mathcal{P}$. Without loss of generality, we may assume that C is a cell of $\mathcal{P}$. Then $C \in \mathcal{P}_2$, and $C \notin \mathcal{P}_1$, since C is properly contained in two maximal inner intervals: one containing the cells C and E and the other containing C and F. Then C is still a cell of $\mathcal{P}'$. Moreover, by definition of grid polyomino, the subpolyominoes displayed in Figure 3.21 are not subpolyominoes of $\mathcal{P}$. Since $\mathcal{P}' \subset \mathcal{P}$, then they are not subpolyominoes of $\mathcal{P}'$ either. By Theorem 3.2.10, $\mathcal{M}$ is a quadratic Gröebner basis for $I_{\mathcal{P}'}$ with respect to $<^i_{\text{grevlex}}$, for all $i \in \{1,\dots,8\}$. By Corollary 3.2.9, the thesis follows. □

In Figure 3.25B, it is shown a subgrid polyomino $\mathcal{P}'$ obtained from the grid polyomino $\mathcal{P}$ displayed in Figure 3.25A by removing some cells in $\mathcal{P}_1$. By Corollary 3.2.15, the ideal $I_{\mathcal{P}'}$ is prime.



(A) A grid polyomino $\mathcal{P}$.



(B) A subgrid polyomino of the grid polyomino in Figure 3.25A.

Figure 3.25: An example of grid polyomino with a related subgrid polyomino

Chapter 4

HILBERT SERIES AND GORENSTEINNESS OF POLYOMINOES

In this chapter, we study the polyomino ideal and the related coordinate ring under the point of view of the Castelnuovo-Mumford regularity, the Hilbert series and the Gorensteinnes. We recall that the above invariants and properties are strictly related, as explained in Section 1.4. In such relations, the Cohen-Macaulayness of $\mathbb{K}[\mathcal{P}]$ plays a fundamental role. Therefore, we study the above properties only for classes of simple polyominoes, in view of Lemma 2.4.3. In fact, it is an open problem to determine whether the coordinate ring $\mathbb{K}[\mathcal{P}]$ is Cohen-Macaulay in the case of non-simple polyominoes. We first discuss the results on Castelnuovo-Mumford regularity and Hilbert series, and then we discuss the Gorensteinnes at the end of this section. In Section 4.1, we focus on L-convex polyominoes, that are widely studied under the point of view of combinatorics (see Section 2.2). For such class of polyominoes we compute the Castelnuovo-Mumford regularity and we prove that it coincides with the rook number of the polyomino, namely the maximum number of non-attacking rooks that can be placed on the polyomino. The latter opens a new horizon in relating the algebraic invariants of polyominoes to the combinatorial properties related to the rook configurations. Let $\mathcal{P}$ be a polyomino and let r_k be the number of ways of arranging k non-attacking rooks on the cells of $\mathcal{P}$. The polynomial

$$r_{\mathcal{P}}(t) = \sum_{k=0}^{r(\mathcal{P})} r_k t^k$$

is called the *rook polynomial* of $\mathcal{P}$ and $r(\mathcal{P})$ is called the *rook number* of $\mathcal{P}$. The set of all rook configurations is a simplicial complex, that we call *rook complex*. In Section 4.2, for the class of *simple thin* polyominoes we prove that for the reduced Hilbert series $h(t)/(1-t)^d$ it holds $h(t) = r_{\mathcal{P}}(t)$. Moreover, this equation does not hold in the case of simple non-thin polyominoes. Nevertheless, by computation, we observe that the polynomial the coefficients of the polynomial $h(t)$ are upper-bounded by the ones of the rook polynomial. The latter leads us to conjecture that there exists another polynomial $\tilde{r}_{\mathcal{P}}(t)$, strictly related to $r_{\mathcal{P}}(t)$, such that $h(t) = \tilde{r}_{\mathcal{P}}(t)$. Such a polynomial is obtained by introducing an equivalence relation on the rook complex. In Section 4.3, we verify computationally the above conjecture

for any polyomino up to rank 11, and we prove the conjecture in the case of parallelogram polyominoes (see Section 2.3). In this discussion, the underlying relation between parallelogram polyominoes and simple planar distributive lattices plays a fundamental role.
For what concerns the Gorensteinnes, for an L-convex polyomino $\mathcal{P}$, we define a sequence of L-convex polyominoes obtained from $\mathcal{P}$ by removing maximal rectangles, called *derived sequence*. We obtain a characterization for the Gorensteinnes of L-convex polyominoes in terms the bounding boxes of the polyominoes of the derived sequence. Furthermore, for simple thin polyominoes, we obtain a characterization of the Gorensteinnes in terms of the *single* cells of $\mathcal{P}$, namely cells belonging to a unique maximal interval. Although the Gorensteinnes of simple planar distributive lattices is characterized, we find a nice combinatorial interpretation of the Gorensteinnes of the parallelogram polyominoes, as we have done for the two classes above. The references for this chapter are [27], [70] and [63].

4.1 CASTELNUOVO-MUMFORD REGULARITY AND GORENSTEINNESS OF L-CONVEX POLYOMINOES

In this section, we study the coordinate ring of L-convex polyominoes, already introduced in Section 2.2, together with its Castelnuovo-Mumford regularity, Gorensteinnes and Cohen-Macaulay type. In particular, in subsection 4.1.1 we prove that if $\mathcal{P}$ is an L-convex polyomino, then there is a natural bipartite graph $F_{\mathcal{P}}$ whose edges correspond to the cells of $\mathcal{P}$. By using this correspondence, we show in Proposition 4.1.1 that there exists a polyomino $\mathcal{P}^*$ which is a Ferrer diagram and such that the bipartite graphs $F_{\mathcal{P}}$ and $F_{\mathcal{P}^*}$ are isomorphic. We call $\mathcal{P}^*$ the Ferrer diagram projected by $\mathcal{P}$. In particular, such Ferrer diagram $\mathcal{P}^*$ has the same horizontal and vertical projections of $\mathcal{P}$ but arranged in descending order. Similarly there exists a bipartite graph $G_{\mathcal{P}}$ whose edges correspond to the coordinates of the vertices of $\mathcal{P}$. By using the intimate relationship between $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ it can be shown that $G_{\mathcal{P}}$ and $G^*_{\mathcal{P}}$ are isomorphic as well, see Corollary 4.1.5. The crucial observation which then follows from these considerations is the result (Theorem 4.1.6) that $\mathbb{K}[\mathcal{P}]$ and $\mathbb{K}[\mathcal{P}^*]$ are isomorphic as standard graded $\mathbb{K}$-algebras. Therefore all algebraic invariants and properties of $\mathbb{K}[\mathcal{P}]$ are shared by $\mathbb{K}[\mathcal{P}^*]$. For many arguments this allows us to assume that $\mathcal{P}$ itself is a Ferrer diagram. Since the coordinate ring of a Ferrer diagram can be identified with the edge ring of a Ferrer graph, results of Corso and Nagel [20] can be used to compute the Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$, denoted by reg $(\mathbb{K}[\mathcal{P}])$. It turns out that reg $(\mathbb{K}[\mathcal{P}])$ has a very nice combinatorial interpretation. Namely, for an L-convex polyomino, reg $(\mathbb{K}[\mathcal{P}])$ is equal to maximal number of non-attacking rooks that can be placed on $\mathcal{P}$, as shown in Theorem 4.1.8. This is the main result of Subsection 4.1.2.

In Subsection 4.1.3, we study the Gorenstein property of L-convex polyominoes. We first observe that if we remove the rectangle of maximal width from $\mathcal{P}$, then the result is again an L-convex polyomino. Repeating this process we obtain a finite sequence of L-convex polyominoes, which we call the derived sequence of $\mathcal{P}$. In Theorem 4.1.11 we then shown that $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if the bounding boxes of the derived sequence of L-convex polyominoes of $\mathcal{P}$ are all squares. For the proof we use again that $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[\mathcal{P}^*]$, and the characterization of Gorenstein stack polyominoes given in Theorem 2.4.4. In addition, under the assumption $\mathbb{K}[\mathcal{P}]$ is not Gorenstein, we show in Theorem 4.1.11 that $\mathbb{K}[\mathcal{P}]$ is Gorenstein on the punctured spectrum if and only if $\mathcal{P}$ is a rectangle, but not a square. Here we use that the coordinate ring of a a Ferrer diagram may be viewed as a Hibi ring. Then we can apply a recent result of Herzog et al [39] which characterizes the Hibi rings which are Gorenstein on the punctured spectrum.

Finally, in Subsection 4.1.4 we compute the Cohen–Macaulay type of $\mathbb{K}[\mathcal{P}]$ for an L-convex polyomino $\mathcal{P}$. Again we use the fact that $\mathbb{K}[\mathcal{P}^*]$ may be viewed as a Hibi ring (of a suitable poset Q). The number of generators of the canonical module of $\mathbb{K}[\mathcal{P}^*]$, which by definiton is the Cohen–Macaulay type, is described by Miyazaki [58] (based on results of Stanley [78] and Hibi [42]). It is the number of minimal strictly order reversing maps on Q. Then somewhat technical counting arguments provide us in Theorem 4.1.17 with the desired formula.

4.1.1 L-convex polyominoes and Ferrer diagrams

A *Ferrer graph* G is a bipartite graph with $V(G) = \{u_1, \dots, u_m\} \sqcup \{v_1, \dots, v_n\}$ such that $\{u_1, v_n\}, \{u_m, v_1\} \in E(G)$ and if $\{u_i, v_j\} \in E(G)$ then $\{u_r, v_s\} \in E(G)$ for all $1 \leq r \leq i$ and for all $1 \leq s \leq j$. Let G be a Ferrer graph and $\mathcal{P}$ be a polyomino such that $H_{\mathcal{P}} = (\deg v_1, \dots, \deg v_n)$, $V_{\mathcal{P}} = (\deg u_1, \dots, \deg u_m)$ and $F_{\mathcal{P}} = G$. Then $\mathcal{P}$ is a Ferrer diagram (see Section 2.2). Note that if $[(0,0),(m,n)]$ is the bounding box of a Ferrer diagram $\mathcal{P}$, then $(0,0),(m,n) \in V(\mathcal{P})$.

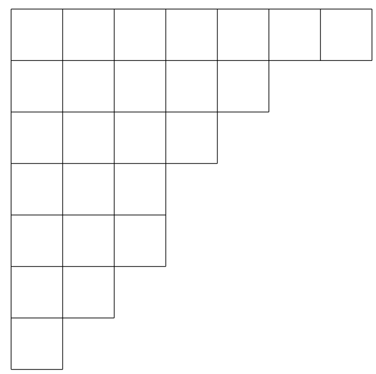


Figure 4.1: Ferrer diagram

Proposition 4.1.1. *Let $\mathcal{P}$ be an L-convex polyomino. Then there exists a Ferrer diagram $\mathcal{P}^*$ such that $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$.*

Proof. Let $F_{\mathcal{P}}$ be the bipartite graph associated to $\mathcal{P}$, with vertex set $V(F_{\mathcal{P}}) = \{X_1,\ldots,X_m\} \sqcup \{Y_1,\ldots,Y_n\}$. We first prove that after a suitable relabelling of vertices of $F_{\mathcal{P}}$, it can be viewed as a Ferrer graph. Let $T_1,T_2,\ldots,T_m$ and $U_1,U_2,\ldots,U_n$ be the relabelling of the vertices of $F_{\mathcal{P}}$ such that $\deg T_1 \geq \deg T_2 \geq \cdots \geq \deg T_m$ and $\deg U_1 \geq \deg U_2 \geq \cdots \geq \deg U_n$. We set $v_i^* = \deg T_i$ for $1 \leq i \leq m$ and $h_j^* = \deg U_j$ for $1 \leq j \leq n$.

Then $v_1^* = n$ and $h_1^* = m$ which implies that $\{T_1,U_n\},\{T_m,U_1\} \in E(F_{\mathcal{P}})$. Furthermore, let $\{T_k,U_l\} \in E(F_{\mathcal{P}})$ for some $1 \leq k \leq m$ and $1 \leq l \leq n$. Then for all $1 \leq r \leq k$ and $1 \leq s \leq l$, we have $v_k^* \leq v_r^*$ and $h_l^* \leq h_s^*$. Therefore, by Theorem 2.2.5.(c), we see that $\{T_r,U_s\} \in E(F_{\mathcal{P}})$ for all $1 \leq r \leq k$ and $1 \leq s \leq l$.

Hence $F_{\mathcal{P}}$ is a Ferrer graph up to relabelling. Let $\mathcal{P}^*$ be the unique polyomino with horizontal and vertical projections $H_{\mathcal{P}^*} = (h_1^*,h_2^*,\ldots,h_n^*)$ and $V_{\mathcal{P}^*} = (v_1^*,v_2^*,\ldots,v_m^*)$, then $\mathcal{P}^*$ is a Ferrer diagram and $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$. $\square$

From the proof of the above proposition, one sees that given an L-convex polyomino $\mathcal{P}$, the Ferrer diagram $\mathcal{P}^*$ such that $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$ is uniquely determined. We refer to $\mathcal{P}^*$ as the Ferrer diagram projected by $\mathcal{P}$.

(A) L-convex polyomino $\mathcal{P}$ (B) The Ferrer diagram $\mathcal{P}^*$ projected by $\mathcal{P}$

Let $r(\mathcal{P}, k)$ be the number of ways of arranging k non-attacking rooks in cells of $\mathcal{P}$. Recall that, for a graph G with n vertices, a k-matching of G is the set of k pairwise disjoint edges in G. Let $p(G, k)$ be the number of k matchings of G. It is a fact, for example see [31, page 56], that $r(\mathcal{P}, k) = p(F_{\mathcal{P}}, k)$. As we showed at the beginning of the chapter, $r(\mathcal{P})$ denotes the maximum number of rooks that can be arranged in $\mathcal{P}$ in non-attacking position, that is $r(\mathcal{P}) = \max_k r(\mathcal{P}, k)$. We have the following

Lemma 4.1.2. *Let $\mathcal{P}$ be an L-convex polyomino and P^* be the Ferrer diagram projected by $\mathcal{P}$. Then $r(\mathcal{P}, k) = r(\mathcal{P}^*, k)$. In particular, $r(\mathcal{P}) = r(\mathcal{P}^*)$.*

Proof. From Proposition 4.1.1, we have $F_{\mathcal{P}} \cong F_{\mathcal{P}^*}$ then $p(F_{\mathcal{P}}, k) = p(F_{\mathcal{P}^*}, k)$. Then by using the theorem on [31, page 56], we see that $r(\mathcal{P}, k) = r(\mathcal{P}^*, k)$. □

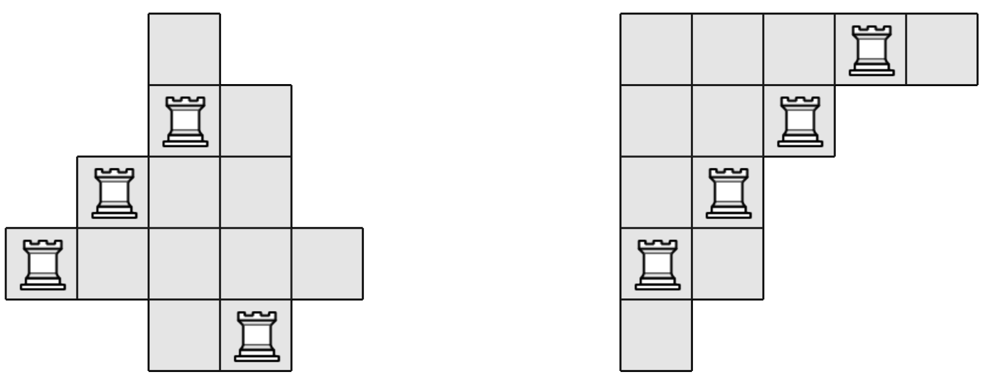


Figure 4.3: Placement of rooks in non-attacking position in $\mathcal{P}$ and $\mathcal{P}^*$.

As described in Section 2.4, we can associate another bipartite graph $G_{\mathcal{P}}$ to $\mathcal{P}$. Set $V(G_{\mathcal{P}}) = \{x_0, \ldots, x_m\} \sqcup \{y_0, \ldots, y_n\}$. To distinguish between $G_{\mathcal{P}}$ and $F_{\mathcal{P}}$, we refer to them as follows:

- The graph $F_{\mathcal{P}}$ is the graph associated to the cells of $\mathcal{P}$.
- The graph $G_{\mathcal{P}}$ is the graph associated to the vertices of $\mathcal{P}$.

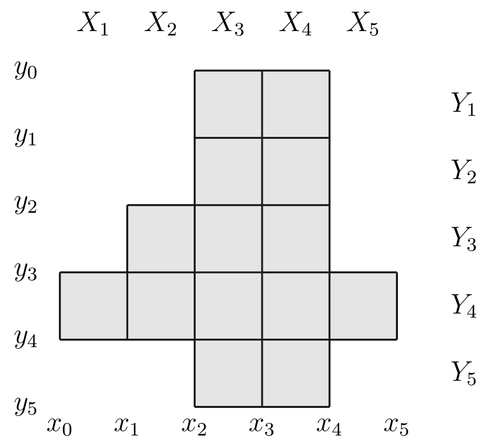
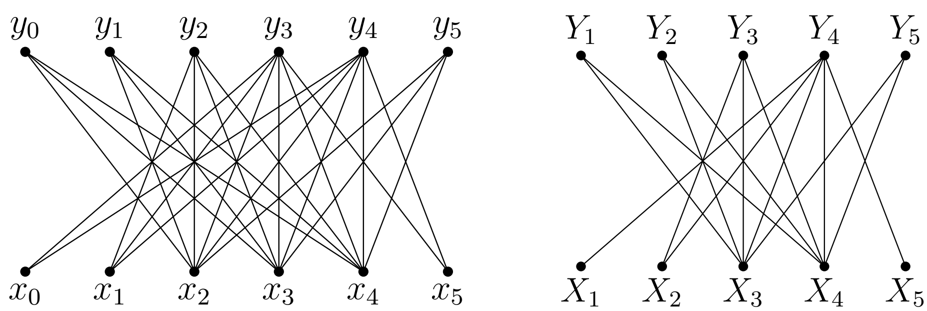


Figure 4.4: The two labellings on $\mathcal{P}$ of Figure 2.12



(A) The bipartite graph $G_{\mathcal{P}}$ of the polyomino $\mathcal{P}$ in Figure 2.12.

(B) The bipartite graph $F_{\mathcal{P}}$ of the polyomino $\mathcal{P}$ in Figure 2.12.

Figure 4.5

The relation between $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ is deducible from the following

Remark 4.1.3 *Let $\mathcal{P}$ be an L-convex polyomino with bounding box $[(0,0),(m,n)]$. Then one can interpret $H_{\mathcal{P}} = (h_1, h_2, \ldots, h_n)$ and $V_{\mathcal{P}} = (v_1, v_2, \ldots, v_m)$ in terms of degrees of vertices of $G_{\mathcal{P}}$ in the following way:*

(i) From Theorem 2.2.5, we know that $V_{\mathcal{P}}$ and $H_{\mathcal{P}}$ are unimodal. Let

$$v_1 \leq v_2 \leq \cdots < v_i = n \geq v_{i+1} \geq \cdots \geq v_m$$

for some $1 \leq i \leq m$. Then $v_j = \deg x_{j-1} - 1$ for all $1 \leq j < i$ and $v_j = \deg x_j - 1$ for

$i \leq j \leq m$. Similarly, by using unimodality of $H_{\mathcal{P}}$, we get

$$h_1 \leq h_2 \leq \cdots < h_i = m \geq h_{i+1} \geq \cdots \geq h_n$$

for some $1 \leq i \leq m$. Then $h_j = \deg y_{j-1} - 1$ for all $1 \leq j < i$ and $h_j = \deg y_j - 1$ for $i \leq j \leq n$.

(ii) As a consequence of (i), if $\mathcal{P}$ is a Ferrer diagram then

$$v_1 \geq v_2 \geq \cdots \geq v_m,$$

$$h_1 \geq h_2 \geq \cdots \geq h_n.$$

Let $G_{\mathcal{P}}$ and $F_{\mathcal{P}}$ be the graphs associated to $\mathcal{P}$ as described above with $V(G_{\mathcal{P}}) = \{x_0, \ldots, x_m\} \sqcup \{y_0, \ldots, y_n\}$ and $V(F_{\mathcal{P}}) = \{X_1, \ldots, X_m\} \sqcup \{Y_1, \ldots, Y_n\}$. Then $v_j = \deg X_j = \deg x_j - 1$ for all $1 \leq j \leq m$, and $h_j = \deg Y_j = \deg y_j - 1$ for all $1 \leq j \leq n$.

Now we obtain

Lemma 4.1.4. *Let $\mathcal{P}$ be an L-convex polyomino and $G_{\mathcal{P}}$ be the graph associated to the vertices of $\mathcal{P}$ with $V(G_{\mathcal{P}}) = \{x_0, \ldots, x_m\} \sqcup \{y_0, \ldots, y_n\}$. Then we have the following:*

(a) *if $\deg x_i < \deg x_{i'}$, then $\{x_{i'}, y_j\} \in E(G_{\mathcal{P}})$ whenever $\{x_i, y_j\} \in E(G_{\mathcal{P}})$.*

(b) *if $\deg y_j < \deg y_{j'}$, then $\{x_i, y_{j'}\} \in E(G_{\mathcal{P}})$ whenever $\{x_i, y_j\} \in E(G_{\mathcal{P}})$.*

Proof. Let $H_{\mathcal{P}} = (h_1, h_2, \ldots, h_n)$ and $V_{\mathcal{P}} = (v_1, v_2, \ldots, v_m)$ be the horizontal and vertical projection of $\mathcal{P}$.

(a): Let $p = \deg x_i < \deg x_{i'} = q$. Then following Remark 4.1.3, $h_s = p - 1$ and $h_t = q - 1$ for some $1 \leq s \neq t \leq n$. Then $h_s < h_t$ and the conclusion follows from Theorem 2.2.5(d).

(b): Let $p = \deg y_i < \deg y_{i'} = q$. Then following Remark 4.1.3, $v_s = p - 1$ and $v_t = q - 1$ for some $1 \leq s \neq t \leq m$. Then $v_s < v_t$ and the the conclusion follows from Theorem 2.2.5(c). $\square$

A result similar to Proposition 4.1.1 holds also for the graph $G_{\mathcal{P}}$.

Corollary 4.1.5. *Let $\mathcal{P}$ be an L-convex polyomino, let $\mathcal{P}^*$ be the Ferrer diagram projected by $\mathcal{P}$. Then $G_{\mathcal{P}} \cong G_{\mathcal{P}^*}$.*

Proof. First, we will show that $G_{\mathcal{P}} \cong H$ where H is a Ferrer graph. Let $V(G_{\mathcal{P}}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$. Similar to the proof of Proposition 4.1.1, we relabel the vertices of $G_{\mathcal{P}}$ as $V(G_{\mathcal{P}}) = \{t_0, \dots, t_m\} \sqcup \{s_0, \dots, s_n\}$ such that $\deg t_0 \geq \deg t_1 \geq \dots \geq \deg t_m$ and $\deg s_0 \geq \deg s_1 \geq \dots \geq \deg s_n$. Let H be the new graph obtained by relabelling of the vertices of $G_{\mathcal{P}}$. Then by using Lemma 4.1.4, we conclude that H is a Ferrer graph.

Now we will show that $H \cong G_{\mathcal{P}^*}$. This is an immediate consequence of Remark 4.1.3(ii). Indeed $V_{\mathcal{P}^*} = (\deg t_1 - 1, \dots, \deg t_m - 1)$ and
$H_{\mathcal{P}^*} = (\deg s_1 - 1, \dots, \deg s_n - 1)$. □

4.1.2 Regularity of L-convex polyominoes

In this subsection, we use the relation between any L-convex polyomino $\mathcal{P}$ and its projected Ferrer diagram to compute the Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$.

Theorem 4.1.6. *Let $\mathcal{P}$ be an L-convex polyomino and let $\mathcal{P}^*$ be the Ferrer diagram projected by $\mathcal{P}$. Then $\mathbb{K}[\mathcal{P}]$ and $\mathbb{K}[\mathcal{P}^*]$ are isomorphic standard graded $\mathbb{K}$-algebras.*

Proof. Since $\mathcal{P}$ is convex, it is known that $\mathbb{K}[\mathcal{P}]$ is isomorphic to the edge ring $\mathbb{K}[G_{\mathcal{P}}]$ of the bipartite graph $G_{\mathcal{P}}$ (see Section 2.4). By Corollary 4.1.5, $G_{\mathcal{P}}$ is isomorphic to $G_{\mathcal{P}^*}$. Hence the assertion follows. □

Theorem 4.1.7. *Let $\mathcal{P}$ be an L-convex polyomino and let $\mathcal{P}^*$ be the Ferrer diagram projected by $\mathcal{P}$. Moreover, let $H_{\mathcal{P}^*} = (h_1, \dots, h_n)$. Then*

$$\text{reg } (\mathbb{K}[\mathcal{P}]) = \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}.$$

Proof. By Theorem 4.1.6, we have $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[\mathcal{P}^*]$. Therefore, it is enough to show that

$$\text{reg } (\mathbb{K}[\mathcal{P}^*]) = \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}.$$

Recall $G_{\mathcal{P}^*}$ is the bipartite graph associated to the vertices of $\mathcal{P}^*$. We may assume that $V(G^*_{\mathcal{P}}) = \{x_0, \dots, x_m\} \sqcup \{y_0, \dots, y_n\}$. Then $\deg y_0 = m+1 \geq 2$ and $\deg x_0 = n+1$. Hence, [20, Proposition 5.7] gives

$$\begin{aligned} \text{reg } (\mathbb{K}[G_{\mathcal{P}^*}]) &= \min\{n, \deg y_j + (j+1) - 3 \mid 1 \leq j \leq n\} \\ &= \min\{n, \deg y_j + j - 2 \mid 1 \leq j \leq n\} \end{aligned}$$

We want to rewrite the formula above in terms of the horizontal projection of $\mathcal{P}^*$. According to Remark 4.1.3.(2), for any $1 \leq j \leq n$ we have $h_j = \deg y_j - 1$. Hence

$$\{\deg y_j + j - 2 \mid 1 \leq j \leq n\} = \{h_j + j - 1 \mid 1 \leq j \leq n\},$$

and the assertion follows. □

Let $\mathcal{P}$ be a Ferrer diagram with horizontal projections $(h_1, \ldots, h_n)$. Then, by using a combinatorial argument, it is easy to see that for any $r \leq n$ the number of ways of placing r rooks in non-attacking position in $\mathcal{P}$ is given by

$$\prod_{i=1}^{r} (h_{r-i+1} - (i-1)). \tag{4.1}$$

By using this fact we obtain

Theorem 4.1.8. *Let $\mathcal{P}$ be an L-convex polyomino. Then*

$$\text{reg } (\mathbb{K}[\mathcal{P}]) = r(\mathcal{P}).$$

Proof. From Lemma 4.1.2 we know that $r(\mathcal{P}) = r(\mathcal{P}^*)$ where $\mathcal{P}^*$ is the Ferrer diagram projected by $\mathcal{P}$. By Theorem 4.1.6, it is enough to show that

$$r(\mathcal{P}^*) = \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}, \tag{4.2}$$

where $(h_1, \ldots, h_n)$ are the horizontal projections of $\mathcal{P}^*$. It follows from Equation (4.1) that $r(\mathcal{P}^*)$ is the greatest integer $r \leq n$ such that each factor of $\prod_{i=1}^{r} (h_{r-i+1} - (i-1))$ is positive. Hence, for any $i \in \{1, \ldots, r\}$ we must have $h_{r-i+1} - (i-1) > 0$. Fix an integer $i \in \{1, \ldots, r\}$. Then we see that

$$h_{r-i+1} - (i-1) > 0 \Leftrightarrow h_{r-i+1} - i + 1 + r - r > 0 \Leftrightarrow r < h_{r-i+1} + (r-i) + 1.$$

Hence we conclude that $r \leq h_{r-i+1} + (r-i)$. Therefore,

$$r(\mathcal{P}^*) = \max\{r \mid r \leq n \text{ and } r \leq \min\{h_{r-i+1} + (r-i) \mid 1 \leq i \leq r\}\}$$

$$= \min\{n, h_j + j - 1 \mid 1 \leq j \leq n\}$$

as requested. □

We observe that, by exchanging the role of rows and columns in $\mathcal{P}^*$, we obtain

$$r(\mathcal{P}^*) = \min\{m, v_j + j - 1 \mid 1 \leq j \leq m\}$$

which is similar to (4.2).

4.1.3 On the Gorenstein property of L-convex polyominoes

In this section, we characterize the L-convex polyominoes having a Gorenstein coordinate ring in terms of their bounding boxes. Such characterization is similar to the one stated in Theorem 2.4.4 for stack polyominoes.

Let $\mathcal{P}$ be a L-convex polyomino with width m. Assume that the unique maximal rectangle of $\mathcal{P}$ with width m, has height d. Then for some positive integer s,

$$H_{\mathcal{P}} = (h_1, \ldots, h_s, m \ldots, m, h_{s+d+1}, \ldots, h_n)$$

with $h_s, h_{s+d+1} < m$. Let $\mathcal{P}_1$ be the collection of cells with $n - d$ rows satisfying the following property: C_{ij} is a cell of $\mathcal{P}$ if and only if C_{ij} is a cell of $\mathcal{P}_1$ for $1 \leq i \leq s$, and for $s + d + 1 \leq i \leq n$, $C_{i-d,j}$ is a cell of $\mathcal{P}_1$.

Lemma 4.1.9. *$\mathcal{P}_1$ is an L-convex polyomino.*

Proof. $\mathcal{P}_1$ could be seen as the polyomino $\mathcal{P}$ from which we remove the maximal rectangle R having width m. Hence, each cell in $\mathcal{P}_1$ corresponds uniquely to a cell in $\mathcal{P}$. Let $C, D \in \mathcal{P}_1$. Then we consider the corresponding cells $C', D' \in \mathcal{P}$. We observe that neither C' nor D' is a cell of R. Since $\mathcal{P}$ is L-convex, there exists a path of cells $\mathcal{C}'$ in $\mathcal{P}$ connecting C' and D' with at most one change of direction.
If no cell of $\mathcal{C}'$ belongs to R, then $\mathcal{C}'$ determines a path of cells $\mathcal{C}$ of $\mathcal{P}_1$ with at most one change of direction connecting C and D.
Otherwise, since neither C' nor D' are cells of R, the path $\mathcal{C}'$ crosses R and the induced path $\mathcal{C}' \cap R$ has no change of direction. Therefore, the path $\mathcal{C}$ in $\mathcal{P}_1$, obtained by cutting off the induced path $\mathcal{C}' \cap R$ from $\mathcal{C}'$, is a path of cells with at most one change of direction connecting C and D. □

If $\mathcal{P}_1 \neq \emptyset$, we may again remove the unique rectangle of maximal width from $\mathcal{P}_1$ to obtain $\mathcal{P}_2$ in a similar way. After a finite number of such steps, say t steps, we arrive at $\mathcal{P}_t$ which is a rectangle. Then $\mathcal{P}_{t+1} = \emptyset$. We set $\mathcal{P}_0 = \mathcal{P}$, and call the sequence $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_t$ the *derived sequence* of L-convex polyominoes of $\mathcal{P}$.

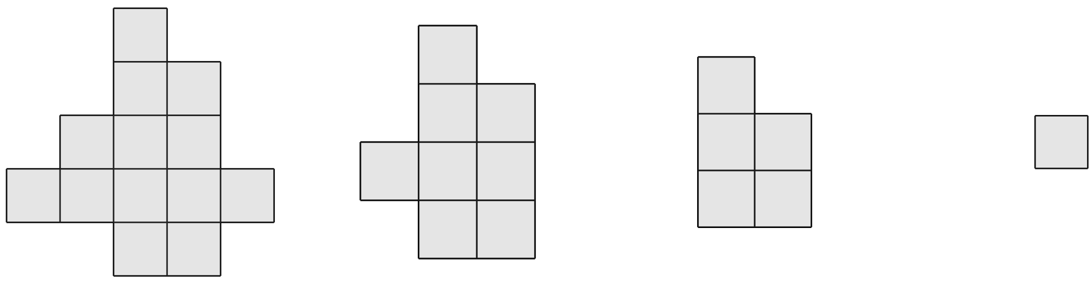


Figure 4.6: The derived sequence of L-convex polyominoes $\mathcal{P}_0 = \mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

Lemma 4.1.10. *Let $\mathcal{P}$ be an L-convex polyomino and $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_t$ be the derived sequence of L-convex polyominoes of $\mathcal{P}$. Let $\mathcal{P}^*$ be the Ferrer diagram projected by $\mathcal{P}$ and let $(\mathcal{P}^*)_0$, $(\mathcal{P}^*)_1$, $\dots, (\mathcal{P}^*)_{t'}$ be the derived sequence of L-convex polyominoes of $\mathcal{P}^*$. Then $t' = t$ and for any $0 \leq k \leq t$ the polyomino $(\mathcal{P}^*)_k$ is the Ferrer diagram projected by $\mathcal{P}_k$. In other words, for all k $(\mathcal{P}^*)_k = (\mathcal{P}_k)^*$.*

Proof. For $k = 0$, the assertion is trivial. We show that $(\mathcal{P}^*)_1$ is the Ferrer diagram projected by $\mathcal{P}_1$. For this aim, assume that the unique rectangular subpolyomino of $\mathcal{P}$ having width m has height $d \in \mathbb{N}$. Let

$$H_{\mathcal{P}} = (h_1, \dots, h_s, m \dots, m, h_{s+d+1}, \dots, h_n)$$

with $h_s, h_{s+d+1} < m$ and let

$$V_{\mathcal{P}} = (d, d, \dots, d, v_{r+1}, \dots, v_{r+l}, d, \dots, d)$$

with $v_{r+1}, v_{r+l} > d$.

From Proposition 4.1.1 it follows that $\mathcal{P}^*$ has a maximal rectangle R^* of width m and height d and

$$H_{\mathcal{P}^*} = (m \dots, m, h_1^*, \dots, h_{n-d}^*)$$

with $m > h_1^* \geq \dots \geq h_{n-d}^*$ and

$$V_{\mathcal{P}^*} = (v_1^*, \dots, v_l^*, d, \dots, d).$$

with $v_1^* \geq \cdots \geq v_l^* > d$. Hence the L-convex polyomino $(\mathcal{P}^*)_1$ is uniquely determined by the projections

$$H_{(\mathcal{P}^*)_1} = (h_1^*, \ldots, h_{n-d}^*) \text{ and } V_{(\mathcal{P}^*)_1} = (v_1^* - d, \ldots, v_l^* - d).$$

On the other hand, $\mathcal{P}_1$ is the L-convex polyomino uniquely determined by the projections $H_{\mathcal{P}_1} = (h_1, \ldots, h_s, h_{s+d+1}, \ldots, h_n)$ and, $V_{\mathcal{P}_1} = (v_{r+1}-d, v_{r+2}-d, \ldots, v_{r+l}-d)$. By reordering the two vectors in a decreasing order, we obtain the Ferrer diagram projected by $\mathcal{P}_1$ which coincides with $(\mathcal{P}^*)_1$. This proves the assertion for $k = 1$. By inductively applying the above argument, the assertion follows for all k. □

Theorem 4.1.11. *Let $\mathcal{P}$ be an L-convex polyomino and let $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_t$ be the derived sequence of L-convex polyominoes of $\mathcal{P}$. Then following conditions are equivalent:*

(a) *$\mathcal{P}$ is Gorenstein.*

(b) *For $0 \leq k \leq t$, the bounding box of $\mathcal{P}_k$ is a square.*

Proof. By Theorem 4.1.6, we have $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[\mathcal{P}^*]$, where $\mathcal{P}^*$ is the Ferrer diagram projected by $\mathcal{P}$. Therefore, $\mathbb{K}[\mathcal{P}]$ is Gorenstein if and only if $\mathbb{K}[\mathcal{P}^*]$ is Gorenstein. Note that $\mathcal{P}^*$ can be viewed as a stack polyomino. Hence it follows from Theorem 2.4.4 that $\mathbb{K}[\mathcal{P}^*]$ is Gorenstein if and only if the bounding box of $(\mathcal{P}^*)_k$ is a square for all $0 \leq k \leq t$. By Lemma 4.1.10, this is the case if and only if the bounding box of $\mathcal{P}_k$ is a square for all $0 \leq k \leq t$. □

The following numerical criteria for the Gorensteinness of $\mathcal{P}$ are an immediate consequence of Theorem 4.1.11.

Corollary 4.1.12. *Let $\mathcal{P}$ be an L-convex polyomino with the vector $H_{\mathcal{P}} = (h_1, h_2, \ldots, h_n)$ of horizontal projections of $\mathcal{P}$ and the vector $V_{\mathcal{P}} = (v_1, v_2, \ldots, v_m)$ of vertical projections of $\mathcal{P}$. We set*

$$\{h_1, h_2, \ldots, h_n\} = \{g_1 < g_2 < \cdots < g_r\} \text{ and } \{v_1, v_2, \ldots, v_m\} = \{w_1 < w_2 < \cdots < w_s\},$$

and let

$$a_i = |\{h_j \;\; h_j = g_i\}| \text{ for } i = 1, \ldots, r, \text{ and } b_i = |\{v_j \;\; v_j = w_i\}| \text{ for } i = 1, \ldots, s.$$

Then the following conditions are equivalent:

(a) $\mathcal{P}$ *is Gorenstein.*

(b) $g_\ell = \sum_{i=1}^{\ell} a_i$ *for* $\ell = 1, \dots, r$.

