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Ph.D. Thesis in Mathematics

Some variational and geometric problems on metric measure spaces

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Abstract

In this Thesis, we analyze three variational and geometric problems, that extend classical Euclidean issues of the calculus of variations to more general classes of spaces. The results we outline are based on the articles [Ved21; MV21] and on a forthcoming joint work with Nicolussi Golo and Serra Cassano. In the first place, in Chapter 1 we provide a general introduction to metric measure spaces and some of their properties.

In Chapter 2 we extend the classical Talenti's comparison theorem for elliptic equations to the setting of $\mathrm{RCD}(K,N)$ spaces: in addition the the generalization of Talenti's inequality, we will prove that the result is rigid, in the sense that equality forces the space to have a symmetric structure, and stable.

Chapter 3 is devoted to the study of the Bernstein problem for intrinsic graphs in the first Heisenberg group $\mathbb{H}^1$: we will show that under mild assumptions on the regularity any stationary and stable solution to the minimal surface equation needs to be intrinsically affine.

Finally, in Chapter 4 we study the dimension and structure of the singular set for p-harmonic maps taking values in a Riemannian manifold.

Contents

Introduction

A very widespread approach in modern research in the Calculus of Variations is to reframe classical Euclidean questions on broader classes of spaces, rather than just Euclidean ones; and in particular, to adapt such problems to specific families of metric measure spaces, such as spaces with a non-vanishing curvature, or Sub-Riemannian spaces. The intention behind this effort is to generalize classical results to new environments, in order to better understand their structures and geometrical properties.

Following this trend, in this thesis we look at three different variational and geometric problems which are fairly well established in the Euclidean setting: namely, the Talenti problem for solutions to elliptic equations, the Bernstein problem for area-minimizing graphs, and the regularity of minimizers of the p-energy. However, each of these problems is here explored in a different setting: respectively, on $\mathrm{RCD}(K,N)$ spaces, on the first Heisenberg group, and for maps taking values on closed Riemannian manifolds.

Simplifying a bit, on a Euclidean domain $\Omega \subset \mathbb{R}^n$ one would study the minimizers of the following functionals:

$$\begin{aligned} \mathcal{J}_f(u) &\doteq \int_\Omega \Big(|\nabla u|^2 - fu \Big)\,\mathrm{d}x \\ \mathcal{A}(u) &\doteq \int_\Omega \sqrt{1+|\nabla u|^2}\,\mathrm{d}x \\ \mathcal{E}_p(u) &\doteq \int_\Omega |\nabla u|^p\,\mathrm{d}x, \end{aligned}$$

defined for any $u : \Omega \to \mathbb{R}$ belonging to a suitable Sobolev space (depending on the relevant boundary conditions), with $f \in L^2(\Omega)$ in the definition of $\mathcal{J}_f$ and $p \in (1,\infty)$ in the definition of $\mathcal{E}_p$. A wide range of very well-known techniques can be applied to the analysis of such functionals – as a starting point, once an admissible class of functions is suitably selected, one can retrieve existence of minimizers through the direct method of calculus of variations, and uniqueness through convexity. Furthermore, to each functional $\mathcal{F}$ of the form above one can associate a *Euler-Lagrange equation* that is satisfied (at least weakly) by minimizers, by expanding the identity $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{F}(u+\varepsilon\varphi)\Big|_{\varepsilon=0} = 0$ for any sufficiently regular test function φ.

Among the many issues connected to the minimization problems above, the Euclidean background of this thesis lies in the three aforementioned problems:

1. *Talenti's comparison problem* for minimizers of $\mathcal{J}_f$ in $W^{1,2}(\Omega)$, which concerns the interaction between elliptic boundary value problems and symmetrization techniques;

2. *Bernstein's problem* for local minimizers of the area functional $\mathcal{A}$: explicitly, the problem of understanding whether local minimizers defined on the whole $\mathbb{R}^n$ are forced to be affine functions;

3. Regularity of the minima of $\mathcal{E}_p$ in $W^{1,p}(\Omega)$ (with fixed boundary conditions, in $\mathbb{R}^n$).

In the following paragraphs, we present a more detailed overview to such problems, as well as an introduction to the non-Euclidean framework where each of them is approached.

New settings: metric measure spaces

In Chapter 1, we introduce the notion of *metric measure space* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, that is a space endowed with a distance that makes the topology complete and separable, and with a measure finite on bounded sets.

Under this quite general assumption, thanks to the work of several authors [HK95; Haj96; Che99; HK00; Sha00; AGS14a] it is possible to develop a consistent version of the first order calculus: indeed, a Lipschitz function u from X to $\mathbb{R}$ can be equipped at any point with a notion of *slope* $|\nabla u|$ which takes the place of the norm of the gradient; one can then define the Sobolev space $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ as the set of functions which can be approximated in L^p by Lipschitz functions with slopes uniformly bounded in L^p.

Within this framework, we select the classes of spaces of our interest: namely, $\mathrm{RCD}(K, N)$ spaces, having Ricci curvature bounded from below by K and dimension bounded from above by N in a *synthetic* sense, see Section 1.3; and Carnot-Carathéodory spaces – and in particular the Heisenberg groups $\mathbb{H}^n$ –, seen here as subsets of $\mathbb{R}^n$ equipped with a family $X_1, \dots, X_m$ of smooth vector fields that induce a metric structure, see Section 1.4.

RCD(K,N) spaces and Talenti's Theorem

Symmetrizations and Talenti's problem in $\mathbb{R}^n$ When dealing with variational problems on subsets of $\mathbb{R}^n$, *symmetrizing* domains and functions often proves useful, as it permits to reduce the problem to the study of spherically symmetric objects.

For domains $\Omega \subset \mathbb{R}^n$, this means considering the unique ball $\Omega^\star$ centered at the origin and having the same Lebesgue measure as Ω; for non-negative functions $f : \Omega \to \mathbb{R}$, the symmetrized $f^\star$ is the unique function on Ω which is radial, decreasing in the radial direction, and equimeasurable with f: superlevels of $f^\star$ have the same measure of the corresponding superlevels of f.

In more precise terms, if we denote by $\mu_f(t) \doteq \mathscr{L}^n(\{f > t\})$ the *distribution function* of f, then the *one-dimensional decreasing rearrangement* $f^\sharp$ is the pseudo-inverse of μ_f, and the *Schwarz-symmetrization* $f^\star$ is obtained by $f^\star(x) \doteq f^\sharp(\omega_n |x|^n)$, where ω_n is the Lebesgue measure of the unit ball.

The class of problems which is in literature named after Giorgio Talenti concerns the interplay between symmetrization techniques and elliptic boundary value problems. The strategy of exploiting symmetrizations to tackle comparison problems related to elliptic partial differential equations carries numerous consequences. At the time when Talenti proved the theorem we outline below, it was already well-known that they could be used to prove that the principal frequency of vibration of an elastic membrane is minimal when the membrane is disk-shaped (see the work of Faber and Krahn). Thanks to the work of several authors (Szegő [Sze30], Pólya and Weinstein [PW50] and Payne [Pay62] among the others), the same strategy proves that the electric condenser (of the form $\Omega_0 \setminus \Omega_1$, with Ω_i bounded domains) with minimal capacity is the spherical annulus, and that the cylindrical beam with maximal torsional rigidity has a disk-shaped section. Among the other consequences, by the use of Talenti's Theorem one can easily extend the latter result to hollow cables: a cable maximizes the torsional rigidity if its section and the hole are concentric circles.

Going into more detail, Talenti's problem asks to compare minimizers of the above-defined functional $\mathcal{J}_f$ among maps in $W_0^{1,2}(\Omega)$, with minimizers of the symmetrized functional

$$\mathcal{J}^\star_{f^\star}(v) \doteq \int_{\Omega^\star} \left(|\nabla v|^2 - f^\star v\right) \mathrm{d}x \tag{1}$$

among maps $v \in W_0^{1,2}(\Omega^\star)$, where $f \in L^2(\Omega)$. Equivalently, looking at the associated Euler-Lagrange equations, one would compare the outcomes of the following procedures:

(a) Solve a Poisson problem of the type

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{2}$$

with $f \in L^2(\Omega)$; then consider the Schwarz symmetrization $u^\star$ of u.

(b) Solve the symmetrized Poisson problem

$$\begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star \subset \mathbb{R}^n \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}. \tag{3}$$

The result of Talenti [Tal76], which builds on previous work of Weinberger [Wei62] and Bandle [Ban76]) and has been since then revisited by several authors (see [Bae19; Lio79; Kes06]) states that the inequality $u^\star \leq v$ holds in $\Omega^\star$, and if equality holds then Ω was already a ball.

The crucial (heuristic) steps in the proof of such result are the following:

- The symmetrized problem $-\Delta v = f^\star$ can be actually reduced to the solution of a ordinary differential equation, leveraging the spherical symmetry of $f^\star$: in particular, the solution v is explicit (clearly depending on $f^\star$).
- Looking at how $u^\star$ is defined starting from $u^\sharp$, it is clear that the relevant estimate to prove is $u^\sharp \leq v^\sharp$. Up to a (pseudo-)inversion, this reduces to estimating $\mu_u(t)$ with an explicit quantity depending on $f^\star$.
- The key comparison at the core of this result is that the measure of the superlevels $\{|u| > t\}$ can be estimated by the measure of the ball having the same perimeter of $\{|u| > t\}$, by means of the *isoperimetric inequality*; in particular

$$\mu_u(t) = \mathscr{L}^n(\{|u| > t\}) \leq C_n \operatorname{Per}(\{|u| > t\})^{\frac{n}{n-1}}. \tag{4}$$

- The final tool needed to complete the argument relies on the *coarea formula* to estimate the perimeter above with the derivative of $t \mapsto \int_{\{|u|>t\}} |\nabla u|$. Notice that nowhere in the previous points it was used that u solves an elliptic problem: by suitably choosing a one-parameter family of test functions in the weak formulation of the problem, we use this information to estimate $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\{|u|>t\}} |\nabla u|$.

As we will see, up to adapting the tools involved, the same scheme can be fruitfully exploited on much more general spaces, namely $\mathrm{RCD}(K, N)$ spaces with $K > 0$ and $N \in (1, \infty)$.

RCD(K,N) spaces In the setting of smooth Riemannian manifolds, lower bounds on the Ricci tensor play a paramount role for a wide variety of results, ranging from results in comparison geometry and isoperimetry (Laplacian bounds for distance functions; Bishop-Gromov volume comparison; Sobolev and Poincaré-Sobolev inequalities; see [Pet16, Chapter 7]), to analytical and PDE-related problems (such as the Li-Yau inequality for solutions to the heat equations, [LY86]).

The introduction of $\mathrm{RCD}(K,N)$ metric measure spaces in the study of geometrical problems stems from the need to extend those results to non-smooth spaces: by Gromov's Precompactness Theorem [Gro07, Theorem 5.3], it is known that the class of Riemannian manifolds with Ricci curvature bounded from below and diameter bounded from above is precompact in the Gromov-Hausdorff topology; however the limit of a converging sequence may fail to be a smooth manifold.

In the series of articles [CC97; CC00a; CC00b] by Cheeger and Colding, it is shown that several of the results mentioned above specifically hold for such limit spaces; building on this observation, in the first of those papers [CC97, Appendix 2] the authors remark the need of finding a *synthetic* condition that encodes "Ricci curvature bounded from below", in the sense that it does "*not depend on the existence of an underlying smooth structure, or indeed, make any reference to the notion of smoothness*".

The response to such need came from the pioneering work of Lott, Villani [LV09] and Sturm [Stu06a; Stu06b], which independently introduced the notion of $\mathrm{CD}(K,N)$ spaces (for $K \in \mathbb{R}$ and $N > 1$) through optimal transport tools, and in particular by looking at the (distorted) convexity of a entropy functional along the geodesics in the Wasserstein space. This approach has its roots in the work of McCann [McC94], Cordero-Erausquin, McCann and Schmuckenschläger [CMS01], Otto and Villani [OV00], Sturm and Von Renesse [RS05] who proved that such properties hold in Euclidean and Riemannian settings, and can be used to prove a number of geometrical inequalities, such as the Brunn-Minkowski and Prékopa-Leindler inequalities.

In this thesis, we will focus on the $\mathrm{RCD}(K,N)$ subclass of $\mathrm{CD}(K,N)$ metric measure spaces, with $K > 0$ and $N \in (1,\infty)$; the "R" in the name stands for *Riemannian*, as a further condition is added in order to single out Riemannian-like spaces and rule out Finsler-like ones. The family of $\mathrm{RCD}(K,N)$ spaces is closed under Gromov-Hausdorff convergence [GMS15]; moreover, many of the results that hold for smooth manifolds with lower bounds on the Ricci tensor can be extended to this new setting. Of special interest to our aim is the fact that a *Lévy-Gromov inequality* holds [CM17; CM18]: in the case where $K > 0$, one can identify a family of (normalized) *model spaces* $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ with curvature bounded from below by K and dimension bounded from above by N; if $E \subset \mathsf{X}$ is any measurable subset of a (normalized) $\mathrm{RCD}(K,N)$ space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ with $\mathfrak{m}(E) = v$, then the comparison

$$(5) \qquad \mathrm{Per}_{\mathsf{X}}(E) \geq \inf\left\{ \mathrm{Per}_{J_{K,N}}(A) \;\middle|\; \begin{array}{c} A \text{ is a measurable subset of the } (K,N)\text{-model space} \\ \text{and } \mathfrak{m}_{K,N}(A) = v \end{array} \right\}$$

holds, where we still denote by Per a suitable notion of perimeter in metric measure spaces [Mir03].

RCD(K,N) version of Talenti's theorem If $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a $\mathrm{RCD}(K,N)$ space, then a weak formulation of the problem (2) on a subset Ω and for a $L^2(\Omega,\mathfrak{m})$ map f is available; moreover, the symmetrization of a set Ω is still available, but now lives in the respective (K,N)-model space $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$. Whenever $\Omega \subset \mathsf{X}$ and $f : \Omega \to \mathbb{R}$ are measurable, one can also retrieve the symmetrized function $f^\star$ as a function from $\Omega^\star \subset J_{K,N}$ to $[0,\infty]$. The Talenti problem thus has a consistent formulation on $\mathrm{RCD}(K,N)$ spaces as well: we have enough structure to consider a weak elliptic problem on $(\mathsf{X},\mathsf{d},\mathfrak{m})$, to symmetrize the solution u, and to compare it with the solution v to the symmetrized problem.

In Chapter 2, and based on the work [MV21], we will show that the inequality $u^\star \leq v$ still holds: going back to the strategy outlined in $\mathbb{R}^n$, we will see that it can be successfully translated in terms of RCD spaces. Indeed, the solution v on the model space still has an explicit solution, derived from a new ordinary differential equation; a version of the Coarea formula holds here as well ([Mir03]); and crucially, the isoperimetric inequality can be replaced by the aforementioned Lévy-Gromov inequality.

Finally, we will show that the result is rigid: once the equality holds between $u^\star$ and v (even at a single point), the space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is forced to have a peculiar structure: namely, it needs to be a *spherical suspension*. What's more, the result is stable as well, in the sense that when $u^\star$ and v are close enough (in L^2), then the space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is arbitrarily close to a spherical suspension in the measured Gromov-Hausdorff topology.

The first Heisenberg group and Bernstein's problem

Bernstein's problem When $\Omega \subset \mathbb{R}^{n-1}$ is an open set with $\mathbf{C}^2$ boundary, it can be proved that the unique classical $\mathbf{C}^2$ solution to the minimal surface equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 0 \tag{6}$$

with fixed boundary conditions minimizes the area functional $\mathcal{A}(\cdot,\Omega)$ among the functions having the same boundary conditions ([Giu84, Theorem 13.8]). A natural question which arises in this context is whether functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$ that satisfy the minimal surface equation in the whole $\mathbb{R}^{n-1}$ (and are thus area minimizers in any bounded domain) are constrained to be affine – in other words, the graph of f is a hyperplane of $\mathbb{R}^n$.

In the two dimensional case (*i.e.*, for functions from $\mathbb{R}^2$ to $\mathbb{R}$), several affirmative proofs are available, the first of which was given by the Russian mathematician Sergej Bernstein, after whom the problem is named [Ber27]. Unluckily, neither Bernstein's proof, nor the subsequent proofs given by means of complex analysis tools (see [Hop50; Mic50; Ber51; Nit57]) have a higher dimension counterpart.

In 1963, a result contained in an article by Fleming in the framework of integral currents gave a new alternative proof for the 2-dimensional Bernstein Theorem which could potentially be extended to higher dimensions [Fle62, Lemma 2.2 and Paragraph 5]; moreover, in 1965 De Giorgi showed that if counterexamples to the Bernstein problem exist in $\mathbb{R}^n$ (*i.e.*, functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$), than there must exist at least a minimal cone in $\mathbb{R}^{n-1}$ [De 65].

By showing that no such minimal cones exist in $\mathbb{R}^k$ with $k \leq 7$, Simons proved that the Bernstein conjecture held for functions from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ with $n \leq 8$. However, Simons himself proposed an example of stable cones in $\mathbb{R}^8$, which turned out to be minimal thanks to a proof by Bombieri, De Giorgi and Giusti [BDG69]. In the same paper, Bombieri, De Giorgi and Giusti provided a counterexample to the Bernstein conjecture for functions from $\mathbb{R}^8 \to \mathbb{R}$, establishing a complete solution of the problem.

The Heisenberg group The simplest examples of non-Euclidean Carnot-Carathéodory spaces (and in particular the simplest non-trivial Carnot groups) are the Heisenberg groups $\mathbb{H}^n$ with $n \geq 1$. Among the many equivalent definitions and descriptions available of the Heisenberg group (see [Ste93] for a complex-analytical approach, [Hal03; Hal13] for the role it plays in quantum physics), we treat it here as the space $\mathbb{R}^{2n+1}$ endowed with a metric induced by a family $\mathbb{X} = \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ of vector fields that satisfy the so-called *Hörmander condition*. If we denote by $\cdot$ the group operation

$$(\mathbf{x}, \mathbf{y}, t) \cdot (\mathbf{x}', \mathbf{y}', t') = (\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', t + t' + \frac{1}{2}(\langle \mathbf{x}, \mathbf{y}' \rangle - \langle \mathbf{x}', \mathbf{y} \rangle)) \tag{7}$$

on $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$, then $(\mathbb{H}^n, \cdot)$ is a *Lie group* and $\mathbb{X}$ arises as a *left-invariant* family with respect to $\cdot$ that generates the whole Lie algebra of the group.

In this setting, one can consider maps between two complementary and *homogeneous* subgroups $\mathbb{W}$ and $\mathbb{V}$ of $\mathbb{H}^n$: in that case, the *intrinsic graph* will be an object lying in the space $\mathbb{H}^n$ and having the structure

$$\mathrm{Graph}_{\mathbb{H}}(f) \doteq \{\xi \cdot f(\xi) \mid \xi \in \omega\}, \qquad \omega \subset \mathbb{W}, \quad f \colon \mathbb{W} \to \mathbb{V}. \tag{8}$$

In particular, in the first Heisenberg group $\mathbb{H}^1 \simeq \mathbb{R}^3$, we will be interested in graphs of maps from the *vertical* plane $\{(0, y, t) \mid y, t \in \mathbb{R}\}$ (which can be thought as $\mathbb{R}^2$) to the *horizontal* line $\{(x, 0, 0) \mid x \in \mathbb{R}\}$. By [ASV06], the *horizontal perimeter* of the subgraph of such maps in an "intrinsic" cylinder $\omega \cdot \mathbb{R}$ is expressed (under mild regularity assumptions) by the area functional

$$\mathcal{A}(f; \omega) \doteq \int_\omega \sqrt{1 + |\nabla^f f|^2} \, \mathrm{d}\mathscr{L}^2, \tag{9}$$

where $\nabla^f f$ is an appropriately defined *intrinsic gradient*. Hence, one can formulate an adapted version of the Bernstein problem for intrinsic graphs of the above type: is it true that any map that is stationary (or stable) with respect to variations in the area functional has a vertical plane as a intrinsic graph?

The Bernstein problem in $\mathbb{H}^1$ The answer to such question varies considerably based on the regularity assumed on the map f, and on the variational assumption: *i.e.*, if we ask that the map is simply stationary (the first variation vanishes, or the minimal surface equation is satisfied), or stable (non-negative second variations), or perimeter minimizing. In Chapter 3, we look at the problem under the *stability* assumption and a regularity which is weaker than Lipschitz but stronger than Sobolev: in this situation, we will prove that the question has an affirmative answer.

The problem in $\mathbb{H}^1$ has a substantially different nature than in the Euclidean $\mathbb{R}^n$, and in fact even different from higher dimensional Heisenberg groups. In this case, indeed, the minimal surface equation takes the form

$$\nabla^f \left(\frac{\nabla^f f}{\sqrt{1 + (\nabla^f f)^2}} \right) = 0, \tag{10}$$

where $\nabla^f \doteq \partial_y + f\partial_t$ is a single vector field. When f is a $\mathbf{C}^2(\mathbb{R}^2, \mathbb{R})$ solution to the minimal surface equation, the authors in [BSV07] showed, by reducing the problem to a *double Burgers'* *equation*, that $\mathbb{R}^2$ is foliated by parabolas of the type

$$\gamma(\upsilon,\tau) = \frac{a(\tau)}{2}\upsilon^2 + b(\tau)\upsilon + \tau \tag{11}$$

so that

$$f(\gamma(\upsilon,\tau),\tau) = a(\tau)\upsilon + b(\tau); \tag{12}$$

in particular, the intrinsic graph of f is ruled by *horizontal lines*. If f is also stable, than the above parabolas have coefficients a and b independent from τ, and the intrinsic graph of f is thus a vertical plane. Thanks to [GR15, Corollary 5.2], it was then proved that the same result holds for $\mathbf{C}^1$ functions, as a consequence of a more general version of the Bernstein Theorem (*i.e.*, holding for arbitrary complete oriented stable surfaces of class $\mathbf{C}^1$ without singular points, see [GR15, Theorem 5.1]).

A careful modification of the strategy above, that exploits suitable *Lagrangian parametrizations* of $\mathbb{R}^2$, makes it possible to extend the result from $\mathbf{C}^2$ to Lipschitz [NS19] and to an even weaker condition, which is the purpose of our Chapter 3.

In particular, we will prove the following:

Theorem 0.1. *Let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}) \cap \mathbf{C}^0(\mathbb{R}^2)$ be such that $\exp(|\nabla f|) \in L^\beta_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$ for some $\beta \geq 1$. If f is stable, then its intrinsic graph is a vertical plane.*

Minimizers of p-harmonic manifold-valued maps

A classical result in the theory of degenerate quasilinear differential equations states that any minimizer $u \in W^{1,p}(\Omega,\mathbb{R})$ of the functional

$$\mathcal{E}_p(u) = \int_\Omega |\nabla u|^p \,\mathrm{d}\mathscr{L}^m \tag{13}$$

with fixed (and regular enough) boundary conditions has locally $\mathbf{C}^{1,\alpha}$ regularity (for the sake of consistency with Chapter 4, Ω is here a subdomain of $\mathbb{R}^m$). This was proved independently by Ural′ceva [Ura68] and Uhlenbeck [Uhl77]; a simpler proof can be found in [Eva82]. Such minimizers solve weakly the Euler-Lagrange equation

$$\Delta_p u \doteq \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0, \tag{14}$$

and are thus consistently called p-harmonic functions.

In Chapter 4, we will treat the problem of minimizing the functional

$$u \mapsto \mathcal{E}_{p,\mathcal{N}}(u) \doteq \int_\Omega \sum_{i=1}^m \left(\sum_{\alpha=1}^N \left(\frac{\partial u^\alpha}{\partial x_i}\right)^2 \right)^{\frac{p}{2}} \mathrm{d}\mathscr{L}^m \tag{15}$$

among maps $u \in W^{1,p}(\Omega,\mathbb{R}^N)$ which take values in a fixed closed Riemannian manifold $\mathcal{N}$. As one immediately sees, the issue of regularity is here much more delicate: the map $x \mapsto \frac{x}{|x|}$ from the m-dimensional ball to the $(m-1)$-dimensional sphere turns out to be a minimizer for the $\mathcal{E}_{2,\mathbb{S}_{m-1}}$ functional, without even being continuous.

Moreover, one could still look at variations in the target space: this can be done by considering $u + \varepsilon\varphi$ with $\varepsilon > 0$ and with φ a smooth test function, and then projecting back onto

$\mathcal{N}$: however, being a critical point with respect to this class of variations does not give any regularity (except on very specific cases, see *e.g.* [Hél90]), as there are examples of maps with this property that are everywhere discontinuous [Riv95].

A way of treating this problem is to look at the dimension, the relevant measure and the geometric structure of the *singular set* $\mathcal{S}(u)$, defined as

$$\mathcal{S}(u) \doteq \{x \in \Omega \mid u \text{ is not continuous at } x\} \tag{16}$$

when u is a minimizer. With this in mind, a series of results were proved in the last decades:

- $p = 2$: in 1982, Schoen and Uhlenbeck proved through a dimension reduction argument that any 2-energy minimizing map is $\mathbf{C}^{0,\alpha}$ outside of a set of Hausdorff dimension at most $m-3$ (see [SU82]). Furthermore, by standard elliptic regularity, the regularity outside the singular set can be improved to $\mathbf{C}^{\infty}$.

- $p \in (1,\infty)$: in 1987, Schoen and Uhlenbeck's result was extended to p-harmonic maps by Hardt and Lin [HL87]. The best regularity one can achieve outside the singular set is $\mathbf{C}^{1,\alpha}$; so any p-energy minimizing map is $\mathbf{C}^{1,\alpha}$ outside of a set of Hausdorff dimension at most $m - \lfloor p \rfloor - 1$. Notice that the case $m \leq p$ was already completely solved here: in this case, there are no singular points, and the map is everywhere $\mathbf{C}^{1,\alpha}$. The only case worth studying is $m > p$.

- $p = 2$: later, Cheeger and Naber [CN13b] proved that the singular set of a 2-minimizing map with energy bounded by Λ satisfies the following estimate:

  $$\mathscr{L}^m(\mathcal{B}_r(\mathcal{S}(u)) \cap B') \leq C(m, \mathcal{N}, \Lambda, \varepsilon) r^{3-\varepsilon} \tag{17}$$

  for any $\varepsilon > 0$; it is assumed that the dimension of the domain is at least 3. This implies that the Minkowski dimension of $\mathcal{S}(u)$ is at most $m-3$, but gives no bound on the Minkowski content. Here a notion of *quantitative stratification* of the singular set was introduced, and the result was obtained through a relatively simple covering of each singular stratum (and by making explicit the link between singular set and stratification).

- $p \in (1,\infty)$: in a more recent paper, Naber, Valtorta and Veronelli [NVV19] extended the estimate (17) to p-minimizing maps: they showed that in this case

  $$\mathscr{L}^m(\mathcal{B}_r(\mathcal{S}(u)) \cap B') \leq C(m, \mathcal{N}, \Lambda, p, \varepsilon) r^{\lfloor p \rfloor + 1 - \varepsilon} \tag{18}$$

  for any $\varepsilon > 0$. It is assumed that the dimension of the domain is greater than p: we have already noticed, however, that this is the only interesting case.

- $p = 2$: finally, Naber and Valtorta [NV17] improved the estimate (17) for 2-minimizing maps, removing the dependence on the parameter ε: assuming that $m > 2$, then the singular set of a 2-minimizing map with energy bounded by Λ satisfies

  $$\mathscr{L}^m(\mathcal{B}_r(\mathcal{S}(u)) \cap B') \leq C(m, \mathcal{N}, \Lambda) r^3; \tag{19}$$

  and thus the upper Minkowski content of $\mathcal{S}(u)$ is bounded by a constant C. Moreover, in the same article they showed that $\mathcal{S}(u)$ is actually $(m-3)$-rectifiable. The main idea to prove both the Minkowski estimate and rectifiability was to replace the simple covering argument of [CN13b] with a more refined one, which makes use of a suitable version of Reifenberg Theorem.

In Chapter 4 (based on the article [Ved21]), we will extend the technique of Naber and Valtorta [NV17] to the case of p-energy minimizing maps, and even to *p-stationary p-harmonic maps* under some assumptions (*i.e.*, maps that are stationary with respect to *internal variations*, or explicitly to one-parameter families of diffeomorphisms in the domain Ω). We will thus prove:

Theorem 0.2. *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a map with p-energy bounded by Λ. Assume that one of the following two conditions hold:*

(m) either u is minimizing (with $\mathcal{N}$ any compact Riemannian manifold with no boundary, see Assumption 4.1 in Chapter 4);

(s) or $\mathcal{N}$ is a homogeneous space with a left-invariant metric, p is not an integer, and u is p-stationary.

Then there exists a constant $C_{\mathcal{S}}(m, \mathcal{N}, \Lambda, p)$ such that for any $r > 0$

$$\mathscr{L}^m(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)) \leq C_{\mathcal{S}} r^{\lfloor p \rfloor + 1}. \tag{20}$$

In particular, the Minkowski dimension of $\mathcal{S}(u)$ is at most $m - \lfloor p \rfloor - 1$, and the upper Minkowski content is bounded by $C_{\mathcal{S}}$.

Furthermore, the singular set $\mathcal{S}(u)$ is $(m - \lfloor p \rfloor - 1)$-rectifiable.

What is relevant about stationary p-harmonic maps is that a *monotonicity* formula holds for a suitably normalized p-energy

$$r \mapsto \vartheta(x, r) \doteq r^{p-m} \int_{B_r(x)} |\nabla u(y)|^p \, \mathrm{d}y \tag{21}$$

for any x (see Theorem 4.32). Moreover, when minimality is added (or additional conditions on $\mathcal{N}$ and p), a classical ε-regularity result holds: there exists a ε_0 such that $\vartheta(x, r) < \varepsilon_0$ implies that u is $\mathbf{C}^{1,\alpha}$-regular in $B_{\frac{r}{2}}(x)$ for some $\alpha > 0$ (see Section 4.3). Finally, we will also make use of a version of the Reifenberg Theorem introduced by Naber and Valtorta in [NV17] (see Section 4.5.2).

Other contributions

In this paragraph, derived from the article [PPV21], we summarize a further result obtained in the field of Multi-marginal Optimal Transportation during the Ph.D. studies, which was not included in this thesis.

Consider a metric space (X, d), and a m-tuple of Borel probability measures $\mu_1, \mu_2, ..., \mu_m$ on it. Given a cost function $c : \mathsf{X}^m \to \mathbb{R}$, we call *Monge's multi-marginal optimal transport problem* the minimization problem

$$\text{minimize} \qquad \int_X c(\mathbf{x}_1, T_2(\mathbf{x}_1) ..., T_m(\mathbf{x}_1)) d\mu_1(\mathbf{x}_1) \tag{22}$$

among $(m-1)$-tuples of mappings $(T_2, ..., T_m)$ with the constraint that $(T_i)_\sharp \mu_1 = \mu_i$, where the subscript $\sharp$ denotes the push-forward of measures. Depending on the specific ambient space (X, d) and on the cost function, a first relevant problem is to determine if the minimum is attained, and in that case if it is unique and has an explicit formulation.

When $m=2$, (22) reduces to the well known classical optimal transport problem of Monge: we refer the reader to the monographs [Vil03; Vil09; San15] for comprehensive surveys. A particularly relevant cost function, in this case, is represented by the squared distance $c(\mathbf{x}_1,\mathbf{x}_2) = \mathsf{d}^2(\mathbf{x}_1,\mathbf{x}_2)$. For this cost, when $\mathsf{X}=\mathbb{R}^n$, a seminal theorem of Brenier [Bre87; Bre91] asserts that there exists a unique minimizer to (22). This result has been extended to a much wider class of geometrical settings, beginning with the work of McCann when X is a Riemannian manifold [McC01].

In the case of the Heisenberg group $\mathbb{H}^n$ endowed with the Carnot-Carathéodory distance d_{cc} (see Section 1.4.1), the existence and uniqueness of an optimal map when $m=2$ was established by Ambrosio and Rigot in [AR04]. In the same article, a representation of such map is given in terms of the *exponential map* of the Lie group $\mathbb{H}^n$, modeled on an analogous representation in the Euclidean and Riemannian setting.

In the multi-marginal case (*i.e.*, when $m\geq 3$), the natural counterpart of the squared distance is for many aspects the cost function

$$c(\mathbf{x}_1,\dots,\mathbf{x}_m) \doteq \inf_{\mathbf{y}\in\mathsf{X}} \sum_{i=1}^m \mathsf{d}^2(\mathbf{x}_i,\mathbf{y}). \tag{23}$$

When this cost is considered on the Euclidean $\mathbb{R}^n$, a pioneering article by Gangbo and Święch [GŚ98] showed that there exists a unique solution to the Monge minimization problem. An analogous result was proved on Riemannian manifolds by Kim and Pass [KP15].

The purpose of the article [PPV21] is to extend, under some additional assumptions, the existence and uniqueness result to the multimarginal transport problem on $(\mathbb{H}^n,\mathsf{d}_{\mathrm{cc}})$ with the cost proposed in Equation (23). The additional assumptions make use of the notion of *barycenter* of a m-tuple: we say that $\mathbf{y}$ is a barycenter for $(\mathbf{x}_1,\dots,\mathbf{x}_m)$ if it realizes the minimum in the definition of the cost: explicitly,

$$\sum_{i=1}^m \mathsf{d}^2(\mathbf{x}_i,\mathbf{y}) = \inf_{\mathbf{z}\in\mathbb{H}^n} \sum_{i=1}^m \mathsf{d}^2(\mathbf{x}_i,\mathbf{z}). \tag{24}$$

Then what we can prove, as a consequence of a more general result, is the following ([PPV21, Corollary 28]):

Theorem 0.3. *Let $\mu_1,\dots,\mu_m$ be compactly supported, absolutely continuous probability measures on $\mathbb{H}^n$, and let c be the cost associated to d_{cc}. Assume that the set*

$$\{\mathbf{y}\in\mathbb{H}^n \mid \mathbf{y} \text{ is a barycenter for } (\mathbf{x}_1,\dots,\mathbf{x}_m) \text{ with } \mathbf{x}_i\in \operatorname{spt}(\mu_i)\} \tag{25}$$

has zero μ_i-measure for all $i=1,\dots,m$. Then there exists a unique $(\mathbb{H}^n)^{m-1}$-valued optimal map in the Monge problem.

Moreover, the same result holds when replacing the Carnot-Carathéodory distance with the *Gauge distance*

$$\mathsf{d}_{\mathrm{g}}([\zeta,t],[\zeta',t']) \doteq \sqrt[4]{|\zeta-\zeta'|^4+(t-t')^2} \tag{26}$$

in the definition of the cost (see [PPV21, Remark 30]). More interestingly, when the squared distance is replaced by the p^{th} power of the distance with $p>2$ in Equation (23), the theorem holds even without the need of the extra assumption on the set of barycenters [PPV21, Paragraph 4.1].

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Chapter 1

Preliminaries

In this preliminary chapter, we recall in the first place some very general facts about metric measure spaces; then, we particularize the results we need to the specific contexts of $\mathrm{RCD}(K,N)$ spaces and Carnot-Carathéodory spaces. Indeed, many geometrical properties of the spaces we work on can be shown to descend from the underlying *metric-measure* structure.

From now on, when we talk about *metric measure spaces*, we mean triples of the type $(\mathsf{X},\mathsf{d},\mathfrak{m})$ where $\mathsf{d}:\mathsf{X}\times\mathsf{X}\to[0,+\infty)$ is a distance on X and $\mathfrak{m}$ is a measure on $\mathscr{B}(\mathsf{X})$, the Borel σ-algebra induced by d. More precisely, unless otherwise stated, in this Chapter we will always make (and keep in the background) the following assumptions:

Assumption 1.1 (Metric measure spaces). $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a metric measure space satisfying:

- (X,d) is a *complete* and *separable* metric space;
- $\mathfrak{m}\geq 0$ is a Borel measure, finite on balls. $\Diamond$

Somewhat surprisingly, this pretty minimal setting already has enough structure to develop a theory of Sobolev functions, and consistent tools of differential calculus. There are at least three different approaches to this problem, which turn out to be (non-trivially) equivalent. For the purposes of this thesis, the most convenient method to define weak differentiability is through approximation by Lipschitz functions, which play the role of the smooth functions in the Euclidean setting.

1.1 General tools on metric measure spaces

First of all, we introduce the tools that are already available once the distance d is defined on X (that is, we do not make use of the *measure* $\mathfrak{m}$).

Notation 1.2 (Balls). In the whole thesis, when (X,d) is a metric space, $x\in\mathsf{X}$ and $r>0$, we will use the notation $B_r(x)$ – or sometimes $B(x,r)$, when the involved expressions make use of typographically heavier symbols – to denote the open metric ball of radius r around x:

$$B_r(x)=B(x,r)\doteq\{y\in\mathsf{X}\mid\mathsf{d}(x,y)<r\}. \tag{1.1}$$

Moreover, if $S \subset \mathsf{X}$, we will instead use the notations $\mathcal{B}_r(S)$ or $\mathcal{B}(x,r)$ to denote the *r-fattening* or *r-neighborhood* of S:

$$\mathcal{B}_r(S) = \mathcal{B}(S,r) \doteq \{y \in \mathsf{X} \mid \mathsf{d}(S,y) < r\}, \tag{1.2}$$

where as usual $\mathsf{d}(S,y) \doteq \inf_{x\in S} \mathsf{d}(x,y)$. ♢

The first object we introduce is the slope of a real-valued function: for smooth functions on $\mathbb{R}^n$, it coincides at any point with the norm of the gradient; however, it is much more general, since it is defined for any function on a metric space (and it is particularly meaningful for locally Lipschitz functions).

Definition 1.3 (Slope). Let (X,d) be a metric space and $u : \mathsf{X} \to \mathbb{R}$ be a real valued function. We define the *slope* of u at the point $x \in \mathsf{X}$ as

$$|\nabla u|(x) \doteq \begin{cases} \limsup_{y\to x} \frac{|u(x)-u(y)|}{\mathsf{d}(x,y)} & \text{if } x \text{ is not isolated} \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$ ♢

The slope functional $u \mapsto |\nabla u|$ is by construction convex:

Lemma 1.4 (Convexity of the slope). *Let $u, v : \mathsf{X} \to \mathbb{R}$ and $\lambda \in (0,1)$. Then*

$$|\nabla(\lambda u + (1-\lambda)v)|(x) \le \lambda|\nabla u|(x) + (1-\lambda)|\nabla v|(x) \qquad \text{for all } x \in \mathsf{X}, \tag{1.4}$$

provided both sides are finite.

Proof. This is an immediate consequence of the triangle inequality and the subadditivity of the limit superior:

$$\begin{aligned} |\nabla(\lambda u + (1-\lambda)v)|(x) &\le \limsup_{y\to x}\left[\lambda\frac{|u(x)-u(y)|}{\mathsf{d}(x,y)} + (1-\lambda)\frac{|v(x)-v(y)|}{\mathsf{d}(x,y)}\right] \le \\ &\le \lambda \limsup_{y\to x}\frac{|u(x)-u(y)|}{\mathsf{d}(x,y)} + (1-\lambda)\limsup_{y\to x}\frac{|v(x)-v(y)|}{\mathsf{d}(x,y)}, \end{aligned} \tag{1.5}$$

which proves the statement. □

A second notion which only needs a *metric* structure to be defined is the notion of *geodesic curve*; for the purpose of this thesis, its importance mainly lies in the role it plays in the definition of RCD spaces (Section 1.3).

Definition 1.5 (Geodesics). We say that a curve $\gamma \in \mathbf{C}^0([0,1],\mathsf{X})$ is a *constant speed geodesic* if $\mathsf{d}(\gamma(s),\gamma(t)) = |s-t|\mathsf{d}(\gamma(0),\gamma(1))$ for any $s,t \in [0,1]$. We denote by $\mathrm{Geo}(\mathsf{X})$ the family of constant speed geodesics on X.

Moreover, for any $t \in [0,1]$, the *evaluation map* e_t is defined on $\mathrm{Geo}(\mathsf{X})$ as

$$\mathrm{e}_t(\gamma) \doteq \gamma(t) \quad \text{for any } \gamma \in \mathrm{Geo}(\mathsf{X}). \tag{1.6}$$ ♢

Metric spaces in which any pair of points admits a geodesic joining them is called a *geodesic space*:

Definition 1.6 (Geodesic space). Let (X,d) be a metric space. We say that (X,d) is a *geodesic space* if for any pair $(x,y) \in \mathsf{X}$ there exists at least a constant speed geodesic $\gamma \in \mathrm{Geo}(\mathsf{X})$ such that $\gamma(0) = x$, $\gamma(1) = y$. ♢

1.1.1 Sobolev spaces on metric measure spaces

The purpose of this paragraph is to define the class of Sobolev functions in the metric-measure setting; as already noted, one has many options to tackle this problem (and see [AG16] for an overview). Here we define the space $H^{1,p}$ through L^p-approximation with Lipschitz functions (see Remark 1.15 for an alternative approach).

Definition 1.7 (Sobolev space and relaxed slope). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a complete separable metric measure space, and let $p \in (1, +\infty)$. We say that $f \in L^p(\mathsf{X}, \mathfrak{m})$ belongs to the Sobolev space $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ if

(1.7)
there exists a sequence of Lipschitz functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Lip_b}(\mathsf{X}, \mathsf{d}) \cap L^p(\mathsf{X}, \mathfrak{m})$ such that

$$\|f_n - f\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0 \quad \text{and} \quad \limsup_{n\to\infty} \int_{\mathsf{X}} |\nabla f_n|^p \,\mathrm{d}\mathfrak{m} < \infty.$$

In this case, we say that $g \in L^p(\mathsf{X}, \mathfrak{m})$ belongs to the *relaxed slope* $\mathrm{RS}(f)$ of f if there exists $\tilde{g} \in L^p(\mathsf{X}, \mathfrak{m})$ and an approximating sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Lip_b}(\mathsf{X}, \mathsf{d}) \cap L^p(\mathsf{X}, \mathfrak{m})$ such that

1. $\tilde{g} \le g$ $\mathfrak{m}$-almost everywhere;
2. $\|f_n - f\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0$ and $|\nabla f_n| \rightharpoonup \tilde{g}$ weakly in $L^p(\mathsf{X}, \mathfrak{m})$. ◊

Lemma 1.8 (Characterization of $\mathrm{RS}(f)$ through strong approximation). *Let $f \in L^p(\mathsf{X}, \mathfrak{m})$. If $g \in \mathrm{RS}(f)$, then there exist $\{\tilde{g}_n\}_{n\in\mathbb{N}}, \tilde{g} \in L^p(\mathsf{X}, \mathfrak{m})$ and $\{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Lip_b}(\mathsf{X}, \mathsf{d}) \cap L^p(\mathsf{X}, \mathfrak{m})$ such that*

1. *$\tilde{g} \le g$ $\mathfrak{m}$-almost everywhere;*
2. *$\|f_n - f\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0$ and $\|\tilde{g}_n - \tilde{g}\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0$;*
3. *$|\nabla f_n| \le \tilde{g}_n$*

Proof. By definition of $\mathrm{RS}(f)$ there exists a sequence $\varphi_n \in \mathrm{Lip_b}(\mathsf{X}, \mathsf{d}) \cap L^p(\mathsf{X}, \mathfrak{m})$ which converges strongly to f and whose slopes converge weakly to a function $\tilde{g} \le g$. By a version of the Mazur's Lemma (see [Bre11, Corollary 3.8 and exercise 3.4.1]), there exists a convex combination of functions in $\bigcup_{n\in\mathbb{N}}\{|\nabla\varphi_n|\}$ which converges *strongly* to $\tilde{g}$: more precisely for any $n \in \mathbb{N}$ one can find a subset of indices $I_n \subset \mathbb{N} \cap [n, \infty)$ and coefficients $\{\lambda_{n,i}\}_{i\in I_n}$ such that $\sum_{i\in I_n} \lambda_{n,i} = 1$ and

$$\tilde{g}_n \doteq \sum_{i\in I_n} \lambda_{n,i} |\nabla\varphi_i| \longrightarrow \tilde{g} \qquad \text{strongly in } L^p(\mathsf{X}, \mathfrak{m}). \tag{1.8}$$

Now it is enough to take

$$f_n \doteq \sum_{i\in I_n} \lambda_{n,i}\varphi_i \tag{1.9}$$

so that

$$\|f_n - f\|_{L^p(\mathsf{X},\mathfrak{m})} \le \sum_{i\in I_n} \lambda_{n,i} \|\varphi_i - f\| \to 0 \tag{1.10}$$

as $n \to \infty$, and $|\nabla f_n| \le \tilde{g}_n$ by convexity of the slope. □

The following lemma contains a few properties of the relaxed slope $\mathrm{RS}(f)$; in particular, the existence of a $\|\cdot\|_{L^p}$-minimizing element will be fundamental.

Lemma 1.9 (Properties of the relaxed slope). *The following properties hold for the relaxed slope:*

(i) a map $f \in L^p(\mathsf{X}, \mathfrak{m})$ belongs to $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ if and only if the relaxed slope of f is not empty;

(ii) if $f \in H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, then $\mathrm{RS}(f)$ is convex in $L^p(\mathsf{X}, \mathfrak{m})$;

(iii) if $f \in H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, then $\mathrm{RS}(f)$ is (strongly, thus also weakly) closed in $L^p(\mathsf{X}, \mathfrak{m})$.

(iv) if $f \in H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, then $\mathrm{RS}(f)$ admits a unique element of minimal L^p norm.

Proof. Part (i): The first assertion follows by the fact that the condition

$$\limsup_{n\to\infty} \int_{\mathsf{X}} |\nabla f_n|^p \, \mathrm{d}\mathfrak{m} < \infty \tag{1.11}$$

guarantees the existence of a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ such that the slopes $|\nabla f_n|$ converge weakly in $L^p(\mathsf{X}, \mathfrak{m})$.

Part (ii): The convexity of $\mathrm{RS}(f)$ follows by the convexity of the slope (Lemma 1.4): let $g, G \in \mathrm{RS}(f)$, with $g \geq \tilde{g}$, $G \geq \tilde{G}$, and f_n, F_n the respective approximating sequences for f such that $|\nabla f_n|$ converges weakly to $\tilde{g}$ and $|\nabla F_n|$ converges weakly to $\tilde{G}$. Let $\lambda \in (0,1)$. Then the sequence $\lambda f_n + (1-\lambda)F_n \in \mathrm{Lip_b}(\mathsf{X}, \mathsf{d}) \cap L^p(\mathsf{X}, \mathfrak{m})$ still converges to f in the strong $L^p(\mathsf{X}, \mathfrak{m})$ sense, and by convexity its slope is bounded from above by $\lambda|\nabla f_n| + (1-\lambda)|\nabla F_n|$. In particular, up to subsequences, $|\nabla(\lambda f_n + (1-\lambda)F_n)|$ converges weakly in $L^p(\mathsf{X}, \mathfrak{m})$ to a function h such that

$$h \leq \lambda\tilde{g} + (1-\lambda)\tilde{G} \leq \lambda g + (1-\lambda)G. \tag{1.12}$$

Thus $\lambda g + (1-\lambda)G \in \mathrm{RS}(f)$.

Part (iii): Let $\{g_m\}_{m\in\mathbb{N}} \subset \mathrm{RS}(f)$, and assume that $g_m \to g$ strongly in L^p as $m \to \infty$. By Lemma 1.8, we can find $\{\tilde{g}_{m,n}\}_{n\in\mathbb{N}}, \tilde{g}_m \in L^p(\mathsf{X}, \mathfrak{m})$ and $\{f_{m,n}\}_{n\in\mathbb{N}} \subset \mathrm{Lip_b}(\mathsf{X}, \mathsf{d}) \cap L^p(\mathsf{X}, \mathfrak{m})$ such that

1. $\tilde{g}_m \leq g_m$;
2. $\|f_{m,n} - f\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0$ and $\|\tilde{g}_{m,n} - \tilde{g}_m\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0$ as $n \to \infty$ for any m;
3. $|\nabla f_{m,n}| \leq \tilde{g}_{m,n}$ for any $m, n \in \mathbb{N}$.

Through a simple diagonal argument, we can find a selection of indices $\{n_m\}_{m\in\mathbb{N}}$ such that both $\|f_{m,n_m} - f\|_{L^p(\mathsf{X},\mathfrak{m})}$ and $\|\tilde{g}_{m,n_m} - \tilde{g}_m\|_{L^p(\mathsf{X},\mathfrak{m})} \to 0$ as $m \to \infty$. Let $F_m \doteq f_{m,n_m}$, $\tilde{G}_m \doteq \tilde{g}_{m,n_m}$. Now

$$\begin{aligned}\limsup_{m\in\mathbb{N}} \||\nabla F_m|\|_{L^p(\mathsf{X},\mathfrak{m})} &\leq \limsup_{m\in\mathbb{N}} \|\tilde{G}_m\|_{L^p(\mathsf{X},\mathfrak{m})} = \\ &= \limsup_{m\in\mathbb{N}} \|\tilde{g}_m\|_{L^p(\mathsf{X},\mathfrak{m})} \leq \limsup_{m\in\mathbb{N}} \|g_m\|_{L^p(\mathsf{X},\mathfrak{m})} = \|g\|_{L^p(\mathsf{X},\mathfrak{m})}.\end{aligned} \tag{1.13}$$

In particular, up to subsequences, $|\nabla F_m|$ converges weakly to a function $\tilde{g} \in L^p(\mathsf{X}, \mathfrak{m})$ and both $\tilde{G}_m$ and $\tilde{g}_m$ converge weakly to a function $\gamma \in L^p(\mathsf{X}, \mathfrak{m})$. Exploiting the inequalities $\tilde{g}_m \leq g_m$

and $|\nabla F_m| \leq \tilde{G}_m$ and the convergence of g_m to g, we have that $\tilde{g} \leq \gamma \leq g$ $\mathfrak{m}$-almost everywhere (indeed, the set $\{u \in L^p(\mathsf{X}, \mathfrak{m}) \mid u \geq 0\}$ is convex and strongly closed, thus it is weakly closed). Summarizing, we have: $\tilde{g} \leq g$ $\mathfrak{m}$-almost everywhere, $F_m \to f$ strongly in L^p and $|\nabla F_m| \rightharpoonup \tilde{g}$; this is exactly the weak characterization of the relaxed slope given in Definition 1.7.

Part (iv): the existence and uniqueness of a element of minimal norm is a direct consequence of the previous points (and the direct method of calculus of variations): if $g_n \in \mathrm{RS}(f)$ is a sequence such that $\lim_n \|g_n\|_{L^p} = \inf\{\|g\|_{L^p} \mid g \in \mathrm{RS}(f)\}$, then up to subsequences it converges weakly to a map $\bar{g}$, which belongs to $\mathrm{RS}(f)$ by weak closedness; moreover, by the properties of weak convergence $\|\bar{g}\|_{L^p} \leq \liminf_n \|g_n\|_{L^p}$ holds. Thus $\bar{g}$ has minimal norm, and it is the unique such element by convexity of the set $\mathrm{RS}(f)$ and of the norm. □

Part (iv) of Lemma 1.9 allows the following definition:

Definition 1.10 (Minimal relaxed slope). Let $u \in H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. We define the *minimal p-relaxed slope* $|\nabla u|_{*,p} \in L^p(\mathsf{X}, \mathfrak{m})$ as the element of $\mathrm{RS}(u)$ with minimal L^p norm. ◊

Proposition 1.11 (Locality properties). *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space, let $p \in (1, +\infty)$ and let $f, \varphi \in H^{1,p}(\mathsf{X}, \mathfrak{m})$. Then:*

(i) $|\nabla f|_{*,p} = |\nabla \varphi|_{*,p}$ *$\mathfrak{m}$-a.e. on the set* $\{f = \varphi\}$*;*

(ii) If $g_1, g_2 \in \mathrm{RS}(f)$*, then also* $\min\{g_1, g_2\} \in \mathrm{RS}(f)$*;*

(iii) In particular, $|\nabla f|_{*,p}$ *is also minimal in the $\mathfrak{m}$-a.e. sense:*

$$|\nabla f|_{*,p} \leq g \qquad \mathfrak{m}\text{-almost everywhere for any } g \in \mathrm{RS}(f). \tag{1.14}$$

Proof. See [AGS14a, Lemma 4.4]. □

Finally, this procedure allows to define an *energy functional* which takes the place of the Dirichlet energy of the smooth setting:

Definition 1.12 (Cheeger energy). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space, let $p \in (1, +\infty)$ and let $f \in L^p(\mathsf{X}, \mathfrak{m})$. The *$p$-Cheeger energy* of f is defined as

$$\mathrm{Ch}_p(f) \doteq \inf\left\{ \liminf_{n\to\infty} \frac{1}{p} \int_{\mathsf{X}} |\nabla f_n|^p \,\mathrm{d}\mathfrak{m} \;\middle|\; \begin{array}{c} f_n \in \mathrm{Lip}(\mathsf{X}) \cap L^p(\mathsf{X}, \mathfrak{m}) \\ \|f_n - f\|_{L^p} \to 0 \end{array} \right\}, \tag{1.15}$$

where $|\nabla f_n|(x)$ is the slope of f_n at the point x. ◊

***Remark* 1.13** (Cheeger energy, Sobolev spaces and relaxed slope)*.* It is trivial to see that the following hold:

- A map $f \in L^p(\mathsf{X}, \mathfrak{m})$ belongs to the Sobolev space $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ if and only if its Cheeger energy is finite;
- If $f \in H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, then its Cheeger energy can be characterized as

$$\mathrm{Ch}_p(f) = \frac{1}{p} \int_{\mathsf{X}} |\nabla f|^p_{*,p} \,\mathrm{d}\mathfrak{m}, \tag{1.16}$$

where $|\nabla f|_{*,p}$ is the minimal p-relaxed slope of f. ◊

Definition 1.14 ($H^{1,p}$ norm)**.** For any map $f \in H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$ we define

$$\|f\|_{H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})} \doteq \Big(\|f\|_{L^p(\mathsf{X},\mathfrak{m})} + p\,\mathrm{Ch}_p(f)\Big)^{\frac{1}{p}} = \Big(\|f\|_{L^p(\mathsf{X},\mathfrak{m})} + \int_{\mathsf{X}} |\nabla f|_{*,p}^p \,\mathrm{d}\mathfrak{m}\Big)^{\frac{1}{p}}. \tag{1.17}$$

The space $H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$, endowed with the norm $\|\cdot\|_{H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})}$ has the structure of a Banach space. ◊

We refer the reader to [HKST15; BB11; Che99; Amb18] for details.

***Remark* 1.15** (Newtonian spaces)**.** We mention here an alternative definition for Sobolev spaces, and an equivalence result which is in fact rather deep: one can define $N^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$ to be the space of functions $f \in L^p(\mathsf{X},\mathfrak{m})$ such that

$$\begin{gathered}\text{there exist } \tilde f \in L^p \text{ and } g \in L^p \text{ with } g \geq 0 \text{ such that}\\ \tilde f = f \ \mathfrak{m}\text{-a.e. and } \left|\tilde f(\gamma_1) - \tilde f(\gamma_0)\right| \leq \int_\gamma g \,\mathrm{d}\sigma \text{ for } \mathrm{Mod}_p\text{-a.e. curve } \gamma;\end{gathered} \tag{1.18}$$

we say that a property holds "for Mod_p-almost every curve" if it fails on a family of curves of zero p-modulus: *i.e.*, there exists a nonnegative Borel function $\varrho \in L^p(\mathsf{X},\mathfrak{m})$ such that $\int_\gamma \varrho \,\mathrm{d}\mathfrak{m} = \infty$ for any curve γ for which the property fails (see [BB11, Proposition 1.37] for this characterization).

Then one can prove what follows:

- Even for functions $f \in N^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$ one can retrieve an object $|\nabla f|_{\mathrm{w}}$ which plays the role of the modulus of the gradient, chosen as the g with minimal L^p-norm in Equation (1.18). Historically, it is called *minimal p-weak upper gradient* in this context.

- Let (X,d) be a complete separable metric space, and $\mathfrak{m}$ be a Borel measure satisfying the condition $\mathfrak{m}(B_r(x)) \leq ae^{br^2}$ for any x, r and for some $a, b > 0$; then $N^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$ *and* $H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$ *actually coincide for all* $p \in (1,\infty)$, and the minimal p-weak upper gradient (for functions in $N^{1,p}$) coincides with the minimal p-relaxed slope (defined for functions in $H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$). A proof of this very deep equivalence result can be found for example in [AG16, Sections 2 and 3]; see also [AGS14a; AGS13].

Since the assumptions which guarantee the equivalence will always be satisfied in the following chapters, from now on we will generally use the more established term *minimal weak upper gradient* in place of *minimal relaxed slope*, and we will adopt the symbol $|\nabla f|_{\mathrm{w}}$, even when we will actually be working with the definition of $H^{1,p}$ space. ◊

***Remark* 1.16.** A priori the minimal p-weak upper gradient may depend on p; however in locally doubling and Poincaré spaces (see Section 1.1.3) it is independent of p by the deep work of Cheeger [Che99]. Again, in the following chapters we will work under this assumption, thus we will be allowed to neglect this possible issue. ◊

The Cheeger functional Ch_2 is not in general a quadratic form. As we will see in Section 1.3, when this happens the geometrical properties of the space $(\mathsf{X},\mathsf{d},\mathfrak{m})$ sensibly improve. We give a name to this condition:

Definition 1.17 (Infinitesimal Hilbertianity). We say that $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is *infinitesimally Hilbertian* if the Cheeger energy Ch_2 defined in (1.15) is a quadratic form on $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ (equivalently, if $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a Hilbert space). In that case, we still denote by Ch the symmetric bilinear form associated to $\mathrm{Ch} = \mathrm{Ch}_2$. ♢

Another class of functions which will be widely used in the sequel – in particular, in the Dirichlet problem of Chapter 2 – is that of Sobolev space with zero boundary values:

Definition 1.18 (The space $H_0^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a complete separable metric measure space, and let $\Omega \subset \mathsf{X}$ be open. We define the space $H_0^{1,p}(\Omega, \mathsf{d}, \mathfrak{m})$ as the closure of $\mathrm{Lip}_c(\Omega, \mathsf{d})$ with respect to the norm of $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$:

$$H_0^{1,p}(\Omega, \mathsf{d}, \mathfrak{m}) \doteq \overline{\mathrm{Lip}_c(\Omega, \mathsf{d})}^{H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})}. \tag{1.19}$$ ♢

As a last definition for this paragraph, we introduce a *local* notion of Sobolev space, which relies on the definition given in [AH18, Definition 2.14]. We specialize to the case $p = 2$, which is the only one we will use: see Section 2.3.2.

Definition 1.19 (Local Sobolev space). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space and let $\Omega \subset \mathsf{X}$ be an open subset. We say that $f \in L^2(\Omega, \mathfrak{m})$ belongs to $H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ if

(a) for any $\varphi \in \mathrm{Lip}_c(\mathsf{X}, \mathsf{d})$ with $\mathrm{spt}(\varphi) \subset \Omega$, it holds $\varphi f \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ (where $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is the global Sobolev space defined in Definition 1.7);

(b) $|\nabla f|_w \in L^2(\Omega, \mathfrak{m})$. ♢

Notice that the property (a), together with the locality properties of the minimal weak upper gradient, guarantees that the condition in (b) is well posed (see again [AH18]).

Notation 1.20 (Classical Sobolev spaces). So far, we have presented the notations $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, $N^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, $H_0^{1,p}(\Omega, \mathsf{d}, \mathfrak{m})$ and $H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ for different classes of Sobolev spaces on metric measure spaces. If $\Omega \subset \mathbb{R}^m$ is an open subset of a Euclidean space, we keep the notation $W^{1,p}(\Omega, \mathbb{R})$ for the classical Sobolev space defined through integration by parts:

$$W^{1,p}(\Omega, \mathbb{R}) \doteq \left\{ u \in L^p(\Omega, \mathscr{L}^m) \;\middle|\; \begin{array}{c} \text{there exists } g_i \in L^p(\Omega, \mathscr{L}^m) \text{ such that} \\ \displaystyle\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}\mathscr{L}^m = -\int_\Omega \varphi g_i \, \mathrm{d}\mathscr{L}^m \\ \text{for all } \varphi \in \mathbf{C}_c^\infty(\Omega, \mathbb{R}) \text{ and all } i = 1, \dots, m. \end{array} \right\} \tag{1.20}$$

The relation between this space and the spaces of type $H^{1,p}$ defined before is given by the classical Meyer-Serrin Theorem (see for example [Eva10, Paragraphs 5.3.2 and 5.3.3]), which states that $\mathbf{C}^\infty(\Omega, \mathbb{R}) \cap W^{1,p}(\Omega, \mathbb{R})$ is dense in $W^{1,p}(\Omega, \mathbb{R})$; moreover, when $\Omega = \mathbb{R}$, the space of compactly supported $\mathbf{C}^\infty$ functions is dense in $W^{1,p}(\mathbb{R}^m, \mathbb{R})$.

Moreover, if $N \geq 2$, we define the space of multi-valued Sobolev functions $W^{1,p}(\Omega, \mathbb{R}^N)$ as

$$W^{1,p}(\Omega, \mathbb{R}^N) = \left\{ u = (u_1, \dots, u_N) : \Omega \to \mathbb{R}^N \;\middle|\; u_i \in W^{1,p}(\Omega, \mathbb{R}) \text{ for all } i \in \{1, \dots, N\} \right\}, \tag{1.21}$$

where $W^{1,p}(\Omega, \mathbb{R})$ is the classical Sobolev space defined in Notation 1.20. This will be used both in Chapter 3 for Sobolev homeomorphism between domains of $\mathbb{R}^N$, and in Chapter 4 for manifold-valued Sobolev mappings. ♢

1.1.2 Wasserstein distance on $\mathscr{P}(\mathsf{X})$

The aim of this short paragraph is to give a very basic introduction to the optimal transport tools needed in the definition of curvature-dimension conditions (Section 1.3); we refer the reader to the classical references [AGS08; Vil09] for a much more extended treatise.

Notation 1.21 (Probability measures). For any metric space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$, we denote by $\mathscr{P}(\mathsf{Y})$ the space of Borel probability measures on Y; moreover, we denote by $\mathscr{P}_2(\mathsf{Y})$ the space of Borel probability measures with *finite second moment*, *i.e.*:

$$\mathscr{P}_2(\mathsf{Y}) \doteq \left\{ \mu \in \mathscr{P}(\mathsf{Y}) \;\middle|\; \int_\mathsf{X} \mathsf{d}_\mathsf{Y}^2(x, x_o)\,\mathrm{d}\mu(x) < +\infty \right\}, \tag{1.22}$$

where $x_o \in \mathsf{Y}$ is any fixed point. ◊

Definition 1.22 (Wasserstein distance). The *Wasserstein distance* W_2 on the space $\mathscr{P}_2(\mathsf{X})$ is defined as

$$W_2(\mu_0, \mu_1) \doteq \inf \left\{ \int_{\mathsf{X}\times\mathsf{X}} \mathsf{d}^2(x, y)\,\mathrm{d}\gamma(x, y) \;\middle|\; \gamma \in \mathscr{P}(\mathsf{X}\times\mathsf{X}), \pi^{(0)}_\sharp \gamma = \mu_0, \pi^{(1)}_\sharp \gamma = \mu_1 \right\}, \tag{1.23}$$

for any $\mu_0, \mu_1 \in \mathscr{P}_2(\mathsf{X})$, where $\pi^{(0)}$ is the projection on the first component, $\pi^{(1)}$ is the projection on the second component, and the subscript $\sharp$ indicates the pushforward of the measure. ◊

***Remark* 1.23.** It can be proved that $(\mathscr{P}_2(\mathsf{X}), W_2)$ is a metric space [AG13, Theorem 3.2]; moreover, under our assumption of completeness and separability, it is a geodesic space whenever (X, d) itself is geodesic [AG13, Theorem 3.10]. ◊

Definition 1.24 (Dynamical optimal plans). For any pair of measures μ_0, μ_1 in $\mathscr{P}_2(\mathsf{X})$, the set of *dynamical optimal plans* are defined as

$$\mathrm{OptGeo}(\mu_0, \mu_1) \doteq \left\{ \nu \in \mathscr{P}(\mathrm{Geo}(\mathsf{X})) \;\middle|\; \begin{array}{c} (\mathrm{e}_0, \mathrm{e}_1)_\sharp \nu \text{ realizes the minimum} \\ \text{in Equation (1.23)} \end{array} \right\}. \tag{1.24}$$ ◊

1.1.3 Doubling and Poincaré spaces

A fruitful approach to the study of the geometry of metric measure spaces is to look for conditions that allow to exploit analogies with the smooth setting; and to do so while keeping a high level of generality at the same time.

After the work of several authors [BB11; HK00; HKM06], it is now well established that requiring the space to be *doubling* and *Poincaré* bears several helpful consequences. Although they will mostly be kept in the background in the following chapters – and implied by more restrictive conditions, see Propositions 1.66, 1.67 and 1.85 –, we will need them in several different points (characterization of pmG convergence in Section 1.1.4, existence of solutions for uniformly elliptic problems in Chapter 2).

Definition 1.25 (Doubling measures). We say that the metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is *doubling* if there exists a constant $C_{\mathrm{db}} > 0$ such that

$$\mathfrak{m}(B(x, 2r)) \leq C_{\mathrm{db}}\mathfrak{m}(B(x, r)) \tag{1.25}$$

for all points $x \in \mathsf{X}$ and all radii $r > 0$. ◊

Since we will need it in Remark 1.71, we state here a first consequence of the doubling property alone (see [BB11, Proposition 3.1] for a proof)

Lemma 1.26. *Any complete metric space endowed with a doubling measure is proper (*i.e.*, all closed bounded sets are compact).*

Proof. Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be doubling, and let $\Omega \subset \mathsf{X}$ with $D \doteq \operatorname{diam} \Omega < \infty$, where $\operatorname{diam} \Omega \doteq \sup_{x,y,\in\Omega} \mathsf{d}(x, y)$. We will show Ω is totally bounded.

For any $\varepsilon > 0$, the number of disjoint balls of radius ε contained in Ω is bounded by $C(D, \varepsilon, C_{\mathrm{db}}) \doteq C_{\mathrm{db}}^{\lceil \log_2 \frac{D}{\varepsilon} \rceil}$: indeed, let $\{B_\varepsilon(x_j)\}_{j=1}^N$ be disjoint balls with $B_\varepsilon(x_j) \subset \Omega$; let $\hat{\jmath}$ be such that $\mathfrak{m}(B_\varepsilon(x_{\hat{\jmath}})) = \min_{j=1,\dots,N} \mathfrak{m}(B_\varepsilon(x_j))$. Then applying the doubling property $\lceil \log_2 \frac{D}{\varepsilon} \rceil$ times we obtain:

$$\begin{aligned}
\mathfrak{m}(B_\varepsilon(x_{\hat{\jmath}})) &\le \frac{1}{N} \sum_{j=1}^N \mathfrak{m}(B_\varepsilon(x_j)) \le \\
&\le \frac{1}{N} \mathfrak{m}(\Omega) \le \frac{1}{N} \mathfrak{m}(B_D(x_{\hat{\jmath}})) \le \frac{1}{N} C(D, \varepsilon, C_{\mathrm{db}}) \mathfrak{m}(B_\varepsilon(x_{\hat{\jmath}})).
\end{aligned} \tag{1.26}$$

As a consequence, if $\varepsilon > 0$, we can build by induction a finite family of balls $\left\{B_{\frac{\varepsilon}{2}}(x_k)\right\}_{k=1}^M$, all contained in Ω, such that any $y \in \Omega$ belongs to some $B_\varepsilon(x_k)$: in particular, Ω is totally bounded.

If Ω is also closed, by the completeness of the ambient space X it is compact ([Fol99, Theorem 0.25]). □

***Remark* 1.27.** An interesting consequence of the doubling property alone is that if $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a doubling space (with $\mathfrak{m}$ finite on bounded sets), then the Sobolev space $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is reflexive and separable for any $p \in (1, \infty)$, and Lipschitz functions with bounded support are dense in $H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. This was proved in [ACD15, Corollary 41 and Proposition 42]. ♢

Definition 1.28 (Poincaré)**.** Let $q, p \ge 1$. We say that a metric measure space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *supports a (weak) (q, p)-Poincaré inequality* if there exist constants $C_{\mathrm{PI}} > 0$ and $\lambda \ge 1$ such that for any u integrable on X, any ball $B \subset \mathsf{X}$ and any upper gradient g of u it holds that

$$\left(\fint_B \left| u - \fint_B u \right|^q \mathrm{d}\mathfrak{m} \right)^{\frac{1}{q}} \le C_{\mathrm{PI}} \operatorname{diam}(B) \left(\fint_{\lambda B} g^p \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{p}}. \tag{1.27}$$

We say that $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ satisfies a strong Poincaré inequality if the above condition holds with $\lambda = 1$. ♢

Notice that, by the characterization in [BB11, Proposition 4.13], we could equivalently ask that Equation (1.27) holds for (any measurable u and) any p-weak upper gradient – in particular, it holds for the minimal weak upper gradient $|\nabla u|_{\mathrm{w}}$ of functions $u \in H^{1,p}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$.

We collect in the following proposition some properties that hold for any doubling and Poincaré space:

Proposition 1.29. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space which satisfies the doubling condition and supports a $(1, p)$-Poincaré inequality. Then:*

(i) X *is connected;*

(ii) $\mathfrak{m}$ *has no atoms:* $\mathfrak{m}(\{x\}) = 0$ *for any* $x \in \mathsf{X}$.

***Remark* 1.30** (References for the proof)**.** The proofs can be found in [BB11, Chapter 4]: in particular, (i) is Proposition 4.2 therein, (ii) is Corollary 4.3. ♢

Finally, we state here a corollary to the Poincaré inequality – also known in the smooth setting with the name of Poincaré inequality – which will be explicitly used in Remark 2.3 in the context of uniformly elliptic operators on $\mathrm{RCD}(K,N)$ spaces.

Proposition 1.31 (Poincaré inequality for $H_0^{1,p}$)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be a (complete, separable) metric measure space which supports a* (p,p)*-Poincaré inequality. Let* $\Omega \subset \mathsf{X}$ *be a bounded set such that* $\mathfrak{m}(\Omega) < \mathfrak{m}(\mathsf{X})$*. There exists a constant* $C_0 > 0$ *such that for any* $u \in H_0^{1,p}(\Omega, \mathsf{d}, \mathfrak{m})$ *the inequality*

$$\int_\Omega |u|^p \, \mathrm{d}\mathfrak{m} \le C_0 \int_\Omega |\nabla u|_{\mathrm{w}}^p \, \mathrm{d}\mathfrak{m} \tag{1.28}$$

holds. In particular,

$$\|u\|^p_{H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})} \le \tilde{C}_0 \int_\Omega |\nabla u|_{\mathrm{w}}^p \, \mathrm{d}\mathfrak{m} \qquad \text{for any } u \in H_0^{1,p}(\Omega, \mathsf{d}, \mathfrak{m}) \tag{1.29}$$

for some constant $\tilde{C}_0$*.*

The proposition is proved in [BB11, Corollary 5.54]. Notice that the condition $\mathfrak{m}(\Omega) < \mathfrak{m}(\mathsf{X})$ could be replaced by the weaker condition that $\mathsf{X} \setminus \Omega$ has *positive p-capacity*; however, we will not need this notion in the future.

1.1.4 Convergence of spaces

In order to perform limiting arguments on the class of metric measure space, one can endow such class (or suitable subclasses) with an appropriate topology. In the following scheme, we do so; again, we always implicitly refer to complete, separable metric spaces.

1. *Hausdorff distance:* Let (X, d) be a metric space, and let $A, B \subset \mathsf{X}$. The *Hausdorff distance* between A and B is defined as

$$\mathsf{d}_{\mathrm{H}}(A,B) \doteq \inf\{\varepsilon > 0 \mid A \subset \mathcal{B}_\varepsilon(B) \text{ and } B \subset \mathcal{B}_\varepsilon(A)\}; \tag{1.30}$$

   recall that $\mathcal{B}_\varepsilon(E)$ is the ε-neighborhood of the set E.

2. *Gromov-Hausdorff distance:* Building on the notion of Hausdorff distance, one can construct a notion of distance between metric spaces: if $(\mathsf{X}, \mathsf{d}_\mathsf{X})$ and $(\mathsf{Y}, \mathsf{d}_\mathsf{Y})$ are metric spaces, the *Gromov-Hausdorff distance* between them is defined as

$$\mathsf{d}_{\mathrm{GH}}((\mathsf{X},\mathsf{d}_\mathsf{X}),(\mathsf{Y},\mathsf{d}_\mathsf{Y})) \doteq \inf\left\{ \mathsf{d}_{\mathrm{H}}(f(\mathsf{X}), g(\mathsf{Y})) \;\middle|\; \begin{array}{c} (\mathsf{Z},\mathsf{d}_\mathsf{Z}) \text{ metric spaces,} \\ f : \mathsf{X} \hookrightarrow \mathsf{Z} \text{ and } g : \mathsf{Y} \hookrightarrow \mathsf{Z} \\ \text{isometric embeddings} \end{array} \right\}. \tag{1.31}$$

This notion was introduced by Gromov in [Gro81b; Gro81a]; a complete introduction to this theory can be found in [BBI01, Chapter 7]. It can be shown that d_{GH} defines a finite distance on the space of isometry classes of metric measure spaces ([BBI01, Theorem 7.3.30]). Quoting a quite helpful heuristics from [Vil09, Remark 27.7], "*two spaces are close in Gromov–Hausdorff topology if they look the same to a short-sighted person*".

3. *Gromov-Hausdorff convergence:* For a sequence of *compact* metric spaces $\mathfrak{X}_n \doteq (\mathsf{X}_n, \mathsf{d}_n)$, and a further *compact* metric space $\mathfrak{X}_\infty \doteq (\mathsf{X}_\infty, \mathsf{d}_\infty)$, we say that $\{\mathfrak{X}_n\}_{n\in\mathbb{N}}$ *Gromov-Hausdorff converges* to $\mathfrak{X}_\infty$ if $\lim_{n\to\infty} \mathsf{d}_{\mathrm{GH}}(\mathfrak{X}_n, \mathfrak{X}_\infty) = 0$.

4. GH *convergence and ε-isometries:* It can be shown that $\mathfrak{X}_n \xrightarrow[\mathrm{GH}]{n\to\infty} \mathfrak{X}_\infty$ is equivalent to the existence of a sequence of real numbers $\varepsilon_n \downarrow 0$ and a sequence of ε_n-isometries $f_n : \mathsf{X}_n \to \mathsf{X}_\infty$; *i.e.*,

$$\sup_{x,y\in\mathsf{X}_n} |\mathsf{d}_n(x,y) - \mathsf{d}_\infty(f_n(x), f_n(y))| \le \varepsilon_n \quad \text{and} \quad \mathsf{X}_\infty \subset \mathcal{B}(f_n(\mathsf{X}_n), \varepsilon_n) \tag{1.32}$$

(see [Vil09, Definition 27.6]).

5. *Pointed Gromov-Hausdorff convergence:* For *non-compact* metric spaces, a notion which proves to be more useful than the standard Gromov-Hausdorff convergence is the *pointed Gromov-Hausdorff convergence*, which can be thought as "Gromov-Hausdorff convergence of balls around a point"; we refer the reader to [BBI01, Definition 8.1.1]. Let $\mathfrak{X}_n \doteq (\mathsf{X}_n, \mathsf{d}_n)$ and $\mathfrak{X}_\infty \doteq (\mathsf{X}_\infty, \mathsf{d}_\infty)$ be metric spaces, and let $x_n \in \mathsf{X}_n$, $x_\infty \in \mathsf{X}_\infty$. We say that the pointed metric spaces $(\mathfrak{X}_n, x_n)$ converge to $(\mathfrak{X}_\infty, x_\infty)$ (and use the term pGH-convergence) if the following holds: fix any $\varepsilon, R > 0$; for any $n \in \mathbb{N}$ greater than a suitable $N = N(\varepsilon, R)$, there exists a Borel map $f_n^{R,\varepsilon} : B_R(x_n) \to \mathsf{X}_\infty$ such that

$$f_n^{R,\varepsilon}(x_n) = x_\infty; \tag{1.33}$$

$$\sup_{x,y\in B_R(x_n)} \Big|\mathsf{d}_n(x,y) - \mathsf{d}_\infty(f_n^{R,\varepsilon}(x), f_n^{R,\varepsilon}(y))\Big| \le \varepsilon; \tag{1.34}$$

$$B_{R-\varepsilon}(x_\infty) \subset \mathcal{B}\Big(f_n^{R,\varepsilon}(B_R(x_n)), \varepsilon\Big). \tag{1.35}$$

6. *Measured Gromov-Hausdorff convergence:* up until now, there was no reference measure involved; when considering *compact metric measure spaces*, the natural notion of convergence can be again characterized in a simple way through ε-isometries: we say that $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n) \xrightarrow{\mathrm{mGH}} (\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)$ if there exist a sequence of real numbers $\varepsilon_n \downarrow 0$ and a sequence of measurable ε_n-isometries $f_n : \mathsf{X}_n \to \mathsf{X}_\infty$ such that $(f_n)_\sharp \mathfrak{m}_n \rightharpoonup \mathfrak{m}_\infty$ (weakly in the topology of measures).

7. *Pointed measured Gromov-Hausdorff convergence:* The case of noncompact metric measure spaces needs again different tools than in the compact setting. In order to consider the convergence of pointed metric measure spaces, an approach proposed by Fukaya [Fuk87] is to look simultaneously at pointed Gromov-Hausdorff convergence (in the non-compact version) and weak convergence of measures. That is: we say that $(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n) \xrightarrow{\mathrm{pmGH}} (\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ if

   - $(\mathsf{X}_n, \mathsf{d}_n, x_n) \xrightarrow{\mathrm{pGH}} (\mathsf{X}_\infty, \mathsf{d}_\infty, x_\infty)$;

- the measures $\left(f_n^{R,\varepsilon}\right)_\sharp(\mathfrak{m}_n \llcorner B_R(x_n))$ weakly converge to $\mathfrak{m}_\infty \llcorner B_R(x_\infty)$ for a.e. $R > 0$, where $f_n^{R,\varepsilon}$ are the maps from the definition of pGH convergence.

8. *Pointed measure Gromov convergence:* In [GMS15], an alternative notion of convergence of pointed metric measure spaces was proposed; of the four characterizations given therein, we are interested in the "extrinsic notion" (Definition 3.9 therein), although one can show it is equivalent to a completely intrinsic one. Firstly, we say that two metric measure spaces $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y})$, $(\mathsf{Z}, \mathsf{d}_\mathsf{Z}, \mathfrak{m}_\mathsf{Z})$ are *isomorphic* if there exists an isometric embedding $\iota : \mathrm{spt}(\mathfrak{m}_\mathsf{Y}) \to \mathsf{Z}$ such that $\iota_\sharp(\mathfrak{m}_\mathsf{Y}) = \mathfrak{m}_\mathsf{Z}$. Let now $\{(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)\}_{n\in\mathbb{N}}$ and $(\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty, x_\infty)$ be pointed metric measure spaces, and let $\{\mathcal{X}_n\}_{n\in\mathbb{N}}$ and $\mathcal{X}_\infty$ be the respective equivalence classes. We say that $\mathcal{X}_n \xrightarrow{\text{pmG}} \mathcal{X}_\infty$ ("pointed measure Gromov converge") provided there exists a (complete, separable) metric space $\left(\hat{\mathsf{X}}, \hat{\mathsf{d}}\right)$ and isometric embeddings $\iota_n : \mathsf{X}_n \to \hat{\mathsf{X}}$, $\iota_\infty : \mathsf{X}_\infty \to \hat{\mathsf{X}}$ such that $\iota_n(x_n) \to \iota_\infty(x_\infty)$ and

$$\int \varphi \, \mathrm{d}(\iota_n)_\sharp(\mathfrak{m}_n) \longrightarrow \int \varphi \, \mathrm{d}(\iota_\infty)_\sharp(\mathfrak{m}_\infty) \tag{1.36}$$

for every $\varphi \in \mathbf{C}^0_\mathrm{b}(\hat{\mathsf{X}})$ with bounded support.

9. In the same article ([GMS15, Theorem 3.30]), it was proved that pmGH convergence of pointed metric measure spaces $\{(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n, x_n)\}_{n\in\mathbb{N}}$ implies pmG convergence of the corresponding equivalence classes; the converse holds (Theorem 3.33 in [GMS15]) whenever the metric measure spaces $\{(\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n)\}_{n\in\mathbb{N}}$ are all doubling with the same doubling constant $C_\mathrm{db} > 0$ (and the support of the limit measure coincides with the whole limit space).

10. The topology that induces the mGH convergence (for compact metric measure spaces) is metrizable when restricted to spaces of uniformly bounded diameter (see for example [Vil09, Chapter 27] and [GMS15]): thus, in that case, one can construct a metric d_mGH such that

$$\begin{gathered} (\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n) \xrightarrow[\text{mGH}]{n\to\infty} (\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty) \\ \text{if and only if} \\ \mathsf{d}_\mathrm{mGH}((\mathsf{X}_n, \mathsf{d}_n, \mathfrak{m}_n), (\mathsf{X}_\infty, \mathsf{d}_\infty, \mathfrak{m}_\infty)) \xrightarrow{n\to\infty} 0. \end{gathered} \tag{1.37}$$

In particular, for a sequence of compact, uniformly doubling metric measure spaces with uniformly bounded diameter (such as $\mathrm{RCD}(K,N)$ spaces with fixed $K > 0$, see Section 1.3), the pmG convergence – which is the most manageable in terms of computations – implies the convergence with respect to the metric d_mGH.

***Remark* 1.32.** A well-known result of Gromov (see [Gro07, Theorem 5.3]) states that, if $K \in \mathbb{R}$, then the family of pointed n-dimensional Riemannian manifolds with Ricci curvature bounded from below by K is *precompact* in the pGH topology. However, it is not closed with respect with this convergence: the limit space may fail to have a Riemannian structure. In Section 1.3 we will define a class of spaces – satisfying the so-called $\mathrm{RCD}(K,N)$ condition – that represents a better class in which to look for limits of Riemannian manifolds. $\lozenge$

1.1.5 Perimeter and coarea formula on metric measure spaces

In this Section, we introduce a notion of *perimeter* on metric measure spaces which extends the classical one on $\mathbb{R}^n$, and we state a general form of the coarea formula. For the sake of completeness, we do this passing through the metric definition of BV_* functions, which was first introduced in the articles [Amb01; Amb02; Mir03]; an overview of the different equivalent definitions can be found in [AD14], and in [AG16, Chapter 5].

Definition 1.33 (BV_*, total variation). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We define the space $\mathrm{BV}_*(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ as

$$\mathrm{BV}_*(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \doteq \left\{ f \in L^1(\mathsf{X}, \mathfrak{m}) \;\middle|\; \begin{array}{c} \text{there exist } \{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}) \text{ such that} \\ f_n \xrightarrow{L^1} f \text{ and } \limsup_{n\to\infty} \int_{\mathsf{X}} |\nabla f_n| \, \mathrm{d}\mathfrak{m} < +\infty \end{array} \right\}. \tag{1.38}$$

Here $\mathrm{Lip}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d})$ is the family of functions $f : \mathsf{X} \to \mathbb{R}$ such that for any $x \in \mathsf{X}$ there exists $r > 0$ such that $f \in \mathrm{Lip}(B_r(x), \mathsf{d})$: if (X, d) is locally compact, it coincides with the family of functions that are Lipschitz on any compact set; if (X, d) is compact it is simply $\mathrm{Lip}(\mathsf{X}, \mathsf{d})$ (this is the case of $\mathrm{RCD}(K, N)$ spaces with $K > 0$).

If $f \in \mathrm{BV}_*(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ and $A \subset \mathsf{X}$ is open, then we define the *total variation* of f in A as

$$|\mathrm{D}f|_*(A) \doteq \inf \left\{ \liminf_{n\to\infty} \int_A |\nabla f_n| \, \mathrm{d}\mathfrak{m} \;\middle|\; \{f_n\}_{n\in\mathbb{N}} \subset \mathrm{Lip}_{\mathrm{loc}}(\mathsf{X}, \mathsf{d}) \text{ and } f_n \xrightarrow{L^1(A,\mathfrak{m})} f \right\}. \tag{1.39}$$ ◊

The following proposition was proved in [Mir03]:

Proposition 1.34. *Let $f \in \mathrm{BV}_*(\mathsf{X}, \mathsf{d}, \mathfrak{m})$. The map $A \mapsto |\mathrm{D}f|_*(A)$ is the restriction to open sets of a unique finite Borel measure on X, which we still denote by $|\mathrm{D}f|_*(\cdot)$.*

Definition 1.35 (Perimeter). Let $E \in \mathscr{B}(\mathsf{X})$, where $\mathscr{B}(\mathsf{X})$ denotes the class of Borel sets of (X, d). We define the *perimeter measure* in E as $\mathrm{Per}(E; \cdot) \doteq |\mathrm{D}\chi_E|_*$, where χ_E is the characteristic function of E. In particular, for any open $A \subset \mathsf{X}$,

$$\mathrm{Per}(E; A) \doteq \inf \left\{ \liminf_{n\to\infty} \int_A |\nabla f_n| \, \mathrm{d}\mathfrak{m} \;\middle|\; f_n \in \mathrm{Lip}_{\mathrm{loc}}(A) \text{ and } f_n \to \chi_E \text{ in } L^1(A, \mathfrak{m}) \right\}.$$

If $\mathrm{Per}(E; \mathsf{X}) < \infty$, we say that E is a *set of finite perimeter*.
Thanks to Proposition 1.34, when E is a fixed set of finite perimeter, the map $A \mapsto \mathrm{Per}(E; A)$ is the restriction to open sets of a finite Borel measure on X, which can be characterized as

$$\mathrm{Per}(E; B) \doteq \inf\{\mathrm{Per}(E; A) \mid A \text{ open}, A \supset B\}. \tag{1.40}$$ ◊

For a given volume v, the isoperimetric profile of a metric measure space at v is defined through a minimization of the perimeter:

Definition 1.36 (Isoperimetric profile). Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathsf{X}) = 1$. The *isoperimetric profile* $\mathcal{I} = \mathcal{I}_{(\mathsf{X},\mathsf{d},\mathfrak{m})} : [0, 1] \to [0, +\infty)$ is defined as

$$\mathcal{I}_{(\mathsf{X},\mathsf{d},\mathfrak{m})}(v) \doteq \inf\{\mathrm{Per}(E) \mid E \in \mathscr{B}(\mathsf{X}), \mathfrak{m}(E) = v\}, \quad v \in [0, 1]. \tag{1.41}$$ ◊

***Remark* 1.37** (Outer Minkowski content)**.** In the literature, the isoperimetric profile of a metric measure space is sometimes defined with the *outer Minkowski content* in place of the perimeter; *i.e.*, with the quantity

$$\mathfrak{m}^+(E) \doteq \liminf_{\varepsilon \downarrow 0} \frac{\mathfrak{m}(\mathcal{B}_\varepsilon(E)) - \mathfrak{m}(E)}{\varepsilon}. \tag{1.42}$$

For the purposes of this thesis, the two approaches are equivalent (see [CM18; CM17]). A version of the Minkowski content in $\mathbb{R}^n$ will appear again in Chapter 4. ♢

We now state a suitable version of the coarea formula for BV_* functions on metric measure spaces, which was first proved in [Mir03, Proposition 4.2 and Remark 4.3]; see also the discussion in [AG16, Chapter 5]. In Proposition 1.70, we'll state a version adapted to the $\mathrm{RCD}(K,N)$ case.

Proposition 1.38 (Coarea formula)**.** *Let $u \in \mathrm{BV}_*(\mathsf{X},\mathsf{d},\mathfrak{m})$. For almost every $t \in \mathbb{R}$, $\{u > t\}$ has finite perimeter, and*

$$\int_{-\infty}^{+\infty} \mathrm{Per}(\{u > t\}; A)\,\mathrm{d}t = |\mathrm{D}u|_*(A) \qquad \text{for all } A \in \mathscr{B}(\mathsf{X}). \tag{1.43}$$

Moreover, if $f : \mathsf{X} \to \mathbb{R}$ is Borel measurable,

$$\int_{-\infty}^{+\infty} \left(\int_A v(x) \mathrm{d}\,\mathrm{Per}(\{u > t\}; A)(x) \right) \mathrm{d}t = \int_A v(x)\,\mathrm{d}|\mathrm{D}u|_*(x). \qquad \text{for all } A \in \mathscr{B}(\mathsf{X}). \tag{1.44}$$

1.1.6 Rearrangements and symmetrizations

As a last step before specializing to particular classes of metric measure spaces, we recall here the classical definition of *decreasing rearrangement* appearing in the theory of symmetrizations (see for example [Kes06]). These notions will play a role in Section 1.2.3 and Section 1.3.1, and thus in the proof of the Talenti Theorem 2.14.

Throughout the section $(\mathsf{X}, \mathfrak{m})$ will be a measure space with $\mathfrak{m}(\mathsf{X}) = 1$, and $\Omega \subset \mathsf{X}$ will be an open subset.

Definition 1.39 (Distribution function)**.** Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define its *distribution function* $\mu = \mu_u : [0, +\infty) \to [0, \mathfrak{m}(\Omega)]$ as

$$\mu(t) \doteq \mathfrak{m}(\{|u| > t\}). \tag{1.45}$$

♢

Definition 1.40 (Decreasing rearrangement $u^\sharp$)**.** Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define $u^\sharp : [0, \mathfrak{m}(\Omega)] \to [0, \infty]$ as

$$u^\sharp(s) \doteq \begin{cases} \operatorname{ess\,sup} |u| & \text{if } s = 0 \\ \inf\{t \in [0, +\infty) \mid \mu_u(t) < s\} & \text{if } s > 0 \end{cases}. \tag{1.46}$$

♢

The decreasing rearrangement $u^\sharp$ plays the role of a generalized inverse of the distribution function $\mu = \mu_u$:

- if μ is continuous at $\bar{t}$ with $\mu(\bar{t}) = \bar{s}$, and μ is not constant in any interval of the type $[\bar{t}, \bar{t} + \delta)$ with $\delta > 0$, then $u^\sharp(\bar{s}) = \bar{t}$;

- if μ is continuous at $\bar{t}$ with $\mu(\bar{t}) = \bar{s}$, and $[\bar{t}, \bar{t} + \bar{\delta})$ is the largest interval of this type on which μ is constant, then $u^\sharp(\bar{s}) = \bar{t} + \bar{\delta}$;
- if μ has a jump discontinuity at $\bar{t}$, with $\lim_{\tau \to \bar{t}^\pm} \mu(\tau) = \bar{s}^\pm$, then $u^\sharp(s) = \bar{t}$ for any $s \in (\bar{s}^-, \bar{s}^+]$.

As the name itself suggests, $u^\sharp$ can be easily shown to be non-increasing; moreover, it is by definition left-continuous.

***Remark* 1.41.** The choice of defining a *non-negative* decreasing rearrangement $u^\sharp$ (*i.e.*, of considering the superlevels of *the absolute value* of u in the definition of μ_u) is not the only meaningful one: for example, the textbook [Kes06], which deals with similar problems, makes the opposite choice (see [Kes06, Remark 1.1.2] therein). As we will see, our choice will be helpful in the proof of the Talenti Theorem 2.14; however, in the paragraph 2.2.1 we will introduce the alternative definition with the name of *signed decreasing rearrangement.* $\Diamond$

We prove here a lemma which only needs the very definition of decreasing rearrangement:

Lemma 1.42. *Let $\Omega \subset \mathsf{X}$ have finite measure; let $f : \Omega \to \mathbb{R}$ be integrable and let $E \subset \Omega$ be measurable. Then:*

$$\int_E f \,\mathrm{d}\mathfrak{m} \leq \int_0^{\mathfrak{m}(E)} f^\sharp(s)\,\mathrm{d}s.$$

Moreover, if f is non-negative, equality holds if and only if $(f|_E)^\sharp \equiv (f^\sharp)\big|_{[0,\mathfrak{m}(E)]}$.

Proof. The proof is analogous to the one proposed in [Kes06, Chap. 1] in Euclidean setting, we report it briefly for the reader's convenience. Preliminarily, we observe that

$$\int_E f \,\mathrm{d}\mathfrak{m} \leq \int_E |f| \,\mathrm{d}\mathfrak{m} \quad \text{and} \quad f^\sharp = |f|^\sharp,$$

thus we can assume without loss of generality that f is non-negative.
First notice that, by equimeasurability,

$$\int_E f \,\mathrm{d}\mathfrak{m} = \int_0^{\mathfrak{m}(E)} (f|_E)^\sharp(s)\,\mathrm{d}s. \tag{1.47}$$

Moreover, for any $t \in \mathbb{R}$, we have:

$$\left\{x \in E \;\middle|\; f|_E > t\right\} = E \cap \{x \in \Omega \mid f > t\} \subset \{x \in \Omega \mid f > t\},$$

thus whenever $s < \mathfrak{m}(E)$:

$$\left\{t > 0 \;\middle|\; \mathfrak{m}(f|_E > t) < s\right\} \supset \{t > 0 \mid \mathfrak{m}(f > t) < s\}.$$

As a consequence, taking the infimum of the two sets in the previous inclusion, we get the inequality

$$(f|_E)^\sharp(s) \leq f^\sharp(s),$$

which gives, together with Equation (1.47), the desired result. $\square$

1.2 Weighted intervals

In this section, we specialize some of the notions from Section 1.1 to the case where $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a *weighted interval*, and develop some of the tools needed in the sequel in the framework of *model spaces* (see Section 1.3.1). Some of the results proposed in this Section are a generalization of results proved for model spaces in [MV21].

The precise setting is the following:

Assumption 1.43. The setting for the whole present section will be the (complete, separable) metric measure space $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$, where:

- $\mathfrak{I} \doteq [\mathfrak{a}, \mathfrak{b}] \subset \mathbb{R}$ is a closed bounded interval, and d_{eu} is the standard Euclidean distance on it;
- $h : \mathfrak{I} \to [0, \infty)$ will be a density, with $\int_{\mathfrak{a}}^{\mathfrak{b}} h \, \mathrm{d}\mathscr{L}^1 = 1$;
- $\mathfrak{m}_h \doteq h\mathscr{L}^1 \llcorner \mathfrak{I}$ is a probability measure on $\mathfrak{I}$.

We will denote by $H : [\mathfrak{a}, \mathfrak{b}] \to [0, 1]$ the distribution function $H(x) \doteq \int_{\mathfrak{a}}^{x} h \, \mathrm{d}x = \mathfrak{m}_h([\mathfrak{a}, x])$. Moreover, we'll denote by $\mathring{\mathfrak{I}} = (\mathfrak{a}, \mathfrak{b})$ the interior of $\mathfrak{I}$. ♢

The (global) Sobolev space $H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$, introduced in Definition 1.7, admits a handy characterization:

Proposition 1.44. *Let $\mathfrak{I}$, h and $\mathfrak{m}_h$ be as in Assumption 1.43. Let $p \in (1, \infty)$. Assume that $h^{\frac{1}{1-p}} \in L^1_{\mathrm{loc}}(\mathring{\mathfrak{I}})$. Then the following holds:*

(i) The Sobolev space $H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ can be characterized as:

$$H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h) = \left\{ v \in L^p(\mathfrak{I}, \mathfrak{m}_h) \,\middle|\, \begin{array}{c} \exists \tilde{v} \in \mathrm{AC}_{\mathrm{loc}}(\mathring{\mathfrak{I}}) \text{ such that} \\ v = \tilde{v} \text{ a.e. and } \tilde{v}' \in L^p(\mathfrak{I}, \mathfrak{m}_h) \end{array} \right\}, \tag{1.48}$$

and the minimal weak upper gradient of a function $v \in H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ coincides $\mathfrak{m}$-a.e. with $|v'|$.

(ii) Let $\Omega \subset \mathfrak{I}$ be open relatively to $\mathfrak{I}$. Then the local Sobolev space $H^{1,p}_0(\Omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ coincides with

$$\overline{\mathbf{C}^\infty_{\mathrm{c}}(\Omega)}^{H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)}. \tag{1.49}$$

***Remark* 1.45 (References for the proof).** We summarize here some useful references for the previous result:

- The inclusion "$\supset$" in Equation (1.48) is rather easy, once one knows the equivalence between $H^{1,p}$ and $N^{1,p}$ (Remark 1.15).
- For a proof of "$\subset$", see [BBS20, Proposition 1.3], which deals with the equality $|\nabla v|_{\mathrm{w}} = |v'|$ at almost every internal point of $\mathfrak{I}$.
- Equation (1.49) follows from the approximation in $H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ of Lipschitz functions with $\mathbf{C}^\infty$ functions, which is proved for example in [HKM06, Lemma 1.11].

- Notice that this result takes this quite simple form thanks to the fact that we are dealing with a *one-dimensional* weighted space which admits an *ad hoc* argument (namely, the argument in [BBS20, Section 6]). The theory of Sobolev spaces on weighted Euclidean domains is quite rich: see [BB11, Appendix A.2] and [HKST15, Section 14.2] for the unbounded case with a weight h which makes $\mathfrak{m}_h$ *doubling* and *Poincaré*; other related results can be found in [BBK06; HKM06; Kil94]; a more general approach to weighted Sobolev spaces on non Euclidean metric measure spaces can be found in [APS19].

$\diamondsuit$

Lemma 1.46. *Let $\mathfrak{I}$, h and $\mathfrak{m}_h$ be as in Assumption 1.43. Let $p \in (1, \infty)$.*

Assume that $\Omega \subset \mathfrak{I}$ is a open subinterval (relatively to $\mathfrak{I}$). If $u \in H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h) \cap \mathbf{C}^0(\mathfrak{I})$ is supported in $\bar{\Omega}$, and $u = 0$ on the boundary of Ω, then $u \in H_0^{1,p}(\Omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$.

Proof. The proof is analogous to the one of [HKM06, Lemma 1.26]: for any $\varepsilon > 0$, the function $u_\varepsilon \doteq (u - \varepsilon)^+$ belongs to $H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ and has support compactly contained in Ω, thus it belongs to $H_0^{1,p}(\Omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ (see [HKM06, Lemma 1.25(i)]). Moreover, $u_\varepsilon \to u$ in $H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$: since $H_0^{1,p}(\Omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ is a closed subset of $H^{1,p}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$, this implies the statement of the lemma. $\square$

***Remark* 1.47.** The application of Lemma 1.46 we are interested in is the case where $\Omega = [\mathfrak{a}, x)$ with $x \in \mathring{\mathfrak{I}} = (\mathfrak{a}, \mathfrak{b})$. $\diamondsuit$

1.2.1 Weighted intervals: isoperimetric profile

In Proposition 1.50, we will analyse the isoperimetric profile of a class of weighted intervals. Let us define the class of density functions involved.

Definition 1.48 (Log-concave)**.** We say that $h : \mathring{\mathfrak{I}} \to \mathbb{R}_+$ is *strictly log-concave* in $\mathring{\mathfrak{I}}$ if $x \mapsto \log h(x)$ is strictly concave in $\mathring{\mathfrak{I}}$. If $h \in \mathbf{C}^1(\mathring{\mathfrak{I}})$, it is equivalent to ask that $(\log h)'$ is strictly increasing in $\mathring{\mathfrak{I}}$. $\diamondsuit$

Lemma 1.49 (Properties of log-concave functions)**.** *Let $\mathfrak{I}$, h and $\mathfrak{m}_h$ be as in Assumption 1.43. Assume the following: $h(\mathfrak{a}) = h(\mathfrak{b}) = 0$ and $h > 0$ in $\mathring{\mathfrak{I}}$; h belongs to $\mathbf{C}^1(\mathring{\mathfrak{I}}) \cap \mathbf{C}^0(\mathfrak{I})$; and h is strictly log-concave in $\mathring{\mathfrak{I}}$. Then:*

(i) h admits a unique maximum at a point $x_h \in \mathring{\mathfrak{I}}$;

(ii) $\lim_{x\to\mathfrak{a}} \frac{h'}{h} = +\infty$ and $\lim_{x\to\mathfrak{b}} \frac{h'}{h} = -\infty$.

Proof. The map $x \mapsto \log h(x)$ is $\mathbf{C}^1$ in $\mathring{\mathfrak{I}}$, and satisfies

$$\lim_{x\to\mathfrak{a}} \log h(x) = \lim_{x\to\mathfrak{b}} \log h(x) = -\infty. \tag{1.50}$$

Thus it admits at least a maximum point, and by concavity it is unique. Moreover, $(\log h)' = \frac{h'}{h}$ is strictly increasing in $\mathring{\mathfrak{I}}$: by Equation (1.50) this implies that $\lim_{x\to\mathfrak{a}} \frac{h'}{h} = +\infty$ and $\lim_{x\to\mathfrak{b}} \frac{h'}{h} = -\infty$. $\square$

Proposition 1.50 (Isoperimetric profile of $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$). *Let $\mathfrak{I}$, h and $\mathfrak{m}_h$ be as in Assumption 1.43. Assume the following: $h(\mathfrak{a}) = h(\mathfrak{b}) = 0$ and $h > 0$ in $\mathring{\mathfrak{I}}$; h belongs to $\mathbf{C}^1(\mathring{\mathfrak{I}}) \cap \mathbf{C}^0(\mathfrak{I})$; and h is strictly log-concave in $\mathring{\mathfrak{I}}$. Then the isoperimetric profile $\mathcal{I}_h$ of the weighted interval $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ is given by the following formula:*

$$\mathcal{I}_h(v) = \min\Big\{h(H^{-1}(v)), h(H^{-1}(1-v))\Big\}, \quad v \in [0,1].$$

Moreover, the inf *in Equation (1.41) is attained at one of the intervals*

$$(\mathfrak{a}, H^{-1}(v)) \quad \text{and} \quad (H^{-1}(1-v), \mathfrak{b}), \tag{1.51}$$

possibly both if their perimeters coincide.

In other words: $\mathcal{I}_h(v)$ coincides with the density function computed either at the point x such that $\mathfrak{m}_h([\mathfrak{a}, x]) = v$ or at the point x such that $\mathfrak{m}_h([x, \mathfrak{b}]) = v$.

Proof. The proof is a slight modification of [Bob96], we include it here for the reader's convenience. Thanks to [CM18, Proposition 3.1], we know that if E has finite perimeter in $\mathfrak{I}$, then it is $\mathfrak{m}_h$-equivalent to a countable union of closed disjoint intervals, i.e. there exists a sequence of pairwise disjoint intervals $\{[a_i, b_i]\}_{i\in\mathbb{N}}$ such that $[a_i, b_i] \subset \mathfrak{I}$ and

$$\mathfrak{m}_h\left(E \triangle \bigcup_{i\in\mathbb{N}} [a_i, b_i]\right) = 0, \tag{1.52}$$

thus it suffices to consider such unions. Moreover, by the same result, if Equation (1.52) holds then one has:

$$\mathrm{Per}(E) = \sum_{i=0}^{\infty} (h(a_i) + h(b_i)).$$

Step 1. We claim that one of the intervals in Equation (1.51) is minimal among the class of closed intervals. Let $v \in (0,1)$; notice that the problem trivializes at 0 and 1. We denote by $f_v : (\mathfrak{a}, H^{-1}(1-v)) \to \mathfrak{I}$ the function defined by

$$f_v(x) \doteq H^{-1}(H(x) + v),$$

that is: $f_v(x)$ is the unique element of $\mathfrak{I}$ such that the interval $(x, f_v(x))$ has $\mathfrak{m}_h$-measure v. Notice that $H(f_v(x)) - H(x) = v$, thus $h(f_v(x)) f_v'(x) = h(x)$.

Moreover, we denote by $p_v : (\mathfrak{a}, H^{-1}(1-v)) \to (0, +\infty)$ the function

$$p_v(x) \doteq \mathrm{Per}((x, f_v(x))) = h(x) + h(f_v(x)).$$

By differentiating with respect to x, one finds:

$$p_v'(x) = h'(x) + h'(f_v(x)) f_v'(x) = h(x)\Big(\frac{h'(x)}{h(x)} + \frac{h'(f_v(x))}{h(f_v(x))}\Big).$$

By definition of (strict) log-concavity the map $z \mapsto \frac{h'(z)}{h(z)}$ is always strictly decreasing; on the other hand, $f_v(\cdot)$ is strictly increasing. As a consequence, the map $x \to \frac{p_v'(x)}{h(x)}$ is strictly decreasing; moreover, by Lemma 1.49, it tends to $+\infty$ when $x \downarrow 0$, while it tends to $-\infty$ when $x \uparrow H^{-1}(1-v)$.

This means there exists a value x_v such that $p_v' > 0$ on $(\mathfrak{a}, x_v)$ and $p_v' < 0$ on $(x_v, H^{-1}(1-v))$; this means precisely that the minimum is attained at one of the intervals in Equation (1.51).

We also notice that

$$x_v < x_h \quad \text{and} \quad f_v(x_v) > x_h \tag{1.53}$$

must hold, where x_h is the unique maximum of h (see Lemma 1.49). Indeed, at x_v it holds that

$$0 = p_v'(x_v) = h'(x_v) + h'(f_v(x_v))f_v'(x_v). \tag{1.54}$$

In particular, since $f_v' > 0$ everywhere, $h'(x_v)$ and $h'(f_v(x_v))$ have opposite signs. But h has x_h as its unique maximum point, thus it must lay between x_v and $f_v(x_v)$.

Step 2. We claim that one of the intervals in Equation (1.51) is also minimal among finite unions of closed intervals. Let now

$$E = \bigcup_{i=1}^{n} [a_i, b_i], \quad n \geq 2,$$

with $a_1 \geq \mathfrak{a}$, $b_n \leq \mathfrak{b}$, and $b_{i-1} < a_i < b_i < a_{i+1}$. Denote by v_i the measure $\mathfrak{m}_h([a_i, b_i])$.

We will move each interval to the *left* or to the *right*, keeping the measure constant and lowering the perimeter. Notice that at least one of the following conditions holds true:

$$a_1 < x_{v_1} \qquad \text{or} \qquad a_n > x_{v_n};$$

indeed, if $a_1 \geq x_{v_1}$, then $a_n > b_1 \geq f(x_{v_1}) > x_h > x_{v_n}$ (here Equation (1.53) has been used). Up to a reflection, we can assume without loss of generality that $a_1 < x_{v_1}$. Then we define E_0 as

$$E_0 \doteq [\mathfrak{a}, f_{v_1}(\mathfrak{a})] \cup \bigcup_{i=2}^{n} [a_i, b_i].$$

E_0 now has the same measure as E and smaller perimeter. If $n = 2$, we skip to the end of the procedure; if otherwise $n > 2$, we proceed inductively in the following way: at each step $1 \leq j \leq n-2$, the set E_{j-1} will be the union of $n+1-j$ closed intervals:

$$E_{j-1} = \bigcup_{i=1}^{n+1-j} [a_i^j, b_i^j], \qquad v_i^j \doteq \mathfrak{m}_h\Big([a_i^j, b_i^j]\Big),$$

with $a_1^j = \mathfrak{a}$. We consider the second of those intervals:

- if $a_2^j \leq x_{v_2^j}$, then we replace $[\mathfrak{a}, b_1^j]$ and $[a_2^j, b_2^j]$ with $[\mathfrak{a}, f_{v_1^j + v_2^j}(\mathfrak{a})]$.
- if $a_2^j > x_{v_2^j}$, then we replace $[a_2^j, b_2^j]$ and $[a_3^j, b_3^j]$ with $\Big[f^{-1}_{v_2^j + v_3^j}(b_3^j), b_3^j\Big]$.

The new set E_j is a union of $n-j$ closed intervals, having the same $\mathfrak{m}_h$-measure of E_{j-1} and smaller or equal perimeter.

At the end of the procedure, we are left with the union of two intervals; applying the same argument once again, the final set $\tilde{E}$ is either the interval $[\mathfrak{a}, f_v(\mathfrak{a})]$ (in which case the claim is proven), or a union of type $[\mathfrak{a}, \tilde{b}] \cup [\tilde{a}, \mathfrak{b}]$. In the latter case, however, we can repeat the above argument for the interval $[\tilde{b}, \tilde{a}]$ and the measure $1-v$: we move it to the left or to the right applying the same criterion as before, and take the complementary in $\mathfrak{I}$. This is an interval of the same type as Equation (1.51), with the same measure of E but lower perimeter.

Step 3. Finally, we show that one of the intervals in Equation (1.51) is also minimal among countable unions of disjoint intervals. Assume $E = \bigcup_{i\in\mathbb{N}}[a_i, b_i]$. Since E has finite perimeter, the only accumulation points for the a_i's can be $\mathfrak{a}$ and $\mathfrak{b}$. Assume $\mathfrak{a}$ is an accumulation point; fix $\tilde{b} \in (\mathfrak{a}, x_h)$ and let $I \doteq \left\{ i \in \mathbb{N} \,\middle|\, b_i \le \tilde{b} \right\}$. Let $\bar{E} \doteq \bigcup_{i\in I}[a_i, b_i]$ and $\bar{v} \doteq \mathfrak{m}_h(\bar{E})$. The set

$$[\mathfrak{a}, f_{\bar{v}}(\mathfrak{a})] \cup \bigcup_{i\in\mathbb{N}\setminus I} [a_i, b_i]$$

has the same measure and lower perimeter than E. Repeating, if necessary, the procedure at $\mathfrak{b}$, we find a set which is a finite union of closed intervals and lowers the perimeter of E, so we can recover the result from Step 2. □

1.2.2 Weighted intervals: Poisson problem

Definition 1.51 (Laplacian on the weighted interval)**.** Let again $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ be as in Assumption 1.43. Assume that $h(\mathfrak{a}) = h(\mathfrak{b}) = 0$ and h belongs to $\mathbf{C}^1(\mathring{\mathfrak{I}}) \cap \mathbf{C}^0(\mathfrak{I})$. We define the *weighted Laplacian*

$$\Delta_h : \mathbf{C}^2(\mathring{\mathfrak{I}}) \cap \mathbf{C}^1(\mathfrak{I}) \to \mathbf{C}^0(\mathring{\mathfrak{I}}) \tag{1.55}$$

on the interval $\mathring{\mathfrak{I}}$ as:

$$\Delta_h \eta \doteq \eta'' + (\log(h))'\eta' = \eta'' + \frac{h'}{h}\eta'. \tag{1.56}$$

Notice that, for any $\eta \in \mathbf{C}^2(\mathring{\mathfrak{I}}) \cap \mathbf{C}^1(\mathfrak{I})$ and any function $\varphi \in \mathbf{C}^1(\mathring{\mathfrak{I}}) \cap \mathbf{C}^0(\mathfrak{I})$, using that $h = 0$ on $\partial\mathfrak{I}$, one has

$$\int_{\mathfrak{I}} \eta'\varphi' \,\mathrm{d}\mathfrak{m}_{K,N} = -\int_{\mathfrak{I}} (\varphi\eta'' h + \varphi\eta' h') \,\mathrm{d}\mathscr{L}^1 = -\int_{\mathfrak{I}} \varphi \Delta_{K,N}\eta \,\mathrm{d}\mathfrak{m}_{K,N},$$

consistently with Definition 1.51.

Accordingly with Definition 1.51, given an interval $\Omega \subset \mathfrak{I}$, open in the topology of $\mathfrak{I}$, and $f \in L^2(\Omega, \mathfrak{m}_h)$, we say that a function w is a weak solution to $-\Delta_h w = f$ in Ω (with appropriate boundary conditions) if it solves

$$-w'' - \frac{h'}{h} w' = f \quad \text{in } \Omega$$

in a distributional sense. In particular, we will be interested in the following Dirichlet problem:

Definition 1.52. Let $\Omega \doteq [\mathfrak{a}, r_1)$ with $\mathfrak{a} < r_1 < \mathfrak{b}$ and let $f \in L^2(\Omega, \mathfrak{m}_h)$. We say that $w \in H^{1,2}(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ is a weak solution to

$$\begin{cases} -\Delta_h w = f & \text{in } \Omega = [\mathfrak{a}, r_1) \\ w(r_1) = 0 \end{cases} \tag{1.57}$$

if:

(i) $\displaystyle\int_{[\mathfrak{a},r_1]} w'\varphi' \,\mathrm{d}\mathfrak{m}_h = \int_{[\mathfrak{a},r_1]} f\varphi \,\mathrm{d}\mathfrak{m}_h$ for any $\varphi \in \mathbf{C}^\infty_{\mathrm{c}}([\mathfrak{a},r_1))$;

(ii) Boundary condition: $w \in H^{1,2}_0([\mathfrak{a},r_1),\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_h)$, where the latter space is the closure of the family $\mathbf{C}^\infty_{\mathrm{c}}([\mathfrak{a},r_1))$ in the topology of $H^{1,2}(\Omega,\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_h)$ (see Proposition 1.44). $\Diamond$

In the next proposition, we give an explicit solution to the problem in (1.57).

Proposition 1.53. *Assume the following: $h(\mathfrak{a}) = h(\mathfrak{b}) = 0$ and $h > 0$ in $\mathring{\mathfrak{I}}$; h belongs to $\mathbf{C}^1(\mathring{\mathfrak{I}}) \cap \mathbf{C}^0(\mathfrak{I})$; and h is strictly log-concave in $\mathring{\mathfrak{I}}$. Let $\Omega \doteq [\mathfrak{a},r_1)$ with $\mathfrak{a} < r_1 < \mathfrak{b}$. Let $f \in L^2(\Omega,\mathfrak{m}_h)$. The problem in Equation (1.57) admits a unique weak solution $w \in H^{1,2}_0(\Omega,\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_h)$, which can be represented as*

$$w(\varrho) = \int_\varrho^{r_1} \frac{1}{h(r)} \int_{\mathfrak{a}}^r f(s)\,\mathrm{d}\mathfrak{m}_h(s)\,\mathrm{d}r, \quad \forall \varrho \in [\mathfrak{a},r_1], \tag{1.58}$$

Proof. We first show that a weak solution must coincide with the function in Equation (1.58), and then we prove that such function is actually a solution to Equation (1.57).

Step 1. Let $w \in H^{1,2}_0(\Omega,\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_h)$ be a weak solution to Equation (1.57). We prove that the weak derivative of w coincides $\mathfrak{m}_h$-a.e. with the function

$$g(x) \doteq -\frac{1}{h(x)} \int_{\mathfrak{a}}^x f(s)\,\mathrm{d}\mathfrak{m}_h(s).$$

Indeed, for any test function $\varphi \in \mathbf{C}^\infty_{\mathrm{c}}([\mathfrak{a},r_1))$ one has, by the Fubini-Tonelli Theorem:

$$\begin{aligned}\int_\Omega (-g(x))\varphi'(x)\,\mathrm{d}\mathfrak{m}_h(x) &= \int_{\mathfrak{a}}^{r_1}\left(\int_{\mathfrak{a}}^{r_1} \chi_{[\mathfrak{a},x]}(s) f(s)\frac{\varphi'(x)}{h(x)}\,\mathrm{d}\mathfrak{m}_h(s)\right)\mathrm{d}\mathfrak{m}_h(x)\\ &= \int_{\mathfrak{a}}^{r_1} f(s)\left(\int_s^{r_1} \varphi'(x)\,\mathrm{d}\mathscr{L}^1(x)\right)\mathrm{d}\mathfrak{m}_h(s) =\\ &= -\int_{\mathfrak{a}}^{r_1} f(s)\varphi(s)\,\mathrm{d}\mathfrak{m}_h(s).\end{aligned} \tag{1.59}$$

Thus, since w is a weak solution to Equation (1.57), for any $\varphi \in \mathbf{C}^\infty_{\mathrm{c}}([\mathfrak{a},r_1))$

$$\int_\Omega [g(x) - w'(x)]h(x)\varphi'(x)\,\mathrm{d}\mathscr{L}^1(x) = 0. \tag{1.60}$$

By a classical result (see for example [Bre11, Lemma 8.1]), there exists a constant $C \in \mathbb{R}$ such that $w'(x)h(x) = g(x)h(x) + C$ for $\mathfrak{m}_h$-a.e. $x \in \Omega$. This however implies that for any $\varphi \in \mathbf{C}^\infty_{\mathrm{c}}([\mathfrak{a},r_1))$

$$0 = C\int_{\mathfrak{a}}^{r_1} \varphi'(x)\,\mathrm{d}\mathscr{L}^1(x) = C\varphi(\mathfrak{a}),$$

hence $C = 0$.

Now w is a $H^{1,2}(\Omega,\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_h)$ function, thus in particular it belongs to the classical Sobolev space $H^{1,2}((\mathfrak{a}+\varepsilon,r_1),\mathsf{d}_{\mathrm{eu}},\mathscr{L}^1)$ for any $\varepsilon > 0$ (because of the assumptions $h > 0$ in $\mathring{\mathfrak{I}}$ and $h \in \mathbf{C}^0(\mathfrak{I})$); moreover, w satisfies $w' = g$ a.e. and $w(r_1) = 0$. Thus, by well known results about Sobolev functions on intervals (see [Bre11, Theorem 8.2], w coincides with the function in Equation (1.58) for $\mathfrak{m}_h$-a.e. $\varrho \in \Omega$.

STEP 2: Let now w be defined as in Equation (1.58). Since the integrand is continuous on $(\mathfrak{a}, r_1]$, w is a $\mathbf{C}^1$ function on $(\mathfrak{a}, r_1]$ (with $w(r_1) = 0$). By straightforward computations, we show that w and w' are $L^2(\Omega, \mathfrak{m}_h)$ functions. Indeed, by Hölder inequality we have that

$$\int_{\mathfrak{a}}^{r} |f(s)| \,\mathrm{d}\mathfrak{m}_h(s) \leq \|f\|_{L^2(\Omega,\mathfrak{m}_h)} H(r)^{\frac{1}{2}}. \tag{1.61}$$

In particular,

$$|w(\varrho)| \leq \|f\|_{L^2(\Omega,\mathfrak{m}_h)} \int_{\varrho}^{r_1} \frac{H^{\frac{1}{2}}(r)}{h(r)} \,\mathrm{d}r \tag{1.62}$$

$$|w'(\varrho)| \leq \|f\|_{L^2(\Omega,\mathfrak{m}_h)} \frac{H^{\frac{1}{2}}(\varrho)}{h(\varrho)}, \tag{1.63}$$

and thus, using the Jensen inequality and the Tonelli Theorem,

$$\|w\|^2_{L^2(\Omega,\mathfrak{m}_h)} \leq \|f\|^2_{L^2(\Omega,\mathfrak{m}_h)} \int_{\mathfrak{a}}^{r_1} \int_{\mathfrak{a}}^{r_1} \frac{H(r)h(\varrho)}{h^2(r)} \chi_{[\varrho,r_1]}(r) \,\mathrm{d}\varrho \,\mathrm{d}r = \|f\|^2_{L^2(\Omega,\mathfrak{m}_h)} \int_{\mathfrak{a}}^{r_1} \left(\frac{H(r)}{h(r)}\right)^2 \mathrm{d}r \tag{1.64}$$

$$\|w'\|^2_{L^2(\Omega,\mathfrak{m}_h)} \leq \|f\|^2_{L^2(\Omega,\mathfrak{m}_h)} \int_{\mathfrak{a}}^{r_1} \frac{H(r)}{h(r)} \,\mathrm{d}r, \tag{1.65}$$

which are both finite by assumption: indeed, $\lim_{r\downarrow 0} \frac{H(r)}{h(r)} = 0$ by Lemma 1.49.

Secondly, the boundary condition in Definition 1.52 is satisfied (*i.e.*, $w \in H^{1,2}_0(\Omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$): this is a consequence of Lemma 1.46 and the fact that $w(r_1) = 0$.

Finally, by tracing back the identity in Equation (1.59), the very same argument shows that w is a weak solution to Equation (1.57). □

Corollary 1.54. *Let the assumptions of Proposition 1.53 hold. If, in addition, the density function h is symmetric with respect to the middle point of $\mathfrak{I}$, then w can be also expressed as*

$$w(\varrho) = \int_{H(\varrho)}^{H(r_1)} \frac{1}{\mathcal{I}_h^2(\sigma)} \int_{\mathfrak{a}}^{\sigma} f \circ H^{-1}(t) \,\mathrm{d}t \,\mathrm{d}\sigma, \quad \forall \varrho \in [\mathfrak{a}, r_1]. \tag{1.66}$$

Proof. We show that the expressions in Equation (1.58) and Equation (1.66) coincide:

$$\begin{aligned}\int_{\varrho}^{r_1} \frac{1}{h(r)} \int_{\mathfrak{a}}^{r} f(s) \,\mathrm{d}\mathfrak{m}_h(s) \,\mathrm{d}r &= \int_{\varrho}^{r_1} \frac{1}{h^2(r)} \left(\int_0^r f(s)h(s) \,\mathrm{d}s\right) h(r) \,\mathrm{d}r \\ &= \int_{H(\varrho)}^{H(r_1)} \frac{1}{h^2(H^{-1}(\sigma))} \int_{\mathfrak{a}}^{\sigma} f \circ H^{-1}(t) \,\mathrm{d}t \,\mathrm{d}\sigma\end{aligned}$$

and by Proposition 1.50 it holds that $\mathcal{I} = h \circ H^{-1}$. We have used the change of variables $t = H(s)$ in the internal integral and the change of variables $\sigma = H(r)$ in the external integral. □

1.2.3 Weighted intervals: rearrangements

Let u be a measurable function defined on a measured space $(\mathsf{X}, \mathfrak{m})$, and let $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ be a weighted interval. The decreasing rearrangement $u^\sharp$ of u defined in Section 1.1.6 can be "reparametrized" so that it becomes equimeasurable with u in the measure space $(\mathfrak{I}, \mathfrak{m}_h)$:

Definition 1.55 (Equimeasurable decreasing rearrangement)**.** Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathsf{X}) = 1$; let $\Omega \subset \mathsf{X}$ be a Borel subset with measure $\mathfrak{m}(\Omega) = v \in [0,1]$ and $u : \Omega \to \mathbb{R}$ be a Borel measurable function. Let us fix a weighted interval $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$, with $\mathfrak{I} \doteq [0, \mathfrak{b}]$, and let $R_{h,v} \in \mathfrak{I}$ be such that $\mathfrak{m}_h([0, R_{h,v}]) = v$ (*i.e.*, $R_{h,v} \doteq H^{-1}(v)$). We define the *equimeasurable decreasing rearrangement* $u_h^\star : [0, R_{h,v}] \to [0, \infty]$ as $u_h^\star \doteq u^\sharp \circ H$; explicitly:

$$u_h^\star(x) \doteq u^\sharp(\mathfrak{m}_h([0,x])). \tag{1.67}$$

$\Diamond$

***Remark* 1.56.** Being the composition of H, which is increasing, and $u^\sharp$, which is non-increasing, $u_h^\star$ is still a non-increasing function. $\Diamond$

We state here a collection of useful facts concerning the decreasing rearrangement of a function: these are quite standard and can be found for instance in [Kes06, Chapter 1] in the context of Euclidean spaces (grounding on a slightly different definition of μ_u: namely, the one we will call μ_u^{sg} in Definition 2.16); the proofs contained there still work with very few straightforward modifications.

Proposition 1.57. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be a metric measure space with $\mathfrak{m}(\mathsf{X}) = 1$; let $\Omega \subset \mathsf{X}$ be a Borel subset with measure $\mathfrak{m}(\Omega) = v \in [0,1]$. Let us fix a weighted interval $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$, with $\mathfrak{I} \doteq [0, \mathfrak{b}]$, and let $R_{h,v} \doteq H^{-1}(v)$. Let $u : \Omega \to \mathbb{R}$ be a Borel measurable function. Let $u^\sharp : [0, v] \to [0, \infty]$ be its decreasing rearrangement and $u_h^\star : [0, H^{-1}(v)] \to [0, \infty]$ be its equimeasurable decreasing rearrangement on $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$. Then:*

(a) *u, $u^\sharp$ and $u_h^\star$ are* equimeasurable, *in the sense that*

$$\mathfrak{m}(\{|u| > t\}) = \mathscr{L}^1(\{u^\sharp > t\}) = \mathfrak{m}_h(\{u_h^\star > t\})$$

for all $t > 0$. The same identities hold true with the symbols $\geq$, $<$, $\leq$ instead of $>$.

(b) *If $u \in L^p(\Omega, \mathfrak{m})$ for some $1 \leq p \leq \infty$, then $u^\sharp \in L^p([0, v], \mathscr{L}^1)$ and $u_h^\star \in L^p([0, R_{h,v}], \mathfrak{m}_h)$. The converse implications also hold. In that case, moreover,*

$$\|u\|_{L^p(\Omega,\mathfrak{m})} = \|u^\sharp\|_{L^p([0,v],\mathscr{L}^1)} = \|u_h^\star\|_{L^p([0,R_{h,v}],\mathfrak{m}_h)}.$$

(c) *If $u, v \in L^p(\Omega, \mathfrak{m})$ for some $1 \leq p \leq \infty$, then*

$$\|u_h^\star - v^\star\|_{L^p([0,R_{h,v}],\mathfrak{m}_h)} = \|u^\sharp - v^\sharp\|_{L^p([0,v],\mathscr{L}^1)} \leq \|u - v\|_{L^p(\Omega,\mathfrak{m})}.$$

Moreover, it is easy to show that even in this setting the classical Hardy-Littlewood inequality holds (see [Kes06, Theorem 1.2.2])

Proposition 1.58 (Hardy-Littlewood inequality)**.** *Let X and $\Omega \subset \mathsf{X}$ be as before. Let $u \in L^p(\Omega, \mathfrak{m})$ and $v \in L^q(\Omega, \mathfrak{m})$, with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_\Omega uv \, \mathrm{d}\mathfrak{m} \leq \int_0^{\mathfrak{m}(\Omega)} u^\sharp v^\sharp \mathrm{d}\mathscr{L}^1 = \int_0^{H^{-1}(\mathfrak{m}(\Omega))} u^\star v^\star \mathrm{d}\mathfrak{m}_h. \tag{1.68}$$

Finally, we give a (necessary and) sufficient condition for a function to coincide with its equimeasurable decreasing rearrangement.

Lemma 1.59. *Let $(\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ be a weighted interval and $\varphi : \mathfrak{I} \to [0, +\infty)$ be a non-increasing and non-negative function. Then $\varphi_h^\star(x) = \varphi(x)$ for all $x \in \mathfrak{I} \setminus L$, where L is a countable set.*

Proof. The claim is equivalent to showing that $\varphi^\sharp = \varphi \circ H^{-1}$ except on a countable set, that is:

$$\inf\{t \mid \mathfrak{m}_h(\{\varphi > t\}) < s\} = \varphi \circ H^{-1}(s), \qquad s \in [0,1] \tag{1.69}$$

out of a countable set. Let $L \subset \mathfrak{I}$ be the set of points where φ is not left continuous (which is countable since φ is nonincreasing), and fix any $s \in [0,1] \setminus H(L)$.
If $\mathfrak{m}_h(\{\varphi > t\}) < s$ for some t, then $\mathfrak{m}_h(\{\varphi > t\}) < \mathfrak{m}_h([0, H^{-1}(s)])$ and thus

$$\{\varphi > t\} \subsetneq [0, H^{-1}(s)].$$

We infer that $\varphi(H^{-1}(s)) \leq t$, and thus

$$\varphi \circ H^{-1}(s) \leq \inf\{t \mid \mathfrak{m}_h(\{\varphi > t\}) < s\}, \quad \forall s \in [0,1] \setminus H(L). \tag{1.70}$$

Assume by contradiction that the inequality in (1.70) is strict for some $s_0 \in (0,1] \setminus H(L)$. Then there exists $\varepsilon > 0$ such that

$$\varphi \circ H^{-1}(s_0) + \varepsilon < \inf\{t \mid \mathfrak{m}_h(\{\varphi > t\}) < s_0\}. \tag{1.71}$$

Since by assumption φ is left-continuous at $H^{-1}(s_0)$, we can find $\sigma < s_0$ such that

$$\varphi(H^{-1}(\sigma)) < \varphi(H^{-1}(s_0)) + \varepsilon. \tag{1.72}$$

Since $\{\varphi > \varphi(H^{-1}(\sigma))\} \subset [0, H^{-1}(\sigma)]$, we infer that

$$\mathfrak{m}_h(\{\varphi > \varphi(H^{-1}(\sigma))\}) \leq \sigma < s_0. \tag{1.73}$$

The combination of (1.71), (1.72) and (1.73) yields the contradiction

$$\varphi(H^{-1}(\sigma)) < \inf\{t \mid \mathfrak{m}_h(\{\varphi > t\}) < s_0\} \leq \varphi(H^{-1}(\sigma)). \tag{1.74}$$

This concludes the proof. □

1.3 RCD(*K*,*N*) spaces

In the classical framework of geometric analysis, Riemannian manifolds with a lower bound on the Ricci curvature constitute the natural setting for a plethora of results: if $\mathcal{M}$ is a manifold with $\mathrm{Ric} \geq K$, on $\mathcal{M}$ the Bishop-Gromov Theorem holds (volume comparison between balls in $\mathcal{M}$ and balls in a model space, see [Pet16, Lemma 7.1.4]), as well as a Lévy-Gromov inequality (perimeter-volume ration comparison), a Cheeger-Gromoll Splitting Theorem (if $K \geq 0$ and $\mathcal{M}$ contains a line, then $\mathcal{M}$ can be split in a product $\mathcal{N} \times \mathbb{R}$ as a Riemannian manifold), and many other useful results.

Moreover, as already mentioned, the family of manifolds such that $\mathrm{Ric} \geq K$ is precompact in the pmGH topology for any $K \in \mathbb{R}$; however, even if $\{\mathcal{M}_n, g_n, x_n\}$ is a converging sequence of pointed Riemannian manifold, it has been shown that singularities can emerge in the limit (see for example [CCT02]).

In this section, we define a class of metric measure spaces that satisfy a condition that encodes "Ricci curvature bounded from below, dimension bounded from above"; this class will include manifolds with lower bounds on the Ricci curvature, will be closed under pmGH limits, and allows to generalize many of the Riemannian results.

First, we give some preliminary definitions: we define a notion of *dimension-dependent entropy* for measures in $\mathscr{P}(\mathsf{X})$ and a family of *distortion coefficients*. We will say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the above mentioned condition if and only if the entropy is "(K,N)-geodesically convex" in the Wasserstein space $\mathscr{P}(\mathsf{X})$, where we adopt a distorted version of convexity which makes use of the distortion coefficients.

Definition 1.60 (Rényi entropy). Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. The *Rényi entropy* functional $\mathcal{E}_N : \mathscr{P}(\mathsf{X}) \to [0,\infty]$ is defined as

$$\mathcal{E}_N(\mu) \doteq \int_{\mathsf{X}} \varrho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}, \quad \text{where } \mu = \varrho\mathfrak{m} + \mu^s \text{ and } \mu^s \perp \mathfrak{m}. \tag{1.75}$$ $\Diamond$

Definition 1.61 (Distortion coefficients). For any $\vartheta > 0$ and $t \in [0,1]$, the *distortion coefficients* are defined as

$$\tau^{(t)}_{K,N}(\vartheta) \doteq t^{\frac{1}{N}} \sigma^{(t)}_{K,N}(\vartheta)^{\frac{N-1}{N}}, \tag{1.76}$$

where

$$\sigma^{(t)}_{K,N}(\vartheta) \doteq \begin{cases} \infty & \text{if } K\vartheta^2 \geq N\pi^2 \\ \frac{\sin(t\vartheta\sqrt{K/N})}{\sin(\vartheta\sqrt{K/N})} & \text{if } 0 < K\vartheta^2 < N\pi^2 \\ t & \text{if } K\vartheta^2 < 0 \text{ and } N = 0, \text{ or if } K\vartheta^2 = 0 \\ \frac{\sinh(t\vartheta\sqrt{K/N})}{\sinh(\vartheta\sqrt{K/N})} & \text{if } K\vartheta^2 \leq 0 \text{ and } N > 0 \end{cases}. \tag{1.77}$$ $\Diamond$

We are now in force to define the curvature-dimension condition $\mathrm{CD}(K,N)$:

Definition 1.62 ($\mathrm{CD}(K,N)$ and $\mathrm{CD}^*(K,N)$). Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. We say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ verifies the $\mathrm{CD}(K,N)$ condition for some $K \in \mathbb{R}$, $N \in (1,\infty)$ if: for any pair of probability measures $\mu_0,\mu_1 \in \mathscr{P}(\mathsf{X})$ with bounded support and with $\mu_0,\mu_1 \ll \mathfrak{m}$, there exists $\nu \in \mathrm{OptGeo}(\mu_0,\mu_1)$ and an optimal plan $\pi \in \mathscr{P}(\mathsf{X}\times\mathsf{X})$ such that $\mu_t \doteq (\mathrm{e}_t)_\sharp\nu \ll \mathfrak{m}$ and

$$\mathcal{E}_{N'}(\mu_t) \geq \int \left[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\varrho_0^{-\frac{1}{N'}} + \tau^{(t)}_{K,N'}(\mathsf{d}(x,y))\varrho_1^{-\frac{1}{N'}}\right]\mathrm{d}\pi(x,y) \tag{1.78}$$

for any $N' \geq N$, $t \in [0,1]$.

We say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ verifies the $\mathrm{CD}^*(K,N)$ condition if the above inequality holds with the coefficients $\sigma_{K,N'}$ in place of $\tau_{K,N'}$:

$$\mathcal{E}_{N'}(\mu_t) \geq \int \left[\sigma^{(1-t)}_{K,N'}(\mathsf{d}(x,y))\varrho_0^{-\frac{1}{N'}} + \sigma^{(t)}_{K,N'}(\mathsf{d}(x,y))\varrho_1^{-\frac{1}{N'}}\right]\mathrm{d}\pi(x,y). \tag{1.79}$$ $\Diamond$

Definition 1.63 ($\mathrm{RCD}(K,N)$ spaces). We say that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ satisfies the $\mathrm{RCD}(K,N)$ *condition* (respectively, the $\mathrm{RCD}^*(K,N)$ condition) if it satisfies the $\mathrm{CD}(K,N)$ condition (respectively, the $\mathrm{RCD}^*(K,N)$ condition) and it is infinitesimally Hilbertian (see Definition 1.17). $\Diamond$

***Remark* 1.64** (RCD $\Leftrightarrow$ RCD*)**.** The condition $\mathrm{CD}(K,N)$ is stronger than $\mathrm{CD}^*(K,N)$. In [CM21, Corollary 13.7], however, the authors proved that the conditions $\mathrm{RCD}(K,N)$ and $\mathrm{RCD}^*(K,N)$ are actually equivalent (*i.e.*, equivalence holds when the infinitesimally hilbertianity is added) if the measure is finite, which will be the case in Chapter 2. Thus, from now on, we will always adopt the terminology $\mathrm{RCD}(K,N)$ to indicate them. $\diamondsuit$

***Remark* 1.65** (Scaling properties and standard normalizations)**.** From the very definitions, it is not difficult to check that for any λ and $c>0$ the following implication holds

$$(\mathsf{X},\mathsf{d},\mathfrak{m}) \text{ is an } \mathrm{RCD}(K,N) \text{ space} \quad\Longrightarrow\quad (\mathsf{X},\lambda\mathsf{d},c\mathfrak{m}) \text{ is an } \mathrm{RCD}\Big(\lambda^{-2}K,N\Big) \text{ space.} \tag{1.80}$$

If $K>0$, the Bonnet-Myers Theorem (proved for $\mathrm{CD}(K,N)$ spaces in [Stu06b]) implies that (X,d) is compact with $\mathfrak{m}(\mathsf{X})\in(0,\infty)$. Thanks to the scaling property (1.80), up to constant scalings, we will be often allowed to assume $\mathfrak{m}(\mathsf{X})=1$ and $K=N-1$ (see Section 2.3.1). $\diamondsuit$

For the following result, see [Stu06b, Theorem 2.3, Corollary 2.4] and [Vil09, Theorem 18.8]. For $K>0$ the quantity $\mathfrak{s}_{K,N}$ coincides up to a constant to the density $h_{K,N}$ which will appear in Section 1.3.1, and $\mathfrak{v}_{K,N}$ is thus up to a constant the volume of a ball around the tip in the model space $(J_{K,N},\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{K,N})$.

Proposition 1.66 (Bishop-Gromov, doubling)**.** *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying the $\mathrm{CD}(K,N)$ condition for $K\in\mathbb{R}$ and $N\in(1,\infty)$. Denote by $\mathfrak{s}_{K,N}(t)$ the quantity*

$$\mathfrak{s}_{K,N}(t) \doteq \begin{cases} \left(\sin\left(\sqrt{\frac{K}{N-1}}t\right)\right)^{N-1} & \text{if } K>0 \\ t^{N-1} & \text{if } K=0 \\ \left(\sinh\left(\sqrt{\frac{|K|}{N-1}}t\right)\right)^{N-1} & \text{if } K<0 \end{cases} \tag{1.81}$$

and by $\mathfrak{v}_{K,N}(r)\doteq\int_0^r \mathfrak{s}_{K,N}(t)\,\mathrm{d}t$. Then the map

$$r\mapsto \frac{\mathfrak{m}(B_r(x))}{\mathfrak{v}_{K,N}(r)} \tag{1.82}$$

is non-decreasing for any $x\in\mathsf{X}$. In particular, if $K\geq 0$, then $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is doubling.

Proposition 1.67. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a $\mathrm{CD}(0,N)$ space for some $N\in(0,\infty)$. Then $(\mathsf{X},\mathsf{d},\mathfrak{m})$ supports the (weak) $(1,1)$-Poincaré inequality*

$$⨍_B\left|u-⨍_B u\right|\mathrm{d}\mathfrak{m}\leq 2^{N+1}\,\mathrm{diam}(B)⨍_{2B} g\,\mathrm{d}\mathfrak{m} \tag{1.83}$$

for any u integrable on X, any ball $B\subset\mathsf{X}$ and any upper gradient g of u. As a further consequence, it also satisfies a (weak) (p,p)-inequality for any $p\geq 1$.

Proof. The $(1,1)$-Poincaré inequality is proved in [Raj12, Theorem 2]. Then by the Hölder inequality a $(1,p)$-Poincaré inequality is satisfied for any $p\geq1$; then by [HK00, Theorem 5.1], together with the Bishop-Gromov inequality (Proposition 1.66), a (p,p)-inequality holds. $\square$

***Remark* 1.68** (Riemannian curvatures). In the next Remark 1.69, we will see how the above definitions interact with smooth ambient spaces. Before doing that, we briefly recall some Riemannian notions; we refer to [Pet16, Chapter 3] or [Car92, Chapter 2] for the underlying definitions.

If $(\mathcal{M}, g)$ is a n-dimensional Riemannian manifold, we denote by $\mathcal{T}(\mathcal{M})$ the set of smooth vector fields on $\mathcal{M}$. By the Fundamental Theorem of Riemannian Geometry (or Levi-Civita Theorem, [Pet16, Theorem 2.2.2, Chapter 2]) there exists a unique *affine connection* $\nabla : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ which is both *symmetric* and *compatible with the Riemannian metric*. This is called the *Levi-Civita connection* on $(\mathcal{M}, g)$.

The Levi-Civita connection ∇ satisfies the following *Koszul identity*: whenever $U, V, W \in \mathcal{T}(\mathcal{M})$, one has

$$\begin{aligned} 2g(\nabla_U V, W) = U(g(V,W)) + V(g(W,U)) - W(g(U,V)) + \\ - g(W,[V,U]) - g(U,[V,W]) - g(V,[U,W]). \end{aligned} \tag{1.84}$$

We then define the Riemannian *curvature tensor* as the $(1,3)$-tensor defined by

$$R_g(U,V)W \doteq \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W, \tag{1.85}$$

and the Ricci tensor $\mathrm{Ric}_g(V,W)$ as the trace of $U \mapsto R_g(U,V)W$. We use the same symbol Ric_g for the quadratic form associated to Ric_g, *i.e.* the quadratic form $V \mapsto \mathrm{Ric}_g(V,V)$.

It can be shown that the Ricci curvature Ric_g at the point $p \in \mathcal{M}$ can be expressed as

$$\mathrm{Ric}_g(v,v) = \sum_{i=2}^n \sec(v, e_i) = \sum_{i=2}^n g_p(R_p(e_i, v)v, e_i) \tag{1.86}$$

whenever $v \in T_p\mathcal{M}$ is a unit tangent vector and $\{v, e_2, \dots, e_n\}$ is a orthonormal basis for the tangent space $T_p\mathcal{M}$ and sec is the sectional curvature. ◊

***Remark* 1.69** (When is a weighted manifold/interval an RCD space?). Let us assume that

$$(\mathsf{X}, \mathsf{d}, \mathfrak{m}) = (\mathcal{M}, \mathsf{d}_g, h\,\mathrm{vol}_g) \tag{1.87}$$

is the metric measure space induced by a weighted n-dimensional Riemannian manifold $(\mathcal{M}, g)$ – that is d_g and vol_g are the distance and the volume form induced by the metric g – with a $\mathbf{C}^2$ density function $h : \mathcal{M} \to (0, \infty)$. Then, by [Stu06b, Theorem 1.7], $(\mathcal{M}, \mathsf{d}_g, h\,\mathrm{vol}_g)$ is a $\mathrm{RCD}(K,N)$ space with $N \geq n$ if and only if $\mathrm{Ric}_{g,h,N} - Kg$ is positive semidefinite, where

$$\mathrm{Ric}_{g,h,N} \doteq \mathrm{Ric}_g - (N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}}; \tag{1.88}$$

here Ric_g is the classical (smooth) Ricci curvature and ∇^2 is the Hessian. When h is constant and $N = n$, this reduces to the classical $\mathrm{Ric}_g \geq Kg$.

When $(\mathsf{X}, \mathsf{d}, \mathfrak{m}) = (\mathfrak{I}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_h)$ as in Section 1.2, in particular, the condition becomes

$$\left(h^{\frac{1}{N-1}}\right)'' + \frac{K}{N-1} h^{\frac{1}{N-1}} \geq 0; \tag{1.89}$$

this can be used to show that the model spaces in Section 1.3.1 are actually $\mathrm{RCD}(K,N)$. ◊

Next, we adapt Proposition 1.38 to the case we will need in Chapter 2.

Proposition 1.70 (Coarea formula, RCD version)**.** *Let* $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathrm{RCD}(K, N)$ *space for some* $K \geq 0$, $N \in (1, \infty)$. *Let* $\Omega \subset \mathsf{X}$ *be an open domain and* $u : \Omega \to \mathbb{R}$ *be a non-negative function in* $W_0^{1,2}(\Omega)$. *Then for any* $t > 0$

$$\int_{\{u>t\}} |\nabla u|_{\mathrm{w}} \,\mathrm{d}\mathfrak{m} = \int_t^\infty \mathrm{Per}(\{u > r\}) \,\mathrm{d}r. \tag{1.90}$$

More generally, for any Borel function $f : \Omega \to \mathbb{R}$ *and for any* $t > 0$, *it holds that*

$$\int_{\{u>t\}} f |\nabla u|_{\mathrm{w}} \,\mathrm{d}\mathfrak{m} = \int_t^\infty \left(\int f \,\mathrm{d}\,\mathrm{Per}(\{u > r\}) \right) \mathrm{d}r.$$

***Remark* 1.71.** Notice that Proposition 1.70 follows from the BV version (Proposition 1.38), combined with [GH16, Remark 3.5]; indeed, the $\mathrm{CD}(K, N)$ condition with $K \geq 0$, $N \in (1, \infty)$ implies properness of the space: $\mathrm{CD}(K, N)$ spaces are doubling by Proposition 1.66, and doubling spaces are proper by Lemma 1.26. ♢

1.3.1 Model spaces with fixed curvature and dimension

Here we introduce a particular class of weighted intervals, which will be at the core of Chapter 2: for any $K > 0$ and $N \in (1, \infty)$, we define an interval $J_{K,N}$ and weight function $h_{K,N}$ such that the corresponding metric measure space models a space with curvature below by K and dimension bounded above by N.