(c) $w_\ell = \sum_{i=1}^{\ell} b_i$ *for* $\ell = 1, \dots, s$.

Theorem 4.1.13. *Let* $\mathcal{P}$ *be L-convex polyominoes such that* $\mathbb{K}[\mathcal{P}]$ *is not Gorenstein. Then following are equivalent:*

(a) $\mathbb{K}[\mathcal{P}]$ *is Gorenstein on the punctured spectrum.*

(b) $\mathcal{P}$ *is not a square and* $\mathbb{K}[\mathcal{P}]$ *has an isolated singularity*

(c) $\mathcal{P}$ *is rectangle, but not a square.*

Before we start the proof of the theorem, we note that if $\mathcal{P}$ is a Ferrer diagram, then $\mathbb{K}[\mathcal{P}]$ can be viewed as a Hibi ring. Even if Hibi rings are discussed in Section 1.5, we recall their construction in the following. Let $Q = \{v_1, \dots, v_n\}$ be a finite poset and let $\mathbb{K}$ be a field. The Hibi ring over the field $\mathbb{K}$ associated to Q, which we denote by $\mathbb{K}[Q] \subset \mathbb{K}[y, x_1, \dots, x_n]$, is defined as follows. The $\mathbb{K}$-algebra $\mathbb{K}[Q]$ is generated by the monomials $yx_I := y\prod_{v_i \in I} x_i$ for every $I \in \mathcal{I}(Q)$, that is

$$\mathbb{K}[Q] := \mathbb{K}[yx_I | I \in \mathcal{I}(Q)].$$

The algebra $\mathbb{K}[Q]$ is standard graded if we set $\deg(yx_I) = 1$ for all $I \in \mathcal{I}(Q)$. Here $\mathcal{I}(Q)$ is the set of poset ideals of Q. The poset ideals of Q are just the subset $I \subset Q$ with the property that if $p \in Q$ and $q \leq p$, then $q \in Q$.

Let $\mathcal{P}$ be a Ferrer diagram with maximal horizontal edge intervals $\{H_0, \dots, H_n\}$, numbered increasingly from the bottom to the top, and maximal vertical edge intervals $\{V_0, \dots, V_m\}$, numbered increasingly from the left to the right. We let Q be the poset on the set $\{H_1, \dots, H_n, V_1, \dots, V_m\}$ consisting of two chains $H_1 < \dots < H_n$ and $V_1 < \dots < V_m$ and the covering relations $H_i < V_j$, if H_i intersects V_j in a way such that there is no $0 \leq i' < i$ for which $H_{i'}$ intersects V_j, and j is the smallest integer with this property.

Lemma 4.1.14. *The standard graded* $\mathbb{K}$*-algebras* $\mathbb{K}[Q]$ *and* $\mathbb{K}[\mathcal{P}]$ *are isomorphic.*

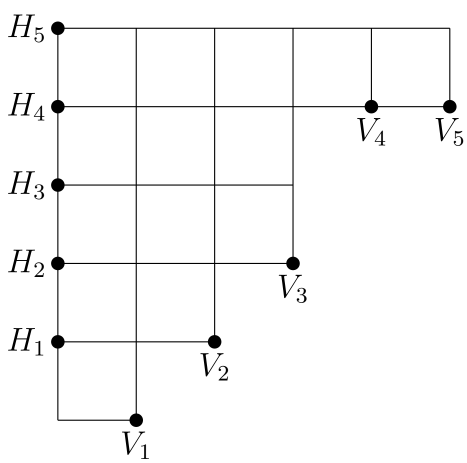
Proof. We may assume that the interval $[(0,0),(m,n)]$ is the bounding box of the Ferrer diagram $\mathcal{P}$. It follows from the definition of Ferrer diagrams that $(0,0)$ and (m,n) belong to $V(\mathcal{P})$. For any two vertices $a = (i, j)$ and $b = (k, l)$ of $\mathcal{P}$ we define the meet $a \wedge b = (\min\{i, k\}, \min\{j, l\})$ and the join $a \vee b = (\max\{i, k\}, \max\{j, l\})$. With this operations of meet and join, $\mathcal{P}$ is a distributive lattice. An element c of this lattice is called join-irreducible, if $c \neq (0, 0)$ and whenever $a \wedge b = c$, then $a = c$ or $b = c$. By Birkhoff's fundamental structure theorem [6], any finite distributive lattice is the ideal lattice of the poset of its join irreducible elements. The join irreducible elements of $\mathcal{P}$ can be described as follows:

1. Every $a_j = (0, j)$ with $1 \leq j \leq n$ is a join irreducible element in $\mathcal{P}$ and $a_1 < a_2 < \ldots < a_n$.
2. Let $(i, k) \in V(\mathcal{P})$ with $1 \leq i \leq m$. Then (i, k) is a join irreducible if $(i, k-1) \notin V(\mathcal{P})$. It shows that in each vertical edge interval $V_1, \ldots, V_m$ of $\mathcal{P}$, there is exactly one join irreducible element. We denote by b_i, the unique join irreducible element in V_i with $1 \leq i \leq m$. Then $b_1 < b_2 < \ldots < b_m$.

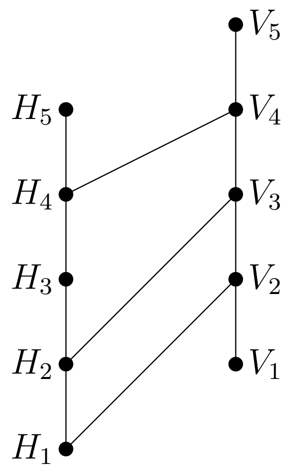
In the poset of join irreducible elements of $\mathcal{P}$, we have two chains $a_1 < a_2 < \ldots < a_n$ and $b_1 < b_2 < \ldots < b_m$, and the covering relations $a_j = (0, j) < b_i = (i, k)$ if $j = k$ and b_i is the minimal element with this property. Then, it follows that the poset of join irreducible elements of $\mathcal{P}$ is exactly the poset Q described above. Thus the elements $a \in \mathcal{P}$ are in bijection with the poset ideals of Q. In fact, the poset ideal $I_a \in \mathcal{I}(Q)$ corresponding to a is the set of join irreducible elements $q \in Q$ with $q \leq a$. Thus we have a surjective $\mathbb{K}$-algebra homomorphism

$$\varphi\, S = \mathbb{K}[x_a \;\; a \in \mathcal{P}] \to \mathbb{K}[Q] = \mathbb{K}[yx_{I_a} \;\; I_a \in \mathcal{I}(Q)]$$

. As shown by Hibi [42] (see also [36, Theorem 10.1.3]), $\mathrm{Ker}(\varphi)$ is generated by the relations $x_a x_b - x_{a\wedge b}x_{a\vee b}$. This shows that $\mathrm{Ker}(\varphi) = I_{\mathcal{P}}$, as desired. $\square$



(A) Ferrer diagram



(B) Poset of join-irreducible elements

Let Q be a poset. The Hasse diagram of Q, viewed as a graph, decomposes into connected components. The corresponding posets $Q_1, \dots, Q_r$ are called the *connected components* of Q.

Now for the proof of Theorem 4.1.13 will use the following results

Theorem 4.1.15. *Let Q be a finite poset and let $Q_1, \dots Q_r$ be the connected components of Q.*

(a) *([42, page 105])* $\mathbb{K}[Q]$ *is Gorenstein if and only if Q is pure (i.e. all maximal chains in Q have the same length).*

(b) *([39, Corollary 3.5])* $\mathbb{K}[Q]$ *is Gorenstein on the punctured spectrum if and only if each Q_i is pure.*

Proof of Theorem 4.1.13. Since $\mathbb{K}[\mathcal{P}] \cong \mathbb{K}[\mathcal{P}^*]$ and since $\mathcal{P}$ is a rectangle if and only if $\mathcal{P}^*$ is a rectangle, we may assume that $\mathcal{P}$ is a Ferrer diagram.

Let Q be the poset such that $\mathbb{K}[Q] \cong \mathbb{K}[\mathcal{P}]$, and assume that $\mathbb{K}[Q]$ is Gorenstein on the punctured spectrum. Then each component of Q is pure, by Theorem 4.1.15(b). Since we assume that $\mathbb{K}[Q]$ is not Gorenstein, Theorem 4.1.15(a) implies that Q is not connected. It follows from the description of Q in terms of its Ferrer diagram $\mathcal{P}$ that $\mathcal{P}$ has no inner corner. In other words, $\mathcal{P}$ is a rectangle. By Theorem 4.1.11 it cannot be a square. This yields $(a) \implies (b)$. The implication $(c) \implies (b)$ follows from [13, Theorem 2.6], and $(b) \implies (a)$ is trivial. □

4.1.4 The Cohen–Macaulay type of L-convex polyominoes.

In this subsection, we give a general formula for the Cohen–Macaulay type of the coordinate ring of an L-convex polyomino. To illustrate our result, we first consider the special case of an L-convex polyomino with just two maximal rectangles.

Proposition 4.1.16. *Let $\mathcal{P}$ be an L-convex polyomino whose maximal rectangles are R_1 having size $m \times s$ and R_2 having size $t \times n$ with $s < n$ and $t < m$. Let $r = \max\{n, m, n + m - (s+t)\}$. Then*

$$\operatorname{type}(\mathbb{K}[\mathcal{P}]) = \begin{cases} \sum_{i=m-t}^{m-(n-s)} \binom{i}{s}\binom{m-i-1}{n-s-1} & \text{if } r = m \\ \sum_{i=m-t}^{s} \binom{i-1}{m-t-1}\binom{n-i}{t} & \text{if } r = n \\ \binom{n-s}{t}\binom{m-t}{s} & \text{if } r = n + m - (s+t) \end{cases}$$

Proof. Let $\mathcal{P}^*$ be the Ferrer diagram projected by $\mathcal{P}$ and let Q be the poset of the join-irreducible elements associated to $\mathcal{P}^*$. It consists of the two chains $V_1 < \cdots < V_m$ and $H_1 < \cdots < H_n$, and the cover relation $H_{n-s} < V_{t+1}$. We have $|Q| = m + n$, and $r = \operatorname{rank} Q + 1$. We compute the number of minimal generators of the canonical module $\omega_{\mathbb{K}[\mathcal{P}^*]}$. For this purpose, let $\widehat{Q}$ be the poset obtained from Q by adding the elements $-\infty$ and ∞ with $\infty > p$ and $-\infty < p$ for all $p \in Q$, and let $\mathcal{T}(\widehat{Q})$ be the set of integer valued functions $\nu : \widehat{Q} \to \mathbb{Z}_{\geq 0}$ with $\nu(\infty) = 0$ and $\nu(p) < \nu(q)$ for all $p > q$. By using a result of Stanley [78], Hibi shows in [42, (3.3)] that the monomials of the form

$$y^{\nu(-\infty)} \prod_{p \in Q} x_p^{\nu(p)}$$

for $\nu \in \mathcal{T}(\widehat{Q})$ form a $\mathbb{K}$-basis for $\omega_{\mathbb{K}[\mathcal{P}^*]}$. By using [56, Corollary 2.4], we have that the number of generators of $\omega_{\mathbb{K}[\mathcal{P}^*]}$ is the number of minimal maps $\nu \in \mathcal{T}(\widehat{Q})$ with respect to the order given in [56, Page 5]. In fact, $\nu \leq \mu$ for $\nu, \mu \in T(\widehat{Q})$ if $\mu - \nu$ is decreasing. We observe that the minimal maps ν necessarily assign the numbers $1, \ldots, r$ to the vertices of a maximal chain of Q in reversed order, hence we have to find the possible values for the remaining $|Q| - r = m + n - r$ elements, depending on r. We distinguish three cases:

(a) $r = m$;

(b) $r = n$;

(c) $r = (n-s) + (m-t)$

In the case (a), the maximal chain is $V_1 < \cdots < V_m$. Hence we must take $\nu(V_{m-i+1}) = i$ for $i \in \{1, \ldots, m\}$. We have to determine how many vectors $(a_1, \ldots, a_n)$ with integers entries $0 < a_1 < \cdots < a_n$ satisfy $m - t < a_{s+1} < r - (n-s) = m - (n-s) + 2$, where the left inequality follows from the cover relation, while the right inequality follows from the fact that $a_{s+2} < \cdots < a_n < m+1$ are determined. Therefore, fixed $i = a_{s+1}$, there are $\binom{i-1}{s}$ ways to choose the values $a_1, \ldots, a_s$ in the range $\{1, \ldots, i-1\}$. Moreover, there are $\binom{m-i}{n-s-1}$ ways to choose $a_{s+2}, \ldots, a_n$ in the range $\{i+1, \ldots, m\}$. Hence we conclude

$$\text{type}(\mathbb{K}[\mathcal{P}]) = \sum_{i=m-t+1}^{m-(n-s)+1} \binom{i-1}{s}\binom{m-i}{n-s-1} = \sum_{i=m-t}^{m-(n-s)} \binom{i}{s}\binom{m-i-1}{n-s-1}.$$

In the case (b), we assign to each element of the chain $H_1, \ldots, H_n$ a number in $\{1, \ldots, n\}$ in strictly decreasing order. We have to determine how many vectors $(b_1, \ldots, b_m)$ with integers entries $0 < b_1 < \cdots < b_m$ satisfy $m-t-1 < b_{m-t} < s+1$, where the left inequality follows from the fact that $0 < b_1 < \cdots < b_{m-t-1}$, while the rightmost inequality follows from the cover relation. Therefore, for $i = b_{m-t}$, there are $\binom{i-1}{m-t-1}$ ways to choose the values $b_1, \ldots, b_{m-t-1}$ in the range $\{1, \ldots, i-1\}$. Moreover, there are $\binom{n-i}{t}$ ways to choose $b_{m-t+1}, \ldots, b_m$ in the range $\{i+1, \ldots, n\}$. Hence we conclude

$$\text{type}(\mathbb{K}[\mathcal{P}]) = \sum_{i=m-t}^{s} \binom{i-1}{m-t-1}\binom{n-i}{t}.$$

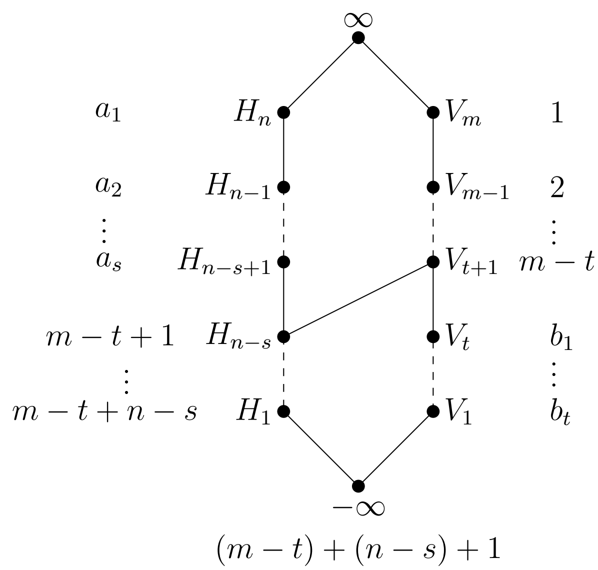


Figure 4.8: We count the number of minimal maps assigning $1 < \cdots < m-t+n-s$ to $V_m > \cdots > V_{t+1} > H_{n-s} > \cdots > H_1$.

In the case (c), we assign to each element of the chain $H_1, \ldots, H_{n-s}, V_{t+1}, \ldots, V_m$ a number in $\{1, \ldots, (m+n)-(s+t)\}$ in strictly decreasing order. We have to determine how many vectors $(a_1, \ldots, a_s, b_1, \ldots b_t)$ with integers entries $0 < a_1 < \cdots < a_s$, $m-t < b_1 < \cdots < b_t$ satisfy $a_s < m-t+1$ and $b_1 > m-t$ (see Figure 4.8). Therefore, there are $\binom{m-t}{s}$ ways to choose the values $a_1, \ldots, a_s$ in the range $\{1, \ldots, m-t\}$ and there are $\binom{n-s}{t}$ ways to choose $b_1, \ldots, b_t$ in the range $\{m-t+1, \ldots, m-t+n-s\}$. Hence in this case, we conclude

$$\operatorname{type}(\mathbb{K}[\mathcal{P}]) = \binom{n-s}{t}\binom{m-t}{s}.$$

□

Now we consider the general case.

Theorem 4.1.17. *Let $\mathcal{P}$ be an L-convex polyomino whose maximal rectangles are $\{R_i\}_{i=1,\ldots,t}$. For $i = 1, \ldots, t$, let $c_i \times d_i$ be the size of R_i with $d_1 = n$ and $c_t = m$ and*

$c_i < c_j$ and $d_i > d_j$ for $i < j$. Let $r = \max\{n, m, \{n+m-(c_i+d_{i+1})\}_{i=1,\ldots,t-1}\}$. Then

$$\operatorname{type}(\mathbb{K}[\mathcal{P}]) = \begin{cases} A & \text{if } r = m \\ B & \text{if } r = n \\ A_h B_h & \text{if } r = n+m-(c_h+d_{h+1}) \end{cases}$$

where

$$A = \sum_{i_1,\ldots,i_{t-1}} \binom{i_1 - 1}{d_t} \prod_{k=2}^{t-1} \binom{i_k - i_{k-1} - 1}{d_{t-k+1} - d_{t-k+2} - 1} \binom{m - i_{t-1}}{n - d_2 - 1}$$

with

$$m - c_{t-j} + 1 \le i_j \le m - (n - d_{t+1-j}) + 1 \quad \text{for} \quad 1 \le j \le t-1,$$

and

$$B = \sum_{i_1,\ldots,i_{t-1}} \binom{i_1 - 1}{m - c_{t-1} - 1} \prod_{k=2}^{t-1} \binom{i_k - i_{k-1} - 1}{c_{t-k+1} - c_{t-k} - 1} \binom{m - i_{t-1}}{c_1}$$

with $m - c_{t-1} \le i_1 \le d_t$ and

$$i_{j-1} + c_{t-j+1} - c_{t-j} \le i_j \le d_{t-j+1} \quad \text{for} \quad 2 \le j \le t-1.$$

Moreover,

$$A_h = \sum_{i_1,\ldots,i_{t-h-1}} \binom{i_1 - 1}{d_t} \prod_{k=2}^{t-h-1} \binom{i_k - i_{k-1} - 1}{d_{t-k+1} - d_{t-k+2} - 1} \binom{m - c_h - i_{t-h-1}}{d_{h+1} - d_{h+2} - 1}$$

with

$$m - c_{t-j} + 1 \le i_j \le m - c_h - (d_{h+1} - d_{t-j+1}) + 1 \quad \text{for} \quad 1 \le j \le t-h-1.$$

for $h = 1, \ldots, t-2$ and $A_{t-1} = \binom{m-c_{t-1}}{d_t}$, and

$$B_h = \sum_{i_1,\ldots,i_{h-1}} \binom{i_1 - 1}{c_h - c_{h-1} - 1} \prod_{k=2}^{h-1} \binom{i_k - i_{k-1} - 1}{c_{h-k+1} - c_{h-k} - 1} \binom{m - c_h + n - d_{h+1} - i_{h-1}}{c_1}$$

with $m - c_{h-1} \le i_1 \le m - c_h + (d_h - d_{h+1})$ and

$$i_{j-1} + (c_{h-j+1} - c_{h-j}) \le i_j \le m - c_h + (d_{h-j+1} - d_{h+1}) \quad \text{for} \quad 2 \le j \le h-1,$$

for $h = 2, \ldots, t-1$ and $B_1 = \binom{n-d_2}{c_1}$.

Proof. Firstly observe that in the general case the cover relations are $H_{n-d_{i+1}} < V_{c_i+1}$ for $i = 1, \ldots, t-1$. We just generalize the ideas of Proposition 4.1.16. We distinguish three cases:

(a) $r = m$;

(b) $r = n$;

(c) $r = (n - d_{h+1}) + (m - c_h)$ for some $k = 1, \ldots, t-1$.

In the case (a), we assign to each element of the chain $V_1, \cdots, V_m$ a number in $\{1, \ldots, m\}$ in decreasing order. We have to determine how many vectors $(a_1, \ldots, a_n)$ with integers entries $0 < a_1 < \cdots < a_n < m+1$ satisfy

$$m - c_{t-k} < a_{d_{t-k+1}+1} < m - (n - d_{t-k+1}) + 2 \quad \text{for } k = 1, \ldots, t-1,$$

where the left inequality follows from the cover relations, while the right inequality follows from the fact that $a_{d_{t-k+1}+2} < a_{d_{t-k+1}+3} < \cdots < a_n < m+1$. Therefore, fixed $i_1 = a_{d_t+1}$ there are $\binom{i_1-1}{d_t}$ ways to choose the values $a_1, \ldots, a_{d_t}$ in the range $\{1, \ldots, i_1 - 1\}$. Moreover, for $2 \leq k \leq t-1$ and fixed $i_k = a_{d_{t-k+1}+1}$, there are $\binom{i_k - i_{k-1} - 1}{d_{t-k+1} - d_{t-k+2} - 1}$ ways to choose the values $a_{d_{t+k+2}+2}, \ldots, a_{d_{t+k+1}}$ in the range $\{i_{k-1}+1, \ldots, i_k - 1\}$. Finally, there are $\binom{m - i_{t-1}}{n - d_2 - 1}$ ways to choose $a_{d_2+2}, \ldots, a_n$ in the range $\{i_{t-1}+1, \ldots, m\}$. Hence in this case, we conclude that type($\mathbb{K}[\mathcal{P}]$) is A.

In the case (b), we assign to each element of the chain $H_1, \ldots, H_n$ a number in $\{1, \ldots, n\}$ in decreasing order. We have to determine how many vectors $(b_1, \ldots, b_m)$ with integers entries $0 < b_1 < \cdots < b_m$ satisfy

$$m - c_{t-1} - 1 < b_{m-c_{t-1}} < d_t + 1$$

and

$$b_{m-c_{t-k+1}} + (c_{t-k+1} - c_{t-k}) - 1 < b_{m-c_{t-k}} < d_{t-k+1} + 1 \text{ for } k = 2, \ldots, t-1,$$

where the left inequalities follow from the fact that $b_{m-c_{t-k+1}+1} < \cdots < b_{m-c_{t-k}-1}$, while the right inequalities follow from the cover relations. Therefore for fixed $i_1 = b_{m-c_{t-1}}$ there are $\binom{i_1-1}{m-c_{t-1}-1}$ ways to choose the values $b_1, \ldots, b_{m-c_{t-1}-1}$ in the range $\{1, \ldots, i_1 - 1\}$. Moreover, for $2 \leq k \leq t-1$ and fixed $i_k = b_{m-c_{t-k}}$, there are $\binom{i_k - i_{k-1} - 1}{c_{t-k+1} - c_{t-k} - 1}$ ways to choose the values $b_{m-c_{t-k+1}+1}, \ldots, b_{m-c_{t-k}-1}$ in the range $\{i_{k-1}+1, \ldots, i_k-1\}$. Finally, there are $\binom{n-i_{t-1}}{c_1}$ ways to choose $b_{m-c_1+1}, \ldots, b_m$ in the range $\{i_{t-1}+1, \ldots, n\}$. Hence in this case, we conclude that type($\mathbb{K}[\mathcal{P}]$) is B.

In the case (c), fix $h \in \{1, \ldots, t-1\}$. We assign to each element of the chain $H_1, \ldots,$ $H_{n-d_{h+1}}, V_{c_h+1}, \ldots, V_m$ a number in $\{1, \ldots, (m+n)-(c_h+d_{h+1})\}$ in decreasing order. Let $\widetilde{m} = m - c_h$, $\tilde{n} = n - d_{h+1}$. We have to determine how many vectors $(a_1, \ldots, a_{d_{h+1}}, b_1, \ldots b_{c_h})$ with integers entries $0 < a_1 < \cdots < a_{d_{h+1}}$, $\widetilde{m} < b_1 < \cdots < b_{c_h}$ satisfy

$$m - c_{t-k} < a_{d_{t-k+1}+1} < \widetilde{m} - (d_{h+1} - d_{t-k+1}) + 2 \quad \text{for } k = 1, \ldots, t-h-1$$

$$m - c_{h-1} - 1 < b_{c_h - c_{h-1}} < \widetilde{m} + (d_h - d_{h+1}) + 1,$$

$$b_{c_h - c_{h-k+1}} + (c_{h-k+1} - c_{h-k}) - 1 < b_{c_h - c_{h-k}} < \widetilde{m} + (d_{h-k+1} - d_{h+1}) + 1 \text{ for } k = 2, \ldots, h-1.$$

For fixed $i_1 = a_{d_t+1}$ there are $\binom{i_1-1}{d_t}$ ways to choose the values $a_1, \ldots, a_{d_t}$ in the range $\{1, \ldots, i_1-1\}$. Moreover, for $2 \le k \le t-h-1$ and fixed $i_k = a_{d_{t-k+1}+1}$, there are $\binom{i_k - i_{k-1} - 1}{d_{t-k+1} - d_{t-k+2} - 1}$ ways to choose the values $a_{d_{t+k+2}+2}, \ldots, a_{d_{t+k+1}}$ in the range $\{i_{k-1}+1, \ldots, i_k-1\}$. Furthermore, there are $\binom{\widetilde{m} - i_{t-h-1}}{d_{h+1} - d_{h+2} - 1}$ ways to choose $a_{d_{h+2}+2}, \ldots, a_{d_{h+1}}$ in the range $\{i_{t-h-1}+1, \ldots, \widetilde{m}\}$.

For fixed $j_1 = b_{c_h - c_{h-1}}$ there are $\binom{j_1 - 1}{c_h - c_{h-1} - 1}$ ways to choose the values $b_1, \ldots, b_{c_h - c_{h-1} - 1}$ in the range $\{1, \ldots, j_1 - 1\}$. Moreover, for $2 \le k \le h$ and fixed $j_k = b_{c_h - c_{h-k}}$ there are $\binom{j_k - j_{k-1} - 1}{c_{h-k+1} - c_{h-k} - 1}$ ways to choose the values $b_{c_h - c_{h-k+1}+1}, \ldots, b_{c_h - c_{h-k} - 1}$ in the range $\{j_{k-1}+1, \ldots, j_k - 1\}$. Finally, there are $\binom{\widetilde{m} + \tilde{n} - j_{h-1}}{c_1}$ ways to choose $b_{c_h - c_1 + 1}, \ldots, b_{c_h}$ in the range $\{j_{h-1}+1, \ldots, \widetilde{m} + \tilde{n}\}$. Hence in this case, we conclude that type($\mathbb{K}[\mathcal{P}]$) is $A_h \cdot B_h$.

Observe that the formula for the a_i makes sense only if $1 \le h \le t-2$. For $h = t-1$, we have to choose the numbers

$$a_1, \ldots, a_{d_t}$$

among the values $\{1, \ldots, m - c_{t-1}\}$, hence $A_{t-1} = \binom{m - c_{t-1}}{d_t}$. Furthermore observe that the formula for the b_i makes sense only if $2 \le h \le t-1$. For $h = 1$, we have to choose the numbers

$$b_1, \ldots, b_{c_1}$$

among the values $\{m - c_1 + 1, \ldots, (m - c_1) + (n - d_2)\}$, hence $B_1 = \binom{n - d_2}{c_1}$. □

We observe that Theorem 4.1.11 can also deduced from Theorem 4.1.17.

The following example demonstrates Theorem 4.1.17.

Example 4.1.18 *Let $\mathcal{P}$ be the Ferrer diagram in Figure 4.9.*

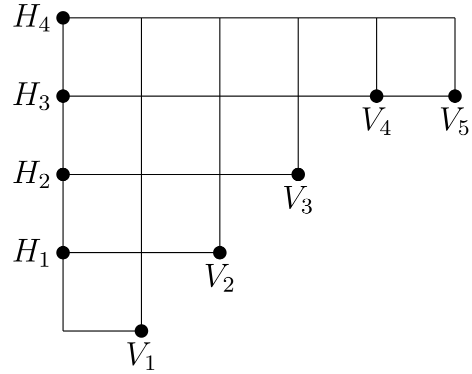


Figure 4.9

We have $t = 4$ maximal rectangles whose sizes are $\{c_i \times d_i\}_{i=1,\dots,4}$ with

$$c_1 = 1 \quad c_2 = 2 \quad c_3 = 3 \quad c_4 = m = 5$$
$$d_1 = n = 4 \quad d_2 = 3 \quad d_3 = 2 \quad d_4 = 1.$$

There are 4 maximal chains in the poset Q corresponding to $\mathcal{P}$ containing 5 vertices. For example, the chain $V_1, \dots, V_5$ and the chain H_1, H_2, V_3, V_4, V_5, that correspond to the cases $r = m$ and $r = (n - d_3) + (m - c_2)$, hence $h = 2$. We are going to compute A and A_2B_2 as in Theorem 4.1.17. We have

$$A = \sum_{i_1=3}^{3} \sum_{i_2=4}^{4} \sum_{i_3=5}^{5} \binom{i_1 - 1}{1} \binom{i_2 - i_1 - 1}{2 - 1 - 1} \binom{i_3 - i_2 - 1}{3 - 2 - 1} \binom{5 - i_3}{4 - 3 - 1} = 2,$$

while

$$A_2 = \sum_{i_1=3}^{3} \binom{i_1 - 1}{1} \binom{5 - 2 - i_1}{2 - 1 - 1} = 2$$

and

$$B_2 = \sum_{i_1=4}^{4} \binom{i_1 - 1}{2 - 1 - 1} \binom{5 - i_1}{1} = 1,$$

yielding

$$A_2B_2 = 2.$$

In conclusion we want to point out that for L-convex polyominoes, important algebraic invariants, like the Castelnuovo-Mumford regularity, the Cohen-Macaulay type, and algebraic properties, like being Gorenstein, are now completely understood and have a nice combinatorial interpretation. It is still a challenge to prove similar results when the polyomino is k-convex for $k > 1$, rather than just L-convex.

4.2 HILBERT SERIES AND GORENSTEINNESS OF SIMPLE THIN POLYOMINOES

In this section, we compare two generating functions associated with polyominoes: the Hilbert series of $\mathbb{K}[\mathcal{P}]$ and the rook polynomial of $\mathcal{P}$ (see [71, Chapter 7]).

In Section 4.1, we proved that, for an L-convex polyomino $\mathcal{P}$, the Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$ is equal to $r(\mathcal{P})$. Starting from this result, we consider the Hilbert-Poincaré series of simple polyominoes as a nice object to grasp the above equality and other fundamental invariants by using elementary proofs.

Recall from Section 3.2 that a polyomino $\mathcal{P}$ is thin if $\mathcal{P}$ does not contain the square tetromino as a subpolyomino. One of the main results of this Section is the following

Theorem 4.2.1. *Let $\mathcal{P}$ be a simple thin polyomino such that the reduced Hilbert-Poincaré series of $\mathbb{K}[\mathcal{P}]$ is*

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{h(t)}{(1-t)^d}.$$

Then $h(t)$ is the rook polynomial of $\mathcal{P}$.

In particular it follows that the Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$ is $r(\mathcal{P})$ and the multiplicity of $\mathbb{K}[\mathcal{P}]$ is $r_{\mathcal{P}}(1)$. Theorem 4.2.1 gives us information on the Hilbert series and the Castelnuovo-Mumford regularity of the toric ring related to the bipartite graph $G_{\mathcal{P}}$ induced in a natural way by a simple polyomino $\mathcal{P}$ (see Section 2.4). The condition that $\mathcal{P}$ is thin translates to the condition that the bipartite graph $G_{\mathcal{P}}$ does not contain $K_{3,3}$ as a subgraph, where $K_{3,3}$ is the complete bipartite graph with two parts of equal size 3.

An open question is to give a complete characterization of the Gorensteinness of the algebra $\mathbb{K}[\mathcal{P}]$ when $\mathcal{P}$ is a simple polyomino. Some partial results in this direction are discussed in Section 2.4 and Section 4.1. The other main result of this Section is Theorem

4.2.18, in which we classify the simple thin polyominoes $\mathcal{P}$ having a Gorenstein algebra $\mathbb{K}[\mathcal{P}]$, due to the geometric properties of $\mathcal{P}$. At the end we present a conjecture and an open question.

4.2.1 Hilbert series of simple thin polyominoes

In this subsection we compute the Hilbert series of simple thin polyominoes in relation with their rook polynomial. We start with the following

Definition 4.2.2 *Let C and D be two cells of $\mathbb{N}^2$ such that $e(C) \leq e(D)$. We call the set*

$$[C, D] = \{F \in \mathbb{N}^2 : e(F) \in [e(C), e(D)]\}$$

interval of cells. *If $e(C)$ and $e(D)$ lie either on the same vertical edge interval or on the same horizontal edge interval, we call $[C, D]$ a* cell interval. *We call $[C, D]$* inner interval of cells *of $\mathcal{P}$ if any cell in $[C, D]$ is a cell of $\mathcal{P}$.*

Lemma 4.2.3. *Let $\mathcal{P}$ be a simple thin polyomino. Then any maximal inner interval I of cells of $\mathcal{P}$ is a cell interval, and for any maximal inner interval $J \neq I$ such that $V(I) \cap V(J) \neq \varnothing$, I and J have either one cell, one edge or one vertex in common.*

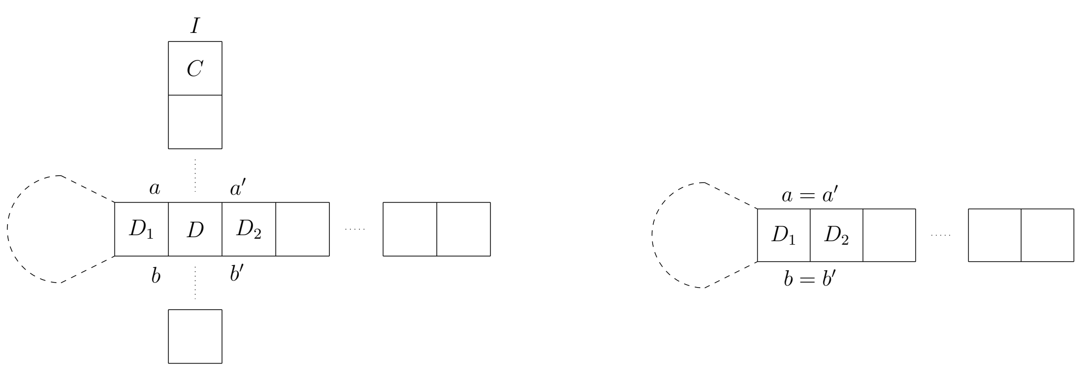
Proof. Since $\mathcal{P}$ does not contain a square tetromino, then also any maximal inner interval of $\mathcal{P}$ does not contain a square tetromino, namely it is a cell interval.

Let I, J be two maximal inner intervals of $\mathcal{P}$ such that $V(I) \cap V(J) \neq \varnothing$. By contradiction, we consider the following two cases: I and J have two or more edges in common, not belonging to the same cell, and I and J have two or more cells in common. In the first case, without loss of generality $V(I) \cap V(J) = [(i, j), (k, j)]$ with $k > i + 1$. Therefore, the cells whose left lower corners are $(i, j-1), (i+1, j-1), (i, j), (i+1, j)$ form a square tetromino, that is a contradiction.
In the second case, $I \cup J$ is a maximal inner interval strictly containing I and J, and this is a contradiction. The assertion follows. □

From now on, we will briefly call inner intervals the inner intervals of cells of a polyomino $\mathcal{P}$. In the following we define the simple polyominoes $\mathcal{P}'$ and $\mathcal{P}''$ obtainable from a simple (thin) polyomino $\mathcal{P}$. The latter are fundamental for the computation of the Hilbert series.

Definition 4.2.4 (Polyomino $\mathcal{P}'$) *Let $\mathcal{P}$ be a simple polyomino. We say that a cell C of $\mathcal{P}$ is a* leaf *if there exists an edge $\{u, v\}$ of C such that $\{u, v\} \cap V(\mathcal{P} \setminus \{C\}) = \varnothing$. We call the vertices u and v* leaf corners *of C. We define the polyomino $\mathcal{P}'$ as the polyomino $\mathcal{P} \setminus \{C\}$.*

Definition 4.2.5 (Polyomino $\mathcal{P}''$) *Let $\mathcal{P}$ be a simple thin polyomino and let I be a maximal inner interval of $\mathcal{P}$. We say that $\mathcal{P}$ is* collapsible *in I if there exists one and only one maximal inner interval J of $\mathcal{P}$ intersecting I in a cell, and $\mathcal{P} = \mathcal{P}_1 \sqcup I \sqcup \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two polyominoes such that $\mathcal{P}_2$ is either empty or a cell interval. When $\mathcal{P}_2$ is empty, I is called a* tail*. When $\mathcal{P}_2$ is a cell interval, I is called an* endcut*. We define the polyomino $\mathcal{P}''$ as follows. Let D be the cell such that $I \cap J = \{D\}$, and let $\{a, b, a', b'\}$ be the corners of D where $a, b \in V(\mathcal{P}_1)$ and $a', b' \in V(\mathcal{P}_2)$. We define $\mathcal{P}''$ as the polyomino obtained from $\mathcal{P} \setminus I$ by the identification of the vertices a and b of $\mathcal{P}_1$ with the vertices a' and b' of $\mathcal{P}_2$, respectively, due to the translation of the cell interval $\mathcal{P}_2$ (see Figure 4.10).*



(A) A simple thin polyomino $\mathcal{P}$ which is collapsible in the endcut I

(B) The polyomino $\mathcal{P}''$ after the collapsing of $\mathcal{P}$ on I

Figure 4.10: The collapsing operation on a simple thin polyomino $\mathcal{P}$

Remark 4.2.6 *Let $\mathcal{P}$ be a simple thin polyomino collapsible in I with leaf C. We observe that $r(\mathcal{P}') \in \{r(\mathcal{P}), r(\mathcal{P}) - 1\}$ and $r(\mathcal{P}'') = r(\mathcal{P}) - 1$. For example, if $\mathcal{P}$ is the polyomino in Figure 4.13 and we consider the leaf C_1, then $r(\mathcal{P}')$ is equal to $r(\mathcal{P}) - 1$. On the other hand,* *if $\mathcal{P}$ is the polyomino in Figure 4.11 and we consider the leaf containing u and v, then $r(\mathcal{P}')$ is equal to $r(\mathcal{P})$. In both cases, we have $r(\mathcal{P}'') = r(\mathcal{P}) - 1$. In general, if C belongs to any set of $r(\mathcal{P})$ non-attacking rooks, then any set of non-attacking rooks of maximal cardinality in $\mathcal{P}'$ has $r(\mathcal{P}) - 1$ elements. Otherwise, there exists some set of non-attacking rooks of maximal cardinality in $\mathcal{P}'$ having $r(\mathcal{P})$ elements. Moreover, any set of $r(\mathcal{P})$ non-attacking rooks has an element on I, that is $r(\mathcal{P}'') = r(\mathcal{P}) - 1$.*

We now want to prove that any simple thin polyomino is collapsible in some inner interval I. For this aim, we first prove the following

Lemma 4.2.7. *Let $\mathcal{P}$ be a simple thin polyomino that is not a cell interval. Then there exists a maximal inner interval I of $\mathcal{P}$ for which there exists one and only one maximal inner interval J of $\mathcal{P}$ intersecting I in a cell.*

Proof. Since $\mathcal{P}$ is simple and thin, we observe that for any two cells C and D of $\mathcal{P}$ there is a unique path of cells connecting C and D.