Definition 1.72 (Model space)**.** Let $K > 0$ and $N \in (1, \infty)$. Define:

(a) the interval $J_{K,N} \doteq \left[0, \pi\sqrt{\frac{N-1}{K}}\right]$;

(b) the probability density function $h_{K,N}$ on $J_{K,N}$:

$$h_{K,N}(t) \doteq \frac{1}{c_{K,N}} \sin^{N-1}\left(t\sqrt{\frac{K}{N-1}}\right), \tag{1.91}$$

where $c_{K,N}$ is the normalizing constant

$$c_{K,N} \doteq \int_{J_{K,N}} \sin^{N-1}\left(t\sqrt{\frac{K}{N-1}}\right) \mathrm{d}\mathscr{L}^1(t); \tag{1.92}$$

(c) the measure $\mathfrak{m}_{K,N} \doteq h_{K,N}\mathscr{L}^1 \llcorner J_{K,N}$.

With these notations, we call *model space of curvature* K *and dimension* N the metric measure space $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$, where d_{eu} is the standard Euclidean distance. For the sake of convenience, we will also denote by $H_{K,N}$ the cumulative distribution function of $\mathfrak{m}_{K,N}$, *i.e.*:

$$H_{K,N}(x) \doteq \mathfrak{m}_{K,N}([0, x]) = \int_0^x h_{K,N}(t) \,\mathrm{d}t.$$

Finally, we also denote by $D_{K,N}$ the diameter of $J_{K,N}$, *i.e.* $D_{K,N} \doteq \pi\sqrt{\frac{N-1}{K}}$, and by $\mathcal{I}_{K,N}$ the isoperimetric profile of $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$. ♢

***Remark* 1.73.** From Equation (1.89) and the definition (1.91) it is immediate that the space $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ is a $\mathrm{RCD}(K,N)$ space. $\Diamond$

The following lemma is an elementary consequence of the definitions of $h_{K,N}$ and $H_{K,N}$:

Lemma 1.74. *Let $K > 0$ and $N \in (1,\infty)$ be fixed. Then:*

1. *If $\gamma_1(K,N) \doteq \frac{1}{c_{K,N}}\left(\frac{K}{N-1}\right)^{\frac{N-1}{2}}$, then*

$$\lim_{t\to 0^+}\frac{h_{K,N}(t)}{t^{N-1}} = \gamma_1(K,N), \quad \text{and} \quad h_{K,N}(t) \le \gamma_1(K,N)t^{N-1} \quad \forall t \in J_{K,N}.$$

Moreover, for any $r_1 \in \left(0, \pi\sqrt{\frac{N-1}{K}}\right)$ there exists $C = C(r_1,K,N) > 0$ such that

$$h_{K,N}(t) \ge C\,t^{N-1}, \quad \forall t \in (0,r_1).$$

2. *$H_{K,N}$ is invertible on $J_{K,N}$; moreover, if $\gamma_2(K,N) \doteq \frac{\gamma_1(K,N)}{N}$:*

$$\lim_{t\to 0^+}\frac{H_{K,N}(t)}{t^N} = \gamma_2(K,N), \qquad \text{and} \quad H_{K,N}(t) \le \gamma_2(K,N)t^N \quad \forall t \in J_{K,N};$$
$$\lim_{t\to 0^+}\frac{H^{-1}_{K,N}(t)}{t^{\frac{1}{N}}} = \frac{1}{\gamma_2(K,N)^{\frac{1}{N}}}, \qquad \text{and} \quad H^{-1}_{K,N}(t) \ge \frac{1}{\gamma_2(K,N)^{\frac{1}{N}}}t^{\frac{1}{N}} \quad \forall t \in (0,1).$$

Let us particularize the definitions of Section 1.1.6 to the case of a function defined on a $\mathrm{RCD}(K,N)$ space, with the rearrangement being defined on the corresponding (K,N)-model space. Notice that the condition $\mathrm{CD}(K,N)$ on curvature and dimension, together with the assumption that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is essentially non-branching, would be enough to ensure a Pólya-Szegő inequality, as shown in [MS19].

Definition 1.75 ((K,N)-Schwarz symmetrization)**.** Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying the $\mathrm{RCD}(K,N)$ condition for some $K > 0$ and $N \in (1,\infty)$. Let $\Omega \subset \mathsf{X}$ be a Borel subset with measure $\mathfrak{m}(\Omega) = v \in [0,1]$ and $u : \Omega \to \mathbb{R}$ be a Borel measurable function. Let $R_{K,N,v} \doteq H^{-1}_{K,N}(v) \in [0, D_{K,N}]$, so that $\mathfrak{m}_{K,N}([0,R_{K,N,v}]) = v$. Then we define:

(i) The (K,N)-Schwarz symmetrization of Ω as the interval $\Omega^\star_{K,N} \doteq [0, R_{K,N,v}) \subset J_{K,N}$.

(ii) The (K,N)-Schwarz symmetrization $u^\star_{K,N} : \Omega^\star_{K,N} \to [0,\infty]$ as the equimeasurable decreasing rearrangement of u on $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$. In particular: $u^\star_{K,N} \doteq u^\sharp \circ H_{K,N}$, and explicitly:

$$u^\star_{K,N}(x) \doteq u^\sharp(\mathfrak{m}_{K,N}([0,x])). \tag{1.93}$$

When the context is clear (and for the whole Chapter 2), we will drop everywhere the subscript "K,N". $\Diamond$

***Remark* 1.76.** With the Definitions 1.72 and 1.75, all the results of Sections 1.1.6, 1.2.1 and 1.2.2 are valid in $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$: in particular the explicit formula for the isoperimetric profile

(Proposition 1.50) in the form $\mathcal{I}_{K,N}(v) = h_{K,N}(H^{-1}_{K,N}(v))$ for all $v \in [0,1]$, the explicit solution to the Poisson problem (Equations (1.58) and (1.66)), the sufficient condition for coincidence with the rearrangement (Lemma 1.59). Indeed, $h_{K,N}$ is log-concave: the map $z \mapsto (\log h_{K,N})'(z) = \frac{h'_{K,N}(z)}{h_{K,N}(z)}$ coincides with

$$z \mapsto \sqrt{K(N-1)} \cot\left(\sqrt{\tfrac{K}{N-1}} z\right),$$

thus it is always decreasing. Moreover, $h_{K,N}$ is always positive in $\mathring{J}_{K,N}$, it is symmetric with respect to the center of the interval, it belongs to $\mathbf{C}^1(J_{K,N})$ (in particular, $h^{-1} \in L^\infty_{\mathrm{loc}}(\mathring{J}_{K,N})$), and it takes the value zero at the extrema. $\Diamond$

***Remark* 1.77** (Interpretations)**.** When $N \in \mathbb{N}$, $\mathfrak{m}_{K,N}([0,x))$ represents the measure of the geodesic ball of radius x on the N-dimensional sphere of Ricci curvature K, endowed with the canonical metric. Notice however that Definition 1.72 makes sense when N is not a natural number as well.

This interpretation also gives an intuition behind the choice of boundary conditions in the Poisson problem (Definition 1.52). When N is an integer, we think of $(J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ as the sphere $\mathbb{S} = \mathbb{S}^N_K$ of dimension N and Ricci curvature K. Consider a geodesic ball $B_{r_1}(p) \subset \mathbb{S}$; we look for radial solutions $\hat{w}(x) = w(\mathsf{d}(x,p))$ of the Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{S}} \hat{w}(x) = f(\mathsf{d}(x,p)) & \text{on } B_{r_1}(p) \\ \hat{w} = 0 & \text{on } \partial B_{r_1}(p) \end{cases},$$

where $\Delta_{\mathbb{S}}$ is the classical Laplace-Beltrami operator on $\mathbb{S}$. Then the condition $w(r_1) = 0$ comes from the Dirichlet condition on $\partial B_{r_1}(p)$. $\Diamond$

1.3.2 The Lévy-Gromov inequality

Consider a N-dimensional Riemannian manifold $(\mathcal{M}, g)$ with Ricci curvature bounded from below by $K > 0$, and an open subset $\Omega \subset \mathcal{M}$; let, moreover, $\left(\mathbb{S}^N_K, \mathrm{vol}_{\mathbb{S}}\right)$ be the N-dimensional sphere of constant Ricci curvature K, and $B \subset \mathbb{S}^N_K$ be a ball satisfying $\frac{\mathrm{vol}_g(E)}{\mathrm{vol}_g(\mathcal{M})} = \frac{\mathrm{vol}_{\mathbb{S}}(B)}{\mathrm{vol}_{\mathbb{S}}(\mathbb{S}^N_K)}$. In this setting, the Lévy-Gromov inequality

$$\frac{\mathrm{Per}(E;\mathcal{M})}{\mathrm{vol}_g(\mathcal{M})} \geq \frac{\mathrm{Per}(B;\mathbb{S}^N_K)}{\mathrm{vol}_{\mathbb{S}}(\mathbb{S}^N_K)} \tag{1.94}$$

holds (see [Gro07, Appendix C]). In other words, if $\mathcal{I}_{\mathbb{S}^N_K}$ is the isoperimetric profile of the sphere (as a metric measure space with the usual metric and normalized volume measure), then

$$\frac{\mathrm{Per}(E;\mathcal{M})}{\mathrm{vol}_g(\mathcal{M})} \geq \mathcal{I}_{\mathbb{S}^N_K}\left(\frac{\mathrm{vol}_g(E)}{\mathrm{vol}_g(\mathcal{M})}\right). \tag{1.95}$$

An $\mathrm{RCD}(K,N)$ version of the Lévy-Gromov isoperimetric inequality was firstly obtained by Cavalletti and Mondino in [CM17] for the outer Minkowski content (see Remark 1.37) and then in [CM18] for the perimeter; the argument adopted there is out of the scope of this thesis, and is based on a technique of localization via L^1-optimal transportation.

Below, we state the result in the form that involves the perimeter:

Proposition 1.78 (Lévy-Gromov inequality). *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ metric measure space with $K>0$ and $N\in(1,\infty)$. Then for any $E\in\mathscr{B}(\mathsf{X})$*

$$\mathrm{Per}(E)\geq\mathcal{I}_{K,N}(\mathfrak{m}(E)). \tag{1.96}$$

In particular, the isoperimetric profile of $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is bounded from below by $\mathcal{I}_{K,N}$.

1.3.3 Compactness of RCD(K,N) spaces

As already mentioned, the class of $\mathrm{RCD}(K,N)$ metric measure spaces with fixed $K>0$ is compact with respect to the mGH convergence. This will be particularly useful in the compactness argument for the stability of the Talenti Theorem (Section 2.3.1): we state here a self-contained version.

Proposition 1.79 (Compactness and stability of $\mathrm{RCD}(K,N)$ sequences). *Fix $K>0$ and $N\in(1,\infty)$. Every sequence of $\mathrm{RCD}(K,N)$ spaces $\{(\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i)\}_{i\in\mathbb{N}}$ admits a subsequence which converges in the* mGH *sense to a metric measure space $(\mathsf{X},\mathsf{d},\mathfrak{m})$, and $(\mathsf{X},\mathsf{d},\mathfrak{m})$ itself satisfies the* $\mathrm{RCD}(K,N)$ *condition. The statement still holds for $K\leq 0$, up to replacing* mGH *convergence with the* pmGH *one.*

***Remark* 1.80.** The result follows from these observations:

- A classical precompactness result of Gromov [Gro07, Proposition 5.2] – already used in Remark 1.32 – states that a sequence $\{(\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i)\}_{i\in\mathbb{N}}$ of metric measure spaces is precompact if and only if *the maximum number $N(\varepsilon,R)$ of disjoint balls of radius ε that fit in a ball $B_R(x)\subset\mathsf{X}_i$ of radius R is uniformly bounded for all the spaces in the sequence.* We have already seen in the proof of Lemma 1.26 that this holds for uniformly doubling metric measure spaces: and this happens for $\mathrm{RCD}(K,N)$ spaces by Proposition 1.66.
- The fact that the class of $\mathrm{RCD}(K,N)$ spaces is stable under mGH convergence follows from stability of the $\mathrm{CD}(K,N)$ class (see [LV09, Section 5.3]), from the stability of the $\mathrm{RCD}(K,\infty)$ class under pmG convergence (see [GMS15, Theorem 7.2]) and from the already mentioned equivalence of pmG and pmGH convergence for $\mathrm{RCD}(K,N)$ spaces (see the previous Section 1.1.4 on the Convergence of Spaces, and [GMS15, Section 3.5]). ◊

1.4 Carnot-Carathéodory spaces and the Heisenberg group

Consider an open connected subset Ω of $\mathbb{R}^n$, with $n\geq 2$, and let $X_1,\dots,X_m$ – with $1\leq m\leq n$ – be a family of vector fields in Ω with locally Lipschitz coefficient; we'll often use the symbol $\mathbb{X}$ to indicate the family $\{X_1,\dots,X_m\}$ as a whole. Such a family induces a metric space structure on Ω, on which – roughly speaking – the only available paths are those whose tangent at any point belongs to the space generated by $\mathbb{X}$.

Let us introduce some basic definitions:

Definition 1.81 (Basic sub-Riemannian notions). Let $\mathbb{X}$ be a family of vector fields as before; denote by $c_{ij}\in\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ be the coefficients such that

$$X_j(x)=\sum_{i=1}^{n}c_{ij}(x)\frac{\partial}{\partial x_i}. \tag{1.97}$$

$\mathbb{X}$-*admissible curves:* Let $I \subset \mathbb{R}$ be an interval. We will say that a Lipschitz curve $\gamma : I \to \Omega$ is $\mathbb{X}$-*admissible* if there exist bounded measurable functions $h_1, \dots, h_m \in L^\infty(I, \mathbb{R})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = \sum_{j=1}^{m} h_j(t) X_j(\gamma_t); \tag{1.98}$$

if, in addition, $|(h_1, \dots, h_m)| \le 1$ almost everywhere, we say that γ is $\mathbb{X}$-*subunit.*

Carnot-Carathéodory distance: For any pair of points $x, y \in \Omega$, we define

$$\mathsf{d}_{\mathrm{cc}}(x, y) \doteq \inf\left\{ T > 0 \;\middle|\; \begin{array}{c}\text{there exists a } \mathbb{X}\text{-subunit curve } \gamma : [0,T] \to \Omega \\ \text{such that } \gamma(0) = x,\ \gamma(1) = y\end{array} \right\}. \tag{1.99}$$

It can be shown that d_{cc} is an *extended* distance (in the sense that it can take the value $+\infty$). When it is a true distance (*i.e.*, when $\mathsf{d}_{\mathrm{cc}} < +\infty$ for any pair of points), we call $(\Omega, \mathsf{d}_{\mathrm{cc}})$ a *Carnot-Carathéodory space*, or CC-space in brief.

Horizontal gradient: If $u \in L^1_{\mathrm{loc}}(\Omega, \mathscr{L}^n)$, we denote by $\mathbb{X}u$ the vector $\mathbb{X}u \doteq (X_1 u, \dots X_m u)$, where we see each X_j as a differential operator; in other words, $X_j u$ is the linear combination of distributional derivatives

$$X_j u \doteq \sum_{i=1}^{n} c_{ij} \frac{\partial u}{\partial x_j}. \tag{1.100}$$

Sobolev space $W^{1,p}_{\mathbb{X}}$: We define the Sobolev space $W^{1,p}_{\mathbb{X}}(\Omega)$ as

$$W^{1,p}_{\mathbb{X}}(\Omega) \doteq \{u \in L^p(\Omega, \mathscr{L}^n) \mid X_j u \in L^p(\Omega, \mathscr{L}^n) \text{ for any } j \in \{1, \dots, n\}\}, \tag{1.101}$$

which is, when endowed with the norm

$$\|u\|_{W^{1,p}_{\mathbb{X}}(\Omega)} \doteq \|u\|_{L^p(\Omega,\mathscr{L}^n)} + \sum_{j=1}^{m} \|X_j u\|_{L^p(\Omega,\mathscr{L}^n)}, \tag{1.102}$$

a Banach space. $\lozenge$

The connection between the Sobolev space $W^{1,p}_{\mathbb{X}}$ defined in Equation (1.101) and the Sobolev spaces of $H^{1,p}$-type introduced in Section 1.1.1 is given by the following Meyers-Serrin-type result, which can be found in [HK00, Theorem 11.9]:

Proposition 1.82 ($W^{1,p}_{\mathbb{X}} = H^{1,p}_{\mathbb{X}}$). *Let Ω and $\mathbb{X}$ be as before, and assume that $(\Omega, \mathsf{d}_{\mathrm{cc}})$ is a CC-space. If $u \in W^{1,p}_{\mathbb{X}}(\Omega)$, then there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathbf{C}^\infty(\Omega)$ such that $u_k \to u$ and $X_j u_k \to X_j u$ (for any j) in in $L^p(\Omega)$.*

For general families of vector fields $\mathbb{X} = \{X_1, \dots, X_m\}$, the extended distance d_{cc} may not be finite – a trivial example is the family consisting of $X_1 = \frac{\partial}{\partial x_1}$ alone, as a vector field in the plane $\mathbb{R}^2$. Below we state a classical result, discovered independently by W.L. Chow and P.K. Rashevskii, which ensures that this does not happen. Let us first introduce the *Hörmander condition.*

Definition 1.83 (Hörmander condition). Let now $\Omega \subset \mathbb{R}^n$ be an open connected set. Let $\mathbb{X} = \{X_1, \dots, X_m\}$ be a family of vector fields in $\mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ (actually, $\mathbf{C}^\infty$ on a neighborhood of Ω would be enough).

We formally define the *Lie bracket* $[X_j, X_k]$ as the vector field $X_jX_k - X_kX_j$; precisely, if $X_j(x) = \sum_{i=1}^n c_{ij}(x)\frac{\partial}{\partial x_i}$ for any $j \in \{1, \dots, m\}$ and $x \in \Omega$, then

$$[X_j, X_k](x) \doteq \sum_{i=1}^n \left(\sum_{\ell=1}^n (c_{ij}(x)\frac{\partial c_{\ell k}(x)}{\partial x_i} - c_{ik}(x)\frac{\partial c_{\ell j}(x)}{\partial x_i}) \right) \frac{\partial}{\partial x_i}. \tag{1.103}$$

We denote by $\mathcal{L}\{X_1, \dots, X_m\}(x)$ the linear subspace of $\mathbb{R}^n$ generated by the X_i's and their iterated Lie brackets at each $x \in \Omega$.

We say that the family $\mathbb{X}$ satisfies the *Hörmander condition* in Ω if the rank of the subspace $\mathcal{L}\{X_1, \dots, X_m\}(x)$ is n at each point of Ω (*i.e.*, if $\mathcal{L}\{X_1, \dots, X_m\}(x) = \mathbb{R}^n$ at each point). ♢

Theorem 1.84 (Chow-Rashevskii). *Let $\mathbb{X} = \{X_1, \dots, X_m\}$ be a family of vector fields belonging to $\mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$. If the family $\mathbb{X}$ satisfies the Hörmander condition in Ω, then any pair of points $x, y \in \Omega$ can be joined by a $\mathbb{X}$-admissible curve; in particular, $\mathsf{d}_{cc}(x, y) < +\infty$.*

Moreover, if for any $x \in \Omega$ the linear subspace $\mathcal{L}\{X_1, \dots, X_m\}(x)$ is generated by iterated Lie brackets of length at most ℓ – that is, by vector fields of type

$$X_{j_1}, [X_{j_1}, X_{j_2}], \dots, [X_{j_1}, X_{j_2}, [\dots, X_{j_\ell}], \dots] \tag{1.104}$$

with $j_1, \dots, j_\ell = 1, \dots, m$ –, then for any compact $K \Subset \Omega$ there exist constants $C_1, C_2 > 0$ such that

$$C_1|x - y| \leq \mathsf{d}_{cc}(x, y) \leq C_2|x - y| \qquad \text{for all } x, y \in K. \tag{1.105}$$

A proof of (the first part of) the Chow-Rashevskii Theorem can be found in [Bel96, Theorem 2.4, and the whole Section 2.1] or in [ABB20, Section 3.2.1]; the estimate in Equation (1.105) is due to Nagel, Stein and Wainger and can be found in [NSW85, Proposition 1.1].

Any Carnot-Carathéodory space induced by a family of vector fields satisfying the Hörmander condition, has varoius geometric properties when seen from the perspective of metric measure spaces:

Proposition 1.85. *Let $\mathbb{X} = \{X_1, \dots, X_m\}$ be a family of vector fields in $\mathbf{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$ that satisfies the Hörmander condition in Ω. The Carnot-Caratheodory space $(\Omega, \mathsf{d}_{cc}, \mathscr{L}^n)$ associated to $\mathbb{X}$ is locally doubling and Poincaré.*

The fact that the space is locally doubling is again a consequence of the work of Nagel, Stein and Wainger: we refer to [NSW85, Theorem 1]. The Poincaré-type inequality is instead due to Jerison [Jer86]; we also refer to [HK00, Theorem 11.20] and references therein for a more articulate discussion on the matter.

Finally, inspired by the Euclidean definition of perimeter ([EG15, Section 5.1]), one can adapt such notion to the sub-Riemannian setting:

Definition 1.86 (sub-Riemannian perimeter). Let $\mathbb{X} = \{X_1, \dots, X_m\}$ be $\mathbf{C}^\infty$ vector fields on Ω.

For each X_j, we denote by X_j^* the adjoint operator of X_j in $L^2(\Omega)$, *i.e.* the unique operator such that:

$$\int_\Omega uX_jv \, \mathrm{d}\mathscr{L}^n = \int_\Omega vX_j^*u \, \mathrm{d}\mathscr{L}^n \qquad \text{for all } u, v \in \mathbf{C}_0^1(\Omega); \tag{1.106}$$

we define the *horizontal divergence* $\operatorname{div}_{\mathbb{X}}$ of a vector field $\varphi \in \mathbf{C}^1_c(\Omega \in \mathbb{R}^m)$ as

$$\operatorname{div}_{\mathbb{X}} \varphi \doteq -\sum_{j=1}^{m} X_j^* \varphi_j. \tag{1.107}$$

Finally, mimicking the Euclidean definition, for any measurable $E \subset \mathbb{R}^n$ we define its $\mathbb{X}$-*perimeter* in Ω as

$$\operatorname{Per}_{\mathbb{X}}(E;\Omega) \doteq \sup\left\{ \int_E \operatorname{div}_{\mathbb{X}} \varphi \;\middle|\; \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^{2n} \text{ with } |\varphi|_{\mathbb{R}^{2n}} \le 1) \right\}. \tag{1.108}$$ ♢

It was proved in [AGM15] that this notion of perimeter actually coincides with the one inherited from the metric measure space structure $(\Omega, \mathsf{d}_{\mathrm{cc}}, \mathscr{L}^n)$ through Definition 1.35.

1.4.1 The Heisenberg group

In this Section, we will define the *Heisenberg group* $\mathbb{H}^n$ as the space $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ endowed with a particular group operation, and more precisely as the Carnot-Carathéodory space induced by this structure.

Recall that we call a *Lie group* any group which is also equipped with a differentiable manifold structure, so that the map $(x, y) \mapsto x \cdot y^{-1}$ is differentiable.

Let us first fix some notations: we will represent a point p in $\mathbb{R}^{2n+1}$ as $p = [\mathbf{x}, \mathbf{y}, t]$ with $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$. Alternatively and equivalently, we can think of p as a point in $\mathbb{C}^n \times \mathbb{R}$ with $p = [\mathbf{z}, t]$, $\mathbf{z} = \mathbf{x} + i\mathbf{y}$.

Let $\alpha \in \mathbb{R} \setminus \{0\}$, and consider the following operation: for any $p, p' \in \mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ with $p = [\mathbf{z}, t]$, $p' = [\mathbf{z}', t']$,

$$\begin{aligned} p \cdot p' &\doteq [\mathbf{z} + \mathbf{z}', t + t' + \alpha \Im \langle \overline{\mathbf{z}}, \mathbf{z}' \rangle] = \\ &= (\mathbf{x} + \mathbf{x}', \mathbf{y} + \mathbf{y}', t + t' + \alpha(\langle \mathbf{x}, \mathbf{y}' \rangle - \langle \mathbf{x}', \mathbf{y} \rangle)). \end{aligned} \tag{1.109}$$

It can be proved that, for any choice of $\alpha \neq 0$, the space $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ endowed with this operation has a group structure, with group identity $o = [\mathbf{0}, 0]$ and $[\mathbf{z}, t]^{-1} = [-\mathbf{z}, -t]$. We will (temporarily) use the symbol $\mathbb{H}^n_\alpha$ to indicate the Lie group obtained by endowing such space with the above defined α-dependent group law.

Definition 1.87 (Left translation and left invariance). For any $p \in \mathbb{H}^n$, the *left-translation* map $\tau_p : \mathbb{H}^n \to \mathbb{H}^n$ is defined as

$$\tau_p(q) \doteq p \cdot q. \tag{1.110}$$

We say that a vector field $X \in \mathbf{C}^\infty(\mathbb{R}^{2n+1}, \mathbb{R}^{2n+1})$ is *left invariant* if $\mathrm{d}\tau_p X = X$ for any $p \in \mathbb{H}^n$. ♢

Notice that, if $p = (\mathbf{x}, \mathbf{y}, t)$, the differential $\mathrm{d}\tau_p$ is (at any point) the linear map whose matrix in the canonical coordinates of $\mathbb{R}^{2n+1}$ is represented as

$$L_p = \left(\begin{array}{c|c|c} \mathbf{1}_{n\times n} & \mathbf{0}_{n\times n} & \mathbf{0}_{n\times 1} \\ \hline \mathbf{0}_{n\times n} & \mathbf{1}_{n\times n} & \mathbf{0}_{n\times 1} \\ \hline -\alpha \mathbf{y}^\intercal & \alpha \mathbf{x}^\intercal & 1 \end{array} \right) \tag{1.111}$$

In particular, one can immediately see that the left-invariant vector fields obtained by translating the canonical basis at the origin are expressed as

$$X_i \doteq \frac{\partial}{\partial x_i} - \alpha y_i \frac{\partial}{\partial t}, \quad Y_i \doteq \frac{\partial}{\partial y_i} + \alpha x_i \frac{\partial}{\partial t} \quad \text{for } i \in \{1, \dots, n\}; \quad T \doteq \frac{\partial}{\partial t} \tag{1.112}$$

in the canonical coordinate system.

***Remark* 1.88.** It is readily seen that the relations

$$[X_i, X_j] = 0 = [Y_i, Y_j] \qquad \text{for all } i, j \in \{1, \dots, n\} \tag{1.113}$$

$$[X_i, Y_j] = 0 \qquad \text{for all } i \neq j \in \{1, \dots, n\} \tag{1.114}$$

$$[X_i, X_i] = 2\alpha T \qquad \text{for all } i \in \{1, \dots, n\}. \tag{1.115}$$

hold. In particular, the family of left invariant vector fields $\mathbb{X} \doteq \{X_1, \dots, X_n, Y_1, \dots, Y_n\}$ induces a Carnot-Carathéodory structure, since it satisfies the Hörmander condition (Definition 1.83) and thus the Chow-Rashevskii Theorem 1.84. More precisely, the Heisenberg group is a particular instance of a *Carnot group*, a concept which will not be deepened in this thesis outside the context of the Heisenberg group. ♢

We finally make a choice on the value of $\alpha \neq 0$ we will adopt: such a choice is only dictated by the ease of computations; other choices are frequently found in literature (typically $\alpha = -2$).

Definition 1.89 (Heisenberg group). Let $n \geq 1$. We define the Heisenberg group $\mathbb{H}^n$ as the Lie group obtained by endowing $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ with the group law in Equation (1.109) with $\alpha = \frac{1}{2}$ (and by extension the CC-space $(\mathbb{H}^n, \mathsf{d}_{\mathrm{cc}}, \mathscr{L}^{2n+1})$ induced by it). The space $\mathbb{H}^n$ admits at any point a basis of left invariant vector fields represented by

$$X_i \doteq \frac{\partial}{\partial x_i} - \frac{1}{2} y_i \frac{\partial}{\partial t} \qquad \text{for } i \in \{1, \dots, n\} \tag{1.116}$$

$$Y_i \doteq \frac{\partial}{\partial y_i} + \frac{1}{2} x_i \frac{\partial}{\partial t} \qquad \text{for } i \in \{1, \dots, n\} \tag{1.117}$$

$$T \doteq \frac{\partial}{\partial t} \tag{1.118}$$

which satisfy the relation $[X_i, Y_i] = T$ for any $i \in \{1, \dots, n\}$ (all the other commutators being zero).

We will denote by $H\mathbb{H}^n$ the *horizontal subbundle* of $\mathbb{H}^n$, *i.e.* the subbundle of the tangent bundle spanned by the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$. ♢

The Heisenberg group is equipped with a family of *non-isotropic dilations*:

Definition 1.90 (Dilations). On the Heisenberg group $\mathbb{H}^n$ we define the family of automorphisms $\{\delta_\lambda\}_{\lambda > 0}$ as

$$\delta_\lambda((\mathbf{x}, \mathbf{y}, t)) \doteq (\lambda \mathbf{x}, \lambda \mathbf{y}, \lambda^2 t) \tag{1.119}$$

for any $(\mathbf{x}, \mathbf{y}, t) \in \mathbb{H}^n$. ♢

Then the following lemma holds for the Carnot-Carathéodory distance on $\mathbb{H}^n$:

Lemma 1.91. *The Carnot-Carathéodory distance* d_{cc} *on* $\mathbb{H}^n$ *is left invariant, in the sense that* $\mathsf{d}_{cc}(g \cdot p, g \cdot q) = \mathsf{d}_{cc}(p, q)$ *for any* $g, p, q \in \mathbb{H}^n$*, and it is homogeneous, in the sense that* $\mathsf{d}_{cc}(\delta_\lambda(p), \delta_\lambda(q)) = \mathsf{d}_{cc}(p, q)$ *for any* $p, q \in \mathbb{H}^n$ *and* $\lambda > 0$.

Notation 1.92. In Chapter 3, we will often use the symbol $\mathrm{Per}_{\mathbb{H}}$ to denote the sub-Riemannian perimeter (Equation (1.108)) in the sub-Riemannian Heisenberg group $\mathbb{H}^n$. $\Diamond$

***Remark* 1.93** ($\mathbb{H}^n$ as the Gromov-Hausdorff limit of Riemannian manifolds)**.** The notions of Gromov-Hausdorff type convergence of metric (measure) spaces from Section 1.1.4 can be used to realize $(\mathbb{H}^n, \mathsf{d}_{cc})$ as the limit of Riemannian manifolds. Indeed, for any $\varepsilon > 0$ consider the metric tensor g_ε on $\mathbb{R}^{2n+1}$ for which the family $\{X_1, \dots, X_n, Y_1, \dots, Y_n, T_\varepsilon\}$ is a orthonormal basis, where $T_\varepsilon \doteq \sqrt{\varepsilon} T$. Let then d_ε be the distance associated to g_ε through

$$\mathsf{d}_\varepsilon(x, y) \doteq \inf\left\{ \int_0^1 \|\gamma'(t)\|_\varepsilon \, \mathrm{d}t \;\middle|\; \begin{array}{l} \gamma : [0,1] \to \mathbb{R}^{2n+1} \text{ is a curve} \\ \text{with } \gamma(0) = x,\ \gamma(1) = y \end{array} \right\}, \tag{1.120}$$

where $\|\cdot\|_\varepsilon$ is the norm associated to g_ε – *i.e.*, in particular,

$$\|\gamma'(t)\|_\varepsilon^2 = \sum_{j=1}^n \Big(\gamma_j'(t)^2 + \gamma_{n+j}'(t)^2 \Big) + \gamma_{2n+1}'(t)^2 \tag{1.121}$$

whenever

$$\gamma'(t) = \sum_{j=1}^n \Big(\gamma_j'(t) X_j(\gamma(t)) + \gamma_{n+j}'(t) Y_j(\gamma(t)) \Big) + \gamma_{2n+1}'(t) T_\varepsilon(\gamma(t)). \tag{1.122}$$

In [Cap+07, Section 2.4], the authors prove that the metric spaces $(\mathbb{R}^{2n+1}, \mathsf{d}_\varepsilon)$ converge in the pointed Gromov-Hausdorff sense to $(\mathbb{H}^n, \mathsf{d})$ (by taking, for instance, the origin as the selected point).

This approach to the Heisenberg group proves fruitful from many points of view: for example, again in [Cap+07], geodesics in the space $(\mathbb{H}^n, \mathsf{d}_{cc})$ are recovered as limits of Riemannian geodesics in $(\mathbb{R}^{2n+1}, \mathsf{d}_\varepsilon)$; as a further example, notions of intrinsic curvature (for curves) and mean curvature (for surfaces) where proposed in [BTV17; BTV20] as limits of the corresponding Riemannian ones. $\Diamond$

***Remark* 1.94** (Is $\mathbb{H}^n$ a $\mathrm{RCD}(K, N)$ space?)**.** In the above Remark 1.93, we have seen that $(\mathbb{H}^n, \mathsf{d}_{cc})$ can be obtained as a Gromov-Hausdorff limit of Riemannian spaces (and in particular, the corresponding metric measure space $(\mathbb{H}^n, \mathsf{d}_{cc}, \mathscr{L}^{2n+1})$ is a pointed *measured* Gromov-Hausdorff limit of the same manifolds with the Lebesgue measures). In sight of Remark 1.32 and Proposition 1.79 one can wonder if the approximants have Ricci curvature bounded from below, or more generally if such a space satisfies a $\mathrm{CD}(K, N)$ condition as in Section 1.3.

The Ricci curvatures of the approximanting Riemannian manifolds $(\mathbb{R}^{2n+1}, g_\varepsilon)$ can be computed explicitly (this is completely done in [Cap+07, Paragraph 2.4.2] for the case $n = 1$): unfortunately, it can be shown that $\mathrm{Ric}_{g_\varepsilon}(T_\varepsilon, T_\varepsilon)$ is not uniformly bounded from below when $\varepsilon \downarrow 0$ (while $g_\varepsilon(T_\varepsilon, T_\varepsilon) = 1$ by the definition of g_ε). Indeed, by Equation (1.86), at each point $p \in \mathbb{R}^{2n+1}$ one can compute

$$\mathrm{Ric}_{g_\varepsilon}(T_\varepsilon, T_\varepsilon) = \sum_{j=1}^n (\sec(T_\varepsilon, X_j) + \sec(T_\varepsilon, Y_j)); \tag{1.123}$$

here by the properties of the Levi-Civita connection and the fact that $[T_\varepsilon, X_j] = [T_\varepsilon, Y_j] = 0$ for any $j = 1, \dots, n$ it holds

$$\begin{aligned}\sec(T_\varepsilon, X_j) = g_\varepsilon(R_{g_\varepsilon}(T_\varepsilon, X_j)X_j, T_\varepsilon) &= g_\varepsilon\Big(\nabla_{T_\varepsilon}\nabla_{X_j}X_j - \nabla_{X_j}\nabla_{T_\varepsilon}X_j - \nabla_{[T_\varepsilon, X_j]}, T_\varepsilon\Big)\\ &= -g_\varepsilon\Big(\nabla_{X_j}\nabla_{T_\varepsilon}X_j, T_\varepsilon\Big)\end{aligned} \tag{1.124}$$

and analogously

$$\sec(T_\varepsilon, Y_j) = -g_\varepsilon\Big(\nabla_{Y_j}\nabla_{T_\varepsilon}Y_j, T_\varepsilon\Big). \tag{1.125}$$

Using the Koszul identity 1.84 – and recalling that the X_j's, the Y_j's and T_ε form an orthonormal basis, so the first three terms in the formula are null –, one can see that

$$\nabla_{T_\varepsilon}X_j = -\frac{1}{2\sqrt{\varepsilon}}Y_j \quad \text{and} \quad \nabla_{T_\varepsilon}Y_j = \frac{1}{2\sqrt{\varepsilon}}X_j; \tag{1.126}$$

the same Koszul formula applied once again gives

$$g_\varepsilon\Big(\nabla_{X_j}\nabla_{T_\varepsilon}X_j, T_\varepsilon\Big) = \frac{1}{4\sqrt{\varepsilon}}g_\varepsilon([X_j, Y_j], T_\varepsilon) = \frac{1}{4\varepsilon} \tag{1.127}$$

$$g_\varepsilon\Big(\nabla_{Y_j}\nabla_{T_\varepsilon}Y_j, T_\varepsilon\Big) = -\frac{1}{4\sqrt{\varepsilon}}g_\varepsilon([Y_j, X_j], T_\varepsilon) = \frac{1}{4\varepsilon}. \tag{1.128}$$

Inserting this into Equation (1.123), one gets that

$$\mathrm{Ric}_{g_\varepsilon}(T_\varepsilon, T_\varepsilon) = -\frac{n}{2\varepsilon}; \tag{1.129}$$

thus the Gromov-Hausdorff approximants of $\mathbb{H}^n$ do *not* have uniformly bounded Ricci curvatures.

Worse than that, in [Jui09] the author proves that *no* $\mathrm{CD}(K, N)$ *condition can hold for the Heisenberg group* $(\mathbb{H}^n, \mathsf{d}_{\mathrm{cc}}, \mathscr{L}^{2n+1})$. This is obtained by constructing a pair of compact sets in $\mathbb{H}^n$ that make the $(0, N)$-*Brunn-Minkowski inequality* fail for any N ([Jui09, Lemma 3.1]); instead, such inequality always holds on $\mathrm{CD}(0, N)$ spaces by [Stu06b, Proposition 2.1]. Using the scaling properties of CD spaces and $\mathbb{H}^n$, one can show that $(\mathbb{H}^n, \mathsf{d}_{\mathrm{cc}}, \mathscr{L}^{2n+1})$ can not be $\mathrm{CD}(K, N)$ with $K < 0$ either (see [Jui09, Remark 3.3]).

It is worth to mention that, up to replacing the distortion coefficients of Definition 1.61 with suitable coefficients adapted to the Heisenberg setting, one can still recover on $\mathbb{H}^n$ a entropy inequality in analogy with Equation (1.79) ([BKS18, Theorem 1.2]); following this argument, one can find a suitable form of the aforementioned Brunn-Minkowski inequality that holds in $\mathbb{H}^n$ ([BKS18, Corollary 4.2]). ◊

Finally, for maps defined on domains of $\mathbb{H}^n$, an adjusted notion of regularity can be given, which takes into consideration the privileged directions $X_1, \dots, X_n, Y_1, \dots, Y_n$:

Definition 1.95 (Horizontally $\mathbf{C}^1$ functions)**.** Let $\Omega \subset \mathbb{H}^n$ be an open set, and let $f \in \mathbf{C}^0(\Omega)$. We call *horizontal gradient* of f the distribution $\nabla_{\mathbb{H}}f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$. Then f is said to be of class $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ its horizontal gradient $\nabla_{\mathbb{H}}f$ is represented by a continuous function. ◊

Building on the definition of $\mathbf{C}^1_{\mathbb{H}}$ functions, one can adapt to $\mathbb{H}^n$ the Euclidean notion of "regular surface" by considering level sets of $\mathbf{C}^1_{\mathbb{H}}$ maps:

Definition 1.96 ($\mathbb{H}$-regular surface). We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular surface* if for every $p \in S$ there exist a neighborhood $\Omega \subset \mathbb{H}^n$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ such that $\nabla_{\mathbb{H}} f \neq 0$ in Ω and $S \cap \Omega = \{q \in \Omega \mid f(q) = 0\}$.

In that case, we define the *horizontal normal* ν_S at the point $p \in S \cap \Omega$ as

$$\nu_S(p) \doteq \frac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|}. \tag{1.130}$$ ◊

Even though we will not need it, let us point out that for $\mathbb{H}$-regular surfaces a version of the Implicit Function Theorem is available, in the sense that $\mathbb{H}$-regular functions can be locally seen as the *intrinsic graphs* of functions defined on subsets of $\{(\mathbf{x}, \mathbf{y}, t) \in \mathbb{H}^n \mid x_1 = 0\}$ (see [FSS01, Theorem 6.5]).

Chapter 2

Symmetrizations and uniformly elliptic equations on RCD spaces

In this Chapter, we overview the results obtained in a joint work with Andrea Mondino [MV21] on a Talenti-type comparison Theorem on $\mathrm{RCD}(K,N)$ spaces. Furthermore, an easy extension to the case *with obstacle* is added in Section 2.2.1.

In the Euclidean space, the Schwarz symmetrization of a set $\Omega \subset \mathbb{R}^n$ is defined as the unique ball $\Omega^\star$ centered at the origin and having the same Lebesgue measure of Ω. If $f : \Omega \to \mathbb{R}$ is a measurable function, its symmetrization $f^\star$ is defined on $\Omega^\star$ as the unique radial function which is non-increasing in the radial direction and is equimeasurable with f (in the sense that all the super levels $\{f^\star > t\}$ have the same Lebesgue measure as $\{|f| > t\}$).

In the study of geometric and variational problems, Schwarz symmetrizations often prove useful: indeed, exploiting their properties, one can frequently simplify a complex problem by reducing it to the study of spherically symmetric objects. Specifically, the notion of Schwarz symmetrization of a function plays a notable role in proving results such as the Rayleigh-Faber-Krahn Inequality, as well as several variational inequalities for differential boundary problems.

A classical problem, historically known as the "Talenti comparison theorem", is to compare the outcomes of the following procedures:

(a) Solve a Poisson problem of the type

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{2.1}$$

with $f \in L^2(\Omega)$; then consider the Schwarz symmetrization $u^\star$ of u.

(b) Solve the symmetrized Poisson problem

$$\begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star \subset \mathbb{R}^n \\ v = 0 & \text{on } \partial\Omega^\star \end{cases}. \tag{2.2}$$

A result of Talenti [Tal76] (which builds on the work of Weinberger [Wei62] and Bandle [Ban76]) states that the inequality $u^\star \le v$ holds in $\Omega^\star$, and if equality holds then Ω was already a ball. We refer the reader to [Ban80; Bae19; Lio79; Kes06; PS51] for different proofs and related topics.

In this Chapter, we set up the same problem on $\mathrm{RCD}(K,N)$ spaces with $K>0$ and $N \in (1,\infty)$, and we show that an analogous estimate holds; moreover, the result is rigid, in the sense that equality at just one point forces the space to assume a particular structure (a spherical suspension, see Definition 2.24); and it is stable, in the sense that if $u^\star$ and v are close enough in L^2-norm, then the space is arbitrarily close to a spherical suspension in measured Gromov-Hausdorff topology. Since N-dimensional Riemannian manifolds with Ricci curvature bounded from below by $K>0$ are in particular $\mathrm{RCD}(K,N)$ spaces, a smooth version of these results can be obtained from the RCD version: we refer the reader to [MV21, Section 6].

Recall that, by Remark 1.65, if $K>0$ a generalized version of the Bonnet-Myers theorem implies that $\mathrm{spt}(\mathfrak{m})$ is compact and thus $\mathfrak{m}(\mathsf{X})<\infty$: up to a constant normalization of the measure, we can thus assume that $\mathfrak{m}(\mathsf{X})=1$. Thus, the setting for this Chapter will be the following:

Assumption 2.1. For the whole Chapter 2, $(\mathsf{X},\mathsf{d},\mathfrak{m})$ will be a metric measure space verifying the $\mathrm{RCD}(K,N)$ condition for some $K>0$ and $N\in(1,\infty)$ (see Definition 1.63). We will always assume that $\mathfrak{m}(\mathsf{X})=1$.

Moreover, $\Omega\subset\mathsf{X}$ will be an open connected subset with $\mathfrak{m}(\mathsf{X}\setminus\Omega)>0$. ◊

Recall that the definition of (K,N)-Schwarz symmetrization can be found in Definition 1.75: domains and functions on $\mathrm{RCD}(K,N)$ spaces can be symmetrized, and the symmetrizations now live on the model space $(J_{K,N},\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{K,N})$.

2.1 Elliptic problems on RCD(K,N) spaces

In the next sections, we will deal with arbitrary uniformly elliptic operators: we summerize here the properties we need.

Assumption 2.2 (Uniformly elliptic operators)**.** From now on we will assume $\mathcal{E}: L^2(\mathsf{X},\mathfrak{m})\times L^2(\mathsf{X},\mathfrak{m})\to[-\infty,\infty]$ to be a non-negative definite bilinear form such that the following properties hold:

(a) Strong locality: $\mathcal{E}(u,v)=0$ whenever $u(x)(v(x)+c)=0$ for $\mathfrak{m}$-a.e. $x\in\mathsf{X}$, for some constant $c\in\mathbb{R}$.

(b) α-uniform ellipticity: there exists $\alpha>0$ such that for any $u\in L^2(\mathsf{X},\mathfrak{m})$

$$\mathcal{E}(u,u)\geq\alpha\,\mathrm{Ch}(u,u), \tag{2.3}$$

where $\mathrm{Ch}=\mathrm{Ch}_2$ is the 2-Cheeger energy defined in Definition 1.12.

(c) $\mathcal{E}$ is of order 1: there exists $\beta>0$ such that $\mathcal{E}(u,u)\leq\beta\|u\|^2_{H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})}$ for every $u\in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$. ◊

***Remark* 2.3.** Notice that, since $\mathrm{RCD}(K,N)$ spaces satisfy a (local) Poincaré inequality, Proposition 1.31 holds, and thus Equation (2.3) implies that

$$\mathcal{E}(u,u)\geq C_\Omega\|u\|^2_{H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})}\qquad\text{for all } u\in H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m}) \tag{2.4}$$

for some constant $C_\Omega>0$. ◊

Definition 2.4. Let $\mathcal{E}$ be as in Assumption 2.2. Let $\Omega \subset \mathsf{X}$ be an open subset such that $\mathfrak{m}(\Omega) < \mathfrak{m}(\mathsf{X})$, and let $f \in L^2(\Omega, \mathfrak{m})$. Let $\mathcal{K} \subset H_0^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ be a closed convex subset of $H_0^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$. Consider the following problems:

(MP) **Minimization problem:** find $u \in \mathcal{K}$ which minimizes the functional

$$\mathcal{J}_{\mathcal{E},f}(v) \doteq \frac{1}{2}\mathcal{E}(v,v) - \int_\Omega f v \, \mathrm{d}\mathfrak{m} \tag{2.5}$$

among functions $v \in \mathcal{K}$.

(WP) **Weak problem:** find $u \in \mathcal{K}$ such that

$$\mathcal{E}(u, v-u) \geq \int_\Omega f(v-u)\, \mathrm{d}\mathfrak{m} \qquad \text{for all } v \in \mathcal{K}. \tag{2.6}$$

◊

As a first step, we recap what can be said about the existence and uniqueness of solutions for the above defined problems; for the sake of completeness, we sketch a simple proof for the case where $\mathcal{E}$ is symmetric. A complete reference for these issues in a more general setting can be found in [KS80, Chapter II, Sections 1 and 2].

Proposition 2.5. *Let $\mathcal{E}, \Omega, f, \mathcal{K}$ be as in Definition 2.4. Then:*

(i) If $\mathcal{E}$ is symmetric, then any solution to the minimization problem (MP) solves the weak problem (WP).

(ii) If $\mathcal{E}$ is symmetric, then the minimization problem (MP) admits at least a solution (and thus (WP) also does).

(iii) The weak problem (WP) admits at most a solution (and thus (MP) also does, when $\mathcal{E}$ is symmetric).

Proof. Statement (i): Let $u \in \mathcal{K}$ be a minimum for the functional $\mathcal{J}_{\mathcal{E},f}$, and let $v \in \mathcal{K}$ be a test function. Consider, for any $\varepsilon \in (0,1)$, the function $v_\varepsilon \doteq u + \varepsilon(v-u)$, which belongs to $\mathcal{K}$ by convexity. By minimality of u,

$$\frac{1}{2}\mathcal{E}(v_\varepsilon, v_\varepsilon) - \int_\Omega f v_\varepsilon \, \mathrm{d}\mathfrak{m} \geq \frac{1}{2}\mathcal{E}(u,u) - \int_\Omega f u \, \mathrm{d}\mathfrak{m}. \tag{2.7}$$

The bilinearity and symmetry of $\mathcal{E}$ thus imply that

$$\frac{1}{2}\varepsilon \cdot 2\mathcal{E}(u, v-u) + \frac{1}{2}\varepsilon^2 \mathcal{E}(v-u, v-u) - \varepsilon \int_\Omega f(v-u)\, \mathrm{d}\mathfrak{m} \geq 0; \tag{2.8}$$

rearranging and dividing by $\varepsilon > 0$,

$$\mathcal{E}(u, v-u) - \int_\Omega f(v-u)\, \mathrm{d}\mathfrak{m} \geq -\frac{1}{2}\varepsilon \mathcal{E}(v-u, v-u). \tag{2.9}$$

Since this holds for any $\varepsilon \in (0,1)$, *(i)* follows.

Statement (ii): The proof for the existence of a minimum is an instance of the classical direct method of the calculus of variations. Let

$$d \doteq \inf\{\mathcal{J}_{\mathcal{E},f}(u) \mid u \in \mathcal{K}\}, \tag{2.10}$$

and notice that $d > -\infty$ because (by Equation (2.4) and Hölder inequality) for any $u \in \mathcal{K}$

$$\mathcal{J}_{\mathcal{E},f}(u) \geq \frac{1}{2}\Big[C_\Omega \|u\|^2_{L^2(\Omega,\mathfrak{m})} - 2\|f\|_{L^2(\Omega,\mathfrak{m})}\|u\|_{L^2(\Omega,\mathfrak{m})}\Big] \geq \frac{1}{2C_\Omega}\|f\|_{L^2(\Omega,\mathfrak{m})} > -\infty. \tag{2.11}$$

Fix a minimizing sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that

$$\lim_{n\to\infty} \mathcal{J}_{\mathcal{E},f}(u_n) = d; \tag{2.12}$$

it is easy to see that $\{u_n\}_n$ is bounded in $H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$, thus up to subsequences it converges to a function u weakly in $H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})$; by closedness of $\mathcal{K}$, $u \in \mathcal{K}$ (by the Mazur's Lemma, weak and strong closedness are equivalent because of the convexity of $\mathcal{K}$). Moreover, by coercivity on $H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m})$, the functional $v \mapsto \mathcal{E}(v,v)$ is weakly lower semicontinuous on $H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m})$ as a consequence of the Riesz Representation Theorem; by continuity of $v \mapsto \int_\Omega fv\,\mathrm{d}\mathfrak{m}$, then, the functional $\mathcal{J}_{\mathcal{E},f}$ is lower semicontinuous: thus

$$\mathcal{J}_{\mathcal{E},f}(u) \leq \liminf \mathcal{J}_{\mathcal{E},f}(u_n) = \inf\{\mathcal{J}_{\mathcal{E},f}(u) \mid u \in \mathcal{K}\}. \tag{2.13}$$

Statement (iii): Let $u_1, u_2 \in \mathcal{K}$ be two solutions to the weak problem (WP). We can use u_1 as a test function for u_2 and vice versa:

$$\mathcal{E}(u_1, u_2 - u_1) \geq \int_\Omega f(u_2 - u_1)\,\mathrm{d}\mathfrak{m} \quad \text{and} \quad \mathcal{E}(u_2, u_1 - u_2) \geq \int_\Omega f(u_1 - u_2)\,\mathrm{d}\mathfrak{m}; \tag{2.14}$$

adding up, and considering the uniform ellipticity assumption,

$$C_\Omega \|u_1 - u_2\|_{H^{1,2}(\mathsf{X},\mathsf{d},\mathfrak{m})} \leq \mathcal{E}(u_1 - u_2, u_1 - u_2) \leq 0. \tag{2.15}$$

In particular, $u_1 = u_2$. □

Proposition 2.6. *The weak problem (WP) admits a unique solution even when $\mathcal{E}$ is not symmetric.*

Proof. We refer the reader to [KS80, Lemma 2.2]. □

Lemma 2.7. *Consider the problem in Definition 2.4. If the family $\mathcal{K}$ coincides with the whole $H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m})$, then the weak problem (WP) is equivalent to*

(WP′) *find $u \in \mathcal{K}$ such that*

$$\mathcal{E}(u,w) = \int_\Omega fw\,\mathrm{d}\mathfrak{m} \qquad \textit{for all } w \in H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m}). \tag{2.16}$$

Proof. Let $u \in H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m})$ be the unique solution to the weak problem, and let w be a test function in $H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m})$. Both $v_+ \doteq u + w$ and $v_- \doteq u - w$ can be used in Equation (2.6): thus we have that

$$\begin{aligned} \mathcal{E}(u,w) &\geq \int_\Omega fw\,\mathrm{d}\mathfrak{m} \\ \mathcal{E}(u,-w) &\geq \int_\Omega f(-w)\,\mathrm{d}\mathfrak{m}, \end{aligned} \tag{2.17}$$

which implies, by bilinearity of $\mathcal{E}$, the thesis. The converse is trivial. □

2.2 A Talenti-type comparison theorem for RCD(K,N) spaces

In this Section, we prove a version of Talenti's comparison theorem in the RCD setting; the proof follows the same scheme as the one given in [Kes06, Section 3.1] for the Euclidean setting.

Definition 2.8 (Domain of $\mathcal{L}_{\mathcal{E}}$)**.** Let $\mathcal{E}$ be a uniformly elliptic bilinear form as in Assumption 2.2. We define the domain of $\mathcal{L}_{\mathcal{E}}$ as the set

$$D_\Omega(\mathcal{L}_{\mathcal{E}}) \doteq \left\{ u \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}) \;\middle|\; \begin{array}{c} \exists f \in L^2(\Omega, \mathfrak{m}) \text{ such that} \\ \mathcal{E}(u,v) = \int_\Omega f v \, \mathrm{d}\mathfrak{m} \text{ for all } v \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}) \end{array} \right\}. \tag{2.18}$$

If $u \in D_\Omega(\mathcal{L}_{\mathcal{E}})$ and f satisfies the condition in Equation (2.18), we write $-\mathcal{L}_{\mathcal{E}}(u) = f$. ♢

The problem we are interested in, in this Section, is the one coming from Proposition 2.6:

Definition 2.9 (Dirichlet problem on $(\Omega, \mathsf{d}, \mathfrak{m})$)**.** Let $\mathcal{E}$ be a uniformly elliptic bilinear form as in Assumption 2.2; let $\Omega \subset \mathsf{X}$ be an open domain and let $f \in L^2(\Omega, \mathfrak{m})$. We say that a function $u \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a weak solution to the Dirichlet problem

$$\begin{cases} -\mathcal{L}_{\mathcal{E}}(u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.19}$$

if $u \in H^{1,2}_0(\Omega, \mathsf{d}, \mathfrak{m})$ and

$$\mathcal{E}(u,v) = \int_\Omega f v \, \mathrm{d}\mathfrak{m}, \qquad \text{for any } v \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}). \tag{2.20}$$

By Proposition 2.6 (and Lemma 2.7), this Dirichlet problem admits a unique solution $u \in H^{1,2}_0(\Omega, \mathsf{d}, \mathfrak{m})$. ♢

***Remark* 2.10.** An alternative (but slightly less general) approach would be to adopt the language of differential calculus on metric measure spaces, as introduced for example in [Gig18]. In particular, let A be an element of the $L^2(\mathsf{X})$-normed $L^\infty(\mathsf{X})$-module $L^2(T^*\mathsf{X}) \otimes L^2(T^*\mathsf{X})$ and assume it is concentrated on Ω. Assume there exists $\alpha > 0$ such that for any $X \in L^2(T\mathsf{X})|_\Omega$

$$A(X,X) \doteq A(X \otimes X) \geq \alpha |X|^2, \tag{2.21}$$

where we have denoted by $|\cdot|$ the pointwise norm of X, and by $L^2(T\mathsf{X})|_\Omega$ the sub-module of the tangent module whose elements are concentrated on Ω. Recall now that for an infinitesimally Hilbertian metric measure space X and a function $u \in H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ we can define the gradient $\nabla u \in L^2(T\mathsf{X})$ as the image of the differential $\mathrm{d}u \in L^2(T^*\mathsf{X})$ through the canonical isomorphism between the two L^∞-modules. If we denote by $\mathcal{E}_A : H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}) \times H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}) \to \mathbb{R}$ the bilinear form defined by

$$\mathcal{E}_A(u,v) \doteq \int_\Omega A(\nabla u, \nabla v) \, \mathrm{d}\mathfrak{m}, \tag{2.22}$$

then for any $f \in L^2(\Omega, \mathfrak{m})$, we say that u is a weak solution to the equation $-\mathcal{L}_{\mathcal{E}_A}(u) = f$ if

$$\mathcal{E}_A(u,v) = \int_\Omega A(\nabla u, \nabla v) \, \mathrm{d}\mathfrak{m} = \int_\Omega f v \, \mathrm{d}\mathfrak{m}, \quad \forall v \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}). \tag{2.23}$$

♢

Before passing to the proof of the main comparison theorem, we establish few auxiliary results. We begin with a simple lemma which only requires $(\mathsf{X},\mathfrak{m})$ to be a measure space and $\Omega\subset\mathsf{X}$ to be measurable with finite measure.

Lemma 2.11. *Let $f,u\in L^2(\Omega,\mathfrak{m})$, with $\Omega\subset\mathsf{X}$ measurable domain with finite measure. Define*

$$F(t)\doteq\int_{\{u>t\}}(u-t)\,f\,\mathrm{d}\mathfrak{m},\quad\forall t\in\mathbb{R}. \tag{2.24}$$

Then F is differentiable out of a countable set $C\subset\mathbb{R}$, and

$$F'(t)=-\int_{\{u>t\}}f\,\mathrm{d}\mathfrak{m},\quad\forall t\in\mathbb{R}\setminus C. \tag{2.25}$$

Proof. The proof is quite standard, however we recall it for the reader's convenience.
First of all notice that $\mathfrak{m}(\{u=t\})>0$ for an at most countable set $C\in\mathbb{R}$. Let $t\in\mathbb{R}\setminus C$ and $h>0$. Then

$$\begin{aligned}F(t+h)-F(t)&=\int_{\{u>t+h\}}(u-t)\,f\,\mathrm{d}\mathfrak{m}-h\int_{\{u>t+h\}}f\,\mathrm{d}\mathfrak{m}\\&\quad-\left[\int_{\{u>t+h\}}(u-t)\,f\,\mathrm{d}\mathfrak{m}+\int_{\{t<u\le t+h\}}(u-t)\,f\,\mathrm{d}\mathfrak{m}\right]\\&=-h\int_{\{u>t+h\}}f\,\mathrm{d}\mathfrak{m}-\int_{\{t<u\le t+h\}}(u-t)\,f\,\mathrm{d}\mathfrak{m},\end{aligned} \tag{2.26}$$

which implies

$$\left|\frac{F(t+h)-F(t)}{h}+\int_{\{u>t+h\}}f\,\mathrm{d}\mathfrak{m}\right|\le\int_{\{t<u\le t+h\}}|f|\,\mathrm{d}\mathfrak{m}. \tag{2.27}$$

The right hand side converges to 0 by Hölder inequality and continuity of the measure, recalling that $\mathfrak{m}(\{u=t\})=0$. An analogous procedure works for $F(t-h)-F(t)$: we find

$$\left|\frac{F(t-h)-F(t)}{-h}+\int_{\{u>t-h\}}f\,\mathrm{d}\mathfrak{m}\right|\le\int_{\{t-h<u\le t\}}|f|\,\mathrm{d}\mathfrak{m}; \tag{2.28}$$

taking the limit as $h\to0$, this gives the claimed identity. □

The next step toward a Talenti Theorem is an estimate which involves the α-uniform ellipticity of $\mathcal{E}$ and a suitable choice of test functions in the Dirichlet problem Equation (2.20).

Lemma 2.12. *Let $\Omega\subset\mathsf{X}$ be an open domain with finite measure, $\mathcal{E}$ be as in Assumption 2.2 and $f\in L^2(\Omega,\mathfrak{m})$. Let $u\in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ be a weak solution to $-\mathcal{L}_{\mathcal{E}}(u)=f$. Then for $\mathscr{L}^1$-a.e. $t>0$ it holds:*

$$\left(-\frac{d}{dt}\int_{\{|u|>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\right)^2\le-\frac{1}{\alpha}\mu'(t)\int_{\{|u|>t\}}|f|\,\mathrm{d}\mathfrak{m}, \tag{2.29}$$

where $\mu=\mu_u$ is the distribution function of u and $|\nabla u|_{\mathrm{w}}$ denotes the minimal 2-weak upper gradient of u.

Proof. Let $t > 0$ be fixed, and consider the following test function:

$$v_t \doteq (u-t)^+ - (u+t)^- = \begin{cases} u - t = |u| - t & \text{if } u > t \\ 0 & \text{if } |u| \le t \\ u + t = -(|u| - t) & \text{if } u < -t \end{cases}. \tag{2.30}$$

It is easy to see that v_t still belongs to the space $H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$, thus it can be used as a test function in Equation (2.20) to obtain

$$\mathcal{E}(u, v_t) = \int_\Omega f v_t \, d\mathfrak{m} = \int_{\{u>t\}} (u-t) \, f \, d\mathfrak{m} - \int_{\{-u>t\}} (-u-t) \, f \, d\mathfrak{m}. \tag{2.31}$$

By applying Lemma 2.11 we obtain that, for $\mathscr{L}^1$-a.e. $t > 0$, $t \mapsto \mathcal{E}(u, v_t)$ is differentiable with

$$-\frac{d}{dt}\mathcal{E}(u, v_t) = \int_{\{u>t\}} f \, d\mathfrak{m} - \int_{\{u<-t\}} f \, d\mathfrak{m} \le \int_{\{|u|>t\}} |f| \, d\mathfrak{m}. \tag{2.32}$$

For fixed $t > 0$ and $h > 0$, by bilinearity of $\mathcal{E}$ it holds that

$$\mathcal{E}(u, v_{t+h}) - \mathcal{E}(u, v_t) = \mathcal{E}(u, v_{t+h} - v_t). \tag{2.33}$$

Moreover, we can explicitly write

$$v_{t+h} - v_t = -\operatorname{sgn}(u)\Big[(|u| - t)\chi_{\{t<|u|\le t+h\}} + h\chi_{\{|u|>t+h\}}\Big] = \begin{cases} h & \text{if } u < -t-h \\ -(u+t) & \text{if } -t-h \le u < -t \\ 0 & \text{if } |u| \le t \\ -(u-t) & \text{if } t < u \le t+h \\ -h & \text{if } u > t+h \end{cases}. \tag{2.34}$$

Notice that, by strong locality and bilinearity of $\mathcal{E}$, for any $B \in \mathscr{B}(\mathsf{X})$

$$0 = \mathcal{E}(u\chi_B, \chi_B) + \mathcal{E}(u\chi_{\mathsf{X}\setminus B}, \chi_B) = \mathcal{E}(u, \chi_B). \tag{2.35}$$

In particular, it follows from Equations (2.33) to (2.35) that

$$\begin{aligned} \frac{\mathcal{E}(u, v_{t+h}) - \mathcal{E}(u, v_t)}{h} &= -\frac{1}{h}\Big[\mathcal{E}\Big(u, (u+t)\chi_{\{-t-h\le u<-t\}}\Big) + \mathcal{E}\Big(u, (u-t)\chi_{\{t<u\le t+h\}}\Big)\Big] \\ &= -\frac{1}{h}\Big\{\mathcal{E}\Big(u\chi_{\{-t-h\le u<-t\}}, u\chi_{\{-t-h\le u<-t\}}\Big) + \\ &\quad + \mathcal{E}\Big(u\chi_{\{t<u\le t+h\}}, u\chi_{\{t<u\le t+h\}}\Big)\Big\}. \end{aligned} \tag{2.36}$$

By α-uniform ellipticity, then, the following estimate holds true:

$$-\frac{1}{\alpha}\frac{\mathcal{E}(u, v_{t+h}) - \mathcal{E}(u, v_t)}{h} \ge \frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|_{\mathrm{w}}^2 \, d\mathfrak{m}. \tag{2.37}$$

Consequently, the following chain of inequalities holds for all $t \in \mathbb{R}$ and $h > 0$:

$$\begin{aligned} \left(\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|_{\mathrm{w}} \, d\mathfrak{m}\right)^2 &\le \left(\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|_{\mathrm{w}}^2 \, d\mathfrak{m}\right)\left(\frac{\mathfrak{m}(\{t < |u| \le t+h\})}{h}\right) \\ &\le \frac{1}{\alpha}\left(-\frac{\mathcal{E}(u, v_{t+h}) - \mathcal{E}(u, v_t)}{h}\right)\left(-\frac{\mu(t+h) - \mu(t)}{h}\right). \end{aligned} \tag{2.38}$$

Hence, if t is a differentiability point for $t \mapsto \mathcal{E}(u, v_t)$, letting $h \to 0$ and using Equation (2.32) we get exactly the desired result. $\square$

A further estimate is needed: here is where the Coarea Formula (1.90) and the Lévy-Gromov inequality (1.96) are exploited.

Corollary 2.13. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K>0$, $N\in(1,\infty)$. Let $\Omega\subset\mathsf{X}$ be an open domain and $u:\Omega\to\mathbb{R}$ be a function in $H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$. Then the map*

$$t\mapsto\int_{\{|u|>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\tag{2.39}$$

is absolutely continuous and

$$-\frac{d}{dt}\left(\int_{\{|u|>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\right)\geq\mathcal{I}_{K,N}(\mathfrak{m}(\{|u|>t\}))=\mathcal{I}_{K,N}(\mu(t)).\tag{2.40}$$

Proof. The absolute continuity of $t\mapsto\int_{\{|u|>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}$ is an immediate consequence of the identity

$$\int_{\{|u|>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}=\int_t^{\infty}\mathrm{Per}(\{|u|>r\})\,\mathrm{d}r\tag{2.41}$$

in Proposition 1.70. Differentiating such a formula, we get that for almost every $t>0$ one has

$$-\frac{d}{dt}\left(\int_{\{|u|>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\right)=\mathrm{Per}(\{|u|>t\})\tag{2.42}$$

and thus we can use the Lévy-Gromov inequality this is greater than or equal to $\mathcal{I}_{K,N}(\mathfrak{m}(\{|u|>t\}))=\mathcal{I}_{K,N}(\mu_u(t))$. □

We have now the tools needed to prove our first main result.

Theorem 2.14 (A Talenti-type comparison for $\mathrm{RCD}(K,N)$ spaces)**.** *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be space satisfying the $\mathrm{RCD}(K,N)$ condition for some $K>0$, $N\in(1,\infty)$, with $\mathfrak{m}(\mathsf{X})=1$, and let $\Omega\subset\mathsf{X}$ be an open domain with measure $\mathfrak{m}(\Omega)=v\in(0,1)$. Let $f\in L^2(\Omega,\mathfrak{m})$. Let $\mathcal{E}$ be a α-uniformly elliptic bilinear form as in Assumption 2.2 and assume that $u\in H^{1,2}_0(\Omega)$ is a weak solution to the equation $-\mathcal{L}_{\mathcal{E}}(u)=f$.*

Let also $w\in H^{1,2}(\Omega^\star,\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{K,N})$ be a weak solution (as in Definition 1.52) to the problem

$$\begin{cases}-\alpha\Delta_{K,N}w=f^\star & \text{in }\Omega^\star\\ w(r_1)=0\end{cases},\tag{2.43}$$

where $\Omega^\star=[0,r_v)$, $r_v\doteq H^{-1}_{K,N}(v)>0$ is such that $\mathfrak{m}_{K,N}([0,r_v))=\mathfrak{m}(\Omega)$, and $f^\star$ is the Schwarz symmetrization of f. Then

1. *$u^\star(x)\leq w(x)$, for every $x\in[0,r_v]$.*
2. *For any $1\leq q\leq 2$, the following L^q-gradient estimate holds:*

$$\int_\Omega|\nabla u|^q_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\leq\int_0^{r_v}|w'(\varrho)|^q\,\mathrm{d}\mathfrak{m}_{K,N}(\varrho).\tag{2.44}$$

***Remark* 2.15.** The Dirichlet problem in Equation (2.43) can be explicitly rewritten as

$$\begin{cases} -w'' - \dfrac{h'_{K,N}}{h_{K,N}} w' = \dfrac{1}{\alpha} f^\star & \text{in } \Omega^\star \\ w(H_{K,N}^{-1}(\mathfrak{m}(\Omega))) = 0 \end{cases}, \tag{2.45}$$

by the definition of $\Delta_{K,N}$ and $H_{K,N}$. ◊

Proof. We split the proof of the two statements:

Proof of 1. By combining Lemma 1.42, Lemma 2.12 and Corollary 2.13, we obtain the following chain of inequalities:

$$\begin{aligned} \mathcal{I}_{K,N}(\mu(t))^2 &\le \left(-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|_{\mathrm{w}} \,\mathrm{d}\mathfrak{m}\right)^2 \le -\frac{1}{\alpha}\mu'(t)\int_{\{|u|>t\}} |f| \,\mathrm{d}\mathfrak{m} \\ &\le -\frac{1}{\alpha}\mu'(t)\int_0^{\mu(t)} f^\sharp(s)\,\mathrm{d}s \end{aligned} \tag{2.46}$$

for almost every $t > 0$, which can be rewritten as

$$1 \le -\frac{\mu'(t)}{\alpha\,\mathcal{I}_{K,N}(\mu(t))^2}\int_0^{\mu(t)} f^\sharp(s)\,\mathrm{d}s \tag{2.47}$$

for almost every $t \in (0, M)$, where $M = \operatorname{ess\,sup} u$. For $\xi > 0$ let

$$F(\xi) \doteq \int_0^\xi f^\sharp(s)\,\mathrm{d}s. \tag{2.48}$$

Let now $0 \le \tau' < \tau \le M$. Integrating Equation (2.47) from τ' to τ we get

$$\tau - \tau' \le \frac{1}{\alpha}\int_{\tau'}^{\tau} \frac{F(\mu(t))}{\mathcal{I}_{K,N}(\mu(t))^2}(-\mu'(t))\,\mathrm{d}t, \qquad 0 \le \tau' < \tau \le M. \tag{2.49}$$

Using the change of variables $\xi = \mu(t)$ on the intervals where μ is absolutely continuous, and observing that the integrand is non-negative, we obtain

$$\tau - \tau' \le \frac{1}{\alpha}\int_{\mu(\tau)}^{\mu(\tau')} \frac{F(\xi)}{\mathcal{I}_{K,N}(\xi)^2}\,\mathrm{d}\xi, \qquad 0 \le \tau' < \tau \le M. \tag{2.50}$$

Let us fix $s \in (0, \mu(0))$ and let $\eta > 0$ be a small enough parameter (that will eventually tend to 0); consider $\tau' = 0$ and $\tau = u^\sharp(s) - \eta$. Notice that, since $u^\sharp(s)$ is the infimum of the $\tilde\tau$ such that $\mu(\tilde\tau) < s$, we have that $\mu(\tau) \ge s$. Using again the non-negativity of the integrand, for any $\eta > 0$ we obtain that

$$u^\sharp(s) - \eta \le \frac{1}{\alpha}\int_s^{\mu(0)} \frac{F(\xi)}{\mathcal{I}_{K,N}(\xi)^2}\,\mathrm{d}\xi, \qquad \forall s \in (0, \mu(0)). \tag{2.51}$$

Letting $\eta \downarrow 0$ and enlarging the integration interval, we get:

$$u^\sharp(s) \le \frac{1}{\alpha}\int_s^{\mathfrak{m}(\Omega)} \frac{1}{\mathcal{I}_{K,N}(\xi)^2}\int_0^\xi f^\sharp(t)\,\mathrm{d}t\,\mathrm{d}\xi, \qquad \forall s \in (0, \mathfrak{m}(\Omega)). \tag{2.52}$$

Notice that on $(\mu(0), \mathfrak{m}(\Omega))$ the function $u^\sharp$ vanishes. Finally, by the definition of the symmetrized function $u^\star = u^\sharp \circ H_{K,N}$, we obtain

$$u^\star(x) \le \frac{1}{\alpha} \int_{H_{K,N}(x)}^{\mathfrak{m}(\Omega)} \frac{1}{\mathcal{I}_{K,N}(\xi)^2} \int_0^\xi f^\star(H_{K,N}^{-1}(t))\,\mathrm{d}t\,\mathrm{d}\xi, \qquad \forall x \in J_{K,N}. \tag{2.53}$$

Now we can recognize that the right hand side coincides with the characterization of w we obtained in Equation (1.66), since r_v was chosen so that $H_{K,N}(r_v) = \mathfrak{m}(\Omega)$. Note that, since the integrand is non-negative, w is non-increasing, and takes the value zero at r_v.

Proof of 2. We start by noticing that

$$\int_\Omega |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} = \int_{\{|u|>0\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m}, \tag{2.54}$$

since $|\nabla u|_{\mathrm{w}} = 0$ $\mathfrak{m}$-a.e. on $\{u = \kappa\}$ for any $\kappa \in \mathbb{R}$. Let $M := \operatorname{ess\,sup}_\Omega |u|$; fix $t > 0$ and $0 < h < M - t$. By using the Hölder inequality (with exponents $\frac{2}{q}$ and $\frac{2}{2-q}$) one gets

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} \le \left(\frac{1}{h}\int_{\{t<|u|\le t+h\}} |\nabla u|_{\mathrm{w}}^2 \,\mathrm{d}\mathfrak{m}\right)^{\frac{q}{2}} \left(\frac{\mathfrak{m}(\{t<|u|\le t+h\})}{h}\right)^{\frac{2-q}{2}}. \tag{2.55}$$

By the very same computations we already performed in Lemma 2.12, exploiting the test functions $v_t \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ defined in Equation (2.30) (see Equations (2.32), (2.37) and (2.38)), we can let h tend to zero in Equation (2.55) and obtain that the map

$$t \mapsto \int_{\{|u|>t\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} \tag{2.56}$$

is absolutely continuous on $(0, M)$ and thus

$$\int_\Omega |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} = \int_0^M -\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m}\,\mathrm{d}t; \tag{2.57}$$

moreover

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} \le \left(\frac{1}{\alpha}\int_{\{|u|>t\}} f\,\mathrm{d}\mathfrak{m}\right)^{\frac{q}{2}} (-\mu'(t))^{\frac{2-q}{2}}. \tag{2.58}$$

Let us now adopt again the notation

$$F(\xi) \doteq \int_0^\xi f^\sharp(s)\,\mathrm{d}s, \tag{2.59}$$

as in Equation (2.48). Exploiting again Lemma 1.42, we get:

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} \le \left(\frac{F(\mu(t))}{\alpha}\right)^{\frac{q}{2}} (-\mu'(t))^{\frac{2-q}{2}} \tag{2.60}$$

for almost every t. In order to obtain a clean term $\mu'(t)$ at the right hand side, we multiply both sides of Equation (2.60) with the respective sides of Equation (2.47) raised at the power $\frac{q}{2}$. This gives, for almost every $t \in (0, M)$:

$$-\frac{d}{dt}\int_{\{|u|>t\}} |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} \le \left(\frac{F(\mu(t))}{\alpha \mathcal{I}_{K,N}(\mu(t))}\right)^{q} (-\mu'(t)). \tag{2.61}$$

Inserting this last inequality in Equation (2.57) and changing the variables as usual with $\xi = \mu(t)$, the following estimate holds:

$$\int_\Omega |\nabla u|_{\mathrm{w}}^q \,\mathrm{d}\mathfrak{m} \le \int_0^{\mathfrak{m}(\Omega)} \left(\frac{F(\xi)}{\alpha \mathcal{I}_{K,N}(\xi)} \right)^q \mathrm{d}\xi. \tag{2.62}$$

Finally, we recall that w has an explicit expression we can differentiate: by differentiating Equation (1.58) (with datum $\frac{f^\star}{\alpha}$), we find for all $\varrho \in (0, r_v)$

$$w'(\varrho) = -\frac{1}{h_{K,N}(\varrho)} \int_0^{H_{K,N}(\varrho)} \frac{1}{\alpha} f^\star(H_{K,N}^{-1}(t)) \,\mathrm{d}t = -\frac{F(H_{K,N}(\varrho))}{\alpha h_{K,N}(\varrho)}. \tag{2.63}$$

Thus, the following identity holds true:

$$\int_0^{r_v} |w'(\varrho)|^q \,\mathrm{d}\mathfrak{m}_{K,N} = \int_0^{r_v} \left(\frac{F(H_{K,N}(\varrho))}{\alpha h_{K,N}(\varrho)} \right)^q h_{K,N}(\varrho) \,\mathrm{d}\varrho = \int_0^{\mathfrak{m}(\Omega)} \left(\frac{F(\xi)}{\alpha \mathcal{I}_{K,N}(\xi)} \right)^q \mathrm{d}\xi, \tag{2.64}$$

where we have used the change of variables $\xi = H_{K,N}(\varrho)$ and the characterization of the isoperimetric profile $\mathcal{I}_{K,N}(\xi) = h_{K,N}(H_{K,N}^{-1}(\xi))$. Comparing with Equation (2.62), we obtain the claimed L^q-gradient estimate. □

2.2.1 Notes for the obstacle case

A variant of the problem approached in this Section 2.2 is the following:

(OB) Find $u \in \mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m})$ such that

$$\mathcal{E}(u, v-u) \ge \int_\Omega f(v-u) \,\mathrm{d}\mathfrak{m} \qquad \text{for all } v \in \mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m}) \tag{2.65}$$

where $\mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m})$ is the family

$$\mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m}) \doteq \left\{ w \in H_0^{1,2}(\Omega, \mathsf{d}, \mathfrak{m}) \,\middle|\, w \ge 0 \text{ a.e. in } \Omega \right\}. \tag{2.66}$$

Then this new problem, which is known in the literature as (an instance of) the obstacle problem, still admits a unique solution, by Proposition 2.6: indeed, $\mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m})$ is convex and strongly closed. We refer to [KS80, Chapter II, Section 6] for a description of the problem in $\mathbb{R}^n$.

In order to explore how this problem interacts with symmetrizations, let us give a slightly different notion of the *decreasing rearrangement* introduced in Definition 1.40:

Definition 2.16 (Signed decreasing rearrangement). Let $f : \Omega \to \mathbb{R}$ be a measurable function. We define $\mu^{\mathrm{sg}} = \mu_f^{\mathrm{sg}} : \mathbb{R} \to [0, \mathfrak{m}(\Omega)]$ as

$$\mu^{\mathrm{sg}}(t) \doteq \mathfrak{m}(\{f > t\}). \tag{2.67}$$

We define $f^{\mathrm{sg}\sharp} : [0, \mathfrak{m}(\Omega)] \to [-\infty, \infty]$ as

$$f^{\mathrm{sg}\sharp}(s) \doteq \begin{cases} \operatorname{ess\,sup} f & \text{if } s = 0 \\ \inf\left\{ t \in \mathbb{R} \,\middle|\, \mu_f^{\mathrm{sg}}(t) < s \right\} & \text{if } s > 0 \end{cases}. \tag{2.68}$$

In practice, we remove the absolute value from the definition of the distribution function of u, and then consider the pseudo-inverse of this new function. The relationship between $f^\sharp$ and $f^{\mathrm{sg}\sharp}$ is expressed by $f^\sharp = |f|^{\mathrm{sg}\sharp}$. ◊

Moreover, we give a signed definition of the (K,N)-Schwarz symmetrization (Definition 1.55 and Equation (1.93)):

Definition 2.17 (Signed (K,N)-Schwarz symmetrization). Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be a metric measure space satisfying the $\mathrm{RCD}(K,N)$ condition for some $K>0$ and $N\in(1,\infty)$. Let $\Omega\subset\mathsf{X}$ be a Borel subset and $u:\Omega\to\mathbb{R}$ be a Borel measurable function. Let $\Omega^\star_{K,N}$ be the (K,N)-Schwarz symmetrization of Ω. Then we define the signed (K,N)-Schwarz symmetrization $u^{\mathrm{sg}\star}_{K,N}:\Omega^\star_{K,N}\to\mathbb{R}$ as:

$$u^{\mathrm{sg}\star}_{K,N}(x)\doteq u^{\mathrm{sg}\sharp}(\mathfrak{m}_{K,N}([0,x])) \tag{2.69}$$

for any $x\in\Omega^\star_{K,N}$. ◊

Some of the results that held for the problem $-\mathcal{L}_{\mathcal{E}}(u)=f$ have an analogous version here. In particular, Lemma 2.12 becomes:

Lemma 2.18. *Let $\Omega\subset\mathsf{X}$ be an open domain with finite measure, $\mathcal{E}$ be as in Assumption 2.2 and $f\in L^2(\Omega,\mathfrak{m})$. Let $u\in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ be a solution to (OB). Then for $\mathscr{L}^1$-a.e. $t>0$ it holds:*

$$\left(-\frac{d}{dt}\int_{\{u>t\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\right)^2\le-\frac{1}{\alpha}\mu'(t)\int_{\{u>t\}}f\,\mathrm{d}\mathfrak{m}. \tag{2.70}$$

Notice that the real difference with Lemma 2.12 lies in the lack of the absolute value of f in the integral at the right hand side (u is already non-negative by the assumption $u\in\mathcal{K}_{\mathrm{ob}}(\Omega,\mathsf{d},\mathfrak{m})$).

Proof. The proof follows the same scheme of Lemma 2.12. Let $t>0$ be fixed, and consider now the functions $v_t\doteq(u-t)^+\in\mathcal{K}_{\mathrm{ob}}(\Omega,\mathsf{d},\mathfrak{m})$. Both $w_t^+\doteq u+v_t$ and $w_t^-\doteq u-v_t$ belong to $\mathcal{K}_{\mathrm{ob}}(\Omega,\mathsf{d},\mathfrak{m})$, thus they can be used as test functions in Equation (2.65): by the same method as in the proof of Lemma 2.7, then, it holds that (for these particular functions v_t):

$$\mathcal{E}(u,v_t)=\int_\Omega f v_t\,\mathrm{d}\mathfrak{m}=\int_{\{u>t\}}(u-t)\,f\,\mathrm{d}\mathfrak{m}. \tag{2.71}$$

By applying Lemma 2.11 we obtain that, for $\mathscr{L}^1$-a.e. $t>0$, $t\mapsto\mathcal{E}(u,v_t)$ is differentiable with

$$-\frac{d}{dt}\mathcal{E}(u,v_t)=\int_{\{u>t\}}f\,\mathrm{d}\mathfrak{m}. \tag{2.72}$$

Moreover, for any fixed $t>0$ and $h>0$, we can replicate the argument from Equation (2.33) to Equation (2.37) (noting that the pieces where $u<0$ vanish here). Consequently, as in Equation (2.38) we get for all $t\in\mathbb{R}$ and $h>0$:

$$\left(\frac{1}{h}\int_{\{t<u\le t+h\}}|\nabla u|_{\mathrm{w}}\,\mathrm{d}\mathfrak{m}\right)^2\le\frac{1}{\alpha}\left(-\frac{\mathcal{E}(u,v_{t+h})-\mathcal{E}(u,v_t)}{h}\right)\left(-\frac{\mu(t+h)-\mu(t)}{h}\right). \tag{2.73}$$

Hence, if t is a differentiability point for $t\mapsto\mathcal{E}(u,v_t)$, letting $h\to0$ and using Equation (2.72) we obtain the new result. □

Lemma 2.19. *Let $\mathcal{E}=\mathrm{Ch}$, and let u be a solution to (OB). If u has a continuous representative $\tilde{u}\in\mathbf{C}^0(\Omega)$, then u satisfies $-\Delta u=f$ on the open set $\tilde{\Omega}\doteq\{\tilde{u}>0\}$.*

Proof. Assume without loss of generality that $u \in \mathbf{C}^0(\Omega)$. Let $\varphi \in \mathrm{Lip}_c(\tilde{\Omega})$. Then u admits a minimum $\delta > 0$ in $\mathrm{spt}(\varphi)$; consider then the test functions $u \pm L\varphi$, with $L \doteq \frac{\delta}{\max_\Omega \varphi}$: they both belong to $\mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m})$, thus we get

$$\mathrm{Ch}(u, \pm L\varphi) \geq \int_\Omega (\pm L\varphi) f \,\mathrm{d}\mathfrak{m}. \tag{2.74}$$

By bilinearity of Ch, this implies (as in Lemma 2.7) that

$$\mathrm{Ch}(u, \varphi) = \int_{\tilde{\Omega}} f\varphi \,\mathrm{d}\mathfrak{m} \qquad \text{for any } \varphi \in \mathrm{Lip}_c(\tilde{\Omega}); \tag{2.75}$$

By the definition of $H^{1,2}_0(\tilde{\Omega}, \mathsf{d}, \mathfrak{m})$ as the closure of $\mathrm{Lip}_c(\tilde{\Omega})$ in the topology of $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, the previous relation holds for any $\varphi \in H^{1,2}_0(\tilde{\Omega}, \mathsf{d}, \mathfrak{m})$ □

Proposition 2.20 (The "model space" case). *Let $\omega = [0, r_1) \Subset J_{K,N}$, $g \in L^2(\omega, \mathfrak{m}_{K,N})$ be a non-increasing function, and assume that $v \in \mathcal{K}_{\mathrm{ob}}(\omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ is the unique solution to the obstacle problem*

$$\mathrm{Ch}(v, \varphi - v) = \int_\omega v'(\varphi - v)' \,\mathrm{d}\mathfrak{m}_{K,N} \geq \int_\omega g(\varphi - v) \,\mathrm{d}\mathfrak{m}_{K,N} \qquad \textit{for all } \varphi \in \mathcal{K}_{\mathrm{ob}}(\omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N}). \tag{2.76}$$

Then v has the form:

$$v(\varrho) = \begin{cases} \int_\varrho^{\bar{x}} \frac{1}{h_{K,N}(r)} \int_0^r g(x) \,\mathrm{d}\mathfrak{m}_{K,N}(x) \,\mathrm{d}r & \textit{if } \varrho \in [0, \bar{x}) \\ 0 & \textit{if } \varrho \in [\bar{x}, r_1) \end{cases} \tag{2.77}$$

where

$$\bar{x} \doteq \inf\left\{ \xi \in (0, r_1) \;\middle|\; \int_0^\xi g \,\mathrm{d}\mathfrak{m}_{K,N} < 0 \right\}. \tag{2.78}$$

***Remark* 2.21.** Notice that $\bar{x}$ is allowed to take the values 0 (when $g < 0$ $\mathfrak{m}_{K,N}$-almost everywhere, and in that case $v \equiv 0$ is trivially a solution) and r_1 (when $\int_\omega f \,\mathrm{d}\mathfrak{m}_{K,N} > 0$).

Furthermore, if $\bar{x} > 0$, then the map $G(r) \doteq \int_0^r g \,\mathrm{d}\mathfrak{m}_{K,N}$ is strictly negative for $r \in (\bar{x}, r_1)$ (by definition), and strictly positive for $r \in (0, \bar{x})$: this depends on the fact that g is non-increasing (and thus G is concave) and that $G > 0$ in a right neighborhood of 0. ◊

Proof. By Proposition 2.5, the solution v is the unique minimizer in $\mathcal{K}_{\mathrm{ob}}(\omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ of the functional

$$w \mapsto \int_\omega \left[\frac{1}{2}(w')^2 - gw \right] \mathrm{d}\mathfrak{m}_{K,N}; \tag{2.79}$$

by the Pólya-Szegő inequality (see Theorem 2.26 in the next section) and the Hardy-Littlewood one (see Proposition 1.58), using the fact that g is non-increasing, it holds that

$$\int_\omega \left[\frac{1}{2}((v^{\mathrm{sg}\star})')^2 - gv^{\mathrm{sg}\star} \right] \mathrm{d}\mathfrak{m}_{K,N} \leq \int_\omega \left[\frac{1}{2}(v')^2 - gv \right] \mathrm{d}\mathfrak{m}_{K,N}, \tag{2.80}$$

so we obtain that $v = v^{\mathrm{sg}\star}$ and thus v is non-increasing as well.

Notice that by the characterization in Proposition 1.44 v is continuous: on the one hand, this implies that the set $\{v > 0\}$ is an interval of type $[0, s)$; on the other hand, we can apply Lemma 2.19, which ensures that v coincides in $\omega_s \doteq [0, s)$ with the solution to the Dirichlet problem in that same domain ω_s, with boundary datum $v(s) = 0$.

Let now

$$G(r) \doteq \int_0^r g(x)\, \mathrm{d}\mathfrak{m}_{K,N}(x) \tag{2.81}$$

for any $r \in [0, r_1]$ and

$$v_s(\varrho) \doteq \left(\int_\varrho^s \frac{G(r)}{h_{K,N}(r)}\, \mathrm{d}r \right) \chi_{[0,s)}(\varrho) \tag{2.82}$$

for any $s, \varrho \in [0, r_1]$. By the previous argument and by Equation (1.58), v coincides with v_σ for some $\sigma \in [0, r_1]$. Trivially, $\sigma \leq \bar{x}$, otherwise v would assume negative values for ϱ close to σ.

Assume by contradiction that $\sigma < \bar{x}$ strictly. We can use $v_{\bar{x}} \in \mathcal{K}_{\mathrm{ob}}$ as a test function in Equation (2.80): hence we have, for any $\varrho \in [0, r_1]$:

$$v_{\bar{x}}(\varrho) - v_\sigma(\varrho) = \begin{cases} \int_\sigma^{\bar{x}} \frac{G}{h_{K,N}}\, \mathrm{d}r & \text{if } \varrho < \sigma \\ \int_\varrho^{\bar{x}} \frac{G}{h_{K,N}}\, \mathrm{d}r & \text{if } \sigma \leq \varrho < \bar{x} \\ 0 & \text{if } \varrho \geq \bar{x} \end{cases} \tag{2.83}$$

$$v'_\sigma(\varrho) = -\frac{G}{h_{K,N}}(\varrho)\chi_{(0,\sigma)}(\varrho) \tag{2.84}$$

$$v'_{\bar{x}}(\varrho) - v'_\sigma(\varrho) = -\frac{G}{h_{K,N}}(\varrho)\chi_{(\sigma,\bar{x})}(\varrho). \tag{2.85}$$

Thus the relation (2.76) with $v = v_\sigma$ and $\varphi = v_{\bar{x}}$ has 0 as a left hand side (because $v'_\sigma(v'_{\bar{x}} - v'_\sigma) = 0$ almost everywhere), while on the right hand side we have by the Fubini-Tonelli Theorem:

(2.86)

$$\begin{aligned} &\int_\omega g(v_{\bar{x}} - v_\sigma)\, \mathrm{d}\mathfrak{m}_{K,N} = \\ &\quad = \int_0^\sigma g(\varrho) \left(\int_\sigma^{\bar{x}} \frac{G(r)}{h_{K,N}(r)}\, \mathrm{d}r \right) \mathrm{d}\mathfrak{m}_{K,N}(\varrho) + \int_\sigma^{\bar{x}} g(\varrho) \left(\int_\varrho^{\bar{x}} \frac{G}{h_{K,N}}\, \mathrm{d}r \right) \mathrm{d}\mathfrak{m}_{K,N}(\varrho) = \\ &\quad = \int_\sigma^{\bar{x}} \frac{G(r)}{h_{K,N}(r)} \int_0^\sigma g(\varrho)\, \mathrm{d}\mathfrak{m}_{K,N}(\varrho)\, \mathrm{d}r + \int_\sigma^{\bar{x}} \int_\sigma^{\bar{x}} \frac{G(r)}{h_{K,N}(r)} g(\varrho) \chi_{(\varrho,\bar{x})}(r)\, \mathrm{d}r\, \mathrm{d}\mathfrak{m}_{K,N} = \\ &\quad = \int_\sigma^{\bar{x}} \frac{G(r)}{h_{K,N}(r)} \int_0^\sigma g(\varrho)\, \mathrm{d}\mathfrak{m}_{K,N}(\varrho)\, \mathrm{d}r + \int_\sigma^{\bar{x}} \frac{G(r)}{h_{K,N}(r)} \int_\sigma^r g(\varrho)\, \mathrm{d}\mathfrak{m}_{K,N}(\varrho)\, \mathrm{d}r = \\ &\quad = \int_\sigma^{\bar{x}} \frac{G(r)}{h_{K,N}(r)} \int_0^r g(\varrho)\, \mathrm{d}\mathfrak{m}_{K,N}(\varrho)\, \mathrm{d}r = \int_\sigma^{\bar{x}} \frac{G^2(r)}{h_{K,N}(r)}\, \mathrm{d}r, \end{aligned}$$

which is *strictly* positive by Remark 2.21. This contradicts the fact that v_σ solves Equation (2.76). $\square$

Theorem 2.22 (Talenti with obstacle). *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K, N)$ space for some $K > 0$, $N \in (1, \infty)$, with $\mathfrak{m}(\mathsf{X}) = 1$, and let $\Omega \subset \mathsf{X}$ be an open domain with measure $\mathfrak{m}(\Omega) = v \in (0, 1)$.*

Let $f \in L^2(\Omega, \mathfrak{m})$. Let $\mathcal{E}$ be a α-uniformly elliptic bilinear form as in Assumption 2.2. Assume that $u \in \mathcal{K}_{\mathrm{ob}}(\Omega, \mathsf{d}, \mathfrak{m})$ is a solution to the obstacle problem.

Let also $v \in \mathcal{K}_{\mathrm{ob}}(\Omega^\star, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ be a solution to the symmetrized obstacle problem

$$\int_{\Omega^\star} v'(\varphi - v)' \,\mathrm{d}\mathfrak{m}_{K,N} \geq \int_{\Omega^\star} f^{\mathrm{sg}\star}(\varphi - v)\,\mathrm{d}\mathfrak{m}_{K,N} \qquad \textit{for all } \varphi \in \mathcal{K}_{\mathrm{ob}}(\omega, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N}). \tag{2.87}$$

Then it holds that $\{u^\star > 0\} \subset \{v > 0\}$, and the inequality $u^\star \leq v$ holds almost everywhere.

Proof. The proof follows almost verbatim the one of Theorem 2.14.

Without loss of generality, we can assume that $\mathfrak{m}(\{f > 0\}) > 0$, otherwise both problems have the constant function 0 as the unique solution.

The function v is already known thanks to Proposition 2.20: if we let

$$G(r) \doteq \int_0^r f^{\mathrm{sg}\star}\,\mathrm{d}\mathfrak{m}_{K,N}, \tag{2.88}$$

then $v(x) = \chi_{[0,\bar{x})}(x) \int_x^{\bar{x}} \frac{G(r)}{h_{K,N}(r)}\,\mathrm{d}r$, with

$$\bar{x} \doteq \inf\left\{ s \in (0, r_1) \;\middle|\; \int_0^s f^{\mathrm{sg}\star}\,\mathrm{d}\mathfrak{m}_{K,N} < 0 \right\}. \tag{2.89}$$

In order to estimate u, we can again combine a suitable version of Lemma 1.42 (with $f^{\mathrm{sg}\sharp}$ in place of $f^\sharp$), Lemma 2.18 (in place of Lemma 2.12) and Corollary 2.13, to obtain:

$$\mathcal{I}_{K,N}(\mu(t))^2 \leq -\frac{1}{\alpha}\mu'(t) \int_0^{\mu(t)} f^{\mathrm{sg}\sharp}(s)\,\mathrm{d}s \tag{2.90}$$

for almost every $t > 0$, where μ is again the distribution function of u (which is non-negative, thus $\mu_u = \mu_u^{\mathrm{sg}}$). Let us define

$$F^{\mathrm{sg}}(\xi) \doteq \int_0^\xi f^{\mathrm{sg}\sharp}(s)\,\mathrm{d}s, \tag{2.91}$$

and notice that $F = G \circ H_{K,N}^{-1}$.

On the one hand, Equation (2.90) already implies that $F^{\mathrm{sg}}(\mathfrak{m}(\{u > 0\})) \geq 0$, thus by Proposition 2.20

$$\mathfrak{m}(\{u > 0\}) \leq \mathfrak{m}_{K,N}(\{v > 0\}) \quad \Rightarrow \quad \{u^\star > 0\} \subset \{v > 0\}. \tag{2.92}$$

On the other hand, by concavity of F^{sg} (and the fact that $F(0) > 0$ by the assumption $\mathfrak{m}(\{f > 0\}) > 0$) we get that $F^{\mathrm{sg}} > 0$ for all $\xi \in (0, \mu(0))$; we can repeat the argument from Equation (2.47) to Equation (2.51) (and let again $\eta \downarrow 0$) to obtain that

$$u^\sharp(s) \leq \frac{1}{\alpha} \int_s^{\mu(0)} \frac{F^{\mathrm{sg}}(\xi)}{\mathcal{I}_{K,N}(\xi)^2}\,\mathrm{d}\xi, \qquad \forall s \in (0, \mu(0)). \tag{2.93}$$

By definition of $\bar{x}$, the integrand in the previous equation is nonnegative for any $\xi < H_{K,N}(\bar{x})$, thus we can enlarge again the domain of integration:

$$u^\sharp(s) \leq \frac{1}{\alpha} \int_s^{H_{K,N}(\bar{x})} \frac{F^{\mathrm{sg}}(\xi)}{\mathcal{I}_{K,N}(\xi)^2}\,\mathrm{d}\xi, \qquad \forall s \in (0, \mu(0)), \tag{2.94}$$

which implies, for any x such that $x < H_{K,N}^{-1}(\mathfrak{m}(\{u > 0\}))$,

$$u^\star(x) \leq \frac{1}{\alpha} \int_{H_{K,N}(x)}^{H_{K,N}(\bar{x})} \frac{F^{\mathrm{sg}}(\xi)}{\mathcal{I}_{K,N}(\xi)^2}\, \mathrm{d}\xi = \frac{1}{\alpha} \int_x^{\bar{x}} \frac{G(r)}{h_{K,N}(r)}\, \mathrm{d}r. \tag{2.95}$$

In particular, on the set $\{u^\star > 0\}$, the estimate $u^\star \leq v$ holds by Proposition 2.20, while on the remaining part of the interval it holds trivially because $u \equiv 0$ and $v \geq 0$. □

2.3 Rigidity and Stability

2.3.1 Rigidity in the Talenti-type theorem

Let $u \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ and $w \in H^{1,2}([0, r_v), \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ be as in Theorem 2.14. The next problem we want to approach is the equality case, that is, what we can say about the original metric measure space when $u^\star = w$; in fact, we will prove that if the equality is attained at least at one point, then the metric measure space is forced to have a particular structure, namely it is a spherical suspension. We recall that, in the Euclidean case $\Omega \subset \mathbb{R}^n$, the condition $u^\star = w$ forces Ω to be a ball and both u and f to be radial.

In order to tackle this question, we recall the definition of a spherical suspension, we state the Rigidity Theorem for the Lévy-Gromov inequality (as proved in [CM18]) and the Pólya-Szegő Theorem for $\mathrm{RCD}(K, N)$ spaces, which was proved in [MS19].

Definition 2.23 (Warped product)**.** Let $(B, \mathsf{d}_B, \mathfrak{m}_B)$ and $(F, \mathsf{d}_F, \mathfrak{m}_F)$ be geodesic metric measure spaces and $f : B \to [0, \infty)$ be a Lipschitz function. Let d be the pseudo-distance on $B \times F$ defined by

$$\mathsf{d}((p, x), (q, y)) \doteq \inf\{L(\gamma) \mid \gamma(0) = (p, x), \gamma(1) = (q, y)\}, \tag{2.96}$$

where, for any absolutely continuous curve $\gamma = (\gamma_B, \gamma_F) : [0, 1] \to B \times F$,

$$L(\gamma) \doteq \int_0^1 \left(|\gamma_B'|^2 + (f \circ \gamma_B)^2 |\gamma_F'|^2 \right)^{\frac{1}{2}} \mathrm{d}t. \tag{2.97}$$

Given $N \geq 1$, we define $B \times_f^N F$ to be the metric measure space

$$((B \times F)/\sim, \mathsf{d}, \mathfrak{m}), \tag{2.98}$$

where $\sim$ is the equivalence relation associated to the pseudo-distance d and $\mathfrak{m} \doteq f^N \mathfrak{m}_B \otimes \mathfrak{m}_F$. ◊

Definition 2.24 (Spherical suspensions)**.** We say that an $\mathrm{RCD}(N-1, N)$ space $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a *spherical suspension* if it is isomorphic to a warped product $[0, \pi] \times_{\sin}^{N-1} \mathsf{Y}$ for an $\mathrm{RCD}(N-2, N-1)$ space $(\mathsf{Y}, \mathsf{d}_\mathsf{Y}, \mathfrak{m}_\mathsf{Y})$ with $\mathfrak{m}_\mathsf{Y}(\mathsf{Y}) = 1$. ◊

Just for simplicity, the following results are stated in the case of $\mathrm{RCD}(N-1, N)$ spaces; indeed when $K > 0$ it is not restrictive to assume $K = N - 1$ by (1.80). Notice, moreover, that this assumption only affects the Rigidity statements, while the Pólya-Szegő inequality holds in the very same form for general $K > 0$.

Theorem 2.25 (Rigidity for Lévy-Gromov, [CM18]). *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an* $\mathrm{RCD}(N-1,N)$ *space for some $N\in[2,+\infty)$, with $\mathfrak{m}(\mathsf{X})=1$. Assume there exists $\bar{v}\in(0,1)$ such that $\mathcal{I}_{(\mathsf{X},\mathsf{d},\mathfrak{m})}(\bar{v})=\mathcal{I}_{N-1,N}(\bar{v})$. Then $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a spherical suspension:* i.e., *there exists an* $\mathrm{RCD}(N-2,N-1)$ *space $(\mathsf{Y},\mathsf{d}_{\mathsf{Y}},\mathfrak{m}_{\mathsf{Y}})$ with $\mathfrak{m}_{\mathsf{Y}}(\mathsf{Y})=1$ such that*

$$(\mathsf{X},\mathsf{d},\mathfrak{m}) \textit{ is isomorphic as a metric measure space to } [0,\pi]\times_{\sin}^{N-1}\mathsf{Y}. \tag{2.99}$$

Theorem 2.26 (Pólya-Szegő for $\mathrm{RCD}(N-1,N)$ spaces, [MS19]). *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(N-1,N)$ space for some $N\in[2,+\infty)$, with $\mathfrak{m}(\mathsf{X})=1$. Let $\Omega\subset\mathsf{X}$ be an open subset with measure $\mathfrak{m}(\Omega)=v\in(0,1)$ and let $r_v\in(0,\pi)$ such that $\mathfrak{m}_{N-1,N}([0,r_v])=v$. Then, for every $p\in(1,\infty)$, the following hold:*

(i) Pólya-Szegő comparison*: for any $u\in W_0^{1,p}(\Omega)$, it holds that $u^\star(r_v)=0$ and*

$$\int_0^{r_v}|\nabla u^\star|^p\,\mathrm{d}\mathfrak{m}_{N-1,N}\leq\int_\Omega|\nabla u|^p\,\mathrm{d}\mathfrak{m}. \tag{2.100}$$

(ii) Rigidity*: if there exists $u\in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ such that $u\not\equiv 0$, and u attains equality in Equation (2.100), then $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a spherical suspension.*

(iii) Rigidity for Lipschitz functions*: if there exists $u\in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})\cap\mathrm{Lip}(\Omega)$ with $u\not\equiv 0$ and $\nabla u\neq 0$ $\mathfrak{m}$-a.e. in $\mathrm{spt}(u)$, achieving equality in Equation (2.100), then $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a spherical suspension and u is radial: that is, u is of the form $u=g(\mathsf{d}(\cdot,x_0))$, with x_0 being the tip of a spherical suspension structure of X, and $g:[0,\pi]\to\mathbb{R}$ satisfying $|g|=u^\star$.*

The following rigidity result for the Talenti-type comparison theorem will build on top of the rigidity in the Lévy-Gromov and Pólya-Szegő inequalities.

Theorem 2.27 (Rigidity for Talenti in RCD). *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(N-1,N)$ space for some $N\in[2,\infty)$, with $\mathfrak{m}(\mathsf{X})=1$, and let $\Omega\subset\mathsf{X}$ be an open domain with measure $\mathfrak{m}(\Omega)=v\in(0,1)$. Let $f\in L^2(\Omega,\mathfrak{m})$, with $f\not\equiv 0$. Let $\mathcal{E}$ be a α-uniformly elliptic bilinear form as in Assumption 2.2 and assume that $u\in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ is a weak solution to the equation $-\mathcal{L}_{\mathcal{E}}(u)=f$. Let also $w\in H^{1,2}(I,\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{N-1,N})$ be a solution to the problem*

$$\begin{cases}-\alpha\Delta_{N-1,N}w=f^\star & \textit{in } I\\ \qquad w(r_v)=0\end{cases} \tag{2.101}$$

where $I=[0,r_v)$, $r_v\in(0,\pi)$ is such that $\mathfrak{m}_{N-1,N}([0,r_v))=v$, and $f^\star$ is the Schwarz symmetrization of f. Assume that $u^\star(\bar{x})=w(\bar{x})$ for a point $\bar{x}\in[0,r_v)$. Then:

1. *$u^\star=w$ in the whole interval $[\bar{x},r_v]$;*
2. *$(\mathsf{X},\mathsf{d},\mathfrak{m})$ is a spherical suspension,* i.e. *there exists an $\mathrm{RCD}(N-2,N-1)$ space $(\mathsf{Y},\mathsf{d}_{\mathsf{Y}},\mathfrak{m}_{\mathsf{Y}})$ with $\mathfrak{m}_{\mathsf{Y}}(\mathsf{Y})=1$ such that $(\mathsf{X},\mathsf{d},\mathfrak{m})$ is isomorphic as a metric measure space to $[0,\pi]\times_{\sin}^{N-1}\mathsf{Y}$;*
3. *if $\bar{x}=0$, $u\in\mathrm{Lip}(\Omega)$ and $|\nabla u|_{\mathrm{w}}\neq 0$ $\mathfrak{m}$-a.e. in $\mathrm{spt}(u)$, then u is radial: that is, u is of the form $u=g(\mathsf{d}(\cdot,x_0))$, with x_0 being the tip of a spherical suspension structure of X, and $g:[0,\pi]\to\mathbb{R}$ satisfying $|g|=u^\star$.*

In order to establish Theorem 2.27, we first prove a preliminary lemma which will also be useful in Section 2.3.2: in the same setting of the Talenti-type Theorem, the difference $w - u^\star$ is non-increasing.

Lemma 2.28. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$, Ω, f, $\mathcal{E}$, u and w be as in Theorem 2.14. Then the map $x \mapsto w(x) - u^\star(x)$ is non-increasing on $[0, r_v]$.*

Proof of Lemma 2.28. Since $w - u^\star = (w^\sharp - u^\sharp) \circ H_{K,N}$, with $H_{K,N}$ strictly increasing, it is enough to show that $w^\sharp - u^\sharp$ is non-increasing in $[0, \mathfrak{m}(\Omega)]$.
Recall that the function $w^\sharp \colon [0, \mathfrak{m}(\Omega)] \to \mathbb{R}$ can be expressed as:

$$w^\sharp(s) = \frac{1}{\alpha} \int_s^{\mathfrak{m}(\Omega)} \frac{F(\xi)}{\mathcal{I}^2_{K,N}(\xi)} \, \mathrm{d}\xi$$

where $F(\xi) \doteq \int_0^\xi f^\sharp(s) \, \mathrm{d}s$ as usual. As a preliminary observation, notice that this explicit representation gives some useful information on the regularity and behavior of $w^\sharp$ itself: indeed, $w^\sharp$ is a continuously differentiable function on $(0, \mathfrak{m}(\Omega))$, and it strictly decreasing in $[0, r_v]$ (since $f \not\equiv 0$). Moreover, as a consequence of the Pólya-Szegő Theorem, $u^\star$ belongs to $H^{1,2}_0([0, r_v), \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$ and it is thus locally absolutely continuous in the interior $(0, r_v)$; the same conclusion thus holds for $u^\sharp$. Hence the result is proved if we can show that

$$(w^\sharp - u^\sharp)' \leq 0 \text{ almost everywhere in } (0, \mathfrak{m}(\Omega)). \tag{2.102}$$

By the continuity of $u^\sharp$ and by the definition of symmetrization, we have that $u^\sharp(\mu(t)) = t$ for all $t \in (0, M)$ (*i.e.* μ is the right inverse of $u^\sharp$), where $M \doteq \sup u$. In particular, $(u^\sharp \circ \mu)' \equiv 1$ in $(0, M)$. On the other hand, $(w^\sharp \circ \mu)' \geq 1$ a.e. in $(0, M)$ by Equation (2.47). Hence,

$$\Big[(w^\sharp)' \circ \mu - (u^\sharp)' \circ \mu\Big] \mu' \geq 0 \qquad \text{a.e. in } (0, M).$$

Moreover, μ' is strictly negative a.e. in $(0, M)$, again by the fact that $[(u^\sharp)' \circ \mu]\mu' = 1$ almost everywhere. This shows that in fact

$$\Big(w^\sharp\Big)'(\mu(t)) \leq \Big(u^\sharp\Big)'(\mu(t)) \qquad \text{for a.e. } t \in (0, M). \tag{2.103}$$

Now, let $s_0 \in (0, \mathfrak{m}(\Omega))$ be a point where $u^\sharp$ is differentiable. If $(u^\sharp)'(s_0) = 0$, then trivially $(w^\sharp - u^\sharp)'(s_0) < 0$ so the inequality (2.102) is proved. Otherwise, $u^\sharp(s) > u^\sharp(s_0)$ for any $s < s_0$ (by the fact that $u^\sharp$ is monotone), so in particular $s_0 = \mu(u^\sharp(s_0))$: by Equation (2.103), $(w^\sharp - u^\sharp)'(s_0) \leq 0$. □

Proof of Theorem 2.27. The first statement ($u^\star = w$ in $[\bar{x}, r_v]$) is a direct consequence of the monotonicity of $w - u^\star$ (Lemma 2.28), of the assumption $w(\bar{x}) = u^\star(\bar{x})$ and of the Talenti inequality $w - u^\star \geq 0$ in $[0, r_v]$.

This also implies that $\mu(t) = \nu(t)$ for any $t \in (0, u^\star(\bar{x}))$, where ν is the distribution function of w; hence, for any such t, equality holds in Equation (2.46), in particular in the Lévy-Gromov inequality: all the superlevels $\{|u| > t\}$ satisfy $\mathcal{I}_{N-1,N}(\mathfrak{m}(\{|u| > t\})) = \mathrm{Per}(\{|u| > t\})$. By the rigidity in the Lévy-Gromov inequality, this implies that $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ is a spherical suspension.

Assume now $\bar{x} = 0$ (thus $u^\star = w$ in $[0, r_v]$), $u \in \mathrm{Lip}(\Omega)$ and $|\nabla u|_{\mathrm{w}} \neq 0$ $\mathfrak{m}$-almost everywhere in $\mathrm{spt}(u)$. Putting together the gradient comparison inequality (2.44) (with $q = 2$) and the Pólya-Szegő inequality (Equation (2.100)), we find

$$\int_0^{r_1} |\nabla u^\star|^2 \, \mathrm{d}\mathfrak{m}_{N-1,N} \leq \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathfrak{m} \leq \int_0^{r_1} |\nabla w|^2 \, \mathrm{d}\mathfrak{m}_{N-1,N}. \tag{2.104}$$

The equality assumption, however, implies that the first and the last expressions coincide: thus, equality in the Pólya-Szegő inequality is achieved. By rigidity in the Pólya-Szegő inequality, then, u is radial. □

2.3.2 Stability

In this Section, we will prove a stable version of the rigidity result (Theorem 2.27); we only consider the case where $\mathcal{E} = \mathrm{Ch}$, so that $\mathcal{L}_{\mathcal{E}}$ is the Laplacian. We first need to recall some results on the convergence of metric measure spaces and of functions defined therein.

Assumption 2.29. From now on, the following assumptions will be made:

Spaces: $\{\mathcal{X}_i\}_{i\in\mathbb{N}} = \{(\mathsf{X}_i, \mathsf{d}_i, x_i, \mathfrak{m}_i)\}_{i\in\mathbb{N}}$ and $\mathcal{X} = (\mathsf{X}, \mathsf{d}, x, \mathfrak{m})$ will be pointed metric measure spaces satisfying the $\mathrm{RCD}(N-1, N)$ condition for some $N \geq 2$, with $\mathfrak{m}_i(\mathsf{X}_i) = 1$, $\mathfrak{m}(\mathsf{X}) = 1$.

Convergence of spaces: we will assume that $\mathcal{X}_i$ converge in the pmGH sense to $\mathcal{X}$; by the already cited [GMS15, Section 3.5], pmGH convergence coincides in our setting with pmG convergence; thanks to the discussion in Section 1.1.4, the following conditions hold:

(GH1) X_i and X are all contained in a common metric space (Y, d), with $\mathsf{d}_i = \mathsf{d}|_{\mathsf{X}_i \times \mathsf{X}_i}$, and $x_i \to x$;

(GH2) $\mathrm{spt}\, \mathfrak{m}_i = \mathsf{X}_i$ and $\mathrm{spt}\, \mathfrak{m} = \mathsf{X}$;

(GH3) The measures $\mathfrak{m}_i$ narrowly converge to $\mathfrak{m}$:

$$\lim_{i\to\infty} \int_{\mathsf{Y}} \varphi \, \mathrm{d}\mathfrak{m}_i = \int_{\mathsf{Y}} \varphi \, \mathrm{d}\mathfrak{m} \quad \text{for all } \varphi \in \mathbf{C}_{\mathrm{b}}(\mathsf{Y}), \tag{2.105}$$

where $\mathbf{C}_{\mathrm{b}}(\mathsf{Y})$ is the space of continuous and bounded functions on (Y, d). ♢

***Remark* 2.30** (L^2 functions)**.** Assume that $B_{R_i}(x_i)$ and $B_R(x)$ are metric balls in X_i and X respectively. Let $f_i \in L^2(B_{R_i}(x_i), \mathfrak{m}_i)$ and $f \in L^2(B_R(x), \mathfrak{m})$ be L^2 functions on such balls; by extending such functions to be 0 out of the balls on which they are defined, we can equivalently assume $f_i \in L^2(\mathsf{X}_i, \mathfrak{m}_i)$ and $f \in L^2(\mathsf{X}, \mathfrak{m})$; by the assumption that the spaces X_i and X are contained in Y, up to a further extension we actually have $f_i \in L^2(\mathsf{Y}, \mathfrak{m}_i)$ and $f \in L^2(\mathsf{Y}, \mathfrak{m})$. ♢

Definition 2.31 (Convergence of L^2 functions)**.** Let $f_i \in L^2(B_{R_i}(x_i), \mathfrak{m}_i)$ and $f \in L^2(B_R(x), \mathfrak{m})$ as in Remark 2.30. Following [GMS15, Definition 6.1], we say that:

(a) $f_i \rightharpoonup f$ in the *weak* L^2 sense if

$$\lim_{i\to\infty} \int_{\mathsf{Y}} \varphi f_i \, \mathrm{d}\mathfrak{m}_i = \int_{\mathsf{Y}} \varphi f \, \mathrm{d}\mathfrak{m} \quad \text{for all } \varphi \in \mathbf{C}_{\mathrm{b}}(\mathsf{Y})$$

$$\sup_i \|f_i\|_{L^2\left(B_{R_i}(x_i), \mathfrak{m}_i\right)} < \infty.$$

(b) $f_i \to f$ in the *strong* L^2 sense if, in addition,

$$\lim_{i\to\infty} \|f_i\|_{L^2(B_{R_i}(x_i),\mathfrak{m}_i)} = \|f\|_{L^2(B_R(x),\mathfrak{m})}. \qquad \Diamond$$

Moreover, the following definition from [AH18, Definition 2.6] will be needed:

Definition 2.32 (Weak $H^{1,2}$ convergence). Let $f_i \in H^{1,2}(B_{R_i}(x_i),\mathfrak{m}_i)$ and $f \in H^{1,2}(B_R(x),\mathfrak{m})$. We say that the f_i's are *weakly convergent in* $H^{1,2}$ to f if $f_i \rightharpoonup f$ in the weak L^2 sense and $\sup_i \mathrm{Ch}^i(f)$ is finite, where Ch^i is the Cheeger energy with respect to the metric measure structure of $\mathcal{X}_i$. $\Diamond$

In order to obtain the stability result, we establish a series of auxiliary lemmas of independent interest. We start by showing that L^2-strong convergence of maps implies the pointwise convergence of the distribution functions to the distribution function of the limit.

Lemma 2.33 (Convergence of distribution functions). *Let $\mathcal{X}_i \xrightarrow{\mathrm{pmGH}} \mathcal{X}$ be pointed metric measure spaces satisfying Assumption 2.29. Let $B_{R_i}(x_i)$ and $B_R(x)$ be metric balls in X_i and X respectively, and let $f_i \in L^2(B_{R_i}(x_i),\mathfrak{m}_i)$ and $f \in L^2(B_R(x),\mathfrak{m})$. Assume $\mu_i \doteq \mu_{f_i}$ and $\mu \doteq \mu_f$ are the distribution functions of f_i and f respectively. If $f_i \to f$ L^2-strongly, then $\mu_i(t)$ converges to $\mu(t)$ for every $t \in (0,+\infty)\setminus C$, where C is a countable set.*

Proof. Let us fix $t \in (0,+\infty)$. We need to show that (except for a countable number of such t)

$$\lim_{i\to\infty} \mathfrak{m}_i(\{|f_i| > t\}) = \mathfrak{m}(\{|f| > t\}). \tag{2.106}$$

Notice that

$$\{x \in \mathsf{X}_i \mid |f_i(x)| > t\} = \{x \in \mathsf{X}_i \mid (x,|f_i(x)|) \in \mathsf{Y}\times(t,+\infty)\}$$
$$\{x \in \mathsf{X} \mid |f(x)| > t\} = \{x \in \mathsf{X} \mid (x,|f(x)|) \in \mathsf{Y}\times(t,+\infty)\}.$$

Given a map $g : \mathsf{Y} \to \mathbb{R}$, we denote by $\boldsymbol{i}\times g : \mathsf{Y} \to \mathsf{Y}\times\mathbb{R}$ the map $\boldsymbol{i}\times g(x) \doteq (x, g(x))$; by the argument above, it holds that

$$\{x \in \mathsf{X}_i \mid |f_i(x)| > t\} = (\boldsymbol{i}\times|f_i|)^{-1}(\mathsf{Y}\times(t,+\infty))$$
$$\{x \in \mathsf{X} \mid |f(x)| > t\} = (\boldsymbol{i}\times|f|)^{-1}(\mathsf{Y}\times(t,+\infty)).$$

Define ν_i and ν to be the following push-forward measures on $\mathsf{Y}\times\mathbb{R}$

$$\nu_i \doteq (\boldsymbol{i}\times|f_i|)_\sharp\mathfrak{m}_i, \quad \nu \doteq (\boldsymbol{i}\times|f|)_\sharp\mathfrak{m}. \tag{2.107}$$

Our goal (Equation (2.106)) is equivalent to show that

$$\lim_{i\to\infty} \nu_i(\mathsf{Y}\times(t,+\infty)) = \nu(\mathsf{Y}\times(t,+\infty)). \tag{2.108}$$

Notice that the topological boundary of $\mathsf{Y}\times(t,+\infty)$ is $\mathsf{Y}\times\{t\}$, which is ν-negligible for all but a countable set of $t>0$ by the finiteness of $\mathfrak{m}$:

$$\nu(\mathsf{Y}\times\{t\}) = \mathfrak{m}((\boldsymbol{i}\times|f|)^{-1}(\mathsf{Y}\times\{t\})) = \mathfrak{m}(\{|f| = t\}). \tag{2.109}$$

Thus, it is sufficient to show that the measures ν_i converge narrowly to ν in $\mathsf{Y}\times\mathbb{R}$. To this aim, notice that for every $\varphi \in \mathbf{C}_{\mathrm{b}}(\mathsf{Y}\times\mathbb{R})$, one has

$$\int_{\mathsf{Y}\times\mathbb{R}} \varphi(x,s)\,\mathrm{d}\nu_i = \int_{\mathsf{Y}} \varphi(x,|f_i(x)|)\,\mathrm{d}\mathfrak{m}_i, \qquad \int_{\mathsf{Y}\times\mathbb{R}} \varphi(x,s)\,\mathrm{d}\nu = \int_{\mathsf{Y}} \varphi(x,|f(x)|)\,\mathrm{d}\mathfrak{m}.$$

Arguing as in [AGS14b, Theorem 5.4.4*(iii)*] (see also [GMS15, Equation (6.6)]), one can show that the items in the left converge to the one in the right. This proves the statement. □

In [AH18, Theorem 4.2], a variant of the following proposition was established. The proof contained therein can be straightforwardly adapted to the present case.

Proposition 2.34 (Compactness of local Sobolev functions). *Let $\mathcal{X}_i \xrightarrow{\mathrm{pmGH}} \mathcal{X}$ be pointed metric measure spaces satisfying Assumption 2.29. Let $R_i \to R$ be a convergent sequence of radii with $R_i, R > 0$. Let $f_i \in H^{1,2}(B_{R_i}(x_i),\mathsf{d},\mathfrak{m}_i)$ have bounded $H^{1,2}$-norm: $\sup_i \|f_i\|_{H^{1,2}} < +\infty$. Then there exists a function $f \in H^{1,2}(B_R(x),\mathsf{d},\mathfrak{m})$ such that $\{f_i\}_{i\in\mathbb{N}}$ converges L^2-strongly to f, up to a subsequence.*

***Remark* 2.35.** The aforementioned [AH18, Theorem 4.2] deals with the same problem, but with $R_i = R$ for all i. A quick way to deduce our case from this is to look at the “rescaled” spaces $\mathcal{X}_i$ as $\tilde{\mathcal{X}}_i \doteq \left(\mathsf{X}_i, \frac{R}{R_i}\mathsf{d}_i, x_i, \mathfrak{m}_i\right)$: for any $\varrho, \varepsilon > 0$, let $F_i^{\varrho,\varepsilon}$ be the maps that arise in the definition of the pmGH convergence of the $\mathcal{X}_i$'s to $\mathcal{X}$ (see Section 1.1.4).

Then the maps

$$\tilde{F}_i^{\varrho,\varepsilon} \doteq F_i^{\frac{R_i}{R}\varrho,\frac{\varepsilon}{2}} \tag{2.110}$$

provide the convergence of $\tilde{\mathcal{X}}_i$ to $\mathcal{X}$ (the condition $R_i \to R$ is essential for Equation (1.34)). Moreover, the functions f_i from Proposition 2.34 still have bounded $H^{1,2}\left(\mathsf{X}_i, \frac{R}{R_i}\mathsf{d}_i, \mathfrak{m}_i\right)$ norm in the new balls $B_R(x_i)$ (*i.e.*, the balls with respect to the new rescaled distance); thus the result of [AH18] can be applied. Finally, the L^2-strong convergence is preserved by the rescaling (it does not depend on the distance).

The same observation can be done for the results from [AH18] used in Lemma 2.38. ◊

The next step is to prove that L^2-strong convergence of functions with bounded $H^{1,2}$-norms implies L^2-strong convergence of the symmetrizations.

Lemma 2.36. *Let $\mathcal{X}_i, \mathcal{X}, R_i, R, f_i$ satisfy the assumptions of Proposition 2.34, and let f_i converge in the strong L^2 sense to $f \in H^{1,2}(B_R(x),\mathsf{d},\mathfrak{m})$. Then, up to subsequences, the $f_i^\star$ converge to $f^\star$ in the strong $L^2(J_{N-1,N},\mathfrak{m}_{N-1,N})$ sense.*

Proof. By Proposition 1.57 and by the Pólya-Szegő inequality (2.100), the norms of the functions $f_i^\star$ in $H^{1,2}(J_{N-1,N},\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{N-1,N})$ are bounded by $C \doteq \sup_i \|f_i\|_{H^{1,2}\left(B_{R_i}(x_i),\mathsf{d},\mathfrak{m}_i\right)} < \infty$, which implies that the $f_i^\star$ also converge (up to subsequences) to a function g in the strong $L^2(J_{N-1,N},\mathfrak{m}_{N-1,N})$ sense. It remains to prove that $f^\star = g$ (at least $\mathfrak{m}_{N-1,N}$-almost everywhere).

By Lemma 2.33, the distribution functions μ_{f_i} converge pointwise to μ_f out of a countable set; similarly, $\mu_{f_i^\star}$ converge to μ_g out of a countable set. By equi-measurability of f_i and $f_i^\star$, however, we have that $\mu_{f_i} = \mu_{f_i^\star}$, thus $\mu_f = \mu_g$ out of a countable set. Since both μ_f and μ_g

are non-increasing and continuous, it follows that $\mu_f \equiv \mu_g$ and thus $f^\sharp \equiv g^\sharp$, which in turn implies $f^\star \equiv g^\star$. Now g was the L^2-limit of a sequence of non-increasing functions, thus it is non-increasing itself. By Lemma 1.59, we conclude that $f^\star = g^\star = g$ out of a countable set. □

In view of what we seek to achieve in Lemma 2.38, we need the next elementary convergence result, which we shortly prove for the sake of completeness.

Lemma 2.37. *Let $\mathcal{X}_i \overset{\text{pmGH}}{\longrightarrow} \mathcal{X}$ be pointed metric measure spaces satisfying Assumption 2.29. Let $R_i \to R$ be such that $\mathfrak{m}_i(B_{R_i}(x_i)) = v \in (0,1)$ for all $i \in \mathbb{N}$. Then $\mathfrak{m}(B_R(x)) = v$.*

Proof. For any $\varepsilon > 0$, the inclusions $B_{R-\varepsilon}(x) \subset B_{R_i}(x_i) \subset B_{R+\varepsilon}(x)$ hold for i large enough. Thus, by weak convergence of the measures, we have for any $\varepsilon > 0$:

$$\mathfrak{m}(B_{R-\varepsilon}(x)) \le \liminf \mathfrak{m}_i(B_{R_i}(x_i)) = v, \quad \mathfrak{m}(B_{R+\varepsilon}(x)) \ge \limsup \mathfrak{m}_i(B_{R_i}(x_i)) = v.$$

Moreover, the following holds (because the space is length):

$$\bigcup_{\varepsilon>0} B_{R-\varepsilon}(x) = B_R(x) \subset \overline{B_R(x)} = \bigcap_{\varepsilon>0} B_{R+\varepsilon}(x). \tag{2.111}$$

Combining these two facts, and the fact that $\mathfrak{m}(\partial B_R(x)) = 0$ for every $R > 0$ (which is true on $\mathrm{RCD}(K,N)$ spaces), implies the statement. □

The next lemma analyses the convergence of solutions to the Poisson problem.

Lemma 2.38. *Let $\mathcal{X}_i \overset{\text{pmGH}}{\longrightarrow} \mathcal{X}$ be pointed metric measure spaces satisfying Assumption 2.29. Let $R_i \to R$ be such that $\mathfrak{m}_i(B_{R_i}(x_i)) = v \in (0,1)$ for all $i \in \mathbb{N}$. Let $f_i \in H^{1,2}(B_{R_i}(x_i), \mathsf{d}, \mathfrak{m}_i)$ with*

$$\sup_i \|f_i\|_{H^{1,2}\left(B_{R_i}(x_i), \mathsf{d}, \mathfrak{m}_i\right)} < \infty. \tag{2.112}$$

Assume that $u_i \in H^{1,2}(B_{R_i}(x_i), \mathsf{d}, \mathfrak{m}_i)$ are weak solutions to

$$\begin{cases} -\Delta u_i = f_i & \text{in } B_{R_i}(x_i) \\ u_i = 0 & \text{on } \partial B_{R_i}(x_i) \end{cases} \tag{2.113}$$

and $w_i \in H^{1,2}(J_{N-1,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{N-1,N})$ are weak solutions to

$$\begin{cases} -\Delta_{N-1,N} w_i = f_i^\star & \text{in } [0, r_v) \\ w_i = 0 & \text{at } r_v \end{cases} \tag{2.114}$$

where $r_v \doteq H_{N-1,N}^{-1}(v)$. Then, up to extracting a subsequence:

(i) f_i converges in L^2-strong to a function $f \in L^2(B_R(x))$ with $\mathfrak{m}(B_R(x)) = v$; $f_i^\star$ converges in L^2-strong to $f^\star$;

(ii) u_i converges in L^2-strong to a weak solution u of

$$\begin{cases} -\Delta u = f & \text{in } B_R(x) \\ u = 0 & \text{on } \partial B_R(x) \end{cases} \tag{2.115}$$

(iii) w_i *converges in* L^2*-strong to a weak solution* w *of*

$$\begin{cases} -\Delta_{N-1,N} w = f^\star & \text{in } (0, r_v) \\ w = 0 & \text{at } r_v. \end{cases} \tag{2.116}$$

Proof. Assertion *(i)* is granted by Lemma 2.36 and Lemma 2.37. In Equation (2.64), the following identity was proved:

$$\int_0^{r_v} |w_i'(\varrho)|^2 \,\mathrm{d}\mathfrak{m}_{N-1,N} = \int_0^v \left(\frac{F_i(\xi)}{h_{N-1,N}(H_{N-1,N}^{-1}(\xi))} \right)^2 \mathrm{d}\xi, \tag{2.117}$$

where as usual $F_i(\xi) \doteq \int_0^\xi f_i^\sharp(t)\mathrm{d}t$. Notice that for any $\xi \in (0, v)$

$$F_i(\xi)^2 = \left(\int_0^\xi 1 \cdot f_i^\sharp(t)\,\mathrm{d}t \right)^2 \leq \Big(\|1\|_{L^2(0,\xi)} \|f_i^\sharp\|_{L^2(0,\xi)} \Big)^2 \leq \xi \|f_i\|^2_{L^2(B_{R_i}(x_i))} \leq C^2 \xi, \tag{2.118}$$

where $C \doteq \sup_i \|f_i\|_{H^{1,2}\left(B_{R_i}(x_i), \mathsf{d}, \mathfrak{m}_i\right)}$. Thus we have:

$$\|w_i'\|_{L^2((0,r_v),\mathfrak{m}_{N-1,N})} \leq C^2 c_{N-1,N}^2 \int_0^v \frac{\xi\,\mathrm{d}\xi}{\sin^{2N-2}(H_{N-1,N}^{-1}(\xi))}, \tag{2.119}$$

where $c_{N-1,N} > 0$ is the constant appearing in the definition of $h_{N-1,N}$. Since $H_{N-1,N}(\xi)$ is of the same order as $\xi \mapsto \xi^N$ near 0, the integrand at the right hand side is asymptotic to $\xi^{1-\frac{2N-2}{N}} = \xi^{-1+\frac{2}{N}}$ when $\xi \to 0$. In particular, the integral is finite and only depends on N and v: the L^2-norm of w_i' is thus uniformly bounded:

$$\|w_i'\|_{L^2((0,r_v),\mathfrak{m}_{N-1,N})} \leq C^2 c_{N-1,N}^2 \kappa(N, v). \tag{2.120}$$

By the Poincaré inequality, the norms $\|w_i\|_{H^{1,2}([0,r_1),\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{N-1,N})}$ are also uniformly bounded. Using the Talenti-type Theorem 2.14 with the associated gradient comparison (2.44), we infer that the norms

$$\|u_i\|_{H^{1,2}(B_{R_i}(x_i),\mathsf{d},\mathfrak{m}_i)} \tag{2.121}$$

are uniformly bounded as well. Thus, by Proposition 2.34 (and up to subsequences), the u_i's converge in L^2-strong to a function u and the w_i's converge in L^2 strong to a function w; moreover, by Lemma 2.36, $u_i^\star$ converges in L^2 strong to $u^\star$.

In order to prove point *(ii)* (and, analogously, point *(iii)*), we apply [AH18, Corollary 4.3]. To this aim, observe that (up to subsequences) we can assume that u_i converges to u also weakly in $H^{1,2}$ by [AH18, Proposition 3.1] (see also Remark 2.35). Moreover, every $\psi \in H_0^{1,2}(B_R(x), \mathsf{d}, \mathfrak{m})$ can be recovered as the strong $H^{1,2}$ limit of a sequence of functions $\psi_i \in H_0^{1,2}(B_{R_i}(x_i), \mathsf{d}, \mathfrak{m}_i)$ by [AH18, Lemma 2.10]. Therefore we have:

- $\mathrm{Ch}(u_i, \psi_i) = \int_{\mathsf{Y}} f_i \psi_i \,\mathrm{d}\mathfrak{m}_i$ by the definition of u_i as a weak solution of the Poisson problem;
- $\lim_{i\to\infty} \mathrm{Ch}(u_i, \psi_i) = \mathrm{Ch}(u, \psi)$ by [AH18, Corollary 4.3];
- $\lim_{i\to\infty} \int_{\mathsf{Y}} f_i \psi_i \,\mathrm{d}\mathfrak{m}_i = \int_{\mathsf{Y}} f\psi \,\mathrm{d}\mathfrak{m}$ by [GMS15, Equation (6.7)].

In particular,

$$\int_{\mathsf{Y}} f\psi \,\mathrm{d}\mathfrak{m} = \mathrm{Ch}(u,\psi), \tag{2.122}$$

thus u is a weak solution of Equation (2.115). An analogous argument proves statement *(iii)*. □

We finally have the tools to prove a stability result, by considering a contradicting sequence, applying a compactness argument, and exploiting the already proven rigidity result on the limit space.

Theorem 2.39 (Stability in the Talenti-type theorem). *For every $\varepsilon > 0, N \in [2,\infty), v \in (0,1)$ and $0 < c_l \le c_u < \infty$ there exists $\delta = \delta(\varepsilon, N, v, \frac{c_l}{c_u}) > 0$ such that the following statement holds. Assume:*

(i) $(\mathsf{X},\mathsf{d},\mathfrak{m})$ *is a* $\mathrm{RCD}(N-1,N)$ *metric space with* $\mathfrak{m}(\mathsf{X}) = 1$ *and* $\Omega = B_R(x) \subset \mathsf{X}$ *is an open ball with* $\mathfrak{m}(\Omega) = v$*;*

(ii) $f \in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ *with* $c_l \le \|f\|_{L^2(\Omega,\mathsf{d},\mathfrak{m})} \le \|f\|_{H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})} \le c_u$*;*

(iii) $u \in H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ *weakly solves* $-\Delta u = f$*;*

(iv) $w \in H^{1,2}([0,r_v),\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{N-1,N})$ *weakly solves* $-\Delta_{N-1,N} w = f^\star$*, with* $w(r_v) = 0$ *and* $r_v \doteq H^{-1}_{N-1,N}(v)$*.*

If $\|u^\star - w\|_{L^2((0,r_v),\mathfrak{m}_{N-1,N})} < \delta$*, then there exists a spherical suspension* $(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})$ *such that*

$$\mathsf{d}_{\mathrm{mGH}}((\mathsf{X},\mathsf{d},\mathfrak{m}),(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})) < \varepsilon. \tag{2.123}$$

Proof. We argue by contradiction: assume there exist $\bar\varepsilon, \bar N, \bar v, \bar c$ such that for any $i \in \mathbb{N}$ we can find an $\mathrm{RCD}(\bar N - 1, \bar N)$ space $(\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i)$, a ball $\Omega_i = B_{R_i}(x_i) \subset \mathsf{X}_i$, and functions $f_i \in H^{1,2}(\Omega_i,\mathsf{d}_i,\mathfrak{m}_i)$, $u_i \in H^{1,2}_0(\Omega_i,\mathsf{d}_i,\mathfrak{m}_i)$, $w_i \in H^{1,2}([0,r_v),\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{N-1,N})$ such that for any $i \in \mathbb{N}$

$$-\Delta u_i = f_i \text{ weakly}, \quad -\Delta_{N-1,N} w_i = f_i^\star \text{ weakly, with } w_i(r_v) = 0,$$
$$\bar c \le \|f\|_{L^2(\Omega,\mathsf{d},\mathfrak{m})} \le \|f\|_{H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})} \le 1, \quad \|u_i^\star - w_i\|_{L^2((0,r_v),\mathfrak{m}_{N-1,N})} < \frac{1}{i},$$

and moreover

$$\inf\Big\{\mathsf{d}_{\mathrm{mGH}}((\mathsf{X}_i,\mathsf{d}_i,\mathfrak{m}_i),(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})) \;\Big|\; (\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}}) \text{ is a spherical suspension}\Big\} \ge \bar\varepsilon. \tag{2.124}$$

Up to subsequences, we can assume that:

- $(\mathsf{X}_i,\mathsf{d}_i,x_i,\mathfrak{m}_i)$ converge to an $\mathrm{RCD}(N-1,N)$ space $(\mathsf{X},\mathsf{d},x,\mathfrak{m})$, and Assumption 2.29 is satisfied (see Proposition 1.79); moreover, $\mathfrak{m}(B_R(x)) = v$ by Lemma 2.37;
- f_i, u_i and w_i satisfy the conclusions of Lemma 2.38: that is, f_i converges in L^2-strong to a function $f \in L^2(B_R(x))$; $f_i^\star$ converges in L^2-strong to $f^\star$; u_i converges in L^2-strong to a weak solution u of $-\Delta u = f$ in $B_R(x)$ (with zero boundary condition); w_i converges in L^2-strong to a weak solution w of $-\Delta_{N-1,N} w = f^\star$ in $[0,r_v)$ with $w(r_v) = 0$.

Notice that

$$\mathsf{d}_{\mathrm{mGH}}((\mathsf{X},\mathsf{d},\mathfrak{m}),(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})) \geq \bar{\varepsilon} \tag{2.125}$$

for any spherical suspension $(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})$, by Equation (2.124). However, by the L^2-strong convergence of $u_i^\star$ to $u^\star$ (Lemma 2.36) and the L^2-strong convergence of w_i to w, one has

$$\|u^\star - w\|_{L^2([0,r_v),\mathfrak{m}_{N-1,N})} = \lim_{i\to\infty} \|u_i^\star - w_i\|_{L^2([0,r_v),\mathfrak{m}_{N-1,N})} = 0, \tag{2.126}$$

which implies that $u^\star = w$. Moreover, since f is the L^2-strong limit of the f_i's, it has L^2-norm bounded from below by $\bar{c}$, thus it is different from 0 on a non-negligible set. By the rigidity in the Talenti-type comparison (Theorem 2.27), $(\mathsf{X},\mathsf{d},\mathfrak{m})$ needs to be a spherical suspension, contradicting (2.125). □

Corollary 2.40. *For every $\varepsilon > 0, N \in [2,\infty), v \in (0,1), 0 < c_l \leq c_u < \infty$ there exists $\delta_1 = \delta_1(\varepsilon, N, v, \frac{c_l}{c_u}) > 0$ such that the following statement holds. Assume that the conditions (i)-(iv) of Theorem 2.39 hold. If $w(0) - u^\star(0) < \delta_1$, then there exists a spherical suspension $(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})$ such that*

$$\mathsf{d}_{\mathrm{mGH}}((\mathsf{X},\mathsf{d},\mathfrak{m}),(\mathsf{Z},\mathsf{d}_{\mathsf{Z}},\mathfrak{m}_{\mathsf{Z}})) < \varepsilon. \tag{2.127}$$

Proof. By Lemma 2.28, $w - u^\star$ is non-increasing (and non-negative) in $[0, r_v]$. Thus

$$\|u^\star - w\|_{L^2((0,r_v),\mathfrak{m}_{N-1,N})} \leq (w(0) - u^\star(0))\sqrt{v}.$$

The results follows from Theorem 2.39 with $\delta_1 = \frac{\delta}{\sqrt{v}}$. □

2.4 Applications

2.4.1 Improved Sobolev embeddings

In this paragraph, we apply the Talenti-type comparison Theorem 2.14 to we deduce a series of Sobolev-type inequalities in the framework of $\mathrm{RCD}(K,N)$ spaces (compare with [Kes06, Section 3.3] for the Euclidean setting).

Theorem 2.41. *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K > 0$, $N \in (1,\infty)$, with $\mathfrak{m}(\mathsf{X}) = 1$. Let $\Omega \subset \mathsf{X}$ be an open domain with measure $v \doteq \mathfrak{m}(\Omega) \in (0,1)$. Let $u : \Omega \to \mathbb{R}$ be a function in $H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$ and $f \in L^2(\Omega,\mathfrak{m})$. Assume that u is a weak solution to the equation $-\mathcal{L}_{\mathcal{E}}(u) = f$, where $\mathcal{E}$ is an α-uniformly elliptic bilinear form as in Assumption 2.2. Then the following statements hold:*

1. *If $f \in L^p(\Omega,\mathfrak{m})$ with $\frac{N}{2} < p \leq \infty$, then $u \in L^\infty(\Omega,\mathfrak{m})$ and*

$$\begin{gathered}\|u\|_{L^\infty(\Omega,\mathfrak{m})} \leq \frac{c_1(K,N,v,p)}{\alpha}\|f\|_{L^p(\Omega,\mathfrak{m})},\\ \text{with } c_1(K,N,v,p) \doteq \int_0^v \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi < \infty,\end{gathered} \tag{2.128}$$

where we adopt the convention that $\frac{1}{p} = 0$ if $p = \infty$.

2. *If $f \in L^p(\Omega, \mathfrak{m})$ with $2 \le p \le \frac{N}{2}$, and $q \ge 1$ is such that $q\left(\frac{1}{p} - \frac{2}{N}\right) < 1$, then $u \in L^q(\Omega, \mathfrak{m})$ and*

$$\|u\|_{L^q(\Omega,\mathfrak{m})} \le \frac{c_2(K,N,v,p,q)}{\alpha}\|f\|_{L^p(\Omega,\mathfrak{m})}, \tag{2.129}$$
$$\text{with} \quad c_2(K,N,v,p,q) \doteq \left(\int_0^v \left(\int_s^v \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi\right)^q \mathrm{d}s\right)^{\frac{1}{q}} < \infty.$$

Proof. By Theorem 2.14, $u^\sharp$ satisfies the following inequality (see Equation (2.52)):

$$0 \le u^\sharp(s) \le \frac{1}{\alpha}\int_s^{\mathfrak{m}(\Omega)} \frac{1}{\mathcal{I}_{K,N}(\xi)^2}\int_0^\xi f^\sharp(t)\,\mathrm{d}t\,\mathrm{d}\xi, \qquad \forall s \in (0, \mathfrak{m}(\Omega)). \tag{2.130}$$

If $f \in L^p(\Omega, \mathfrak{m})$ for some $p \in [2, \infty]$, then by Hölder inequality, by equimeasurability of f and $f^\sharp$ (Proposition 1.57), and by the characterization of the isoperimetric profile on $J_{K,N}$ (Proposition 1.50 and Remark 1.76),

$$u^\sharp(s) \le \frac{1}{\alpha}\|f\|_{L^p(\Omega,\mathfrak{m})}\int_s^{\mathfrak{m}(\Omega)} \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi, \tag{2.131}$$

with the convention that $\frac{1}{p} = 0$ if $p = \infty$. Now by the estimates on $h_{K,N}$ (Lemma 1.74) there exist constants $C_0 > 0$ and $C_1 > 0$ only depending on $K > 0$, $N \in (1, \infty)$ and $v = \mathfrak{m}(\Omega) \in (0, 1)$ such that for all $\xi \in [0, \mathfrak{m}(\Omega)]$

$$h_{K,N}(H_{K,N}^{-1}(\xi)) \ge C_0(H_{K,N}^{-1}(\xi))^{N-1} \ge C_1\xi^{\frac{N-1}{N}}. \tag{2.132}$$

We can thus draw the following conclusions:

Case 1: If $\frac{N}{2} < p \le \infty$, then

$$\int_s^{\mathfrak{m}(\Omega)} \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi \le \frac{1}{C_1^2}\int_0^{\mathfrak{m}(\Omega)} \xi^{\frac{2}{N}-\frac{1}{p}-1}\,\mathrm{d}\xi = \frac{v^{\frac{2}{N}-\frac{1}{p}}}{\left(\frac{2}{N}-\frac{1}{p}\right)C_1^2} < \infty, \quad \forall s \in [0, v]. \tag{2.133}$$

By Equation (2.131) and by equimeasurability of u and $u^\sharp$, this implies Equation (2.128).