By contradiction, assume that for any maximal inner interval of $\mathcal{P}$ there are at least two maximal inner intervals intersecting it in one cell. We show that there exist two different paths connecting two given cells. For this aim, let I be a maximal inner interval of $\mathcal{P}$. There exist I_1 and J such that $I_1 \cap I$ and $I_1 \cap J$ are cells of $\mathcal{P}$. Furthermore, there exists $I_2 \neq I$ intersecting I_1 in one cell. By using the same argument, we find a sequence of inner intervals $I_1, I_2, \dots$ of $\mathcal{P}$ such that I_j and I_{j+1} have a cell in common. Since the number of inner intervals of $\mathcal{P}$ is finite, then there exists k such that $I_k = J$, and hence there are two paths connecting a cell C of $I \setminus I \cap J$ with a cell D of $J \setminus I \cap J$, one passing through $I_1, \dots, I_{k-1}$ and one passing through the cell $I \cap J$. This is a contradiction and the assertion follows. □

Proposition 4.2.8. *Let $\mathcal{P}$ be a simple thin polyomino that is not a cell interval. Then $\mathcal{P}$ is collapsible in some maximal inner interval I.*

Proof. If $\mathcal{P}$ has a tail, then the assertion follows. Therefore, assume that $\mathcal{P}$ does not contain tails.

By contradiction, assume that $\mathcal{P}$ has no endcuts. From Lemma 4.2.7, there exists a maximal inner interval I_1 of $\mathcal{P}$ for which there exists one and only one inner interval J_1 of $\mathcal{P}$ intersecting I_1 in one cell. Let $\mathcal{P} = \mathcal{P}_1 \sqcup I_1 \sqcup \mathcal{P}_2$. Since I_1 is not an endcut, then $\mathcal{P}_2$ is a simple thin polyomino that is not a cell interval. Moreover, rank $\mathcal{P}_2 <$ rank $\mathcal{P}$. Again from Lemma 4.2.7, there exists an inner interval I_2 in $\mathcal{P}_2$ for which there exists one and only one inner interval J_2 of $\mathcal{P}$ intersecting I_2 in one cell. We write $\mathcal{P} = \mathcal{P}_3 \sqcup I_2 \sqcup \mathcal{P}_4$, with $\mathcal{P}_1 \subset \mathcal{P}_3$. We repeat the same argument for the simple thin polyomino $\mathcal{P}_4$ with rank $\mathcal{P}_4 <$ rank $\mathcal{P}_2$. By proceeding in this way, since the rank $\mathcal{P}$ is finite, at the end we find an inner interval I_k for which $\mathcal{P} = \mathcal{P}_{2k-1} \sqcup I_k \sqcup \mathcal{P}_{2k}$ such that rank $\mathcal{P}_{2k} = 0$, namely I_k is a tail, that is a contradiction. $\square$

We observe that the interval I in Lemma 4.2.7 in which $\mathcal{P}$ is collapsible has one leaf C.

Lemma 4.2.9. *Let $\mathcal{P}$ be a simple polyomino with a leaf C having leaf corners u and v, and let $\mathcal{P}'$ be as in Definition 4.2.4. Then $((I_{\mathcal{P}}, x_u) : x_v) = I_{\mathcal{P}'} + J$ where J is a monomial ideal generated in degree one.*

Proof. Since C is a leaf of $\mathcal{P}$, then there exists a maximal cell interval I of $\mathcal{P}$ such that $C \in I$. Let $E = \{u_1, u_2, \dots, u_r, u\}$ and $F = \{v_1, \dots, v_r, v\}$ be the edge intervals of length $r+1$ of I. We observe that the ideal $I_{\mathcal{P}}$ is generated by the inner 2-minors of $\mathcal{P}' = \mathcal{P} \setminus \{C\}$ and by the inner 2-minors of I whose inner intervals contain the cell C, namely

$$I_{\mathcal{P}} = I_{\mathcal{P}'} + (\{x_v x_{u_i} - x_u x_{v_i}\}_{i=1,\dots,r}).$$

Then

$$(I_{\mathcal{P}}, x_u) = I_{\mathcal{P}'} + (\{x_v x_{u_i}\}_{i=1,\dots,r}) + (x_u).$$

The thesis follows if we prove that $(I_{\mathcal{P}}, x_u) : x_v \subseteq I_{\mathcal{P}'} + (x_{u_1}, \dots, x_{u_r}, x_u)$, since the other inclusion is trivial. If $f \in (I_{\mathcal{P}}, x_u) : x_v$, then $x_v f \in I_{\mathcal{P}'} + (\{x_v x_{u_i}\}_{i=1,\dots,r}) + (x_u)$, that is

$$x_v f = g + x_v g' + x_u g''$$

where $g \in I_{\mathcal{P}'}$, $g' \in (x_{u_1}, \dots, x_{u_r})$ and, $g'' \in R$. That is, $x_v(f - g') \in I_{\mathcal{P}'} + (x_u)$ and $f - g' \in (I_{\mathcal{P}'} + (x_u)) : x_v$. Since $\mathcal{P}'$ is simple, then $I_{\mathcal{P}'}$ is prime, and since x_u is not a variable of $I_{\mathcal{P}'}$, then also $I_{\mathcal{P}'} + (x_u)$ is prime. Therefore, since $x_v \notin I_{\mathcal{P}'} + (x_u)$, then $f - g' \in I_{\mathcal{P}'} + (x_u)$ and the assertion follows. □

Remark 4.2.10 *By using the notation of Lemma 4.2.9, we want to remark that the ideal in the statement has different behaviours, depending on the choice of u and v. Let $\mathcal{P}$ be the simple thin polyomino in Figure 4.11, namely the skew tetromino.*

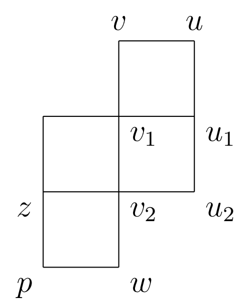


Figure 4.11: The skew tetromino

Since $x_v x_{u_2} - x_u x_{v_2} \in I_{\mathcal{P}}$, then $x_u x_{v_2} \in (I_{\mathcal{P}}, x_v)$ and $x_{v_2} \in (I_{\mathcal{P}}, x_v) : x_u$. Therefore, since $x_p x_{v_2} - x_w x_z \in I_{\mathcal{P}}$, then $x_w x_z \in (I_{\mathcal{P}}, x_v) : x_u$, namely $(I_{\mathcal{P}}, x_v) : x_u$ has a monomial generator of degree 2. Nevertheless, the ideal $(I_{\mathcal{P}}, x_u) : x_v$ has no monomial generators of degree greater than 1.

Lemma 4.2.11. *Let $\mathcal{P}$ be a simple thin polyomino, collapsible in I that has r cells, and let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}', \mathcal{P}''$ be as in Definitions 4.2.4 and 4.2.5. Let C be a leaf of I with leaf corners u and v, and assume that $E = \{u_1, u_2, \ldots, u_r, u\}$ is the edge interval of I such that $E \cap V(\mathcal{P}_1) = \varnothing$. Then $R/(I_{\mathcal{P}}, x_u, x_v) \cong \mathbb{K}[\mathcal{P}']$ and $R/((I_{\mathcal{P}}, x_u) : x_v) \cong \mathbb{K}[\mathcal{P}''] \otimes \mathbb{K}[y_1, \ldots, y_{r-1}]$.*

Proof. Let $F = \{v_1, \ldots, v_r, v\}$ be the other edge interval of I of length $r + 1$. By the proof of Lemma 4.2.9, we have

$$I_{\mathcal{P}} = I_{\mathcal{P}'} + (\{x_v x_{u_i} - x_u x_{v_i}\}_{i=1,\ldots,r}).$$

and

$$(I_{\mathcal{P}}, x_u) = I_{\mathcal{P}'} + (\{x_v x_{u_i}\}_{i=1,\ldots,r}, x_u).$$

Since $\{u, v\} \cap V(\mathcal{P}') = \varnothing$, then $(I_{\mathcal{P}}, x_u, x_v) = (I_{\mathcal{P}'}, x_u, x_v)$, that is $R/(I_{\mathcal{P}}, x_u, x_v) \cong \mathbb{K}[\mathcal{P}']$.

Now let $I'' = ((I_{\mathcal{P}}, x_u) : x_v)$. By the proof of Lemma 4.2.9, it arises $I'' = I_{\mathcal{P}'} + (x_{u_1}, \dots, x_{u_r}, x_u)$. Let us consider J and D as in Definition 4.2.5, with $V(D) = \{u_k,\, u_{k+1},\, v_k,\, v_{k+1}\}$. We can split J into the cell intervals J_1 and J_2, such that $J_1 \subseteq \mathcal{P}_1$, $\mathcal{P}_2 = J_2$, and the cell D. Since the variables $x_{u_1}, \dots, x_{u_r},\, x_u$ are generators of I'', then all of the inner 2-minors of the interval I, and all of the inner 2-minors of J having corners on u_k, u_{k+1}, are redundant. Since $\mathcal{P}_2$ is either empty or a cell interval, then the edge E is a maximal edge interval of $\mathcal{P}$ (see also Remark 4.2.10). We want to prove that I'' has no minimal monomial generators of degree greater than 1. By Lemma 4.2.9, assume that there exists a minimal generator $x_w x_z \in I''$, with $w, z \notin \{u_1, \dots, u_r, u\} = E$. That is there exists $i \in \{1, \dots, r\}$ and $p \in V(\mathcal{P})$ such that $g = x_w x_z - x_{u_i} x_p$ is an inner 2-minor of $\mathcal{P}$. That is one between w and z, say w, lies on the same edge interval containing the u_i's and $w \notin E$, namely $E \cup \{w\}$ is an edge interval of $\mathcal{P}$ containing E, that is E is not a maximal, contradiction.
If $\mathcal{P}_2$ is empty, from Definition 4.2.5 we have $\mathcal{P}'' = \mathcal{P} \setminus I = \mathcal{P}_1$. Since $E \cap V(\mathcal{P}_1) = \varnothing$, then $I'' = I_{\mathcal{P}_1} + (x_{u_1}, \dots, x_{u_r}, x_u)$, $V(\mathcal{P}'') \cap F = \{v_k, v_{k+1}\}$, and therefore

$$R/I'' \cong \mathbb{K}[\mathcal{P}''] \otimes \mathbb{K}[x_{v_1}, \dots, x_{v_{k-1}}, x_{v_{k+2}} \dots, x_{v_r}, x_v]$$

and the assertion follows.
Otherwise, let $\mathcal{P}''$ be the polyomino arising from the translation of the edge $\{u_k, u_{k+1}\}$ on the edge $\{v_k, v_{k+1}\}$. We want to prove that $I'' = I_{\mathcal{P}''} + (x_{u_1}, \dots,\, x_{u_r}, x_u)$.

Let $f = f^+ - f^- \in I''$ be an irreducible binomial and let

$$V(f) = \{v \in V(\mathcal{P}) \mid x_v | f^+ \text{ or } x_v | f^-\}.$$

One of the following is true

(a) $V(f) \subseteq V(\mathcal{P}_1)$ or $V(f) \subseteq V(\mathcal{P}_2) \setminus \{u_k, u_{k+1}\}$;

(b) $|V(f) \cap V(\mathcal{P}_1)| = |V(f) \cap V(\mathcal{P}_2) \setminus \{u_k, u_{k+1}\}| = 2$.

In case (a) we have $f \in I_{\mathcal{P}''}$.
In case (b), since J is the unique maximal cell interval having non-empty intersection with both $\mathcal{P}_1$ and $\mathcal{P}_2$, we have that $|V(f) \cap V(J_1)| = |V(f) \cap V(J_2) \setminus \{u_k, u_{k+1}\}| = 2$. Since $J_1 \cup J_2$ is a maximal cell interval of $\mathcal{P}''$, then $f \in I_{\mathcal{P}''}$. The latter proves $I'' \subseteq I_{\mathcal{P}''} + (x_{u_1}, \dots, x_{u_r}, x_u)$. Similarly the other inclusion follows, due to the fact that an inner interval in $\mathcal{P}''$ is either

an inner interval of $\mathcal{P}_1$, of $\mathcal{P}_2$ (up to the translation defined in Definition 4.2.5), or it is contained in $J_1 \cup J_2$. Lastly, since $V(\mathcal{P}'') \cap F = \{v_k, v_{k+1}\}$, then

$$R/I'' \cong \mathbb{K}[\mathcal{P}''] \otimes \mathbb{K}[x_{v_1}, \dots, x_{v_{k-1}}, x_{v_{k+2}} \dots, x_{v_r}, x_v]$$

□

Corollary 4.2.12. *Let $\mathcal{P}$ be a simple thin polyomino, collapsible in I that has r cells, with $\mathcal{P}'$ and $\mathcal{P}''$ as in Definitions 4.2.4 and 4.2.5. Then*

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{1}{1-t}\left(\mathrm{HP}_{\mathbb{K}[\mathcal{P}']}(t) + \frac{t}{(1-t)^{r-1}} \cdot \mathrm{HP}_{\mathbb{K}[\mathcal{P}'']}(t)\right)$$

Proof. Let C be a leaf of I and let u and v be the leaf corners of C with u satisfying the hypotheses of Lemma 4.2.11. We take the following short exact sequence:

$$0 \longrightarrow R/(I_{\mathcal{P}} : x_u) \longrightarrow R/I_{\mathcal{P}} \longrightarrow R/(I_{\mathcal{P}}, x_u) \longrightarrow 0$$

Since $\mathcal{P}$ is simple, then from Lemma 2.4.3 $I_{\mathcal{P}}$ is prime, that is $(I_{\mathcal{P}} : x_u) = I_{\mathcal{P}}$. Therefore, by Proposition 1.3.2.(2) we have

$$\mathrm{HP}_{R/I_{\mathcal{P}}}(t) = \frac{1}{1-t}\mathrm{HP}_{R/(I_{\mathcal{P}}, x_u)}(t).$$

We study the Hilbert series of $R/(I_{\mathcal{P}}, x_u)$. By applying Proposition 1.3.2 to the following short exact sequence:

$$0 \longrightarrow R/((I_{\mathcal{P}}, x_u) : x_v) \longrightarrow R/(I_{\mathcal{P}}, x_u) \longrightarrow R/(I_{\mathcal{P}}, x_u, x_v) \longrightarrow 0$$

we get

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{1}{1-t}\left(\mathrm{HP}_{R/(I_{\mathcal{P}}, x_u, x_v)}(t) + t \cdot \mathrm{HP}_{R/((I_{\mathcal{P}}, x_u) : x_v)}(t)\right).$$

Furthermore, by Lemma 4.2.11, we have

1. $R/(I_{\mathcal{P}}, x_u, x_v) \cong \mathbb{K}[\mathcal{P}']$;
2. $R/((I_{\mathcal{P}}, x_u) : x_v) \cong \mathbb{K}[\mathcal{P}''] \otimes \mathbb{K}[y_1, \dots, y_{r-1}]$.

It is well known that

$$\mathrm{HP}_{\mathbb{K}[y_1, \dots, y_n]}(t) = \frac{1}{(1-t)^n}$$

and

$$\mathrm{HP}_{A \otimes B}(t) = \mathrm{HP}_A(t) \cdot \mathrm{HP}_B(t),$$

that is

$$\mathrm{HP}_{R/((I_{\mathcal{P}},x_u):x_v)}(t) = \frac{1}{(1-t)^{r-1}} \cdot \mathrm{HP}_{\mathbb{K}[\mathcal{P}'']}(t)$$

and the assertion follows. □

Let $\mathcal{P}$ be a cell interval with $\operatorname{rank}\mathcal{P} = r$. The ideal $I_{\mathcal{P}}$ can be seen as the determinantal ideal of a $2 \times (r+1)$ matrix. The resolution of the above ideal is well-known (see [13, 24]), as well as its Hilbert series. For the sake of completeness, we give the following result

Lemma 4.2.13. *Let $\mathcal{P}$ be a cell interval with* $\operatorname{rank}\mathcal{P} = r$. *Then*

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{1+rt}{(1-t)^{r+2}}.$$

Proof. By [24, Corollary 6.2], $I_{\mathcal{P}}$ has linear resolution, and $\beta_{i,i+1} = i\binom{r+1}{i+1}$ for $i = 1, \ldots, r$. It is well-known that if M is an R-module, then

$$\mathrm{HP}_M(t) = \frac{1}{(1-t)^n} \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i \beta_{ij} t^j.$$

That is, the Hilbert series of $\mathbb{K}[\mathcal{P}]$ is

$$\frac{1 + \sum_{i=1}^{r-1} (-1)^i i \binom{r+1}{i+1} t^{i+1} + (-1)^r r t^{r+1}}{(1-t)^{2r+2}}. \qquad (*)$$

We study the coefficient $i\binom{r+1}{i+1}$ for $2 \leq i \leq r-1$.

$$i\binom{r+1}{i+1} = (i+1)\binom{r+1}{i+1} - \binom{r+1}{i+1} =$$
$$= (r+1)\binom{r}{i} - \binom{r+1}{i+1} = r\binom{r}{i} - \binom{r}{i+1}.$$

Hence the numerator of Equation $(*)$ becomes

$$1 + \sum_{i=1}^{r-1} (-1)^i \left(r\binom{r}{i} - \binom{r}{i+1} \right) t^{i+1} + (-1)^r r t^{r+1} =$$
$$= 1 + \sum_{i=2}^{r} (-1)^i \binom{r}{i} t^i + \sum_{i=1}^{r} (-1)^i r \binom{r}{i} t^{i+1} - rt + rt =$$
$$(1-t)^r + rt(1-t)^r.$$

That is

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{(1+rt)(1-t)^r}{(1-t)^{2r+2}},$$

and the assertion follows. □

We now state the main theorem (see also Examples 4.2.19 and 4.2.20).

Theorem 4.2.14. *Let $\mathcal{P}$ be a simple thin polyomino with*

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{h(t)}{(1-t)^d}.$$

Then $h(t)$ is the rook polynomial of $\mathcal{P}$.

Proof. Let $I_1, \dots I_s$ be the maximal inner intervals of $\mathcal{P}$. We proceed by induction on $p = \text{rank } \mathcal{P}$.

If $p = 1$, then $\mathcal{P}$ consists of one cell and by Lemma 4.2.13, the statement follows.

Let $p > 1$ and assume the thesis true for any polyomino with rank less than or equal to $p-1$. If $s = 1$, then $\mathcal{P}$ is a cell interval and from Lemma 4.2.13 we have

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{1+pt}{(1-t)^{p+2}}.$$

The polynomial $1 + pt$ is the rook polynomial of a cell interval having p cells, that is the assertion follows. If $s > 1$, then $\mathcal{P}$ is not a cell interval, that is, from Proposition 4.2.8, $\mathcal{P}$ is collapsible in some maximal inner interval I. Assume that I has r cells. In order to apply Corollary 4.2.12, we focus on $\mathrm{HP}_{\mathbb{K}[\mathcal{P}']}(t)$ and $\mathrm{HP}_{\mathbb{K}[\mathcal{P}'']}(t)$. The polyomino $\mathcal{P}'$ has $p-1$ cells, while the polyomino $\mathcal{P}''$ has $p-r$ cells. Hence, from the inductive hypothesis we have

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}']}(t) = \frac{\sum_{i=0}^{a} r'_i t^i}{(1-t)^{d_1}},$$

where $a = r(\mathcal{P})$ with $r'_a \geq 0$ due to Remark 4.2.6, and $\sum_{i=0}^{a} r'_i t^i$ is the rook polynomial of $\mathcal{P}'$, and

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}'']}(t) = \frac{\sum_{i=0}^{b} r''_i t^i}{(1-t)^{d_2}}.$$

where $b = r(\mathcal{P}'') = r(\mathcal{P}) - 1$ due to Remark 4.2.6, and $\sum_{i=0}^{b} r''_i t^i$ is the rook polynomial of $\mathcal{P}''$. From Corollary 4.2.12 we get that $\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t)$ is equal to

$$\frac{1}{1-t}\left(\frac{\sum_{i=0}^{a} r'_i t^i}{(1-t)^{d_1}} + \frac{1}{(1-t)^{r-1}}\frac{\sum_{i=0}^{b} r''_i t^{i+1}}{(1-t)^{d_2}}\right) = \frac{\sum_{i=0}^{a} r'_i t^i}{(1-t)^{d_1+1}} + \frac{\sum_{i=0}^{b} r''_i t^{i+1}}{(1-t)^{d_2+r}}$$

We first show that $d_1+1=d_2+r=n-p$, where $n=|V(\mathcal{P})|$. Since $\mathcal{P}'$ is the polyomino having $n-2$ vertices and $p-1$ cells, then from Lemma 2.4.3 we have $(n-2)-(p-1)=n-p-1$. Moreover, since I is on the $2r+2$ vertices $\{x_1,\dots,x_r,x,y_1,\dots,y_r,y\}$ but y_k,y_{k+1} for some k are corners of one cell of $\mathcal{P}\setminus I$, then $\mathcal{P}''$ is the polyomino having $n-2r$ vertices and $p-r$ cells, hence from Lemma 2.4.3 $d_2+r-1=(n-2r)-(p-r)+r-1=n-p-1$. That is

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t)=\frac{1+\sum_{i=1}^{r(\mathcal{P})}(r'_i+r''_{i-1})t^i}{(1-t)^d}$$

For $1\leq i\leq r(\mathcal{P})$, $r_i=r'_i+r''_{i-1}$. In fact, r_i is the number of ways of placing i non-attacking rooks on all of the cells of $\mathcal{P}$, whereas r'_i is the number of ways of placing i non-attacking rooks on the simple thin polyomino $\mathcal{P}'$, namely the number of ways of placing i non-attacking rooks on the cells $D\neq C$ of $\mathcal{P}$, and r''_{i-1} is the number of ways of placing $i-1$ non-attacking rooks on the simple thin polyomino $\mathcal{P}''$, namely the number of ways of placing $i-1$ non-attacking rooks on the cells D of $\mathcal{P}$ such that $D\notin I$, given that the i-th rook is placed on the cell C, hence the thesis follows. $\square$

We immediately deduce the following

Corollary 4.2.15. *Let $\mathcal{P}$ be a simple thin polyomino. Then the Castelnuovo-Mumford regularity is $r(\mathcal{P})$ and the multiplicity of $\mathbb{K}[\mathcal{P}]$ is $r_{\mathcal{P}}(1)$.*

Remark 4.2.16 *In general the equality $h(t)=r_{\mathcal{P}}(t)$ does not hold for any simple polyomino $\mathcal{P}$. Let $\mathcal{P}$ be the square tetromino. Then, $\mathbb{K}[\mathcal{P}]$ is the toric ring related to the complete bipartite graph $K_{3,3}$ and from [83, Lemma 2.2] we have*

$$h(t)=1+4t+t^2 \text{ and } r_{\mathcal{P}}(t)=1+4t+2t^2.$$

Even though the two polynomials are different, they have the same degree, that is $\operatorname{reg}\mathbb{K}[\mathcal{P}]=r(\mathcal{P})$ *also in this case.*

4.2.2 Gorenstein simple thin polyominoes

In this subsection we characterize the Gorenstein simple thin polyominoes. We start with a fundamental definition for our goal.

Definition 4.2.17 *Let $\mathcal{P}$ be a simple thin polyomino. A cell C of $\mathcal{P}$ is* single *if there exists a unique maximal inner interval of $\mathcal{P}$ containing C. If any maximal inner interval of $\mathcal{P}$ has exactly one single cell, we say that $\mathcal{P}$ has the* S-property.

Let $\mathcal{C}$ be the set of the single cells of a simple thin polyomino. We set $\mathcal{D}$ as the collection of cells $\mathcal{P} \setminus \mathcal{C}$. In particular since $\mathcal{P}$ is thin, then any cell of $\mathcal{D}$ belongs exactly to two maximal inner intervals of $\mathcal{P}$.

Theorem 4.2.18. *Let $\mathcal{P}$ be a simple thin polyomino, $I_1, \ldots, I_s$ be its maximal inner intervals, and let $r_{\mathcal{P}}(t) = \sum_{k=0}^{s} r_k t^k$ be its rook polynomial. Then the following conditions are equivalent:*

(a) $\mathbb{K}[\mathcal{P}]$ is Gorenstein;

(b) $\forall i = 0, \ldots, s$ we have $r_i = r_{s-i}$;

(c) $\mathcal{P}$ satisfies the S-property.

Proof. (a)$\Leftrightarrow$(b): By combining Theorem 1.4.1 and Theorem 4.2.14, for a simple thin polyomino $\mathcal{P}$, the Cohen-Macaulay domain $\mathbb{K}[\mathcal{P}] = R/I_{\mathcal{P}}$ is Gorenstein if and only if $\forall i = 0, \ldots, s$ we have $r_i = r_{s-i}$, and the assertion follows.
(c)$\Rightarrow$(b): Since $\mathcal{P}$ satisfies the S-property, then any maximal inner interval I of $\mathcal{P}$ contains a unique single cell C. Therefore, let $\mathcal{C} = \{C_1, \ldots C_s\}$ be the set of the single cells of $\mathcal{P}$, and let $I_1, \ldots, I_s$ be the maximal inner intervals of $\mathcal{P}$ such that $C_i \in I_i$. We set $\mathcal{D} = \mathcal{P} \setminus \mathcal{C}$. As we have observed above, any cell of $\mathcal{D}$ is the intersection of two maximal inner intervals of $\mathcal{P}$, and we denote by D_{jk} the cell of $\mathcal{D}$ in the intersection of I_j and I_k.

Let $\mathbf{i}$ be a subset of $[s] = \{1, 2, \ldots, s\}$ of cardinality l, and let $\mathbf{jk} = \{\{j_1, k_1\}, \ldots, \{j_m, k_m\}\}$ with $j_t, k_t \in [s]$ for $1 \leq t \leq m$. We denote by $\mathcal{C}_{\mathbf{i}} = \{C_i \in \mathcal{C} : \ i \in \mathbf{i}\}$ and by $\mathcal{D}_{\mathbf{jk}} = \{D_{jk} \in \mathcal{D} : \{j, k\} \in \mathbf{jk}\}$.

Let $\mathbf{j} = \{j_1, \ldots j_m\}$ and $\mathbf{k} = \{k_1, \ldots k_m\}$ be such that $\mathbf{j} \cap \mathbf{k} = \varnothing$ and let $\mathbf{i}$ be such that $\mathbf{i} \cap (\mathbf{j} \sqcup \mathbf{k}) = \varnothing$ then

$$\mathcal{C}_{\mathbf{i}} \cup \mathcal{D}_{\mathbf{jk}} \tag{4.3}$$

induces a set of $d = l + m$ non-attacking rooks, and any set of non-attacking rooks of cardinality d can be written in the form (4.3), and this configuration is unique because a set $\mathbf{jk}$ identifies a unique subset of $\mathcal{D}$ and thanks to the S-property a set $\mathbf{i} \subset [\mathbf{s}]$ identifies a unique subset of $\mathcal{C}$. Our goal is to prove that for any configuration (4.3) of d non-attacking rooks there exists a unique configuration of the form (4.3) of $s-d$ non-attacking rooks. Let $\overline{\mathcal{C}}_{\mathbf{i}\cup\mathbf{i}\cup\mathbf{k}} = \mathcal{C} \setminus (\mathcal{C}_{\mathbf{i}} \cup \mathcal{C}_{\mathbf{j}} \cup \mathcal{C}_{\mathbf{k}})$, and since $\mathbf{i} \cap (\mathbf{j} \cup \mathbf{k}) = \varnothing$, then $|\overline{\mathcal{C}}_{\mathbf{i}\cup\mathbf{j}\cup\mathbf{k}}| = s-(l+2m)$. From the configuration of cardinality d in (4.3), we retrieve the following configuration of cardinality $s-d$,

$$\overline{\mathcal{C}}_{\mathbf{i}\cup\mathbf{j}\cup\mathbf{k}} \cup \mathcal{D}_{\mathbf{jk}}. \tag{4.4}$$

In fact, $s-(l+2m)+m = s-d$ and the configuration (4.4) satisfies the properties of configuration (4.3), and the configuration (4.4) is uniquely determined by (4.3) because $\mathcal{D}_{\mathbf{jk}}$ is fixed, and once we set $\mathcal{C}_{\mathbf{i}}$ and $\mathbf{j} \cup \mathbf{k}$, the complement set $\overline{\mathcal{C}}_{\mathbf{i}\cup\mathbf{j}\cup\mathbf{k}}$ is unique.

(b)⇒(c): By contraposition, assume that $\mathcal{P}$ does not satisfy the S-property, that is there exists an inner interval I of $\mathcal{P}$ having q single cells with $q \neq 1$. We want to prove that either $r_s > r_0 = 1$ or $r_{s-1} > r_1 = \operatorname{rank} \mathcal{P}$.

Let $q > 1$, and let C, C' be two single cells of I. Any set $\mathcal{C}$ of s non-attacking rooks contains a single cell C'' of I such that either $C'' \neq C$ or $C'' \neq C'$. In both cases the sets $\mathcal{C} \setminus \{C''\} \cup C$ and $\mathcal{C} \setminus \{C''\} \cup C'$ are two distinct sets of s non-attacking rooks, that is $r_s > 1$, and it is a contradiction.

Hence, from now on we assume that in $\mathcal{P}$ do not exist maximal inner intervals with two or more single cells. That is, any maximal inner interval of $\mathcal{P}$ has either 0 or 1 single cells and in particular we assume $q = 0$. Let $\mathcal{C}$ be a set of s non-attacking rooks of $\mathcal{P}$. In this case one of the following is true:

1. any inner interval J intersecting I in a cell D contains a cell $C \neq D$ such that $C \in \mathcal{C}$, in particular $I \cap \mathcal{C} = \varnothing$;
2. there exists an inner interval J intersecting I in a cell $D \in \mathcal{C}$.

In case (1), $(\mathcal{C} \setminus \{C\}) \cup \{D\}$ is a set of s non-attacking rooks different from $\mathcal{C}$, that is $r_s > 1$, and it is a contradiction.

In case (2), we want to show $r_{s-1} > r_1$. Let E be a cell of $\mathcal{P}$. If $E \in \mathcal{C}$, then $\mathcal{C} \setminus \{E\}$ is a set of $s-1$ non-attacking rooks. If $E \notin \mathcal{C}$, then E is not single, that is E is the intersection of two cell intervals I_1 and I_2. From the maximality of $\mathcal{C}$, there exist two cells

$F \in I_1$ and $G \in I_2$ with $F, G \in \mathcal{C}$, and $\mathcal{C} \setminus \{F, G\} \cup \{E\}$ is a set of $s-1$ non-attacking rooks. Hence $r_{s-1} \geq r_1$.

The hypothesis (2) implies that there exist some cells A, B, C_1, C_2 of $\mathcal{P}$ such that the polyomino $\mathcal{Q}$ in Figure 4.12 is a subpolyomino of $\mathcal{P}$ (up to rotations and reflections). In fact, without loss of generality assume that A is a cell of I and B is a cell of J. Since I has no single cells there exists an inner interval J' intersecting I in A. Moreover, if the cell B is single, then $B \in \mathcal{C}$ and this contradicts (2). Hence there exists an inner interval J'' intersecting J in B.

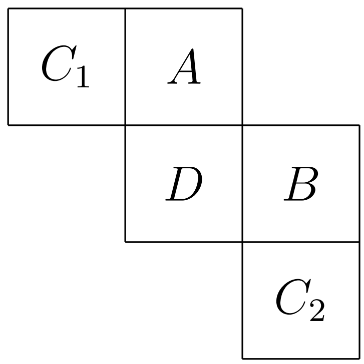


Figure 4.12: A simple thin polyomino $\mathcal{Q}$ that does not satisfy the S-property

Let F and G be the cells of $\mathcal{C}$ that belong to J' and J'', respectively. We consider the following sets of $s-1$ non-attacking rooks:

$$\mathcal{C} \setminus \{F, D\} \cup \{A\}, \mathcal{C} \setminus \{G, D\} \cup \{B\},\ \mathcal{C} \setminus \{F, G, D\} \cup \{A, B\}.$$

The first two were mentioned in the discussion above, while the third one increases the number r_{s-1}. Hence $r_{s-1} > r_1$, that is a contradiction.

□

Example 4.2.19 *Let $\mathcal{P}$ be the polyomino in Figure 4.13.*

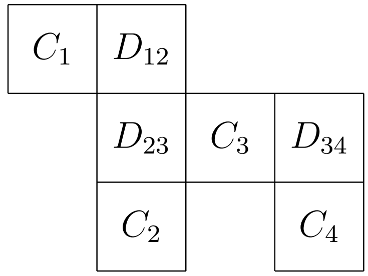


Figure 4.13: A simple thin polyomino satisfying the S-property

We see that $\mathcal{P}$ has 4 maximal inner intervals and a single cell for any of these ones, that is $\mathcal{P}$ satisfies the S-property. We want to compute the Hilbert series of $\mathbb{K}[\mathcal{P}]$. It is easy to see that $r(\mathcal{P}) = 4$. According to Theorem 4.2.14, the Hilbert series of $\mathbb{K}[\mathcal{P}]$ is

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{\sum_{i=0}^{4} r_i t^i}{(1-t)^d}$$

where $d = |V(\mathcal{P})| - \mathrm{rank}\ \mathcal{P} = 16 - 7 = 9$. We compute r_i, namely the number of sets of i non-attacking rooks for $i = 0, \dots, 4$.

- $i = 0$. $\varnothing$;
- $i = 1$. $\{C_1\}, \{C_2\}, \{C_3\}, \{C_4\}, \{D_{12}\}, \{D_{23}\}, \{D_{34}\}$;
- $i = 2$. $\{C_1, D_{23}\}, \{C_1, C_2\}, \{C_1, C_3\}, \{C_1, D_{34}\}, \{C_1, C_4\}, \{D_{12}, C_3\}$, $\{D_{12},\ D_{34}\}, \{D_{12}, C_4\}, \{C_2, C_3\}, \{C_2, D_{34}\}, \{C_2, C_4\}, \{D_{23}, C_4\}, \{C_3, C_4\}$;
- $i = 3$. $\{C_1, C_2, C_3\}, \{C_1, C_2, C_4\}, \{C_1, C_3, C_4\}, \{C_2, C_3, C_4\}, \{C_1, C_2, D_{34}\}$, $\{C_1, D_{23}, C_4\}$ $\{D_{12}, C_3, C_4\}$;
- $i = 4$. $\{C_1, C_2, C_3, C_4\}$.

It follows

$$r_0 = 1,\ r_1 = 7,\ r_2 = 13,\ r_3 = 7,\ r_4 = 1,$$

that is

$$\mathrm{HP}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{1 + 7t + 13t^2 + 7t^3 + t^4}{(1-t)^9}$$

and according to Theorem 1.4.1, $\mathbb{K}[\mathcal{P}]$ is Gorenstein.

Example 4.2.20 *In the notation of Theorem 4.2.18, we highlight that the condition $r_s = 1$ is not sufficient to guarantee that the polynomial has symmetric coefficients. In fact, let* *us consider the polyomino $\mathcal{Q}$ in Figure 4.12. The rook number of $\mathcal{Q}$ is 3 and the rook polynomial of $\mathcal{Q}$ is*

$$1 + 5t + 6t^2 + t^3,$$

in fact, the sets of i non-attacking rooks are

- $i = 0$. $\varnothing$;
- $i = 1$. $\{A\}, \{B\}, \{C_1\}, \{D\}, \{C_2\}$;
- $i = 2$. $\{C_1, D\}, \{C_1, C_2\}, \{D, C_2\}, \{B, C_1\}, \{A, C_2\}, \{A, B\}$;
- $i = 3$. $\{C_1, D, C_2\}$;

As already noted in the proof of Theorem 4.2.18 the fact that $r_2 > r_1$ depends on the set $\{A, B\}$.