Case 2: If $p = \frac{N}{2}$ and $q \ge 1$, then

$$\left(\int_s^{\mathfrak{m}(\Omega)} \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi\right)^q \le \frac{1}{C_1^{2q}}(\log v - \log s)^q \tag{2.134}$$

and thus

$$\|u\|_{L^q(\Omega,\mathfrak{m})}^q = \|u^\sharp\|_{L^q((0,v))}^q \le \frac{\|f\|_{L^p(\Omega,\mathfrak{m})}^q}{\alpha^q}\int_0^v\left(\int_s^v \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi\right)^q \mathrm{d}s \tag{2.135}$$
$$\le \frac{\|f\|_{L^p(\Omega,\mathfrak{m})}^q}{C_1^{2q}\alpha^q}\int_0^1 (-\log s)^q\,\mathrm{d}s < \infty.$$

CASE 3: If $2 \le p < \frac{N}{2}$ and $q \ge 1$, with $q\left(\frac{1}{p} - \frac{2}{N}\right) < 1$, then

$$\left(\int_s^{\mathfrak{m}(\Omega)} \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi\right)^q \le \frac{1}{C_1^{2q}\left(\frac{1}{p} - \frac{2}{N}\right)^q}\left(s^{\frac{2}{N}-\frac{1}{p}} - v^{\frac{2}{N}-\frac{1}{p}}\right)^q \tag{2.136}$$

and thus

$$\begin{aligned} \|u\|^q_{L^q(\Omega,\mathfrak{m})} = \|u^\sharp\|^q_{L^q((0,v))} &\le \frac{\|f\|^q_{L^p(\Omega,\mathfrak{m})}}{\alpha^q}\int_0^v\left(\int_s^v \frac{\xi^{1-\frac{1}{p}}}{h_{K,N}(H_{K,N}^{-1}(\xi))^2}\,\mathrm{d}\xi\right)^q \mathrm{d}s \\ &\le \frac{\|f\|^q_{L^p(\Omega,\mathfrak{m})}}{C_1^{2q}\alpha^q\left(\frac{1}{p} - \frac{2}{N}\right)^q}\int_0^v \left(s^{\frac{2}{N}-\frac{1}{p}} - v^{\frac{2}{N}-\frac{1}{p}}\right)^q \mathrm{d}s < \infty. \end{aligned} \tag{2.137}$$

Let now $2 \le p \le \infty$. If we define $D_{\Omega,p}(\mathcal{L}_\mathcal{E})$ to be the space

$$D_{\Omega,p}(\mathcal{L}_\mathcal{E}) \doteq \{u \in D_\Omega(\mathcal{L}_\mathcal{E}) \mid \mathcal{L}_\mathcal{E}(u) \in L^p(\Omega,\mathfrak{m})\}, \tag{2.138}$$

where $D_\Omega(\mathcal{L}_\mathcal{E})$ is the space defined in Definition 2.8, then Theorem 2.41 can be restated as follows:

Corollary 2.42 (Improved Sobolev embeddings). *Let $(\mathsf{X},\mathsf{d},\mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K > 0$, $N \in (1,\infty)$, with $\mathfrak{m}(\mathsf{X}) = 1$. Let $\Omega \subset \mathsf{X}$ be an open domain with measure $v \doteq \mathfrak{m}(\Omega) \in (0,1)$ and $u : \Omega \to \mathbb{R}$ be a function in $H^{1,2}(\Omega,\mathsf{d},\mathfrak{m})$. Let $\mathcal{E}$ be a bilinear form satisfying Assumption 2.2 with uniform ellipticity parameter α.*

(a) If $\frac{N}{2} < p \le \infty$, then $H^{1,2}(\Omega,\mathsf{d},\mathfrak{m}) \cap D_{\Omega,p}(\mathcal{L}_\mathcal{E}) \subset L^\infty(\Omega)$ with

$$\|u\|_{L^\infty(\Omega,\mathfrak{m})} \le C(K,N,v,p,\alpha)\|\mathcal{L}_\mathcal{E}(u)\|_{L^p(\Omega,\mathfrak{m})}. \tag{2.139}$$

(b) If $2 \le p \le \frac{N}{2}$ and $1 \le q < \left(\frac{1}{p} - \frac{2}{N}\right)^{-1}$, then $H^{1,2}(\Omega,\mathsf{d},\mathfrak{m}) \cap D_{\Omega,p}(\mathcal{L}_\mathcal{E}) \subset L^q(\Omega)$ with

$$\|u\|_{L^q(\Omega,\mathfrak{m})} \le C(K,N,v,p,q,\alpha)\|\mathcal{L}_\mathcal{E}(u)\|_{L^p(\Omega,\mathfrak{m})}. \tag{2.140}$$

2.4.2 An alternative proof for the RCD(*K*,*N*) version of Rayleigh-Faber-Krahn-Bérard-Meyer comparison theorem

A classical application of the theory of symmetrizations is the proof of the conjecture proposed by Lord Rayleigh: the principal frequency of vibration of a membrane of given area is minimal when the shape of the membrane is a disk. In the Euclidean setting, this was proved by Faber [Fab23] and Krahn [Kra25] using symmetrizations.

An analogous result was proved for the p-Laplacian by Mondino and Semola [MS19] in the general setting of essentially non-branching $\mathrm{CD}(K,N)$ spaces (for $K > 0$), as a consequence of a Pólya-Szegő type inequality.

We give below an alternative proof in case $p = 2$, based instead on Talenti's comparison theorem for RCD spaces.

Firstly, we recall the notions of *first eigenfunction* and *first eigenvalue* of the Laplacian:

Definition 2.43. Let $\Omega \subset \mathsf{X}$ be an open domain. For any non-zero function $w \in H^{1,2}(\Omega, \mathsf{d}, \mathfrak{m})$ we define the Rayleigh quotient to be

$$\mathcal{R}_\Omega(w) \doteq \frac{\int_\Omega |\nabla w|_{\mathrm{w}}^2 \, \mathrm{d}\mathfrak{m}}{\int_\Omega w^2 \, \mathrm{d}\mathfrak{m}}. \tag{2.141}$$

We say that:

(i) $\lambda_\Omega \doteq \inf\left\{\mathcal{R}_\Omega(w) \,\middle|\, w \in H_0^{1,2}(\Omega), w \not\equiv 0\right\}$ is the first eigenvalue of the Laplacian in Ω with Dirichlet homogeneous conditions;

(ii) $u \in H_0^{1,2}(\Omega)$ is a first eigenfunction of the Laplacian in Ω (with Dirichlet homogeneous conditions) if it minimizes $\mathcal{R}_\Omega$ among functions $w \in H_0^{1,2}(\Omega)$, $w \not\equiv 0$ (that is, $\mathcal{R}_\Omega(u) = \lambda_\Omega$).

When $(\mathsf{X}, \mathsf{d}, \mathfrak{m}) = (J_{K,N}, \mathsf{d}_{\mathrm{eu}}, \mathfrak{m}_{K,N})$, $v \in (0,1)$, and $\Omega = [0, H_{K,N}^{-1}(v))$, we will denote the first eigenvalue with $\lambda_{K,N,v}$. ♢

Theorem 2.44. *Let $(\mathsf{X}, \mathsf{d}, \mathfrak{m})$ be an $\mathrm{RCD}(K,N)$ space for some $K > 0$, $N \in (1, \infty)$, and let $\Omega \subset \mathsf{X}$ be an open domain with measure $v \doteq \mathfrak{m}(\Omega) \in (0,1)$. Then:*

(i) $\lambda(\Omega) \geq \lambda_{K,N,v}$;

(ii) There exists a unique first eigenfunction of the Laplacian in Ω, up to multiplication by a constant; such an eigenfunction can be chosen to be strictly positive and continuous in Ω;

(iii) If u is a positive first eigenfunction, then $0 < u^\star \leq w$ in $[0, r_v)$, where $r_v \doteq H_{K,N}^{-1}(v)$ and w is a solution to $-\Delta_{K,N} w = \lambda_\Omega u^\star$ in $[0, r_v)$ with $w(r_v) = 0$.

Proof. Step 1: A first eigenfunction exists. This was already proved for example in [MS19, Theorem 4.3], but we recall here the argument: let $\{u_n\}_n$ be a minimizing sequence for $\mathcal{R}_\Omega$ with $u_n \in H_0^{1,2}(\Omega)$, $\|u_n\|_{L^2(\Omega,\mathfrak{m})} = 1$ and $\int_\Omega |\nabla u|_{\mathrm{w}}^2 \, \mathrm{d}\mathfrak{m} \searrow \lambda_\Omega$. Since the embedding $H^{1,2}(\mathsf{X}, \mathsf{d}, \mathfrak{m}) \subset L^2(\mathsf{X}, \mathfrak{m})$ is compact for an $\mathrm{RCD}(K,N)$ space with $K > 0$, $N \in (1, \infty)$ (see [GMS15, Proposition 6.7]), the sequence u_n converges to a function $u \in H_0^{1,2}(\Omega)$ in the strong $L^2(\Omega, \mathfrak{m})$ sense. Thus $\|u\|_{L^2(\Omega,\mathfrak{m}} = 1$; moreover, from the very definition of λ_Ω and by the L^2-lower semicontinuity of the Cheeger energy, it holds that

$$\lambda_\Omega \leq \int_\Omega |\nabla u|_{\mathrm{w}}^2 \, \mathrm{d}\mathfrak{m} \leq \liminf_{n\to\infty} \int_\Omega |\nabla u_n|_{\mathrm{w}}^2 \, \mathrm{d}\mathfrak{m} = \lambda_\Omega. \tag{2.142}$$

Thus u is a first eigenfunction.

Step 2: Any first eigenfunction $u \in H_0^{1,2}(\Omega)$ actually weakly solves

$$\begin{cases} -\Delta u = \lambda_\Omega u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{2.143}$$

This relies on a standard variational argument: for any $w \in H_0^{1,2}(\Omega)$, we can explicitly compute the derivative of $\varepsilon \mapsto \mathcal{R}_\Omega(u + \varepsilon w)$ at $\varepsilon = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{R}_\Omega(u + \varepsilon w) \Big|_{\varepsilon=0} = 2 \frac{\mathrm{Ch}(u,w) - \lambda_\Omega \int_\Omega uw \, \mathrm{d}\mathfrak{m}}{\|u\|_{L^2(\Omega)}}. \tag{2.144}$$

Since this derivative must vanish, the identity $\mathrm{Ch}(u,w) = \lambda_\Omega \int_\Omega uw \,\mathrm{d}\mathfrak{m}$ needs to hold for any $w \in H_0^{1,2}(\Omega)$, which proves that u weakly solves Equation (2.143).

Step 3: Since $\mathrm{RCD}(K,N)$ spaces are locally doubling and Poincaré (see [Stu06b; Raj12]), the Harnack-type results proved in [LMP06] hold in these spaces: hence any first eigenfunction is continuous (Theorem 5.1 therein), and is strictly positive in Ω up to multiplying by a constant (Corollary 5.7 and Corollary 5.8 therein). The same results also imply the uniqueness of the first eigenfunction up to a multiplicative constant: if u_1 and u_2 are two first eigenfunctions with $\frac{u_1}{u_2}$ non-constant, then there exists $\gamma > 0$ such that $u_1 - \gamma u_2$ is a first eigenfunction that changes sign in Ω.

Step 4: Let now w be a solution to $-\Delta_{K,N} w = \lambda_\Omega u^\star$ in $[0, r_v)$ with $w(r_v) = 0$. By the definition of w and by using w itself as a test function, it holds that $\int_0^{r_v} |\nabla w|_{\mathrm{w}}^2 \,\mathrm{d}\mathfrak{m}_{K,N} = \lambda_\Omega \int_0^{r_v} u^\star w \,\mathrm{d}\mathfrak{m}_{K,N}$; by the Talenti-type theorem it holds that $0 < u^\star \leq w$. Thus

$$\lambda_{K,N,v} \leq \frac{\int_0^{r_v} |\nabla w|_{\mathrm{w}}^2 \,\mathrm{d}\mathfrak{m}_{K,N}}{\int_0^{r_v} w^2 \,\mathrm{d}\mathfrak{m}_{K,N}} = \lambda_\Omega \frac{\int_0^{r_v} u^\star w \,\mathrm{d}\mathfrak{m}_{K,N}}{\int_0^{r_v} w^2 \,\mathrm{d}\mathfrak{m}_{K,N}} \leq \lambda_\Omega \frac{\int_0^{r_v} w^2 \,\mathrm{d}\mathfrak{m}_{K,N}}{\int_0^{r_v} w^2 \,\mathrm{d}\mathfrak{m}_{K,N}} = \lambda_\Omega. \tag{2.145}$$ □

Chapter 3

Minimizers of the area functional in the first Heisenberg group

In this chapter, we prove a version of the Bernstein Theorem for stable intrinsic graphs in $\mathbb{H}^1$ under an assumption on the regularity which is lower than Lipschitz – thus more general than the version currently available. Such result is based on a joint work with Sebastiano Nicolussi Golo and Francesco Serra Cassano, still unpublished at the moment of the writing of this thesis; and relies on a work by Ambrosio, Nicolussi Golo and Serra Cassano ([ANS], still unpublished itself) on the Sobolev regularity of the flow of an ordinary differential equation under suitable assumptions on the underlying vector field.

Around the year 1915, Sergej Bernstein proved that any solution to the planar minimal surface equation defined on the whole plane

$$\partial_x \frac{\partial_x \psi}{\sqrt{1+|\nabla\psi|^2}} + \partial_y \frac{\partial_y \psi}{\sqrt{1+|\nabla\psi|^2}} = 0 \tag{3.1}$$

is necessarily affine [Ber27]. As the techniques proposed by Bernstein are specifically dependent on the dimension 2 of the domain, this raised the problem of determining if such statement holds in higher dimension: precisely, to understand whether solutions to the minimal surface equation defined on the whole $\mathbb{R}^n$ have hyperplanes as graphs.

A complete solution to this issue came after several contributions between the 1920s and the 1960s, and is embodied in the following theorem:

Theorem 3.1 (Euclidean Bernstein Theorem for graphs). *Let $n \geq 2$. The following statements hold:*

(i) If $n \leq 8$, then any solution $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ to the (classical) minimal surface equation

$$\operatorname{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = 0 \tag{3.2}$$

is affine.

(ii) If $n \geq 9$, then there exist analytic solutions $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ to the (classical) minimal surface equation which are not affine.

The problem for maps with graph in $\mathbb{H}^1$ needs to be reformulated in order to take care of the specific structure of the space: in this chapter, we will focus on maps defined on a *vertical* plane $\mathbb{W}$ of $\mathbb{H}^1$ and taking values in a *horizontal* line $\mathbb{V}$ complementary to the vertical plane.

In this new setting, the area functional for smooth functions $f \in \mathbf{C}^2(\mathbb{W}, \mathbb{V})$ takes the form

$$\mathcal{A}(f; \mathcal{U}) \doteq \int_{\mathcal{U}} \sqrt{1 + |\nabla^f f|^2} \, \mathrm{d}\mathscr{L}^2 \tag{3.3}$$

where $\nabla^f = \partial_y + f\partial_t$, so that the minimal surface equation derived from the first variation of $\mathcal{A}$ has the form

$$\nabla^f \left(\frac{\nabla^f f}{\sqrt{1 + |\nabla^f f|^2}} \right) = 0. \tag{3.4}$$

Unfortunately, the lack of convexity of the area functional in $\mathbb{H}^1$ implies that solutions to Equation (3.4) are not automatically minimizers of the area, and there are explicit examples of smooth solutions which are not affine: this suggests that the *second variation* of $\mathcal{A}$ should be considered as well. We will say that a map f (not necessarily smooth) is *stable* if $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathcal{A}(f + \varepsilon\varphi; \mathcal{U}) = 0$ and $\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\mathcal{A}(f + \varepsilon\varphi; \mathcal{U}) \geq 0$ for any bounded domain $\mathcal{U}$ and any test function $\varphi \in \mathbf{C}^\infty_{\mathrm{c}}(\mathcal{U})$.

The question that interests us is thus the following:

(3.5) Are all stable maps f from $\mathbb{W}$ to $\mathbb{V}$ affine?

As we will see in Section 3.1.2, this question admits different answers based on the regularity we allow f to have; our Theorem 3.17 gives a positive answer under a condition weaker than Lipschitz. A first positive answer was given in [BSV07] for $\mathbf{C}^2$ functions, by showing that the minimal surface equation can be rephrased in terms of a *double Burgers' operator*, by studying the new equation through its characteristic curves, and by selecting among the solutions those which are also stable. An extension to $\mathbf{C}^1$ functions was proved in [GR15, Corollary 5.2] in a more general framework. The same technique as in [BSV07], suitably adapted to compensate the lack of regularity, has been used in [NS19] for the Lipschitz case, and will also be used in this chapter.

3.1 The first Heisenberg group

Particularizing the definitions of Section 1.4.1 to the case where $n = 1$, the first Heisenberg group $\mathbb{H}^1$ can be represented as the group structure obtained by endowing $\mathbb{R}^3$ with the (non-commutative) operation

$$(x, y, t) \cdot (x', y', t') \doteq \left(x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y) \right), \qquad \text{for all } (x, y, t), (x', y', t') \in \mathbb{R}^3. \tag{3.6}$$

Definition 3.2 (Vertical planes). In this context, we say that $\Gamma \subset \mathbb{H}^1$ is a *vertical plane* if it is of the form

$$\Gamma = \{(x, y, t) \mid x = ay + b\} \qquad \text{for some } a, b \in \mathbb{R}. \tag{3.7}$$ ◊

Notation 3.3. In the whole chapter, we will denote by $\mathbb{W}$ and $\mathbb{V}$ the following *complementary* and *homogeneous* subgroups of $\mathbb{H}^1$:

$$\mathbb{W} \doteq \left\{(0,y,t) \in \mathbb{H}^1 \,\middle|\, (y,t) \in \mathbb{R}^2\right\}; \qquad \mathbb{V} \doteq \left\{(x,0,0) \in \mathbb{H}^1 \,\middle|\, x \in \mathbb{R}\right\}. \tag{3.8}$$

Here by complementary we mean that $\mathbb{H}^1$ can be decomposed as $\mathbb{W} \cdot \mathbb{V}$, where $\cdot$ is the group operation of $\mathbb{H}^1$. Homogeneous means that $\delta_\lambda(\mathbb{W}) \subset \mathbb{W}$ and $\delta_\lambda(\mathbb{V}) \subset \mathbb{V}$ for any $\lambda > 0$, where δ_λ are the dilations defined in Definition 1.90. $\Diamond$

In general, if $\omega \subset \mathbb{W}$ is open and $g : \omega \to \mathbb{V}$ is a map, the intrinsic graph of g is given by

$$\Gamma_g \doteq \{(0,y,t) \cdot g(0,y,t) \mid (0,y,t) \in \omega\}. \tag{3.9}$$

from now on, however, we will prefer to identify the *vertical plane* $\mathbb{W}$ with $\mathbb{R}^2 = \mathbb{R}_y \times \mathbb{R}_t$ and its complementary $\mathbb{V}$ with $\mathbb{R}$; thus we have:

Definition 3.4 (Intrinsic graphs and subgraphs in $\mathbb{H}^1$)**.** For a function $f : \omega \to \mathbb{R}$, $\omega \subset \mathbb{R}^2$, the *intrinsic graph* is defined as the subset

$$\begin{aligned}\Gamma_f &\doteq \left\{(0,y,t) \cdot (f(y,t),0,0) \,\middle|\, (y,t) \in \omega \subset \mathbb{R}^2\right\} = \\ &= \left\{\left(f(y,t), y, t - \frac{1}{2} y f(y,t)\right) \,\middle|\, (y,t) \in \omega \subset \mathbb{R}^2\right\}\end{aligned} \tag{3.10}$$

of $\mathbb{H}^1$.

The *intrinsic subgraph* is defined as the subset

$$E_f \doteq \left\{(0,y,t) \cdot (x,0,0) \,\middle|\, (y,t) \in \omega \subset \mathbb{R}^2, x < f(y,t)\right\} \tag{3.11}$$

of $\mathbb{H}^1$. $\Diamond$

***Remark* 3.5.** Let us notice explicitly the easy fact that the intrinsic graph of a function f is a vertical plane as in Definition 3.2 if and only if f has the form $f(y,t) = ay + b$ for some $a, b \in \mathbb{R}$ (and if and only if E_f is a *vertical halfspace*). $\Diamond$

As in the classical Bernstein problem, we are interested in functions whose subgraph has locally minimal $\mathbb{H}$-perimeter (see Definition 1.86 and Notation 1.92):

Definition 3.6. Let $\omega \subset \mathbb{R}^2$ and $f : \omega \to \mathbb{R}$. We say that f is $\mathbb{H}$-*perimeter minimizing* in the intrinsic cylinder $\omega \cdot \mathbb{R}$ if its intrinsic subgraph E_f is a minimizer of the perimeter in Ω: *i.e.*,

$$\mathrm{Per}_{\mathbb{H}}(E_f, \tilde{\Omega}) \leq \mathrm{Per}_{\mathbb{H}}(T; \tilde{\Omega}) \tag{3.12}$$

for any open set $\tilde{\Omega}$ compactly contained in $\omega \cdot \mathbb{R}$ and any measurable $T \subset \omega \cdot \mathbb{R}$ such that $T \triangle E_f \subset \tilde{\Omega}$. $\Diamond$

3.1.1 Area functional, stationariety and stability

Next, we introduce one of the most fundamental notions of this chapter, the intrinsic gradient of a function defined on the subspace $\mathbb{W}$: it will appear in the characterization of $\mathbb{H}$-regular intrinsic graphs (Theorem 3.9), as well as in the expression of the area functional.

Definition 3.7. Let ω be an open subset of $\mathbb{R}^2 = \mathbb{R}_y \times \mathbb{R}_t$.

1. The intrinsic gradient operator $f \mapsto \nabla^f f$ maps any function $f \in L^1_{\mathrm{loc}}(\omega)$ to the distribution defined by

$$\langle \nabla^f f, \psi \rangle \doteq \int_\omega \left(f \partial_y \psi + \frac{1}{2} f^2 \partial_t \psi \right) \mathrm{d}\mathscr{L}^2(y,t) \qquad \text{for all } \psi \in \mathbf{C}^\infty_{\mathrm{c}}(\omega). \tag{3.13}$$

2. Moreover, whenever $f \in L^\infty_{\mathrm{loc}}$, we define $\nabla^f : W^{1,1}_{\mathrm{loc}}(\omega) \to L^1_{\mathrm{loc}}(\omega)$ as the first order linear differential operator defined by

$$\nabla^f \varphi \doteq \partial_y \varphi + f \partial_t \varphi \qquad \text{for all } \varphi \in W^{1,1}_{\mathrm{loc}}(\omega). \tag{3.14}$$

3. In a completely equivalent way to Equation (3.14), when $f \in \mathbf{C}^0(\omega)$ we will often implicitly use the notation ∇^f to indicate the continuous vector field $\nabla^f : \omega \to \mathbb{R}^2$ defined on ω by $\nabla^f(y,t) = (1, f(y,t))$.

It is clear that whenever $f \in W^{1,1}_{\mathrm{loc}}(\omega) \cap L^\infty_{\mathrm{loc}}(\omega)$, the distribution $\nabla^f f$ defined in Equation (3.13) is represented by the L^1_{loc} function $\nabla^f f$ defined in Equation (3.14), so there is no ambiguity in this double definition. ◊

***Remark* 3.8.** Despite them being related by the next Theorem 3.9, the intrinsic gradient just defined is an object of substantially different nature from the horizontal gradient in $\mathbb{H}^1$ as defined in Definition 1.95: while the latter is defined for functions on open domains of the whole $\mathbb{H}^1$, the former acts on maps defined on open domains of $\mathbb{W} \simeq \mathbb{R}^2$. ◊

Exploiting the intrinsic gradient defined above, one can characterize $\mathbb{H}$-regular surfaces defined by intrinsic graphs, in analogy with the Euclidean case. Indeed, the following theorem can be obtained combining [ASV06] and [BS10, Theorem 1.2] in the case $n = 1$:

Theorem 3.9. *Let $\omega \subset \mathbb{R}^2$ be open, and let $f : \omega \to \mathbb{R}$ be a continuous function. Let Γ_f be the intrinsic graph of f. Then the following are equivalent:*

(a) Γ_f is an $\mathbb{H}$-regular surface, and $\nu^1_{\Gamma_f}(p) < 0$ for all $p \in S$, where $\nu_{\Gamma_f} = \left(\nu^1_{\Gamma_f}, \nu^2_{\Gamma_f}\right)$ is the horizontal normal to ν_{Γ_f}.

(b) The distribution $\nabla^f f$ defined in Equation (3.13) is represented by a continuous function.

We can now give the following definition (which, again, differs substantially from the definition of $\mathbf{C}^1_{\mathbb{H}}$ given in Definition 1.95):

Definition 3.10 (Intrinsic $\mathbf{C}^1$)**.** Let $\omega \subset \mathbb{W} \simeq \mathbb{R}^2$ be an open set, and let $f : \omega \to \mathbb{R}$. We say that f is *intrisically* $\mathbf{C}^1$, and we write $f \in \mathbf{C}^1_{\mathbb{W}}(\omega)$, whenever one of the equivalent conditions (a), (b) of Theorem 3.9 is satisfied. ◊

For a large class of functions, the $\mathbb{H}$-perimeter of the intrinsic subgraph and the horizontal normal to the intrinsic graph admit a manageable representation, which reminds of the Euclidean counterpart. The following theorem, which deals with the case of $\mathbf{C}^1_{\mathbb{W}}$ functions, was first proved in [ASV06, Theorem 1.2].

Theorem 3.11 (Perimeter of intrisic subgraphs). *Let $\omega \subset \mathbb{R}^2$ be open, and let $f \in \mathbf{C}^1_{\mathbb{W}}(\omega)$ be an intrinsically $\mathbf{C}^1$ function. Let Γ_f and E_f be the intrinsic graph and the subgraph of f, respectively; let $F : \omega \to E_f$ be defined as $f(A) := A \cdot f(A)$ for all $A \in \omega$. Then E_f is a set of locally finite $\mathbb{H}$-perimeter in $\omega \cdot \mathbb{R}$ and we have:*

$$\nu_{E_f}(p) = \left(-\frac{1}{\sqrt{1+|\nabla^f f|^2}}, \frac{\nabla^f f}{\sqrt{1+|\nabla^f f|^2}}\right)\left(F^{-1}(p)\right) \text{ for all } p \in \Gamma_f \tag{3.15}$$

$$\mathrm{Per}_{\mathbb{H}}(E_f; \omega \cdot \mathbb{R}) = \int_\omega \sqrt{1+|\nabla^f f|^2}\, \mathrm{d}\mathscr{L}^2 \tag{3.16}$$

where $\mathrm{Per}_{\mathbb{H}}(\cdot; \omega \cdot \mathbb{R})$ is the subriemannian perimeter of $\mathbb{H}^1$ (see Notation 1.92) in the cylinder $\omega \cdot \mathbb{R}$.

In [MSV08, Definition 3.1], the authors present two classes of "intrinsic Sobolev" functions $W^{1,1}_{\mathbb{W},\mathrm{loc}}(\omega)$ and $W^{1,1}_{\mathbb{W},T,\mathrm{loc}}(\omega)$, satisfying the inclusions

$$W^{1,1}_{\mathbb{W},T,\mathrm{loc}}(\omega) \subset W^{1,1}_{\mathbb{W},\mathrm{loc}}(\omega) \subset L^2_{\mathrm{loc}}(\omega), \tag{3.17}$$

and such that:

- For any function in the (larger) class $W^{1,1}_{\mathbb{W},\mathrm{loc}}(\omega)$, the perimeter formula Equation (3.16) still holds in any $\mathcal{U}$ compactly contained in ω [MSV08, Theorem 3.4].
- For any function in the (smaller) class $W^{1,1}_{\mathbb{W},T,\mathrm{loc}}(\omega)$, a weak formulation for the first and second variation of the *area functional* is available [MSV08, Theorem 3.5].

Moreover, in the same paper, the further inclusion $W^{1,1}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega) \subset W^{1,1}_{\mathbb{W},T,\mathrm{loc}}(\omega)$ is proved [MSV08, Proposition 3.6]. Since this is the class of functions we need, we can finally give the following definitions and results.

Definition 3.12. Let $\omega \subset \mathbb{R}^2$ be an open set, and $f : \omega \to \mathbb{R}$ be a $W^{1,1}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$ function. For any bounded measurable subset $\mathcal{U} \subset \omega$, we define the area of f in $\mathcal{U}$ as

$$\mathcal{A}(f; \mathcal{U}) \doteq \int_{\mathcal{U}} \sqrt{1+|\nabla^f f|^2}\, \mathrm{d}\mathscr{L}^2, \tag{3.18}$$

where $\nabla^f f$ is the L^1_{loc} function defined in Definition 3.7. ◊

Proposition 3.13. *Let $\omega \subset \mathbb{R}^2$ be open, and let $f \in W^{1,1}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$. Then for any $\mathcal{U} \Subset \omega$ the perimeter of the subgraph E_f in $\mathcal{U} \cdot \mathbb{R}$ is represented by the formula*

$$\mathrm{Per}_{\mathbb{H}}(E_f; \mathcal{U} \cdot \mathbb{R}) = \mathcal{A}(f; \mathcal{U}) = \int_{\mathcal{U}} \sqrt{1+|\nabla^f f|^2}\, \mathrm{d}\mathscr{L}^2. \tag{3.19}$$

Moreover, if f is $\mathbb{H}$-perimeter minimizing, then for any $\psi \in \mathbf{C}^\infty_c(\omega)$

$$\mathrm{IV}_f(\varphi) \doteq \int_\omega \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}}\left(\nabla^f \varphi + (\partial_t f)\varphi\right) \mathrm{d}\mathscr{L}^2 = 0 \tag{3.20}$$

and

$$\mathrm{I\!IV}_f(\varphi) \doteq \int_\omega \left[\frac{\left(\nabla^f\varphi + (\partial_t f)\varphi\right)^2}{(1+(\nabla^f f)^2)^{\frac{3}{2}}} + \frac{(\nabla^f f)\partial_t(\varphi^2)}{(1+(\nabla^f f)^2)^{\frac{1}{2}}} \right] \mathrm{d}\mathscr{L}^2 \geq 0 \tag{3.21}$$

hold.

Finally, if the Sobolev regularity of f is better than $W^{1,1}_{\mathrm{loc}}$, we can enlarge the class of test functions we can consider in the first and second variation formulas:

Lemma 3.14. *Let $\omega \subset \mathbb{R}^2$ be open, and let $f \in W^{1,q}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$ with $q \in (2,\infty)$ be a $\mathbb{H}$-perimeter minimizer. Then the formulas Equation (3.20) and Equation (3.21) hold for any $\mathcal{U} \Subset \omega$ and any $\varphi \in W^{1,2}_0(\mathcal{U})$.*

Proof. Let us fix $\mathcal{U} \Subset \omega$ and $\varphi \in W^{1,2}_0(\mathcal{U})$. Let $\{\varphi_j\}_{j\in\mathbb{N}} \subset \mathbf{C}^\infty_{\mathrm{c}}(\mathcal{U})$ be a sequence of smooth functions such that $\varphi_j \to \varphi$ in $W^{1,2}(\mathcal{U})$.

Then we have, by using the Hölder inequality, that:

$$\left\|\nabla^f\varphi_j - \nabla^f\varphi\right\|_{L^2(\mathcal{U})} \leq \|\partial_y(\varphi_j-\varphi)\|_{L^2(\mathcal{U})} + \|f\|_{L^\infty(\mathcal{U})}\|\partial_t(\varphi_j-\varphi)\|_{L^2(\mathcal{U})} \xrightarrow{j\to\infty} 0 \tag{3.22}$$

and

$$\|(\partial_t f)(\varphi_j-\varphi)\|_{L^1(\mathcal{U})} \leq \|\partial_t f\|_{L^q(\mathcal{U})}\|\varphi_j-\varphi\|_{L^2(\omega)} \xrightarrow{j\to\infty} 0, \tag{3.23}$$

so that $\mathrm{IV}_f(\varphi_j) \to \mathrm{IV}_f(\varphi)$ as $j\to\infty$; notice indeed that the quotient $\frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}}$ is bounded. In particular, Equation (3.20) holds for φ.

Moreover, $\mathrm{I\!IV}_f(\varphi)$ can be written as

$$\begin{aligned}\mathrm{I\!IV}_f(\varphi) = \int_\omega \Bigg[&\frac{1}{(1+(\nabla^f f)^2)^{\frac{3}{2}}}(\nabla^f\varphi)^2 + 2\frac{\partial_t f}{(1+(\nabla^f f)^2)^{\frac{3}{2}}}(\varphi\nabla^f\varphi) + \\ &+ \frac{(\partial_t f)^2}{(1+(\nabla^f f)^2)^{\frac{3}{2}}}\varphi^2 + \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}}\varphi\partial_t\varphi \Bigg] \mathrm{d}\mathscr{L}^2,\end{aligned} \tag{3.24}$$

where the following relations hold:

- $(\nabla^f\varphi_j)^2 \xrightarrow{j\to\infty} (\nabla^f\varphi)^2$ in $L^1(\mathcal{U})$, again by Equation (3.22);
- $(\partial_t f)\varphi_j\nabla^f\varphi_j \xrightarrow{j\to\infty} (\partial_t f)\varphi\nabla^f\varphi$ in $L^1(\mathcal{U})$, by the Morrey's inequality (which ensures that $\varphi_j \to \varphi$ in any $L^r(\mathcal{U})$ space with $r \in [1,\infty)$, see [Bre11, Corollary 9.11]), the Hölder inequality and the Dominated Convergence Theorem;
- $(\partial_t f)^2\varphi_j^2 \xrightarrow{j\to\infty} (\partial_t f)^2\varphi^2$ in $L^1(\mathcal{U})$, by the same argument as the previous point;
- $\varphi_j\partial_t\varphi_j \xrightarrow{j\to\infty} \varphi\partial_t\varphi$ in $L^1(\mathcal{U})$, by the $W^{1,2}$ strong convergence of φ_j to φ.

In particular, $\mathrm{I\!IV}_f(\varphi_j) \to \mathrm{I\!IV}_f(\varphi)$ as $j\to\infty$. $\square$

The previous result makes the following definition possible:

Definition 3.15. Let $\omega \subset \mathbb{R}^2$ be an open set, and let $f : \omega \to \mathbb{R}$ belong to $W^{1,q}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$ for some $q > 2$. We say that:

- f is **stationary** in ω if

$$\int_{\mathcal{U}} \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}} \Big(\nabla^f \varphi + (\partial_t f)\varphi\Big)\, \mathrm{d}\mathscr{L}^2 = 0 \tag{3.25 | 1VF}$$

  for any $\mathcal{U} \Subset \omega$ and any $\varphi \in W^{1,2}_0(\mathcal{U})$; sometimes we will also say that f satisfies the *weak minimal surface equation for intrinsic graphs.*

- f is **stable** in ω if it is stationary and

$$\int_{\mathcal{U}} \left[\frac{\Big(\nabla^f \varphi + (\partial_t f)\varphi\Big)^2}{(1+(\nabla^f f)^2)^{\frac{3}{2}}} + \frac{(\nabla^f f)\partial_t(\varphi^2)}{(1+(\nabla^f f)^2)^{\frac{1}{2}}} \right] \mathrm{d}\mathscr{L}^2 \geq 0 \tag{3.26 | 2VF}$$

  for any $\mathcal{U} \Subset \omega$ and any $\varphi \in W^{1,2}_0(\mathcal{U})$.

Thanks to Lemma 3.14, whenever f is $\mathbb{H}$-perimeter minimizing, it is both stationary and stable. ♢

***Remark* 3.16** (Case $n \geq 2$)**.** Let us point out that the definitions and the results introduced in this paragraph can be extended to the higher-dimensional Heisenberg groups $\mathbb{H}^n$ with $n \geq 2$: in that case, the operator ∇^f is replaced by a family of $2n-1$ differential operators; then one can find a characterization of maps from $\mathbb{R}^{2n}$ to $\mathbb{R}$ whose intrinsic graph is a $\mathbb{H}$-regular surface in terms of this new intrinsic operator, as well as a formula for the perimeter and first and second variation formulas (see for example [ASV06; MSV08]). ♢

3.1.2 The Bernstein problem(s) in $\mathbb{H}^1$

In the context of the first Heisenberg group, the expression *Bernstein-type problems* refers to a wide range of questions regarding functions defined on the whole $\mathbb{R}^2$, which take the following form:

$$\begin{gathered} \text{if } f : \mathbb{R}^2 \to \mathbb{R} \text{ belongs to a suitable class of functions } \mathcal{X}(\mathbb{R}^2,\mathbb{R}), \\ \text{and } f \text{ is stationary/stable/}\mathbb{H}\text{-perimeter minimizing,} \\ \text{is it true that the intrinsic graph of } f \text{ is a vertical plane?} \end{gathered} \tag{3.27 | BP}$$

In Section 3.5 we will prove the following result, concerning the case of *stable* functions:

Theorem 3.17. *Let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}) \cap \mathbf{C}^0(\mathbb{R}^2)$ be such that $\exp(|\nabla^f f|) \in L^{\beta}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$ for some $\beta \geq 1$. If f is stable, then its intrinsic graph is a vertical plane.*

***Remark* 3.18** (State of art)**.** To the best of our knowledge, the state of art on the general Bernstein problem (3.27 | BP) at the writing of this thesis is the following:

- Even when restricting to $\mathcal{X}(\mathbb{R}^2,\mathbb{R}) = \mathbf{C}^2(\mathbb{R}^2,\mathbb{R})$, the answer is negative for the case of maps which are only stationary: indeed, in [DGN08] it is proved that the map

$$\varphi_\alpha(y,t) \doteq -\frac{\alpha y t}{1+2\alpha y^2}, \qquad (y,t)\in\mathbb{R}^2 \tag{3.28}$$

  is stationary for any $\alpha > 0$, but it is not stable (nor $\mathbb{H}$-perimeter minimizing) and its intrinsic graph is not a vertical plane. This phenomenon does not appear in the Euclidean setting, where any solution to Equation (3.25 | 1VF) is a minimizer by convexity.

- For stable maps, Nicolussi Golo and Serra Cassano proved in [NS19, Theorem 1.1] that the problem (3.27 | BP) has an affirmative answer when $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^2)$ – thus, for a smaller class than Theorem 3.17 – by generalizing the techniques introduced in [BSV07] for the case of $\mathbf{C}^2$ functions.

- On the other hand, in the same paper they find a counterexample for the conjecture in the Sobolev class $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ with $p\in[1,3)$, and even a counterexample in $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ with $\mathbf{C}^1_{\mathbb{W}}$ regularity (see [NS19, Sections 7 and 8]).

- When the class $\mathcal{X}(\mathbb{R}^2,\mathbb{R})$ is too large, the answer to (3.27 | BP) is negative even for $\mathbb{H}$-perimeter minimizers: indeed, in [MSV08, Section 2] it is shown by a calibration argument that the map

$$\varphi(y,t) \doteq -\operatorname{sgn}(t)\sqrt{|t|}, \qquad (y,t)\in\mathbb{R}^2 \tag{3.29}$$

  is $\mathbb{H}$-perimeter minimizing. On the positive side, in addition to the results that hold for stable maps, Young recently proved in [You21] that any area minimizing intrinsic graph which is ruled by horizontal lines is a vertical plane – here *horizontal lines* mean curves of the type $\{p\cdot(\lambda x,\lambda y,0) \mid \lambda\in\mathbb{R}\}$, with $p\in\mathbb{H}^1$ and $(x,y)\ni\mathbb{R}^2$. This in particular shows that the stable counterexamples of [NS19] are not actually area-minimizers. ◊

***Remark* 3.19** (Other version of the Bernstein problem)**.** Just as in the Euclidean setting, several variations of the above-described problem are possible:

- Instead of looking at intrinsic graphs, one can generalize to the case of complete regular surfaces which minimize the $\mathbb{H}$-perimeter. Even for $\mathbf{C}^2$-regular surfaces in $\mathbb{H}^1\simeq\mathbb{R}^3$ (in the Euclidean sense), the Bernstein conjecture turns out to be false; however, perimeter minimizers are completely classified: a complete, oriented, connected $\mathbf{C}^2$-surface in $\mathbb{H}^1$ minimizes the $\mathbb{H}$-perimeter if and only if it is either a Euclidean plane or congruent to the hyperbolic paraboloid $t = 2xy$ (see [HRR10]). If, in addition, we ask that S is a ($\mathbf{C}^1$) surface such that the *characteristic set*

$$\mathrm{Char}(S) \doteq \left\{ p\in S \;\middle|\; H\mathbb{H}^1_p \subset T_pS \right\} \tag{3.30}$$

  is empty, then S must be a vertical plane (see [GR15]); here $H\mathbb{H}^1$ is the *horizontal subbundle* of $\mathbb{H}^1$, and T_pS is the tangent space to S at p.

- Another possibility is to look at *t-graphs*, *i.e.* graphs of functions defined on the horizontal plane $\{(x,y,0)\in\mathbb{H}^1 \mid (x,y)\in\mathbb{R}^2\}$ ([Che+05; GP02]). In this case, the above-mentioned result for general surfaces provides a smooth counterexample to the Bernstein conjecture: the map $\varphi(x, y) = 2xy$ has a minimizing t-graph without being affine. Moreover, by lowering the regularity, the class of minimizing t-graphs grows considerably: we refer to [Rit09] for several counterexamples with Lipschitz or even $\mathbf{C}^{1,1}$ regularity.

- Finally, let us mention that the Bernstein problem has been partially explored in the higher dimensional Heisenberg groups: for t-graphs, the conjecture is false (see [SV20, Paragraph 4.1]). On the Heisenberg groups $\mathbb{H}^n$ with $n \geq 5$ one can find counterexamples to the problema for intrinsic graphs (*i.e.*, minimal intrinsic graphs which are not vertical planes) by adapting the Euclidean counterexamples in high dimensions (see [BSV07, Section 5.2]); the cases $n = 2, 3, 4$ remain, to our knowledge, open. $\lozenge$

3.1.3 Homeomorphic Lagrangian parametrizations

Consider a stable map f defined on a subset ω of $\mathbb{R}^2$. The strategy in the next sections will be to find a "reparametrizion" $\Psi : \tilde{\omega} \to \omega$ of the domain ω such that horizontal segments in $\tilde{\omega}$ are mapped to integral curves of ∇^f. A thorough introduction to *Lagrangian parametrizations* in this setting can be found for example in [BCS15].

Remark **3.20** (Beware of the notation)**.** For the sake of clarity: in the following paragraphs, the map f will frequently be thought as a "time-dependent vector field" on $\mathbb{R}$; however, there might be some confusion with the sets of coordinates adopted: here the "time variable" is y, the "space variable" is t (in order to be consistent with the usual notation in the Heisenberg group $\mathbb{H}^1$).

When this point of view is adopted, the vector field $\nabla^f : \omega \to \mathbb{R}^2$ plays the role of the time-independent vector field associated to f. $\lozenge$

Notation 3.21. Let $A \subset \mathbb{R}^2$. In what follows, for any $r \in \mathbb{R}$ we define

$$A_{1,r} \doteq \{t \in \mathbb{R} \mid (r, t) \in A\} \quad \text{and} \quad A_{2,r} \doteq \{y \in \mathbb{R} \mid (y, r) \in A\} \tag{3.31}$$

to be the sections of A with r fixed. Moreover, we'll denote by $\pi_1(y, t) \doteq y$ and $\pi_2(y, t) \doteq t$ the projections on the first and second coordinate respectively. $\lozenge$

Definition 3.22 ((Bi-Sobolev) Lagrangian homeomorphism)**.** Let $\omega, \tilde{\omega} \subset \mathbb{R}^2$ be open sets. We say that $\Psi : \tilde{\omega} \to \omega$ is a Lagrangian homeomorphism associated to ∇^f if:

1. Ψ is a homeomorphism;
2. Ψ has the form $\Psi(\upsilon, \tau) = (\upsilon, \chi(\upsilon, \tau))$ with χ continuous and $\chi(\upsilon, \cdot)$ non-decreasing for any υ;
3. For any $\tau \in \mathbb{R}$, the map $\Psi(\cdot, \tau)$ is absolutely continuous on subintervals of $\tilde{\omega}_{2,\tau}$ and satisfies

$$\partial_\upsilon \Psi(\upsilon, \tau) = \nabla^f(\Psi(\upsilon, \tau)) \qquad (\text{or equivalently, } \partial_\upsilon \chi(\upsilon, \tau) = f(\upsilon, \chi(\upsilon, \tau))) \tag{3.32}$$

for a.e. υ.

We will say that Ψ is a locally ($\bar{p}$-)bi-Sobolev Lagrangian homeomorphism if both $\Psi \in W^{1,\bar{p}}_{\text{loc}}(\tilde{\omega}, \omega)$ and $\Psi^{-1} \in W^{1,\bar{p}}_{\text{loc}}(\omega, \tilde{\omega})$ for some $\bar{p} \geq 1$ (see Notation 1.20 for the classical definition of multi-valued Sobolev functions). $\lozenge$

***Remark* 3.23.** Notice that bijectivity and continuity of Ψ are enough to ensure that Ψ^{-1} is also continuous, by Brouwer's Invariance of Domain Theorem, and thus that Ψ is indeed a homeomorphism. $\lozenge$

3.2 Useful properties for bi-Sobolev parametrizations

As a first step towards the Bernstein problem for intrinsic graphs, in the forthcoming Section 3.3 we will analyze the consequences of the weak minimal surface equation for intrinsic graphs Equation (3.25 | 1VF) on a domain $\omega \subset \mathbb{R}^2$, under a Sobolev regularity assumption on f, and assuming the existence of a locally bi-Sobolev Lagrangian parametrization of ω. In particular, we will work under the following assumption.

Assumption 3.24. Let $\omega \subset \mathbb{R}^2$ be an open connected set and $f \in \mathbf{C}^0(\omega; \mathbb{R})$. Assume that the following holds:

- **Sobolev regularity of f:** there exists $\bar{q} > 2$ such that f belongs to the local Sobolev space $W^{1,\bar{q}}_{\mathrm{loc}}(\omega, \mathbb{R})$.
- **Parametrization of ω:** there exists $\bar{p} \geq 1$ such that $\omega \subset \mathbb{R}^2$ admits a locally bi-Sobolev Lagrangian parametrization $\Psi\colon \tilde{\omega} \to \omega$ associated to ∇^f, of the form $\Psi(\upsilon, \tau) = (\upsilon, \chi(\upsilon, \tau))$, with Sobolev exponent $\bar{p} \geq 1$.

The Sobolev regularity of Ψ is equivalent to the condition that $\chi\colon \tilde{\omega} \to \mathbb{R}$ belongs to the Sobolev space $W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega})$. As the argument develops, we will be more precise on the lower bounds we'll need on $\bar{p}$ and $\bar{q}$ and on the relationship between them. $\lozenge$

Since Ψ is a bi-Sobolev homeomorphism, it holds that for any $\bar{\upsilon}$ the function $\chi_{\bar{\upsilon}} \doteq \chi(\bar{\upsilon}, \cdot)$ is invertible. Moreover, for almost every $\bar{\upsilon}$ the map $\chi_{\bar{\upsilon}}$ belongs to $W^{1,p}_{\mathrm{loc}}(\tilde{\omega}_{1,\bar{\upsilon}}))$ as a consequence the Fubini Theorem (see [EG15, Theorem 4.21]). Moreover, Ψ^{-1} can be expressed as

$$\Psi^{-1}(y,t) = \Big(y, \chi_y^{-1}(t)\Big). \tag{3.33}$$

Finally, the $\bar{p}$-Sobolev integrability of Ψ^{-1} is equivalent to the condition that $(y,t) \mapsto \chi_y^{-1}(t)$ belongs to the Sobolev space $W^{1,\bar{p}}_{\mathrm{loc}}(\omega)$.

Notation 3.25. From now on, we will frequently denote by χ_υ the map from $\tilde{\omega}_{1,\upsilon}$ to $\omega_{1,\upsilon}$ defined by $\chi_\upsilon(\tau) \doteq \chi(\upsilon, \tau)$; consistently, χ_υ^{-1} will be the inverse of such map. $\lozenge$

Notation 3.26. In this Chapter, we will sometimes use the symbol $\mathrm{D}\Psi$ to denote the *weak* differential of Ψ: *i.e.*, the function in $L^p(\tilde{\omega}, \mathbb{R}^{2\times 2})$ whose components satisfy the "integration by parts" formula (1.20). By classical results, at the points where Ψ is classically differentiable, $\mathrm{D}\Psi$ coincides with the classical differential $\nabla\Psi$ (see [HK14, Theorem A.15, Corollary A.16]). Moreover, we will soon put ourselves in the condition of having differentiability almost everywhere (see Proposition 3.31, part 3). $\lozenge$

Definition 3.27. Let $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}; \omega)$. We denote by J_Ψ the *Jacobian determinant* of Ψ, *i.e.* the determinant of the weak differential $\mathrm{D}\Psi$. $\lozenge$

3.2.1 Area formula and consequences

First of all, we show that the Sobolev assumption on Ψ guarantees the validity of a suitable form of the so-called *area formula* – a tool that will prove useful to perform change of variables inside integrals.

Definition 3.28 (Lusin (N)-condition). We say that a map $\Phi : \Omega \to \mathbb{R}^n$ defined on an open set $\Omega \subset \mathbb{R}^n$ satisfies the Lusin (N) condition if

$$\text{for each } E \subset \Omega \text{ such that } \mathscr{L}^n(E) = 0, \text{ we have } \mathscr{L}^n(\Phi(E)) = 0. \tag{3.34 | N}$$ ◊

Proposition 3.29 (Lusin condition for Sobolev homeomorphisms). *Let $\omega \subset \mathbb{R}^2$ be open, $f \in \mathbf{C}^0(\omega)$, and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega};\omega)$ be a $\bar{p}$-Sobolev Lagrangian homeomorphism associated to ∇^f. Then Ψ satisfies the Lusin (N) condition.*

Proof. Let $E \subset \tilde{\omega}$ have $\mathscr{L}^2$-measure zero. Recall that Ψ can be written as $\Psi(\upsilon,\tau) = (\upsilon, \chi(\upsilon,\tau))$, with $\chi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega})$.

By the already mentioned classical result [EG15, Theorem 4.21 *(i)*] on Sobolev functions restricted to lines (and by continuity of χ), for almost every $\upsilon \in \pi_1(\tilde{\omega})$ the map $\chi_\upsilon = \chi(\upsilon,\cdot)$ belongs to $W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}_{1,\upsilon})$. By local absolute continuity, χ_υ then satisfies the Lusin (N) condition for almost every υ; moreover, $\mathscr{L}^1(E_{1,\upsilon}) = 0$ for almost every υ. Thus $\mathscr{L}^1(\chi_\upsilon(E_{1,\upsilon})) = 0$ for almost every υ, which in turn by the Fubini-Tonelli Theorem implies that $\mathscr{L}^2(\Psi(E)) = 0$, since the representation

$$\Psi(E) = \{(\upsilon, \chi_\upsilon(\tau)) \mid (\upsilon,\tau) \in E\} = \bigcup_{\upsilon \in \mathbb{R}} \{\upsilon\} \times \chi_\upsilon(E_{1,\upsilon}) \tag{3.35}$$

holds. □

Thanks to the validity of the Lusin (N)-condition, one can prove a particularly manageable version of the Area Formula for Lagrangian homeomorphisms:

Proposition 3.30 (Area formula). *Let $\omega \subset \mathbb{R}^2$ be open, $f \in \mathbf{C}^0(\omega)$, and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega};\omega)$ be a $\bar{p}$-Sobolev Lagrangian homeomorphism associated to ∇^f. Then the area formula*

$$\int_{\tilde{\omega}} (\eta \circ \Psi) \partial_\tau \chi \,\mathrm{d}\upsilon\,\mathrm{d}\tau = \int_\omega \eta \,\mathrm{d}y\,\mathrm{d}t \tag{3.36 | AF}$$

holds for any non-negative (or summable) Borel function $\eta : \omega \to \mathbb{R}$.

Proof. By Proposition 3.29, Ψ satisfies the Lusin (N) condition; by [HK14, Theorem A.35], it holds that for any non-negative Borel function η

$$\int_{\tilde{\omega}} \eta(\Psi(\upsilon,\tau)) |J_\Psi(\upsilon,\tau)| \,\mathrm{d}\upsilon\,\mathrm{d}\tau = \int_\omega \eta(y,t) N(\Psi,\tilde{\omega},(y,t)) \,\mathrm{d}y\,\mathrm{d}t, \tag{3.37}$$

where J_Ψ is the Jacobian determinant and N is the multiplicity function (*i.e.*, the cardinality of $\Psi^{-1}(y,t) \cap \tilde{\omega}$). Since Ψ is a homeomorphism, $N \equiv 1$ in ω; moreover, the Jacobian determinant can be explicitly computed, since

$$\mathrm{D}\Psi(\upsilon,\tau) = \begin{pmatrix} 1 & 0 \\ 0 & \partial_\tau \chi(\upsilon,\tau) \end{pmatrix}, \tag{3.38}$$

with $\partial_\tau\chi$ being almost everywhere greater than or equal to 0 (because $\chi(\upsilon,\cdot) \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}_{1,\upsilon})$ for almost every υ, and it is non-increasing for every υ). This implies the validity of the formula as stated in Equation (3.36 | AF). □

In the following proposition, we state a couple of useful properties of Ψ under Assumption 3.24.

Proposition 3.31 (Properties of Ψ). *Let $\omega \subset \mathbb{R}^2$ be open, $f \in \mathbf{C}^0(\omega)$, and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega};\omega)$ be a locally $\bar{p}$-bi-Sobolev Lagrangian homeomorphism associated to ∇^f. Then:*

1. *If $\bar{p} \geq 1$, then $\partial_\tau\chi(\upsilon,\tau) > 0$ for almost every $(\upsilon,\tau) \in \tilde{\omega}$.*
2. *If $\bar{p} \geq 1$, then $\frac{1}{|\partial_\tau\chi|^{\bar{p}-1}} \in L^1_{\mathrm{loc}}(\tilde{\omega})$.*
3. *If $\bar{p} > 2$, then Ψ is almost everywhere differentiable and the classical differential coincides with the weak one.*
4. *If Ψ is differentiable at $x \in \tilde{\omega}$ and $J_\psi(x) > 0$, then Ψ^{-1} is differentiable at $\Psi(x)$ and $\nabla\Psi^{-1}(\Psi(x)) = \nabla\Psi(x)^{-1}$.*

Proof. As already noted in the proof of Proposition 3.30, $\partial_\tau\chi(\upsilon,\tau) \geq 0$ almost everywhere in $\tilde{\omega}$. Moreover, since Ψ^{-1} is also $\bar{p}$-Sobolev, $\chi_\upsilon^{-1} \in W^{1,\bar{p}}_{\mathrm{loc}}(\omega_{1,\upsilon})$ also holds for almost every υ; thus χ_υ^{-1} satisfies the Lusin (N) condition for every υ out of a negligible set N. For any such υ, let

$$A_\upsilon \doteq \{\tau \in \tilde{\omega}_{1,\upsilon} \mid (\chi_\upsilon)' = 0\}. \tag{3.39}$$

By absolute continuity of χ_υ,

$$\mathscr{L}^1(\chi_\upsilon(A_\upsilon)) = \int_{A_\upsilon} (\chi_\upsilon)' \,\mathrm{d}\mathscr{L}^1 = 0; \tag{3.40}$$

since χ_υ^{-1} satisfies the Lusin (N) condition, $\mathscr{L}^1(A_\upsilon) = 0$; as this holds for almost every υ, the first statement follows.

Concerning the L^1 summability of $|\partial_\tau\chi|^{1-\bar{p}}$, observe that, for any measurable set K compactly contained in $\tilde{\omega}$, the chain of identities

(3.41)
$$\begin{aligned}\int_K \frac{1}{|\partial_\tau\chi(\upsilon,\tau)|^{\bar{p}-1}} \mathrm{d}\mathscr{L}^2(\upsilon,\tau) &= \int_K \frac{|J_\Psi(\upsilon,\tau)|}{|\partial_\tau\chi(\upsilon,\tau)|^{\bar{p}}} \mathrm{d}\mathscr{L}^2(\upsilon,\tau) = \\ &= \int_{\Psi(K)} \frac{1}{|\partial_\tau\chi(\Psi^{-1}(y,t))|^{\bar{p}}} \mathrm{d}\mathscr{L}^2(y,t) = \int_{\Psi(K)} \left|\frac{\mathrm{d}}{\mathrm{d}t}\chi_y^{-1}(t)\right|^{\bar{p}} \mathrm{d}\mathscr{L}^2(y,t)\end{aligned}$$

holds by the Area Formula (3.36 | AF). By the local Sobolev regularity of Ψ^{-1}, the last term is finite.

The last two statements in the proposition, regarding the almost everywhere differentiability of Ψ and the differential of the inverse, are classical results: the first can be found in [EG15, Theorem 6.5], the second one in [HK14, Lemma A.29]. □

3.2.2 Maps of finite distortion and regularity of the composition

This paragraph is dedicated to a sequence of technical intermediate result, which will carry as a consequence the Sobolev regularity of the composition of suitable Sobolev functions, as well as a chain rule formula. The following definitions are borrowed from [HK14, Definition 1.11 and page 91].

Definition 3.32 (q-distortion function)**.** Let $\Omega \subset \mathbb{R}^n$ be an open connected set, and let $\Phi : \Omega \to \mathbb{R}^n$. We say that Φ has finite distortion if $\Phi \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, $J_\Phi \in L^1_{\mathrm{loc}}(\Omega)$, and there exists $K : \Omega \to [1, \infty]$ satisfying

$$\|\mathrm{D}\Phi(x)\|^n \leq K(x) J_\Phi(x) \text{ and } K(x) < \infty \text{ for a.e. } x \in \Omega. \tag{3.42}$$

Here $\|\cdot\|$ represents the operator norm and J_Φ is the Jacobian determinant.
For such a map Φ and a value $q \in [1, \infty)$, we define the q-distortion function as

$$K^\Phi_q(x) \doteq \begin{cases} \dfrac{\|\mathrm{D}\Phi(x)\|^q}{|J_\Phi(x)|} & \text{if } J_\Phi(x) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{3.43}$$

◊

Definition 3.33 (Continuity of composition operator between Sobolev spaces)**.** Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets, $\Phi : \Omega_1 \to \Omega_2$ and $1 \leq p \leq q < \infty$. Let the composition (with Φ) operator be defined as $T_\Phi u \doteq u \circ \Phi$ for any map $u : \Omega \to \mathbb{R}$. We say that T_Φ is continuous from $W^{1,q}_{\mathrm{loc}}(\Omega_2)$ to $W^{1,p}_{\mathrm{loc}}(\Omega_1)$ if

$$\begin{gathered} T_\Phi u \in W^{1,p}_{\mathrm{loc}}(\Omega_1) \text{ whenever } u \in W^{1,q}_{\mathrm{loc}}(\Omega_2), \\ \text{and } \|D[T_\Phi u]\|_{L^p(\Omega_1)} \leq C \|Du\|_{L^q(\Omega_2)}. \end{gathered} \tag{3.44}$$

Analogously, we say that T_Φ is continuous from $W^{1,q}_{\mathrm{loc}}(\Omega_2) \cap \mathbf{C}(\Omega_2)$ to $W^{1,p}_{\mathrm{loc}}(\Omega_1)$ if Equation (3.44) holds whenever $W^{1,q}_{\mathrm{loc}}(\Omega_2) \cap \mathbf{C}(\Omega_2)$. ◊

In our setting, $\Psi : \tilde{\omega} \to \omega$ is a homeomorphism between subsets of $\mathbb{R}^2$, with $\Psi, \Psi^{-1} \in W^{1,\bar{p}}$ and $J_\Psi = \partial_\tau \chi > 0$ almost everywhere. In this case, the following lemma holds; since it is a consequence of a result contained in the unpublished (at the writing of this thesis) [ANS], we give here a short proof based on the same computations.

Lemma 3.34. *Let $\Psi : \tilde{\omega} \to \omega$ be a locally bi-Sobolev homeomorphism with Sobolev exponent $\bar{p} > 2$. Assume that $J_\Psi > 0$ almost everywhere. For any $q > 0$, let $r_{\bar{p}}(q) = r(q)$ be defined as*

$$r_{\bar{p}}(q) \doteq \left(\frac{q}{\bar{p}} + \frac{2}{\bar{p} - 2} \right)^{-1} > 0. \tag{3.45}$$

Then Ψ has finite distortion and $(K^\Psi_q)^r \in L^1_{\mathrm{loc}}(\tilde{\omega})$.

Proof. By our assumptions, we can apply [HK14, Theorem 1.6], which guarantees that Ψ has finite distortion.

Let now $\mathcal{U} \Subset \tilde{\omega}$, and notice that by definition of K^Ψ_1

$$\int_{\mathcal{U}} \left| K^\Psi_q(x) \right|^r \mathrm{d}x = \int_{\mathcal{U}} \|\mathrm{D}\Psi(x)\|^{qr} |J_\Psi(x)|^{-r} \mathrm{d}x; \tag{3.46}$$

we are omitting the dependence of r on q and $\bar{p}$. Let $\alpha \doteq \frac{\bar{p}}{qr}$, and notice that

$$\alpha = 1 + \frac{\bar{p}}{q}\frac{2}{\bar{p}-2} > 1; \tag{3.47}$$

we can thus use α and its Hölder conjugate $\beta \doteq \alpha^\star$ as exponents in the Hölder inequality, so that

$$\int_{\mathcal{U}} \left|K_q^\Psi(x)\right|^r \mathrm{d}x \le \left(\int_{\mathcal{U}} \|\mathrm{D}\Psi(x)\|^{\bar{p}} \mathrm{d}x\right)^{\frac{1}{\alpha}} \left(\int_{\mathcal{U}} |J_\Psi(x)|^{-r\beta} \mathrm{d}x\right)^{\frac{1}{\beta}}. \tag{3.48}$$

The first term at the right hand side is finite, by the assumption that $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}, \omega)$. On the other hand, by the Area Formula (Proposition 3.30) and the almost everywhere differentiability of Ψ and Ψ^{-1} (parts 3 and 4 of Proposition 3.31), one has:

$$\begin{aligned}\int_{\mathcal{U}} |J_\Psi|^{-r\beta} \mathrm{d}x &= \int_{\mathcal{U}} |J_{\Psi^{-1}}(\Psi(x))|^{r\beta+1} |J_\Psi(x)| \,\mathrm{d}x = \\ &= \int_{\Psi(\mathcal{U})} |J_{\Psi^{-1}}(y)|^{r\beta+1} \mathrm{d}y \le \int_{\Psi(\mathcal{U})} \left\|\mathrm{D}\Psi^{-1}(y)\right\|^{2(r\beta+1)}.\end{aligned} \tag{3.49}$$

Notice that we have also used the Hadamard inequality $|\det M| \le \|M\|^n$ that holds for any real $n \times n$ matrix M, and the fact that

$$\frac{1}{\beta} + \frac{rq}{\bar{p}} = 1 \quad \Rightarrow \quad \frac{1}{\beta r} = \frac{1}{r} - \frac{q}{\bar{p}} = \frac{2}{\bar{p}-2} > 0. \tag{3.50}$$

By the same computation, one shows that $2(r\beta + 1) = \bar{p}$, so that the right hand side of Equation (3.49) is also finite. $\square$

Moreover, for homeomorphisms that satisfy the condition $K_q^\Psi \in L^r$ for some q and r, the following proposition holds:

Proposition 3.35. *Let $\tilde{\omega}, \omega \subset \mathbb{R}^2$ be open sets, and let $\Psi \in W^{1,1}_{\mathrm{loc}}(\tilde{\omega}, \omega)$ be a homeomorphism of finite distortion. Let $2 \le \alpha < q$. If $(K_q^\psi)^{\frac{\alpha}{q-\alpha}} \in L^1_{\mathrm{loc}}\tilde{\omega}$, then T_Ψ is continuous from $W^{1,q}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$ to $W^{1,\alpha}(\tilde{\omega})$, and the chain rule $D(u \circ \Psi)(x) = Du(\Psi(x)) \cdot \mathrm{D}\Psi(x)$ holds for almost every $x \in \tilde{\omega}$, for any $u \in W^{1,q}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$.*

Therefore, in our case the following observations hold:

- **Summability of K_q^Ψ**: if $\bar{p} > 2$, we can apply Lemma 3.34 with any $q > 0$ and obtain that $K_q^\Psi \in L^r_{\mathrm{loc}}(\tilde{\omega})$ with

$$r = r(q) \doteq \left(\frac{q}{\bar{p}} + \frac{2}{\bar{p}-2}\right)^{-1} > 0. \tag{3.51}$$

- **Sobolev regularity of composition with Ψ**: assume $\bar{p} > 2$, and let $q \ge \bar{p}$ be an arbitrary number. We can apply the Regularity of Composition Proposition 3.35; the exponent which is denoted by α therein is here chosen to be

$$p = p_{\bar{p}}(q) \doteq \left(\frac{1}{\bar{p}} + \frac{1}{q}\frac{\bar{p}}{\bar{p}-2}\right)^{-1}; \tag{3.52}$$

indeed, in this case, the identity $\frac{p_{\bar{p}}(q)}{q - p_{\bar{p}}(q)} = r_{\bar{p}}(q)$ holds for any $q > 0$, where $r_{\bar{p}}(q) > 0$ is the exponent defined in Lemma 3.34.

Thanks to these observations, we obtain what follows: the composition operator associated to a locally bi-Sobolev homeomorphisms maps (high enough) Sobolev spaces to Sobolev spaces.

Proposition 3.36 (Regularity of composition). *Let $\tilde{\omega}, \omega \subset \mathbb{R}^2$ be open sets, and let $\Psi : \tilde{\omega} \to \omega$ be a locally bi-Sobolev homeomorphism with Sobolev exponent $\bar{p} > 2$. Assume that*

$$q \geq \max\left\{\bar{p}, \frac{\bar{p}^2}{(\bar{p}-1)(\bar{p}-2)}\right\}. \tag{3.53}$$

Then the operator T_Ψ associated to the composition with Ψ is continuous from $W^{1,q}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$ to $W^{1,p(q)}_{\mathrm{loc}}(\tilde{\omega})$, where

$$p(q) \doteq \left(\frac{1}{\bar{p}} + \frac{1}{q}\frac{\bar{p}}{\bar{p}-2}\right)^{-1}. \tag{3.54}$$

In particular, if $u \in \mathbf{C}^0(\omega)$ belongs to $W^{1,q}_{\mathrm{loc}}(\omega)$ for any $q \in [1, \infty)$, then $u \circ \Psi \in W^{1,p}_{\mathrm{loc}}(\tilde{\omega})$ for any $1 \leq p < \bar{p}$.

Finally, if $u \in W^{1,q}_{\mathrm{loc}}(\omega) \cap \mathbf{C}^0(\omega)$ for with q satisfying Equation (3.53), then the chain rule holds for $u \circ \Psi$.

***Remark* 3.37.** For the sake of clearness: the assumption $q \geq \frac{\bar{p}^2}{(\bar{p}-1)(\bar{p}-2)}$ is only needed in order to ensure that $p(q) \geq 1$. Moreover, the condition is already contained in $q \geq \bar{p}$ whenever $\bar{p}$ is big enough (namely, $\bar{p} \geq 2 + \sqrt{2}$). ♢

As a corollary, we can apply Proposition 3.36 to the case of bi-Sobolev Lagrangian homeomorphisms associated to a vector field ∇^f:

Corollary 3.38 (Sobolev regularity of $\partial_\upsilon \chi$). *Let $\omega \subset \mathbb{R}^2$, $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\omega)$ and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}; \omega)$ satisfy Assumption 3.24, with the Sobolev exponents of Ψ and f satisfying*

$$\bar{p} > 2 \qquad \text{and} \qquad \bar{q} \geq \max\left\{\bar{p}, \frac{\bar{p}^2}{(\bar{p}-1)(\bar{p}-2)}\right\}. \tag{3.55}$$

Then the map $\partial_\upsilon \chi = f \circ \Psi : \tilde{\omega} \to \mathbb{R}$ belongs to $W^{1,p(\bar{q})}(\tilde{\omega})$, where $p(\bar{q})$ is defined as in Equation (3.54).

Finally, as a further consequence, we write explicitly some formulas descending from the chain rule for Sobolev functions, in order to be able to recall them later:

Corollary 3.39 (Chain rule). *Let $\omega \subset \mathbb{R}^2$, $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\omega)$ and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}; \omega)$ satisfy Assumption 3.24, with the Sobolev exponents of Ψ and f satisfying $\bar{p} > 2$ and $\bar{q} \geq \max\left\{\bar{p}, \frac{\bar{p}^2}{(\bar{p}-1)(\bar{p}-2)}\right\}$. Then:*

1. *$\partial_\upsilon \chi \in W^{1,p(\bar{q})}(\tilde{\omega})$ satisfies*

$$\partial^2_\upsilon \chi(\upsilon, \tau) = \partial_\upsilon f(\upsilon, \chi(\upsilon, \tau)) = (\nabla^f f) \circ \Psi(\upsilon, \tau), \tag{3.56}$$

$$\partial_\tau \partial_\upsilon \chi(\upsilon, \tau) = \partial_\tau f(\upsilon, \chi(\upsilon, \tau)) = [(\partial_t f \circ \Psi) \cdot \partial_\tau \chi](\upsilon, \tau). \tag{3.57}$$

2. *$\partial_\upsilon \partial_\tau \chi = \partial_\tau \partial_\upsilon \chi$ holds as an equality between distributional derivatives.*

3. *More generally, if* $\varphi \in W^{1,q}_{\mathrm{loc}}(\omega)$ *with* $q \geq \max\left\{\bar{p}, \frac{\bar{p}^2}{(\bar{p}-1)(\bar{p}-2)}\right\}$, *then*

$$\partial_\upsilon(\varphi \circ \Psi) = (\nabla^f \varphi) \circ \Psi \in L^{p(q)}_{\mathrm{loc}}(\tilde{\omega}), \tag{3.58}$$

$$\partial_\tau(\varphi \circ \Psi) = (\partial_t \varphi \circ \Psi) \cdot \partial_\tau \chi \in L^{p(q)}_{\mathrm{loc}}(\tilde{\omega}), \tag{3.59}$$

so that in particular,

$$(\partial_t \varphi) \circ \Psi = \frac{\partial_\tau(\varphi \circ \Psi)}{\partial_\tau \chi}, \tag{3.60}$$

$$(\partial_y \varphi) \circ \Psi = \partial_\upsilon(\varphi \circ \Psi) - (f \circ \Psi)\frac{\partial_\tau(\varphi \circ \Psi)}{\partial_\tau \chi} \tag{3.61}$$

also hold, and they both belong to $L^{\varrho(q)}_{\mathrm{loc}}(\tilde{\omega})$ *if there exists* $\varrho(q) \geq 1$ *satisfying* $\frac{1}{\varrho(q)} = \frac{1}{\bar{p}-1} + \frac{1}{p(q)}$.

Proof. This is an easy consequence of the chain rule established in Proposition 3.36, and by the fact that $\partial_\tau \chi$ is almost everywhere positive (by Proposition 3.31) and its reciprocal belongs to $L^{\bar{p}-1}_{\mathrm{loc}}(\tilde{\omega})$ (again Proposition 3.31). □

3.3 First variation

In this section, we explore the consequences of the first variation formula (3.25 | 1VF) alone under the Assumption 3.24: as a first step, we obtain some results on a general domain ω. Then, in Section 3.3.2, we analyze the case of functions defined on the whole $\mathbb{R}^n$, again satisfying (locally) the weak minimal surface equation and admitting a bi-Sobolev Lagrangian parametrization.

3.3.1 Local results

Let again $\omega \subset \mathbb{R}^2$, $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\omega)$ and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega}; \omega)$ be as in Assumption 3.24; recall that $\Psi(\upsilon, \tau) = (\upsilon, \chi(\upsilon, \tau))$. Assume, in addition, that $\bar{p} > 2$ and $\bar{q} \geq \max\left\{\bar{p}, \frac{\bar{p}^2}{(\bar{p}-1)(\bar{p}-2)}\right\}$, in order to be allowed to apply Corollary 3.38.

Definition 3.40 (Regularizations). From now on, $\{\varrho_\varepsilon\}_{\varepsilon>0} \in \mathbf{C}^\infty_{\mathrm{c}}(\mathbb{R}^2)$ will be the standard family of mollifiers on $\mathbb{R}^2$; in particular, ϱ_ε is supported on the ball $B_\varepsilon(0)$ of radius ε. For any ε is small enough, we will consider the regularized function $\chi_\varepsilon \doteq \chi * \varrho_\varepsilon$, *i.e.*:

$$\chi_\varepsilon(\upsilon, \tau) = \chi * \varrho_\varepsilon(\upsilon, \tau) = \int_{\tilde{\omega}} \chi(\upsilon', \tau')\varrho_\varepsilon(\upsilon - \upsilon', \tau - \tau')\, \mathrm{d}\mathscr{L}^2(\upsilon', \tau'), \tag{3.62}$$

defined for any pair $(\upsilon, \tau) \in \tilde{\omega}$ such that $\mathsf{d}((\upsilon, \tau), \partial\tilde{\omega}) > \varepsilon$. If $K \Subset \tilde{\omega}$ is compactly contained in $\tilde{\omega}$, and $\varepsilon_K \doteq \mathsf{d}(K, \partial\tilde{\omega})$, then χ_ε belongs to $\mathbf{C}^\infty(K)$ for any $0 < \varepsilon < \varepsilon_K$. ♢

***Remark* 3.41** (Properties of the regularizations)**.** The following properties hold true for the regularized functions χ_ε in a set $K \Subset \tilde{\omega}$, whenever $\varepsilon \in (0, \varepsilon_K)$:

1. $\partial_\tau \chi_\varepsilon = (\partial_\tau \chi) * \varrho_\varepsilon$; $\partial_\tau \partial_\upsilon \chi_\varepsilon = (\partial_\tau \partial_\upsilon \chi) * \varrho_\varepsilon$; indeed, both χ and $\partial_\upsilon \chi$ belong to $W^{1,1}_{\mathrm{loc}}(\tilde{\omega})$ under our assumptions (see Corollary 3.38), and standard properties of the regularizations imply that the regularization of weak derivatives coincide with the derivative of regularizations;

2. $\chi_\varepsilon \to \chi$ in $W^{1,\bar p}(K)$; moreover, $\partial_\upsilon \chi_\varepsilon \to \partial_\upsilon \chi$ in $W^{1,p}(K)$ for any $p \in [1, p(\bar q)]$ (see Corollary 3.38);
3. $\partial_\tau \chi_\varepsilon > 0$ holds *everywhere* in K: indeed, for $(\upsilon,\tau) \in K$ and $\varepsilon \in (0, \varepsilon_K)$,

$$\partial_\tau(\chi_\varepsilon)(\upsilon,\tau) = \int_{B_\varepsilon(0)} \partial_\tau \chi(\upsilon-\upsilon', \tau-\tau')\varrho_\varepsilon(\upsilon',\tau')\,\mathrm{d}\mathscr{L}^2(\upsilon',\tau'), \tag{3.63}$$

   and $\partial_\tau\chi > 0$ almost everywhere, while $\varrho_\varepsilon > 0$ in $B_\varepsilon(0)$;
4. $0 < \frac{1}{\partial_\tau \chi_\varepsilon} \le \left(\frac{1}{\partial_\tau\chi}\right)_\varepsilon$ pointwise in K. Indeed, by Hölder inequality, for any $x = (\upsilon,\tau) \in K$,

$$\begin{aligned} 1 = \|\varrho_\varepsilon\|_{L^1(B_\varepsilon(0))} &\le \left\|\sqrt{\partial_\tau\chi(x-\cdot)\varrho_\varepsilon(\cdot)}\right\|_{L^2(B_\varepsilon(0))} \left\|\sqrt{\frac{\varrho_\varepsilon(\cdot)}{\partial_\tau\chi(x-\cdot)}}\right\|_{L^2(B_\varepsilon(0))} = \\ &= \sqrt{\partial_\tau\chi_\varepsilon(x)} \cdot \sqrt{\left(\frac{1}{\partial_\tau\chi}\right)_\varepsilon(x)} \end{aligned} \tag{3.64}$$

5. By the last property and by the fact that $\left(\frac{1}{\partial_\tau\chi}\right)_\varepsilon \to \frac{1}{\partial_\tau\chi}$ in $L^{\bar p-1}(K)$, it holds that

$$\limsup_{\varepsilon\downarrow 0} \left\|\frac{1}{\partial_\tau\chi_\varepsilon}\right\|_{L^{\bar p-1}(K)} \le \left\|\frac{1}{\partial_\tau\chi}\right\|_{L^{\bar p-1}(K)}. \tag{3.65}$$ ◊

Notation 3.42. Following the notation adopted in [NS19], for any map $u : \omega \to \mathbb{R}$ we will denote by $\tilde u$ the composition $u \circ \Psi : \tilde\omega \to \mathbb{R}$. ◊

Theorem 3.43. *Let $\omega \subset \mathbb{R}^2$, $f \in W^{1,\bar q}_{\mathrm{loc}}(\omega)$ and $\Psi \in W^{1,\bar p}_{\mathrm{loc}}(\tilde\omega;\omega)$ be as in Assumption 3.24. Assume that the Sobolev exponents $\bar p$ and $\bar q$ satisfy*

$$2 < \bar p \le \bar q \qquad \text{and} \qquad \frac{1}{\bar p} + \frac{1}{\frac{\bar p-1}{2}} + \frac{1}{p(\bar q)} \le 1. \tag{3.66 | PQ}$$

Assume that f is stationary. Let $\sigma \in [1,\infty]$ be such that

$$\sigma \ge 2\left(\frac{\bar p}{\bar p-2}\right)^2 \qquad \text{and} \qquad \frac{1}{\sigma} \le 1 - \left(\frac{1}{\bar p} + \frac{1}{\frac{\bar p-1}{2}} + \frac{1}{p(\bar q)}\right). \tag{3.67}$$

Then

$$\int_{\tilde\omega} \frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\partial_\upsilon\vartheta\,\mathrm{d}\mathscr{L}^2(\upsilon,\tau) = 0 \tag{3.68}$$

for any $\vartheta \in W^{1,\sigma}(\tilde\omega) \cap \mathbf{C}^0_c(\tilde\omega)$.

***Remark* 3.44.** The technical condition (3.66 | PQ) is stronger than the assumption needed in Proposition 3.36. Moreover, by the definition of $p(\bar q)$, the inequality on the right can be restated as

$$\frac{2}{\bar p} + \frac{2}{\bar p-1} + \frac{\bar p}{\bar q}\frac{1}{\bar p-2} \le 1; \tag{3.69}$$

in particular, if $\bar p$ is large enough it is already implied by $\bar p \le \bar q$. ◊

Proof. We first show an adapted version of the first variation formula (3.25 | 1VF) that exploits the existence of the parametrization Ψ; then, we apply it to a suitably chosen class of test functions.

Step 1. Let us first fix a test function $\tilde{\varphi} \in W^{1,\sigma}(\tilde{\omega}) \cap \mathbf{C}^0_c(\tilde{\omega})$. Let $\varphi \doteq \tilde{\varphi} \circ \Psi^{-1}$. By Proposition 3.36 and by the first bound of Equation (3.67), the new test function φ belongs to the space $W^{1,2}(\omega) \cap \mathbf{C}^0_c(\omega)$. Hence we can apply the first variation formula (3.25 | 1VF) to deduce that $\tilde{\varphi}$ satisfies

$$\int_\omega \frac{\nabla^f f}{\sqrt{1+(\nabla^f f)^2}} \Big(\nabla^f\Big(\tilde{\varphi}\circ\Psi^{-1}\Big) + (\partial_t f)\Big(\tilde{\varphi}\circ\Psi^{-1}\Big)\Big)\, \mathrm{d}\mathscr{L}^2 = 0. \tag{3.70}$$

Now, again by Proposition 3.36, the chain rule can be applied to $\tilde{\varphi}\circ\Psi^{-1}$; by Corollary 3.39, $\nabla^f f \circ \Psi = \partial^2_\upsilon \chi$ and $\partial_t f \circ \Psi = \frac{\partial_\tau(f\circ\Psi)}{\partial_\tau \chi}$; finally, applying the Area Formula (3.36 | AF) we get

$$\int_{\tilde{\omega}} \frac{\partial^2_\upsilon \chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}} \Big[\partial_\tau\chi\partial_\upsilon\tilde{\varphi} + \tilde{\varphi}\partial_\tau\tilde{f}\Big]\, \mathrm{d}\mathscr{L}^2(\upsilon,\tau) = 0 \qquad \text{for any } \tilde{\varphi} \in W^{1,\sigma}(\tilde{\omega}) \cap \mathbf{C}^0_c(\tilde{\omega}). \tag{3.71}$$

Step 2. Now fix a function $\vartheta \in W^{1,\sigma}(\tilde{\omega}) \cap \mathbf{C}^0_c(\tilde{\omega})$. Let $K \doteq \operatorname{spt}\vartheta \Subset \tilde{\omega}$. Fix $\varepsilon_0 \doteq \frac{1}{2}\mathsf{d}(K, \partial\tilde{\omega})$ and the ε_0 neighborhood of K, *i.e.*

$$K_0 \doteq \Big\{(\upsilon,\tau)\in\mathbb{R}^2 \;\Big|\; \mathsf{d}((\upsilon,\tau),K) \le \varepsilon_0\Big\} \Subset \tilde{\omega}. \tag{3.72}$$

For any $\varepsilon \in (0,\varepsilon_0)$, consider the regularized function $\chi_\varepsilon = \chi * \varrho_\varepsilon$, which belongs to $\mathbf{C}^\infty(K_0)$. Our goal is to use $\tilde{\varphi}_\varepsilon \doteq \frac{\vartheta}{\partial_\tau\chi_\varepsilon}$ with $\varepsilon > 0$ small enough as test functions in Equation (3.71), and then let $\varepsilon \downarrow 0$.

Notice that:

- $\tilde{\varphi}_\varepsilon = \frac{\vartheta}{\partial_\tau\chi_\varepsilon}$ belongs to the space of admissible test functions: indeed, $\frac{1}{\partial_\tau\chi_\varepsilon} \in \mathbf{C}^\infty(K_0)$, because $\partial_\tau\chi > 0$ everywhere in K_0.
- We can compute:

$$\partial_\tau\chi\partial_\upsilon\tilde{\varphi}_\varepsilon + \tilde{\varphi}_\varepsilon\partial_\tau\tilde{f} = \partial_\upsilon\vartheta\frac{\partial_\tau\chi}{\partial_\tau\chi_\varepsilon} + \vartheta\frac{\partial_\tau\partial_\upsilon\chi(\partial_\tau\chi_\varepsilon - \partial_\tau\chi) + \partial_\tau\chi(\partial_\tau\partial_\upsilon\chi - \partial_\tau\partial_\upsilon\chi_\varepsilon)}{(\partial_\tau\chi_\varepsilon)^2}. \tag{3.73}$$

Let us prove that

$$\frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\Big[\partial_\tau\chi\partial_\upsilon\tilde{\varphi}_\varepsilon + \tilde{\varphi}_\varepsilon\partial_\tau\tilde{f}\Big] \to \frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\partial_\upsilon\vartheta \qquad \text{in } L^1(\tilde{\omega}) \text{ as } \varepsilon\to 0. \tag{3.74}$$

Step 3. Indeed, by Equation (3.73):

$$\begin{aligned}&\left\|\frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\Big[\partial_\tau\chi\partial_\upsilon\tilde{\varphi}_\varepsilon + \tilde{\varphi}_\varepsilon\partial_\tau\tilde{f} - \partial_\upsilon\vartheta\Big]\right\|_{L^1} = \\ &= \left\|\frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\left[\partial_\upsilon\vartheta\frac{\partial_\tau\chi - \partial_\tau\chi_\varepsilon}{\partial_\tau\chi_\varepsilon} + \vartheta\partial_\tau\partial_\upsilon\chi\frac{\partial_\tau\chi_\varepsilon - \partial_\tau\chi}{(\partial_\tau\chi_\varepsilon)^2} + \vartheta\partial_\tau\chi\frac{\partial_\tau\partial_\upsilon\chi - \partial_\tau\partial_\upsilon\chi_\varepsilon}{(\partial_\tau\chi_\varepsilon)^2}\right]\right\|_{L^1},\end{aligned} \tag{3.75}$$

thus, by Hölder inequality and triangle inequality, the left hand side is smaller than or equal to

$$\left\|\frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\right\|_{L^\infty}\left[\left\|\partial_\upsilon\vartheta\frac{\partial_\tau\chi - \partial_\tau\chi_\varepsilon}{\partial_\tau\chi_\varepsilon}\right\|_{L^1} + \left\|\vartheta\partial_\tau\partial_\upsilon\chi\frac{\partial_\tau\chi_\varepsilon - \partial_\tau\chi}{(\partial_\tau\chi_\varepsilon)^2}\right\|_{L^1} + \left\|\vartheta\partial_\tau\chi\frac{\partial_\tau\partial_\upsilon\chi - \partial_\tau\partial_\upsilon\chi_\varepsilon}{(\partial_\tau\chi_\varepsilon)^2}\right\|_{L^1}\right] \tag{3.76}$$

Exploiting the L^σ integrability of ϑ and $\partial_\upsilon\vartheta$, and then using again the Hölder inequality (once with exponents $(\sigma,\bar p,\bar p-1)$, then twice with exponents $(\sigma,\bar p,p(q),\frac{\bar p-1}{2})$), one obtains

$$\left\|\partial_\upsilon\vartheta\frac{\partial_\tau\chi-\partial_\tau\chi_\varepsilon}{\partial_\tau\chi_\varepsilon}\right\|_{L^1}\le\|\partial_\upsilon\vartheta\|_{L^\sigma}\|\partial_\tau\chi-\partial_\tau\chi_\varepsilon\|_{L^{\bar p}}\left\|\frac{1}{\partial_\tau\chi_\varepsilon}\right\|_{L^{\bar p-1}} \tag{3.77}$$

$$\left\|\vartheta\partial_\tau\partial_\upsilon\chi\frac{\partial_\tau\chi_\varepsilon-\partial_\tau\chi}{(\partial_\tau\chi_\varepsilon)^2}\right\|_{L^1}\le C\|\vartheta\|_{L^\sigma}\|\partial_\tau\partial_\upsilon\chi\|_{L^{p(q)}}\|\partial_\tau\chi_\varepsilon-\partial_\tau\chi\|_{L^{\bar p}}\left\|\frac{1}{\partial_\tau\chi_\varepsilon}\right\|^2_{L^{\bar p-1}} \tag{3.78}$$

$$\left\|\vartheta\partial_\tau\chi\frac{\partial_\tau\partial_\upsilon\chi-\partial_\tau\partial_\upsilon\chi_\varepsilon}{(\partial_\tau\chi_\varepsilon)^2}\right\|_{L^1}\le C\|\vartheta\|_{L^\sigma}\|\partial_\tau\chi\|_{L^{\bar p}}\|\partial_\tau\partial_\upsilon\chi-\partial_\tau\partial_\upsilon\chi_\varepsilon\|_{L^{p(\bar q)}}\left\|\frac{1}{\partial_\tau\chi_\varepsilon}\right\|^2_{L^{\bar p-1}}, \tag{3.79}$$

where (3.78) and (3.79) hold thanks to the fact that $\frac{1}{\sigma}+\frac{1}{\bar p}+\frac{1}{p(\bar q)}+\frac{1}{\frac{\bar p-1}{2}}\le 1$.

Collecting all the information obtained until now, letting $\varepsilon\downarrow 0$ and exploiting the estimate

$$\limsup_{\varepsilon\downarrow 0}\left\|\frac{1}{\partial_\tau\chi_\varepsilon}\right\|_{L^{\bar p-1}(K)}\le\left\|\frac{1}{\partial_\tau\chi}\right\|_{L^{\bar p-1}(K)}, \tag{3.80}$$

we get that

$$0=\int_{\tilde\omega}\frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\left[\partial_\tau\chi\partial_\upsilon\tilde\varphi_\varepsilon+\tilde\varphi_\varepsilon\partial_\tau\tilde f\right]\mathrm{d}\mathscr{L}^2(\upsilon,\tau)\xrightarrow{\varepsilon\to 0}\int_{\tilde\omega}\frac{\partial^2_\upsilon\chi}{\sqrt{1+(\partial^2_\upsilon\chi)^2}}\partial_\upsilon\vartheta\,\mathrm{d}\mathscr{L}^2(\upsilon,\tau), \tag{3.81}$$

which is the statement we wanted to prove. □

Let us now state a quite easy lemma that ensures we can usefully exploit the identity (3.68).

Lemma 3.45. *Let $\tilde\omega\subset\mathbb{R}^2$ be an open domain such that $\tilde\omega_{2,\tau}$ is convex for any $\tau\in\pi_2(\tilde\omega)$. Let $h\in L^1(\tilde\omega)$. If*

$$\int_{\tilde\omega}h(\upsilon,\tau)\partial_\upsilon\vartheta(\upsilon,\tau)\,\mathrm{d}\mathscr{L}^2(\upsilon,\tau)=0\qquad\text{for any }\vartheta\in\mathbf{C}^\infty_{\mathrm{c}}(\tilde\omega), \tag{3.82}$$

then (up to a modification on a negligible set) h does not depend on the first variable; i.e., *there exists $g:\pi_2(\tilde\omega)\to\mathbb{R}$ such that $h(\upsilon,\tau)=g(\tau)$ for $\mathscr{L}^2$-almost every $(\upsilon,\tau)\in\tilde\omega$ (here $\pi_2(\upsilon,\tau)=\tau$ is the projection on the second coordinate).*

Proof. Let $\varrho_\varepsilon\in\mathbf{C}^\infty_{\mathrm{c}}(\mathbb{R}^2)$ be the standard family of mollifiers as before. Let $h_\varepsilon\doteq h*\varrho_\varepsilon$; we have already observed that $h_\varepsilon\in\mathbf{C}^\infty(\tilde\omega_\varepsilon)$, where $\tilde\omega_\varepsilon$ is the set of points $(\upsilon,\tau)\in\tilde\omega$ such that $\mathsf{d}((\upsilon,\tau),\partial\tilde\omega)>\varepsilon$; moreover, $h_\varepsilon\to h$ almost everywhere in $\tilde\omega$.
By Equation (3.82), the distributional derivative $\partial_\upsilon h$ is zero; thus $\partial_\upsilon h_\varepsilon=(\partial_\upsilon h)*\varrho_\varepsilon=0$; in particular, since this last term is a classical partial derivative of a smooth function, for any $(\upsilon,\tau)\in\tilde\omega_\varepsilon$ it holds that $\partial_\upsilon h_\varepsilon(\upsilon,\tau)=0$. This fact, together with the smoothness of h_ε, implies that h_ε is independent of υ, *i.e.* $h_\varepsilon(\upsilon,\tau)=g_\varepsilon(\tau)$ for some function g_ε.
As already noted, for almost every $(\upsilon,\tau)\in\tilde\omega$

$$h(\upsilon,\tau)=\lim_{\varepsilon\downarrow 0}h_\varepsilon(\upsilon,\tau)=\lim_{\varepsilon\downarrow 0}g_\varepsilon(\tau), \tag{3.83}$$

which allows to conclude. □

Exploiting together the first variation formula on the reparametrized domain Theorem 3.43 and the easy Lemma 3.45, we obtain a first consequence of the stationariety:

Corollary 3.46. *Let $\omega \subset \mathbb{R}^2$, $f \in W^{1,\bar q}_{\mathrm{loc}}(\omega)$ and $\Psi \in W^{1,\bar p}_{\mathrm{loc}}(\tilde\omega;\omega)$ be as in Assumption 3.24, with $\tilde\omega = \Upsilon \times T$, where Υ and T are open intervals. Assume that the Sobolev exponents $\bar p$ and $\bar q$ satisfy Equation (3.66 | PQ). Assume that f is stationary.*

Then for any fixed $\hat\upsilon \in (\upsilon_1, \upsilon_2)$ the map χ can be written in $\tilde\omega$ as

$$\chi(\upsilon,\tau) = \frac{(\upsilon-\hat\upsilon)^2}{2}\nabla^f f(\hat\upsilon,\chi(\hat\upsilon,\tau)) + (\upsilon-\hat\upsilon) f(\hat\upsilon,\chi(\hat\upsilon,\tau)) + \chi(\hat\upsilon,\tau) \tag{3.84}$$

with $\nabla^f f(\hat\upsilon,\chi(\hat\upsilon,\cdot)) \in \mathbf{C}^0(T)$. In particular, if $\hat\upsilon$ belongs to the (full-measure) set

$$\Upsilon_0 \doteq \left\{ \upsilon \in \Upsilon \;\middle|\; \begin{array}{r} \chi_\upsilon = \chi(\upsilon,\cdot) \in W^{1,\bar p}_{\mathrm{loc}}(T) \\ \chi_\upsilon^{-1}(\cdot) \in W^{1,\bar p}_{\mathrm{loc}}(\chi_\upsilon(T)) \\ f\circ\Psi(\upsilon,\cdot) = \partial_\upsilon \chi(\upsilon,\cdot) \in W^{1,p(\bar q)}_{\mathrm{loc}}(T) \end{array} \right\}, \tag{3.85}$$

then χ can be written as

$$\chi(\upsilon,\tau) = \frac{a(\tau)}{2}(\upsilon-\hat\upsilon)^2 + b(\tau)(\upsilon-\hat\upsilon) + c(\tau) \tag{3.86}$$

in $\tilde\omega$, with $c : (\tau_1,\tau_2) \to c((\tau_1,\tau_2))$ locally $\bar p$-bi-Sobolev homeomorphism, $c(\tau) = \chi(\hat\upsilon,\tau)$, and

$$a(\tau) \doteq \nabla^f f(\hat\upsilon, c(\tau)) \in W^{1,p(\bar q)}((\tau_1,\tau_2);\mathbb{R}), \qquad b(\tau) \doteq f(\hat\upsilon, c(\tau)) \in W^{1,p(\bar q)}((\tau_1,\tau_2);\mathbb{R}). \tag{3.87}$$

If we further assume that f is well-posed as a time-dependent vector field, then Ψ is the unique such parametrization satisfying $\Psi(\hat\upsilon,\tau) = c(\tau)$ for any $\tau \in T$.

***Remark* 3.47.** Recall that $f \in \mathbf{C}^0(\Upsilon\times T)$ is a *well-posed* time-dependent vector field if for any $(\hat\upsilon,\hat\tau) \in \Upsilon \times T$ there exists $\varepsilon > 0$ such that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\upsilon}\gamma(\upsilon) = f(\upsilon,\gamma(\upsilon)) \\ \gamma(\hat\upsilon) = \hat\tau \end{cases} \tag{3.88}$$

admits a unique $\mathbf{C}^1$ solution on $(\hat\upsilon-\varepsilon,\hat\upsilon+\varepsilon)$. ◊

Proof. Let us denote $\tilde\omega = \Upsilon\times T$, with $\Upsilon = (\upsilon_1,\upsilon_2)$, $T = (\tau_1,\tau_2)$.

Step 1. By local Sobolev regularity of f, Ψ and Ψ^{-1}, by Corollary 3.38 and by the Fubini-Tonelli Theorem, the set Υ_0 has full measure in Υ. On the other hand, the set

$$T_0 \doteq \left\{ \tau \in T \;\middle|\; \chi(\cdot,\tau) \in W^{1,\bar p}_{\mathrm{loc}}(\Upsilon) \text{ and } \partial_\upsilon\chi(\cdot,\tau) \in W^{1,p(\bar q)}_{\mathrm{loc}}(\Upsilon) \right\} \tag{3.89}$$

has also full measure in T, by the same argument.

Step 2. The stationariety of f implies that Theorem 3.43 can be applied. By Lemma 3.45 and Equation (3.68), there exists $g : T \to (-1,1)$ such that for almost every $(\upsilon,\tau) \in \tilde\omega$

$$\frac{\partial^2_\upsilon \chi(\upsilon,\tau)}{\sqrt{1+(\partial^2_\upsilon\chi(\upsilon,\tau))^2}} = g(\tau); \tag{3.90}$$

this easily implies that for any such (υ,τ)

$$\partial^2_\upsilon\chi(\upsilon,\tau) = a(\tau) \doteq \frac{g(\tau)}{\sqrt{1-g^2(\tau)}} \tag{3.91}$$

holds.

Step 3. For any $\tau \in T_0$ let

$$\Upsilon_\tau \doteq \{\upsilon \in \Upsilon \mid \text{Equation (3.91) holds at } (\upsilon, \tau)\}, \tag{3.92}$$

and let

$$T_1 \doteq \left\{\tau \in T_0 \;\middle|\; \mathscr{L}^1(\Upsilon \setminus \Upsilon_\tau) = 0\right\}; \tag{3.93}$$

by Step 2, T_1 has full measure in T.

Step 4. Fix $\hat{\upsilon} \in \Upsilon$. For any $\tau \in T_1$, by Sobolev regularity (and continuity) of $\partial_\upsilon \chi(\cdot, \tau)$ it holds that

$$\partial_\upsilon \chi(\upsilon, \tau) = \partial_\upsilon \chi(\hat{\upsilon}, \tau) + (\upsilon - \hat{\upsilon}) a(\tau) \tag{3.94}$$

for every $\upsilon \in \Upsilon$, and by Sobolev regularity (and continuity) of $\chi(\cdot, \tau)$

$$\chi(\upsilon, \tau) = \chi(\hat{\upsilon}, \tau) + (\upsilon - \hat{\upsilon}) \partial_\upsilon \chi(\hat{\upsilon}, \tau) + \frac{(\upsilon - \hat{\upsilon})^2}{2} a(\tau) \tag{3.95}$$

holds for every $\upsilon \in \Upsilon$.

Step 5. If $\hat{\upsilon}' \in \Upsilon \setminus \{\hat{\upsilon}\}$, for any $\tau \in T_1$ it holds that

$$a(\tau) = \frac{2}{(\hat{\upsilon}' - \hat{\upsilon})^2} [\chi(\hat{\upsilon}', \tau) - \chi(\hat{\upsilon}, \tau) - (\hat{\upsilon}' - \hat{\upsilon}) \partial_\upsilon \chi(\hat{\upsilon}, \tau)]; \tag{3.96}$$

thus a coincides in T_1 with a function which is continuous in the whole T. As a consequence, by the continuity of χ, Equation (3.95) holds for any $(\upsilon, \tau) \in \Upsilon \times T$. By the chain rule (Corollary 3.38), $a = \partial_\upsilon^2 \chi(\hat{\upsilon}, \cdot)$ coincides with $\nabla^f f(\hat{\upsilon}, \chi(\hat{\upsilon}, \tau))$.

Step 6. As a further consequence, if $\hat{\upsilon} \in \Upsilon_0$, then again by Equation (3.96) (choosing also $\hat{\upsilon}' \in \Upsilon_0$) it holds that $a \in W^{1,p(\bar{q})}_{\mathrm{loc}}(\Upsilon)$. Concluding, if $\hat{\upsilon} \in \Upsilon_0$ we have shown that Equation (3.86) holds for all $(\upsilon, \tau) \in \tilde{\omega}$ with:

- $c(\tau) = \chi(\hat{\upsilon}, \tau)$ (which is locally $\bar{p}$-bi-Sobolev by our choice of $\hat{\upsilon}$ in Υ_0);
- $b(\tau) = \partial_\upsilon \chi(\hat{\upsilon}, \tau) = f(\hat{\upsilon}, \chi(\hat{\upsilon}, \tau)) = f(\hat{\upsilon}, c(\tau))$ (which is $p(\bar{q})$-Sobolev, again by our choice of $\hat{\upsilon}$);
- $a(\tau) = \partial_\upsilon^2 \chi(\hat{\upsilon}, \tau) = (\nabla^f f)(\hat{\upsilon}, c(\tau))$, which is $p(\bar{q})$-Sobolev by Equation (3.96).

This completes the proof. □

As a consequence, stationary functions under the Sobolev Assumption 3.24 are intrinsically $\mathbf{C}^1$:

Corollary 3.48. *Let the assumptions be the same as in Corollary 3.46. Then $\nabla^f f \in \mathbf{C}^0(\omega)$, and thus $f \in \mathbf{C}^1_{\mathbb{W}}(\omega)$.*

Proof. We can write $\nabla^f f$ as

$$\nabla^f f = (\nabla^f f \circ \Psi) \circ \Psi^{-1}; \tag{3.97}$$

and now Ψ^{-1} is continuous by assumption, and $\nabla^f f \circ \Psi = \partial_\upsilon^2 \chi$ by Corollary 3.39. By the previous Corollary 3.46, $\partial_\upsilon^2 \chi(\upsilon, \tau) = a(\tau)$ is continuous, thus the assertion follows. □

The goal of the future Section 3.3.2 will be to retrieve a global parametrization of the whole $\mathbb{R}^2$, starting from local parametrizations of rectangles having the form of the one obtained in Corollary 3.46. In sight of this, it would be useful to perform further reparametrizations so that the term $c(\tau)$ therein is a predetermined function (for example, $c(\tau) = \tau$), while still maintaining the Sobolev regularity of all the functions involved (with possibly lower exponent). The following results go in this direction.

Lemma 3.49 (One-dim. Sobolev composition). *Let $I, J \subset \mathbb{R}$ be open intervals, let $\alpha \geq 1$ and $\beta > 1$; let $h \in W^{1,\alpha}_{\mathrm{loc}}(I, J)$ be non-decreasing and $u \in W^{1,\beta}_{\mathrm{loc}}(J, \mathbb{R})$. Then for $\gamma = \gamma(\alpha, \beta)$ defined by*

$$\gamma(\alpha, \beta) \doteq \frac{\alpha\beta}{\alpha + \beta - 1} \in [1, \beta) \tag{3.98}$$

it holds that $u \circ h \in W^{1,\gamma}_{\mathrm{loc}}(I, \mathbb{R})$.

Proof. Fix a compact set $K \Subset I$. The map $u \circ h$ is bounded and absolutely continuous in K, and the chain rule holds for $u \circ h$ (see for example [SV69, Corollary 4]). By assumption, the integral $\int_K |h'|^\alpha$ is finite; moreover, by the change of variables formula (see again [SV69, Corollary 6]),

$$\int_K |u' \circ h|^\beta |h'| = \int_K \Big(|u'|^\beta \circ h\Big) h' = \int_{h(K)} |u'|^\beta < \infty. \tag{3.99}$$

For any $\gamma \in [1, \beta)$, by the Hölder inequality (with exponents $\frac{\beta}{\gamma}$ and its Hölder conjugate),

$$\int_K |(u \circ h)'|^\gamma = \int_K |u' \circ h|^\gamma |h'|^{\frac{\gamma}{\beta}} |h'|^{\gamma(1 - \frac{1}{\beta})} \leq \left(\int_K |u' \circ h|^\beta |h'|\right)^{\frac{\gamma}{\beta}} \left(\int_K |h'|^{\gamma \frac{\beta - 1}{\beta - \gamma}}\right)^{1 - \frac{\gamma}{\beta}}. \tag{3.100}$$

With our choice of $\gamma(\alpha, \beta)$, both factors are finite. □

Lemma 3.50 (Reparametrization). *Let $\Psi : \tilde{\omega} \to \omega$ be a locally $\bar{p}$-bi-Sobolev homeomorphism with $\tilde{\omega} = \Upsilon \times T = (\upsilon_1, \upsilon_2) \times (\tau_1, \tau_2)$ and*

$$\Psi(\upsilon, \tau) = \left(\upsilon, \frac{a(\tau)}{2}(\upsilon - \hat{\upsilon})^2 + b(\tau)(\upsilon - \hat{\upsilon}) + c(\tau)\right) \tag{3.101}$$

in $\tilde{\omega}$, with $c : (\tau_1, \tau_2) \to c((\tau_1, \tau_2))$ locally $\bar{p}$-bi-Sobolev homeomorphism and Sobolev coefficients $a \in W^{1,p(\bar{q})}((\tau_1, \tau_2); \mathbb{R})$, $b \in W^{1,p(\bar{q})}((\tau_1, \tau_2); \mathbb{R})$. Let $\psi : \hat{T} \to T$ be a locally $\bar{p}$-bi-Sobolev non-decreasing homeomorphism. Then

$$\hat{\Psi}(\upsilon, \tau) \doteq \Psi(\upsilon, \psi^{-1}(\tau)) = \left(\upsilon, \frac{\hat{a}(\tau)}{2}(\upsilon - \hat{\upsilon})^2 + \hat{b}(\tau)(\upsilon - \hat{\upsilon}) + \hat{c}(\tau)\right) \tag{3.102}$$

is a locally bi-Sobolev homeomorphism with Sobolev exponent

$$\hat{p} \doteq \left(\frac{1}{\bar{p}} + \frac{1}{\bar{p} - 2}\right)^{-1}, \tag{3.103}$$

and $\hat{a}, \hat{b}, \hat{c} \in W^{1,\gamma}_{\mathrm{loc}}(\hat{T})$, γ being the Sobolev exponent coming from Lemma 3.49 with $\alpha = \bar{p}$, $\beta = p(\bar{q})$.

Proof. Define $\Phi(\upsilon,\tau) \doteq (\upsilon,\psi^{-1}(\tau))$ so that $\hat{\Psi} \doteq \Psi \circ \Phi$. Both Φ and Ψ are, by definition, locally $\bar{p}$-bi-Sobolev homeomorphisms; applying Proposition 3.36 to both $\hat{\Psi}$ and $\hat{\Psi}^{-1}$, we deduce that $\hat{\Psi}$ is a locally $\hat{p}$-bi-Sobolev homeomorphism. Moreover, by the previous Lemma 3.49 on the composition of one-dimensional Sobolev mappings, the statement regarding $\hat{a},\hat{b},\hat{c}$ holds. □

Corollary 3.51 (Lagrangian reparametrization). *Let $\omega \subset \mathbb{R}^2$, $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\omega)$ and $\Psi \in W^{1,\bar{p}}_{\mathrm{loc}}(\tilde{\omega};\omega)$ be as in Assumption 3.24, with $\tilde{\omega} = \Upsilon \times T$, Υ and T open intervals. Assume that the Sobolev exponents $\bar{p}$ and $\bar{q}$ satisfy Equation (3.66 | PQ) and $\bar{p} \geq 2+\sqrt{2}$. Assume that f is stationary, and well-posed as a time-dependent vector field.*

For any $\hat{\upsilon} \in \Upsilon_0$ (where Υ_0 is defined in Equation (3.85)), there exists a unique Lagrangian parametrization $\hat{\Psi} : \varpi \to \omega$ of ω of the form

$$\hat{\Psi}(\upsilon,\tau) = (\upsilon,\hat{\chi}(\upsilon,\tau)) = \left(\upsilon, \frac{\hat{a}(\tau)}{2}(\upsilon-\hat{\upsilon})^2 + \hat{b}(\tau)(\upsilon-\hat{\upsilon}) + \tau\right), \qquad \varpi = \Upsilon \times \hat{T}, \tag{3.104}$$

with

$$\hat{a}(\tau) = \nabla^f f(\hat{\upsilon},\tau), \qquad \hat{b}(\tau) = f(\hat{\upsilon},\tau), \qquad a,b \in W^{1,\gamma}_{\mathrm{loc}}(\hat{T}), \tag{3.105}$$

γ being the Sobolev exponent coming from Lemma 3.49 with $\alpha = \bar{p}$, $\beta = p(\bar{q})$. Moreover, $\hat{\Psi}$ turns out to be a locally bi-Sobolev Lagrangian homeomorphism with Sobolev exponent

$$\hat{p} \doteq \left(\frac{1}{\bar{p}} + \frac{1}{\bar{p}-2}\right)^{-1} \geq 1. \tag{3.106}$$

Proof. Let $\hat{\upsilon} \in \Upsilon_0$. Define $\hat{T} \doteq \chi_{\hat{\upsilon}}^{-1}(T)$ and $\psi = \chi_{\hat{\upsilon}}$, and apply Lemma 3.50: we obtain a new locally bi-Sobolev homeomorphism $\hat{\Psi}$, with Sobolev exponent $\hat{p}$ (greater than or equal to 1 by our assumption on $\bar{p}$). It is immediate to show that $\partial_\upsilon \hat{\Psi}(\upsilon,\tau) = \nabla^f(\hat{\Psi}(\upsilon,\tau))$ holds for Ψ as well; the uniqueness follows by the constraint $\hat{\Psi}(\hat{\upsilon},\tau) = \tau$ for all τ, by the requirement that $\hat{\Psi}$ is a Lagrangian parametrization, and by the well-posedness of f (which ensures that trajectories in ω do not meet).
Finally, by Corollary 3.46, χ could be written as

$$\chi(\upsilon,\tau) = \frac{a(\tau)}{2}(\upsilon-\hat{\upsilon})^2 + b(\tau)(\upsilon-\hat{\upsilon}) + \chi_{\hat{\upsilon}}(\tau), \tag{3.107}$$

with $a,b \in W^{1,p(\bar{q})}_{\mathrm{loc}}(T)$; by the representation of a and b given in Corollary 3.46, and by Lemma 3.50, Equation (3.105) holds. □

3.3.2 Global results

The results stated until now involved a fixed domain $\omega \subset \mathbb{R}^2$ together with a stationary Sobolev map f on ω; moreover, we had set an *a priori* assumption on the existence of a locally bi-Sobolev homeomorphic Lagrangian parametrization of ω associated to ∇^f. The next step is to move to maps defined on the whole $\mathbb{R}^2$; we still make an *a priori* assumption on the (local) existence of Lagrangian parametrizations, which will be justified in Section 3.5.

Assumption 3.52. We will assume that the following conditions hold:

- **Sobolev regularity of f**: f is a map defined on $\mathbb{R}^2$ such that $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}) \cap \mathbf{C}^0(\mathbb{R}^2)$ for some $\bar{q} > 2$;
- **Well-posedness of f**: f is well-posed as a time-dependent vector field on $\mathbb{R}$;
- **Existence of parametrizations "in small"**: there exist $\bar{p} \geq 1$ and $\ell > 0$ such that any open bounded connected subset $\omega \subset \mathbb{R}^2$ of width smaller than ℓ (that is, $\mathscr{L}^1(\pi_1(\omega)) < \ell$) admits a locally bi-Sobolev Lagrangian homeomorphism associated to ∇^f with Sobolev exponent $\bar{p}$. $\Diamond$

Thanks to a result proved in the article (to appear) [ANS], we'll be able to retrieve in Section 3.5 a condition that guarantees this assumption is satisfied. However, if f is also stationary, this assumption alone is enough to constrain the intrinsic graph of f to be ruled by horizontal lines:

Theorem 3.53. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy Assumption 3.52, with $\bar{q}$ and $\bar{p}$ satisfying Equation (3.66 | PQ) and $\bar{p} \geq 2 + \sqrt{2}$. Assume that f is stationary in $\mathbb{R}^2$. Then there exists $\hat{\upsilon} \in \mathbb{R}$ such that*

$$\Psi(\upsilon, \tau) \doteq \left(\upsilon, \frac{a(\tau)}{2}(\upsilon - \hat{\upsilon})^2 + b(\tau)(\upsilon - \hat{\upsilon}) + \tau\right), \qquad \text{with} \quad \begin{cases} a(\tau) \doteq \nabla^f f(\hat{\upsilon}, \tau) \\ b(\tau) \doteq f(\hat{\upsilon}, \tau) \end{cases}, \tag{3.108 | Ψ}$$

defines a locally $\hat{p}$-bi-Sobolev Lagrangian homeomorphism associated to ∇^f mapping the whole $\mathbb{R}^2$ onto itself, with γ-Sobolev coefficients a and b. The Sobolev exponents $\hat{p}$ and γ are both defined in Lemma 3.50.

Proof. We proceed as follows: we first show that for any $(\upsilon_0, \tau_0) \in \mathbb{R}^2$ the Cauchy problem

$$\begin{cases} \dot{\gamma}(\upsilon) = f(\upsilon, \gamma(\upsilon)) \\ \gamma(\upsilon_0) = \tau_0 \end{cases} \tag{3.109}$$

admits a solution globally defined in time; such a solution is a quadratic polynomial, and the coefficients are Sobolev-regular in τ_0 when υ_0 is fixed and suitably chosen.

STEP 1. Fix $(\upsilon_0, \tau_0) \in \mathbb{R}^2$. By assumption, every narrow enough neighborhood ω_0 of (υ_0, τ_0) admits a locally bi-Sobolev Lagrangian parametrization $\Psi_0 : \tilde{\omega}_0 \to \omega_0$ associated to ∇^f; up to restricting $\tilde{\omega}_0$ and ω_0 we can assume $\tilde{\omega}_0 = \Upsilon \times T$ is a rectangle. By Corollary 3.46, such Ψ_0 has the form $(\upsilon, a(\tau)\upsilon^2 + b(\tau)\upsilon + c(\tau))$. In particular, the unique local solution to Equation (3.109) is a quadratic polynomial:

$$\gamma(\upsilon) = a_0\upsilon^2 + b_0\upsilon + c_0. \tag{3.110}$$

Assume by contradiction that the maximal interval $I(\upsilon_0, \tau_0)$ for the Cauchy Problem (3.109) is (right) bounded by $\sup I(\upsilon_0, \tau_0) = M$: let

$$(\upsilon_1, \tau_1) = \left(\upsilon_1, a_0\upsilon_1^2 + b_0\upsilon_1 + c_0\right) \in \left(M - \frac{\ell}{4}, M\right) \tag{3.111}$$

be a point belonging to the trajectory of γ and close to the supremum of $I(\upsilon_0, \tau_0)$. Fix a neighborhood $\mathcal{U}$ of (υ_1, τ_1) that contains the whole trait of curve

$$(\upsilon, a_0\upsilon^2 + b_0\upsilon + c_0), \qquad \upsilon \in \left(M - \frac{\ell}{4}, M + \frac{\ell}{4}\right) \tag{3.112}$$

but is narrower than ℓ. Now $\mathcal{U}$ also admits a Lagrangian parametrization associated to f; again by Corollary 3.46, the second component of this new parametrization still needs to be a quadratic polynomial in υ (at least in a thin tubular neighborhood of the trajectory (3.112)). By the well-posedness of f, the trajectory passing through (υ_1, τ_1) coincides with (3.112): this implies that the polynomial γ in Equation (3.110) is a solution to the Cauchy Problem (3.109) at least up to $M + \frac{\ell}{4}$, contradicting the definition of M as the supremum of the maximal interval.

Step 2. By assumption, any set of the form $A^k \doteq (0, \frac{\ell}{2}) \times (-k, k)$ admits a locally bi-Sobolev Lagrangian parametrization $\Psi^k(\upsilon,\tau) = (\upsilon, \chi^k(\upsilon,\tau))$; we can find a suitable $\hat{\upsilon}$ which makes each $\chi^k_{\hat{\upsilon}}$ $\bar{p}$-bi-Sobolev and each $(\chi^k_{\hat{\upsilon}})'$ $p(\bar{q})$-Sobolev; thus for any k, by Corollary 3.51, we can find a (uniquely determined) reparametrization of a strip containing $\{\hat{\upsilon}\} \times (-\frac{k}{2}, \frac{k}{2})$ having the form of Equation (3.108 | Ψ). Notice that now a and b are globally defined in $\mathbb{R}$ and locally Sobolev with exponent γ, while Ψ is locally $\hat{p}$-bi-Sobolev.

Step 3. Once $\hat{\upsilon}$ is selected as in the previous step, the map Ψ in Equation (3.108 | Ψ) is continuous in the whole $\mathbb{R}^2$ (by the continuity of the coefficients in the quadratic representation); the curves $\upsilon \mapsto \Psi(\upsilon,\tau)$ are global solutions to the differential equation in (3.109); Ψ in surjective (by existence of solutions for (3.109) for any initial datum) and injective (by uniqueness of solution); thus it is a Lagrangian homeomorphism associated to ∇^f by Brouwer's Invariance of Domain Theorem.

Step 4. Finally, for any bounded domain ω having width smaller than ℓ, there exists a locally $\bar{p}$-bi-Sobolev Lagrangian homeomorphism as in Corollary 3.46; up to reparametrizing through a locally $\bar{p}$-bi-Sobolev homeomorphism of the type $(\upsilon,\tau) \to (\upsilon, c^{-1}(\tau))$, it must coincide with Ψ. In particular, by Lemma 3.50, the restriction of Ψ to $\Psi^{-1}(\omega)$ is locally bi-Sobolev with Sobolev exponent $\hat{p}$. □

3.4 Second variation

In this section, we discuss the consequences of stability under the Assumption 3.52 of existence of local Sobolev Lagrangian homeomorphisms.

First of all, we can replicate [NS19, Lemma 5.3]; half of it comes free of charge from [Gol18]. Observe that no stability or stationariety are needed at this stage: the main content of the statement is that if the integral curves of ∇^f describe parabolic trajectories, then by uniqueness the defining coefficients of two such trajectories are related to each other.

Lemma 3.54. *Let $a, b \in \mathbf{C}^0(\mathbb{R})$ and let $f \in \mathbf{C}^0(\mathbb{R}^2;\mathbb{R})$ be a well-posed time-dependent vector field. Define*

$$\chi(\upsilon,\tau) \doteq \frac{a(\tau)}{2}\upsilon^2 + b(\tau)\upsilon + \tau. \tag{3.113}$$

Assume that $\Psi\colon (\upsilon,\tau) \mapsto (\upsilon, \chi(\upsilon,\tau))$ is a Lagrangian homeomorphism from $\mathbb{R}^2$ to $\Psi(\mathbb{R}^2)$ associated to ∇^f. Then:

(i) For all $\tau_1 \neq \tau_2 \in \mathbb{R}$,

$$\begin{aligned} &\textit{either} \quad a(\tau_1) = a(\tau_2) \ \textit{and} \ b(\tau_1) = b(\tau_2) \\ &\textit{or} \qquad 2\frac{a(\tau_1) - a(\tau_2)}{\tau_1 - \tau_2} > \left(\frac{b(\tau_1) - b(\tau_2)}{\tau_1 - \tau_2}\right)^2. \end{aligned} \tag{3.114}$$

(ii) *Assume in addition that* $a, b \in W^{1,1}_{\mathrm{loc}}(\mathbb{R})$ *and* Ψ *is locally bi-Sobolev of exponent* $\bar{p} > \frac{3}{2}$. *Then for a.e.* $\tau \in \mathbb{R}$ *we have*

$$\begin{aligned} &\textit{either} \quad a'(\tau) = b'(\tau) = 0 \\ &\textit{or} \qquad 2a'(\tau) > b'(\tau)^2. \end{aligned} \tag{3.115}$$

Proof. Part *(i)* is proved in [Gol18, Lemma 3.3] as a consequence of the definition of Ψ as a homeomorphism (in particular, injectivity).
In order to prove *(ii)* we proceed in a similar fashion to [NS19, Lemma 5.3]. By the assumption that a, b belong to $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, they are almost everywhere differentiable in $\mathbb{R}$ with derivatives in L^1_{loc}; if $\hat{\tau} \in \mathbb{R}$ is a differentiability point for both a and b, then:

- by the first part the inequality $2a'(\hat{\tau}) \geq b'(\hat{\tau})^2$ is granted, thus it only remains to show that equality holds only if both terms are zero;
- if $2a'(\hat{\tau}) = b'(\hat{\tau})^2$, then for any $\upsilon \in \mathbb{R}$ we have:

$$\partial_\tau \chi(\upsilon, \hat{\tau}) = \frac{b'(\hat{\tau})^2}{4}\upsilon^2 + b'(\hat{\tau})\upsilon + 1 = \left(\frac{b'(\hat{\tau})}{2}\upsilon + 1\right)^2, \tag{3.116}$$

  which is null at $\upsilon = -\frac{2}{b'(\hat{\tau})}$.

Let now E_k be the set defined by

$$E_k \doteq \left\{ \tau \in [-2k, 2k] \;\middle|\; \begin{array}{l} a \text{ and } b \text{ are differentiable at } \tau, \\ 2a'(\tau) = b'(\tau)^2 \geq k^{-2} \end{array} \right\} \qquad \text{for } k \in \mathbb{N}, \tag{3.117}$$

and $\tilde{E}_k \doteq [-2k, 2k] \times E_k$. The set E_k is obtained as the intersection of the differentiability sets of a and b (which are measurable and have full measure) and the sets $(2a' - (b')^2)^{-1}(\{0\})$ and $(a')^{-1}\left(\left[\frac{1}{2k^2}, \infty\right)\right)$, which are measurable themselves because $a', b' \in L^1_{\mathrm{loc}}$. Then by the Fubini-Tonelli theorem

$$\int_{\tilde{E}_k} \frac{1}{|\partial_\tau \chi(\upsilon, \tau)|^{\bar{p}-1}} \,\mathrm{d}\upsilon\,\mathrm{d}\tau = \int_{E_k} \left(\int_{-2k}^{2k} \frac{1}{\left|\frac{b'(\tau)}{2}\upsilon + 1\right|^{2\bar{p}-2}} \,\mathrm{d}\upsilon \right) \mathrm{d}\tau. \tag{3.118}$$

Now we can observe what follows:

- Since $\left| -\frac{2}{b'(\tau)} \right| \leq |2k|$ for any $\tau \in E_k$ by the definition of E_k itself, the internal integral at the right hand side of (3.118) diverges for any $\tau \in E_k$ (because $\bar{p} \geq \frac{3}{2}$);
- Moreover, by the second part Proposition 3.31 and the Sobolev assumption on Ψ, we know that the left hand side of (3.118) is finite.

Hence, we conclude that $\mathscr{L}^1(E_k) = 0$ for all $k \in \mathbb{N}$; as a consequence, the set

$$E \doteq \left\{ \tau \in \mathbb{R} \;\middle|\; \begin{array}{l} a \text{ and } b \text{ are differentiable at } \tau, \\ 2a'(\hat{\tau}) = b'(\hat{\tau})^2 \neq 0 \end{array} \right\} = \bigcup_{k \in \mathbb{N}} E_k \tag{3.119}$$

is $\mathscr{L}^1$-negligible as well. $\square$

Next, we show how stability interacts with the existence of Lagrangian paramterizations. The results is a new inequality derived from the second variation formula (Equation (3.124)).

Lemma 3.55 (Integration by parts). *Let $h \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n) \cap \mathbf{C}^0_c(\mathbb{R}^n)$. Let $i \in \{1, \dots, n\}$. Let $g \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$ be such that for almost every $\hat{\mathbf{x}}_i \doteq (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \mathbb{R}^{n-1}$ the map $g_{\hat{\mathbf{x}}_i} \doteq g(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$, and $\partial_{x_i} g \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} h(\partial_{x_i} g)\, \mathrm{d}\mathscr{L}^n = -\int_{\mathbb{R}^n} (\partial_{x_i} h) g \, \mathrm{d}\mathscr{L}^n \tag{3.120}$$

holds.

Proof. By a classical result ([EG15, Theorem 4.21 *(i)*]), for almost every $\hat{\mathbf{x}}_i \in \mathbb{R}^{n-1}$ the map $h_{\hat{\mathbf{x}}_i} \doteq h(x_1, \dots, x_{i-1}, \cdot, x_{i+1}, \dots, x_n)$ belongs to $W^{1,1}_{\mathrm{loc}}(\mathbb{R})$; thus $\partial_{x_i}(hg)$ exists almost everywhere $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} \partial_{x_i}(hg) \mathrm{d}\mathscr{L}^n = \int_{\mathbb{R}^n} h(\partial_{x_i} g) + \int_{\mathbb{R}^n} (\partial_{x_i} h) g \, \mathrm{d}\mathscr{L}^n; \tag{3.121}$$

as a further consequence, $\partial_{x_i}(hg) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Now by Fubini's Theorem

$$\int_{\mathbb{R}^n} \partial_{x_i}(hg) \mathrm{d}\mathscr{L}^n = \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} (h_{\hat{\mathbf{x}}_i} g_{\hat{\mathbf{x}}_i})' \, \mathrm{d}x_i \right) \mathrm{d}\hat{\mathbf{x}}_i, \tag{3.122}$$

and the internal integral equals 0, by the local absolute continuity of $h_{\hat{\mathbf{x}}_i} g_{\hat{\mathbf{x}}_i}$ and the fact that h has compact support. $\square$

Lemma 3.56. *Let $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\mathbb{R}^2) \cap \mathbf{C}^0(\mathbb{R}^2; \mathbb{R})$, and let $\Psi : \mathbb{R}^2 \to \mathbb{R}^2$ be a locally $\hat{p}$-bi-Sobolev Lagrangian homeomorphism associated to ∇^f, with $\hat{p} > 2$. Assume that Ψ has the form*

$$\Psi(\upsilon, \tau) = \left(\upsilon, \frac{a(\tau)}{2}(\upsilon - \hat{\upsilon})^2 + b(\tau)(\upsilon - \hat{\upsilon}) + \tau \right) \tag{3.123}$$

*with $\hat{\upsilon} \in \mathbb{R}$ and $a, b \in W^{1,\gamma}_{\mathrm{loc}}(\mathbb{R})$. Finally, assume that f is stable (*i.e.*, it is stationary and satisfies Equation (3.26 | 2VF)). Then for all $\tilde{\varphi} \in W^{1,\sigma}(\mathbb{R}^2) \cap \mathbf{C}^0_c(\mathbb{R}^2)$ it holds that*

(3.124)

$$\int_{\mathbb{R}^2} \left[(\partial_\upsilon \tilde{\varphi})^2 \frac{\frac{a'}{2}(\upsilon - \hat{\upsilon})^2 + b'(\upsilon - \hat{\upsilon}) + 1}{(1 + a^2)^{\frac{3}{2}}} - \tilde{\varphi}^2 \frac{2a' - (b')^2}{\left(\frac{a'}{2}(\upsilon - \hat{\upsilon})^2 + b'(\upsilon - \hat{\upsilon}) + 1\right)(1 + a^2)^{\frac{3}{2}}} \right] \mathrm{d}\upsilon \, \mathrm{d}\tau \geq 0,$$

where σ is chosen so that $\sigma \geq 2\left(\frac{\hat{p}}{\hat{p}-2}\right)^2$

Proof. The proof follows essentially the same lines and computations as [NS19, Lemma 5.4], up to making sure that the involved tools are still exploitable in the Sobolev setting. We can assume without loss of generality that $\hat{\upsilon} = 0$.

Step 1. Let $\tilde{\varphi} \in W^{1,\sigma}(\mathbb{R}^2) \cap \mathbf{C}^0_c(\mathbb{R}^2)$; the map $\varphi \doteq \tilde{\varphi} \circ \Psi^{-1}$ belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2) \cap \mathbf{C}^0_c(\mathbb{R}^2)$ by Proposition 3.36, thus it can be used as a test function in Equation (3.26 | 2VF). In particular, using the Area Formula (3.36 | AF),

$$\int_{\mathbb{R}^2} \left[\left(\frac{\left(\nabla^f \varphi + (\partial_t f) \varphi \right)^2}{(1 + (\nabla^f f)^2)^{\frac{3}{2}}} + \frac{(\nabla^f f) \partial_t (\varphi^2)}{(1 + (\nabla^f f)^2)^{\frac{1}{2}}} \right) \circ \Psi \right] (\partial_\tau \chi) \, \mathrm{d}\mathscr{L}^2 \geq 0 \tag{3.125}$$

Here we exploit the formulas in Corollary 3.39 and find that:

$$\begin{aligned} &\left(\nabla^f \varphi\right) \circ \Psi = \partial_\upsilon \tilde{\varphi} && \partial_t f \circ \Psi = \frac{\partial_\tau \partial_\upsilon \chi}{\partial_\tau \chi} = \frac{a'\upsilon + b'}{\frac{a'}{2}\upsilon^2 + b'\upsilon + 1} \\ &\left(\partial_t \varphi^2\right) \circ \Psi = 2\tilde{\varphi}\frac{\partial_\tau \tilde{\varphi}}{\partial_\tau \chi} = \frac{\partial_\tau(\tilde{\varphi}^2)}{\partial_\tau \chi} && \left(\nabla^f f\right) \circ \Psi = \partial_\upsilon^2 \chi = a(\tau) \end{aligned} \tag{3.126}$$

so that the previous inequality becomes

(3.127)
$$0 \geq \int_{\mathbb{R}^2} \frac{\partial_\tau \chi}{(1+a^2)^{\frac{3}{2}}} \left[(\partial_\upsilon \tilde{\varphi})^2 + \frac{a'\upsilon + b'}{\partial_\tau \chi} \partial_\upsilon(\tilde{\varphi}^2) + \frac{(a'\upsilon + b')^2}{(\partial_\tau \chi)^2} \tilde{\varphi}^2 + a(1+a^2)\frac{\partial_\tau(\tilde{\varphi}^2)}{\partial_\tau \chi} \right] \mathrm{d}\mathscr{L}^2 =$$

(3.128)
$$= \int_{\mathbb{R}^2} \left[\frac{\partial_\tau \chi}{(1+a^2)^{\frac{3}{2}}} \left((\partial_\upsilon \tilde{\varphi})^2 + \frac{(a'\upsilon + b')^2}{(\partial_\tau \chi)^2} \right) + \partial_\upsilon(\tilde{\varphi}^2) \frac{a'\upsilon + b'}{(1+a^2)^{\frac{3}{2}}} + \partial_\tau(\tilde{\varphi}^2) \frac{a}{(1+a^2)^{\frac{1}{2}}} \right] \mathrm{d}\mathscr{L}^2 =$$

(3.129)
$$= \int_{\mathbb{R}^2} \left[\frac{\partial_\tau \chi}{(1+a^2)^{\frac{3}{2}}} \left((\partial_\upsilon \tilde{\varphi})^2 + \frac{(a'\upsilon + b')^2}{(\partial_\tau \chi)^2} \right) - \tilde{\varphi}^2 \frac{a'}{(1+a^2)^{\frac{3}{2}}} - \tilde{\varphi}^2 \frac{a'}{(1+a^2)^{\frac{3}{2}}} \right] \mathrm{d}\mathscr{L}^2 =$$

(3.130)
$$= \int_{\mathbb{R}^2} (\partial_\upsilon \tilde{\varphi})^2 \frac{\frac{a'}{2}\upsilon^2 + b'\upsilon + 1}{(1+a^2)^{\frac{3}{2}}} + \frac{(a'\upsilon + b')^2 - 2a'\left(\frac{a'}{2}\upsilon^2 + b'\upsilon + 1\right)}{(1+a^2)^{\frac{3}{2}}\left(\frac{a'}{2}\upsilon^2 + b'\upsilon + 1\right)} \mathrm{d}\mathscr{L}^2$$

where the equality in (3.129) comes from the integration by parts (Lemma 3.55); up to rearranging the terms in the last line, the statement is proved. □

Corollary 3.57. *Let $a, b \in W^{1,\gamma}_{\mathrm{loc}}(\mathbb{R})$ be such that Equation (3.124) holds for any $\tilde{\varphi} \in W^{1,\sigma}(\mathbb{R}^2) \cap \mathbf{C}^0_c(\mathbb{R}^2)$. Then for almost every $\hat{\tau} \in \mathbb{R}$ and for every $\psi \in W^{1,\sigma}(\mathbb{R}) \cap \mathbf{C}^0_c(\mathbb{R})$ it holds that*

$$\begin{aligned} &\int_{\mathbb{R}} (\psi'(\upsilon))^2 \left(\frac{a'(\hat{\tau})}{2}(\upsilon - \hat{\upsilon})^2 + b'(\hat{\tau})(\upsilon - \hat{\upsilon}) + 1 \right) \mathrm{d}\upsilon \geq \\ &\qquad \geq \left(2a'(\hat{\tau}) - (b'(\hat{\tau}))^2\right) \int_{\mathbb{R}} \frac{(\psi(\upsilon))^2}{\frac{a'(\hat{\tau})}{2}(\upsilon - \hat{\upsilon})^2 + b'(\hat{\tau})(\upsilon - \hat{\upsilon}) + 1} \mathrm{d}\upsilon. \end{aligned} \tag{3.131}$$

Proof. For simplicity, denote by λ the function

$$\lambda(\upsilon, \tau) \doteq \frac{a'(\tau)}{2}(\upsilon - \hat{\upsilon})^2 + b'(\tau)(\upsilon - \hat{\upsilon}) + 1, \tag{3.132}$$

defined for every $\upsilon \in \mathbb{R}$ and $\tau \in \mathbb{R}$ up to modifications on a negligible set. Let $\varrho_\varepsilon \in \mathbf{C}^\infty_c$ be the standard uni-dimensional mollifier for any $\varepsilon > 0$; fix $\hat{\tau} \in \mathbb{R}$ such that a and b are differentiable at $\hat{\tau}$, and $\psi \in W^{1,\sigma}(\mathbb{R}) \cap \mathbf{C}^0_c(\mathbb{R})$ and let

$$\tilde{\varphi}_\varepsilon(\upsilon, \tau) \doteq \psi(\upsilon)\sqrt{\varrho_\varepsilon(\tau - \hat{\tau})}. \tag{3.133}$$

so that $\partial_\upsilon \tilde{\varphi}_\varepsilon(\upsilon, \tau) = \psi'(\upsilon)\sqrt{\varrho_\varepsilon(\tau - \hat{\tau})}$.

By Equation (3.124) and the Fubini-Tonelli Theorem, it holds that
(3.134)
$$\int_{\mathbb{R}} \varrho_\varepsilon(\tau-\hat\tau)\left(\frac{1}{(1+a^2)^{\frac{3}{2}}}\int_{\mathbb{R}}\Big(\psi'(\upsilon)^2\Big)\lambda(\upsilon,\tau)\,\mathrm{d}\upsilon\right)\mathrm{d}\tau \geq \int_{\mathbb{R}} \varrho_\varepsilon(\tau-\hat\tau)\left(\frac{2a'-(b')^2}{(1+a^2)^{\frac{3}{2}}}\int_{\mathbb{R}}\frac{\psi(\upsilon)^2}{\lambda(\upsilon,\tau)}\,\mathrm{d}\upsilon\right)\mathrm{d}\tau.$$

Letting $\varepsilon \downarrow 0$, and recalling the standard properties of pointwise convergence of the regularizations,

$$\frac{1}{(1+a^2)^{\frac{3}{2}}}\int_{\mathbb{R}}\Big(\psi'(\upsilon)^2\Big)\lambda(\upsilon,\tau)\,\mathrm{d}\upsilon = \frac{2a'-(b')^2}{(1+a^2)^{\frac{3}{2}}}\int_{\mathbb{R}}\frac{\psi(\upsilon)^2}{\lambda(\upsilon,\tau)}\,\mathrm{d}\upsilon \tag{3.135}$$

holds, which is exactly the needed identity. □

As a last step towards the Bernstein-type Theorem 3.59, we recall a lemma from [BSV07] which clarifies how Equation (3.131) will be used:

Lemma 3.58. *Let $A, B \in \mathbb{R}$ be such that $B^2 \leq 2A$, and set $h(\upsilon) \doteq \frac{A}{2}\upsilon^2 + B\upsilon + 1$. If*

$$\int_{\mathbb{R}} \psi'(\upsilon)^2 h(\upsilon)\,\mathrm{d}\upsilon \geq (2A-B^2)\int_{\mathbb{R}}\frac{\psi(\upsilon)^2}{h(\upsilon)}\,\mathrm{d}\upsilon \quad \textit{for every } \psi \in \mathbf{C}^1_c(\mathbb{R}), \tag{3.136}$$

then $B^2 = 2A$.

Proof. See [BSV07, p. 45]. □

Finally, we state a Bernstein-type Theorem for stable maps, in the most general form available from the results in this section.

Theorem 3.59. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy Assumption 3.52, with $\bar q$ and $\bar p$ satisfying Equation (3.66 | PQ) and $\bar p > 3+\sqrt{5}$. Assume that f is stable in $\mathbb{R}^2$. Then there exist $\hat\upsilon \in \mathbb{R}$ and $a, b \in \mathbb{R}$ such that*

$$\Psi(\upsilon,\tau) \doteq \left(\upsilon, \frac{a}{2}(\upsilon-\hat\upsilon)^2 + b(\upsilon-\hat\upsilon)+\tau\right), \tag{3.137}$$

defines a Lagrangian homeomorphism associated to ∇^f mapping the whole $\mathbb{R}^2$ onto itself. In particular, $\nabla^f f$ is constant, and the intrinsic graph of f is a vertical intrinsic plane.

Proof. First of all, by stationariety, f admits a $\hat p$-Sobolev Lagrangian homeomorphism which has the form of Equation (3.108 | Ψ) with γ-Sobolev coefficients a and b (Theorem 3.53). By our assumption on $\bar p$, the new Sobolev coefficient $\hat p$ turns out to be greater than 2.
Up to translations in the first coordinate, by Lemmas 3.56 and 3.58 and Corollary 3.57, for almost every $\tau \in \mathbb{R}$ it holds that $b'(\tau)^2 = a'(\tau)$. By Lemma 3.54 (again up to translating $\hat\upsilon$ to 0), then a and b must satisfy $a'(\tau) = b'(\tau) \equiv 0$ for almost every $\tau \in \mathbb{R}$. Since a and b are absolutely continuous, this implies they are constant in $\mathbb{R}$. □

3.5 Regularity of f

In this final Section, we retrieve a condition that ensures the validity of Assumption 3.52. As already mentioned, we make use of a result proved by Ambrosio, Nicolussi Golo and Serra Cassano in [ANS], which has not yet been published at the writing of this thesis and has been personally communicated to the author.

Let us first introduce the condition we will need: no *a priori* existence on parametrizations is required.

Assumption 3.60. We assume that the following hold:

- **Sobolev regularity of f:** $f \in W^{1,\bar{q}}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}) \cap \mathbf{C}^0(\mathbb{R}^2)$ for some $\bar{q} > 2$;
- **Exponential summability for $\partial_t f$:** there exists $\beta \in [1,\infty)$ such that

$$\exp(|\partial_t f(\cdot,\cdot)|) \in L^{\beta}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}); \tag{3.138}$$

  equivalently, $\exp(|\beta\,\partial_t f(\cdot,\cdot)|) \in L^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R})$. ◊

***Remark* 3.61.** It is easy to see that the second condition implies that in fact $\partial_t f \in L^q_{\mathrm{loc}}(\mathbb{R}^2)$ for any $q \in [1,\infty)$. Moreover, if $M(\beta,q) \doteq \inf\left\{x > 0 \,\middle|\, x^q \le e^{\beta x}\right\}$ and $K \subset \mathbb{R}^2$ is a compact set, then the estimate

$$\int_K |\partial_t f(y,t)|^q \,\mathrm{d}y\,\mathrm{d}t \le M(\beta,q)^q \mathscr{L}^2(K) + \|\exp(|\partial_t f(\cdot,\cdot)|)\|_{L^\beta(K)} \tag{3.139}$$

holds.

By the same observation, notice that if $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{R}) \cap \mathbf{C}^0(\mathbb{R}^2)$ is such that $\exp(|\nabla f|) \in L^{\beta}_{\mathrm{loc}}(\mathbb{R}^2)$ for some $\beta \ge 1$, then both conditions in Assumption 3.60 are satisfied. ◊

The exponential summability condition on $\partial_t f$ implies that f is well posed as a time-dependent vector field. Although this is somewhat known in literature, we give here a self-contained proof based on the Osgood criterion, which provides the uniqueness for solutions ([Har02, Chapter III, Corollary 6.2]), while the existence is ensured by the continuity of f; we first introduce such criterion in the form we need it. Again, the reader should be aware that the notation might be misleading: the variable y should be thought as the *time* variable (see Remark 3.20).

Lemma 3.62 (Osgood criterion). *Let $I,\Omega \Subset \mathbb{R}$ and let $f \in \mathbf{C}^0(I\times\Omega,\mathbb{R})$. Assume that there exist $\varphi \in L^1(I,[0,+\infty))$ and a modulus of continuity $\xi \in \mathbf{C}^0([0,\infty),[0,\infty))$ such that*

$$\int_0^1 \frac{\mathrm{d}\delta}{\xi(\delta)} = \infty, \quad \xi(0) = 0, \quad \textit{and} \quad \xi(\delta) > 0 \textit{ for all } \delta > 0 \tag{3.140}$$

and

$$|f(y,t_1) - f(y,t_2)| \le \varphi(y)\xi(|t_1 - t_2|) \tag{3.141}$$

for all $y \in I$, $t_1,t_2 \in \Omega$. Then f is well posed.

Well-posedness under Assumption 3.60 follows. The same result can be extended to weaker conditions on the integrability of $\partial_t f$ ([ANS]).

Proposition 3.63 (Well posedness under Assumption 3.60). *Let $I, \Omega \Subset \mathbb{R}$ be bounded open intervals and let $f \in W^{1,1}(I \times \Omega; \mathbb{R}) \cap \mathbf{C}^0(I \times \Omega)$ be such that*

$$\int_{I \times \Omega} \exp(\beta |\partial_t f|(y,t)) \, \mathrm{d}\mathscr{L}^2 < \infty \tag{3.142}$$

for some $\beta \geq 1$. Then f is well posed as a time-dependent vector field.

Proof. By the absolute continuity of $f(y, \cdot)$ on almost every line ([EG15, Theorem 4.21]), for almost every y and for every $t_1 < t_2 \in \Omega$ one has

$$\frac{|f(y,t_1) - f(y,t_2)|}{|t_1 - t_2|} \leq \frac{1}{|t_1 - t_2|} \int_{t_1}^{t_2} |\partial_t f(y,t)| \, \mathrm{d}t. \tag{3.143}$$

Then by the Jensen Inequality it holds that:

$$\exp\left(\beta \frac{|f(y,t_1) - f(y,t_2)|}{|t_1 - t_2|}\right) \leq \frac{1}{|t_1 - t_2|} \int_{t_1}^{t_2} \exp(\beta |\partial_t f(y,t)|) \, \mathrm{d}t \tag{3.144}$$

and by algebraic computations

$$\beta \frac{|f(y,t_1) - f(y,t_2)|}{|t_1 - t_2|} \leq \log\left(\frac{2|\Omega|}{|t_1 - t_2|} \int_\Omega \frac{1}{2|\Omega|} \sup\{4, \exp(\beta |\partial_t f(y,t)|)\} \, \mathrm{d}t\right), \tag{3.145}$$

where we denote by $|\Omega|$ the length $\mathscr{L}^1(\Omega)$ of Ω. Now notice that both $\frac{2|\Omega|}{|t_1 - t_2|}$ and the integral at the right hand side of Equation (3.145) are greater than or equal to 2; and there exists $K > 0$ such that

$$\log(ab) \leq K a \log b \text{ for any } a, b \geq 2 \tag{3.146}$$

– in other words, $(a,b) \mapsto \frac{\log(ab)}{a \log b}$ is bounded from above on $[2, \infty)^2$. Thus, if we let

$$\xi(\delta) \doteq -\beta \delta \log \delta \tag{3.147}$$

$$\varphi(y) \doteq \frac{1}{2|\Omega|} \int_\Omega \sup\{4, \exp \beta |\partial_t f(y,t)|\} \, \mathrm{d}t, \tag{3.148}$$

then $\varphi \in L^1(I)$ by the assumption (3.142) and the Fubini-Tonelli Theorem; moreover, in that case, Equation (3.145) gives exactly

$$|f(y,t_1) - f(y,t_2)| \leq \xi(|t_1 - t_2|) \varphi(y), \tag{3.149}$$

which is what we need to apply the Osgood criterion (Lemma 3.62). □

The following result we state is really the only one we need from [ANS], and is here exploited to make sure that the flow of the non-autonomous vector field f has Sobolev regularity. Let us first precise what we mean by *flow* of a well-posed vector field; everything will be particularized to the case of one spatial dimension.