To conclude the Section, we want to remark that among the thin polyominoes that are not simple, namely multiply-connected, there are some non-prime ones, so that we can not directly retrieve the Cohen-Macaulayness of $\mathbb{K}[\mathcal{P}]$. Nevertheless, due to Theorem 4.2.14 and Remark 4.2.16, we conjecture the following.

Conjecture 4.2.21 *Let $\mathcal{P}$ be a polyomino. Then $\mathcal{P}$ is thin if and only if $r_{\mathcal{P}}(t) = h(t)$.*

Moreover, due to Theorem 4.2.14 and Theorem 4.1.8, we ask the following.

Question 4.2.22 *Let $\mathcal{P}$ be a polyomino. Then* $\operatorname{reg} \mathbb{K}[\mathcal{P}] = r(\mathcal{P})$*?*

4.3 HILBERT SERIES OF PARALLELOGRAM POLYOMINOES

In this Section, we make a further step in the study of the Hilbert series and Castelnuovo-Mumford regularity, as done in Sections 4.1 and 4.2. We focus on parallelogram polyominoes (see 2.3. In Subsection 4.3.1, the relationship between polyominoes and distributive lattices is explored. In particular, in Proposition 4.3.2, we prove that the parallelogram polyominoes are simple planar distributive lattices. In particular, the join-meet ideal and polyomino ideal of a simple planar distributive lattice is the same, see Remark 4.3.3. This identification allows us to use the existing knowledge on Hibi rings arising from simple planar distributive lattices and translate it in terms of their structure as coordinate rings of parallelogram polyominoes. Moreover, from Section 4.1, we know that the polyomino ideals of L-convex polyominoes can be interpreted as polyomino ideals of certain Ferrer diagrams. The Ferrer diagrams are a special subclass of parallelogram polyominoes. Therefore, the results provided in subsequent sections hold for L-convex polyominoes, which in general, do not have a structure of a simple planar distributive lattice.

In Subsection 4.3.2, we study the Hilbert series of parallelogram polyominoes. In Sections 4.1 and 4.2, we linked the Castelnuovo-Mumford regularity of $\mathbb{K}[\mathcal{P}]$ to the maximum number of non-attacking rooks that can be placed on the polyomino, for the classes of L-convex polyominoes and simple thin polyominoes. In particular, Theorem 4.2.14 motivates us to study the relation between the Hilbert series and the rook polynomial for simple non-thin polyominoes. Recently, another paper in this direction has been written by Kummini and Veer [49]. In Subection 4.3.2, we introduce an equivalence relation on the rook complex of a simple polyomino $\mathcal{P}$. We conjecture that the number of equivalence classes of k non-attacking rooks arrangements coincides with the coefficient h_k of the polynomial $h(t)$ in the reduced Hilbert series. We prove that Conjecture 4.3.5 holds true for the class of parallelogram polyominoes. Moreover, by using a computational approach, we prove that Conjecture 4.3.5 holds true for any simple polyomino having at most 11 cells. In [64], we provide an implementation in `Macaulay2` [33] and Java for such computations.

Even though the Gorenstein ladder determinantal rings and the Gorenstein Hibi rings are completely characterized, in subsection 4.3.3 we give a combinatorial characterization of Gorenstein parallelogram polyominoes that is analogous to the characterizations given in Sections 4.1 and 4.2 for L-convex and simple thin polyominoes, respectively. Such characterization involves the intersections of the maximal rectangles of the parallelogram polyominoes. It is well-known that every parallelogram polyomino can be uniquely represented as a Motzkin path (see Section 2.3). In Corollary 4.3.29, we give a characterization of the Motzkin paths which represent Gorenstein parallelogram polyominoes.

4.3.1 The relationship between polyominoes and distributive lattices

In this Subsection, we talk about the polyominoes arising from simple planar distributive lattices, already introduced in Section 1.5. Note that any simple planar distributive lattice L can be identified as a convex polyomino. Moreover, we have the following

Proposition 4.3.1. *Let $\mathcal{P}$ be a convex polyomino with bounding box* $[(0,0),(m,n)]$. *If* $(0,0),(m,n) \in V(\mathcal{P})$*, then $V(\mathcal{P})$ determines a simple planar distributive lattice.*

Proof. First we show that $V(\mathcal{P})$ is a sublattice of $\mathbb{N}^2$. Let $a, b \in V(\mathcal{P})$ be two incomparable elements. We need to show that $a \vee b$ and $a \wedge b$ belong to $V(\mathcal{P})$. Let $a = (i,j)$ and $b = (k,l)$. Since a and b are incomparable, we may assume that $i < k$ and $j > l$. Then $a \vee b = (k,j)$ and $a \wedge b = (i,l)$. First we claim that $a \wedge b = (i,l) \in V(\mathcal{P})$. On the contrary, suppose that $a \wedge b = (i,l) \notin V(\mathcal{P})$. By using the convexity of $\mathcal{P}$ and applying [62, Lemma 1.1], it follows that $(i,p),(q,l) \notin V(\mathcal{P})$, for any $p \leq l$ and $q \leq i$. Since, $\mathcal{P}$ is a polyomino, and hence connected, there must exist a path in $V(\mathcal{P})$ from $(0,0)$ to (i,j). However, any possible path in $V(\mathcal{P})$ from $(0,0)$ to (i,j) must either contain a vertex (i,p) with $p \leq l$ or a vertex (q,l) with $q \leq i$, a contradiction. This yields $a \wedge b = (i,l) \in V(\mathcal{P})$. A similar argument can be applied to conclude that $a \vee b = (k,j) \in V(\mathcal{P})$. Moreover, the assertion that $\mathcal{P}$ is simple and planar as a distributive lattice, follows directly from the definition of polyominoes. □

If a polyomino $\mathcal{P}$ admits a structure of a distributive lattice on $V(\mathcal{P})$, then instead of $V(\mathcal{P})$, we refer to $\mathcal{P}$ as a distributive lattice. One can observe that every parallelogram polyomino $\mathcal{P}$ (see Section 2.3) is a simple planar distributive lattice, as shown in the following proposition.

Proposition 4.3.2. *A finite collection of cells $\mathcal{P}$ is a parallelogram polyomino if and only if $\mathcal{P}$ is a simple planar distributive lattice.*

Proof. Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ be a parallelogram polyomino. By a translation, we may assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ meet at $(0,0)$ and (m,n). The definition of parallelogram polyomino together with Proposition 4.3.1 yields that $\mathcal{P}$ is a simple planar distributive lattice.

To show the converse, assume that $\mathcal{P}$ is a simple planar distributive lattice with bounding box $[(0,0),(m,n)]$. It follows from the definition of simple planar distributive lattice that $\mathcal{P}$ is convex polyomino. Note that rank $(\mathcal{P}) = m+n$ as a lattice. Let $\mathfrak{m}_0$: $x_0 < x_1 < \cdots < x_{m+n-1} < x_{m+n}$ be the maximal chain of $\mathcal{P}$ with $x_t = (i_t, j_t)$ for all $0 \leq t \leq m+n$ satisfying the following property: $(i_0, j_0) = (0,0)$, $(i_{m+n}, j_{m+n}) = (m,n)$, and for any $(k, \ell) \in V(\mathcal{P})$ with $k = i_t$ for some t, if $l \geq j_t$ then $(k, l) = (i_s, j_s)$ for some $s \geq t$. We call such an $\mathfrak{m}_0$ the uppermost chain of $\mathcal{P}$. Similarly, one can define the lower most chain $\mathfrak{m}'_0$ of $\mathcal{P}$. Then it can be easily seen that $\mathcal{P}$ is a parallelogram polyomino determined by $(\mathfrak{m}_0, \mathfrak{m}'_0)$. □

The following remark plays a vital role in subsequent text.

Remark 4.3.3 *Let L be a simple planar distributive lattice and $a, b \in L$ be two incomparable elements in L. Let $c = a \vee b$ and $d = a \wedge b$. Then a and b determine an inner interval in L with diagonal corners c and d and antidiagonal corners a and b. Therefore, a typical generator $f_{ab} = x_a x_b - x_{a\vee b} x_{a\wedge b} = x_a x_b - x_c x_d$ of the join-meet ideal of L is also an inner 2-minor of L. Similarly, any inner 2-minor of L can be interpreted as a relation arising from two incomparable elements and their meet and join. This shows that the join-meet ideal and polyomino ideals of L coincide.*

4.3.2 Hilbert series and rook complex of simple polyominoes

In this subsection, we give a conjecture about the Hilbert series of the coordinate ring of simple polyominoes in terms of some rook arrangements on their cells.

We already mentioned at the beginning of the chapter that the set of the configurations of pairwise non-attacking rooks is a simplicial complex. We now give a detailed explanation. We observe that for any cell $C \in \mathcal{P}$, the set $\{C\}$ is a 1 non-attacking rook. Moreover, for any set of non-attacking rooks $F \subset \mathcal{P}$, the subset $G \subset F$ is also a set of non-attacking rooks. This yields that the set $\mathcal{R}$ of sets of non-attacking rooks is a simplicial complex and

$$\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_{r(\mathcal{P})},$$

where for any $k = 0, \ldots, r(\mathcal{P})$, $\mathcal{R}_k$ contains the sets of k non-attacking rooks, with $\mathcal{R}_0 = \varnothing$. Set $r_k = |\mathcal{R}_k|$. Next, we introduce an equivalence relation on the set $\mathcal{R}_k$ for $2 \leq k \leq r(\mathcal{P})$. For this aim, we define the following.

Definition 4.3.4 *Two non-attacking rooks R_1 and R_2 of $\mathcal{P}$ are* switching rooks *if they are diagonal (resp. antidiagonal) cells of a rectangle of $\mathcal{P}$. Let R'_1 and R'_2 be the antidiagonal (resp. diagonal). Observe that if $F \in \mathcal{R}$ and $R_1, R_2 \in F$ are switching rooks, then the set*

$F' \setminus \{R_1, R_2\} \cup \{R'_1, R'_2\} \in \mathcal{R}$. The replacement of R_1 and R_2 by R'_1 and R'_2 is called switch *of R_1 and R_2.*

There exists a natural equivalence relation $\sim$ on $\mathcal{R}_k$ given as: $F_1, F_2 \in \mathcal{R}_k$ are equivalent *if one can obtain F_2 from F_1 after some switches. We define the quotient set*

$$\widetilde{\mathcal{R}}_k = \mathcal{R}_k / \sim .$$

We observe that the rook number $r(\mathcal{P})$ does not change. We define the polynomial

$$\widetilde{r}_{\mathcal{P}}(t) = \sum_{k=0}^{r(\mathcal{P})} |\widetilde{\mathcal{R}}_k| t^k.$$

With the notation introduced above, we state the following:

Conjecture 4.3.5 *Let $\mathcal{P}$ a simple polyomino. Then $h(t) = \widetilde{r}_{\mathcal{P}}(t)$.*

The following example depicts the construction of a rook complex $\mathcal{R}$ of a polyomino and the quotient set $\widetilde{\mathcal{R}} := \mathcal{R}/\sim$.

Example 4.3.6 *We describe $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ for the simple polyomino $\mathcal{P}$ in Figure 4.14. The polyomino $\mathcal{P}$ consists of seven cells labelled as A, B, C, D, E, F, G and $r(\mathcal{P}) = 3$. The rook complex $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ of $\mathcal{P}$ is given below.*

$\mathcal{R}_0 = \{\varnothing\}$

$\mathcal{R}_1 = \{ \{A\}, \{B\}, \{C\}, \{D\}, \{E\}, \{F\}, \{G\}\}$

$\mathcal{R}_2 = \{\{A, D\}, \{A, E\}, \{A, G\}, \{B, C\}, \{B, E\}, \{B, F\}, \{B, G\}, \{C, G\}, \{D, F\}, \{D, G\}, \{E, F\}, \{F, G\}\}$

$\mathcal{R}_3 = \{\{A, D, G\}, \{B, C, G\}, \{B, E, F\}, \{B, F, G\}, \{D, F, G\}\}$.

This gives,

$$r_{\mathcal{P}}(t) = 1 + 7t + 12t^2 + 5t^3$$

We observe that A and D are switching rooks and they can be switched with B and C. Then

$$\{A, D\} \sim \{B, C\},\ \{A, D, G\} \sim \{B, C, G\}$$

and

$$\widetilde{r}_{\mathcal{P}}(t) = 1 + 7t + 11t^2 + 4t^3.$$

By using `Macaulay2`*, one can see that* $h(t) = \widetilde{r}_{\mathcal{P}}(t)$.

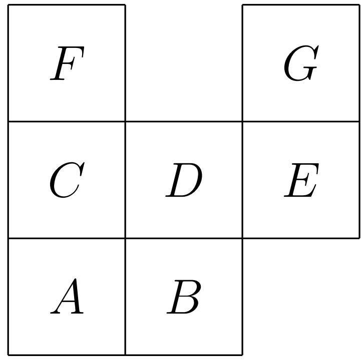


Figure 4.14: A simple polyomino

As proved in Section 4.2, Conjecture 4.3.5 holds true for the class of simple thin polyominoes. In fact, by Definition 3.2.7, a thin polyomino does not contain a square tetromino as a subpolyomino. Therefore, a simple thin polyomino $\mathcal{P}$ does not contain switching rooks and $\widetilde{r}_{\mathcal{P}}(t) = r_{\mathcal{P}}(t) = h(t)$. Moreover, by computational approach we obtain the following

Theorem 4.3.7. *Let* $\mathcal{P}$ *be a simple polyomino with* $\operatorname{rank} \mathcal{P} \leq 11$*. Then* $h(t) = \widetilde{r}_{\mathcal{P}}(t)$*.*

Proof. To prove the claim we have implemented a computer program that, for a fixed number n, performs the following steps:

(S1) compute the set of all the simple polyominoes of rank n;

(S2) for any polyomino in (S1) compute the polynomial $h(t)$;

(S3) for any polyomino in (S1) compute the polynomial $\widetilde{r}_{\mathcal{P}}(t)$;

(S4) check whether the polynomial from step (S2) is equal to the polynomial from step (S3).

In particular, for step (S1) we slightly modified the implementation given in [55]. For step (S2) we used the `Macaulay2` functions for the Hilbert series. For step (S3), we constructed the rook complex $\mathcal{R}$ as the independence complex of the graph G with $V(G) = \{C : C \in \mathcal{P}\}$ and

$$E(G) = \{\{C, D\} : \text{the cells } C \text{ and } D \text{ lie on the same row or column}\},$$

and then, by introducing the equivalence relation, we constructed $\widetilde{\mathcal{R}}$. Finally we refer the reader to [64] for a complete description of the algorithm that we used. $\square$

Now we will prove Conjecture 4.3.5 for parallelogram polyominoes.

Theorem 4.3.8. *Let $\mathcal{P}$ a parallelogram polyomino. Then $h(t) = \widetilde{r}_{\mathcal{P}}(t)$.*

From Proposition 4.3.2, we know that a parallelogram polyomino can be seen as a simple planar distributive lattice. Furthermore, Remark 4.3.3 shows that the join-meet ideal and polyomino ideal of a simple planar distributive lattice coincides. To achieve our aim, we will first recall some notions related to simple planar distributive lattices and their Hilbert series.

Let $x, y \in L$ such that y covers x, that is, $x < y$ and there is no $z \in L$ such that $x < z < y$. Then the edge between x and y in the Hasse diagram of L can be represented by $x \to y$. Recall from [5] that an edge-labeling λ of L is an integer labelling of the edges in Hasse diagram of L. Each chain in L, say $\mathfrak{c} : x_0 \to x_1 \to x_2 \to \ldots \to x_k$ can be labelled by a k-tuple $\lambda(\mathfrak{c}) = (\lambda(x_0 \to x_1), \lambda(x_1 \to x_2), \ldots, \lambda(x_{k-1} \to x_k))$. One can compare two such k-tuples $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_k)$ lexicographically, that is, $(a_1, \ldots, a_k) <_{\text{lex}} (b_1, \ldots, b_k)$, if the most-left nonzero component of the vector $(a_1 - b_1, \ldots, a_k - b_k)$ is positive.

Definition 4.3.9 *[5, Definition 2.1] An edge labelling λ of L is called EL-labelling if for every interval $[x, y]$ in L, λ satisfies the following:*

(i) there is a unique chain $\mathfrak{c} : x = x_0 \to x_1 \to x_2 \to \ldots \to x_k = y$ such hat $\lambda(x_0 \to x_1) \leq \lambda(x_1 \to x_2) \leq \ldots \leq \lambda(x_{k-1} \to x_k))$.

(ii) for every other chain $\mathfrak{b} : x = y_0 \to y_1 \to y_2 \to \ldots \to y_k = y$, we have $\lambda(\mathfrak{b}) >_{\text{lex}} \lambda(\mathfrak{c})$.

In Figure 4.16, we give an illustration of *EL*-labeling λ.

Let rank $(L) = d + 1$. Then for each maximal chain $\mathfrak{m} : \min L = x_0 \to x_1 \to x_2 \to \ldots \to x_{d+1} = \max(L)$, the *descent set* of $\mathfrak{m}$ is $D(\mathfrak{m}) = \{i : \lambda(x_{i-1} \to x_i) > \lambda(x_i \to x_{i+1})\}$. Then by following [5, Theorem 2.2], for any $S \subset [d]$, we set $\beta(S)$ to be the number of maximal chains $\mathfrak{m}$ in L such that $D(\mathfrak{m}) = S$. It is known from [5], that

$$\mathrm{HS}_{\mathbb{K}[L]}(t) = \frac{h(t)}{(1-t)^{d+2}}$$

where

$$h(t) = \sum_{S \subset [d]} \beta(S) t^{|S|}.$$

Our main goal is to interpret $h(t)$ in terms of $\widetilde{r}_{\mathcal{P}}(t)$.

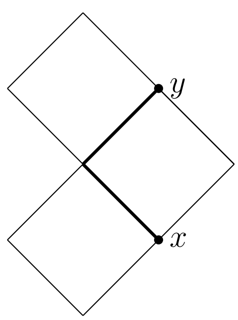
In the following, we recall the definition of uppermost chain (already used in the proof of Proposition 4.3.2), adding a nice EL-labelling.

Definition 4.3.10 *Let L be a simple planar distributive lattice of rank $d+1$.*

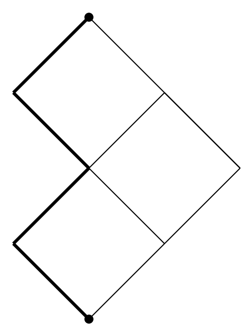
(i) Let $\mathfrak{m}_0 : x_0 < x_1 < \cdots < x_d < x_{d+1}$ be the maximal chain of L with $x_t = (i_t, j_t)$ for all $0 \le t \le d+1$ satisfying the following property: $(i_0, j_0) = (0,0)$, $(i_{d+1}, j_{d+1}) = \max L$, and for any $(k, \ell) \in L$ with $k = i_t$ for some t, if $\ell \ge j_t$ then $(k, \ell) = (i_s, j_s)$ for some $s \ge t$. We call such an $\mathfrak{m}_0$ the uppermost chain of L. We label the edges of $\mathfrak{m}_0$ by $\lambda(x_t \to x_{t+1}) = t+1$ for $0 \le t \le d$.

(ii) Let $x, y \in L$ such that $x < y$. Then the uppermost chain *from x to y in L is the uppermost chain of the sublattice $L \cap [x, y]$.*

Figure 4.15.(I) illustrates an example of an uppermost chain between two elements x, y of L, while Figure 4.15.(II) illustrates an example of an uppermost chain of a lattice L. The uppermost chains are indicated by thick lines.



(A) The uppermost chain between x and y.



(B) The uppermost chain of L

Figure 4.15: Two examples of uppermost chains

In [28], the following *EL*-labelling is defined for simple planar distributive lattices.

Definition 4.3.11 *We label all the edges in the Hasse diagram of L as follows. If $i_{t+1} = i_t + 1$, in other words if $x_t \to x_{t+1}$, is a horizontal edge, then we label by $t+1$ all the edges* *of L of the form $(i_t, j) \to (i_{t+1}, j)$. If $j_{t+1} = j_t + 1$, that is, if $x_t \to x_{t+1}$ is a vertical edge, then we label by $t+1$ all the edges of L of the form $(i, j_t) \to (i, j_{t+1})$. In [28, Proposition 6], it is shown that λ is an* EL*-labelling.*

In Figure 4.16, we use Definition 4.3.11 for the *EL*-labeling λ. The chain marked with thick line is the uppermost chain of L.

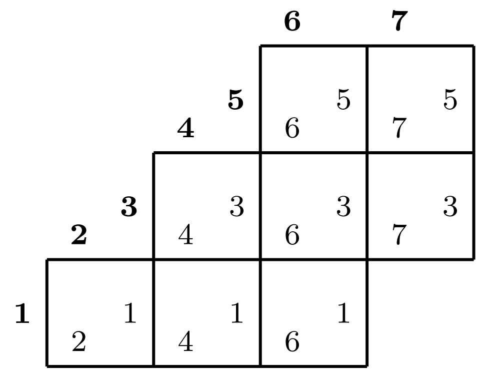


Figure 4.16: The EL-labelling for a parallelogram polyomino

Throughout the following text, we will follow the *EL*-labelling given in Definition 4.3.10. The following remarks are immediate consequences of Definition 4.3.10 (see Figure 4.16).

Remark 4.3.12 *(i) Let $(i, j) \to (i, j+1)$ and $(k, l) \to (k, l+1)$ be two edges in L with $i \leq k$ and $j+1 \leq l$. Then $\lambda((i, j) \to (i, j+1)) < \lambda((k, l) \to (k, l+1))$.*

(ii) Let $(i, j) \to (i+1, j)$ and $(k, l) \to (k+1, l)$ be two edges in L with $i+1 \leq k$ and $j \leq l$. Then $\lambda((i, j) \to (i+1, j)) < \lambda((k, l) \to (k+1, l))$.

(iii) Let $(i, j) \to (i, j+1)$ and $(k, l) \to (k+1, l)$ be two edges in L with $i \leq k$ and $j+1 \leq l$. Then $\lambda((i, j) \to (i, j+1)) < \lambda((k, l) \to (k+1, l))$.

(iv) Let $(i, j) \to (i+1, j)$ and $(i+1, j) \to (i+1, j+1)$ be two edges in L. If $(i, j+1) \notin L$, then $(i, j) \to (i+1, j)$ and $(i+1, j) \to (i+1, j+1)$ appear in the uppermost chain of L and $\lambda((i, j) \to (i+1, j)) < \lambda((i+1, j) \to (i+1, j+1))$. However, if $(i, j+1) \in L$, then $(i, j) \to (i, j+1)$ and $(i, j+1) \to (i+1, j+1)$ are edges in L. Moreover, due to (3) we have $\lambda((i, j) \to (i, j+1)) < \lambda((i, j+1) \to (i+1, j+1))$. Following the

Definition 4.3.10, we have

$$\lambda((i,j) \to (i,j+1)) = \lambda((i+1,j) \to (i+1,j+1))$$

and

$$\lambda((i,j+1) \to (i+1,j+1)) = \lambda((i,j) \to (i+1,j))$$

which gives

$$\lambda((i,j) \to (i+1,j)) > \lambda((i+1,j) \to (i+1,j+1))$$

(v) *From (1)–(4), we can compute the descent set of a maximal chain $\mathfrak{m}$ in L. If $\mathfrak{m}$ contains edges of the form $(i,j) \to (i+1,j)$ and $(i+1,j) \to (i+1,j+1)$ and $(i,j+1) \in L$, then we have a descent at $(i+1,j)$.*

(vi) *Let $x = (i,j), y = (p,q) \in L$ with $i < p$ and $j < q$ and let $\mathfrak{c} : x = x_0 < x_1 < \ldots < x_l = y$ be the uppermost chain between x and y. It follows that if $(i,j+1) \in L$, then $x_1 = (i,j+1)$. That is, in $\mathfrak{c}$ there are no descents. Similarly one proves that in an uppermost chain there are no descents.*

The following definition is needed for the Proposition 4.3.14.

Definition 4.3.13 *Let $C = [(i,j),(i+1,j+1)]$ be a cell in a simple planar distributive lattice L. Then the lower left corner (i,j) of C is denoted by $l(C)$. Given any maximal chain $\mathfrak{m}$ in a simple planar distributive lattice L, we say that $\mathfrak{m}$ has a descent at cell C if $\mathfrak{m}$ passes through the edges $(i,j) \to (i+1,j)$ and $(i+1,j) \to (i+1,j+1)$.*

Proposition 4.3.14. *Let L be a simple planar distributive lattice. Then the following are equivalent.*

1. *There exists a maximal chain $\mathfrak{m}$ in L with $|D(\mathfrak{m})| = r$.*
2. *There exists C_1, C_2, $\ldots$, C_r cells of L with $l(C_k) = (i_k, j_k)$ for $1 \le k \le r$*

$$i_1 < i_2 < \ldots < i_r \text{ and } j_1 < j_2 < \ldots < j_r.$$

Observe that the chain $\mathfrak{m}$ has descents exactly at $C_1, \ldots C_r$.

Proof. (1) $\Rightarrow$ (2) Let $\mathfrak{m} : x_0 < x_1 < \cdots < x_d < x_{d+1}$ be a maximal chain with descent set $D(\mathfrak{m}) = \{l_1, \ldots, l_r\}$ with $l_1 < \ldots < l_r$. Then $x_{l_1} < x_{l_2} < \ldots < x_{l_r}$. From Remark 4.3.12.(5)

and Definition 4.3.13, for any $i \in \{1, \dots, r\}$ there exists a cell C_i such that lower right corner and the lower left corner of C_i are x_{l_i} and $x_{l_i - 1}$, respectively. We now prove that for any $k = 1, \dots, r-1$ we have $i_k < i_{k+1}$ and $j_k < j_{k+1}$. From the fact that, $x_{l_k - 1} < x_{l_{k+1}-1}$ it follows that $i_k \leq i_{k+1}$. By contraposition, assume that $i_k = i_{k+1}$ for some k, then we have that $x_{l_k - 1} = (i_k, j_k)$, $x_{l_k} = (i_k + 1, j_k)$ and $x_{l_{k+1}-1} = (i_k, j_{k+1})$. That is, $x_{l_k} \not< x_{l_{k+1}-1}$, hence $\mathfrak{m}$ is not a chain and this is a contradiction. Hence $i_k < i_{k+1}$ and similarly $j_k < j_{k+1}$.

(2) $\Rightarrow$ (1) For a cell C in L, let $r(C)$ be its lower right corner. Let $\mathfrak{c}_0$ be the uppermost chain between $(0,0)$ and (i_1, j_1) and for $1 \leq k \leq r-1$ let $\mathfrak{c}_k$ be the uppermost chain between $r(C_k) = (i_k + 1, j_k)$ and $l(C_{k+1}) = (i_{k+1}, j_{k+1})$ and let $\mathfrak{c}_r$ be the uppermost chain between $r(C_r)$ and (m, n). From the concatenation of $\mathfrak{c}_0\mathfrak{c}_1 \dots \mathfrak{c}_r$, we obtain in a natural way a maximal chain $\mathfrak{m}$ of L. We prove that $\mathfrak{m}$ has descent at $C_1, C_2, \dots, C_r$. We fix $k \in \{1, \dots, r\}$. Since $(i_k, j_k) \to (i_k + 1, j_k) \in E(\mathfrak{m})$, it is sufficient to prove $(i_k + 1, j_k) \to (i_k + 1, j_k + 1) \in E(\mathfrak{m})$. The assertion follows from the inequalities on $i_1 < \dots < i_r$ and $j_1 < \dots < j_r$ and Remark 4.3.12.(6) applied to the uppermost chain $\mathfrak{c}_{k+1}$. Therefore, $\mathfrak{m}$ has descent at C_k. This completes the proof. $\square$

In order to prove Theorem 4.3.8, we premise the following lemma which shows that given any set of non-attacking rooks in a parallelogram polyomino, one can find an equivalent set of non-attacking rooks whose lower left corners appear in a chain.

Lemma 4.3.15. *Let $\mathcal{P}$ be a parallelogram polyomino and let $F = \{A_1, \dots, A_d\} \in \mathcal{R}$. Then there exists $G = \{B_1, \dots, B_d\} \in \mathcal{R}$ with $l(B_k) = (i_k, j_k)$ for $1 \leq k \leq d$ such that*

$$i_1 < i_2 < \dots < i_d \text{ and } j_1 < j_2 < \dots < j_d$$

and $F \sim G$.

Proof. Let $l(A_i) = (x_i, y_i)$ for $i = 1, \dots, d$. We prove the assertion by applying induction on d.

Let $d = 2$ and assume that $\mathcal{A} = \{A_1, A_2\}$ is labelled such that $x_1 < x_2$. If $y_1 < y_2$ then the statement holds trivially. If $y_1 > y_2$, then by using the assumption that $\mathcal{P}$ is a parallelogram polyomino and hence a distributive lattice, we conclude that the join $b_1 = (x_2, y_1)$ and the meet $b_2 = (x_1, y_2)$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ belong to $V(\mathcal{P})$. In particular, b_1

and b_2 are lower left corners of some cells of $\mathcal{P}$ and A_1 and A_2 are antidiagonal cells of a rectangle of $\mathcal{P}$. Let B_1 and B_2 be the cells having lower left corners b_1 and b_2, respectively. It follows that the set $\{B_1, B_2\}$ satisfies the assertion. Now, let $d > 2$ and assume that the assertion is true for all of the sets containing $d-1$ non-attacking rooks. We label the elements of $\mathcal{A}$ in a way such that $x_1 < x_2 < \ldots < x_d$. Let $k \in \{1, \ldots, d\}$ such that $y_k < y_i$ for any $i \neq k$. If $k = 1$, then we set $B_1 = A_1$ and we apply the inductive hypothesis on the set $\{A_2, \ldots, A_d\}$ to get the desired result.
If $k > 1$, then let B_1 and C_k be the cells whose lower left corners are respectively (x_1, y_k) and (x_k, y_1). Then $\{A_2, \ldots, A_{k-1}, C_k, A_{k+1}, \ldots, A_d\}$ is a set of $d-1$ non-attacking rooks and by applying the inductive hypothesis the assertion follows. □

Now we state the proof of Theorem 4.3.8.

Proof of Theorem 4.3.8. Let

$$\mathrm{HS}_{\mathbb{K}[\mathcal{P}]}(t) = \frac{\sum_k h_k t^k}{(1-t)^{\dim \mathbb{K}[\mathcal{P}]}}.$$

We show that for any k one has $\tilde{r}_k = h_k$. For $k = 0, 1$ one has $\tilde{r}_k = h_k$.
For $k \geq 2$, by Proposition 4.3.14 the maximal chains with descent set of cardinality k in $\mathcal{P}$ seen as a planar distributive lattice are in bijection with the sets F of non-attacking rooks $B_1, \ldots, B_k$ with $l(B_\ell) = (i_\ell, j_\ell)$ for $1 \leq \ell \leq k$ such that $i_1 < i_2 < \ldots < i_k$ and $j_1 < j_2 < \ldots < j_k$. Thanks to Lemma 4.3.15, such sets F are the representatives of the equivalence classes of $\mathcal{R}_k/\sim$, that is $\tilde{r}_k = h_k$. □

As a consequence of Theorem 4.3.8, we observe that the Conjecture 4.3.5 holds for L-convex polyominoes, too. As stated in Section 4.1, the coordinate ring of an L-convex polyomino is isomorphic to the coordinate ring of a suitable Ferrer diagram. Then the conclusion follows from the fact that every Ferrer diagram is a particular parallelogram polyomino. We also note that the Hilbert series of Ferrer diagram was given in [20].

From Lemma 2.4.3, it follows that for a parallelogram polyomino $\mathcal{P}$ the coordinate ring $\mathbb{K}[\mathcal{P}]$ is a Cohen-Macaulay domain. Furthermore, from $\deg h(t) = \operatorname{reg} \mathbb{K}[\mathcal{P}]$, we obtain the following corollary of Theorem 4.3.8.

Corollary 4.3.16. *Let $\mathcal{P}$ be a parallelogram polyomino. Then* $\operatorname{reg} \mathbb{K}[\mathcal{P}] = r(\mathcal{P})$.

4.3.3 Gorenstein parallelogram polyominoes

Given a polyomino $\mathcal{P}$, we call $\mathcal{P}$ Gorenstein if $\mathbb{K}[\mathcal{P}]$ is Gorenstein. In this Subsection we discuss the Gorenstein parallelogram polyominoes. Although the Gorenstein distributive lattices are completely characterized in [42], we plan to give a combinatorial interpretation of the Gorenstein parallelogram polyominoes in the language of polyominoes. Our aim is to compare the conditions on a parallelogram polyomino to be Gorenstein with the conditions found in Section 4.1 for L-convex polyominoes and in Section 4.2 for simple thin polyominoes.

Let $\mathcal{M}(\mathcal{P})$ be the set of the maximal rectangles of $\mathcal{P}$. We generalize Definition 4.2.17 with the following.

Definition 4.3.17 *Let S be a rectangular (resp. square) subpolyomino of a parallelogram polyomino $\mathcal{P}$. Then S is said to be* single *if there exists a unique maximal rectangle $R \in \mathcal{M}(\mathcal{P})$ such that $S \subseteq R$ and $S \cap R' = \varnothing$ for all $R' \in \mathcal{M}(\mathcal{P})$ with $R' \neq R$. We say that $\mathcal{P}$ has the S-*property *if each maximal rectangle R of $\mathcal{P}$ has a unique single square.*

To see an illustration of the above definition, consider the parallelogram polyomino $\mathcal{P}$ given in Figure 4.17.(i). $\mathcal{P}$ has six maximal rectangles

$$\{A,B\},\{B,C,E\},\{C,D,E,F,\},\{D,F,H\},\{E,F,G\},\{F,G,H,I\}$$

The maximal rectangle $\{A,B\}$ has A as its single square, and the maximal rectangle $\{F,G,H,I\}$ has I as its single square. However, other rectangles do not have a single square or a single rectangle because each of their cells belong to other rectangles as well. The maximal rectangles $\{A,B\}$ and $\{F,G,H,I\}$ are special in a sense that $\{A,B\}$ is the unique maximal rectangle containing $\min(\mathcal{P})$ (as a distributive lattice) and $\{F,G,H,I\}$ is the unique maximal rectangle containing $\max(\mathcal{P})$.

Next, we prove that if a maximal rectangle R in a parallelogram polyomino $\mathcal{P}$ contains either $\min(\mathcal{P})$ or $\max(\mathcal{P})$ then R must contain a single rectangle. Given a parallelogram polyomino $\mathcal{P}$, we set $\min(\mathcal{P}) = (0,0)$ throughout the following text.

Lemma 4.3.18. *Let $\mathcal{P}$ be a parallelogram polyomino. Then there exists a unique $R \in$*

$\mathcal{M}(\mathcal{P})$ *such that* $(0,0) \in V(R)$. *In particular, the maximal rectangle* R *has a single rectangle.*

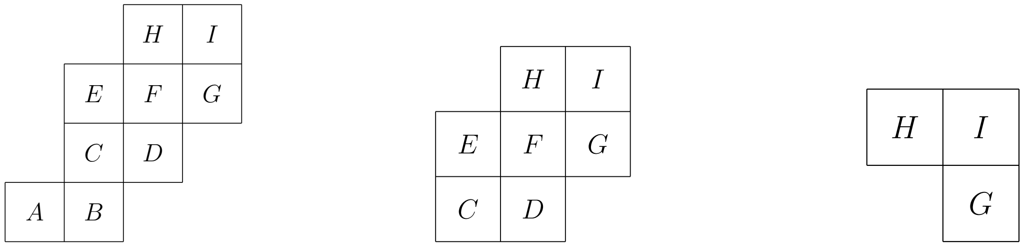
Proof. By contraposition, assume that there are two distinct maximal rectangles R, R' of $\mathcal{P}$, such that $(0,0) \in V(R) \cap V(R')$. Let $a, b, c, d \in V(\mathcal{P})$ be such that $V(R) = [(0,0),(a,b)]$ and $V(R') = [(0,0),(c,d)]$. Since R and R' are distinct, without loss of generality, we may assume that $a < c$ and $b > d$. From Proposition 4.3.2, it follows that $\mathcal{P}$ is a simple planar distributive lattice. Therefore, $(c,b) \in V(\mathcal{P})$ because it is the join of (a,b) and (c,d). This shows that the rectangle $\widetilde{R}$ with $V(\widetilde{R}) = [(0,0),(c,b)]$ contains both R and R', a contradiction to the maximality of R and R'. Therefore, we conclude that there exists a unique maximal rectangle R that contains $(0,0)$. In addition, we obtain that the cell with lower left corner $(0,0)$ only belongs to R. This shows that R must have a single rectangle. □

In the following text, for a given parallelogram polyomino $\mathcal{P}$, the unique maximal rectangle of $\mathcal{P}$ containing $\min(\mathcal{P}) = (0,0)$ is denoted by R_0. Let $\mathcal{P}'$ be a subpolyomino of $\mathcal{P}$. Then $\mathcal{P} \setminus \mathcal{P}'$ is a collection of cells obtained by removing all cells of $\mathcal{P}'$ from $\mathcal{P}$. Next, we introduce a new family of parallelogram polyominoes.