Definition 3.64 (Flow of a vector field). Let $I, \Omega \Subset \mathbb{R}$ and let $f \in \mathbf{C}^0(I \times \Omega, \mathbb{R})$ be a well-posed non-autonomous vector field. For any pair $(\upsilon_0, \tau_0) \in I \times \Omega$, let I_{υ_0, τ_0} be the maximal interval on which the solution $\gamma_{\upsilon_0, \tau_0}$ to

$$\begin{cases} \dfrac{\mathrm{d}\gamma}{\mathrm{d}\upsilon} = f(\upsilon, \gamma(\upsilon)) \\ \gamma(\upsilon_0) = \tau_0 \end{cases} \tag{3.150}$$

is defined. For any $\upsilon, \upsilon_0 \in I$, we denote by $\Omega_{\upsilon,\upsilon_0} \subset \Omega$ the set of points τ_0 such that the solution starting from τ_0 at time υ_0 still exists at time υ:

$$\Omega_{\upsilon,\upsilon_0} \doteq \{\tau_0 \in \Omega \mid I_{\upsilon_0,\tau_0} \ni \upsilon\}. \tag{3.151}$$

Then we define $X_{\upsilon,\upsilon_0} \colon \Omega_{\upsilon,\upsilon_0} \to \Omega$ as

$$X_{\upsilon,\upsilon_0}(\tau_0) \doteq \gamma_{\upsilon_0,\tau_0}(\upsilon) \tag{3.152}$$

and the *flow* of f as

$$X(\upsilon;\upsilon_0,\tau_0) \doteq X_{\upsilon,\upsilon_0}(\tau_0) \tag{3.153}$$

for any $\upsilon, \upsilon_0, \tau_0$ for which it is defined. ◊

The regularity result we need is here stated in its local form:

Theorem 3.65 (Regularity of the flow). *Let $I \subset \mathbb{R}$ be a bounded interval and $\Omega \subset \mathbb{R}$ be a bounded open set. Let $f \in L^1_{\mathrm{loc}}(I, W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R})) \cap \mathbf{C}^0_{\mathrm{c}}(I \times \Omega, \mathbb{R})$ be a compactly supported time-dependent vector field defined on $I \times \Omega$. Denote by ℓ the length of I.*

Assume that there exists some $p > 1$ such that

$$\int_I \int_\Omega \exp\left(\frac{\ell p^2}{p-1}|\partial_\tau f(\upsilon,\tau)|\right) \mathrm{d}\tau\, \mathrm{d}\upsilon < \infty. \tag{3.154}$$

Then (f is well posed and)

$$X_{\upsilon,\upsilon_0} \in W^{1,p}(\Omega,\mathbb{R}) \quad \textit{with} \quad \int_\Omega |\partial_\tau X(\upsilon;\upsilon_0,\tau)|^p \,\mathrm{d}\tau \le \Lambda_p \tag{3.155}$$

for any $\upsilon, \upsilon_0 \in I$, where

$$\Lambda_p \doteq \ell^{\frac{1}{1-p}} \int_I \int_\Omega \exp\left(\frac{\ell p^2}{p-1}|\partial_\tau f(\upsilon,\tau)|\right) \mathrm{d}\tau\, \mathrm{d}\upsilon < \infty. \tag{3.156}$$

Thanks to Theorem 3.65, then, we find:

Proposition 3.66. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ satisfy Assumption 3.60. Then it satisfies Assumption 3.52. More precisely, for any $\bar p > 1$ there exists $\ell > 0$ such that any open bounded connected subset $\omega \subset \mathbb{R}^2$ of width smaller than ℓ (that is, $\mathscr{L}^1(\pi_1(\omega)) < \ell$) admits a locally bi-Sobolev Lagrangian homeomorphism associated to ∇^f with Sobolev exponent $\bar p$.*

Proof. Let $\bar p > 1$ be fixed, and let

$$\ell \doteq \frac{\bar p - 1}{\bar p^2}\beta. \tag{3.157}$$

Fix a bounded connected domain $\omega \subset \mathbb{R}^2$ of width smaller than ℓ, and let $I, \Omega \subset \mathbb{R}$ be bounded intervals such that $|I| \le \ell$ and ω is compactly contained in $I \times \Omega$.

Step 1. Let $\varphi \in \mathbf{C}^\infty_{\mathrm{c}}(I \times \Omega)$ be a smooth cut-off function such that $0 \le \varphi \le 1$, $\varphi \equiv 1$ on ω, and φ has compact support in $I \times \Omega$; consider $\hat f \doteq \varphi f$ in $I \times \mathbb{R}$. It is now enough to prove that $I \times \Omega$ admits a locally $\bar{p}$-bi-Sobolev Lagrangian homeomorphism Ψ associated to $\nabla^{\hat{f}}$: in that case, the restriction of such a homeomorphism to $\Psi^{-1}(\omega)$ is a locally bi-Sobolev Lagrangian homeomorphism associated to ∇^f which parametrizes ω. Notice that $\partial_t \hat{f}$ satisfies

$$\exp\Big(|\partial_t \hat{f}|\Big) \leq \exp\Big(\|\partial_t \varphi\|_\infty \|f\|_{L^\infty(I\times\Omega)}\Big) \exp(|\partial_t f|) \in L^\beta_{\mathrm{loc}}(I \times \Omega) \tag{3.158}$$

and $\|\hat{f}\|_{L^\infty(I\times\Omega)} = \|f\|_{L^\infty(\omega)}$.

Step 2. By Equation (3.158) and by Proposition 3.63, the non-autonomous vector field $\hat{f}$ is well posed; since it is also bounded in $I \times \Omega$, the solution to the Cauchy problem

$$\begin{cases} \dot{\gamma}(\upsilon) = \hat{f}(\upsilon, \gamma(\upsilon)) \\ \gamma(\upsilon_0) = \tau_0 \end{cases} \tag{3.159 | CP}$$

is defined on the whole interval I for any pair $(\upsilon_0, \tau_0) \in I \times \Omega$. Let us select a fixed time $\hat{\upsilon} \in I$. Using the same notations as in Definition 3.64, for any $\upsilon \in I$ and $\tau \in \Omega$ we define:

$$\chi(\upsilon, \tau) \doteq X(\upsilon; \hat{\upsilon}, \tau) = X_{\upsilon,\hat{\upsilon}}(\tau), \tag{3.160}$$

$$\Psi(\upsilon, \tau) \doteq (\upsilon, \chi(\upsilon, \tau)); \tag{3.161}$$

namely, $\chi(\upsilon, \tau)$ is the evaluation at time υ of the solution to Equation (3.159 | CP) with initial time $\hat{\upsilon}$ and initial position τ. Let us show that $\Psi \colon I \times \Omega \to I \times \Omega$ is a locally $\bar{p}$-bi-Sobolev Lagrangian homeomorphism between $I \times \Omega$ and itself.

Step 3. Ψ is bijective: this is guaranteed by the uniqueness of the solution to Equation (3.159 | CP) and by the global existence for any initial datum. Notice that for any $(\upsilon, \tau) \in I \times \Omega$ the image $\Psi(\upsilon, \tau)$ still belongs to $I \times \Omega$ thanks to the fact that $\hat{f}$ has compact support.

Step 4. For any $\upsilon \in I$, $\tau \mapsto \chi(\upsilon, \tau)$ is a locally $\bar{p}$-Sobolev homeomorphism of Ω onto Ω; moreover, for any $\upsilon \in I$ it holds that

$$\int_\Omega |\partial_\tau \chi(\upsilon, \tau)|^{\bar{p}} \,\mathrm{d}\tau \leq \Lambda_p(\Omega) \tag{3.162}$$

where Λ_p is a constant only depending on Ω (and thus independent of υ). This is a direct consequence of the previous Theorem 3.65: indeed, by Assumption 3.60 (and in particular (3.138)) and our choice of ℓ (Equation (3.157)), it holds that $\exp\Big(\frac{\ell \bar{p}^2}{\bar{p}-1}|\partial_t \hat{f}|\Big)$ lies in $L^1(I \times \Omega)$. Thus, Equation (3.155) is satisfied on Ω.

Step 5. The map $\tau \mapsto \chi(\upsilon, \tau)$ is invertible for any $\upsilon \in I$, and its inverse is $\chi_\upsilon^{-1}(\tau) = X(\hat{\upsilon}; \upsilon, \tau)$: this follows again by the uniqueness of the solution to Equation (3.159 | CP). In particular, following the same argument of the previous point, χ_υ^{-1} is locally $\bar{p}$-Sobolev; and thus $\chi(\upsilon, \cdot)$ is a locally $\bar{p}$-bi-Sobolev homeomorphism.

Step 6. The map $\upsilon \to \chi(\upsilon, \tau)$ is Lipschitz in I for any τ, with Lipschitz constant bounded by $\|f\|_{L^\infty(\omega)}$. Indeed,

$$|\chi(\upsilon, \tau) - \chi(\upsilon', \tau)| = \left| \int_{\upsilon'}^{\upsilon} \hat{f}(s, \chi(s, \tau)) \,\mathrm{d}s \right| \leq \|f\|_{L^\infty(\omega)} |\upsilon - \upsilon'|. \tag{3.163}$$

An analogous argument shows that $\upsilon \mapsto \chi_\upsilon^{-1}(\tau)$ is also Lipschitz with the same constant.

Step 7. The map $\chi : I \times \Omega \to \Omega$ is continuous: indeed, if $(\upsilon, \tau), (\upsilon', \tau') \in I \times \Omega$, then

$$|\chi(\upsilon, \tau) - \chi(\upsilon', \tau')| \leq |\chi(\upsilon, \tau) - \chi(\upsilon, \tau')| + \|f\|_{L^\infty(\Omega)} |\upsilon - \upsilon'| \tag{3.164}$$

and the conclusion follows by the continuity of $\chi(\upsilon,\cdot)$.

STEP 8. The map $(\upsilon,\tau)\mapsto\Psi(\upsilon,\tau)$ is locally $\bar p$-bi-Sobolev. We show that it is $\bar p$-Sobolev, since the regularity of Ψ^{-1} works similarly. The following upper bounds

$$\|\chi(\upsilon,\cdot)\|_{W^{1,\bar p}(\Omega)}\le\Lambda_p(\Omega)\qquad\text{for any } \upsilon\in I, \tag{3.165}$$

$$\|\chi(\cdot,\tau)\|_{W^{1,\infty}(I)}\le\|f\|_{L^\infty(\omega)}\qquad\text{for any } \tau\in\mathbb{R} \tag{3.166}$$

hold (uniformly in υ and τ respectively); as a consequence, it is easy to see that the derivatives of such functions are actually the weak partial derivatives of χ (see [HK14, Theorem A.15]), and for any $\hat\omega\Subset I\times\mathbb{R}$ we have that $\partial_\tau\chi\in L^{\bar p}(\hat\omega)$, $\partial_\upsilon\chi\in L^\infty(\hat\omega)$. Moreover, by standard properties of the operator norm, the weak differential of Ψ satisfies

$$\|\mathrm{D}\Psi(\upsilon,\tau)\|_{\mathrm{op}}^{\bar p}\le\Big(1+|\partial_\tau\chi(\upsilon,\tau)|^2+|\partial_\upsilon\chi(\upsilon,\tau)|^2\Big)^{\frac{\bar p}{2}}\le C(\bar p)\Big(1+|\partial_\tau\chi(\upsilon,\tau)|^{\bar p}+|\partial_\upsilon\chi(\upsilon,\tau)|^{\bar p}\Big), \tag{3.167}$$

thus for any $\hat\omega\Subset I\times\mathbb{R}$ the norm $\|\mathrm{D}\Psi\|_{L^{\bar p}(\hat\omega)}$ is finite.

STEP 9. Collecting all the information we proved so far, Ψ is a continuous (STEP 7) and bijective (STEP 3) map from $I\times\mathbb{R}$ to itself; thus it is a homeomorphism (by the Invariance of Domain); it is locally $\bar p$-bi-Sobolev (STEP 8); by the definition of χ (STEP 2), $\chi(\upsilon,\cdot)$ is non-decreasing for any $\upsilon\in I$ and χ solves $\partial_\upsilon\chi(\upsilon,\tau)=\hat f(\upsilon,\chi(\upsilon,\tau))$, thus Ψ is actually a Lagrangian homeomorphism associated to $\hat f$. □

Corollary 3.67. *Let $f:\mathbb{R}^2\to\mathbb{R}$ satisfy Assumption 3.60, with $\bar q$ high enough. If f is stable, then its intrinsic graph is an intrinsic plane.*

Proof. The statement is a consequence of Proposition 3.66 and Theorem 3.59; the only thing to notice is that one needs $\bar q$ to be sufficiently big so that there exists $\bar p\in(3+\sqrt5,\bar q]$ such that $\bar p,\bar q$ satisfy Equation (3.66 | PQ). □

Chapter 4

Minimizers of the p-energy among maps taking values on a Riemannian manifold

The main issue of this chapter will be the problem of estimating the dimension of the *singular set* of a map taking values in a Riemannian manifold, and minimizing the p-Dirichlet energy. The strategies and proofs will be based on the article [Ved21]: the main difference in the results obtained is that here we explicit the case of *stationary* maps under specific additional assumptions on p and on the target (assumptions borrowed from [TW95], which we use extensively).

It is clear that, contrary to the case of real-valued p-harmonic functions (or even $\mathbb{R}^N$-valued), no global smoothness result can hold: the simple example of the projection map $x \mapsto \frac{x}{|x|}$ to the sphere shows that there exist p-energy minimizing maps that have discontinuities. We should also take in consideration that *external variations* alone are not sufficient to achieve *any* regularity result: a result of Rivière [Riv95] shows that one can build *weakly p-harmonic* maps from the n-dimensional ball to the $(n-1)$-dimensional sphere that are discontinuous everywhere.

Our result will be stated in terms of the Minkowski (or Hausdorff) dimension of the *singular set*, which is bounded by $m - \lfloor p \rfloor - 1$; moreover, we will retrieve an upper bound on the appropriate Minkowski content (or Hausdorff measure) of the singular set, and its rectifiability. Our strategy is based on the scheme introduced in [NV17], which in turn refines the strategy of [CN13b].

4.1 Setting

Assumption 4.1 (Setting). The setting (and consequent notations for spaces, dimensions and parameters) will be the following:

- $\Omega \subset \mathbb{R}^m$ is an open connected domain which contains a large enough ball $B_{\bar{R}}(0)$. The size of the needed radius $\bar{R}$ will be the result of a sequence of intermediate lemmas, thus it will be better explained as the argument develops: for the moment, let us just fix the condition $\bar{R} > 0$.

- $(\mathcal{N}, h_{\mathcal{N}})$ is an n-dimensional closed (compact with no boundary) Riemannian manifold. Thanks to the celebrated Nash Embedding Theorem (see [Nas54]), we can assume that $\mathcal{N}$ isometrically embeds in a Euclidean space $\mathbb{R}^N$, with N high enough.

- u is a map from Ω to $\mathcal{N}$.
- $p \in (1, m)$, where m is the dimension of the domain. $\Diamond$

When u is a map between Ω and $\mathcal{N}$, we define its *singular set* as follows:

Definition 4.2 (Singular set). Let $u : \Omega \to \mathcal{N}$. Then the *singular set* of u is defined as the subset

$$\mathcal{S}(u) \doteq \{x \in \Omega \mid u \text{ is not continuous at } x\} \tag{4.1}$$

of Ω. $\Diamond$

***Remark* 4.3.** A bit of clarification regarding Assumption 4.1:

- The overall goal of this chapter will be to estimate from above the Hausdorff dimension of $\mathcal{S}(u) \cap B_1(0)$, and study its k-rectifiability, when u is a p-harmonic map. We thus allow ourselves to be vague about the assumptions on the domain Ω (Assumption 4.1) because the problem we will address is of *local* nature: up to rescaling, there's no harm in the assumption we have made – *i.e.*, that $\Omega \supset B_{\bar{R}}(0)$ with $\bar{R} > 4$.
- More generally, again by the locality of the problem, the argument we will use also works when $\Omega \subset \mathcal{M}$, an m-dimensional Riemannian manifold.
- In the literature, alternative assumptions on the target manifold $\mathcal{N}$ can be found; for our purposes, instead of considering an abstract closed manifold, it would be sufficient to ask that $\mathcal{N}$ is a (not necessarily closed) Riemannian manifold embedded in $\mathbb{R}^N$ and that u takes values in a fixed compact subset of $\mathcal{N}$. $\Diamond$

Before discussing various definitions regarding p-harmonicity, let us fix some notations on manifold-valued Sobolev mappings.

Definition 4.4 (Sobolev maps into $\mathcal{N}$). Let $\Omega, \mathcal{N}, p$ be as in Assumption 4.1, with $\mathcal{N}$ embedded in $\mathbb{R}^N$.

The manifold-valued Sobolev space $W^{1,p}(\Omega, \mathcal{N})$ is defined as

$$W^{1,p}(\Omega, \mathcal{N}) \doteq \left\{ u \in W^{1,p}\left(\Omega, \mathbb{R}^N\right) \,\middle|\, u(x) \in \mathcal{N} \text{ for almost every } x \in \Omega \right\}, \tag{4.2}$$

where $W^{1,p}\left(\Omega, \mathbb{R}^N\right)$ is the class of multi-valued Sobolev maps from Notation 1.20. $\Diamond$

For Sobolev maps in $W^{1,p}(\Omega, \mathcal{N})$, we define the p-Dirichlet energy as follows:

Definition 4.5 (p-energy). The p-energy functional $\mathcal{E}_p : W^{1,p}(\Omega, \mathcal{N}) \to \mathbb{R}$ is defined as

$$\mathcal{E}_p(u) \doteq \int_\Omega |\nabla u|^p \, \mathrm{d}x = \int_\Omega \left(\sum_{i=1}^m \sum_{\alpha=1}^N \left(\frac{\partial u^\alpha}{\partial x_i}(x) \right)^2 \right)^{\frac{p}{2}} \mathrm{d}x. \tag{4.3}$$ $\Diamond$

***Remark* 4.6** (The Riemannian p-energy). Up to adapting some details, the definition of $\mathcal{E}_p$ also works in the case where Ω is a domain in a Riemannian manifold $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$ is an arbitrary (not necessarily embedded) Riemannian manifold. Indeed, in this case, one can write

$$\mathcal{E}_p(u) \doteq \int_\Omega |\nabla u(x)|^p_{\mathcal{N}} \,\mathrm{d}\,\mathrm{vol}_g, \tag{4.4}$$

where vol_g is the volume form associated to the metric g. In a local coordinate chart, we would have:

$$|\nabla u(x)| = \sqrt{g^{ij}(x)\left\langle \frac{\partial u}{\partial x_i}(x), \frac{\partial u}{\partial x_j}(x)\right\rangle_{\mathcal{N}}}, \qquad \mathrm{d}\,\mathrm{vol}_g = \sqrt{\det g}\,\mathrm{d}\mathscr{L}^m, \tag{4.5}$$

where the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{N}}$ is the Riemannian scalar product in the target $\mathcal{N}$. $\Diamond$

The aim of this chapter is to study the maps which are critical points or minimizers of the p-energy functional. The results we will describe work under general conditions in the case of minimality; under stronger assumptions on $\mathcal{N}$, it will instead be enough to assume criticality with respect to two classes of variations:

Definition 4.7. Let $u \in W^{1,p}(\Omega, \mathcal{N})$. We say that:

1. u is a *weakly p-harmonic* map if it is a critical point of the p-energy functional with respect to *external variations*, *i.e.*: for any $\xi \in \mathbf{C}^\infty_c(\Omega, \mathbb{R}^N)$, it holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\nabla(\Pi_{\mathcal{N}}(u + t\xi))|^p \bigg|_{t=0} = 0, \tag{4.6}$$

   where $\Pi_{\mathcal{N}}$ is the nearest-point projection onto $\mathcal{N}$, defined on a tubular neighborhood of $\mathcal{N}$ itself.

2. u is a *stationary p-harmonic* map if it is weakly p-harmonic and it is a critical point of the p-energy functional with respect to compact variations in the domain (*internal variations*). Explicitly: let $\Phi = \{\varphi_t\}_{t\in I}$ be any smooth family of diffeomorphisms of Ω, with I open interval containing 0; assume that $\varphi_0 \equiv \mathrm{id}_\Omega$, and that there exists a compact set $K \subset \Omega$ such that $\varphi_t|_{\Omega\setminus K} = \mathrm{id}_{\Omega\setminus K}$ for any $t \in I$; then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_p(u \circ \varphi_t)\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla(u \circ \varphi_t)(x)|^p dx\bigg|_{t=0} = 0. \tag{4.7}$$

3. u is a p-energy minimizing map if $\mathcal{E}_p(u) \le \mathcal{E}_p(v)$ for any compact set $K \subset \Omega$ and for any $v \in W^{1,p}(\Omega, \mathcal{N})$ such that $u|_{\Omega\setminus K} \equiv v|_{\Omega\setminus K}$ (more precisely: $u = v$ almost everywhere in $\Omega \setminus K$). $\Diamond$

Before making some initial considerations about the above definitions, let us establish a notation for the blow up of a map:

Notation 4.8 (Blow ups). If $\Omega \subset \mathbb{R}^m$ is open, $B_r(x) \subset \Omega$ and $u : \Omega \to \mathcal{N}$, we denote by $\frac{\Omega - x}{r}$ the set

$$\frac{\Omega - x}{r} \doteq \{y \in \mathbb{R}^m \mid x + ry \in \Omega\} \supset B_1(0); \tag{4.8}$$

moreover, we define the *blow-up* of u (centered at x, with scale r) as the map $T_{x,r}u : \frac{\Omega - x}{r} \to \mathcal{N}$ defined as

$$T_{x,r}u(y) = u(x + ry). \tag{4.9}$$

◊

***Remark* 4.9** (Blow up of p-harmonic maps)**.** It is trivial to see that each of the three properties listed in the above Definition 4.7 is stable under blow ups: if $u \in W^{1,p}(\Omega, \mathcal{N})$ is weakly p-harmonic (resp. p-stationary, resp. p-energy minimizing) in Ω, then $T_{x,r}u \in W^{1,p}\left(\frac{\Omega - x}{r}, \mathcal{N}\right)$ is weakly p-harmonic (resp. p-stationary, resp. p-energy minimizing) in $\frac{\Omega - x}{r}$. ◊

***Remark* 4.10.** The class of p-energy minimizers clearly contains that of stationary p-harmonic ones, which by definition contains that of weakly p-harmonic maps. The condition (4.7) without the requirement of weak harmonicity is sometimes referred to as *weak Noether $p-$harmonicity* [Hél02]; if u belongs to $\mathbf{C}^2(\Omega, \mathcal{N})$ and is weakly p-harmonic, then it is also stationary: indeed any internal variation can be retrieved as an external one (see for example [Hél02, Theorem 1.3.6 and Paragraph 1.4.5]). ◊

***Remark* 4.11** (The case $p \geq m$)**.** Notice that:

- If $p > m$, then any map in $W^{1,p}(\Omega, \mathcal{N})$ is Hölder continuous by the Sobolev embedding theorem. In particular $\mathcal{S}(u) = \varnothing$ whenever $u \in W^{1,p}(\Omega, \mathcal{N})$.
- Even in the equality case, it has been proven (for example in [NVV19, Theorem 2.19]) that p-energy minimizers have no singular points.

In sight of this, the assumption $1 < p < m$ we initially made is not restrictive and captures all the interesting cases. ◊

Notation 4.12 (Second fundamental form, stress p-energy tensor)**.** By taking admissible variations in Definition 4.7, one can show that weak p-harmonicity and p-stationariety can be associated to suitable Euler-Lagrange equations (Theorem 4.13): let us fix some notation which will be useful in the next Theorem 4.13:

- We denote by $\mathcal{A}$ the second fundamental form of the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^N$: it is defined as the unique map that associates a pair of tangent vector fields $(X, Y) \in \mathcal{T}(\mathcal{N}) \times \mathcal{T}(\mathcal{N})$ to a normal vector field $\mathcal{A}(X, Y) \in (\mathcal{T}(\mathcal{N}))^\perp$ in such a way that

$$\langle \nu, \mathcal{A}(X, Y) \rangle = \langle \nabla_X \nu, Y \rangle \tag{4.10}$$

  for every normal section $\nu : \mathcal{N} \to (\mathcal{T}(\mathcal{N}))^\perp$. With a slight abuse of notation, when $u \in W^{1,p}(\Omega; \mathcal{N})$ we will also denote with $\mathcal{A}$ the operator

$$\mathcal{A}(\nabla u, \nabla u) = \sum_{i=1}^{m} \mathcal{A}(\nabla_i u, \nabla_i u). \tag{4.11}$$

- If $u \in W^{1,p}(\Omega, \mathcal{N})$, we denote by $\mathsf{S}(u) = \mathsf{S}_{ik}(u)$ the *stress p-energy tensor*:

$$\mathsf{S}_{ij}(u) \doteq |\nabla u|^{p-2} \left[\frac{1}{p} |\nabla u|^2 \delta_{ij} - \langle \nabla_i u, \nabla_j u \rangle \right]. \tag{4.12}$$

◊

The following theorem collects the essential information on the Euler-Lagrange equations satisfied by p-harmonic maps. We refer to [Mos05] for proofs and discussions.

Theorem 4.13. *Let $\Omega, \mathcal{N}, u, p$ be as in Assumption 4.1. Then:*

A. If u is weakly p-harmonic, then it satisfies

$$-\Delta_p u = |\nabla u|^{p-2}\mathcal{A}(u)(\nabla u, \nabla u) \tag{4.13 | WH}$$

in the distributional sense, that is:

$$\int_\Omega |\nabla u|^{p-2}\langle \nabla u, \nabla\varphi\rangle \, \mathrm{d}x = -\int_\Omega |\nabla u|^{p-2}\mathcal{A}(u)(\nabla u, \nabla u)\varphi \, \mathrm{d}x, \tag{4.14}$$

for any $\varphi \in \mathbf{C}_c^\infty(\Omega, \mathbb{R}^N)$.

B. If u is stationary p-harmonic, then it satisfies

$$\operatorname{div} \mathsf{S}(u) = 0 \tag{4.15 | SH}$$

in the distributional sense, that is:

$$\int_\Omega |\nabla u|^{p-2} \sum_{i,j=1}^m \left[p\langle \nabla_i u, \nabla_j u\rangle - |\nabla u|^2 \delta_{ij}\right] \frac{\partial X^k}{\partial x_i} \, \mathrm{d}x = 0 \tag{4.16}$$

for any $X \in \mathbf{C}_c^\infty(\Omega, \mathbb{R}^m)$.

4.1.1 The projection map to the sphere

In order to give a motivation for the study of singular sets of p-harmonic maps, we devote this paragraph to a basic example of a p-harmonic map with a point of discontinuity.

Indeed, a notable example of such a phenomenon is the projection of the unit ball on its boundary, namely the map $x \mapsto \frac{x}{|x|}$; which, as one can immediately see, has a singularity at the origin. In this subsection, we will restrict to the case $m = N = n + 1$, and consider the map

$$u_0 : \mathbb{B}^m \to \mathbb{S}^{m-1}, \qquad u(x) = \frac{x}{|x|}. \tag{4.17}$$

The minimality of the p-energy of u_0 has been widely studied [HKL86; CG89; AL88; HLW98; Hon01; Bou06] and a complete picture is still missing (namely, minimality is not yet established for all values of p).

We begin with weak-harmonicity:

Proposition 4.14. *For any $1 \le p < m$, the map u_0 is weakly p-harmonic.*

Proof. All the terms appearing in Equation (4.13 | WH) can be computed by hand: we have, for all $x \neq 0$ and $1 \le j, k \le m$:

$$u_0^k(x) = \frac{x_k}{|x|}, \qquad \text{and thus} \qquad \frac{\partial u_0^k}{\partial x_j}(x) = \frac{|x|^2\delta_{jk} - x_j x_k}{|x|^3}; \tag{4.18}$$

more compactly,

$$\nabla u_0(x) = \frac{|x|^2 \operatorname{id} - x \cdot x^T}{|x|^3}, \qquad \text{and} \qquad |\nabla u_0(x)| = \frac{\sqrt{m-1}}{|x|}. \tag{4.19}$$

As a side note, this shows in particular that $u_0 \in W^{1,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$ for any $p \in [1, m)$. Moreover, one has:

- $\operatorname{div}\Big(|\nabla u_0|^{p-2} \nabla u_0\Big)$ can be computed as

  (4.20)
  $$\begin{aligned}\operatorname{div}\Big(|\nabla u_0|^{p-2} \nabla u_0\Big) &= (m-1)^{\frac{p}{2}-1} \operatorname{div} \frac{|x|^2 \operatorname{id} - x \cdot x^T}{|x|^{p+1}} = \\ &= (m-1)^{\frac{p}{2}-1} \left[\frac{\operatorname{div}(|x|^2 \operatorname{id} - x \cdot x^T)}{|x|^{p+1}} - \frac{\Big(|x|^2 \operatorname{id} - x \cdot x^T\Big) \cdot \nabla\Big(|x|^{p+1}\Big)}{|x|^{2p+2}} \right] = \\ &= (m-1)^{\frac{p}{2}-1} \frac{(1-m)x}{|x|^{p+1}} = -(m-1)^{\frac{p}{2}} \frac{x}{|x|^{p+1}},\end{aligned}$$

  where we have used that $\nabla\Big(|x|^{p+1}\Big) = (p+1)|x|^{p-1}x$.

- The second fundamental form of the sphere $\mathbb{S}^{m-1}$, evaluated at the point $x \in \mathbb{S}^{m-1}$, is given by

  $$\mathcal{A}(X, Y) = \langle X(x), Y(x) \rangle x \qquad \text{for any } X, Y \in \Gamma(T\mathbb{S}^{m-1}). \tag{4.21}$$

  Thus the right hand side of Equation (4.13 | WH) is given by

  $$|\nabla u_0|^{p-2} \mathcal{A}(u_0)(\nabla u_0, \nabla u_0) = |\nabla u_0|^{p-2} |\nabla u_0|^2 u = \frac{(m-1)^{\frac{p}{2}}}{|x|^{p+1}} x. \tag{4.22}$$

This proves the statement. □

The question of minimality is actually more delicate. For $p \in \{1, 2, \ldots, m-1\}$, a direct argument by Avellaneda and Lin is available ([AL88], based on a similar strategy by Lin [Lin87] for the case $p = 2$):

Theorem 4.15. *Let $m \geq 2$, and $p \in \{1, \ldots, m-1\}$. Then the projection map $u_0 : \mathbb{B}^m \to \mathbb{S}^{m-1}$ minimizes the p-energy.*

Sketch of proof. For any subset $I \subset \{1, \ldots, m\}$ and map $u \in \mathbf{C}^\infty(\mathbb{B}^m, \mathbb{R}^m)$, let $\omega_I(u)$ be defined as the differential form

$$\omega_I(u) \doteq \bigwedge_{i=1}^m \beta_i, \qquad \text{with } \beta_i = \begin{cases} du^i & \text{if } i \in I \\ dx_i & \text{if } i \notin I \end{cases}; \tag{4.23}$$

then, for any $u \in \mathbf{C}^\infty(\mathbb{B}^m, \mathbb{R}^m)$ define

$$S_p(u) \doteq \sum_{|I|=p} \omega_I(u). \tag{4.24}$$

Notice that:

1. By density of smooth functions, the definition of S_p admits a natural extension to the case $u \in W^{1,p}(\mathbb{B}^m, \mathbb{R}^m)$.
2. By the Stokes' Theorem,

$$S_p(u) = S_p(v) \text{ whenever } u \text{ and } v \text{ coincide on } \partial\mathbb{B}^m = \mathbb{S}^{m-1}. \tag{4.25}$$

3. $S_p(u)$ is rotationally invariant: if $R \in \mathrm{SO}(m)$, then $S_p(Ru)(Rx) = S_p(u)(x)$.
4. *As a side note:* when $p = 2$, the expression for S_p sensibly simplifies (*cfr.* [Lin87]):

$$\begin{aligned} S_2(u) &= \left[\sum_{1\le i,j\le m}\left(\frac{\partial u^i}{\partial x^i}\frac{\partial u^j}{\partial x^j} - \frac{\partial u^j}{\partial x^i}\frac{\partial u^i}{\partial x^j}\right)\right]\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n = \\ &= \Big[(\operatorname{div} u)^2 - \operatorname{tr}(\nabla u)^2\Big]\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n. \end{aligned} \tag{4.26}$$

Now we first estimate the p-energy of a map v with the integral of $S_p(v)$, we compute the latter when $v \equiv u_0$ on $\mathbb{S}^{m-1}$, then we show that for $v = u_0$ the two expressions coincide.

Step 1. Let us fix $x \in \mathbb{B}^m$ and an arbitrary $v \in W^{1,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$. By rotational invariance, we can assume that $v(x) = (0, \dots, 0, 1)$: this implies in particular that we can assume $\frac{\partial v}{\partial x_n}(x) = 0$. By triangle inequality we have that

$$|S_p(v)(x)| \le \sum_{|I|=p} |\omega_I(v)(x)| \tag{4.27}$$

The norm of a decomposable m-covector of the form $\beta_1 \wedge \cdots \wedge \beta_m$ can be explicitly computed through

$$|\beta_1 \wedge \cdots \wedge \beta_m|^2 = \det(\{\langle\beta_i, \beta_j\rangle\}_{1\le i,j\le m}) = \det(B)^2, \tag{4.28}$$

where B is the matrix having the expressions of the β_i's with respect to the canonical basis of $\mathbb{R}^m$ as rows. By the Hadamard's inequality, then, for any such m-covector we have

$$|\beta_1 \wedge \cdots \wedge \beta_m| \le \prod_{i=1}^m \|\beta_i\|; \tag{4.29}$$

going back to Equation (4.27), this implies

$$|S_p(v)(x)| \le \sum_{1\le i_1<\cdots<i_p\le m} |\mathrm{d}v^{i_1}(x)| \dots |\mathrm{d}v^{i_p}(x)|, \tag{4.30}$$

and by our assumption $v(x) = (0, \dots, 0, 1)$ the summation at the right hand side can be restricted to the indices $1 \le i_1 < \cdots < i_p \le m-1$. By the classical MacLaurin's inequality and the Cauchy-Schwarz inequality, the right hand side can be estimated by

$$\begin{aligned} \sum_{1\le i_1<\cdots<i_p\le m-1} |\mathrm{d}v^{i_1}(x)| \dots |\mathrm{d}v^{i_p}(x)| &\le \frac{1}{(m-1)^p}\binom{m-1}{p}\left(\sum_{i=1}^{m-1}|\mathrm{d}v^i(x)|\right)^p \\ &\le \frac{1}{(m-1)^p}\binom{m-1}{p}(m-1)^{\frac{p}{2}}\left(\sum_{i=1}^{m-1}|\mathrm{d}v^i(x)|^2\right)^{\frac{p}{2}} \\ &= \frac{1}{(m-1)^{\frac{p}{2}}}\binom{m-1}{p}|\nabla v(x)|^p. \end{aligned} \tag{4.31}$$

Joining the estimates in Equation (4.30) and Equation (4.31), and integrating in $\mathbb{B}^m$, we get

$$\int_{\mathbb{B}^m} S_p(v)(x) \le \int_{\mathbb{B}^m} |S_p(v)(x)| \, \mathrm{d}x \le \frac{1}{(m-1)^{\frac{p}{2}}} \binom{m-1}{p} \mathcal{E}_p(v). \tag{4.32}$$

Step 2. Assume now that the map v satisfies $v = u_0 = \mathrm{id}$ on the boundary $\partial\mathbb{B}^m = \mathbb{S}^{m-1}$. By the Stokes' Theorem (Equation (4.25)),

$$\int_{\mathbb{B}^m} S_p(v)(x) = \int_{\mathbb{B}^m} S_p(\mathrm{id})(x) = \int_{\mathbb{B}^m} \binom{m}{p} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m = \binom{m}{p} \omega_m, \tag{4.33}$$

where $\mathrm{id}(x) = x$ is the identity map and ω_m is the volume of $\mathbb{B}^m$. In particular by Equation (4.32), for any such v,

$$\mathcal{E}_p(v) \ge \frac{(m-1)^{\frac{p}{2}}}{\binom{m-1}{p}} \binom{m}{p} \omega_m = (m-1)^{\frac{p}{2}} \frac{m}{m-p} \omega_m. \tag{4.34}$$

Step 3. On the other hand, the p-energy of u_0 can be computed explicitly: since $|\nabla u_0(x)| = \frac{\sqrt{m-1}}{|x|}$, we have

$$\mathcal{E}_p(u_0) = (m-1)^{\frac{p}{2}} \int_{\mathbb{B}^m} \frac{\mathrm{d}x}{|x|^p} = (m-1)^{\frac{p}{2}} \int_0^1 \frac{r^{m-1}\mathscr{H}^{m-1}(\mathbb{S}^{m-1})}{r^p} \, \mathrm{d}r = (m-1)^{\frac{p}{2}} \frac{m\omega_m}{m-p}. \tag{4.35}$$

Thus, for any map $v \in W^{1,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$ which coincides with u_0 on $\partial\mathbb{B}^m$ we have proved that $\mathcal{E}_p(v) \ge \mathcal{E}_p(u_0)$. □

***Remark* 4.16** (What about non-integer p's?)**.** To the best of our knowledge, the minimality of u_0 for the p-energy functional has been proved in the following cases:

- When $p \in \{1, \dots, m-1\}$ is an integer (first in [CG89]);
- When $p \in (m-1, m)$ ([HLW98]);
- When $p \in (2, m - 2\sqrt{m-1}]$ ([Wan98]).

In [Hon01], the author observed that the p-minimality of $u \in W^{1,p}(\mathbb{B}^m, \mathbb{S}^{m-1})$ follows if one can show that u minimizes the *weighted p-energy* $\int_{\mathbb{B}^m} |x|^{2-p} |\nabla u| \, \mathrm{d}x$. In the same paper, the claim that u_0 actually minimizes the weighted energy for any $p \in (1, m)$ is made; however, it was later found in [Bou06] that the proof contained an error, and it is even possible to built a counterexample. The general case thus seems to remain open. ◊

4.1.2 Measures for the singular set

In sight of Section 4.1.1, it makes sense to look for results that, roughly speaking, ensure that the singular set is *not too big*. Our main regularity result about p-harmonic maps can be stated in the form of an estimate on the *dimension* of the singular set. Let us makes this statement more precise.

Definition 4.17 (Hausdorff measure). Let $k \in [0, +\infty)$. We denote by ω_k the quantity

$$\omega_k \doteq \frac{\pi^{\frac{k}{2}}}{\Gamma\left(1 + \frac{k}{2}\right)}, \tag{4.36}$$

where $\Gamma(a) \doteq \int_0^\infty e^{-x} x^{a-1}\, \mathrm{d}x$ is the Euler function – notice that $\omega_k = \mathscr{L}^k(B_1(0))$ when k is an integer.

For any $\delta > 0$ and $S \subset \mathbb{R}^m$, we denote by $\mathscr{H}^k_\delta(S)$ the quantity

$$\mathscr{H}^k_\delta(S) \doteq \inf\left\{ \sum_{i=0}^\infty \omega_k \left(\frac{\operatorname{diam}(S_i)}{2}\right)^k \,\middle|\, S \subset \bigcup_{i=0}^\infty S_i \text{ and } \operatorname{diam}(S_i) < \delta \right\}. \tag{4.37}$$

Then the *k-dimensional Hausdorff measure* of S is defined as

$$\mathscr{H}^k(S) \doteq \lim_{\delta \downarrow 0} \mathscr{H}^k_\delta(S). \tag{4.38}$$

$\mathscr{H}^k$ is then an outer measure on $\mathbb{R}^m$, and hence its restriction to the σ-algebra of the $\mathscr{H}^k$-measurable sets (in the Carathéodory sense) is a measure. ♢

A related but non coincident concept is the Minkowski content of a set:

Definition 4.18. Let $k \in [0, +\infty)$, and let ω_k be defined as in Definition 4.17. Let $S \subset \mathbb{R}^m$ We define the *upper* and *lower k-dimensional Minkowski content* of S respectively as

$$\mathscr{M}^k_*(S) \doteq \liminf_{\varrho \downarrow 0} \frac{\mathscr{L}^m(\mathcal{B}_\varrho(S))}{\omega_{m-k}\varrho^{m-k}} \tag{4.39}$$

$$\mathscr{M}^{k*}(S) \doteq \limsup_{\varrho \downarrow 0} \frac{\mathscr{L}^m(\mathcal{B}_\varrho(S))}{\omega_{m-k}\varrho^{m-k}}. \tag{4.40}$$

When $\mathscr{M}^k_*(S) = \mathscr{M}^{k*}(S)$, we denote by $\mathscr{M}^k(S)$ the common value. ♢

Lemma 4.19. *Let $S \subset \mathbb{R}^m$, and let $0 \leq k < h$.*

(i) If $\mathscr{M}^k_*(S) < \infty$, then $\mathscr{M}^h_*(S) = 0$. The same holds for $\mathscr{M}^{k*}$ and $\mathscr{M}^{h*}$.

(ii) If S is Borel-measurable, the above implication also holds for $\mathscr{H}^k$ and $\mathscr{H}^h$.

This allows to define the Hausdorff and Minkowski dimensions as follows:

Definition 4.20. Let $S \subset \mathbb{R}^m$ be a Borel-measurable set. We define:

$$\dim_{\mathscr{M}_*}(S) \doteq \inf\left\{k \in [0, \infty) \,\middle|\, \mathscr{M}^k_*(S) = 0\right\} \tag{4.41}$$

$$\dim_{\mathscr{M}^*}(S) \doteq \inf\left\{k \in [0, \infty) \,\middle|\, \mathscr{M}^{k*}(S) = 0\right\} \tag{4.42}$$

$$\dim_{\mathscr{H}}(S) \doteq \inf\left\{k \in [0, \infty) \,\middle|\, \mathscr{H}^k(S) = 0\right\}.$$
♢

***Remark* 4.21** ($\mathscr{H}^k$ and $\mathscr{M}^k$). Some remarks are in order:

- The definition of Hausdorff measure can be generalized to any metric space: indeed, the only notion involved in the definition is that of *diameter* of a set, which is available whenever a distance is present.

- The Minkowski content itself has a metric-measure counterpart: in Remark 1.37 we defined an *outer* Minkowski content of sets, which had to be thought as an alternative to the perimeter measure; if $S \subset \mathbb{R}^m$ is a Lebesgue-measurable set with $\mathscr{L}^m(S) = 0$, then the outer Minkowski content of Remark 1.37 coincides with the lower $(m-1)$-dimensional Minkowski content of Definition 4.18.

- The Minkowski content is *not* a measure, as it is not additive on disjoint sets; moreover, simple (fractal) examples show that in general $\mathscr{M}^k_* < \mathscr{M}^{k*}$ can hold strictly (see [Mat95, Section 5.5]). ♢

4.1.3 Rectifiability

A second result we will achieve concerns the *rectifiability* of the singular set. In this paragraph, we give a brief introduction to this notion, again in the Euclidean setting (see [Mat95, Chapter 15]):

Definition 4.22 (k-rectifiability)**.** Let $S \subset \mathbb{R}^m$ be a $\mathscr{H}^k$-measurable set for some $k \in \{1, \dots, m\}$. We say that S is *k-rectifiable* if there exist countably many Lipschitz maps $\{f_i\}_{i\in\mathbb{N}}$ from $\mathbb{R}^k$ to $\mathbb{R}^m$ such that

$$\mathscr{H}^k\left(S \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k)\right) = 0 \tag{4.43}$$

holds. ♢

***Remark* 4.23.** Let us briefly comment the above definition:

(a) First of all, the term "k-rectifiable" is not universally associated in literature to the above notion: some classical sources ([Fed69; AFP00; Mag12]) prefer the expression "countably $\mathscr{H}^k$-rectifiable", and keep the term "k-rectifiable" for the cases in which $S \setminus \bigcup_{i=0}^{\infty} f_i(\mathbb{R}^k)$ is actually empty.

(b) Being k-rectifiable in the above sense is equivalent to being $\mathscr{H}^k$-almost everywhere contained in a countable union of smooth k-dimensional submanifolds of $\mathbb{R}^m$ [Fed69, Theorem 3.2.29]: while this shows the importance of the notion (up to arbitrarily small sets, a k-rectifiable set is a finite union of smooth pieces), this approach is often proves less convenient than working with Lipschitz maps.

(c) In the Euclidean setting, rectifiability carries several useful consequences, such as the existence and uniqueness of approximate tangent spaces [Mat95, Theorem 15.19], suitable forms of the area formula [AFP00, Theorem 2.91] and of the coarea formula [AFP00, Theorem 2.93]. More generally, notions of rectifiability comparable to the above-given one appear ubiquitously in many areas of geometric measure theory – let us just mention that *rectifiable currents* play a fundamental role in the study of the regularity theory for mass-minimizing currents [Alm00; FF60].

(d) Also in the general context of metric measure spaces with curvature bounds from below (see Section 1.3), rectifiability has been widely studied in the recent years; namely, it has been proved that $\mathrm{RCD}(K, N)$ spaces are actually rectifiable in an appropriate sense (see

[CC00b, Section 5] for the Ricci limit case, [MN19, Theorem 1.1] for the general RCD case). $\Diamond$

We also state here an estimate of the Hausdorff measure with the Minkowski content in the case of rectifiable sets: we refer to [AFP00, Proposition 2.101] for a proof.

Lemma 4.24 ($\mathscr{M}^k_* \leq \mathscr{H}^k$). *Let $S \subset \mathbb{R}^m$ be $\mathscr{H}^k$-measurable, with $k \in \{1, \dots, m\}$. If S is k-rectifiable, then $\mathscr{H}^k(S) \leq \mathscr{M}^k_*(S)$.*

4.1.4 Main results for the singular set

Equipped with the definitions given in Sections 4.1.2 and 4.1.3, we can finally state the main result of this Chapter. We need one last definition:

Definition 4.25 (Homogeneous space, left invariant metric). We say that the manifold $\mathcal{N}$ is a *homogeneous space* if it can be realized as a quotient G/H, where G is a connected Lie group and H is a closed subgroup. Recall that a *Lie group* is a group which is also equipped with a differentiable manifold structure, so that the map $(x, y) \mapsto x \cdot y^{-1}$ is differentiable; moreover, we denote by G/H the space of left cosets $\{xH \mid x \in G\}$.

We say that the metric g on the homogeneous space $\mathcal{N} \simeq G/H$ is *left-invariant* if the action of G on G/H induced by the left multiplication acts by isometries (with respect to g). $\Diamond$

Theorem 4.26 (Singular set). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a map with p-energy bounded by Λ. Assume that one of the following two conditions hold:*

(m) either u is minimizing (with $\mathcal{N}$ arbitrary closed Riemannian manifold without boundary);

(s) or $\mathcal{N}$ is a homogeneous space with a left-invariant metric, p is not an integer, and u is p-stationary.

Then there exists a constant $C_{\mathcal{S}}(m, \mathcal{N}, \Lambda, p)$ such that for any $r > 0$

$$\mathscr{L}^m(\mathcal{B}_r(\mathcal{S}(u)) \cap B_1(0)) \leq C_{\mathcal{S}} r^{\lfloor p \rfloor + 1}. \tag{4.44}$$

In particular, the Minkowski dimension of $\mathcal{S}(u)$ is at most $m - \lfloor p \rfloor - 1$, and the upper Minkowski content is bounded by $C_{\mathcal{S}}$.

Furthermore, the singular set $\mathcal{S}(u)$ is $(m - \lfloor p \rfloor - 1)$-rectifiable. In particular, by Lemma 4.24, the k-dimensional Hausdorff measure of $\mathcal{S}(u)$ is also bounded by $C_{\mathcal{S}}$.

The proof will follow from the following two facts, which will be proved in the next sections: $\mathcal{S}(u)$ is contained in a *quantitative singular stratum* (Proposition 4.49); and such stratum satisfies a bound on the Minkowski content analogous to Equation (4.44) and the same rectifiability property (Theorem 4.56).

***Remark* 4.27.** It is worth remarking that no such regularity result can hold if one only assumes weak p-harmonicity: in the case $p = 2$, a famous theorem by Rivière [Riv95] shows that for any $\mathbf{C}^\infty$ boundary datum one can find a weakly 2-harmonic map from the 3-dimensional unit ball to the unit sphere in $\mathbb{R}^3$ such that the singular set coincides with the whole ball $\bar{B}_1(0) \subset \mathbb{R}^3$. $\Diamond$

4.2 Monotonicity formula and cone splitting

Various formulations of the Monotonicity Formula for the normalized energy have been used in the context of p-harmonic maps; the most convenient to introduce here is a *weighted* version, for technical reasons which will become clear in the next Sections.

Assumption 4.28 (Weight). In the sequel, $\psi \in \mathbf{C}_c^\infty([0,\infty))$ is a fixed non-negative function, satisfying

$$\operatorname{spt}(\psi) = [0, t_b], \qquad \psi(0) = 1, \tag{4.45}$$

$$\psi'(t) < 0 \quad \text{in } [0, t_b), \qquad \psi'(t) \le -\xi \quad \text{in } [0, t_a) \tag{4.46}$$

for some fixed numbers $0 < t_a < t_b$ and $\xi > 0$. Moreover, since this will be needed in Section 4.6, we will actually assume $2 < t_a < t_b$: this choice will be better explained in Remark 4.74. When $x, y \in \mathbb{R}^m$ and $r > 0$, we will also denote by $\Psi_{x,r}$ the radial function $\Psi_{x,r}(y) \doteq \psi\left(\frac{|y-x|}{r}\right)$. ♢

Definition 4.29 (Weighted normalized energy). Let u be a $W^{1,p}(\Omega, \mathcal{N})$ map. Let $\psi \in \mathbf{C}_c^\infty([0,\infty))$ be as in Assumption 4.28. For all $x \in \Omega$ and $r > 0$ such that $B_{t_b r}(x) \subset \Omega$, we define the *weighted normalized p-energy* as the function

$$\vartheta_{p,\psi}[u](x,r) \doteq r^{p-m} \int_\Omega \Psi_{x,r}(y) |\nabla u(y)|^p \, \mathrm{d}y = r^{p-m} \int_{B(x,t_b r)} \psi\left(\frac{|y-x|}{r}\right) |\nabla u(y)|^p \, \mathrm{d}y. \tag{4.47}$$

We will always drop the subscripts p and ψ, and most of times also the argument $[u]$, since it will be clear from the context. ♢

***Remark* 4.30** (Classical normalized energy). A common choice in the literature is to define ϑ as in Equation (4.47), but with $\psi = \chi_{[0,1]}$ being the characteristic function of the interval $[0,1]$ (which is not an admissible choice in our definition). The consequences are only of technical nature rather than substantial: we just remark here that for any $B_r(t_b x) \subset \Omega$

$$\psi(1)\vartheta_{p,\chi_{[0,1]}}(x,r) \le \vartheta_{p,\psi}(x,r) \le t_b^{m-p} \vartheta_{p,\chi_{[0,1]}}(x, t_b r); \tag{4.48}$$

in particular, for example, the results of Section 4.3 (obtained in the literature for the classical normalized energy) still work here just by adjusting the constants involved. ♢

Proposition 4.31 (Scale invariance). *If $u \in W^{1,p}(\Omega, \mathcal{N})$, $x \in B_1(0)$ and $r > 0$, the following identity holds:*

$$\vartheta[T_{x,r} u](0,1) = \vartheta[u](x,r). \tag{4.49}$$

As a further consequence, if also $w \in \frac{\Omega - x}{r}$ and $\tau > 0$ is small enough, then:

$$\vartheta[T_{x,r} u](w,\tau) = \vartheta[u](x + rw, r\tau). \tag{4.50}$$

Proof. The result is a rather simple computation: since the identity

$$\nabla(T_{x,r} u)(y) = r \nabla u(x + ry) \tag{4.51}$$

holds for any $y \in \frac{\Omega - x}{r}$, then we have

$$\vartheta[T_{x,r}u](w,\tau) = \tau^{p-m} \int_{\frac{\Omega-x}{r}} \psi\left(\frac{|x+rw-(x+ry)|}{r\tau}\right) r^p |\nabla u(x+ry)|^p \,\mathrm{d}y. \tag{4.52}$$

By the change of variables $z = x + ry$,

$$\vartheta[T_{x,r}u](w,\tau) = (r\tau)^{p-m} \int_\Omega \psi\left(\frac{x+rw-z}{r\tau}\right) |\nabla u(z)|^p \,\mathrm{d}z, \tag{4.53}$$

which is what we need. □

Theorem 4.32 (Monotonicity formula). *Let u be a stationary p-harmonic map, and let ϑ denote the weighted normalized energy as in Definition 4.29 (with fixed weight ψ as in Assumption 4.28). Fix $x \in \Omega$ and $r > 0$ such that $B_{t_b r}(x) \Subset \Omega$. Then $\vartheta(x,\cdot)$ has a derivative at r and the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}r}\vartheta(x,r) = -p r^{p-m-2} \int_\Omega |y-x| \psi'\left(\frac{|y-x|}{r}\right) |\nabla u(y)|^{p-2} |\partial_{r_x(y)} u(y)|^2 \,\mathrm{d}y, \tag{4.54 | MF}$$

where for any y we define $r_x(y) \doteq \frac{y-x}{|y-x|}$ to be the unit vector in the direction connecting x to y.

Proof. We will proceed in two steps.

STEP 1. We first consider the case $x = 0$, $r = 1$; the general case will then follow by scale invariance. In particular, we have to prove the following identity:

$$\frac{\mathrm{d}}{\mathrm{d}r}\vartheta(0,r)\Big|_{r=1} = -p \int_\Omega |y| \psi'(|y|) |\nabla u(y)|^{p-2} \left| \left\langle \nabla u, \frac{y}{|y|} \right\rangle \right|^2 dy. \tag{4.55}$$

The key idea is to find a suitable vector field to plug into the Euler-Lagrange equation (4.16): thus, we consider the following one:

$$Y(y) = \psi(|y|) y \in \mathbf{C}^\infty_c(\mathbb{R}^m, \mathbb{R}^m).$$

A simple computation gives, for $1 \le i, j \le m$,

$$\frac{\partial Y^j}{\partial y^i} = \psi'(|y|) \frac{y_i y_j}{|y|} + \psi(|y|)\delta_{ij}.$$

Then, with this choice of Y, the integral appearing in Equation (4.16) reads:

$$\int_\Omega |\nabla u|^{p-2} \left[p|y|\psi'(|y|) \left| \partial_{\frac{y}{|y|}} u \right|^2 - |y|\psi'(|y|)|\nabla u|^2 + (p-m)\psi(|y|)|\nabla u|^2 \right] dy; \tag{4.56}$$

this follows by a straightforward computation, and by the fact that:

$$\sum_{i,j=1}^m \langle y_i \nabla_i u, y_j \nabla_j u \rangle = |y|^2 \left\langle \sum_{i=1}^m \frac{y_i}{|y|} \nabla_i u, \sum_{j=1}^m \frac{y_j}{|y|} \nabla_j u \right\rangle = |y|^2 \left| \partial_{\frac{y}{|y|}} u(y) \right|^2. \tag{4.57}$$

Now by Equation (4.16) the integral in (4.56) is zero; hence Equation (4.55) follows easily, just by taking the derivative of $\vartheta(0,\cdot)$ at $r = 1$ (and changing the order of integral and derivative):

$$\frac{\mathrm{d}}{\mathrm{d}r}\vartheta(0,r) = (p-m) r^{p-m-1} \int_\Omega \psi\left(\frac{|y|}{r}\right) |\nabla u(y)|^p \, dy + \\ + r^{p-m} \int_\Omega \psi'\left(\frac{|y|}{r}\right)\left(-\frac{|y|}{r^2}\right) |\nabla u|^p \,\mathrm{d}y. \tag{4.58}$$

STEP 2. Consider now the general case: arbitrarily fix $x \in \Omega$ and $\bar{r} > 0$ such that $B_{\bar{r}}(x) \Subset \Omega$. By scale invariance, we know that $\vartheta[u](x, r) = \vartheta[T_{x,r}u](0, 1)$ for all r in a neighborhood of $\bar{r}$. Hence in particular

$$\frac{\mathrm{d}}{\mathrm{d}r}\vartheta[u](x,r)\Big|_{r=\bar{r}} = \frac{\mathrm{d}}{\mathrm{d}r}\vartheta[T_{x,r}u](0,1)\Big|_{r=\bar{r}}. \tag{4.59}$$

Notice that by STEP 1 we have information about the quantity $\frac{\mathrm{d}}{\mathrm{d}s}\vartheta[T_{x,\bar{r}}u](0,s)$ at $s = 1$, which is not directly the information we seek, but is really close. Indeed, a simple computation (which involves nothing more than the definition of $T_{x,r}$) shows that the two quantities are related by

$$\frac{\mathrm{d}}{\mathrm{d}s}\vartheta[T_{x,\bar{r}}u](0,s)\Big|_{s=1} = \bar{r}\frac{\mathrm{d}}{\mathrm{d}r}\vartheta[T_{x,r}u](0,1)\Big|_{r=\bar{r}}. \tag{4.60}$$

Thus we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\vartheta(x,\bar{r}) &= \frac{1}{\bar{r}}\frac{\mathrm{d}}{\mathrm{d}s}\vartheta[T_{x,\bar{r}}u](0,s)\Big|_{s=1} \\ &= -\frac{p}{\bar{r}}\int_{B_{\bar{R}}(0)} |y|\psi'(|y|)|\nabla T_{x,\bar{r}}u(y)|^{p-2}\Big|\partial_{\frac{y}{|y|}}T_{x,\bar{r}}u(y)\Big|^2 dy \\ &= -p\bar{r}^{p-1}\int_{B_{\bar{R}}(0)} |y|\psi'(|y|)|\nabla u(x+\bar{r}y)|^{p-2}\Big|\partial_{\frac{y}{|y|}}u(x+\bar{r}y)\Big|^2 dy. \end{aligned} \tag{4.61}$$

By performing the change of variables $w = x + \bar{r}y$, we obtain exactly the desired result. □

Corollary 4.33. *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a stationary p-harmonic map, and let $y \in B_1(0)$; assume R_1, R_2 and R are radii satisfying $R_2 \le \frac{R}{t_a} < R_1$. Then the following inequality holds for some constant $C_1(m, R, R_1, p)$ (provided both sides are well defined):*

$$\int_{B_R(y)} |\nabla u(z)|^{p-2}|\langle\nabla u(z), y-z\rangle|^2\,\mathrm{d}z \le C_1(\vartheta(y,R_1) - \vartheta(y,R_2)). \tag{4.62}$$

In particular, taking $R_2 = 1$, $R = \max\{2, t_a\}$, and $R_1 > R/t_a$ we also get

$$\int_{B_1(0)} |\nabla u(z)|^{p-2}|\langle\nabla u(z), y-z\rangle|^2\,\mathrm{d}z \le C_2(\vartheta(y,R_1) - \vartheta(y,1)), \tag{4.63}$$

with C_2 depending only on R_1, m, p.

Proof. By Theorem 4.32:

$$\vartheta(y,R_1) - \vartheta(y,R_2) \ge \int_{\frac{R}{t_a}}^{R_1} pr^{p-m-2}\int_{B_R(y)} |z-y|\left[-\psi'\left(\frac{|z-y|}{r}\right)\right]|\nabla u|^{p-2}|\partial_{r_y}u|^2\,\mathrm{d}z\,\mathrm{d}r. \tag{4.64}$$

Since $z \in B_R(y)$ and $r \ge \frac{R}{t_a}$, by the assumptions on ψ made in Assumption 4.28 we have:

$$-\psi'\left(\frac{|z-y|}{r}\right) \ge \xi. \tag{4.65}$$

The two integrals can be then separated and we get:

$$\vartheta(y,R_1) - \vartheta(y,R_2) \ge p\xi\frac{R_1^{p-m-1} - \left(\frac{R}{t_a}\right)^{p-m-1}}{p-m-1}\int_{B_R(y)} |\nabla u(z)|^{p-2}\frac{|\langle\nabla u(z), z-y\rangle|^2}{|z-y|}\,\mathrm{d}z. \tag{4.66}$$

But now notice that $|z - y|^{-1} \ge R^{-1}$: inequality (4.62) is proved. The last statement (Equation (4.63)) follows from the fact that $B_1(0) \subset B_R(y)$, because $y \in B_1(0)$ and $R \ge 2$. □

A straightforward consequence of the monotonicity formula is that the constancy of $\vartheta[u](x,\cdot)$ in an interval implies that u is 0-homogeneous in a ball around x.

Definition 4.34 (0-homogeneity). Let $x \in \mathbb{R}^m$, $R > 0$ and $u : B_R(x) \to \mathcal{N}$. We say that u is a 0-homogeneous map in $B_R(x)$ (with respect to x) if $u(y) = u(z)$ whenever $y, z \in B_R(x)$ and $y - x = \lambda(z - x)$ for some $\lambda > 0$. $\lozenge$

Corollary 4.35. *Let $0 < s < r$; let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a p-stationary harmonic map, and $B_{t_b r}(x) \Subset \Omega$. Let ϑ denote the weighted normalized energy as in Definition 4.29. If*

$$\vartheta(x,r) - \vartheta(x,s) = 0, \tag{4.67}$$

then u is 0-homogeneous in $B_{t_b r}(x)$ (with respect to x).

Moreover, by simple geometric considerations, if a map is 0-homogeneous with respect to different points, then it is invariant along the affine subspace generated by those points: this is a version of the principle known in the literature as *cone-splitting principle* (see [CN13a]). Let us first give some precise definitions.

Notation 4.36 (Grassmannians and affine subspaces). Let $0 \le k \le m$ be an integer. We denote by $\mathbf{G}^k(\mathbb{R}^m)$ the family of k-dimensional *linear* subspaces of $\mathbb{R}^m$, and by $\mathbf{H}^k(\mathbb{R}^m)$ the family of k-dimensional *affine* subspaces. $\lozenge$

Definition 4.37 (Invariance with respect to a subspace). Let $0 \le k \le m$ be an integer and $L \in \mathbf{G}^k(\mathbb{R}^m)$ be a k-dimensional linear subspace. Let $x \in \mathbb{R}^m$ and $R > 0$. We say that a function $u : B_R(x) \to \mathcal{N}$ is L-invariant in $B_R(x)$ if $u(y) = u(z)$ whenever $y, z \in B_R(x)$ and $z - y \in L$. $\lozenge$

The following result is thus a consequence of Corollary 4.35:

Corollary 4.38 (Cone splitting). *Let $0 < s < r$; let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a p-stationary harmonic map, and let ϑ denote the weighted normalized energy as in Definition 4.29. Let $0 \le k \le m$ be an integer. If there exist $k+1$ points $\{x_i\}_{i=0}^k$ such that:*

(a) $x_i \in B_{\frac{1}{2}t_b r}(x_0) \subset \Omega$ for any $i = 1, \dots, k$;

(b) $\{x_i\}_{i=0}^k$ span a k-dimensional affine subspace L;

(c) For all $i = 0, \dots, k$,

$$\vartheta(x_i, r) - \vartheta(x_i, s) = 0; \tag{4.68}$$

then u is 0-homogeneous at any point of $L \cap B_{\frac{1}{2}t_b r}(x_0)$ and L-invariant in $B_{\frac{1}{2}t_b r}(x_0)$.

4.3 ε-regularity and strong compactness

In this section, we first introduce two versions of the so-called *ε-regularity*, a deep tool which will be used in Section 4.4 to link singular stratifications with the singular set; we then analyze limits of p-minimizing and p-stationary maps. Let us stress that here lies the mian difference of this chapter – in terms of tools and results – with the article [Ved21]: therein, the only case tackled is that of p-minimizing maps.

Both results are far from being trivial and build on the work of several authors; we state them here in the concise version that we will need, and then in Remark 4.41 we list a collection of references that provide the different parts of the proof.

Proposition 4.39 (ε-regularity). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ and $B_r(x) \Subset \Omega$.*

(m) There exist two constants $\varepsilon_0 = \varepsilon_0(m, \mathcal{N}, p)$ and $\alpha(m, \mathcal{N}, p)$ such that the following holds. If u is a minimizer for the p-energy and $\vartheta(x,r) < \varepsilon_0$, then u is $\mathbf{C}^{1,\alpha}$-regular in $B_{\frac{r}{2}}(x)$.

(s) Let us assume in addition that $\mathcal{N}$ is a homogeneous space with a left invariant metric. There exist two constants $\varepsilon_0 = \varepsilon_0(m, \mathcal{N}, p)$ and $\alpha(m, \mathcal{N}, p)$ such that the following holds. If u is a p-stationary map and $\vartheta(x,r) < \varepsilon_0$, then u is $\mathbf{C}^{1,\alpha}$-regular in $B_{\frac{r}{2}}(x)$.

Theorem 4.40 (Strong compactness). *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of maps in $W^{1,p}(\Omega, \mathcal{N})$ with bounded p-energy, let $p \in (1, \infty)$ and assume $B_r(x) \Subset \Omega$ is compactly contained in Ω.*

(m) If each u_n is p-energy minimizing, then there exists $\bar{u} \in W^{1,p}(B_r(x), \mathcal{N})$ such that $u_n \to \bar{u}$ in the strong $W^{1,p}(B_r(x), \mathcal{N})$ sense, and $\bar{u}$ is a p-minimizing map.

(s) Let us assume in addition that $\mathcal{N}$ is a homogeneous space with a left invariant metric, and that $p \notin \mathbb{N}$. If each u_n is p-stationary harmonic, then there exists $\bar{u} \in W^{1,p}(B_r(x), \mathcal{N})$ such that $u_n \to \bar{u}$ in the strong $W^{1,p}(B_r(x), \mathcal{N})$ sense, and $\bar{u}$ is a p-stationary map.

***Remark* 4.41** (References and timeline)**.** The ε-regularity result was first established for the case of 2-energy minimizing maps by Schoen and Uhlenbeck in [SU82]; the case of p-minimizers for arbitrary p was proved in [HL87, Corollary 2.7 and Theorem 3.1]. When u is only p-stationary – but even weakly harmonic would be enough at this stage – and under the additional assumptions on $\mathcal{N}$ introduced in Proposition 4.39, the ε-regularity was proved by Toro and Wang in [TW95, Corollary 3.2].

Concerning the result of strong compactness, some crucial steps are the following:

- Weak convergence in $W^{1,p}$ follows trivially by the boundedness of the energy (and the compactness of $\mathcal{N}$) and the Rellich-Kondrachov Compactness Theorem.

- In the case of minimizing maps, the fact that the convergence is actually strong was established by Schoen and Uhlenbeck in [SU82, Proposition 4.6] for the case $p = 2$ and extended by Hardt and Lin in [HL87, Corollary 2.8] to the case of generic p.

- The minimality of the limit map $\bar{u}$ relies on a result of Luckhaus [Luc88]; a concise proof for the case $p = 2$ can be found in [Sim96, Section 2.9].

- The case of stationary maps is more convoluted and relies on a fine analysis of the *defect measure* (*i.e.*, the measure ν such that $|\nabla u_n|^2 \mathrm{d}x \rightharpoonup |\nabla u|^2 \mathrm{d}x + \nu$ in the weak sense of measures): it can is proved in [NVV19, Proposition 3.14] and is partially based on a preceding work of Lin [Lin99]. $\Diamond$

We give here a name to the assumptions on $\mathcal{N}$ which guarantee the validity of a ε-regularity Theorem for p-stationary maps, in order to be able to recall them later:

Notation 4.42 (Additional assumptions on $\mathcal{N}$). We say that $\mathcal{N}$ satisfies the condition (4.69 | εR) if

(4.69 | εR) $\quad (\mathcal{N}, h_{\mathcal{N}})$ is a homogeneous space and $h_{\mathcal{N}}$ is a left-invariant metric. ◊

Given the wide use we will make of the Strong Compactness (Theorem 4.40), especially starting from Section 4.5.4, we also give a name to the assumptions on both u and $\mathcal{N}$ needed to make it work:

Notation 4.43 (Strong compactness). We will say that a map $u \in W^{1,p}(\Omega, \mathcal{N})$ satisfies the condition (4.70 | SC) if

(4.70 | SC) $\quad$ *either* u is p-energy minimizing;
or $\mathcal{N}$ satisfies (4.69 | εR), p is not an integer and u is p-stationary. ◊

4.4 Singular set and quantitative stratifications

In this section we introduce a quantitative generalization of the "invariance with respect to a subspace" notion, given in Definition 4.37.

Notation 4.44 (Directional derivatives). Let $u \in W^{1,p}(\Omega, \mathcal{N})$, and assume $L \in \mathbf{G}^k(\mathbb{R}^m)$ is a k-dimensional linear subspace of $\mathbb{R}^m$. We denote by either $|\nabla_L u|$ or $|\langle \nabla u, L\rangle|$ the quantity

$$|\nabla_L u| \doteq \left(\sum_{i=1}^{k} |\langle \nabla u, v_i\rangle|^2\right)^{\frac{1}{2}}, \tag{4.71}$$

where $\{v_1, \ldots, v_k\}$ is any orthonormal basis of L. ◊

Definition 4.45 (Almost invariance). Let $u \in W^{1,p}(\Omega, \mathcal{N})$, and let $B_r(x) \subset \Omega$. Fix $k \in \{0, \ldots, m\}$ and a parameter $\eta > 0$. We say that u is **(η, k)-invariant in** $B_r(x)$ if there exists a linear subspace $L \in \mathbf{G}^k(\mathbb{R}^m)$ such that

$$r^{p-m} \int_{B_r(x)} |\nabla_L u(y)|^p \, \mathrm{d}y < \eta. \tag{4.72}$$

Equivalently, there exists $L \in \mathbf{G}^k(\mathbb{R}^m)$ such that

$$\int_{B_1(0)} |\nabla_L T_{x,r} u(y)|^p \, \mathrm{d}y < \eta. \tag{4.73}$$

When this condition holds for some η and k, we will generically refer to it as "almost invariance". ◊

An essential class of objects in our analysis is the family of *singular strata* of a function: vaguely speaking, we classify the points of $B_1(0) \subset \Omega$ according to their "degree of almost invariance".