Definition 4.3.19 *A parallelogram polyomino* $\mathcal{P}$ *is said to be* shortenable *if* $\mathcal{P} \setminus R_0$ *is a parallelogram polyomino. Moreover,* $\mathcal{P}$ *is* well-shortenable *if* $\mathcal{P}$ *is shortenable and either* $\mathcal{P} \setminus R_0$ *is a rectangle or* $\mathcal{P} \setminus R_0$ *is a well-shortenable parallelogram polyomino. The sequence of polyominoes* $\{\mathcal{P}_i\}_{i=1,\dots,l}$ *such that* $\mathcal{P}_1 = \mathcal{P} \setminus R_0$, *and* $\mathcal{P}_{i+1} = \mathcal{P}_i \setminus R_i$ *where* R_i *is the unique rectangle containing* $\min(\mathcal{P}_i)$, *is called the* derived sequence *of* $\mathcal{P}$.

We observe that a thin parallelogram polyomino and an L-convex parallelogram polyomino (Ferrer diagram) are well-shortenable. In particular, for a Ferrer diagram the definition of derived sequence coincides with the one of Section 4.1.

Example 4.3.20 *We give an example of a shortenable polyomino that is not well-shortenable. Let* $\mathcal{P}$ *be the parallelogram polyomino in Figure 4.17.(I). We observe that the maximal rectangle* R_0 *of* $\mathcal{P}$ *is the maximal rectangle on the cells* A *and* B, *and the polyomino* $\mathcal{P}_1 = \mathcal{P} \setminus R_0$ *is a parallelogram polyomino (see Figure 4.17.(II)). Then* $\mathcal{P}$ *is* *shortenable. However, the rectangle R_1 on the cells $\{C, D, E, F\}$ in $\mathcal{P}_1$ is such that $\mathcal{P}_1 \setminus R_1$ is not a parallelogram polyomino (without rotation), see Figure 4.17.(III).*



(A) $\mathcal{P}$ (B) $\mathcal{P}_1 = \mathcal{P} \setminus \{A, B\}$ (C) $\mathcal{P}_1 \setminus \{C, D, E, F\}$

Figure 4.17: A shortenable polyomino that is not well-shortenable

In order to characterize the parallelogram polyominoes that are shortenable, we prove the following.

Lemma 4.3.21. *Let $\mathcal{P}$ be a parallelogram polyomino. Assume that R_0 has size $s \times t$ and its single rectangle R has size $s' \times t'$ with $s' < s$ and $t' < t$. Then there exist $R', R'', \widetilde{R} \in \mathcal{M}(\mathcal{P})$ as in Figure 4.18.*

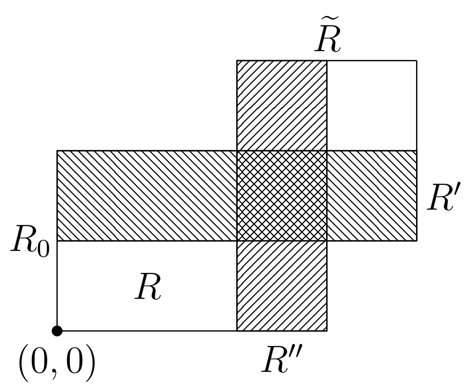


Figure 4.18

Proof. Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$. Since $s' < s$ and $t' < t$, then all cells of R_0 with lower left corner (a, b) with either $s' \leq a$ or $t' \leq b$ belong to some other maximal rectangles of $\mathcal{P}$ as well. Using the fact that $R_0 \in \mathcal{M}(\mathcal{P})$, we observe that $\mathcal{S}_2$ takes a north step at $(s, 0)$. By using the assumptions $s' < s$ and $t' < t$ and R is the single rectangle of R_0, we conclude that $\mathcal{S}_2$ changes the direction from north to east at (s, t'). Then the coordinates of R' are determined by the next north turn of $\mathcal{S}_2$. Similar argument on $\mathcal{S}_1$ shows the existence of R''. The existence of $\tilde{R}$ is guaranteed by the fact that $\mathcal{P}$ is a parallelogram polyomino and hence a distributive lattice, therefore the join of the diagonal corners of R' and R'' must belong to $V(\mathcal{P})$.

□

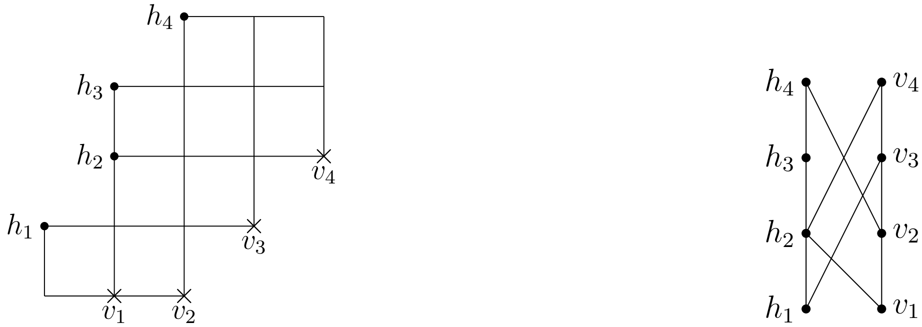
In the following, we give a characterization of parallelogram polyominoes that are shortenable in terms of the size of the single rectangle of R_0.

Lemma 4.3.22. *Let $\mathcal{P}$ be a parallelogram polyomino and assume R_0 has size $s \times t$. Then $\mathcal{P}$ is shortenable if and only if the single rectangle R of R_0 has size $s' \times t'$ with either $s' = s$ or $t' = t$.*

Proof. By contraposition, assume that R has size $s' \times t'$ with $s' < s$ and $t' < t$. From Lemma 4.3.21 there exist in $\mathcal{P}$ the maximal rectangles in Figure 4.18. We consider the polyomino $\mathcal{P}_1 = \mathcal{P} \setminus R_0$. We observe that $(s', t), (s, t') \in V(\mathcal{P}_1)$ with $s' < s$ and $t' < t$, that is a contradiction to the fact that $\mathcal{P}$ is parallelogram.
Conversely, assume that $V(R) = [(0,0), (s', t)]$ with $s' < s$. Then using Proposition 4.3.2, we obtain $\mathcal{P} \setminus R_0$ is the parallelogram polyomino that corresponds to the sublattice $L \cap [(s', t), \max L]$. □

We now want to link the shortenability to the Gorensteinness. Hibi showed in [42, page 105] that given a distributive lattice L, the Hibi ring $\mathbb{K}[L]$ is Gorenstein if and only if the poset P of the join-irreducible elements of L is pure, i.e. all of the maximal chains have the same length. Hence we look at the structure of the poset of the join-irreducible elements of parallelogram polyomino $\mathcal{P}$ that we identify as a distributive lattice.

Let $H_0, H_1, \dots, H_n$ be the maximal edge horizontal intervals of $\mathcal{P}$ and $V_0, V_1, \dots, V_n$ be the maximal edge vertical intervals of $\mathcal{P}$. Note that $H_0 \cap V_0 = \{(0,0)\} = \min L$. Set $h_i = \min(H_i)$ for all $1 \leq i \leq n$ and $v_j = \min(V_i)$ for all $1 \leq j \leq m$ (see Figure 4.19). Then $h_1 \leq h_2 \leq \ldots \leq h_n$ and $v_1 \leq v_2 \leq \ldots \leq v_m$ are two maximal chains of P.

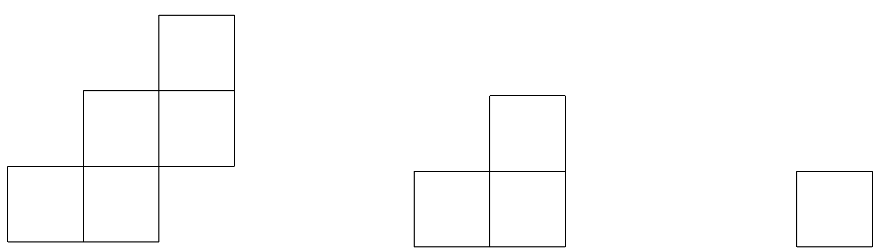


(A) The join-irreducible elements are the minimum of the maximal horizontal and vertical edge intervals

(B) The poset P of join-irreducible elements

Figure 4.19

In Section 4.1, we prove that an L-convex polyomino $\mathcal{P}$ with derived sequence $(\mathcal{P}_k)_{k=1,\dots,t}$ for some t is Gorenstein if and only if the bounding box of any $\mathcal{P}_k$ is a square. For parallelogram polyominoes the latter condition is necessary but not sufficient, as shown in Figure 4.20. The polyomino $\mathcal{P}$ in Figure 4.20 is known to be non-Gorenstein from Theorem 4.2.18, while $\mathcal{P}$, $\mathcal{P}_1$ and $\mathcal{P}_2$ have square bounding boxes.



(A) $\mathcal{P}$ (B) $\mathcal{P}_1$ (C) $\mathcal{P}_2$

Figure 4.20: An example of non-Gorenstein parallelogram polyomino with square bounding boxes

Next, we prove that a Gorenstein parallelogram polyomino is well-shortenable.

Lemma 4.3.23. *Let $\mathcal{P}$ be a parallelogram polyomino. If $\mathcal{P}$ is Gorenstein, then $\mathcal{P}$ is well-shortenable.*

Proof. Let $\mathcal{P}$ be Gorenstein. Then due to [42, page 105], the poset P of join-irreducible elements of $\mathcal{P}$ is pure. Assume that $\mathcal{P}$ is not shortenable. Then by using Lemma 4.3.22, we obtain that if R_0 has size $s \times t$ with $t \leq s$ then the single rectangle of R has size $r \times q$ with $V(R) = [(0,0),(r,q)]$ with $q \leq r$, $r < s$ and $q < t$. From Lemma 4.3.21, we can find the maximal rectangle R' with $V(R') = [(0,q),(u,t)]$ with $u > s$ as shown in Figure 4.21.

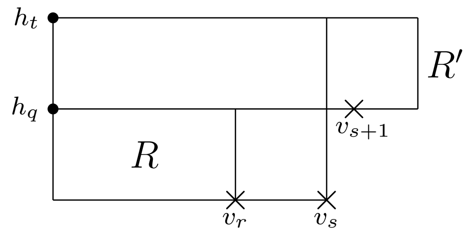


Figure 4.21

We observe that h_q and v_{s+1} correspond to $(0,q)$ and $(s+1,q)$, respectively. The latter implies that $h_q \leq v_{s+1}$. We consider the following maximal chains of the poset P,

$$h_1 \leq h_2 \leq \ldots \leq h_n, \quad h_1 \leq h_2 \leq \ldots \leq h_q \leq v_{s+1} \leq \ldots \leq v_n.$$

The first chain has length n while the second one has length $n-s+q < n$ since $q < t \leq s$. This contradicts the Gorensteinness of $\mathcal{P}$. Therefore, we conclude that $\mathcal{P}$ is shortenable and hence, $\mathcal{P}_1 = \mathcal{P} \setminus R_0$ is a parallelogram polyomino.

To show that $\mathcal{P}$ is well-shortenable, it is enough to show that $\mathcal{P}_1$ is Gorenstein. Indeed, if $\mathcal{P}_1$ is Gorenstein then by following the previous argument, it is shortenable and the conclusion follows by applying the same argument. Let P_1 be the poset of the join-irreducible elements of $\mathcal{P}_1$. Assume that the single rectangle R of R_0 (in $\mathcal{P}$) is such that $V(R) = [(0,0),(r,t)] \subset V(R_0) = [(0,0),(s,t)]$. Then $\min(\mathcal{P}_1) = (r,t)$ and in P we have $v_r \leq h_{t+1}$. If $\mathcal{P}_1$ is not Gorenstein, we exhibit two chains in $\mathcal{P}$ that have different lengths. Let

$$c_1 \leq c_2 \leq \ldots \leq c_l, \quad d_1 \leq d_2 \leq \ldots \leq d_h$$

be two chains with $c_i, d_j \in V(P_1) = \{v_{r+1}, \ldots, v_n, h_{t+1}, \ldots, h_n\}$ and $l \neq h$. If $c_1 = d_1 = h_{t+1}$, then $h_1 \leq \ldots \leq h_t \leq c_1 \leq c_2 \leq \ldots \leq c_l$ and $h_1 \leq \ldots \leq h_t \leq d_1 \leq d_2 \leq \ldots \leq d_h$ are two maximal chains of P having different lengths, a contradiction to the Gorensteinness of $\mathcal{P}$. Similar arguments hold for the case $c_1 = d_1 = v_{r+1}$. We are left with the case $c_1 = h_{t+1}$

and $d_1 = v_{r+1}$. Since $v_r \le h_{t+1}$, then

$$v_1 \le \ldots \le v_r \le c_1 \le c_2 \le \ldots \le c_l, v_1 \le \ldots \le v_r \le d_1 \le d_2 \le \ldots \le d_h$$

are two chains of P having lengths $r+l$ and $r+h$ and since $h \neq l$, then they have different lengths and $\mathcal{P}$ is not Gorenstein, a contradiction. This shows that $\mathcal{P} \setminus R_0$ is Gorenstein and hence shortenable. □

In order to link the Gorensteinness with the S-property, we prove that a parallelogram polyomino with S-property is well shortenable.

Lemma 4.3.24. *Let $\mathcal{P}$ be a parallelogram polyomino with S-property. Then $\mathcal{P}$ is shortenable.*

Proof. Let S be the single square of R_0. Assume that R_0 has size $s \times t$ and S has size $r \times r$ with $r < \min\{s,t\}$. From Lemma 4.3.21, there exist some maximal rectangles R', R'' and $\widetilde{R}$ as in Figure 4.18. We observe that in this case R', R'' are contained in $R_0 \cup \widetilde{R}$, that is they do not have single squares, and it is a contradiction to the fact that $\mathcal{P}$ has the S-property. Therefore, either $r = s$ or $r = t$ and the conclusion follows from Lemma 4.3.22. □

Corollary 4.3.25. *Let $\mathcal{P} \subseteq [m,n]$ be a parallelogram polyomino with S-property, let $R_0, \ldots, R_l$ be the maximal rectangles of $\mathcal{P}$ having single squares $S_0, S_1, \ldots S_l$ of sizes $t_1 \times t_1, \ldots, t_l \times t_l$, respectively. For any $i \in 1, \ldots, l$ let $c_i = \sum_{j=1}^{i} t_j$. Then, we have $V(S_i) \cap V(S_{i+1}) = (c_i, c_i)$ and $m = n = c_l$. Moreover $\mathcal{P}$ is well-shortenable.*

Proof. From Lemma 4.3.24 we have that $V(S_0) = [(0,0),(t_1,t_1)]$ and $\mathcal{P}$ is shortenable. Let $\mathcal{P}_1 = \mathcal{P} \setminus R_0$. From Lemma 4.3.24 applied to $\mathcal{P}_1$, we obtain that S_1 is such that $V(S_1) = [(t_1,t_1),(c_2,c_2)]$. We recursively consider the polyomino $\mathcal{P}_i$ obtained from from $\mathcal{P}_{i-1}$ by removing the rectangle R_i and we obtain from Lemma 4.3.24.that $V(S_{i+1}) = [(c_i,c_i),(c_{i+1},c_{i+1})]$. The polyomino $\mathcal{P}_l$ is a square, that is $c_l = m = n$ and $\mathcal{P}$ is well-shortenable. □

Now we prove the main theorem of this section.

Theorem 4.3.26. *Let $\mathcal{P}$ be a parallelogram polyomino. The following are equivalent:*

(i) $\mathcal{P}$ *is Gorenstein;*

(ii) $\mathcal{P}$ *has the* S*-property.*

Proof. (i)⇒(ii). From Lemma 4.3.23, we have that $\mathcal{P}$ is well shortenable. Moreover, from the proof of Lemma 4.3.23 it arises that all of the polyominoes in the derived sequence are Gorenstein, that is they have square bounding boxes due to the pureness of the poset. In particular, $\mathcal{P} \subset [(0,0),(n,n)]$.

First, we show that the single rectangle R of R_0 is a square. Let P be the poset of the join-irreducible elements of $\mathcal{P}$. Assume that

$$V(R) = [(0,0),(s,t)] \subset V(R_0) = [(0,0),(q,t)]$$

with $s \neq t$. Hence $\min V(\mathcal{P}_1) = (s,t)$ and in P we have $v_s \leq h_{t+1}$. This gives that the two chains

$$v_1 \leq \ldots \leq v_s \leq v_{s+1} \leq \ldots \leq v_n, v_1 \leq \ldots \leq v_s \leq h_{t+1} \leq \ldots \leq h_n$$

have different lengths and this is a contradiction to the assumption that P is Gorenstein. That is $s = t$. Furthermore, we claim that there exists a unique maximal rectangle R_1 containing $\widetilde{R} = R_0 \backslash R$, namely $[(s,0),(q,s)]$. Let $R_1' = [(a,b),(c,d)]$ be a maximal rectangle such that $R_1' \cap \widetilde{R} \neq \varnothing$, that is $s \leq a \leq q$ and $b < s$. From the property of parallelogram polyominoes, we also obtain that $b \geq 0$. If $b > 0$, then the rectangle $[(a,0),(c,d)]$ is a rectangle containing R_1', contradicting its maximality. That is, we have $b = 0$. We observe that $c \leq q$, otherwise the rectangle $[(0,b),(c,s)]$ is a maximal rectangle having non-empty intersection with R, contradiction. Moreover $d > s$, otherwise $R_1' \subseteq \widetilde{R}$. The latter implies that all of the maximal rectangle having non-empty intersection with $\widetilde{R}$ have lower left corner on the edge interval $[(s,0),(q,0)]$. Then, there exists a unique maximal rectangle R_1 with vertices $[(s,0),(q,u)]$ where u is the minimum of the heights of such rectangles. We now show that R_1 has a single rectangle. If this is not the case, then there exists a maximal rectangle R_2 such that $R_1 \subseteq R_0 \cup R_2$ and $V(R_2) = [(s,s),(a,b)]$ with $a > q$, hence $h_s \leq v_{q+1}$ in P. This implies that

$$h_1 \leq \ldots \leq h_s \leq v_{q+1} \leq \ldots \leq v_n, h_1 \leq \ldots \leq h_n$$

are two chains having lengths $n-q+s$ and n, respectively. Since $s < q$, then $n-q+s < n$, contradicting the Gorensteinnes of $\mathcal{P}$. In particular this implies that any maximal rectangle has a single rectangle. By using a similar technique on any polyomino of the derived sequence we obtain that all of the rectangles of $\mathcal{P}$ have a single square.

(ii)⇒(i). We assume that $\mathcal{P}$ has the S-property. To have the Gorensteinness, we have to prove that for any edge in the Hasse diagram of the poset $\mathcal{P}$ of the form $v_s \to h_{t+1}$ (or $h_c \to v_{d+1}$), we have $s = t$ (or $c = d$). We follow the notation of Corollary 4.3.25. From the latter result we obtain that if S_0 has size $t_0 \times t_0$ and S_1 has size $t_1 \times t_1$. Then R_0 has either size $t_0 \times (t_0 + t_1)$ or $(t_0 + t_1) \times t_0$, that is either $h_{t_0} \to v_{t_0+1}$ or $v_{t_0} \to h_{t_0+1}$. Since $\mathcal{P}$ is well-shortenable, we inductively apply the same argument to find that for any $k \in \{1, \dots l\}$ either $h_{c_k} \to v_{c_k+1}$ or $v_{c_k} \to h_{c_k+1}$.
Moreover, assume that $h_r \to v_s$ is an edge of the poset P such that $c_{k-1} + 1 \leq r < c_k$ for some k. It follows that $s > c_k + 1$ and there exists a maximal rectangle in $\mathcal{M}(\mathcal{P})$ of size $a \times b$ with $b = c_k - r$ that has non-empty intersection with S_k. This leads to a contradiction to the fact that S_k is single. □

Now, we give a description of Gorenstein parallelogram polyominoes in terms of the 2-colored Motzkin paths (see Section 2.3). To do this, we need the following terminologies

Definition 4.3.27 *(i) Let $\mathcal{S} : s_1, \dots, s_l$ be a north-east path. A sequence of consecutive north steps (resp. east steps) $s_i, \dots, s_{i+k}$ makes a* maximal block *of length k in $\mathcal{S}$ if either $i = 1$ or s_{i-1} is an east step (resp. north step), and either $i + k = l$ or s_{i+k+1} is a north step (resp. east step). Note that in $\mathcal{S}$, a maximal block of length k of consecutive north steps (resp. east steps) corresponds to a maximal block of k 1s (resp. 0s) in its binary representation.*

(ii) Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ be a parallelogram polyomino. A sequence of consecutive elements $s_1, s_2, \dots, s_l$ of $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) is called a maximal NE-block *if there exists $i \in \{1, \dots l\}$ such that $s_1 \dots s_i$ is a maximal block of north steps (resp. east steps) and $s_{i+1} \dots s_l$ is a maximal block of east steps (resp. north steps).*

For example, for the parallelogram polyomino given in Figure 2.15, the binary representation of $\mathcal{S}_1$ is $u(\mathcal{P}) : 10110100$. The maximal NE-block in $\mathcal{S}_1$ are 10, 110 and 100. The NE-blocks in $\mathcal{S}_1$ determine the corners in $\mathcal{S}_1$. Similarly, the maximal NE-block in $\mathcal{S}_2$ are 001, 01, and 011 and the NE-blocks in $\mathcal{S}_2$ determine the corners in $\mathcal{S}_2$. We emphasize that in $\mathcal{S}_1$ each NE-block starts with a north step, while an NE-block in $\mathcal{S}_2$ starts with an east step.

Theorem 4.3.28. *Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ be a parallelogram polyomino. $\mathcal{P}$ has the S-property if and only if the following conditions hold:*

1. *in $\mathcal{S}_1$, each maximal block of length k of consecutive north steps is followed by a maximal block of length k of consecutive east steps.*
2. *in $\mathcal{S}_2$, each maximal block of length k of consecutive east steps is followed by a maximal block of length k of consecutive north steps.*

Proof. Assume that $\mathcal{P}$ has the S-property. We need to show that $\mathcal{P}$ satisfies the conditions (1) and (2). We proceed by induction on the total number l of maximal rectangles of $\mathcal{P}$. If $l = 1$, then $\mathcal{P}$ itself is a rectangle. Using the assumption that $\mathcal{P}$ has the S-property, we see that $\mathcal{P}$ is in fact a square of size $t \times t$. This shows that binary representations of $\mathcal{S}_1$ and $\mathcal{S}_2$ are given by

$$u(\mathcal{P}) : \underbrace{11\ldots1}_{t \text{ times}}\underbrace{00\ldots0}_{t \text{ times}}, \quad \ell(\mathcal{P}) : \underbrace{00\ldots0}_{t \text{ times}}\underbrace{11\ldots1}_{t \text{ times}}$$

as claimed.

Now assume that $l \geq 2$ and the assertion is true for any parallelogram polyomino with $l - 1$ maximal rectangles. Let the size of R_0 be $s \times t$. Assume that $t < s$, and the case when $t > s$ can be discussed in a similar way. The assumption that $\mathcal{P}$ has the S-property together with Lemma 4.3.22 and 4.3.24 shows that the single square S of R_0 has size $t \times t$. Consider the parallelogram polyomino $\mathcal{P}' = \mathcal{P} \setminus R_0$ given by some paths $(\mathcal{S}'_1, \mathcal{S}'_2)$. Then $\mathcal{P}'$ has the S-property, too. We observe that since S is single square, the path $\mathcal{S}_1$ is of the form

$$\underbrace{11\ldots1}_{t \text{ times}}\underbrace{00\ldots0}_{t \text{ times}}u(\mathcal{P}').$$

where $u(\mathcal{P}')$ is the binary representation of $\mathcal{S}'_1$. By using the inductive hypothesis on $\mathcal{P}'$ we conclude that $\mathcal{P}$ satisfies the condition (1). Moreover, again by using the inductive hypothesis on $\mathcal{P}'$, we see that $\mathcal{S}_2'$ satisfies condition (2). That is, the binary representation $\ell(\mathcal{P}')$ of $\mathcal{S}_2'$ starts with a block of $s-t$ 0s followed by a block of $s-t$ 1s, in particular

$$\underbrace{00\dots0}_{s-t \text{ times}}\beta.$$

Hence $\ell(\mathcal{P})$ is given by

$$\underbrace{00\dots0}_{s \text{ times}}\underbrace{11\dots1}_{t \text{ times}}\beta$$

This shows that $\mathcal{S}_2$ satisfies the condition (2).

To prove the converse, assume that $\mathcal{P}$ satisfies the conditions (1) and (2). We need to show that $\mathcal{P}$ has the S-property. We proceed by induction on the total number $e \geq 2$ of maximal NE-blocks in $\mathcal{S}_1$ and $\mathcal{S}_2$. In other words, we apply the induction on the total number of corners in $\mathcal{S}_1$ and $\mathcal{S}_2$.

For $e = 2$, from the conditions (1) and (2) we get that in $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) the maximal NE-block has size $2t$ for some $t \in \mathbb{N}$. More precisely, $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) has a maximal block of t north-steps (resp. east-steps) followed by a block of t east-steps (resp. north-steps). Then the binary representations of $\mathcal{S}_1$ and $\mathcal{S}_2$ are

$$u(\mathcal{P}) : \underbrace{11\dots1}_{t \text{ times}}\underbrace{00\dots0}_{t \text{ times}} \text{ and } \ell(\mathcal{P}) : \underbrace{00\dots0}_{t \text{ times}}\underbrace{11\dots1}_{t \text{ times}}$$

and the polyomino is a square.

Now, let $e \geq 3$ and assume that any parallelogram polyomino having a total number of maximal NE-blocks equal to $e-1$ has the S-property. Let $\mathcal{P}' = \mathcal{P} \setminus R_0$. It follows from Lemma 4.3.24 that $\mathcal{P}'$ is a parallelogram polyomino. Set $\mathcal{P}' = (\mathcal{S}_1', \mathcal{S}_2')$. To prove that $\mathcal{P}$ has the S-property, it is enough to show that R_0 has a single square and that $\mathcal{P}'$ has the S-property. In particular, we prove that $\mathcal{S}_1'$ and $\mathcal{S}_2'$ satisfy conditions (1) and (2), respectively. Then the conclusion will follow by using inductive hypothesis on $\mathcal{P}'$ and the existence of single square in R_0.

If R_0 has size $s \times t$ with $t < s$, then $u(\mathcal{P})$ begins with a maximal block of t 1s and by using condition (1), there is a maximal block of t $0s$ following it. Therefore, $u(\mathcal{P})$ is of the following form

$$\underbrace{11\dots1}_{t \text{ times}}\underbrace{00\dots0}_{t \text{ times}}u(\mathcal{P}').$$

This shows that R_0 has a single square of size $t \times t$ and $\mathcal{S}_1'$ satisfies the condition (1). Moreover, by using the assumption that $\mathcal{P}$ satisfies condition (2), we obtain that $\ell(\mathcal{P})$ starts with a maximal block of s 0s followed by a maximal block of s 1s. We write

$$\underbrace{00\ldots0}_{s \text{ times}}\underbrace{11\ldots1}_{s \text{ times}}\beta.$$

where β is binary sequence consistent with condition (2). Then $\ell(\mathcal{P}')$ takes the following form

$$\underbrace{00\ldots0}_{s-t \text{ times}} \quad \underbrace{11\ldots1}_{s-t \text{ times}}\beta.$$

which shows that $\mathcal{P}'$ satisfies the condition (2). Moreover, the total number of maximal NE-blocks in $\mathcal{P}'$ is $e-1$ (because one maximal NE-block is at the beginning of $\mathcal{S}_1$). By using the inductive hypothesis, we conclude that $\mathcal{P}'$ has the S-property. Then, it follows that $\mathcal{P}$ has the S-property as well. □

With the help of Theorem 4.3.26, to be able to describe Motzkin paths associated with Gorenstein parallelogram polyominoes, it is enough to see the impact of conditions (1) and (2) of Theorem 4.3.28 on the associated Motzkin paths. Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ be a Gorenstein parallelogram polyomino with associated Motzkin path $\mathcal{M}_{\mathcal{P}}$. Note that in $\mathcal{S}_1$, a maximal block of length k of consecutive north steps corresponds to a combination of k rise and α-colored horizontal steps in $\mathcal{M}_{\mathcal{P}}$. Indeed, this combination of rise and α-colored horizontal steps in $\mathcal{M}_{\mathcal{P}}$ is maximal in a sense that it is followed by either a fall or a β-colored horizontal step. Similarly, a maximal block of length k of consecutive east steps corresponds to a maximal block of a combination of k fall and β-colored horizontal steps in $\mathcal{M}_{\mathcal{P}}$. Hence, the condition (1) of Theorem 4.3.28 translates as: in $\mathcal{M}_{\mathcal{P}}$ each maximal block of a combination of k rise and α-colored horizontal steps must be followed by a maximal block of a combination of k fall and β-colored horizontal steps.

To translate condition (2) for $\mathcal{M}_{\mathcal{P}}$, we consider the reflection of $\mathcal{M}_{\mathcal{P}}$ through the x-axis. We denote this reflection by $\overline{\mathcal{M}_{\mathcal{P}}}$. The reflection $\overline{\mathcal{M}_{\mathcal{P}}}$ corresponds to the coding given in (2.1) applied to the matrix that contains $\ell(\mathcal{P})$ as first row and $u(\mathcal{P})$ as the second row. Then the condition (2) of Theorem 4.3.28 translates as: in $\overline{\mathcal{M}_{\mathcal{P}}}$ each maximal block of a combination of k fall and β-colored horizontal steps must be followed by a maximal block of a combination of k rise and α-colored horizontal steps. We formulate this discussion in the following corollary.

Corollary 4.3.29. *Let $\mathcal{P} = (\mathcal{S}_1, \mathcal{S}_2)$ be a parallelogram polyomino with associated Motzkin path $\mathcal{M}_{\mathcal{P}}$. Let $\overline{\mathcal{M}_{\mathcal{P}}}$ be the reflection of $\mathcal{M}_{\mathcal{P}}$ through x-axis. Then $\mathcal{P}$ is Gorenstein if and only if the following conditions hold:*

1. *in $\mathcal{M}_{\mathcal{P}}$ each maximal block of a combination of k rise and α-colored horizontal steps must be followed by a maximal block of a combination of k fall and β-colored horizontal steps;*
2. *in $\overline{\mathcal{M}_{\mathcal{P}}}$ each maximal block of a combination of k fall and β-colored horizontal steps must be followed by a maximal block of a combination of k rise and α-colored horizontal steps.*

We give an illustration of Corollary 4.3.29 in the following example.

Example 4.3.30 *The Figure 4.22 shows a Gorenstein parallelogram polyomino. The associated Motzkin path $\mathcal{M}_{\mathcal{P}}$ is shown on the left side and its reflection through x-axis is shown on the right side. The Motzkin path $\mathcal{M}_{\mathcal{P}}$ and its reflection satisfy the conditions (1) and (2) of Corollary 4.3.29.*

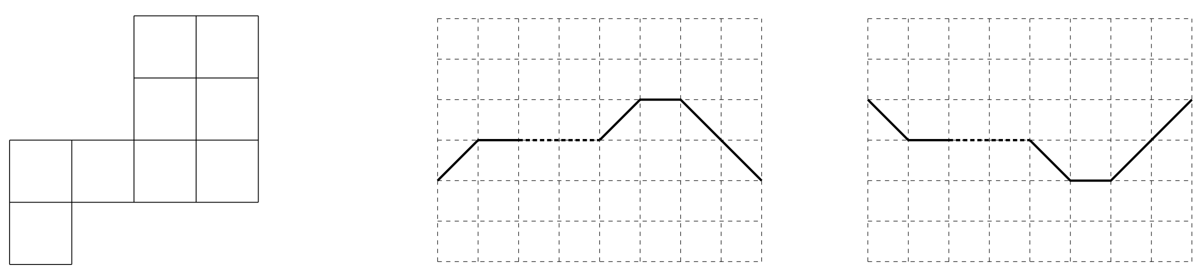


Figure 4.22: A Gorenstein parallelogram polyomino satisfying conditions (1) and (2) of Corollary 4.3.29

The Figure 4.23 shows a non-Gorenstein parallelogram polyomino. The associated Motzkin path $\mathcal{M}_{\mathcal{P}}$ is shown on the left side and its reflection through x-axis is shown on the right side. The Motzkin path $\mathcal{M}_{\mathcal{P}}$ fails the condition (1) of Corollary 4.3.29. However, its reflection satisfies the condition (2) of Corollary 4.3.29.

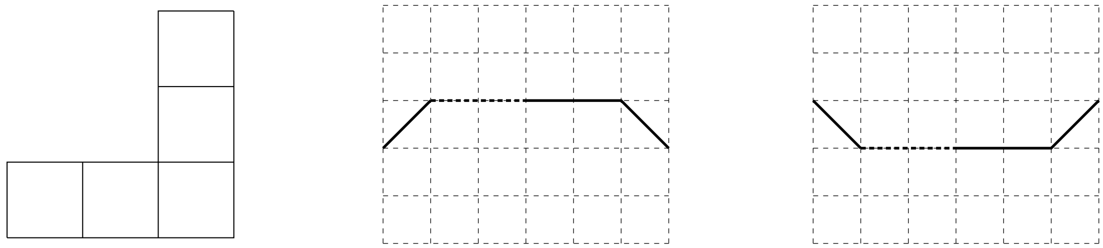


Figure 4.23: A non-Gorenstein parallelogram polyomino satisfying condition (2) of Corollary 4.3.29

CONCLUSION

In conclusion, in this thesis we have studied the algebraic invariants and properties of the ideals associated to polyominoes. Many results have been obtained on this topic. There are several open problems, such as finding a complete classifications of the polyominoes $\mathcal{P}$ having a prime ideal $I_{\mathcal{P}}$ or a Gorenstein $\mathbb{K}[\mathcal{P}]$, determine whether for any polyomino $\mathcal{P}$ the coordinate ring is Cohen-Macaulay and finding precise formulas for the Hilbert series of $\mathbb{K}[\mathcal{P}]$. We have a strong belief that some insights to work in these directions are given in Conjecture 3.1.10, Question 3.1.10 and Conjecture 4.3.5.

Furthermore, finding similar combinatorial conditions for the algebraic invariants the ideals arising from t-minors for $t > 2$ could be of interest for the modern research.

Chapter 5
APPENDIX: FURTHER WORKS

In this chapter, we present two further works done during PhD that do not involve polyominoes. In fact, both of them involve simple graphs and simplicial complexes. In Section 5.1, we enter into the world of monomial ideals associated to graphs, the edge ideals. In particular, we study a class of graphs that arise from $\mathbb{Z}_n$, *circulant graphs.* We characterize the chordal circulant graphs as the ones that are disjoint union of complete graphs and then we give a formula for their induced matching number. Both of these results allow us to study the Castelnuovo-Mumford regularity of the edge ideal of circulant graphs, due to the results in [30, 48]. In Section 5.2, we study some algebraic and combinatorial properties and invariants of the binomial edge ideal of graphs, such as Serre's condition, strongly unmixedness and accessibility that are widely studied in the paper [9]. To keep this chapter detached from the rest of the work, we introduce in the next sections all of the basic notations needed. The references for this chapter are [72] and [50]

5.1 CHORDAL CIRCULANT GRAPHS AND INDUCED MATCHING NUMBER

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $\mathcal{C}$ be a cycle of G. An edge $\{v,w\}$ in $E(G)\setminus E(\mathcal{C})$ with v,w in $V(\mathcal{C})$ is a *chord* of $\mathcal{C}$. A graph G is said to be *chordal* if every cycle has a chord.

We recall that a circulant graph is defined as follows. Let $S\subseteq T:=\{1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor\}$. The *circulant graph* $G:=C_n(S)$ is a simple graph with $V(G)=\mathbb{Z}_n=\{0,\dots,n-1\}$ and $E(G):=\{\{i,j\}\mid |j-i|_n\in S\}$ where $|k|_n=\min\{|k|,n-|k|\}$. Given $i,j\in V(G)$ we call *labelling distance* the number $|i-j|_n$. By abuse of notation we write $C_n(a_1,a_2,\dots,a_s)$ instead of $C_n(\{a_1,a_2,\dots,a_s\})$.

Circulant graphs have been studied under combinatorial ([10, 11]) and algebraic ([71]) points of view. In the former, the authors studied some families of circulants, i.e. the d-th powers of a cycle, namely the circulants $C_n(1,2,\dots,d)$ (that we will analyse in Subsection 5.1.3) and their complements. In the latter, the author studied some properties of the edge ideal of circulants. Let $R = \mathbb{K}[x_0, \ldots, x_{n-1}]$ be the polynomial ring on n variables over a field $\mathbb{K}$. The *edge ideal* of G, denoted by $I(G)$, is the ideal of R generated by all square-free monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Some algebraic properties and invariants of $R/I(G)$ can be derived from combinatorial properties of G. Chordality and the induced matching number have been used to give bounds on the Castelnuovo-Mumford regularity of $R/I(G)$ (see Subsection 5.1.1).

In Subsection 5.1.2 we prove that a circulant graph is chordal if and only if it is either complete or a disjoint union of complete graphs.
In Subsection 5.1.3 we give an explicit formula for the induced matching number of a circulant graph $C_n(S)$ depending on the cardinality and the structure of the set S. Moreover, by using `Macaulay2`, we compare the Castelnuovo-Mumford regularity of $R/I(G)$ with $\nu(G)$, the lower bound of Theorem 5.1.3, when G is the d-th power of a cycle and n is less than or equal to 15. We report the result in Table 5.1.

5.1.1 Preliminaries

In this Subsection we recall some concepts and notation that we will use later on in this chapter.

We recall that the circulant graph $C_n(1, 2, \ldots, \lfloor \frac{n}{2} \rfloor)$ is the complete graph K_n. Moreover, we compute the number of components of a circulant graph with the following

Lemma 5.1.1. *Let $S = \{a_1, \ldots, a_r\}$ be a subset of T and let $G = C_n(S)$ be a circulant graph. Then G has $\gcd(n, a_1, \ldots, a_r)$ disjoint components. In particular, G is connected if and only if $\gcd(n, a_1, \ldots, a_r) = 1$.*

For a proof see [7]. From Lemma 5.1.1 it follows that if $n = dk$, then the disjoint components of $C_n(a_1 d, a_2 d, \ldots, a_s d)$ are d copies of the circulant graph $C_k(a_1, a_2, \ldots, a_s)$.

Let G be a graph. A collection C of edges in G is called an *induced matching* of G if the edges of C are pairwise disjoint and the graph having C has edge set is an induced subgraph of G. The maximum size of an induced matching of G is called *induced matching number* of G and we denote it by $\nu(G)$.

Let G be a graph. The *complement graph* $\bar{G}$ of G is the graph whose vertex set is $V(G)$ and whose edges are the non-edges of G. We conclude the section by stating some known results relating chordality and induced matching number to the Castelnuovo-Mumford regularity. The first one is due to Fröberg ([30, Theorem 1])

Theorem 5.1.2. *Let G be a graph. Then* $\operatorname{reg} R/I(G) \leq 1$ *if and only if $\bar{G}$ is chordal.*

The second one is due to Katzman ([48, Lemma 2.2]).

Theorem 5.1.3. *For any graph G, we have* $\operatorname{reg} R/I(G) \geq \nu(G)$.

When G is the circulant graph $C_n(1)$, namely the cycle on n vertices, we have the following result due to Jacques ([47]).

Theorem 5.1.4. *Let C_n be the n-cycle and let $I = I(C_n)$ be its edge ideal. Let $\nu = \lfloor \frac{n}{3} \rfloor$ denote the induced matching number of C_n. Then*

$$\operatorname{reg} R/I = \begin{cases} \nu & \text{if } n \equiv 0,1 \pmod 3 \\ \nu + 1 & \text{if } n \equiv 2 \pmod 3). \end{cases}$$

5.1.2 Chordality of Circulants

The aim of this section is to prove the following

Theorem 5.1.5. *Let G be a circulant graph. Then G is chordal if and only if there exists $d \geq 1$ such that $n = dm$ and $G = C_n(d, 2d, \ldots, \lfloor \frac{m}{2} \rfloor d)$.*

The $\Leftarrow$) implication is trivial. If $d = 1$, then G is the complete graph K_n, while if $d > 1$, then G is the disjoint union of d complete graphs K_m.

To prove $\Rightarrow$) implication we need some preliminary results.

Lemma 5.1.6. *Let $G = C_n(S)$ be a circulant graph. Let us assume that there exists $a \in S$ with $k = \operatorname{ord}(a) \geq 4$ such that*

$$\left\{a, 2a, \ldots, \left\lfloor \frac{k}{2} \right\rfloor a\right\} \nsubseteq S.$$

Then G is not chordal.

Proof. Since $k \geq 4$, then $\{a\} \subset \{a, 2a, \ldots, \lfloor \frac{k}{2} \rfloor a\}$. If $\left\{a, 2a, \ldots, \left\lfloor \frac{k}{2} \right\rfloor a\right\} \nsubseteq S$, then we have two cases:

(1S) $\{a, 2a, \ldots, ra, (r+t)a\} \subseteq S$ and $(r+1)a, \ldots, (r+t-1)a \notin S$, with $r \geq 1$ and $t \geq 2$;

(2S) $\{a, 2a, \ldots, ra\} \subseteq S$ and $(r+1)a, \ldots, \lfloor \frac{k}{2} \rfloor a \notin S$, with $1 \leq r < \lfloor \frac{k}{2} \rfloor$.

(1S) We want to find a non-chordal cycle of G. We consider the edges $\{0, (r+t)a\}$, $\{0, a\}$, $\{a, (r+1)a\}$ (see Figure 5.1). If $(r+1)a$ is adjacent to $(r+t)a$, then we found a non-chordal cycle of G. Otherwise, we apply the division algorithm to $r+t$ and $r+1$,

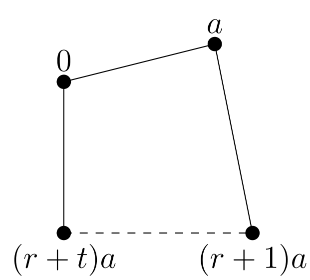


Figure 5.1: Some edges of a non-chordal cycle of G.

that is

$$r + t = (r+1)q + s \quad 0 \leq s \leq r.$$

From the vertex $(r+1)a$ we alternately add a and ra to get the multiples of $(r+1)a$, until $q(r+1)a$. If $s = 0$, then we get $(r+t)a$, otherwise $0 < s \leq r$ and $sa \in S$ so we join $q(r+1)a$ and $(r+t)a$. The above cycle has length greater than or equal to 4 because the vertices $0, a, (r+1)a, (r+t)a$ are different. Furthermore, it is non-chordal because by construction any pair of non-adjacent vertices in the cycle has labelling distance in $\{(r+1)a, \ldots, (r+t-1)a\}$.

(2S) As in case (1S), we want to construct a non-chordal cycle of G. We write $k = \lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil$ and $\lfloor \frac{k}{2} \rfloor = qr + t$ with $0 \leq t \leq r - 1$. Now we write $\lceil \frac{k}{2} \rceil = qr + s$, where

$$s = \begin{cases} t & \text{if } k \text{ even} \\ t+1 & \text{if } k \text{ odd,} \end{cases}$$

and we take the cycle on vertices

$$\left\{0, ra, 2ra, \ldots, qra, \left\lfloor\frac{k}{2}\right\rfloor a, \left\lfloor\frac{k}{2}\right\rfloor a + ra, \left\lfloor\frac{k}{2}\right\rfloor + 2ra, \ldots \left\lfloor\frac{k}{2}\right\rfloor a + qra\right\}. \tag{5.1}$$

Since $r < \lfloor\frac{k}{2}\rfloor$, then $q \geq 1$ and in the case $q = 1$, $s > 0$. That is, the cycle on vertices (5.1) has length at least 4 and it is not chordal because by construction any pair of non-adjacent vertices in the cycle has labelling distance in $\{(r+1)a, \ldots, \lfloor\frac{k}{2}\rfloor a\}$.

In any case G is not chordal and the assertion follows.

□

An immediate consequence of the previous Lemma is

Corollary 5.1.7. *Let $G = C_n(S)$ be a circulant graph. If there exists $a \in S$ with $k = \operatorname{ord}(a) \geq 4$ such that $\gcd(a, n) \notin S$, then G is not chordal.*

Lemma 5.1.8. *Let $G = C_n(S)$ be a circulant graph. If $a_1, \ldots, a_r \in S$ and $\gcd(a_1, \ldots, a_r) \notin S$, then G is not chordal.*

Proof. We proceed by induction on r.

Let $r = 2$ and let $a_1, a_2 \in S$ be such that $c = \gcd(a_1, a_2) \notin S$. We consider

$$a = \gcd(a_1, n),\ b = \gcd(a_2, n),\ d = \gcd(a, b).$$

From Corollary 5.1.7, we have that if one between a, b does not belong to S, then G is not chordal. Hence $a, b \in S$. We have that d divides c and we distinguish two cases. If $d \in S$, since $c = td \notin S$ for some t, then by Lemma 5.1.6 G is not chordal. Therefore, from now on we suppose $d \notin S$. Since a and b divide n, then $\operatorname{lcm}(a, b) = \frac{ab}{d}$ divides n. We want to find a non-chordal cycle of G having length 4. Let $ra + sb = d \pmod n$ be a Bézout identity of a and b. From Lemma 5.1.6, if one between ra and sb is not in S, then G is not chordal. Hence, let us assume $ra, sb \in S$. Now we consider the cycle

$$\{0, ra, ra + sb = d, sb\}.$$

Since $d \notin S$, then the edge $\{0, d\} \notin E(G)$. We distinguish two cases about $ra - sb$. If $ra - sb \notin S$, then the assertion follows.

If $ra - sb \in S$, then we set

$$kd = \gcd(ra - sb, n) \Rightarrow k = \gcd\left(r\left(\frac{a}{d}\right) + s\left(\frac{b}{d}\right), \frac{n}{d}\right).$$

If kd is not in S, then from Corollary 5.1.7 G is not chordal. Hence, we consider $kd \in S$. Since $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$, then $\gcd\left(k, \frac{a}{d}\right) = \gcd\left(k, \frac{b}{d}\right) = 1$, and

$$\gcd\left(k, \frac{ab}{d^2}\right) = 1 \ \Rightarrow \gcd\left(kd, \frac{ab}{d}\right) = d. \tag{5.2}$$

Hence $\text{lcm}\left(kd, \frac{ab}{d}\right) = k\frac{ab}{d}$ divides n. We distinguish two cases. If $k = 1$, then we obtain the contradiction $d \in S$, arising from the assumption $ra - sb \in S$. If $k \neq 1$, then k is a new proper divisor of n. We set $a' = kd$ and $b' = \frac{ab}{d}$, we apply the steps above and we find a k' so that $k'\frac{a'b'}{d}$ divides n, and so on. By applying the steps above to a' and b' a finite number of times, we could either find a k' equal to 1 or we could get new proper divisors of n, that are finite in number. We want to study the case $n = \frac{a'b'}{d}$. Let

$$va' + zb' = d$$

be a Bézout identity, we assume $va' - zb' \in S$, and we set

$$hd = \gcd\left(va' + zb', n\right).$$

We have that $h\frac{a'b'}{d} = hn$ divides n, that is $hn = n$ and $h = 1$. It implies $d \in S$, that is a contradiction arising from the assumption $va' - zb' \in S$. Hence $va' - zb' \notin S$ and $\{0, va', d, zb'\}$ is a non-chordal cycle of G. It ends the induction basis. For the inductive step, we suppose the statement true for $r-1$ and we prove it for r. We have to prove that if $\gcd(a_1, \dots, a_r) \notin S$, then G is not chordal. By inductive hypothesis if $\gcd(a_1, \dots, a_{r-1}) \notin S$, then G will be not chordal. Hence we assume $b = \gcd(a_1, \dots, a_{r-1}) \in S$. By applying the inductive basis to a_r and b,we obtain that G is not chordal. □

Now we are able to complete the proof of Theorem 5.1.5.

Proof of Theorem 5.1.5.⇒). Under the hypothesis that G is chordal, we also assume that G is connected and we prove that $d = 1$, that is $G = K_n$. By contradiction assume that the graph is not complete, namely $G = C_n(a_1, \ldots, a_s)$ with $s < \lfloor \frac{n}{2} \rfloor$. From Lemma 5.1.1, G is connected if and only if $\gcd(a_1, \ldots, a_s, n) = 1$.

Let $b = \gcd(a_1, \ldots, a_s)$.

If $b \notin S$, then from Lemma 5.1.8 G is not chordal. If $b \in S$, we have $1 = \gcd(n, a_1, \ldots, a_s) = \gcd(n, \gcd(a_1, \ldots, a_s)) = \gcd(n, b)$. If $1 \notin S$, then from Lemma 5.1.8, G is not chordal. Then $1 \in S$ and from Lemma 5.1.6 the graph G is not chordal, that is a contradiction. If G is not connected, then it has $a = \gcd(n, S)$ distinct components, each of $m = \mathrm{ord}(a)$ vertices. By Lemma 5.1.6, $S = \{a, 2a, \ldots, \lfloor \frac{m}{2} \rfloor a\}$ and each component is the complete graph K_m. □

Example 5.1.9 *Here we present three examples of non-chordal circulant graphs $C_n(S)$.*

(i) *Take $n = 15$ and $S = \{2, 3, 4, 7\}$. If we take $a = 2$, then $\mathrm{ord}(a) = 15$ and $2a = 4$, $3a = 6$, $n - 4a = 7$, and $n - 6a = 3$. Hence, we are in case (1S) of Lemma 5.1.6 with $S = \{a, 2a, 4a, 6a\}$. We observe that the cycle on vertices*

$$\{0, a, 3a, 4a\} = \{0, 2, 6, 8\}$$

is not chordal because $6 \notin S$.

(ii) *Take $n = 10$, $S = \{3, 4\}$ and $a = 3$. We have $\mathrm{ord}(a) = 10$. Moreover $n - 2a = 4$, hence this is the case (2S) of Lemma 5.1.6 with $S = \{a, 2a\}$. We have $\lfloor \frac{n}{2} \rfloor = \lceil \frac{n}{2} \rceil = 5$, and*

$$5 = qr + t = 2 \cdot 2 + 1.$$

Hence, we take the cycle on vertices

$$\{0, 2a, 4a, 5a, 7a, 9a\} = \{0, 6, 2, 5, 1, 7\}$$

that is not chordal because $1, 2$ and 5 do not belong to S.

(iii) *We take $n = 30$ and $S = \{2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 15\}$. We observe that $\gcd(5, 2) = 1 \notin S$, hence we are in the case of Lemma 5.1.8 with $a_1 = a = 5$ and $a_2 = b = 2$. We observe that $\mathrm{ord}(a) = 6, ord(b) = 15$ and $2a = 10, 3a = 15, b, 2b, \ldots, 7b \in S$. We take a Bezóut identity of a and b*

$$1 = ra + sb = 5 \cdot 1 - 2 \cdot 2.$$

We take the cycle on vertices $\{0, 5, 1, -4\}$. *The quantity* $ra - sb = 5 + 4 = 9$ *belongs to* S *and* $k = \gcd(9, 30) = 3$, *while* $\gcd(k, ab) = \gcd(3, 10) = 1$ *and* $n = abk = 30$. *Hence we write*

$$1 = vab + sk = 10 - 3 \cdot 3,$$

and we take the cycle on vertices $\{0, 10, 1, -9\}$. *The quantity* $10 + 9 = 19$ *does not belong to* S, *hence the cycle above is not chordal.*

5.1.3 Induced matching number of Circulant graphs

In this section we compute the induced matching number for any circulant graph $C_n(S)$. Then we plot a table representing the behaviour of reg $R/I(G)$ with respect to the lower bound described in Theorem 5.1.3, when G is the d-th power of the cycle, namely $G = C_n(1, 2, \ldots, d)$. For the computation we used `Macaulay2` (see [33]).

Definition 5.1.10 *Let* G *be a graph with edge set* $E(G)$. *We say that two edges* e, e' *are* adjacent *if* $e \cap e' = v$ *and* $v \in V(G)$. *We say that* e, e' *are* 2-adjacent *if there exist* $v \in e$ *and* $u \in e'$ *such that* $\{u, v\} \in E(G)$.

Remark 5.1.11 *From Definition 5.1.10, an induced matching of* G *is a subset of* $E(G)$ *where the edges are not pairwise adjacent or 2-adjacent.*

Then we have the following

Theorem 5.1.12. *Let* $G = C_n(S)$ *be a connected circulant graph, let* $s = |S|$ *and let* $r = \min S$. *Then* $\nu(G) = \lfloor \frac{|E(G)|}{t} \rfloor$ *where*

$$t = \begin{cases} s^2 + (|A| + 1)s & \text{if } \frac{n}{2} \notin S \\ s^2 + (|A| + 1)s - 2 & \text{if } \frac{n}{2} \in S, \end{cases}$$

with

$$A = \Big\{ r + a \ : \ a \in S \quad \text{and} \quad r + a \in V(G) \setminus S \Big\}.$$

If G *has* $d = gcd(n, S)$ *components, then* $\nu(G) = d \cdot \nu(C_{n/d}(S'))$, *where* $S' = \{s/d \ : \ s \in S\}$.

Proof. We consider some disjoint subsets of $E(G)$, E_i $i = 1, \ldots, m$ consisting in an edge $e_i = \{u, v = u + s\}$ for an $s \in S$, the edges $\{v, w = v + s\}$ for an $s \in S$ adjacent to e_i, and the edges $\{w, w+s\}$ for an $s \in S$ 2-adjacent to e_i. By suitably choosing the e_i, the $\{e_i\}_{i=1,\dots,m}$ is the biggest induced matching and $m = \nu(G)$. So we have only to count the edges in any set E_i.

We assume that $s = |S|$, $r = \min S$ and $S = \{a_0 = r, a_1, \dots, a_{s-1}\}$, we assume that the edge $e = \{0, r\}$ is in the induced matching, and let E' be the set containing e and the edges adjacent or 2-adjacent to e. The edges adjacent to e are $\{0, a_i\}$ for $i = 1, \dots, s-1$ and $\{r, b_i = r + a_i\}$ for $i = 0, \dots, s-1$. The above edges are all distinct. The edges 2-adjacent to e are $\{a_j, a_j + a_i\}$ for $j \in \{1, \dots, s-1\}, i \in \{0, \dots, s-1\}$ and $\{b_j, b_j + a_i\}$ for $i, j \in \{0, \dots, s-1\}$. The edges above may not be all distinct. In fact, it can happen that some b_j coincides with some a_k, in that case $\{b_j, b_j + a_i\} = \{a_k, a_k + a_i\}$ for any $i \in \{0, \dots, s-1\}$. Then, we only consider $\{b_j, b_j + a_i\}$ for $i \in \{0, \dots, s-1\}$ when $b_j \in A$. To sum up, in the set E' we find:

a) The s edges $\{0, a_i\}$ for $i \in \{0, \dots, s-1\}$;

b) The s^2 edges $\{a_j, a_j + a_i\}$ for $i, j \in \{0, \dots, s-1\}$;

c) The $s \cdot |A|$ edges $\{b, b + a_i\}$ for $i \in \{0, \dots, s-1\}$ and $b \in A$.

If $a_{s-1} = \frac{n}{2}$, then $b_{s-1} = r + a_{s-1} \in A$ and the edges $\{a_{s-1}, a_{s-1} + a_{s-1} = 0\}$ of point b) and $\{b_{s-1}, b_{s-1} + a_{s-1} = r\}$ of point c) are already counted. The assertion follows.

For the case disconnected, let $d = gcd(n, S)$ be the number of disjoint connected components of the graph G. Since the components are disjoint, it turns out that $\nu(G)$ is d times the induced matching number of one component. That component is $C_{n/d}(S')$ where $S' = \{s/d \ : \ s \in S\}$, hence the assertion follows. □

The formula in Theorem 5.1.12 can be written in a compact way when G is the d-th power of a cycle. We set $C_n^d = C_n(\{1, 2, \dots, d\})$.

Corollary 5.1.13. *Let $G = C_n^d$ be the d-th power of a cycle and $d < \lfloor \frac{n}{2} \rfloor$. Then*

$$\nu(G) = \left\lfloor \frac{n}{d+2} \right\rfloor.$$

Proof. We want to apply Theorem 5.1.12, with $s = d$ and $|E(G)| = nd$. We have $r = 1$ and $A = \{d+1\}$. Hence it follows that $t = d^2 + d + d \cdot 1 = d^2 + 2d = d(d+2)$, that is

$$\nu(G) = \left\lfloor \frac{nd}{d(d+2)} \right\rfloor = \left\lfloor \frac{n}{d+2} \right\rfloor.$$

□

In Table 5.1, we compare the values of reg $R/I(C_n^d)$ for $n \leq 15$ and $1 \leq d \leq \lfloor \frac{n}{2} \rfloor$. We highlight that the regularity of $R/I(G)$ is strictly greater than $\nu(G)$ in two different cases:

(1) when $G = C_n$ and $n \equiv 2 \pmod 3$.

(2) when $G = C_n^{\lfloor \frac{n}{2} \rfloor - 1}$ and n is odd.

The two anomalous cases were expected: in case (1), we know from Theorem 5.1.4 that reg $R/I(G) = \nu + 1$; in case (2), $\nu(G) = 1$ while $\bar{G} = C_n(\lfloor \frac{n}{2} \rfloor)$ that is a cycle and hence it is not chordal; hence from Theorem 5.1.2 we know that reg $R/I(G) = 2$. In general, it seems that apart from cases (1) and (2), the Castelnuovo-Mumford regularity of the d-th power of a cycle grips the bound of $\nu(G)$.

G	$\nu(G)$	reg $R/I(G)$	G	$\nu(G)$	reg $R/I(G)$
$C_6(\{1\})$	2	2	$C_{12}(\{1,2,3\})$	2	2
$C_6(\{1,2\})$	1	1	$C_{12}(\{1,2,3,4\})$	2	2
$C_7(\{1\})$	2	2	$C_{12}(\{1,2,3,4,5\})$	1	1
$C_7(\{1,2\})$	1	2	$C_{13}(\{1\})$	4	4
$C_8(\{1\})$	2	3	$C_{13}(\{1,2\})$	3	3
$C_8(\{1,2\})$	2	2	$C_{13}(\{1,2,3\})$	2	2
$C_8(\{1,2,3\})$	1	1	$C_{13}(\{1,2,3,4\})$	2	2
$C_9(\{1\})$	3	3	$C_{13}(\{1,2,3,4,5\})$	1	2
$C_9(\{1,2\})$	2	2	$C_{14}(\{1\})$	4	5
$C_9(\{1,2,3\})$	1	2	$C_{14}(\{1,2\})$	3	3
$C_{10}(\{1\})$	3	3	$C_{14}(\{1,2,3\})$	2	2
$C_{10}(\{1,2\})$	2	2	$C_{14}(\{1,2,3,4\})$	2	2
$C_{10}(\{1,2,3\})$	2	2	$C_{14}(\{1,2,3,4,5\})$	2	2
$C_{10}(\{1,2,3,4\})$	1	1	$C_{14}(\{1,2,3,4,5,6\})$	1	1
$C_{11}(\{1\})$	3	4	$C_{15}(\{1\})$	5	5
$C_{11}(\{1,2\})$	2	2	$C_{15}(\{1,2\})$	3	3
$C_{11}(\{1,2,3\})$	2	2	$C_{15}(\{1,2,3\})$	3	3
$C_{11}(\{1,2,3,4\})$	1	2	$C_{15}(\{1,2,3,4\})$	2	2
$C_{12}(\{1\})$	4	4	$C_{15}(\{1,2,3,4,5\})$	2	2
$C_{12}(\{1,2\})$	3	3	$C_{15}(\{1,2,3,4,5,6\})$	1	2

Table 5.1: The behavior of reg $R/I(G)$ with respect to $\nu(G)$ for $G = C_n^d$.

5.2 (S_2)-CONDITION AND COHEN-MACAULAY BINOMIAL EDGE IDEAL

Binomial edge ideals have been introduced in [37] and, independently, in [60]. They are associated to finite simple graphs, in fact they arise from the 2-minors of a $2 \times n$ matrix related to the edges of a graph with n vertices. The problem of finding a characterization of Cohen–Macaulay binomial edge ideals has been studied intensively by many authors. There are several attempts at this problem available for some families of graphs. Some recent papers in this direction are [8] and [9]. In the latter, the authors introduce two combinatorial properties strictly related to the Cohen-Macaulayness of binomial edge ideals: accessibility and strongly unmixedness. In particular, they prove

$$J_G \text{ strongly unmixed} \implies J_G \text{ Cohen-Macaulay} \implies G \text{ accessible}.$$

In the same article, they show that the three conditions are equivalent for chordal and traceable graphs.

On the other hand, a fundamental condition to describe Cohen-Macaulay modules is the so-called *Serre's condition* (S_r). N. Terai, in [79], translates this condition into nice combinatorial terms for the class of squarefree monomial ideals. In general, for any ideal $I \subseteq S$, it holds true

$$S/I \text{ Cohen-Macaulay} \implies S/I \text{ satisfies Serre's condition } (S_2).$$

The main aim of this Section is to combine all the above-mentioned algebraic and combinatorial notions, showing that

$$S/J_G \text{ satisfies Serre's condition } (S_2) \implies G \text{ accessible},$$

and finding a large family of graphs that satisfies all of them. To reach the goal, in Subsection 5.2.2, we describe the simplicial complex $\Delta_<$ such that $\text{in}(J_G) = I_{\Delta_<}$, for any term order $<$. It is well known that $\text{in}(J_G)$ is a squarefree monomial ideal. In [19], the authors prove that a binomial edge ideal J_G satisfies the Serre's condition (S_2) if and only if $\text{in}(J_G)$ satisfies it, as well. We exploit this fact and the knowledge of $\Delta_<$ to prove that if J_G satisfies (S_2)-condition, then G is accessible, improving the results of [9].

In Subsection 5.2.3, we focus on accessible graphs. In particular, in Proposition 5.2.15 we show that any accessible graph induces, in a natural way, blocks with whiskers that are accessible, too. The latter gives us a sufficient condition for having non-Cohen-Macaulay binomial edge ideals. In literature, many of the examples of non-Cohen-Macaulay J_G are blocks with whiskers (see [67], [68], [8], and [9]). This fact and Proposition 5.2.15 motivate us to study accessible blocks with whiskers. In particular, we identify all the blocks with whiskers having cycle rank 3 (See Figure 5.3) and among them we characterize the accessible ones (see Figures 5.4 and 5.5). This represents a further step in the study of graphs with a given cycle rank, following the works done in [67] and [68], where the author classifies the complete intersection ideals by means of cycle rank (0 in that case), and all the Cohen-Macaulay graphs with cycle rank 1 and 2. Moreover, we observe that the number of blocks with whiskers of a given cycle rank is finite (Lemma 5.2.16 and Lemma 5.2.19). We define a rich family of blocks with whiskers of a given cycle rank that we call *chain of cycles* (see Definition 5.2.24), and we provide necessary conditions for being accessible. Finally, under certain hypotheses on the structure of these graphs (see Setup 5.2.34), we find an infinite subfamily of chain of cycles G for which all the above-mentioned algebraic and combinatorial properties for G and J_G are equivalent (see Theorem 5.2.39).

In the last Subsection, we give a computational classification of all the indecomposable Cohen-Macaulay binomial edge ideals of graphs with at most 12 vertices (see Theorem 5.2.40). This result has been obtained by using a C++ implementation of the algorithms related to the combinatorial properties of accessibility, (S_2)-condition and strongly-unmixedness. The implementation is freely downloadable from the website [51]. This computation and Theorem 5.2.11 lead us to the following.

Conjecture 5.2.1 *Let G be a graph. Then G is accessible if and only if S/J_G satisfies Serre's condition (S_2).*

In [9], the authors conjecture that accessible graphs are the only with Cohen-Macaulay binomial edge ideal. Our computation supports this conjecture. Finally, among the blocks that, after adding suitable whisker, satisfy Theorem 5.2.40 we find two polyhedral graphs, hence Question 5.2.43 naturally arises.

5.2.1 Preliminaries

In this subsection we recall some concepts and notation on graphs, simplicial complexes and binomial edge ideals that we will use in the article (see also [37],[66],[8], [79]).

Throughout this section, all graphs will be finite and simple, namely undirected graphs with no loops nor multiple edges. Let G be a graph with vertex set $[n] = \{1, \dots, n\}$. If $e = \{u, v\} \in E(G)$, with $u, v \in V(G)$, we say that u and v are *adjacent* and the edge e is *incident* with u and v. We denote by $N_G(v)$ (or simply $N(v)$ if G is clear from the context) the set of vertices of G adjacent to v. The *degree* of $v \in V(G)$, denoted $\deg v$, is the number of edges incident with v. An edge $\{u, v\} \in E(G)$, where $\deg v = 1$, is called *whisker* on u. Given $u, v \in V(G)$, a *path* from v to u of length n is a sequence of vertices $v = v_0, \dots, v_n = u \in V(G)$, such that for each $1 \leq i, j \leq n, \{v_{i-1}, v_i\} \in E(G)$ and $v_i \neq v_j$ if $i \neq j$. A subset C of $V(G)$ is called a *clique* of G if for all $u, v \in C$, with $u \neq v$, one has $\{u, v\} \in E(G)$. A *maximal clique* is a clique that cannot be extended by including one more adjacent vertex. A vertex v is called *free vertex* of G if it belongs to only one maximal clique, otherwise it is called an *inner vertex* of G. If $T \subseteq V(G)$, we denote by $G \setminus T$ the induced subgraph of G obtained by removing from G the vertices of T and all the edges incident in them. A set $T \subset V(G)$ is called *cutset* of G if $c_G(T \setminus \{v\}) < c_G(T)$ for each $v \in T$, where $c_G(T)$ (or simply $c(T)$, if the graph is clear from the context) denotes the number of connected components of $G \setminus T$. We denote by $\mathcal{C}(G)$ the set of all cutsets of G. When $T \in \mathcal{C}(G)$ consists of one vertex v, v is called a *cutpoint*. A connected induced subgraph of G that has no cutpoint and is maximal with respect to this property is called a *block*.

A subgraph H of G *spans* G if $V(H) = V(G)$. In a connected graph G, a *chord* of a tree H that spans G is an edge of G not in H. The number of chords of any spanning tree of a connected graph G, denoted by $m(G)$, is called the *cycle rank* of G and it is given by $m(G) = |E(G)| - |V(G)| + 1$.

Let $S = \mathbb{K}[\{x_i, y_j\}_{1 \leq i,j \leq n}]$ be the polynomial ring in $2n$ variables with coefficients in a field $\mathbb{K}$. Define $f_{ij} = x_i y_j - x_j y_i \in S$. The *binomial edge ideal* of G, denoted by J_G, is the ideal generated by all the binomials f_{ij}, for $i < j$ and $\{i, j\} \in E(G)$.

The cutsets of a graph G are essential tools to describe the primary decomposition and several algebraic properties of J_G. Let $T \in \mathcal{C}(G)$ and let $G_1, \dots, G_{c(T)}$ denote the connected components of $G \setminus T$. Let

$$P_T(G) = \left(\bigcup_{i \in T} \{x_i, y_i\}, J_{\tilde{G}_1}, \dots, J_{\tilde{G}_{c(T)}} \right) \subseteq S$$

where $\tilde{G}_i$, for $i = 1, \ldots, c(T)$, denotes the complete graph on $V(G_i)$. It holds

$$J_G = \bigcap_{T \in \mathcal{C}(G)} P_T(G). \tag{5.3}$$

A graph G is *decomposable*, if there exist two subgraphs G_1 and G_2 of G, and a decomposition $G = G_1 \cup G_2$ with $\{v\} = V(G_1) \cap V(G_2)$, where v is a free vertex of G_1 and G_2. If G is not decomposable, we call it *indecomposable*.

Let H be a graph. The *cone* G of v on H is the graph with $V(G) = V(H) \cup \{v\}$ and edges $E(G) = E(H) \cup \{\{v, w\} \mid w \in V(G)\}$.

A cutset T of G is said *accessible* if there exists $t \in T$ such that $T \setminus \{t\} \in \mathcal{C}(G)$. G is said *accessible* if J_G is unmixed and $\mathcal{C}(G)$ is an accessible set system, that is all non-empty cutsets of G are accessible.

To describe the reduced Gröbner basis of J_G, in [37] the following concept has been introduced. Let i and j be two vertices of G with $i < j$. A path $i = i_0, i_1, \ldots, i_r = j$ from i to j is called *admissible* if

(i) $i_k \neq i_\ell$ for $k \neq \ell$;

(ii) for each $k = 1, \ldots, r-1$ one has $i_k < i$ or $i_k > j$;

(iii) for any $\{j_1, \ldots, j_s\}$ *subset*$\{i_1, \ldots, i_r\}$, the sequence $i, j_1, \ldots, j_s, j$ is not a path.

Given an admissible path $\pi : i = i_0, i_1, \ldots, i_r = j$ from i to j, where $i < j$, define the monomial

$$u_\pi = \left(\prod_{i_k > j} x_{i_k}\right)\left(\prod_{i_\ell < i} y_{i_\ell}\right).$$

Theorem 5.2.2. *Let G be a graph on $[n]$. Let $<$ be the lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n > y_1 > \cdots > y_n$. Then the set*

$$\mathcal{G} = \bigcup_{i<j} \{u_\pi f_{ij} \mid \pi \text{ is an admissible path from } i \text{ to } j\}$$

is the reduced Gröbner basis of J_G with respect to $<$.

A finitely generated graded module M over a Noetherian graded $\mathbb{K}$-algebra R is said to satisfy the *Serre's condition* (S_r), or simply M is an (S_r) *module* if, for all $\mathfrak{p} \in \operatorname{Spec}(R)$, the inequality

$$\operatorname{depth} M_\mathfrak{p} \geq \min(r, \dim M_\mathfrak{p})$$

holds true. The Serre's conditions are strictly connected with the Cohen-Macaulayness of a module, in fact M is Cohen–Macaulay if and only if it is an (S_r) module for all $r \geq 1$.

A *simplicial complex* Δ on the set of vertices $[n]$ is a collection of subsets of $[n]$ which is closed under taking subsets, that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of Δ; the *size* of a face F is defined to be $|F|$, that is, the number of elements of F, and its dimension is defined to be $|F| - 1$. The dimension of Δ, which is denoted by $\dim(\Delta)$, is defined to be $d - 1$, where $d = \max\{|F| \mid F \in \Delta\}$. A *facet* of Δ is a maximal face of Δ with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ. A set $N \subseteq [n]$ that does not belong to Δ is called *nonface* of Δ. We say that Δ is *pure* if all facets of Δ have the same size. The link of Δ with respect to a face $F \in \Delta$, denoted by $\mathrm{lk}_\Delta(F)$, is the simplicial complex

$$\mathrm{lk}_\Delta(F) = \{G \subseteq [n] \setminus F \mid G \cup F \in \Delta\}.$$

A simplicial complex Δ is called *connected* if, for every $F, G \in \mathcal{F}(\Delta)$, there exists a sequence of facets $F = F_0, \ldots, F_m = G$ such that, for every $0 \leq i, j \leq m-1$, we have $F_i \cap F_{i+1} \neq \emptyset$ and $F_i \neq F_j$, where $i \neq j$. We say that the sequence $F = F_0, \ldots, F_m = G$ connects F and G.

Let $R = \mathbb{K}[z_1, \ldots, z_k]$ be the polynomial ring in k variables over a field $\mathbb{K}$, and let Δ be a simplicial complex on $[k]$. For every subset $F \subseteq [k]$, we set $z_F = \prod_{i \in F} z_i$. The *Stanley–Reisner ideal of* Δ over $\mathbb{K}$ is the ideal I of R which is generated by those squarefree monomials z_F with $F \notin \Delta$. In other words, $I_\Delta = (z_F \mid F \in \mathcal{N}(\Delta))$, where $\mathcal{N}(\Delta)$ denotes the set of minimal nonfaces of Δ with respect to inclusion. The *Stanley–Reisner ring of* Δ over $\mathbb{K}$, denoted by $\mathbb{K}[\Delta]$, is defined to be $\mathbb{K}[\Delta] = R/I_\Delta$.

A simplicial complex Δ is said to satisfy *Serre's condition* (S_r) *over* $\mathbb{K}$, or simply Δ is an (S_r) simplicial complex over $\mathbb{K}$, if the Stanley–Reisner ring $\mathbb{K}[\Delta]$ of Δ satisfies Serre's condition (S_r). An immediate consequence of [79, Theorem 1.4] is the following result that provides a useful combinatorial tool to check if Δ is (S_2).

Proposition 5.2.3. *Let* $\mathbb{K}$ *be a field and* Δ *a simplicial complex. Then* Δ *is* (S_2) *over* $\mathbb{K}$ *if and only if, for every face* $F \in \Delta$ *with* $\dim(\mathrm{lk}_\Delta(F)) \geq 1$*, the simplicial complex* $\mathrm{lk}_\Delta(F)$ *is connected. In particular, the* (S_2) *property of a simplicial complex is independent from* *the base field.*

5.2.2 Simplicial complex of binomial edge ideals and (S_2)-condition

The aim of this section is to prove that if S/J_G satisfies the Serre's condition (S_2), then G is an accessible graph.

Let $<$ be a monomial order on S and $\mathrm{in}(I)$ denote the initial ideal of an ideal I with respect to $<$. A consequence of [19, Theorem 1.3] is that, if I is an ideal and $\mathrm{in}(I)$ is a square-free monomial ideal, then, for any $r \in \mathbb{N}$, S/I satisfies Serre's condition (S_r) if and only if $S/\mathrm{in}(I)$ does. Since $\mathrm{in}(J_G)$ is square-free (see [19, Section 3.2]), it follows that to study the (S_2) condition for S/J_G it is sufficient to study it for $S/\mathrm{in}(J_G)$.

From now on, we fix the lexicographic order on S induced by $x_1 > x_2 > \cdots > x_n > y_1 > \cdots > y_n$.

Let $T \in \mathcal{C}(G)$ and let $G_1, \cdots, G_{c(T)}$ be the connected components induced by T. By Theorem 5.2.2, it follows immediately

$$\mathrm{in}(J_G) = (x_i y_j u_\pi \mid \pi \text{ is an admissible path from } i \text{ to } j, \text{ with } i < j),$$

and

$$\mathrm{in}(P_T(G)) = \left(\bigcup_{t\in T}\{x_t, y_t\}\right) + \sum_{k=1}^{c(T)} (x_i y_j \mid i, j \in V(G_k) \text{ and } i < j).$$

Moreover, thanks to [18], it holds

$$\mathrm{in}(J_G) = \bigcap_{T\in\mathcal{C}(G)} \mathrm{in}(P_T(G)). \tag{5.4}$$

Define

$$P_T(\mathbf{vv}) = \left(\bigcup_{t\in T}\{x_t, y_t\}\right) + \sum_{k=1}^{c(T)} (\{x_i \mid i \in V(G_k), i < v_k\} \cup \{y_j \mid j \in V(G_k), j > v_k\})$$

where $\mathbf{vv} = (v_1, \ldots, v_{c(T)}) \in V(G_1) \times \cdots \times V(G_{c(T)})$.

Lemma 5.2.4. *Let G be a graph. Let $T \in \mathcal{C}(G)$ and let $G_1, \cdots, G_{c(T)}$ be the connected components induced by T. Then*

$$\mathrm{in}(P_T(G)) = \bigcap_{\mathbf{vv}\in V(G_1)\times\cdots\times V(G_{c(T)})} P_T(\mathbf{vv}).$$

Proof. " $\subseteq$ " Let u be a generator of $\mathrm{in}(P_T(G))$. If $u \in \{x_t, y_t\}$ for $t \in T$, then $u \in P_T(\mathbf{vv})$, for all $\mathbf{vv} \in V(G_1) \times \cdots \times V(G_{c(T)})$. Let $u = x_i y_j$, with $i < j$ and $i, j \in V(G_k)$, for some $k = 1, \ldots, c(T)$, and consider v_k, the k-th component of $\mathbf{vv}$. When $v_k \leq i$, then $y_j \in P_T(\mathbf{vv})$, when $v_k > i$, then $x_i \in P_T(\mathbf{vv})$. Hence, the monomial $x_i y_j \in P_T(\mathbf{vv})$ for all $\mathbf{vv} \in V(G_1) \times \cdots \times V(G_{c(T)})$.

" $\supseteq$ " Let u be a generator of $\bigcap_{\mathbf{vv} \in V(G_1) \times \cdots \times V(G_{c(T)})} P_T(\mathbf{vv})$. If x_t divides u, for some $t \in T$, then $u \in \mathrm{in}(P_T(G))$, as well. Assume that x_t does not divide u, for any $t \in T$. For $k = 1, \ldots, c(T)$, denote $J_k = (x_i y_j \mid i, j \in V(G_k) \text{ and } i < j)$ and $I_{v_k} = (\{x_i \mid i \in V(G_k), i < v_k\} \cup \{y_j \mid j \in V(G_k), j > v_k\})$, for $v_k \in V(G_k)$. Then

$$\mathrm{in}(P_T(G)) = \left(\bigcup_{t \in T} \{x_t, y_t\} \right) + \sum_{k=1}^{c(T)} J_k$$

and

$$P_T(\mathbf{vv}) = \left(\bigcup_{t \in T} \{x_t, y_t\} \right) + \sum_{k=1}^{c(T)} I_{v_k}.$$

Note that I_{v_k} and J_k are both ideals of $S_k = \mathbb{K}[x_i, y_i]_{i \in V(G_k)}$. Moreover, I_{v_k} and I_{v_h}, with $v_k \in G_k$, $v_h \in G_h$ and $k \neq h$, are defined on a disjoint set of variable, and the same holds for the J_k's. It is sufficient to prove that

$$J_k \supseteq \bigcap_{v_k \in V(G_k)} I_{v_k}.$$

Assume that $u \in \bigcap_{v_k \in V(G_k)} I_{v_k}$. Note that u can not be the product of only x_i's (resp. y_j's). Indeed, when $v_k = \min\{a \mid a \in V(G_k)\}$ (resp. $v_k = \max\{b \mid b \in V(G_k)\}$), then no x_i belongs to I_{v_k} (resp. no y_j belongs to I_{v_k}). Now, suppose, by contradiction, that for any $x_i y_j$ that divides u, it holds $i > j$. Set $v_k = \min\{i \mid x_i \text{ divides } u\}$. Then all the x_i's and y_j's that divide u do not belong to I_{v_k}, namely $u \notin I_{v_k}$. It follows that if $x_i y_j$ divides u, then $i < j$ and $u \in J_k$. $\square$

Let $T \in \mathcal{C}(G)$ and let $G_1, \ldots, G_{c(T)}$ denote the connected components of $G \setminus T$. For $i = 1, \ldots, c(T)$, let $|V(G_i)| = m_i$ and $V(G_i) = \{v_1^i, \ldots, v_{m_i}^i\}$. Given $\mathbf{vv} = \left(v_{j_1}^1, \ldots, v_{j_{c(T)}}^{c(T)}\right) \in V(G_1) \times \cdots \times V(G_{c(T)})$, define

$$F(T, \mathbf{vv}) = \bigcup_{i=1}^{c(T)} \left\{ \{y_j \mid j \leq v_{j_i}^i\} \cup \{x_j \mid j \geq v_{j_i}^i\} \right\}.$$

Since $\text{in}(J_G)$ is a squarefree monomial ideal, then there exists a unique simplicial complex $\Delta_<$ such that $\text{in}(J_G) = I_{\Delta_<}$. By Equation (5.4) and Lemma 5.2.4, we obtain the following description of $\Delta_<$.

Corollary 5.2.5. *Let G be a graph. Then $\text{in}(J_G) = I_{\Delta_<}$, where*

$$\mathcal{F}(\Delta_<) = \bigcup_{T \in \mathcal{C}(G)} \{F(T, \mathbf{vv}) : \mathbf{vv} \in V(G_1) \times \cdots \times V(G_{c(T)})\}.$$

For a graded S-module M we denote by $H(t) = \sum_{i=0}^{d} (h_i(M)) t^i / (1-t)^d$ the Hilbert series of M and by $h = (h_0, \ldots, h_d)$ its h-vector. The following result, by a well known formula that relates f-vector with h-vector, gives a way to compute the invariant by $\Delta_<$ as defined above.

Corollary 5.2.6. *The h-vector of $\Delta_<$ is*

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}(\Delta_<).$$

for $k = 0, \ldots, d$.

In [4], authors provide a formula to compute the multiplicity of S/J_G. By knowing $\Delta_<$ such that $\text{in}(J_G) = I_{\Delta_<}$ and by Corollary 5.2.6, one can easily obtain another simple way to get the multiplicity.

In the following, we deeply use the simplicial complex $\Delta_<$ defined in Corollary 5.2.5 to prove that if S/J_G satisfies the Serre's condition (S_2), then the graph G is accessible. Nevertheless, we observe that the simplicial complex is strongly related to the chosen monomial order also for very simple graphs, as the following Example shows.

Example 5.2.7 *Let $G = P_2$ be the path on 3 vertices with $E(G) = \{\{1,2\},\{2,3\}\}$ and fix the lexicographic order on S induced by $x_1 > x_2 > x_3 > y_1 > y_2 > y_3$. Then, $\mathcal{C}(G) = \{\emptyset, \{2\}\}$ and $I_{\Delta_<} = (x_1y_2, x_2y_3)$, where*

$$\begin{aligned}\Delta_< &= \{F(\emptyset,(1)), F(\emptyset,(2)), F(\emptyset,(3)), F(\{2\},(1,3))\} \\ &= \{\{x_1,y_1,x_2,x_3\},\{y_1,x_2,y_2,x_3\},\{y_1,y_2,x_3,y_3\},\{x_1,y_1,x_3,y_3\}\}.\end{aligned}$$

One can immediately observe that all the facets in $\Delta_<$ contain the variables y_1 and x_3. Consider now the same graph but with a different vertex labelling with $E(G) = \{\{1,3\},\{2,3\}\}$. Fix the same term order for S. Then, $\mathcal{C}(G) = \{\emptyset, \{3\}\}$ and $I_{\Delta_<} = (x_1y_3, x_2y_3, x_1y_2x_3)$, where

$$\begin{aligned}\Delta_< &= \{F(\emptyset,(1)), F(\emptyset,(2)), F(\emptyset,(3)), F(\{3\},(1,2))\}\\ &= \{\{x_1,y_1,x_2,x_3\},\{y_1,x_2,y_2,x_3\},\{y_1,y_2,x_3,y_3\},\{x_1,y_1,x_2,y_2\}\}.\end{aligned}$$

In this case, only the variable y_1 is contained in all the facets of $\Delta_<$. This implies that the two simplicial complexes are not isomorphic.

Remark 5.2.8 *Let G be a graph on $[n]$. Let $T \in \mathcal{C}(G)$ and $v \in T$ be a cutpoint of G that induces two connected components, H_1 and H_2. For $i = 1, 2$, let $T_i \subseteq T \cap V(H_i)$. If T_1 and T_2 are cutsets of G, then $T_1 \cup T_2$ is a cutset of G.*

Lemma 5.2.9. *Let G be a graph on $[n]$. Let $T \in \mathcal{C}(G)$ and $v \in T$ be a cutpoint of G that induces two connected components, H_1 and H_2. For $i = 1, 2$, let $T_i = T \cap V(H_i)$. If $S_1 = T_1 \cup \{v\}$ and $S_2 = T_2 \cup \{v\}$ are accessible cutsets of G, then T is an accessible cutset of G.*

Proof. By hypothesis, S_1 and S_2 are accessible, that is there exist $v_1 \in S_1$ and $v_2 \in S_2$ such that $S_1 \setminus \{v_1\}, S_2 \setminus \{v_2\} \in \mathcal{C}(G)$. If $v_1 = v_2 = v$, then, by Remark 5.2.8, $T_1 \cup T_2 = T \setminus \{v\}$ is a cutset of G, namely T is accessible. If at least one between v_1 and v_2 is not v, assume $v_1 \neq v$, then, by Remark 5.2.8, $S_1 \setminus \{v_1\} \cup T_2 = T \setminus \{v_1\}$ is a cutset of G, namely T is accessible. □

Remark 5.2.10 *Let G be a graph and $T \in \mathcal{C}(G)$. If all the cutset T', with $T' \subset T$, are accessible, then T contains a cutpoint. The proof of this fact is the same of [9, Lemma 4.1].*

Theorem 5.2.11. *Let G be a graph such that S/J_G satisfies the Serre's condition (S_2). Then G is an accessible graph.*

Proof. To prove the statement, we suppose that G is not accessible and we show that S/J_G does not satisfy the Serre's condition (S_2). If G is not accessible then J_G is not unmixed or $\mathcal{C}(G)$ is not an accessible set system. If J_G is not unmixed, then it is known that the (S_2)-condition is not satisfied. Hence, we can suppose that J_G is unmixed but $\mathcal{C}(G)$ is not an accessible set system. Let $T \in \mathcal{C}(G)$ be the non-empty cutset with the minimum cardinality such that $T \setminus \{v\} \notin \mathcal{C}(G)$, for every $v \in T$. Let $T = \{w_1, \dots, w_k\}$, with $k > 1$, and $G_1, \dots, G_{k+1}$ be the connected components of $G \setminus T$. For $i = 1, \dots, k+1$, let $|V(G_i)| = m_i$ and $V(G_i) = \{v_1^i, \dots, v_{m_i}^i\}$.

Fix the lexicographic order on S induced by the total order

$$w_1 < \cdots < w_k < v_1^1 < \cdots < v_{m_1}^1 < \cdots < v_1^{k+1} < \cdots < v_{m_{k+1}}^{k+1} \qquad (\star)$$

Thanks to [19, Theorem 1.3], it is sufficient to prove that $S/\mathrm{in}(J_G)$ does not satisfies the Serre's condition (S_2).

Consider $\mathbf{vv} = (v_{m_1}^1, \dots, v_{m_{k+1}}^{k+1}) \in V(G_1) \times \cdots \times V(G_{k+1})$ and

$$F(T, \mathbf{vv}) = \bigcup_{i=1}^{k+1} \left\{ y_{v_1^i}, \dots, y_{v_{m_i}^i}, x_{v_{m_i}^i} \right\} \in \mathcal{F}(\Delta_<).$$

The set

$$F = \bigcup_{i=1}^{k} \left\{ y_{v_1^i}, \dots, y_{v_{m_i}^i} \right\} \cup \left\{ y_{v_1^{k+1}}, \dots, y_{v_{m_{k+1}}^{k+1}}, x_{v_{m_{k+1}}^{k+1}} \right\}$$

is a subset of $F(T, \mathbf{vv})$, that is a face of $\Delta_<$. Consider the link of $\Delta_<$ with respect to F. The sets $A = \{x_{v_{m_1}^1}, \dots, x_{v_{m_k}^k}\}$ and $B = \{y_{w_1}, \dots, y_{w_k}\}$ belong to $\mathrm{lk}_{\Delta_<}(F)$. In fact, thanks to the order $(\star)$, $A \cap F = \emptyset$ and $A \cup F = F(T, \mathbf{vv}) \in \mathcal{F}(\Delta_<)$, whereas, $B \cap F = \emptyset$ and $B \cup F = F(\emptyset, \mathbf{u}) \in \mathcal{F}(\Delta_<)$, where $\mathbf{u} = (v_{m_{k+1}}^{k+1})$. Since $|A| = |B| = k > 1$, it follows $\dim \mathrm{lk}_{\Delta_<}(F) \geq 1$. Assume, by contradiction, that $\mathrm{lk}_{\Delta_<}(F)$ is connected, that is there exists a sequence of facets $A = F_0, F_1, \dots, F_{t+1} = B$ of $\mathrm{lk}_{\Delta_<}(F)$ such that, for every $0 \leq i, j \leq t$, $F_i \cap F_{i+1} \neq \emptyset$ and $F_i \neq F_j$ when $i \neq j$. First of all, suppose that $F_t \cap B = \{y_{w_i}\}$, for some $i = 1, \dots, k$. Without loss of generality, assume $i = 1$. Then there exists $F(T',\bar{\mathbf{v}}) \in \mathcal{F}(\Delta_<)$ such that $F(T',\bar{\mathbf{v}}) = F_t \cup F$. Note that $y_{w_1} \in F(T',\bar{\mathbf{v}})$ but $y_{w_i} \notin F(T',\bar{\mathbf{v}})$, for $i \neq 1$, otherwise $F_t \cap B \supset \{y_{w_1}\}$. Since $y_{w_i} \notin F(T',\bar{\mathbf{v}})$, for $i \neq 1$, and $y_v \in F(T',\bar{\mathbf{v}})$, for $v \in (V(G) \setminus T) \cup \{w_1\}$, that is either $v = w_1$ or $v > w_k$, then $x_{w_i} \notin F(T',\bar{\mathbf{v}})$, for $i \neq 1$. From the fact that $x_{w_i}, y_{w_i} \notin F(T',\bar{\mathbf{v}})$, it follows that $T' = \{w_2, \dots, w_k\}$ and $\bar{\mathbf{v}} = (v_{m_2}^2, \dots, v_{m_{k+1}}^{k+1})$. This implies that $T' = T \setminus \{w_1\} \in \mathcal{C}(G)$, but this is in contradiction with the hypothesis that T is not an accessible cutset.

Now, suppose that $|F_t \cap B| > 1$. Note that $|F_t \cap B| < k$, otherwise $F_t \cap B = B$, that is $F_t = F_{t+1} = B$, which contradicts the hypothesis on F_i. Without loss of generality, assume $F_t \cap B = \{y_{w_1}, \ldots, y_{w_a}\}$, with $1 < a < k$. There exists $F(T'',\bar{\mathbf{v}}') \in \mathcal{F}(\Delta_<)$ such that $F(T'',\bar{\mathbf{v}}') = F_t \cup F$. For $i > a$, it holds $y_{w_i} \notin F_t$, hence $y_{w_i} \notin F(T'',\bar{\mathbf{v}}')$. Since $y_v \in F(T',\bar{\mathbf{v}})$, for every $v > w_k$, then $x_{w_i} \notin F(T',\bar{\mathbf{v}})$, for $i = 1, \ldots, k$. Therefore, $x_{w_i}, y_{w_i} \notin F(T'',\bar{\mathbf{v}}')$ for $i > a$ and $T'' = \{w_{a+1}, \ldots, w_k\}$. By hypothesis, T is the smallest not accessible cutset, then any cutset which is a proper subset of T is accessible. Since $T'' \subset T$, then T'' is accessible and, by Remark 5.2.10, T'' contains a cutpoint, we say w_{a+1}. Then w_{a+1} induces two connected components, H_1 and H_2. Let $T_i = T \cap V(H_i)$, for $i = 1, 2$. For $i = 1, 2$, $T_i \cup \{w_{a+1}\}$ is a cutset of G. By the minimality of T, both $T_1 \cup \{w_{a+1}\}$ and $T_2 \cup \{w_{a+1}\}$ are accessible cutsets of G. By Lemma 5.2.9, also $T = T_1 \cup T_2 \cup \{w_{a+1}\}$ is an accessible cutset, which is a contradiction.

It follows that $\mathrm{lk}_{\Delta_<}(F)$ is not connected, and then $S/\mathrm{in}(J_G)$ does not satisfy the Serre's condition (S_2).

□

Let G be a graph such that J_G is unmixed. The following results state that to verify the Serre's condition (S_2) for S/J_G is not necessary to check the link of all the faces F of $\Delta_<$.

Proposition 5.2.12. *Let G be a graph on $[n]$, with $n \leq 12$, such that J_G is unmixed. For all monomial order $<$ and all $F \in \Delta_<$ such that $\dim F < \lfloor \frac{n+1}{2} \rfloor$ it holds that $\mathrm{lk}_{\Delta_<}(F)$ is connected.*

Proof. We have implemented a computer program, see [51], that checks the Serre's condition (S_2) for S/J_G. By means of it, we have verified that the statement holds. In particular, there exists a unique family of graphs such that $\mathrm{lk}_{\Delta_<}(F)$ is disconnected for $F \in \Delta_<$ with $\dim F = \lfloor \frac{n+1}{2} \rfloor$, that is the one in Example 5.2.13. □

Example 5.2.13 *Let G be a graph on $[n]$ obtained by joining $s+1$ complete graphs $G_1, \ldots, G_{s+1}$ such that $G_1 = \cdots = G_s = K_{s+1}$, if n is odd $G_{s+1} = K_{s+1}$, otherwise $G_{s+1} = K_{s+2}$, and $G_i \cap G_j = H$, where $H = K_s$, for all $1 \leq i < j \leq s+1$. See Figure 5.2 for* *an example, with $n = 7$. We observe that $\mathcal{C}(G) = \{\emptyset, T\}$, where $T = V(H)$. Moreover, J_G is unmixed but G is a block that is not a complete graph, hence J_G is not Cohen-Macaulay by [4]. Fix the lexicographic order on S induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. Let $V(H) = \{n-s+1, \ldots, n\}$, and consider $F = \{y_1, \ldots, y_{n-s}, x_{n-s}\} \in \Delta_<$. Note that $\dim F = n - s = \lfloor \frac{n+1}{2} \rfloor$. The facets of the link of F in $\Delta_<$ are only two: $F(\emptyset, n-s) \setminus F$ and $F(T, (1, \ldots, n-s)) \setminus F$, which are respectively $\{x_{n-s+1}, \ldots, x_n\}$ and $\{x_1, \ldots, x_{n-s-1}\}$ and they are obviously disjoint. It follows that $\mathrm{lk}_{\Delta_<}(F)$ is disconnected.*

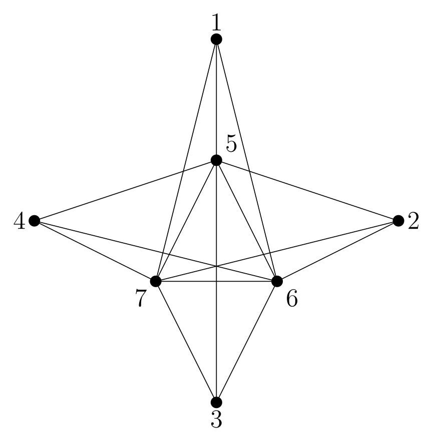


Figure 5.2

Proposition 5.2.14. *Let G be a graph on $[n]$ such that J_G is unmixed. Let $F = \{x_{i_1}, \ldots, x_{i_t}, y_{j_1}, \ldots, y_{j_s}\} \in \Delta_<$, with $1 \leq j_1 < \cdots < j_s < i_1 < \cdots < i_t \leq n$ and $\dim F \leq n-2$. Then $\mathrm{lk}_{\Delta_<}(F)$ is connected.*

Proof. If $F = \emptyset$, then $\mathrm{lk}_{\Delta_<}(F) = \Delta_<$ is connected. In fact, any facets of $\Delta_<$ have a non-empty intersection with a facet $F(\emptyset, v)$, for some $v \in V(G)$, and $F(\emptyset, v_1) \cap F(\emptyset, v_2) \neq \emptyset$, for all $v_1, v_2 \in V(G)$. Hence, assume $F = \{x_{i_1}, \ldots, x_{i_t}, y_{j_1}, \ldots, y_{j_s}\} \in \Delta_<$, with $1 \leq j_1 < \cdots < j_s < i_1 < \cdots < i_t \leq n$ and $\dim F \leq n-2$. Let F_1, F_2 be facets of $\mathrm{lk}_{\Delta_<}(F)$. If $F_1 \cap F_2 \neq \emptyset$, then they are connected and there is nothing to prove. Therefore, we may assume that $F_1 \cap F_2 = \emptyset$. $F \cup F_1$ and $F \cup F_2$ are facets of $\Delta_<$ and both of them contain y_{j_s}. By Corollary 5.2.5, there exist $x_a \in F \cup F_1$ and $x_b \in F \cup F_2$ such that $j_s \leq a, b \leq i_1$. Let $a = \min\{a \mid x_a \in F \cup F_1 \text{ and } j_s \leq a \leq i_1\}$ and $b = \min\{b \mid x_b \in F \cup F_2 \text{ and } j_s \leq b \leq i_1\}$.

Note that, if $a = b = i_1$, then $y_{i_1} \in F \cup F_i$, for $i = 1, 2$, but $y_{i_1} \notin F$, then $y_{i_1} \in F_1 \cap F_2$, which is a contradiction since F_1 and F_2 are supposed to be disjoint. Moreover, if $a, b < i_1$ and $a = b$, then $x_a \in F_1 \cap F_2$, which is a contradiction, as well. Therefore, let $a \neq b$, and, without loss of generality, suppose $a < b$. Consider the facets $F(\emptyset, \mathbf{vv})$, for $a \leq \mathbf{vv} \leq b$, namely $F(\emptyset, \mathbf{vv}) = \{x_i \mid \mathbf{vv} \leq i \leq n\} \cup \{y_j \mid 1 \leq j \leq \mathbf{vv}\}$. Note that, for all $a \leq \mathbf{vv} \leq b$, $F(\emptyset, \mathbf{vv}) \cap F = F$, hence $\bar{F}_{\mathbf{vv}} = F(\emptyset, \mathbf{vv}) \setminus F = \{x_i \mid \mathbf{vv} \leq i \leq n,\ i \neq i_1, \ldots, i_t\} \cup \{y_j \mid 1 \leq j \leq \mathbf{vv},\ j \neq j_1, \ldots, j_s\}$ is a facet of $\mathrm{lk}_{\Delta_<}(F)$. Consider the sequence $F_1, \bar{F}_a, \bar{F}_{a+1}, \ldots, \bar{F}_b, F_2$ of facets of $\mathrm{lk}_{\Delta_<}(F)$. Note that $F_1 \cap \bar{F}_a \supseteq \{x_a\}$ and $\bar{F}_b \cap F_2 \supseteq \{y_b\}$. If $i_1 = j_s + 1$, then $a = j_s$ and $b = i_1$, since $\dim F \leq \dim \Delta_< - 2$, there exists either $i^* > i_1$ such that $x_{i^*} \notin F$ or $j^* < j_s$ such that $y_{j^*} \notin F$. It follows that either $\bar{F}_a \cap \bar{F}_b \supseteq \{x_{i^*}\}$ or $\bar{F}_a \cap \bar{F}_b \supseteq \{y_{j^*}\}$, that is $F_1, \bar{F}_a, \bar{F}_b, F_2$ is a sequence of facets of $\mathrm{lk}_{\Delta_<}(F)$ that connects F_1 and F_2. If $i_1 \neq j_s + 1$ and $a + 1 \neq i_1$, it holds $\bar{F}_a \cap \bar{F}_{a+1} \supseteq \{x_{a+1}\}$ and $\bar{F}_i \cap \bar{F}_{i+1} \supseteq \{y_i\}$ for all $i = a+1, \ldots, b-1$. If $i_1 \neq j_s + 1$ and $a + 1 = b = i_1$, then $\bar{F}_a \cap \bar{F}_b = \{y_{i_1}\}$. Hence, $F_1, \bar{F}_a, \bar{F}_{a+1}, \ldots, \bar{F}_b, F_2$ is a sequence of facets of $\mathrm{lk}_{\Delta_<}(F)$ that connects F_1 and F_2. Therefore, $\mathrm{lk}_{\Delta_<}(F)$ is connected. □

5.2.3 Accessible blocks with whiskers

In this section we study a particular class of accessible graphs. We know from [9, Theorem 4.12] and [4] that if an accessible graph is a block, then it is a complete graph. It arises a natural question:

"Under which hypotheses a block with whiskers is accessible?"

Let G be a connected graph such that J_G is unmixed and B be a block of G. Denote by $W = \{w_1, \ldots, w_r\}$ the set of cutpoints of G which are vertices of B. Then

$$G = B \cup \left(\bigcup_{i=1}^{r} G_i \right) \tag{5.5}$$

where $V(G_i) \cap V(B) = \{w_i\}$ for $i = 1, \ldots, r$, and $B \setminus W, G_1 \setminus \{w_1\}, \ldots, G_r \setminus \{w_r\}$ are the connected components of $G \setminus W$.

By the decomposition (5.5), we define a *block with whiskers*, namely $\bar{B}$, a graph obtained, roughly speaking, by replacing each subgraph G_i with a whisker. That is

1. $V(\bar{B}) = V(B) \cup \{f_1, \ldots, f_r\}$;
2. $E(\bar{B}) = E(B) \cup \{\{w_i, f_i\} \mid i = 1, \ldots, r\}$.

Note that $V(\overline{B}) = V(G)/\sim$, where the relation $\sim$ identifies each vertex of B with itself and, for $i = 1, \ldots, r$, if $a, b \in V(G_i) \setminus \{w_i\}$, then $a \sim b$, and we denote by f_i the equivalence class of $V(G_i) \setminus \{w_i\}$.

Proposition 5.2.15. *Let G be an accessible graph and let B be a block of G. The graph $\overline{B}$ constructed as above is accessible.*

Proof. Let $\pi : V(G) \to V(G)/\sim$ be the canonical projection. Let $T \in \mathcal{C}(\overline{B})$. By construction, for any $i = 1, \ldots, r$ f_i is a free vertex of $\overline{B}$, hence $T \subset V(B)$. Denote by $\bar{\pi}$ the restriction of π to $V(G) \setminus T$. We prove that $\bar{\pi}$ induces a bijection between the connected components of $G \setminus T$ and the ones of $\overline{B} \setminus T$.

Let A be a connected component of $G \setminus T$. For any $i = 1, \ldots, r$, let G_i be the connected component of $G \setminus W$, where W is the set of all the cutpoints of $\overline{B}$. Let $a, b \in A$, and $a, a_1, \ldots, a_\ell, b$ be a path in $V(G) \setminus T$ from a to b. If a and b belong to the same G_i, then $\bar{\pi}(a) = \bar{\pi}(a_j) = \bar{\pi}(b) = f_i$, for all $j = 1, \ldots, \ell$. Therefore, they are obviously connected in $\overline{B} \setminus T$. If $a \in B$, and $b \in G_i$, then there exists j such that $a_j, \ldots, a_\ell \in G_i \cup \{w_i\}$ with, in particular, $a_j = w_i$. Then $\bar{\pi}(a) = a, \bar{\pi}(a_1) = a_1, \ldots, \bar{\pi}(a_{j-1}) = a_{j-1}, f_i$ is a path from $\bar{\pi}(a)$ and $\bar{\pi}(b) = f_i$. The other cases follow by the same argument. Therefore, if A is a connected component of $G \setminus T$, then $\bar{\pi}(A)$ is a connected component of $\overline{B} \setminus T$.

Let D be a connected component of $\overline{B} \setminus T$. Let $c, d \in D$ and let $c, u_1, \ldots, u_\ell, d$ be a path in D from c to d. Note that, by the definitions of path and $\overline{B}$, for $i = 1, \ldots, \ell$, $u_i \in V(B) \setminus T$, that is $\bar{\pi}^{-1}(u_i) = u_i$. If $c = f_j$ (resp. $d = f_j$) for some $j = 1, \ldots, r$, then set $\bar{\pi}^{-1}(c) = v$ (resp. $\bar{\pi}^{-1}(d) = v$), where $v \in V(H_j)$ and $\{w_j, v\} \in E(G)$. Otherwise, $\bar{\pi}^{-1}(c) = c$ (resp. $\bar{\pi}^{-1}(d) = d$). Then, $\bar{\pi}^{-1}(c), u_1, \ldots, u_\ell, \bar{\pi}^{-1}(d)$ is a path in $V(G) \setminus T$. It follows that if D is a connected component of $\overline{B} \setminus T$, then $(D \setminus \{f_j\}_{j \in J}) \cup \bigcup_{j \in J} G_j$ is a connected component of $G \setminus T$, where J is the set of indices such that $f_j \in D$.

The bijection between the connected components of $G \setminus T$ and the ones of $\overline{B} \setminus T$ implies $c_G(T) = c_{\overline{B}}(T)$. Since J_G is unmixed by hypothesis, then $J_{\overline{B}}$ is unmixed, as well. Moreover, if $T \in \mathcal{C}(\overline{B})$, then $T \in \mathcal{C}(G)$. Due to the accessibility of G, there exists a vertex a such that $T \setminus \{a\} \subset V(B)$ is a cutset of G and so, using the bijection, $T \setminus \{a\}$ is a cutset of $\overline{B}$, that $\overline{B}$ is accessible. $\square$

A block with a fixed number of vertices, say n, and minimum number of edges is a cycle C_n. It is useful to connect the degree of the vertices with the cycle rank.

Lemma 5.2.16. *Let G be a connected graph. The cycle rank of G is*

$$m(G) = 1 + \frac{\sum_{v \in V(G)} (\deg v - 2)}{2}.$$

Proof. From ([34, Theorem 4.5(a)]), we know $m(G) = q - p + 1$ where $q = |E(G)|$ and $p = |V(G)|$. We can see

$$2q = \sum_{v \in V(G)} \deg v \qquad \text{and} \qquad p = \sum_{v \in V(G)} 1.$$

So, we conclude that

$$m(G) = q - p + 1 = 1 + \frac{\sum_{v \in V(G)} (\deg v - 2)}{2}.$$

□

By the previous lemma, we observe that fixed a cycle rank of G the number of vertices with degree greater than 2 is bounded, but we do not have any information on the number of vertices v with $\deg v \leq 2$. We will show that under the hypothesis of accessibility this cardinality is bounded, too.

Now we are going to state some general results for accessible blocks that we are going to exploit for the classification of accessible graphs with cycle rank 3 and in Subsection 5.2.4. Let us introduce some notation.

Definition 5.2.17 *Given a block B, we say that a vertex $v \in V(B)$ is pivotal if* $\deg v \geq 3$.

Definition 5.2.18 *Let B be a block and let $a, b \in V(B)$ be two pivotal vertices. A path L_i of length i from a to b and such that any $v \in V(L_i) \setminus \{a, b\}$ is not pivotal is said a line from a to b.*

Lemma 5.2.19. *Let G be an accessible graph and B a block of G. If two pivotal vertices a, b of B are connected by a line L_i, with $i \geq 2$, then a is a cutpoint in $\overline{B}$ and b is not. Moreover, the following conditions hold:*

1. *$i < 4$;*
2. *if $i = 3$, there exists a unique vertex $c \in V(L_i) \setminus \{a, b\}$ which is a cutpoint in $\overline{B}$. In particular, c is such that $\{a, c\} \in E(G)$;*
3. *if $m(G) \geq 3$, there are no other lines L_j from a to b, with $j \in \{2, 3\}$.*

Proof. By Proposition 5.2.15, we can focus on the graph $\overline{B}$ which is accessible, too.

Let a and b be two pivotal vertices of B connected by a line L_i, with $i \geq 2$. We observe that $T = \{a, b\}$ is a cutset of B, and hence of $\overline{B}$. In fact, $B \setminus T$ consists of at least two connected components: $L_i \setminus \{a, b\}$ and $B \setminus L_i$. Since $\overline{B}$ is accessible, at least one between a and b has to be a cutpoint, assume a. Namely, there is a whisker $\{a, f\} \in E(\overline{B})$. Moreover, at most one of them is a cutpoint, otherwise there should be another whisker $\{b, f'\}$ and $c_{\overline{B}}(T) = 4$, namely $\{f\}$, $\{f'\}$, $L_i \setminus \{a, b\}$ and $\overline{B} \setminus (L_i \cup \{f, f'\})$.

From now on, we assume that a is a cutpoint in $\overline{B}$, while b is not.

(1) Let $L_i = a, a_1, \cdots, a_{i-1}, b$ be a line from a to b. Assume $i \geq 4$. $T = \{a, a_2\} \in \mathcal{C}(\overline{B})$ and using the same argument of above, a_2 is not a cutpoint and $\overline{B} \setminus T$ consists of three connected components: $\{f\}$, $\{a_1\}$ and $\overline{B} \setminus (T \cup \{a_1\})$. At the same time, $T' = \{a_2, b\} \in \mathcal{C}(\overline{B})$ but it induces only two connected components: $\{a_3, \ldots, a_{i-1}\}$ and $\overline{B} \setminus (L_i \setminus \{a, a_1\})$, which is a contradiction.

(2) Let $i = 3$ and $L_3 = a, a_1, a_2, b$ be a line from a to b. Since $T = \{a_1, b\} \in \mathcal{C}(\overline{B})$, $\overline{B}$ is accessible and b is not a cutpoint of $\overline{B}$, then a_1 is a cutpoint of $\overline{B}$. Moreover, since $T' = \{a, a_2\} \in \mathcal{C}(\overline{B})$, then a_2 is not a cutpoint otherwise, $c_{\overline{B}}(T) = 4$.

(3) Suppose there are two lines $L'_j \neq L_i$, with $i, j \in \{2, 3\}$, from a to b. Consider the cutset $T = \{a, b\}$. Then, $\overline{B} \setminus T$ consists of at least 4 connected components: $\{f\}$, $L_i \setminus \{a, b\}$, $L'_j \setminus \{a, b\}$, and $\overline{B} \setminus (L_i \cup L'_j)$, which is a contradiction.

□

Lemma 5.2.20. *Let G be an accessible graph and B a block of G. If two pivotal vertices a, b of B are connected by a line L_3, then $\{a, b\} \in E(B)$.*

Proof. It is sufficient to show that the vertices a and b are not separable. By Lemma 5.2.19, a is a cutpoint in $\overline{B}$ and let $\{a, f\} \in E(\overline{B})$ be the whisker on a. Then,

$$G \setminus \{a, b\} = \{f\} \sqcup (L_3 \setminus \{a, b\}) \sqcup H,$$

where H is a non-empty connected component of $G \setminus \{a, b\}$. Assume by contradiction that a and b are separable. Let $L_3 = a, a_1, a_2, b$ be a line from a to b and let T be a minimal cutset that separates a and b. T has vertices in $L_3 \setminus \{a, b\}$ and in H. If $a_1 \in T$, then $T' = (T \setminus \{a_1\}) \cup \{a_2\}$ is a cutset, as well. By Lemma 5.2.19 (2), a_1 is a cutpoint, but a_2 is not. Therefore, $|T| = |T'|$ but $c(T) = c(T') + 1$, which is a contradiction. □

As an application, by means of the implementation described in Subsection 5.2.5, we will prove that the accessible blocks with whiskers of cycle rank 3 are the ones in Figures 5.4 and 5.5. From Lemma 5.2.16, we have a bound on the number of pivotal vertices and, when $m(G) = 3$, it holds

$$\sum_{v \text{ pivotal vertices of } G} (\deg v - 2) = 2\,(m(G) - 1) = 4.$$

All of the possible blocks with cycle rank 3 are showed in Figure 5.3, where the dot points denote pivotal vertices v, the number is $\deg v - 2$ and the dashed line represents a line from a pivotal vertex to another. As regards accessible graphs $\overline{B}$ with $m(\overline{B}) = 3$, they are obtained from the blocks B in Figure 5.3 by adding opportune whiskers. By Lemma 5.2.19, there are no accessible graphs obtained from the blocks in the class of Figure 5.3 (A). In Figures 5.4 and 5.5, all the accessible graphs $\overline{B}$ with $m(\overline{B}) = 3$ are displayed. As regards Figure 5.4, the graphs (1)–(4) are obtained from the ones in Figure 5.3 (B), while the graph (5) from the ones in Figure 5.3 (C). These five graphs are chain of cycles that we characterize in the next section. Finally, the graphs in Figure 5.5 are all obtained from the blocks in Figure 5.3 (D). In particular, they are obtained by the complete graph K_4 substituting any edge by a line L_i, with $i \in 1, 2, 3$, and by adding whiskers in order to have accessibility of the graph. We denote this class of graphs by $\mathcal{K}_4$.

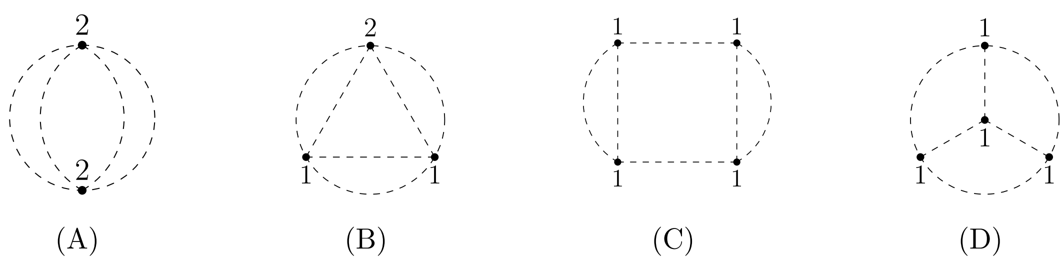


Figure 5.3: All classes of blocks having cycle rank 3.

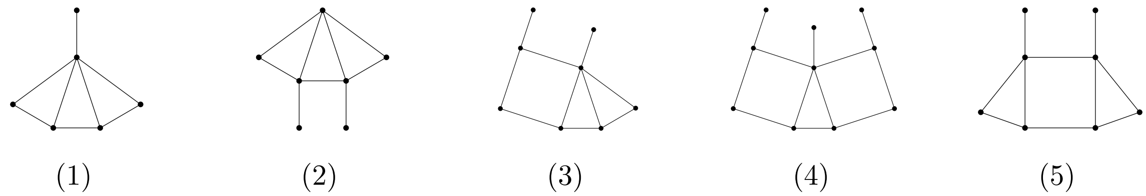


Figure 5.4: The accessible chains of cycles with cycle rank 3.

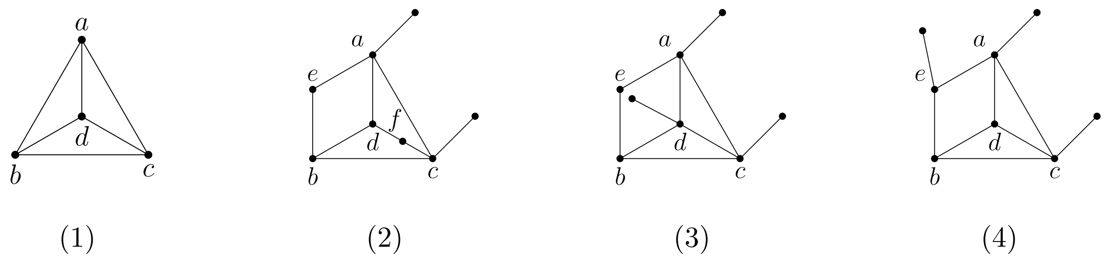


Figure 5.5: The class $\mathcal{K}_4$.

In the next results, by focusing on the lines connecting two pivotal vertices, we exhibit that, starting from blocks belong to the class (D) of Figure 5.3, there are no other possible accessible blocks with whiskers than the graphs (1)–(4) in Figure 5.5.

Lemma 5.2.21. *Let $\overline{B}$ be an accessible graph such that B is a block with $m(B) = 3$ that belongs to the class (D) of Figure 5.3. Then in B there are at most two lines L_2 that have no vertex in common and there is no line L_3.*

Proof. Let a, b, c, d be the pivotal vertices of $\overline{B}$. Without loss of generality, assume that there are two lines L_2 in B having a vertex in common: one from a to b and a second one from a to c. We claim that a has a whisker in $\overline{B}$ and b and c have no whiskers. In fact, $\{a, b\}$ and $\{a, c\} \in \mathcal{C}(\overline{B})$. By Lemma 5.2.19, either a has a whisker or both b and c have whiskers. Moreover, $T = \{a, b, c\} \in \mathcal{C}(\overline{B})$ and if b and c have whiskers $c(T) = 5$. Hence the claim follows.

Let a_1 (resp. a_1') be the vertex of degree 2 in the line L_2 from a to b (resp. to c). Let $T' = \{c, d, a_1\} \in \mathcal{C}(\overline{B})$ and $T'' = \{b, d, a_1'\} \in \mathcal{C}(\overline{B})$. We observe that there are no subsets of T' (resp. T'') disconnecting the block. Hence d, a_1 and a_1' have whiskers. But, for $T''' = \{d, a_1, a_1'\} \in \mathcal{C}(\overline{B})$, it holds $c(T''') = 5$, which is a contradiction.

Finally, suppose by contradiction that we have a line L_3 from a to b. By Lemma 5.2.20, $\{a, b\} \in E(G)$. This implies that the cycle rank of G is greater than 3. $\square$

Corollary 5.2.22. *The accessible graphs $\overline{B}$ such that B belongs to the class in Figure 5.3 (D) are all and only the graphs in $\mathcal{K}_4$ displayed in Figure 5.5.*

Proof. If B has no line L_2, then $\overline{B}$ is a K_4 with or without whiskers (Figure 5.5 (1)).

If B has 2 lines L_2, $\overline{B}$ is a bipartite graph and the only accessible bipartite graph with cycle rank 3 is the one in Figure 5.5 (2).

Suppose B has exactly one line L_2. Assume it is from a to b and denote by e the unique vertex of degree 2 in L_2. Let c and d be the other 2 pivotal vertices. We observe that the non-empty cutsets of B are $\{a, b\}$ and $\{c, d, e\}$. By Lemma 5.2.19, without loss of generality, we may assume that a has a whisker and b has no whisker. Since $\{c, d, e\}$ has cardinality 3 and none of its subsets is a cutset of the block, we have that exactly 2 vertices in $\{c, d, e\}$ have a whisker. That is either both c and d have a whisker, or one whisker is on e and the other one is, without loss of generality, on c. Then the obtained $\overline{B}$ are the non-bipartite and non-complete graphs (3) and (4) in Figure 5.5.

$\square$

5.2.4 Chain of cycles

In this section, we define a new family of graphs, the chain of cycles, and we classify the ones with Cohen-Macaulay binomial edge ideal by means of combinatorial properties.

Given a graph G, we denote by G_v the graph obtained from G by adding edges $\{u, w\}$ to $E(G)$ for all $u, w \in V(G)$ adjacent to v. We recall the following definition given first in [9].

Definition 5.2.23 *Let G be a graph. J_G is strongly unmixed if the connected components of G are complete graphs or if J_G is unmixed and there exists a cutpoint v of G such that $J_{G\setminus\{v\}}$, J_{G_v} and $J_{G_v\setminus\{v\}}$ are strongly unmixed.*

Definition 5.2.24 *Let B be a block with $m(B) = r$ such that $B = \bigcup_{i=1}^{r} D_i$ where D_i are cycles, and if $j = i + 1$ then $E(D_i) \cap E(D_j) = E(P)$, where P is a path, otherwise*

$E(D_i) \cap E(D_j) = \emptyset$. We call B a chain of cycles.

Lemma 5.2.25. *Let $\overline{B}$ be an accessible graph such that $B = \bigcup_{i=1}^{r} D_i$ is a chain of cycles. Then $D_i \in \{C_3, C_4\}$ and $E(D_i) \cap E(D_{i+1})$ is an edge of B.*

Proof. If $r \in \{1, 2\}$, the claim follows by [68]. From now on, assume $r \geq 3$, that is $m(B) \geq 3$.

Let $i = 1$, and let $a, b \in V(D_1) \cap V(D_2)$ be pivotal vertices of B. By Lemma 5.2.19, there is a unique line L_i, with $i \in \{2, 3\}$, from a to b. Hence, we may assume $E(D_1) \cap E(D_2)$ is an edge and D_1 is either C_3 or C_4. By the same argument, D_r has the same property.

Let $i \in \{2, \dots, r-1\}$ and let $a, b \in V(D_i) \cap V(D_{i+1})$ be pivotal vertices of B. $T = \{a, b\}$ is a cutset of $\overline{B}$ and since $\overline{B}$ is accessible, either a or b is a cutpoint in $\overline{B}$. Therefore, $E(D_i) \cap E(D_{i+1})$ is an edge, due to the unmixedness of $J_{\overline{B}}$.

Let $a, b \in V(D_{i-1}) \cap V(D_i)$ and $c, d \in V(D_i) \cap V(D_{i+1})$ be pivotal vertices of B. Let $T = \{a, b\}$ and $T' = \{c, d\}$. Assume that $c \notin T$ and, without loss of generality, L_j is a line from a to c. We will prove that $j = 1$. By contradiction, suppose $j > 1$. Hence $T'' = \{a, c\}$ is a cutset. By Lemma 5.2.19 applied to T, in T' and T'' there are two distinct vertices $u, v \in \{a, b, c\}$ that are cutpoints. We obtain a contradiction since $c_{\overline{B}}(\{u, v\}) = 4$.

It follows that $\{a, c\}$ is an edge and either $b = d$ or $\{b, d\}$ is an edge. That is D_i is either C_3 or C_4. ☐

Remark 5.2.26 *By Lemma 5.2.25 we can relabel the vertices of B so that $V(D_i) \cap V(D_{i+1}) = \{w_i, u_i\}$ and such that if $w_i \neq w_{i+1}$ (resp. $u_i \neq u_{i+1}$) then the edge $\{w_i, w_{i+1}\}$ (resp. $\{u_i, u_{i+1}\}$) belongs to $E(D_{i+1})$ and does not belong to any cycle D_j for $j \neq i + 1$.*

Lemma 5.2.27. *Let $\overline{B}$ be an accessible graph such that $B = \bigcup_{i=1}^{r} D_i$ is a chain of cycles. Following the labelling defined in Remark 5.2.26, every w_i is a cutpoint in $\overline{B}$ and u_i is not a cutpoint in $\overline{B}$.*

Proof. We observe that $\{w_1, u_1\}$ is a cutset of $\overline{B}$. Hence, due to accessibility of $\overline{B}$ either w_1 or u_1 is a cutpoint in $\overline{B}$. Without loss of generality, we may assume w_1 is a cutpoint. We observe that also $\{u_1, w_2\}$, $\{w_1, u_2\}$ are cutsets of $\overline{B}$. Hence, w_2 must be a cutpoint and u_2 cannot be a cutpoint. Applying the same argument for all $\{w_i, u_i\}$, the assertion follows. $\square$

Remark 5.2.28 *From now on, thanks to Lemma 5.2.20 and Lemma 5.2.27, we may consider the following partition of the set of vertices of* B*:*

$$V(B) = W \sqcup U,$$

where W *consists of all the cutpoints of* $\overline{B}$*, and* $U = V(B) \setminus W$*. We observe that the induced subgraphs on* W *and* U *(respectively) are paths.*

Lemma 5.2.29. *Let* $\overline{B}$ *be an accessible graph such that* $B = \bigcup_{i=1}^{r} D_i$ *is a chain of cycles. If* $D_i = C_4$*, then* $D_{i+1} = C_3$*.*

Proof. By contradiction, suppose that D_i and D_{i+1} are both C_4. By Lemma 5.2.27, w_{i-1}, w_i , w_{i+1} are all cutpoints while u_{i-1}, u_i, u_{i+1} are not cutpoints. We can see that $T = \{w_{i-1}, u_i, w_{i+1}\} \in \mathcal{C}(\overline{B})$ and $c(T) = 5$. Contradiction. $\square$

Lemma 5.2.30. *Let* $\overline{B}$ *be an accessible graph such that* $B = \bigcup_{i=1}^{r} D_i$ *is a chain of cycles. Let* $v \in V(B)$ *with* $\deg(v) \geq 5$ *or* $\deg(v) \geq 4$ *if* v *is a vertex of a* C_4*. Then* v *is a cutpoint.*

Proof. By hypothesis, we can identify $T_i = \{v, v_i\} \in \mathcal{C}(B)$ for $i = 1, 2, 3$, with $\{v_1, v_2\}$,$\{v_2, v_3\} \in E(B)$. Since $\overline{B}$ is accessible, we obtain that each T_i contains exactly a cutpoint. By contradiction, assume that v is not a cutpoint. The latter implies that v_1, v_2 and v_3 belong to W, namely they are cutpoints in $\overline{B}$. We observe that $T = \{v, v_1, v_3\} \in \mathcal{C}(\overline{B})$, but $c(T) = 5$. Contradiction. $\square$

Remark 5.2.31 *Let* G *be a graph and let* $v, w \in V(G)$ *with* $v \neq w$*. Then* $(G \setminus \{v\})_w = G_w \setminus \{v\}$*. Clearly* $V((G \setminus \{v\})_w) = V(G_w \setminus v) = V(G \setminus \{v\})$*. We have:*

$$E(G_w \setminus \{v\}) = (E(G) \cup \{\{x, y\} \mid x, y \in N_G(w)\}) \setminus \{\{v, u\} \mid u \in N_{G_w}(v)\}.$$

Moreover, we observe that $N_{G_w}(v)$ *is either equal to* $N_G(v)$ *if* $\{v, w\} \notin E(G)$ *or to* $N_G(v) \cup N_{G\setminus\{v\}}(w)$ *if* $\{v, w\} \in E(G)$*, that is*

$$\begin{aligned} E(G_w \setminus \{v\}) &= (E(G) \setminus \{\{v, u\} \mid u \in N_G(v)\}) \cup \{\{x, y\} \mid x, y \in N_{G\setminus\{v\}}(w)\} \\ &= E((G \setminus \{v\})_w). \end{aligned}$$

Lemma 5.2.32. *Let G be a graph such that J_G is unmixed and let $v \in V(G)$ be a free vertex of G. If $J_{G\setminus\{v\}}$ is strongly unmixed, then J_G is strongly unmixed.*

Proof. We proceed by induction on the cardinality of $\mathcal{C}(G\setminus\{v\})$, hence set $r = |\mathcal{C}(G\setminus\{v\})|$. If $r = 0$, then $G \setminus \{v\}$ is a complete graph. The latter implies that G is a complete graph with or without a whisker, and it is immediate to see that J_G is strongly unmixed.
We assume $r > 0$ and the thesis true for any graph $G \setminus \{v\}$ with $|\mathcal{C}(G \setminus \{v\})| < r$. Let $\{w\} \in \mathcal{C}(G \setminus \{v\})$ such that the binomial edge ideals of $(G \setminus \{v\}) \setminus \{w\}, (G \setminus \{v\})_w$, and $(G\setminus\{v\})_w\setminus\{w\}$ are strongly unmixed. We observe that w is also a cutpoint for G, otherwise $\{v, w\}$ is a cutset for G contradicting the fact that v is a free vertex. Set $H = G \setminus \{v\}$. Since $H \setminus \{w\}$ is strongly unmixed, then it is unmixed and from [9, Proposition 5.2] we have

$$\mathcal{C}(H \setminus \{w\}) = \{S \subset V(H \setminus \{w\}) : S \cup \{w\} \in \mathcal{C}(H)\},$$

and since $\varnothing \in \mathcal{C}(H)$ cannot be expressed as $S\cup\{w\}$, then $|\mathcal{C}(H\setminus\{w\})| < r$. From [9, Lemma 4.5.(1)], we have that $\mathcal{C}(H_w) \subseteq \mathcal{C}(H)$ and $\{w\} \in \mathcal{C}(H) \setminus \mathcal{C}(H_w)$, that is $|\mathcal{C}(H_w)| < r$. From Lemma [9, Lemma 5.5], one has $\mathcal{C}(H_w \setminus \{w\}) \subseteq \mathcal{C}(H_w)$, that is $|\mathcal{C}(H_w \setminus \{w\})| < r$. From Remark 5.2.31, it follows that $(G \setminus \{v\})_w = G_w \setminus \{v\}$ and $(G \setminus \{v\})_w \setminus \{w\} = G_w \setminus \{v, w\}$. By combining the latter with the computation above, one has that from the inductive hypothesis the assertion follows. □

Lemma 5.2.33. *Let G_1 and G_2 be two graphs and let $G = G_1 \cup G_2$ be such that $V(G_1) \cap V(G_2) = \{v\}$, with v free vertex of G_1 and G_2. The following conditions are equivalent:*

1. *J_{G_1} and J_{G_2} are strongly unmixed (resp. G_1 and G_2 are accessible);*
2. *J_G is strongly unmixed (resp. G is accessible).*

Proof. With respect to accessibility the two conditions are equivalent by [66, Proposition 2.6] and [66, Lemma 2.3]. Now we focus on strong unmixedness.

(1)⇒(2). By [66, Proposition 2.6], J_G is unmixed. Let $\{v_1, \dots, v_r\} \subset V(G_1)$ such that v_i is a cutpoint of $H_i = G_1 \setminus \{v_1, \dots, v_{i-1}\}$ and J_{H_i} is strongly unmixed. Let $\{u_1, \dots, u_s\} \subset V(G_2)$ be the set satisfying the same property for G_2.

Since v is a free vertex, it is neither a cutpoint of G_1 nor a cutpoint of G_2. Moreover, for any $a \in G_i$, v remains a free vertex of $G_i \setminus \{a\}$.

We claim that G is strongly unmixed with respect to the sequence of vertices

$$v_1, \dots, v_r, u_1, \dots, u_s,$$

adding v if necessary.

By [66, Proposition 2.6], $G \setminus \{v_1\}$ is decomposable in H_1 and G_2 whose ideals are both unmixed. Hence $J_{H_1 \cup G_2}$ is unmixed, as well. By the same argument, we can remove the remaining vertices $\{v_2, \dots, v_r, u_1, \dots, u_s\}$ obtaining unmixed ideals. Now either all the components are complete graphs or there is only one containing v that is decomposable into 2 complete graphs. In this case, we add v to the sequence of cutpoints.

(2)⇒(1). We proceed by induction on the cardinality r of of $\mathcal{C}(G)$. We observe that $r \geq 2$ since G is decomposable, hence we take $r = 2$ as base case. In this case, $\mathcal{C}(G) = \{\varnothing, \{v\}\}$, v is the unique cutpoint and $G_1 \setminus \{v\}$ and $G_2 \setminus \{v\}$ are both complete graphs, that is G_1 and G_2 are complete graphs and the thesis follows. We assume $r > 2$ and that the thesis holds true for any graph H with $|\mathcal{C}(H)| \leq r - 1$. Since J_G is strongly unmixed, we take a cutpoint w of G such that $J_{G \setminus \{w\}}$, J_{G_w} and $J_{G_w \setminus \{w\}}$ are strongly unmixed. If $w = v$, then we obtain that $J_{G_1 \setminus \{v\}}$ and $J_{G_2 \setminus \{v\}}$ are strongly unmixed, and since v is a free vertex of G_1 and G_2, then the assertion follows from Lemma 5.2.32. If $w \neq v$, we assume without loss of generality that $w \in V(G_1 \setminus \{v\})$. We obtain that $G \setminus \{w\}$ has two connected components, one $H = H_1 \cup G_2$ with $V(H_1) \cap V(G_2) = \{v\}$ and another component H_2. From the strong unmixedness of $J_{G \setminus \{w\}}$ and since $|\mathcal{C}(H)| \leq r - 1$, then from the inductive hypothesis we obtain that J_{H_1}, J_{G_2} and J_{H_2} are strongly unmixed and since $G_1 \setminus \{w\} = H_1 \cup H_2$, then $J_{G_1 \setminus \{w\}}$ is also strongly unmixed. By similar arguments, one can prove that also $J_{(G_1)_w}$ and $J_{(G_1)_w \setminus \{w\}}$ are strongly unmixed, that is J_{G_1} is strongly unmixed.

□

Set-up 5.2.34 *Let $\overline{B}$ be a block with whiskers, where $B = \bigcup_{i=1}^{r} D_i$ is a chain of cycles, satisfying the following properties:*

(i) each $D_i \in \{C_3, C_4\}$;

(ii) if $D_i = C_4$ then $D_{i+1} = C_3$;

(iii) $E(D_i) \cap E(D_{i+1}) = \{\{w_i, u_i\}\}$, where w_i is a cutpoint and u_i is not a cutpoint;

(iv) $\{w_i, w_{i+1}\} \in E(D_{i+i})$ *(resp.* $\{u_i, u_{i+1}\} \in E(D_{i+1})$*) or* $w_i = w_{i+1}$ *(resp.* $u_i = u_{i+1}$*);*

(v) if $D_1 = C_4$ *with* $V(D_1) = \{w_0, w_1, u_0, u_1\}$ *with* $\{w_0, w_1\}, \{u_0, u_1\} \in E(D_1)$ *then* w_0 *and* w_1 *are cutpoints, whereas* u_0 *and* u_1 *are not cutpoints;*

(vi) if $D_r = C_4$ *with* $V(D_r) = \{w_r, w_{r+1}, u_r, u_{r+1}\}$ *with* $\{w_r, w_{r+1}\}, \{u_r, u_{r+1}\} \in E(D_r)$ *then* w_r *and* w_{r+1} *are cutpoints, whereas* u_r *and* u_{r+1} *are not cutpoints;*

(vii) if $v \in V(B)$ *with* $\deg(v) \geq 5$ *or* $\deg(v) \geq 4$ *with* v *a vertex of a* C_4 *then* v *is a cutpoint.*

In Figure 5.6, an example of a graph $\overline{B}$ satisfying Setup 5.2.34 is displayed.

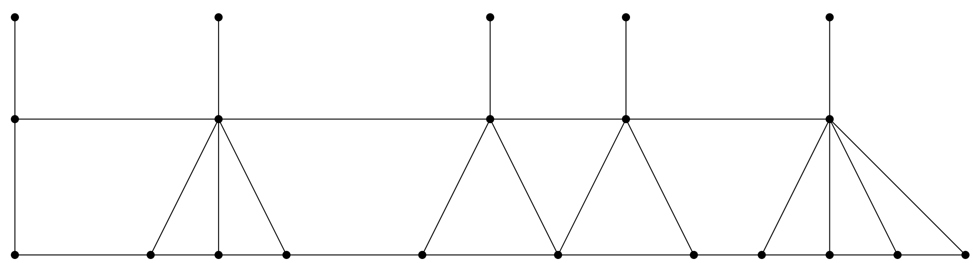


Figure 5.6: A graph $\overline{B}$ satisfying Setup 5.2.34

Lemma 5.2.35. *Let* $\overline{B}$ *be a graph satisfying Setup 5.2.34, and let* $T \in \mathcal{C}(\overline{B})$*. Then for all* $u \in U \cap T$ *there exists* $w \in W \cap T$ *such that* $\{u, w\} \in \mathcal{C}(\overline{B})$*.*

Proof. By contradiction, assume that there exists $u \in T \cap U$ such that any vertex $w \in W$ for which $\{u, w\} \in \mathcal{C}(\overline{B})$ does not belong to T. Let $T' = T \setminus \{u\}$. We prove that $c_{\overline{B}}(T) = c_{\overline{B}}(T')$. Let H be the connected component of $\overline{B} \setminus T'$ containing u. We prove that $H \setminus u$ is connected. Let $v, v' \in V(H \setminus \{u\})$ and let $\pi : v, v_1, \dots, v_\ell, v'$ be a path in H from v to v'. If $u \notin V(\pi)$, then v and v' are connected in $H \setminus \{u\}$ through π. If $u \in V(\pi)$, then $\pi : v, v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_\ell, v'$.

We claim that there exists a path $v_{i-1}, z_1, \dots, z_m, v_{i+1}$ with $\{u, z_j\} \in \mathcal{C}(\overline{B})$ and $z_j \notin T$ for any $j \in \{1, \dots, m\}$. If $v_{i-1}, v_{i+1} \in W$, then $v_{i-1} = w_j$, $v_{i+1} = w_k$ with $j < k$ as in the Setup 5.2.34, hence the vertices $w_{j+1}, \dots, w_{k-1}$ make a path between w_j and w_k. Furthermore being w_j, w_k adjacent to u, then $\{u, w_{j+1}\}, \dots, \{u, w_{k-1}\} \in \mathcal{C}(\overline{B})$ and in particular $w_{j+1}, \dots, w_{k-1} \notin T$. In this case, the claim follows.

Now, we deal with the case v_{i-1} or $v_{i+1} \in U$. Observe that any vertex $u' \in U$ adjacent to u is also adjacent to a vertex $w' \in W$ such that $\{u, w'\} \in \mathcal{C}(\overline{B})$. In fact, let D_k be the cycle containing u and u'. The vertex $w' \neq u$ adjacent to u' that belongs to D_k is such that $\{u, w'\}$ disconnects u' from the rest of the graph. That is, if one or both of v_{i-1}, v_{i+1} are in U, by the previous arguments we find the desired path in W. In any of the above cases, we find that $H \setminus \{u\}$ is connected, that is $T \notin \mathcal{C}(\overline{B})$. Contradiction. □

Corollary 5.2.36. *Let $\overline{B}$ be a graph satisfying Setup 5.2.34, and let $T \in \mathcal{C}(\overline{B})$. Then for any $u \in U \cap T$ we have $T' = T \setminus \{u\} \in \mathcal{C}(\overline{B})$. In particular, $\mathcal{C}(\overline{B})$ is an accessible set system.*

Proof. Let $a \in T'$. If $a \in U$, then from Lemma 5.2.35 there exists $b \in W \cap T$ such that $\{a, b\} \in \mathcal{C}(\overline{B})$. In particular, $b \in T'$ and $c_{\overline{B}}(T') > c_{\overline{B}}(T' \setminus \{a\})$. If $a \in W$, namely a is a cutpoint of $\overline{B}$, then $c_{\overline{B}}(T') > c_{\overline{B}}(T' \setminus \{a\})$.
Furthermore, for any non-empty $T \in \mathcal{C}(\overline{B})$ if $u \in T \cap U \neq \varnothing$, then $T' = T \setminus \{u\} \in \mathcal{C}(\overline{B})$, while if $T \cap U = \varnothing$, then any $w \in T$ is a cutpoint, hence $T \setminus \{w\} \in \mathcal{C}(\overline{B})$. □

Proposition 5.2.37. *Let $\overline{B}$ be a graph satisfying Setup 5.2.34. Then $J_{\overline{B}}$ is unmixed.*

Proof. We prove the statement by induction on r, the number of cycles in $\overline{B}$.

If $r = 1$, then the claim follows. In fact, if $D_1 = C_3$, then $\overline{B}$ is a complete graph with or without whiskers, hence $J_{\overline{B}}$ is unmixed by [66, Proposition 2.6]. If $D_1 = C_4$, then $\overline{B}$ has to satisfy the condition (6) in Setup 5.2.34, and the resulting graph is known to be Cohen-Macaulay and hence unmixed.

Suppose $r > 1$. By induction hypothesis we have that $J_{\overline{B}_k}$ is unmixed with $B_k = \bigcup_{i=k}^{r} D_i$ and $k > 1$.

If $D_1 = C_3$ with $V(D_1) = \{u_0, u_1, w_1\}$ and $E(D_1) \cap E(D_2) = \{\{w_1, u_1\}\}$. Let $T \in \mathcal{C}(\overline{B})$. If $w_1 \notin T$, then T is a cutset for $\overline{B}_2$ and by induction hypothesis the assertion follows. We distinguish the following cases:

1. $w_1 \in T$ and $u_1 \notin T$;
2. $w_1, u_1 \in T$.

(1) Assume $w_1 \in T$ and $u_1 \notin T$. If T is a cutset of $\overline{B}_2$ the number of connected components does not change. In fact, by adding the graph C_3 and removing the vertex w_1

we only obtain that the connected component of $\overline{B}_2 \setminus T$ containing u_1 now contains the graph $D_1 \setminus w_1$. If $T \notin \mathcal{C}(\overline{B}_2)$, we claim that $T' = T \setminus \{w_1\}$ is a cutset of $\overline{B}_2$. We start observing that the connected component of $\overline{B}_2 \setminus T$ containing u_1 contains $D_1 \setminus \{w_1\}$ in $\overline{B} \setminus T$. Since by hypothesis for any $a \in T'$ $c_{\overline{B}}(T) > c_{\overline{B}}(T \setminus \{a\})$ we have that $c_{\overline{B}_2}(T') > c_{\overline{B}_2}(T' \setminus \{a\})$, the claim follows. Hence, by induction hypothesis, $c_{\overline{B}_2}(T') = |T'| + 1$. Let H be the connected component of $\overline{B}_2 \setminus T'$ containing w_1. By adding the vertex w_1 to T', w_1 disconnects H into two connected components: the one containing u_1 and the free vertex attached to w_1.

(2) If $w_1, u_1 \in T$, then there exists $v \in V(\overline{B}_2)$ adjacent to u_1 such that u_1 breaks the connected component H of $\overline{B} \setminus (T \setminus \{u_1\})$ containing u_1 in two, one containing v and one containing u_0. By Setup 5.2.34 (7), the vertices adjacent to u_1 in $\overline{B_2}$ are either w_1 and u or w_1, w, and u. In the former case, since $w_1 \in T$, then $u \notin T$ and $v = u$, otherwise u_1 is a free vertex in $D_1 \setminus w_1$. In the latter case, $u, w \in V(D_3)$, that is $\{u, w\} \in \mathcal{C}(\overline{B}_2)$. We observe that $\{u, w\} \not\subset T$, otherwise u_1 is a free vertex of D_1. The claim follows.

Moreover, from Corollary 5.2.36, $T' = T \setminus \{u_1\}$ is a cutset of $\overline{B}$ such that $w_1 \in T'$ and $u_1 \notin T'$. By applying similar arguments to the case (1) we get that $c_{\overline{B}}(T') = |T'| + 1$ and $T' \cap \{u_1\}$ breaks the component containing u_1 in two: the vertex u_0, and the component containing the vertex v.

If $D_1 = C_4$ with $V(D_1) = \{u_0, w_0, u_1, w_1\}$, then $E(D_1) \cap E(D_2) = \{\{w_1, u_1\}\}$. Let $T \in \mathcal{C}(\overline{B})$. Assume $w_0, w_1 \notin T$, then T is a cutset for $\overline{B}_2$ and by induction hypothesis the assertion follows. We now assume $u_0, u_1 \notin T$ and since $\{w_0, u_1\}$ is the unique cutset of B with cardinality 2 containing w_0, then the cases $w_0 \in T$ and $w_1 \notin T$, $w_0 \notin T$ and $w_1 \in T$, $w_0, w_1 \in T$ are analogous to the cases $w_1 \notin T$ and $w_1 \in T$ of $D_1 = C_3$. In fact, in all of the cases we obtain that $T \setminus \{w_0\}$ is a cutset of $\overline{B}$, that is $c_{\overline{B}}(T \setminus \{w_0\}) = |T \setminus \{w_0\}| + 1$ and the component containing u_0 and f_0 is eventually broken by w_0. We now assume $u_1 \in T$. Observe that from Setup 5.2.34 (2) $D_2 = C_3$ and the vertex $u \in U$ adjacent to u_1 in $\overline{B}_2$ is such that $\{w_1, u\} \in E(\overline{B})$, otherwise u_0, w_1, w_2, u are all adjacent to u_1 contradicting Setup 5.2.34 (7). That is either w_0 or $w_1 \in T$, $u \notin T$, and from Corollary 5.2.36 $T \setminus \{u_1\}$ is a cutset of $\overline{B}$. From the above cases, we obtain $c_{\overline{B}}(T \setminus \{u_1\}) = |T \setminus \{u_1\}| + 1$ and u_1 breaks the component containing u_0 and u_2. If $u_0 \in T$, then, by Lemma 5.2.35, $w_1 \in T$ and $w_0, u_1 \notin T$, that is from Corollary 5.2.36 $T \setminus \{u_0\}$ is a cutset for $\overline{B}$. By the previous cases we obtain $c_{\overline{B}}(T \setminus \{u_0\}) = |T \setminus \{u_0\}| + 1$ and u_0 breaks the component containing w_0 and u_1.

$\square$

Remark 5.2.38 *In Proposition 5.2.37, if we substitute D_1 with a complete graph K_n, with $n \geq 3$, satisfying (3) in the Setup 5.2.34, then $J_{\overline{B}}$ is unmixed.*

Theorem 5.2.39. *Let $\overline{B}$ be a graph. The following conditions are equivalent:*

1. *$\overline{B}$ satisfies Setup 5.2.34;*
2. *$J_{\overline{B}}$ is Cohen-Macaulay;*
3. *$S/J_{\overline{B}}$ is S_2;*
4. *$\overline{B}$ is accessible;*
5. *$J_{\overline{B}}$ is strongly unmixed.*

Proof. We prove the following implications:

$$(5) \implies (2) \implies (3) \implies (4) \implies (1) \implies (5).$$

By [9, Section 5], it holds (5) $\implies$ (2).
It is a well known result that (2) $\implies$ (3).
Theorem 5.2.11 states (3) $\implies$ (4).
By Lemmas 5.2.25, 5.2.29, 5.2.27, 5.2.30, and observing that a C_4 with 2 whiskers satisfying Setup 5.2.34 (e) (or Setup 5.2.34 (f)) is accessible, we have (4) $\implies$ (1).
To prove (1) $\implies$ (5) we proceed by induction on the number s of cutpoints of $\overline{B}$.

Let $s = 1$ and w be the cutpoint of $\overline{B}$. Then $\overline{B}$ is a cone from w to exactly 2 graphs: an isolated vertex and a path. By [66], $J_{\overline{B}}$ is unmixed. Moreover $\overline{B} \setminus \{w\}$ is decomposable into edges, therefore $J_{\overline{B}}$ is strongly unmixed by Lemma 5.2.33, and $\overline{B}_w$ and $\overline{B}_w \setminus \{w\}$ are complete graphs.

Suppose $s > 1$ and we focus on the cycle D_1. Let w be the first cutpoint, namely $w = w_0$ if $D_1 = C_4$ or $w = w_1$ if $D_1 = C_3$. We observe that $\overline{B} \setminus w = \pi \cup \overline{B}_{t+1}$, where $\pi : u_0, u_1, \ldots, u_t$ is a path, $\{u_t\} = V(\pi) \cap V(\overline{B}_{t+1})$, and $B_{t+1} = \bigcup_{i=t+1}^{r} D_i$. If $D_{t+1} = C_3$, then $\pi \cup \overline{B}_{t+1}$ is decomposable in u_t. Note that D_{t+1} cannot be a C_4. In fact, if by contradiction $D_{t+1} = C_4$, then $D_t = C_3$ and u_{t-1}, u_{t+1}, w, w_t are all adjacent to u_t. That is $\deg u_t \geq 4$ obtaining a contradiction and the claim follows. Therefore, by Lemma 5.2.33 and by induction hypothesis, $J_{\overline{B}\setminus w}$ is strongly unmixed.

Now we prove that $J_{\overline{B}_w}$ is strongly unmixed, as well. Suppose $D_t = C_3$ then $\overline{B}_w = K_{t+3} \cup \overline{B}_{t+1}$ with $V(K_{t+3}) \cap V(D_{t+1}) = \{w_t, u_t\}$ and the associated binomial edge ideal is strongly unmixed by induction hypothesis. If $D_t = C_4$ with $V(D_t) = \{w_{t-1}, w_t, u_{t-1}, u_t\}$, then $\overline{B}_w = K_{t+3} \cup D'_t \cup \overline{B}_{t+1}$ where $D'_t = C_3$, $V(K_{t+3}) \cap V(D'_t) = \{u_{t-1}, w_t\}$ and $V(\overline{B}_{t+1}) \cap V(D'_t) = \{w_t, u_t\}$. We observe that $\overline{B}_w$ satisfies Setup 5.2.34 and Remark 5.2.38. By induction hypothesis, the associated binomial edge ideal is strongly unmixed. It is straightforward to observe that $J_{\overline{B}_w\setminus\{w\}}$ is strongly unmixed, too. □

5.2.5 Computation of graphs with $n \in \{2, \dots, 12\}$ vertices

The main aim of this section is to prove, using a computational approach, that for graphs G with at most 12 vertices the three conditions, J_G strongly unmixed, J_G Cohen-Macaulay, and G accessible, holds true as conjectured in [9]. Finally, we discuss some interesting examples obtained by direct computation.

Theorem 5.2.40. *Let G be a graph on $[n]$, with $n \leq 12$. The following conditions are equivalent:*

1. *S/J_G is Cohen-Macaulay;*
2. *S/J_G is S_2;*
3. *G is accessible;*
4. *J_G is strongly unmixed.*

Proof. We know that

$$(4) \implies (1) \implies (2) \implies (3)$$

so, to prove the equivalence it is sufficient to show that $(3) \implies (4)$.

To prove the claim we have implemented a computer program that, for a fixed number n of vertices, performs the following steps (steps *(S2)*, *(S3)* and *(S4)* work on the result of the previous step):

(S1) compute all connected non isomorphic graphs on $[n]$;

(S2) thanks to Lemma 5.2.33, keep only the graphs which are indecomposable and unmixed;

(S3) keep only the ones that are accessible;

(S4) keep only the ones that are strongly unmixed;

(S5) verify that the graphs obtained from step *(S3)* and *(S4)* are the same.

The previous procedure was executed for the graph whose number of vertices is between 2 and 12. Finally, we refer readers to [51] for a complete description of the algorithm that we used.

□

We underline that the computation of the graphs with $n = 12$ vertices has been obtained in a month of computation on a node with 4 CPU Xeon-Gold 5118 having in total 48 cores and 96 threads. All the graphs satisfying the equivalent conditions of Theorem 5.2.40 are downloadable from [51]. Within this set we would like to focus on the graphs shown in the following example.

Example 5.2.41 *By direct computation we obtain the two graphs in Figure 5.7.*

(A) (B)

Figure 5.7: The accessible $\overline{W}_n$.

The graphs in Figure 5.7 (A) and (B) are well known. In fact, the blocks that are not edges are the so-called wheel graphs and they are denoted by W_4 and W_5, respectively. Whereas the blocks with whiskers are called Helm graphs *(see [84]).*

We observe that if $i > 5$ then $J_{\overline{W}_i}$ is not unmixed. In fact, in this case we have at least 6 vertices of degree 4, say $v_1, \ldots, v_6$. Without loss of generality, we may assume that

$\{v_i, v_{i+1}\} \in E(\overline{W}_i)$, *for* $i = 1, \ldots, 5$. *Moreover, assume that* v *is the vertex of degree* i. *We can see that* $T = \{v, v_1, v_3, v_5\}$ *is a cutset such that* $c(T) = 6$.

We recall the following definition.

Definition 5.2.42 *A* polyhedral graph *is a* 3*-connected planar graph.*

The name of polyhedral derives from the fact that it is the graph whose vertices and edges are the ones of a convex polyhedron.

By Example 5.2.41 and Definition 5.2.42, it is natural to ask

Question 5.2.43 *Is it possible to find an infinite family of accessible graphs* $\overline{B}$ *such that* B *is a polyhedral graph?*

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