Definition 4.46 (Quantitative stratification). Let $u \in W^{1,p}(\Omega, \mathcal{N})$. Fix $k \in \{0, \ldots, m\}$, a radius $r > 0$ and a parameter $\eta > 0$. We define the *singular k^{th} stratum* of u, with *scale parameter* r and *closeness parameter* η as the subset of $B_1(0)$ defined by

$$\begin{aligned}\mathcal{S}^k_{\eta,r}(u) &\doteq \{x \in B_1(0) \mid u \text{ is } \underline{\text{not}}\ (\eta, k+1)\text{-invariant in } B_s(x) \text{ for any } s \geq r\} = \\ &= \left\{ x \in B_1(0) \;\middle|\; s^{p-m} \int_{B_s(x)} |\nabla_L u|^p \, \mathrm{d}y \geq \eta \text{ for all } L \in \mathbf{G}^{k+1}(\mathbb{R}^m) \text{ and } s \geq r \right\}.\end{aligned} \tag{4.74}$$

Moreover, we denote by $\mathcal{S}^k_\eta(u)$ the intersection

$$\mathcal{S}^k_\eta(u) \doteq \bigcap_{r>0} \mathcal{S}^k_{\eta,r}(u) = \{x \in B_1(0) \mid u \text{ is } \underline{\text{not}}\ (\eta, k+1)\text{-invariant in } B_s(x) \text{ for any } s > 0\}, \tag{4.75}$$

for any given η and k. ◊

The following relations can be immediately seen to hold:

Lemma 4.47 (Inclusions and trivial strata). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$. If $k' \leq k \in \{0, \ldots, m\}$, $0 < r' \leq r$ and $\eta' \geq \eta > 0$, then*

$$\mathcal{S}^{k'}_{\eta',r'}(u) \subset \mathcal{S}^k_{\eta,r}(u). \tag{4.76}$$

Moreover, for any $\eta > 0$ and $r > 0$, the stratum $\mathcal{S}^m_{\eta,r}(u)$ coincides with the whole $B_1(0)$.

In the following proposition we show that the definition of almost invariance given above implies an "almost 0-homogeneity" condition at a smaller scale. Not only: given a ball $B_r(x)$ where u is $(\bar\delta, k)$-invariant for a suitable $\bar\delta = \bar\delta(\eta)$, all the points in $B_{\frac{1}{2}r}(x)$ are both (η, k)-invariant and almost 0-homogeneous with respect to the aforementioned smaller scale.

Proposition 4.48. *Let $\eta, \Lambda > 0$ be fixed parameters, and $m, \mathcal{N}, p$ as in Assumption 4.1. There exists a constant $\bar\gamma(m, p, \Lambda, \eta)$, with $0 < \bar\gamma < \frac{1}{2}$, such that the following holds. Define $\bar\delta \doteq \bar\gamma^{2(m-p)}$. Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a map with bounded p-energy $\mathcal{E}_p(u) \leq \Lambda$, and let $B_r(x) \subset B_1(0)$. Assume u is $(\bar\delta, k)$-invariant in $B_r(x)$:*

$$r^{p-m} \int_{B_r(x)} |\nabla_L u|^p \, \mathrm{d}z < \bar\delta \qquad \textit{for some } L \in \mathbf{G}^k(\mathbb{R}^m). \tag{4.77}$$

Then for any $y \in B_{\frac{1}{2}r}(x)$ there exists a radius $\bar\gamma r \leq r_y \leq \frac{1}{2}r$ such that u satisfies the following almost invariance *and* almost 0-homogeneity *conditions in $B_{r_y}(y)$:*

$$r_y^{p-m} \int_{B_{r_y}(y)} |\nabla_L u|^p \, \mathrm{d}z < \eta, \tag{4.78}$$

$$\vartheta(y, r_y) - \vartheta\Big(y, \tfrac{1}{2} r_y\Big) < \eta. \tag{4.79}$$

Proof. By scale invariance, it is sufficient to prove the statement for $x = 0$, $r = 1$. Choose $\bar\gamma(m, p, \Lambda, \eta)$ so that

$$\bar\gamma < \min\left\{ 2^{-\frac{\Lambda}{\eta} - 1}, \eta^{\frac{1}{m-p}}, \frac{1}{2} \right\}. \tag{4.80}$$

Consider a point $y \in B_{\frac{1}{2}}(0)$; assume, by contradiction, that for all i such that $2^{-i} \geq \bar{\gamma}$ we have

$$\vartheta\left(y, 2^{-i}\right) - \vartheta\left(y, 2^{-i-1}\right) \geq \eta. \tag{4.81}$$

Then we should have:

$$\Lambda > \vartheta\left(y, \frac{1}{2}\right) \geq \sum_{i=1}^{\lfloor \frac{\Lambda}{\eta}+1 \rfloor} \left(\vartheta\left(y, 2^{-i}\right) - \vartheta\left(y, 2^{-i-1}\right)\right) \geq \left\lfloor \frac{\Lambda}{\eta} + 1 \right\rfloor \eta > \Lambda, \tag{4.82}$$

a contradiction. Thus for any $y \in B_{\frac{1}{2}}(0)$ we have a radius $r_y \in \left[\bar{\gamma}, \frac{1}{2}\right]$ for which (4.79) holds. Moreover, by (4.80) we also have:

$$r_y^{p-m} \int_{B_{r_y}(y)} |\nabla_L u|^p \, \mathrm{d}z \leq \bar{\gamma}^{p-m} \bar{\delta} = \bar{\gamma}^{m-p} < \eta, \tag{4.83}$$

which concludes the proof. □

The bridge between singular stratification and singular set of a p-harmonic map is given by the following proposition, which strongly relies on the ε-regularity Theorem (Proposition 4.39). This result can be seen as a quantitative version of the following known fact ([HL87, Theorem 4.5] for p-minimizers, [TW95, Theorem 2] for p-stationary maps): a p-minimizing map (or p-stationary, under the additional condition Equation (4.69 | εR) on $\mathcal{N}$) which is 0-homogeneous and invariant along a $(m - \lfloor p \rfloor)$-linear subspace must be constant.

Proposition 4.49 (Singular set and stratification). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ and $\Lambda > 0$. Assume that $\mathcal{N}$ and u satisfy the assumption (4.70 | SC) for the strong compactness theorem:*

- *(m) either u is minimizing (with $\mathcal{N}$ arbitrary closed Riemannian manifold without boundary);*
- *(s) or $\mathcal{N}$ is a homogeneous space with a left-invariant metric, p is not an integer, and u is p-stationary.*

Assume that $\mathcal{E}_p(u) < \Lambda$. There exists $\eta = \eta(m, \mathcal{N}, \Lambda, p)$ such that for all $r > 0$ (small) we have

$$\mathcal{S}(u) \cap B_1(0) \subset \mathcal{S}^{m-\lfloor p \rfloor -1}_{\eta, r}(u). \tag{4.84}$$

Proof. We give a proof for the case *(m)*: the case of p-stationary maps follows the same lines, by replacing the respective versions of Proposition 4.39 and theorem 4.40.

For any $i \in \mathbb{N}$, let $\gamma_i \doteq \bar{\gamma}\left(m, \Lambda, p, \frac{1}{i}\right)$ be the constant given by Proposition 4.48 when $\eta = \frac{1}{i}$, and let $\delta_i \doteq \gamma_i^{2(m-p)}$.

We argue by contradiction: assume that for all $i \in \mathbb{N}$ there exists a p-minimizing map u_i with $\mathcal{E}_p(u_i) \leq \Lambda$, a singular point $x_i \in \mathcal{S}(u_i)$, a $r_i > 0$ and a $(m - \lfloor p \rfloor)$-plane L_i such that

$$\int_{B_1(0)} |\nabla_{L_i} T_{x_i, r_i} u_i|^p < \delta_i. \tag{4.85}$$

Up to precomposing with a rotation of the space, we can assume $L_i = L$ for all i, for some affine subspace L. By Proposition 4.48, we have

$$(\alpha_i r_i)^{p-m} \int_{B_{\alpha_i r_i}(x_i)} |\nabla_L u_i|^p \, \mathrm{d}z < \frac{1}{i} \tag{4.86}$$

$$\vartheta[u_i](x_i, \alpha_i r_i) - \vartheta[u_i]\left(x_i, \frac{1}{2}\alpha_i r_i\right) < \frac{1}{i} \tag{4.87}$$

for a sequence $\{\alpha_i\}_i$, with $\gamma_i \le \alpha_i \le \frac{1}{2}$. The maps $T_{x_i,\alpha_i r_i} u_i$ are p-minimizing, and they are uniformly bounded in $W^{1,p}(B_{1+\varepsilon}(0), \mathcal{N})$ for some ε (by compactness of $\mathcal{N}$ and by the bound on the p-energy); thus, up to subsequences, they weakly converge to a map $\tilde{u} \in W^{1,p}(B_{1+\varepsilon}(0), \mathcal{N})$. By Theorem 4.40, the convergence is actually strong in $W^{1,p}(B_1(0), \mathcal{N})$, and $\tilde{u}$ is a p-minimizer in $B_1(0)$. But now by strong $W^{1,p}(B_1(0), \mathcal{N})$ convergence and by (4.86), (4.87) we have

$$\int_{B_1(0)} |\nabla_L \tilde{u}|^p = 0, \tag{4.88}$$

$$\vartheta[\tilde{u}](0,1) - \vartheta[\tilde{u}]\left(0, \frac{1}{2}\right) = 0; \tag{4.89}$$

so $\tilde{u}$ is p-minimizing, $(m - \lfloor p \rfloor)$-invariant on $B_1(0)$ and 0-homogeneous on $B_1(0)$. By [HL87, Theorem 4.5] (or [TW95, Theorem 2] in the p-stationary case), this implies that $\tilde{u}$ is constant on $B_1(0)$: thus in particular $\vartheta[\tilde{u}](0, \cdot) \equiv 0$ in $(0,1)$. However, by the fact that $x_i \in \mathcal{S}(u_i)$, and by the ε-regularity Proposition 4.39, we have:

$$\vartheta[T_{x_i,\alpha_i r_i} u_i](0,s) \ge \varepsilon_0 \qquad \forall s > 0, \tag{4.90}$$

which implies $\vartheta[\tilde{u}](0,s) \ge \varepsilon_0$ by $W^{1,p}$-convergence: we have reached a contradiction. □

Since we will make great use of compactness and limiting arguments, we will need an effective notion of "points in general position" which is preserved when passing to the limit.

Definition 4.50. Given $k+1$ points $\{x_i\}_{i=0}^k$ in $\mathbb{R}^m$ (with $0 \le k \le m$), and $\varrho > 0$, we say that $\{x_i\}_i$ are in ϱ-general position if for all $j = 1, \dots, k$

$$\mathsf{d}(x_j, x_0 + \operatorname{span}\{x_1 - x_0, \dots, x_{j-1} - x_0\}) \ge \varrho. \tag{4.91}$$

We say that a set of points $\mathcal{S}$ spans ϱ-effectively a given k-subspace $L \in \mathbf{H}^k(\mathbb{R}^m)$ if there exist $k+1$ points $\{x_0, x_1, \dots, x_k\} \subset \mathcal{S}$ in ϱ-general position such that

$$L = x_0 + \operatorname{span}\{x_1 - x_0, \dots, x_k - x_0\}. \tag{4.92}$$

◊

***Remark* 4.51.** Let $\{x_i\}_{i=0}^k$ be points in $\mathbb{R}^m$ with $0 \le k \le m$, and let $\varrho > 0$. Denote by $v_0, v_1, \dots, v_k$ the vectors $v_0 \doteq 0$ and $v_j \doteq x_j - x_0$ for any $j = 1, \dots, k$; denote by $V_0, V_1, \dots, V_k$ the linear subspaces $V_j \doteq \operatorname{span}\{v_0, \dots, v_j\}$. The condition in Equation (4.91) is clearly equivalent to:

$$\left| v_j - \pi_{V_{j-1}}(v_j) \right| \ge \varrho \qquad \text{for all } j = 1, \dots, k, \tag{4.93}$$

where π_{V_j} is the orthogonal projection on the subspace V_j. ◊

As we wanted, the notion of ϱ-general position passes to the limit:

Lemma 4.52 (Limits of points in ϱ-general position)**.** *For any $j \in \mathbb{N}$, let $\{x_{ij}\}_{i=0}^k$ be $k+1$ points of $\mathbb{R}^m$ in ϱ-general position, with $\varrho > 0$. Assume that $x_{ij} \xrightarrow{j \to \infty} \bar{x}_i$ for all $i = 0, \dots, k$. Then $\{\bar{x}_i\}_{i=0}^k$ are still in ϱ-general position.*

Proof. Let us introduce, as in Remark 4.51, the following notations:

$$
\begin{aligned}
v_{0j} = \bar v_0 &\doteq 0 && \text{for all } j \in \mathbb{N},\\
v_{ij} \doteq x_{ij} - x_{0j}, \quad \bar v_i &\doteq \bar x_i - \bar x_0 && \text{for all } j \in \mathbb{N},\ i = 1, \dots, k,\\
V_{ij} \doteq \operatorname{span}\{v_{0j}, \dots, v_{ij}\}, \quad \bar V_i &\doteq \operatorname{span}\{\bar v_0, \dots, \bar v_i\} && \text{for all } j \in \mathbb{N},\ i = 0, \dots, k.
\end{aligned}
\tag{4.94}
$$

Then, by Remark 4.51, the statement is proved if we can show that

$$
\lim_{j\to\infty} \left| v_{ij} - \pi_{V_{i-1,j}}(v_{ij}) \right| = \left| \bar v_i - \pi_{\bar V_{i-1}}(\bar v_i) \right| \qquad \text{for all } i \in \{1, \dots, k\}. \tag{4.95}
$$

For $i = 1$, this is trivial since it simply says $\lim_{j\to\infty} |v_{1j}| = |\bar v_1|$. Assume it is true for $i = 1, \dots, \hat\imath$: in particular, $\bar v_1, \dots, \bar v_{\hat\imath}$ are linearly independent. The orthogonal projections on $V_{\hat\imath j}$ and $\bar V_{\hat\imath}$ can be represented for any $v \in \mathbb{R}^m$ by

$$
\pi_{V_{\hat\imath j}}(v) = A_{\hat\imath j} \left(A_{\hat\imath j}^\top A_{\hat\imath j} \right)^{-1} A_{\hat\imath} j^\top v, \qquad \pi_{\bar V_{\hat\imath}}(v) = \bar A_{\hat\imath} \left(\bar A_{\hat\imath}^\top \bar A_{\hat\imath} \right)^{-1} \bar A_{\hat\imath}^\top v, \tag{4.96}
$$

where $A_{\hat\imath j}$ is the matrix whose columns are $v_{1j}, \dots, v_{\hat\imath j}$, $\bar A_{\hat\imath}$ is the matrix whose columns are $\bar v_1, \dots, \bar v_{\hat\imath}$ (the latter being a consequence of the linear independence assumption until $\hat\imath$). Thus

$$
\lim_{j\to\infty} \pi_{V_{\hat\imath j}}(v_{\hat\imath+1,j}) = \pi_{\bar V_{\hat\imath}}(\bar v_{\hat\imath+1}); \tag{4.97}
$$

by continuity of the norm, Equation (4.95) holds for $\hat\imath+1$. By induction, the lemma is proved. $\square$

Lemma 4.53. *Let $\{x_i\}_{i=0}^k$ be points in $\mathbb{R}^m$ with $0 \le k \le m$. Let $\varrho > 0$ be such that the points are in ϱ-general position. Let $D > 0$ be such that $\operatorname{diam}(\{x_1, \dots, x_k\}) \le D$. Denote by $v_0, v_1, \dots, v_k$ the vectors $v_0 \doteq 0$ and $v_j \doteq x_j - x_0$ for any $j = 1, \dots, k$; denote by $V_0, V_1, \dots, V_k$ the linear subspaces $V_j \doteq \operatorname{span}\{v_0, \dots, v_j\}$. Then any $v \in V_k$ can be written as*

$$
v = \sum_{i=1}^k \alpha_i v_i \tag{4.98}
$$

with

$$
|\alpha_i| \le \frac{|v|}{\varrho} \left(1 + \frac{D}{\varrho} \right)^{k-i} \quad \textit{for } i = 1, \dots, k. \tag{4.99}
$$

In particular,

$$
|\alpha_i| \le |v| c_1(m, D, \varrho) \qquad \textit{with } c_1(m, D, \varrho) \doteq \frac{1}{\varrho} \left(1 + \frac{D}{\varrho} \right)^{m-1}. \tag{4.100}
$$

Proof. We proceed by (reverse) induction.

Step 1. Let us estimate α_k. We have:

$$
\langle v, v_k - \pi_{V_{k-1}}(v_k) \rangle = \alpha_k \langle v_k, v_k - \pi_{V_{k-1}}(v_k) \rangle = \alpha_k |v_k - \pi_{V_{k-1}}(v_k)|^2, \tag{4.101}
$$

where we have used the orthogonality of $v_k - \pi_{V_{k-1}}(v_k)$ with the vectors $v_1, \dots, v_{k-1}$ and $\pi_{V_{k-1}}(v_k)$. By the Cauchy-Schwarz inequality and the ϱ-general position of the points $\{x_i\}_{i=0}^k$, then:

$$
\alpha_k \le \frac{|v|}{|v_k - \pi_{V_{k-1}}(v_k)|} \le \frac{|v|}{\varrho}. \tag{4.102}
$$

STEP 2. Assume the statement is true for $i = j+1, \dots, k$ with $j \in \{1, \dots, k-1\}$. Consider the vector $w_j \doteq \sum_{i=1}^{j} \alpha_j v_j = v - \sum_{i=j+1}^{k} \alpha_j v_j$. Then again by orthogonality it holds that:

$$\langle w_j, v_j - \pi_{V_{j-1}}(v_j)\rangle = \alpha_j \Big| v_j - \pi_{V_{j-1}}(v_j)\Big|^2. \tag{4.103}$$

By Cauchy-Schwarz and by the ϱ-general position of the points $\{x_i\}_{i=0}^{k}$,

$$|\alpha_j| \le \frac{|w_j|}{\varrho} = \frac{\Big|v - \sum_{i=j+1}^{k} \alpha_j v_j\Big|}{\varrho} \le \frac{1}{\varrho}\left(|v| + \sum_{i=j+1}^{k} |\alpha_i||v_i|\right). \tag{4.104}$$

Here we can use the inductive assumption and the fact that $|v_i| \le D$:

$$\begin{aligned} |\alpha_j| &\le \frac{1}{\varrho}\left(|v| + \sum_{i=j+1}^{k} \frac{|v|}{\varrho}\left(1 + \frac{D}{\varrho}\right)^{k-i} D\right) = \frac{|v|}{\varrho}\left(1 + \frac{D}{\varrho}\sum_{\ell=0}^{k-j-1}\left(1 + \frac{D}{\varrho}\right)^{\ell}\right) = \\ &= \frac{|v|}{\varrho}\left(1 + \frac{D}{\varrho}\frac{1 - \left(1 + \frac{D}{\varrho}\right)^{k-j}}{-\frac{D}{\varrho}}\right) = \frac{|v|}{\varrho}\left(1 + \frac{D}{\varrho}\right)^{k-j}. \end{aligned} \tag{4.105}$$

This proves the statement. ☐

Lemma 4.54 (Sufficient condition). *Let $\mathcal{S} \subset \mathbb{R}^m$ and let $\varrho > 0$. If $\mathcal{S}$ is not contained in $\mathcal{B}_\varrho(V)$ for any $(k-1)$-dimensional affine subspace $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$, then $\mathcal{S}$ spans ϱ-effectively a k-dimensional affine subspace.*

Proof. We need to show that $\mathcal{S}$ contains $k+1$ points $\{x_i\}_{i=0}^{k}$ in ϱ-general position. By the assumption, $\mathcal{S}$ is not empty, thus there exists a point $x_0 \in \mathcal{S}$. Let $\{x_0, \dots, x_j\}$ be points of $\mathcal{S}$ in ϱ-general position, with $j \in \{0, \dots, k-1\}$: fix any $(k-1)$-dimensional subspace V such that $x_0 + \operatorname{span}\{x_1 - x_0, \dots, x_j - x_0\} \subset V$. Then by assumption there exists a point $x_{j+1} \in \mathcal{S} \setminus \mathcal{B}_\varrho(V)$: in particular,

$$\mathsf{d}(x_{j+1}, x_0 + \operatorname{span}\{x_1 - x_0, \dots, x_j - x_0\}) \ge \mathsf{d}(x_{j+1}, V) \ge \varrho. \tag{4.106}$$

Hence $\{x_0, \dots, x_j, x_{j+1}\}$ are still in ϱ-general position. By induction, this proves the lemma. ☐

The following corollary is a quantitative version of the Cone Splitting Corollary 4.38; the proof is a simple compactness argument, based on Lemma 4.52 and on the Strong Compactness Theorem 4.40.

Corollary 4.55 (Quantitative cone splitting). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ and $\Lambda > 0$. Assume that $\mathcal{N}$ and u satisfy the condition (4.70 | SC) for the strong compactness theorem. Assume that $\mathcal{E}_p(u) < \Lambda$. Let $0 \le k \le m$ be an integer. Fix the constants $\eta, p, \gamma, \varrho > 0$. There exists a constant $\varepsilon > 0$ (depending on $m, \mathcal{N}, \Lambda, \eta, p, \gamma, \varrho$) such that the following holds. If there exist $k+1$ points $\{x_i\}_{i=0}^{k}$ such that:*

(i) $x_i \in B\left(x_0, \frac{1}{2}t_b r\right) \subset \Omega$ for any $i = 1, \dots, k$;

(ii) $\{x_i\}_{i=0}^{k}$ span ϱ-effectively a k-dimensional affine subspace L;

(iii) For all $i = 0, \dots, k$,

$$\vartheta(x_i, r) - \vartheta(x_i, \gamma r) < \varepsilon; \tag{4.107}$$

then $r^{p-m} \int_{B_{t_b r}(x_0)} |\nabla_L u|^p < \eta$.

Proof. Up to rescaling, thanks to the scale invariance of ϑ, we can assume that $r = 1$ and x_0 is the origin. Let us assume by contradiction that there exist:

- sequences of points $\{x_{ij}\}_{j\in\mathbb{N}}$ such that $x_{0j} = 0$ for all $j \in \mathbb{N}$, each x_{ij} with $1 \le i \le k$ belongs to $B\Big(0, \frac{1}{2}t_b\Big)$, and $\{x_{ij}\}_{i=1}^k$ span ϱ-effectively a k-dimensional affine subspace L (which we can assume is the same for any $j \in \mathbb{N}$, up to rotations);
- a sequence of maps $\{u_j\}_{j\in\mathbb{N}}$ that satisfy the condition (4.70 | SC);

and that the inequalities

$$\vartheta[u_j](x_{ij}, 1) - \vartheta[u_j](x_{ij}, \gamma) < \frac{1}{j} \tag{4.108}$$

$$\int_{B_{t_b}(0)} |\nabla_L u_j|^p \, \mathrm{d}x \ge \eta \tag{4.109}$$

hold for any $i \in \{0, \dots, m\}$ and $j \in \mathbb{N}$.

By the Strong Compactness Theorem 4.40, up to subsequences the maps u_j converge in the strong $W^{1,p}$ sense to a stationary map $\bar{u} \in W^{1,p}(\Omega, \mathcal{N})$; moreover, up to further subsequences, for any $i \in \{1, \dots, m\}$ one has $x_{ij} \to \bar{x}_i \in \bar{B}(0, t_b) \cap L$, and the points $\{\bar{x}_i\}_{i=0}^k$ still ϱ-effectively span L by Lemma 4.52. However, by strong $W^{1,p}$ convergence and Equation (4.108) it is easy to see that the limit map $\bar{u}$ satisfies

$$\vartheta[\bar{u}](\bar{x}_i, 1) - \vartheta[\bar{u}](\bar{x}_i, \gamma) = 0 \tag{4.110}$$

$$\int_{B_{t_b}(0)} |\nabla_L \bar{u}|^p \, \mathrm{d}x \ge \eta, \tag{4.111}$$

which contradicts the Cone Splitting lemma 4.38. □

4.4.1 Main results for the singular strata

We are now in the condition to state the main theorem of this chapter, which concerns the regularity and structure of singular strata in general, and has Theorem 4.26 as a corollary:

Theorem 4.56 (Singular strata). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a map with energy bounded by Λ. Let $\eta > 0$ and $1 \le k \le m$. Assume that $\mathcal{N}$ and u satisfy the condition Equation (4.70 | SC) for the strong compactness theorem:*

(m) either u is minimizing (with $\mathcal{N}$ arbitrary closed Riemannian manifold without boundary);

(s) or $\mathcal{N}$ is a homogeneous space with a left-invariant metric, p is not an integer, and u is p-stationary.

Then there exists two constants C_{strat} and δ_0 depending on $m, \mathcal{N}, p, \Lambda, \eta$ such that for any $r > 0$

$$\mathscr{L}^m\Big(\mathcal{B}_r(\mathcal{S}^k_{\eta,\delta_0 r}(u)) \cap B_1(0)\Big) \le C_{\mathrm{strat}} r^{m-k}. \tag{4.112}$$

Moreover, for any $\eta > 0$ and any $0 \le k \le m$, the stratum $\mathcal{S}^k_\eta(u)$ is k-rectifiable.

4.5 Reifenberg-type theorems and other technical results

The next step will be to introduce some more advanced techniques which allow us to analyze the behaviour of each singular stratum *at every scale* r around a point x.

4.5.1 Jones' numbers

In order to state Reifenberg Theorem (in a form which is suited to our context), we first need to recall the definition of Jones' numbers of a measure μ (first appeared in [Jon90]; for a detailed introduction, see [Paj02]): this is a scale-invariant notion which quantifies how close $\operatorname{spt}(\mu)$ is to be contained in an affine k-space (near a given point).

Definition 4.57 (Jones' numbers). Let $x \in B_1(0)$ and $0 < r < 1$. Assume μ is a positive Radon measure on Ω. For any $k \in \{0, \dots, m\}$ we define the *k-dimensional Jones' number* of μ in $B_r(x)$ as

$$\beta_\mu^k(x,r) \doteq \left(r^{-k} \inf \left\{ \int_{B_r(x)} \frac{\mathsf{d}(y,L)^2}{r^2} \, \mathrm{d}\mu(y) \;\middle|\; L \in \mathbf{H}^k(\mathbb{R}^m) \right\} \right)^{\frac{1}{2}}. \tag{4.113}$$

Here $\mathbf{H}^k(\mathbb{R}^m)$ is again the collection of all the k-dimensional affine subspaces of $\mathbb{R}^m$. $\Diamond$

***Remark* 4.58.** The quantity β_μ^k is scale invariant in the following sense. Assume μ is a measure on a ball $B_r(x)$; define the blow up measure $\hat{\mu} = \hat{\mu}_{x,r}^k$ on $B_1(0)$ as

$$\hat{\mu}(A) \doteq r^{-k}\mu(x + rA) \qquad \forall A \subset B_1(0) \text{ measurable.} \tag{4.114}$$

Then it is easy to compute that $\beta_{\hat{\mu}}^k(0,1) = \beta_\mu^k(x,r)$. $\Diamond$

The Jones' numbers admit an explicit representation in terms of the *eigenvalues of the second moment of* μ.

Definition 4.59. Let μ be a measure with support in $B_1(0)$. We define:

(i) the **center of mass** of μ as the point $x_{cm}^\mu \in B_1(0)$ such that

$$x_{cm}^\mu \doteq \fint_{B_1(0)} x \, \mathrm{d}\mu(x); \tag{4.115}$$

(ii) the **second moment** of μ as the bilinear form Q^μ such that for all $v, w \in \mathbb{R}^m$

$$Q^\mu(v,w) \doteq \int_{B_1(0)} [(x - x_{cm}) \cdot v][(x - x_{cm}) \cdot w] \, d\mu(x). \tag{4.116}$$

We will usually drop the superscript μ when it is clear from the context. $\Diamond$

Since Q is symmetric and positive-definite, by the Spectral Theorem the associated matrix (which we still denote by Q) admits an orthonormal basis of eigenvectors, with non-negative eigenvalues. We denote with $\lambda_1^\mu, \dots, \lambda_m^\mu$ the eigenvalues of Q in decreasing order, and with

$v_1^\mu, \dots, v_m^\mu$ the respective eigenvectors (pairwise orthogonal and of norm 1), again dropping the superscripts when they are clear; in particular:

$$\lambda_k v_k = \int_{B_1(0)} [(x - x_{cm}) \cdot v_k](x - x_{cm}) \, \mathrm{d}\mu(x); \tag{4.117}$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m. \tag{4.118}$$

Finally, we denote by V_k^μ (or V^k) the following affine k-plane:

$$V_k^\mu \doteq x_{cm}^\mu + W_k^\mu \tag{4.119}$$

$$W_k^\mu \doteq \mathrm{span}\{v_1^\mu, \dots, v_k^\mu\}. \tag{4.120}$$

We are now ready to characterize β_μ^k:

Lemma 4.60. *Let μ be a measure on $B_1(0)$. The affine space V_k^μ achieves the minimum in the definition of $\beta_\mu^k(0,1)$. Moreover,*

$$\beta_\mu^k(0,1)^2 = \int_{B_1(0)} \mathsf{d}^2(y, V_k^\mu) \, \mathrm{d}\mu(y) = \lambda_{k+1}^\mu + \cdots + \lambda_m^\mu. \tag{4.121}$$

For a proof of this fact, see [NV17, Lemma 7.4] or [NV18, Subsection 6.1]. This is based on the (visually helpful) fact that the eigenvalues λ_k and eigenvectors v_k admit the following characterization:

- λ_1 is the maximum of $\int_{B_1(0)} \langle x - x_{cm}, v \rangle^2 \, \mathrm{d}\mu$ among vectors v of norm 1, and v_1 is any maximizing vector;
- λ_k is the maximum of the same operator among all unit vectors orthogonal to $v_1, \dots, v_{k-1}$, and v_k is any maximizing vector.

4.5.2 Reifenberg-type Theorems

The main hypothesis one needs, in order to apply Reifenberg Theorem in its different forms, is a control on the Jones numbers of a suitable measure (*e.g.*, $\mathscr{H}^k$ restricted to a set). In all the cases, the condition we need takes the following form:

Definition 4.61 (Reifenberg condition)**.** Let μ be a positive Radon measure on Ω, and $k \in \{0, \dots, m\}$. We say that μ satisfies the (k-dimensional) *Reifenberg condition* with constant δ if for any $x \in B_1(0)$ and $0 < r < 1$ we have:

$$\int_{B_r(x)} \int_0^r \beta_\mu^k(y,s)^2 \frac{ds}{s} d\mu(y) < \delta r^k. \tag{4.122 | k-Reif}$$ ◊

As we will clarify in Theorem 4.63, two versions of Reifenberg Theorem are available for our purposes: in one of them (necessary for the rectifiability of a set), one needs to check (4.122 | k-Reif) on the restriction of the Hausdorff measure to the given set; the other one (necessary for volume estimates) makes use of *discrete measures* as the following:

Definition 4.62 (Measure associated to a disjoint family of balls)**.** Assume $\mathcal{C}$ is a (discrete) subset of $B_1(0)$, and $\mathscr{F} = \{B_{r_x}(x)\}_{x\in\mathcal{C}}$ is a collection of *disjoint* balls centered in $\mathcal{C}$, each contained in $B_2(0)$. For any $k \in \{0,\dots,m\}$, we define the following measure associated to $\mathscr{F}$:

$$\mu_{\mathscr{F},k} \doteq \sum_{x\in\mathcal{C}} r_x^k \delta_x, \tag{4.123}$$

where δ_x is the Dirac measure centered at x. ◊

Theorem 4.63 (Reifenberg)**.** *There exist two constants C_R and δ_R such that the following statements hold true.*

(i) Assume $\mathscr{F}$ is a family of disjoint balls with centers in $\mathcal{C} \subset B_1(0)$, each contained in $B_2(0)$. If $\mu_{\mathscr{F},k}$ satisfies the condition (4.122 | k-Reif) with constant δ_R, then

$$\sum_{x\in\mathcal{C}} r_x^k \le C_R. \tag{4.124}$$

(ii) Assume $S \subset B_1(0)$ is a $\mathscr{H}^k$-measurable set. If $\mathscr{H}^k \llcorner S$ satisfies the Reifenberg condition (4.122 | k-Reif) with constant δ_R, then S is k-rectifiable and

$$\mathscr{H}^k \llcorner S(B_r(x)) \le C_R r^k \tag{4.125}$$

for any $x \in S$ and $0 < r < 1$.

The original proof of this version of Reifenberg Theorem can be found in [NV17, Sections 5 and 6]; a more general form of it is contained in [ENV16, Section 2], while similar arguments are developped in [DT12; ENV19; Mi18; Tor95; AT15].

Since we will need it in this form, for the sake of clarity we state here a rescaled version of Theorem 4.63, part (i).

Corollary 4.64 (Reifenberg, rescaled version)**.** *Let $B_{\bar r}(\bar x)$ be a fixed ball. Assume $\mathscr{F}$ is a family of disjoint balls with centers in $\mathcal{C} \subset B_{\bar r}(\bar x)$, each contained in $B_{2\bar r}(\bar x)$. If $\mu = \sum_{x\in\mathcal{C}} r_x^k \delta_x$ satisfies*

$$\int_{B_{\bar r r}(w)} \int_0^{\bar r r} \beta_\mu^k(y,s)^2 \frac{ds}{s} d\mu(y) < \delta_R (\bar r r)^k \tag{4.126}$$

for all $w \in B_{\bar r}(\bar x)$ and all $0 < r < 1$, then

$$\sum_{x\in\mathcal{C}} r_x^k \le C_R \bar r^k. \tag{4.127}$$

Proof. It suffices to apply Theorem 4.63 to the measure $\hat\mu_{\bar x,\bar r} \doteq \bar r^{-k} T_{\bar x,\bar r}\sharp\mu$ introduced in the remark above: by using the change of variable formula for the integral, and exploiting the scale invariance of β^k we obtain the result. □

***Remark* 4.65.** Notice that the constants δ_R and C_R for the rescaled version are the same as in Theorem 4.63, and thus only depending on m. ◊

4.5.3 Estimates on Jones' numbers

The key estimate, linking the Jones' numbers of a measure with the normalized p-energy of a p-minimizing map, is given in the following theorem, which we prove in several steps.

Theorem 4.66 (Estimates on β^k_μ). *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a stationary p-harmonic map. Fix the following constants: $0 < \bar{r} \leq 1, \eta > 0, \sigma > 1, k \in \{1, \dots, m\}$. Let $x \in B_1(0)$ and $r > 0$. Assume u is not $(\eta, k+1)$-invariant in $B_{\bar{r}r}(x)$. There exists a constant $C_3(m, p, \eta, \sigma, \bar{r})$ such that the following estimate*

$$\beta^k_\mu(x,r)^2 \leq C_3 r^{-k} \int_{B_r(x)} (\vartheta(y,\sigma r) - \vartheta(y,r))\, \mathrm{d}\mu(y) \tag{4.128 | JN}$$

holds for any positive Radon measure μ on Ω.

***Remark* 4.67** (The constant σ). When this theorem will be used in Section 4.6, we will assign a precise value to σ; notice that in order that all the expressions involved are meaningful we could need to enlarge the domain Ω (and thus $\bar{R}$) according to σ (see Assumption 4.1). ◊

***Remark* 4.68** (Scale invariance application). By scale invariance (of both β^k_μ and ϑ), it will be enough to prove the estimate for $x = 0$ and $r = 1$. Moreover, since the inequality does not change when μ is multiplied by a constant, we can assume μ is a probability measure on $B_1(0)$. That is: assuming u is not $(\eta, k+1)$-invariant in $B_{\bar{r}}(0)$, we will prove that

$$\beta^k_\mu(0,1)^2 \leq C_3 \int_{B_1(0)} (\vartheta(y,\sigma) - \vartheta(y,1))\, \mathrm{d}\mu(y) \tag{4.129}$$

for any measure μ with $\mu(B_1(0)) = 1$. ◊

***Remark* 4.69.** By Lemma 4.60, $\beta^k_\mu(0,1)^2$ admits an explicit representation as $\lambda^\mu_{k+1} + \dots + \lambda^\mu_m$, where $\lambda^\mu_1 \geq \dots \geq \lambda^\mu_m$ are the eigenvalues of the second moment of μ in decreasing order. We have thus reduced the problem to showing:

$$\lambda^\mu_{k+1} + \dots + \lambda^\mu_m \leq C \int_{B_1(0)} (\vartheta(y,\sigma) - \vartheta(y,1))\, \mathrm{d}\mu(y); \tag{4.130}$$

since the eigenvalues are in decreasing order, we need to show even less:

$$\lambda^\mu_{k+1} \leq C \int_{B_1(0)} (\vartheta(y,\sigma) - \vartheta(y,1))\, \mathrm{d}\mu(y). \tag{4.131}$$ ◊

Lemma 4.70 (Energy along eigenvectors). *Let u be a stationary p-harmonic map, and μ a Radon measure on $B_1(0)$; let $\{\lambda_j\}_j$, $\{v_j\}_j$ be the eigenvalues and eigenvectors of Q^μ, as before. There exists a constant $C_4(m, \mathcal{N}, p)$ such that for all $j = 1, \dots, m$ the following holds:*

$$\lambda_j \int_{B_1(0)} |\nabla u(z)|^{p-2} |\langle \nabla u(z), v_j \rangle|^2\, \mathrm{d}z \leq C_4 \int_{B_1(0)} (\vartheta(y,\sigma) - \vartheta(y,1))\, \mathrm{d}\mu(y). \tag{4.132}$$

Proof. Notice that if $\lambda_j = 0$, the statement is trivial by the Monotonicity Theorem 4.32. We can thus assume that $\lambda_j \geq 0$.

Let us first estimate the quantity $\langle \nabla u(z), v_j\rangle$ appearing in the integral at the left hand side. By the definition of v_j and λ_j as eigenvector and eigenvalue of Q^μ, and by linearity of the integral and the scalar product, we have:

$$\begin{aligned}\lambda_j\langle \nabla u(z), v_j\rangle = \langle \nabla u(z), \lambda_j v_j\rangle = \langle \nabla u(z), \int_{B_1(0)} \langle y - x_{cm}, v_j\rangle (y - x_{cm})\,\mathrm{d}\mu(y)\rangle = \\ = \int_{B_1(0)} \langle y - x_{cm}, v_j\rangle\langle \nabla u(z), y - x_{cm}\rangle\,\mathrm{d}\mu(y).\end{aligned} \tag{4.133}$$

Moreover, by the definition of center of mass and again by linearity,

$$\int_{B_1(0)} \langle y - x_{cm}, v_j\rangle\langle \nabla u(z), x_{cm} - z\rangle\,\mathrm{d}\mu(y) = 0; \tag{4.134}$$

observe indeed that the term $\langle \nabla u(z), x_{cm} - z\rangle$ does not depend on the integration variable y here. Thus, using the information from Equation (4.134) in Equation (4.133), we get:

$$\lambda_j\langle \nabla u(z), v_j\rangle = \int_{B_1(0)} \langle y - x_{cm}, v_j\rangle\langle \nabla u(z), y - z\rangle\,\mathrm{d}\mu(y). \tag{4.135}$$

Taking the squared norms of both sides and applying the Hölder inequality, this gives

$$\begin{aligned}\lambda_j^2|\langle \nabla u(z), v_j\rangle|^2 &\leq \int_{B_1(0)} \langle y - x_{cm}, v_j\rangle^2\,\mathrm{d}\mu(y) \int_{B_1(0)} |\langle \nabla u(z), y - z\rangle|^2\,\mathrm{d}\mu(y) \\ &= \lambda_j \int_{B_1(0)} |\langle \nabla u(z), y - z\rangle|^2\,\mathrm{d}\mu(y),\end{aligned} \tag{4.136}$$

where the definitions of λ_j and v_j have been used once again. Finally, dividing by λ_j, multiplying both sides by $|\nabla u(z)|^{p-2}$, and integrating in $z \in B_1(0)$ with respect to the Lebergue measure, we get:

$$\lambda_j \int_{B_1(0)} |\nabla u(z)|^{p-2}|\langle \nabla u(z), v_j\rangle|^2\,\mathrm{d}z \leq \int_{B_1(0)} |\nabla u(z)|^{p-2} \int_{B_1(0)} |\langle \nabla u(z), y - z\rangle|^2\,\mathrm{d}\mu(y)\,\mathrm{d}z \tag{4.137}$$

Hence, the Tonelli Theorem and a direct application of Corollary 4.33 gives the desired result. □

Now, thanks to the last result, we have an upper bound on the p-energy along the $(k+1)$-plane V_k introduced in Equation (4.119); but this is bounded from below by a constant, by the lack of almost invariance in 0. We have all the ingredients to complete the proof of Theorem 4.66.

Proof of Theorem 4.66. Applying Lemma 4.70 to $\lambda_1, \dots, \lambda_{k+1}$, and recalling that the λ_j's are ordered decreasingly, we get:

$$\begin{aligned}\lambda_{k+1} \int_{B_1(0)} |\nabla u|^{p-2}\left|\left\langle \nabla u, V^\mu_{k+1}\right\rangle\right|^2\,\mathrm{d}z \leq \sum_{j=1}^{k+1} \lambda_j \int_{B_1(0)} |\nabla u|^{p-2}|\langle \nabla u, v_j\rangle|^2\,\mathrm{d}z \leq \\ \leq (k+1)C_4 \int_{B_1(0)} (\vartheta(y,\sigma) - \vartheta(y,1))\,\mathrm{d}\mu(y).\end{aligned} \tag{4.138}$$

On the other hand: since u is not $(\eta, k+1)$-invariant in $B_{\bar{r}}(0)$, and V^μ_{k+1} is a $(k+1)$-plane, we have by definition:

$$\begin{aligned}\int_{B_1(0)} |\nabla u|^{p-2}\left|\left\langle \nabla u, V^\mu_{k+1}\right\rangle\right|^2\,\mathrm{d}z \geq \int_{B_1(0)} \left|\left\langle \nabla u, V^\mu_{k+1}\right\rangle\right|^p\,\mathrm{d}z \geq \\ \geq \int_{B_{\bar{r}}(0)} \left|\left\langle \nabla u, V^\mu_{k+1}\right\rangle\right|^p\,\mathrm{d}z \geq \bar{r}^{m-p}\eta.\end{aligned} \tag{4.139}$$

In particular, putting together Equation (4.138) and Equation (4.139) we obtain:

$$\lambda_{k+1} \leq \frac{(k+1)C_4(\sigma, m, p)}{\eta \bar{r}^{m-p}} \int_{B_1(0)} (\vartheta(y,\sigma) - \vartheta(y,1))\, \mathrm{d}\mu(y), \tag{4.140}$$

which is (4.131). □

4.5.4 A collection of structural lemmas

This section is devoted to building a series of "quantitative" geometric results about p-minimizing mappings (and p-stationary ones, under the further conditions given by Equation (4.70 | SC)), describing the behavior of some special subsets of $\mathcal{S}(u)$. Analogous results for (approximate) 2-harmonic maps can be found in [NV18, Section 4], although stated with some differences.

In Lemmas 4.71 to 4.73, recall that t_a, t_b are the structural constants introduced in Assumption 4.28, describing some particular features of ψ.

In the first lemma we convey this idea: consider the set of points in $B_1(0)$ at which $\vartheta[u]$ satisfies a suitable pinching condition; if it spans ϱ-effectively a k-dimensional plane L, then for some $\delta > 0$ the stratum $\mathcal{S}^k_{\eta,\delta}(u)$ lies inside a fattening of L (see Definition 4.50 for the definition of a set effectively spanning a k-plane). We remark that the proof has been partially reorganized with respect to [Ved21], while maintaining the underlying structure.

Lemma 4.71 (Pinching). *Let $\varrho_1, \eta > 0$, $0 < \lambda_1 < 1$ and $0 < \varsigma < 1$ be (small enough) constants. Define $c = c(\varsigma) \doteq \frac{1}{2}(1-\varsigma)t_a$, where t_a is introduced in Assumption 4.28.*

There exist constants δ_0, ε (depending on $m, p, \mathcal{N}, \Lambda$ and on the parameters just introduced) such that the following holds: for any p-stationary map u with energy bounded by Λ, if the set

$$\mathcal{P} \doteq \{y \in B_{cr}(x) \mid \vartheta(y,r) - \vartheta(y,\lambda_1 r) < \varepsilon\} \tag{4.141}$$

spans $\varrho_1 r$-effectively a k-plane L, then $\mathcal{S}^k_{\eta,\delta_0 r}(u) \cap B_{cr}(x) \subset \mathcal{B}_{\varrho_1 r}(L)$.

In the proof we drop the subscript 1 on ϱ and λ; it was introduced so that lemma is easier to recall when we need it.

Proof. Assume without loss of generality that $x = 0$, $r = 1$. We are thus assuming that

$$\mathcal{P} = \{y \in B_c(0) \mid \vartheta(y,1) - \vartheta(y,\lambda) < \varepsilon\} \quad \text{spans } \varrho\text{-effectively } L \in \mathbf{H}^k(\mathbb{R}^m), \tag{4.142}$$

with λ and ϱ fixed and ε to be chosen. For a fixed point $w \in B_c(0) \setminus \mathcal{B}_\varrho(L)$ we need to show that

$$\tau^{p-m} \int_{B_\tau(w)} |\nabla_V u|^p < \eta \tag{4.143}$$

for some $\tau \geq \delta_0$, with δ_0 (to be determined) depending only on $m, p, \mathcal{N}, \Lambda, \eta, \varrho, \lambda, \varsigma$, and for some $(k+1)$-dimensional plane V.

Step 1. First of all, assume from now on that $\tau < \frac{\varsigma}{2} t_a$ (the choice of δ_0 will be influenced by this). In this way, for any $w \in B_c(0)$, any $y \in \mathcal{P} \subset B_c(0)$, and any $z \in B_\tau(w)$ we have that

$$|z - y| \leq 2c + \tau < (1-\varsigma)t_a + \frac{\varsigma}{2} t_a = \left(1 - \frac{\varsigma}{2}\right) t_a, \tag{4.144}$$

and thus $B_\tau(w) \subset B(y,(1-\frac{\varsigma}{2})t_a)$. Thus one can apply the Monotonicity Formula, and in particular Corollary 4.33 with

$$\frac{R}{t_a} \in (\max\{\lambda,(1-\varsigma)\},1), \quad R_1 = 1, \quad R_2 = \lambda; \tag{4.145}$$

we obtain:

$$\begin{aligned}\int_{B_\tau(w)} |\nabla u(z)|^{p-2} |\langle \nabla u(z), y-z\rangle|^2 \,\mathrm{d}z \leq \int_{B_R(y)} |\nabla u(z)|^{p-2} |\langle \nabla u(z), y-z\rangle|^2 \,\mathrm{d}z \leq \\ \leq C_5(\vartheta(y,1) - \vartheta(y,\lambda)) \leq C_5\varepsilon\end{aligned} \tag{4.146}$$

whenever $w \in B_c(0)$ and $y \in \mathcal{P}$, where $C_5(m,\lambda,\varsigma,p)$ is obtained from $C_1(m,R,R_1,p)$ with the aforesaid choices of R, R_1, R_2.

Step 2. Denote by $y_0,\dots,y_k$ a set of points of $\mathcal{K}$ which ϱ-effectively span L. Define $\hat{L} = \operatorname{span}\{y_1-y_0,\dots,y_k-y_0\}$ to be the linear subspace associated to L. By Lemma 4.53, for any $v \in \hat{L}$ of norm 1, we have that:

$$v = \sum_{i=1}^k \alpha_i(y_i - y_0), \qquad \text{with} \quad |\alpha_i| \leq c_2(m,\varrho,\varsigma), \tag{4.147}$$

where $c_2 \doteq c_1(m,2c(\varsigma),\varrho)$ and c_1 is the coefficient from Lemma 4.53. Hence, by a standard estimate, and by Step 1:

$$\begin{aligned}\int_{B_\tau(w)} |\nabla u(z)|^{p-2} |\nabla_v u(z)|^2 \,\mathrm{d}z \leq 2\sum_i \alpha_i^2 \int_{B_\tau(w)} |\nabla u|^{p-2} |\langle \nabla u, y_i - z\rangle|^2 \,\mathrm{d}z + \\ + 2\Big(\sum_i \alpha_i\Big)^2 \int_{B_\tau(w)} |\nabla u|^{p-2} |\langle \nabla u, z-y_0\rangle|^2 \,\mathrm{d}z \leq \\ \leq C_6(m,\varrho,\varsigma,\lambda,p)\varepsilon.\end{aligned} \tag{4.148}$$

As a consequence, if $\{v_1,\dots,v_k\}$ is an orthonormal basis of $\hat{L}$,

$$\int_{B_\tau(w)} |\nabla u(z)|^{p-2} \sum_{i=1}^k |\nabla_{v_i} u(z)|^2 \,\mathrm{d}z \leq kC_6(m,\varrho,\varsigma,\lambda,p)\varepsilon \leq C_7(m,\varrho,\varsigma,\lambda,p)\varepsilon \tag{4.149}$$

thus, along k directions, we have some information that goes in the direction we need (recall that ε is a constant we still have to choose).

Step 3. For points $w \in B_c(0)$ lying out of $\mathcal{B}_\varrho(L)$, we need to gain another independent direction of smallness, orthogonal to L; to do so, we consider the direction orthogonal to L and passing through w, namely $v_{k+1} \doteq \frac{w-\pi_L(w)}{|w-\pi_L(w)|}$. Then we have:

$$\begin{aligned}\int_{B_\tau(w)} |\nabla u(z)|^{p-2} |\nabla_{v_{k+1}} u(z)|^2 \,\mathrm{d}z = \int_{B_\tau(w)} |\nabla u|^{p-2} \left|\left\langle \nabla u(z), \frac{w-\pi_L(w)}{|w-\pi_L(w)|}\right\rangle\right|^2 \mathrm{d}z \leq \\ \leq \frac{2}{\varrho^2}\Bigg[\int_{B_\tau(w)} |\nabla u|^{p-2} |\langle \nabla u(z), w-z\rangle|^2 \,\mathrm{d}z + \\ + \int_{B_\tau(w)} |\nabla u|^{p-2} |\langle \nabla u(z), z-\pi_L(w)\rangle|^2 \,\mathrm{d}z\Bigg]\end{aligned} \tag{4.150}$$

where we have used the fact that $|w - \pi_L(w)| \geq \varrho$ and the triangle inequality. The first term only needs the Cauchy-Schwarz inequality and the bound on the energy to be estimated:

$$\int_{B_\tau(w)} |\nabla u(z)|^{p-2} |\langle \nabla u(z), w - z\rangle|^2 \, \mathrm{d}z \leq \tau^2 \int_{B_\tau(w)} |\nabla u(z)|^p \, \mathrm{d}z \leq \Lambda C_8 \tau^{2-(p-m)}, \tag{4.151}$$

where C_8 is a "universal" constant only depending on our definition of the weighted normalized energy (see Remark 4.30).

For the second term, we need to use again the Corollary 4.33 to the Monotonicity Formula, as in the previous step: indeed, $\pi_L(w) \in L$, thus by Lemma 4.53

$$\begin{aligned} &\pi_L(w) - y_0 = \sum_{i=1}^k \beta_i (y_i - y_0), \\ &\text{with} \quad |\beta_i| \leq |\pi_L(w) - y_0| c_2(m, \varrho, \varsigma) \leq 2c(\varsigma) c_2(m, \varrho, \varsigma) \doteq C_9(m, \varrho, \varsigma). \end{aligned} \tag{4.152}$$

Thus in particular

$$z - \pi_L(w) = \left(1 - \sum_{i=1}^k \beta_i\right)(z - y_0) + \sum_{i=1}^k \beta_i (z - y_i), \tag{4.153}$$

and

$$\begin{aligned} |\langle \nabla u(z), z - \pi_L(w)\rangle|^2 &\leq \left[\left(1 - \sum_{i=1}^k \beta_i\right)^2 + \sum_{i=1}^k \beta_i^2\right] \sum_{i=0}^k |\langle \nabla u(z), z - y_i\rangle|^2 \leq \\ &\leq C_{10}(m, \varrho, \varsigma) \sum_{i=0}^k |\langle \nabla u(z), z - y_i\rangle|^2. \end{aligned} \tag{4.154}$$

Hence finally by Step 1:

$$\begin{aligned} \int_{B_\tau(w)} |\nabla u|^{p-2} |\langle \nabla u(z), z - \pi_L(w)\rangle|^2 \, \mathrm{d}z &\leq C_{10}(m, \varrho, \varsigma) \sum_{i=0}^k \int_{B_\tau(w)} |\nabla u|^{p-2} |\langle \nabla u(z), z - y_i\rangle|^2 \, \mathrm{d}z \leq \\ &\leq C_{10}(m, \varrho, \varsigma) \sum_{i=0}^k C_1(m, \varsigma, \lambda, p)(\vartheta(y_i, 1) - \vartheta(y_i, \lambda)) \leq \\ &\leq C_{11}(m, \varrho, \varsigma, \lambda, p)\varepsilon \end{aligned} \tag{4.155}$$

and thus

$$\int_{B_\tau(w)} |\nabla u(z)|^{p-2} |\nabla_{v_{k+1}} u(z)|^2 \, \mathrm{d}z \leq \frac{2}{\varrho^2}\left(\Lambda C_8 \tau^{2-(p-m)} + C_{11}(m, \varrho, \varsigma, \lambda, p)\varepsilon\right) \tag{4.156}$$

Step 4. Putting together the previous steps, we consider $V = \hat{L} \oplus v_{k+1} = \operatorname{span}\{v_1, \ldots, v_{k+1}\}$; a simple computation gives:

$$\begin{aligned} \tau^{p-m} \int_{B_\tau(w)} |\nabla_V u|^p \, \mathrm{d}z &\leq \tau^{p-m} \int_{B_\tau(w)} |\nabla u(z)|^{p-2} \sum_{i=1}^{k+1} |\nabla_{v_i} u(z)|^2 \, \mathrm{d}z \leq \\ &\leq C_7(m, \varrho, \varsigma)\varepsilon\tau^{p-m} + \frac{2C_{11}(m, \varrho, \varsigma, \lambda, p)}{\varrho^2}\varepsilon\tau^{p-m} + \frac{2\Lambda C_8}{\varrho^2}\tau^2 \end{aligned} \tag{4.157}$$

Thus, in order to conclude, we only need to choose τ (and thus δ_0) so that the last term is smaller than $\frac{\eta}{2}$ (and of $\frac{\varsigma}{2}t_a$, from STEP 1), and then choose ε such that also the sum of the first two pieces is smaller than $\frac{\eta}{2}$. □

The upcoming lemma says the following: if we have a set of points that satisfy a suitable "pinching condition with high energy" on ϑ, and they effectively span a k-subspace L, then all the points of L inherit a similar pinching condition with high energy. Here we use the (non established) terminology "high energy" with the meaning that *theta* assumes a value that is close to the maximal possible value. This can be seen as a further quantitative version of Corollary 4.38 (it is indeed applied to the limit of a contradicting sequence).

While Lemma 4.71 only required p-stationariety (in the form of the Corollary 4.33 to the Monotonicity Formula), here we will also need to use the Strong Compactness Theorem; this is thus where the assumption (4.70 | SC) comes into play.

Lemma 4.72 (High energy pinching). *Let ϱ_2, λ_2, Λ, $\gamma > 0$. Let $0 < c < \frac{1}{2}t_b$.*

There exists a constant $\delta_1(\varrho_2, \lambda_2, \Lambda, \gamma, c)$ such that the following holds. Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a map that satisfies (4.70 | SC) and has p-energy bounded by Λ; let $E \le \Lambda$ and $\mathcal{S} \subset \Omega$; if $\vartheta(y, r) \le E$ for all $y \in B_{cr}(x) \cap \mathcal{S}$, and the set

$$\mathcal{H} \doteq \{y \in B_{cr}(x) \cap \mathcal{S} \mid \vartheta(y, \lambda_2 r) > E - \delta_1\} \tag{4.158}$$

spans $\varrho_2 r$-effectively a k-space L, then we have

$$\vartheta(z, \lambda_2 r) > E - \gamma \tag{4.159}$$

for all $z \in B_{cr}(x) \cap L$.

In the proof we drop the subscript 2 on ϱ and λ; it was introduced so that lemma is easier to recall when we need it.

Proof. Assume $x = 0$, $r = 1$. If the statement is false, one can find a sequence of maps $\{u_i\}_i$ that satisfy (4.70 | SC), $k+1$ sequences of points $\{y_{ij}\}_{i\in\mathbb{N}}$ in $B_c(0) \cap \mathcal{S}$ (with $j = 1, \ldots, k+1$), and a further sequence $\{z_i\}_i$ in $B_c(0)$ such that:

- $\{y_{ij}\}_{j=1}^{k+1}$ spans ϱ-effectively a k-space L (which can be assumed to be the same for all i).
- $\vartheta[u_i](y_{ij}, \lambda) > E - \frac{1}{i}$, and $\vartheta[u_i](y_{ij}, 1) \le E$.
- $\vartheta[u_i](z_i, \lambda) < E - \gamma$, and $z_i \in L$.

Up to subsequences, by the assumption (4.70 | SC) and Theorem 4.40, $\{u_i\}$ converges in $W^{1,p}$ to a map $\bar{u}$ which is still p-stationary; furthermore, up to further subsequences, $y_{ij} \to \bar{y}_j \in \bar{B}_c(0)$, and $z_i \to \bar{z} \in \bar{B}_c(0) \cap L$. Moreover, the set of points $\{\bar{y}_j\}_{j=1}^{k+1}$ still spans L, and we have $\vartheta[\bar{u}](\bar{y}_j, \lambda) = E$, which implies

$$\vartheta[\bar{u}](\bar{y}_j, 1) - \vartheta[\bar{u}](\bar{y}_j, \lambda) = 0. \tag{4.160}$$

Thus $\bar{u}$ is L-invariant in $\bar{B}_c(0)$ by the Cone Splitting Corollary 4.38; so in particular both $\vartheta[\bar{u}](\bar{z}, \lambda) = E$ and $\vartheta[\bar{u}](\bar{z}, \lambda) \le E - \gamma$ should hold. □

Finally, a result which states that *the lack of almost invariance spreads uniformly along pinched points*. This is yet another quantitative rephrasing of the fact that if $\vartheta(\cdot, r) - \vartheta(\cdot, \lambda r) = 0$ at two different (close) points, then u is invariant along the direction connecting them.

Lemma 4.73 (Lack of invariance near pinched points). *Let $\lambda > 0$, $\sigma_0 \in \left(0, \frac{1}{2}t_b\right)$, $\kappa_0 \in (0,1)$. There exists a constant ε such that the following holds. For any $u \in W^{1,p}(\Omega, \mathcal{N})$ satisfying (4.70 | SC) with energy bounded by Λ, if the following conditions are satisfied by a pair of points x, y:*

1. $|x-y| < \frac{1}{2}t_b r$;
2. $\vartheta(x,r) - \vartheta(x, \lambda r) < \varepsilon$;
3. $\vartheta(y,r) - \vartheta(y, \lambda r) < \varepsilon$;

and u is not (η, k)-invariant in $B_{\sigma r}(x)$ for some $\sigma_0 \le \sigma \le \frac{1}{2}t_b$, then u is not $(\kappa_0 \eta, k)$-invariant in $B_{\sigma r}(y)$.

Proof. Assume $x = 0$, $r = 1$. By contradiction, there exist: a sequence $\{u_i\}_i$ of maps satisfying (4.70 | SC), a sequence $\{y_i\}_i$ of points in $B_{\frac{1}{2}t_b}(0)$ and a sequence $\{\sigma_i\}_i$ in $[\sigma_0, \frac{1}{2}t_b]$ such that:

$$\vartheta[u_i](0,1) - \vartheta[u_i](0,\lambda) < \frac{1}{i}, \tag{4.161}$$

$$\vartheta[u_i](y_i,1) - \vartheta[u_i](y_i,\lambda) < \frac{1}{i}, \tag{4.162}$$

$$\sigma_i^{p-m} \int_{B_{\sigma_i}(0)} |\nabla_L u_i|^p > \eta \qquad \forall L \in \mathbf{G}^k(\mathbb{R}^m), \tag{4.163}$$

$$\sigma_i^{p-m} \int_{B_{\sigma_i}(y_i)} |\nabla_{\tilde{L}} u_i|^p < \kappa_0 \eta \qquad \exists \tilde{L} \in \mathbf{G}^k(\mathbb{R}^m). \tag{4.164}$$

Up to subsequences, they converge, respectively, to a p-stationary map $\bar{u}$ (in $W^{1,p}$), to a point $\bar{y} \in \bar{B}_{\frac{1}{2}t_b}(0)$ and to a number $\bar{\sigma} \in \left[\sigma_0, \frac{1}{2}t_b\right]$. Moreover, due to Equations (4.161) and (4.162) and by the L^p-convergence of gradients, $\bar{u}$ is 0-homogeneous with respect to both $x = 0$ and $\bar{y}$, thus it is invariant along the direction $\frac{\bar{y}}{|\bar{y}|}$ in $B_{\frac{1}{2}t_b}(0)$. Again by the fact that $u_i \to \bar{u}$ in $W^{1,p}(\Omega, \mathcal{N})$, however, we also have that

$$\bar{\sigma}^{p-m} \int_{B_\sigma(0)} |\nabla_{\tilde{L}} \bar{u}|^p \ge \eta \tag{4.165}$$

$$\bar{\sigma}^{p-m} \int_{B_\sigma(\bar{y})} |\nabla_{\tilde{L}} \bar{u}|^p \le \kappa_0 \eta; \tag{4.166}$$

this contradicts the fact that the two left hand sides should be equal (by translation invariance of $\bar{u}$). □

***Remark* 4.74** (Assumptions on ψ, again). The structural constants t_a and t_b we introduced in Assumption 4.28 were broadly used in these last few lemmas. In practice, the key feature (of ψ) we need for our purposes is the possibility to work handily in the ball $B_1(0)$ (so for example we require that 1 is an admissible value for $c(\varsigma)$ in Lemma 4.71). In the end, with the choice $t_a > 2$ done in Assumption 4.28, we can apply Lemmas 4.71 and 4.72 with $c = 1$ and Lemma 4.73 with $|x-y| < r$. ◊

4.6 Covering arguments

In the sequel, if $B = B_r(x)$ and k is a positive number, we denote by kB the ball $B_{kr}(x)$. We first give two useful definitions of "sets of points satisfying a pinched condition"; we have (more or less) already used both of them in Section 4.5.4.

Definition 4.75. Let u be a p-stationary map, $x \in B_1(0)$, $r > 0$. Assume $E, \lambda, \delta > 0$ are fixed, and $\mathcal{S} \subset B_1(0)$. We define

$$\mathcal{H}(x,r) = \mathcal{H}^{\mathcal{S}}_{E,\delta,\lambda}(x,r) \doteq \{y \in B_r(x) \cap \mathcal{S} \mid \vartheta(y, \lambda r) > E - \delta\}$$
$$\mathcal{P}(x,r) = \mathcal{P}^{\mathcal{S}}_{\delta,\lambda}(x,r) \doteq \{y \in B_r(x) \cap \mathcal{S} \mid \vartheta(y,r) - \vartheta(y, \lambda r) < \delta\}.$$

If $B = B_r(x)$, we also denote by $\mathcal{H}_B, \mathcal{P}_B$ the sets $\mathcal{H}(x,r), \mathcal{P}(x,r)$ respectively. $\diamondsuit$

***Remark* 4.76 ($\mathcal{H}$ and $\mathcal{P}$).** Heuristically $\mathcal{H}$ should be thought as a set of pinched points at which $\vartheta(y, \lambda r)$ *has a value which is close to the maximum possible.* We notice that:

(a) If all the parameters appearing are fixed, and u is such that $\vartheta(y,r) \le E$ for all $y \in B_r(x)$, then $\mathcal{H}(x,r) \subset \mathcal{P}(x,r)$;

(b) As a consequence, whenever $\mathcal{H}$ ϱ-effectively spans a k-subspace, also $\mathcal{P}$ trivially does. $\diamondsuit$

Notations and map of the constants. This will be the context for the whole section:

- $m, \mathcal{N}, p, \Lambda$ are fixed as in the previous sections (respectively: dimension of the domain, target manifold, exponent for the energy, upper bound on the p-energy).
- $u \in W^{1,p}(\Omega, \mathcal{N})$ is a p-minimizing harmonic map, or more generally a p-stationary map under the condition (4.70 | SC) for the strong compactness theorem. Moreover, u has p-energy bounded by Λ.
- We let $\varrho \in (0,1)$ be a fixed constant, and $r = \varrho^{\hat{\jmath}}$ for some $\hat{\jmath} \in \mathbb{N}_{\ge 1}$. The radius r will be the scale parameter for the singular stratification, up to a constant. *The constant ϱ will be arbitrary in the first covering, and will be then suitably selected in the construction of the second covering.*
- $\eta > 0$ is the (fixed) closeness parameter for the stratification.
- $k \in \{1, \dots, m\}$ is the dimension parameter for the stratification.
- $\gamma > 0$ is a constant used for the pinching condition on ϑ. *It will be arbitrary in the construction of the first covering, then selected in Proposition 4.80.*
- δ_0 and ε are the constants produced by Lemma 4.71 when $\varrho_1 = \lambda_1 = \frac{1}{5}\varrho$, η is the already chosen closeness parameter and ς is chosen such that $c(\varsigma) = 1$; δ_1 is the constant produced by Lemma 4.72 when $\varrho_2 = \lambda_2 = \frac{1}{5}\varrho$, γ is the arbitrary parameter introduced above and $c = 1$. We fix $\delta = \delta(\varrho, \eta, \gamma) \doteq \min\{\varepsilon, \delta_1\}$.
- $\mathcal{S}$ is a subset of the stratum $\mathcal{S}^k_{\eta,\delta_0 r}(u) \cap B_1(0)$.
- $0 < E \le \Lambda$ is such that $\vartheta(x,1) \le E$ for all $x \in B_1(0) \cap \mathcal{S}$.

The goal of the upcoming constructions will be to build a "controlled" covering of $\mathcal{S}$. In words, the ultimate goal will be to cover $\mathcal{S}$ with balls B satisfying the following:

1. The sum of the k^{th} powers of the radii is bounded by a universal constant.
2. Up to rescaling by a fixed constant, the balls are pairwise disjoint.
3. Either the radius of B is less than or equal to the fixed radius r; or the (normalized) p-energy in B is lower than the "maximal initial p-energy" E by a fixed amount δ (E and δ were introduced in the previous list of constants). In the latter case, we say that B satisfies a uniform energy drop condition (see the below Definition 4.77).

This will be achieved in Proposition 4.82. Once we have this, we can then apply the same reasoning to each of the balls where the p-energy drops uniformly (while keeping the other balls as they are). At each step other balls of radius $\leq r$ are produced, while the p-energy continues to drop uniformly in all the other balls. The procedure lasts a finite number of steps, until there's no energy left: indeed the total initial p-energy was bounded by a fixed constant Λ. This is the content of Section 4.7.1. Let's give a precise definition of energy drop:

Definition 4.77. We say that a ball B with radius r_B satisfies the uniform (λ, δ)-*energy drop condition* if

$$\vartheta(y, \lambda r_B) \leq E - \delta \qquad \text{for all } y \in \mathcal{S} \cap B; \tag{4.167}$$

From now on $\lambda = \frac{1}{5}$ and δ will be fixed (δ as in the Map of the Constants above), so we omit them and simply say "uniform energy drop". The terminology used here hides the fact that we have in mind a fixed map u as in the Notations above. $\Diamond$

At first, we are only able to reach a partial result: we don't manage to fully get a uniform energy drop condition on the balls of the covering; but we can show that, in each ball, the points for which the p-energy does not drop uniformly lie close to a $(k-1)$-plane (so in the end they can be controlled very efficiently). This is the content of the next subsection.

4.6.1 First covering

Recall that $r = \varrho^{\hat{\jmath}}$ is the scale parameter of the singular stratum we are considering. Here $\varrho > 0$ is a fixed parameter and $\hat{\jmath} \in \mathbb{N}$: we are allowed to work with constants which depend on ϱ, but *not* on $\hat{\jmath}$.

Construction of the first covering We construct a covering $\mathcal{F}$ of $\mathcal{S}$ with the following properties:

1. $\mathcal{F} = \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_{\hat{\jmath}-1} \cup \mathcal{E}_{\hat{\jmath}}$. If $B \in \mathcal{G}_h$, then $B = B\Big(x, \varrho^h\Big)$ for some x. If $B \in \mathcal{E}_{\hat{\jmath}}$, then $B = B\Big(x, \varrho^{\hat{\jmath}}\Big)$ for some x. Mnemonic rule: when the construction of $\mathcal{F}$ is complete, the subcovering $\mathcal{E}_{\hat{\jmath}}$ is made of balls with radius *equal* to r, while the subcoverings labeled with $\mathcal{G}$ are made of balls with radius *greater* than r.
2. If $B\Big(x, \varrho^h\Big) \in \mathcal{G}_h$ with $0 \leq h \leq \hat{\jmath} - 1$, then $\mathcal{H}(x, \varrho^h) \subset \mathcal{B}\Big(V, \frac{1}{5}\varrho^{h+1}\Big)$ for some $(k-1)$-affine subspace $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$; here $\mathcal{H} = \mathcal{H}^{\mathcal{S}}_{E,\delta,\frac{1}{5}\varrho}$.

3. If $B, B' \in \mathcal{F}$ and $B \neq B'$, then $\frac{1}{5}B \cap \frac{1}{5}B' = \varnothing$.

4. If $B(x, r_x) \in \mathcal{G}_1 \cup \cdots \cup \mathcal{G}_{\hat{\jmath}-1} \cup \mathcal{E}_{\hat{\jmath}}$, then

$$\vartheta\left(x, \frac{1}{5}r_x\right) > E - \gamma, \tag{4.168}$$

$$x \in \mathcal{S}\left[k; \frac{1}{2}\eta; \delta_0 r_x\right](u). \tag{4.169}$$

The strategy will be to apply inductively the lemmas from Section 4.5.4 at different scales. We thus proceed inductively on $j \in \{0, \ldots, \hat{\jmath}\}$.

Step 1, case A. If $\mathcal{H}(0,1)$ is contained in $\mathcal{B}\left(V, \frac{1}{5}\varrho\right)$ with $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$, then we define $\mathcal{G}_0 \doteq \{B(0,1)\}$. The other subcoverings are left empty, and the process stops here.

Step 1, case B. If $\mathcal{H}(0,1)$ is *not* contained in *any* neighborhood of the form $\mathcal{B}\left(V, \frac{1}{5}\varrho\right)$ with $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$, then:

- by Lemma 4.54 ("Sufficient condition"), $\mathcal{H}(0,1)$ spans $\frac{1}{5}\varrho$-effectively a k-space $L(0,1) \in \mathbf{H}^k(\mathbb{R}^m)$ (and $\mathcal{P}(0,1)$ also does by Remark 4.76);
- thus, by Lemma 4.71 ("Pinching") with $\lambda_1 = \varrho_1 = \frac{1}{5}\varrho$, $\mathcal{S} \cap B(0,1)$ is contained in $\mathcal{B}\left(L(0,1), \frac{1}{5}\varrho\right)$;
- by Lemma 4.72 ("High energy pinching"), with $\lambda_2 = \varrho_2 = \frac{1}{5}\varrho$, for any $z \in L(0,1) \cap B(0,1)$ we have

$$\vartheta\left(z, \frac{1}{5}\varrho\right) > E - \gamma; \tag{4.170}$$

- if γ is small enough (smaller than a constant depending on m, p, η, ϱ), by Lemma 4.73 ("Lack of invariance near pinched points") with $\kappa_0 = \frac{1}{2}$, $\lambda = \frac{1}{5}\varrho$ and $\sigma = \delta_0\varrho$, for any $z \in L(0,1) \cap B(0,1)$ we have $z \in \mathcal{S}\left[k; \frac{1}{2}\eta; \delta_0\varrho\right](u)$ (because $\mathcal{S}[k;\eta;\delta_0\varrho](u) \supset \mathcal{S}[k;\eta;\delta_0 r](u)$).

Cover $\mathcal{S} \cap B(0,1)$ with balls of radius ϱ with centers in $L(0,1)$ and such that $\frac{1}{5}B \cap \frac{1}{5}B' = \varnothing$ if $B \neq B'$. Call $\mathcal{E}_1$ this covering.

If $\hat{\jmath} = 1$, *i.e.* the final radius r we want to reach is ϱ^1, then we can stop here the procedure. Otherwise, assume that $B \doteq B(x, \varrho) \in \mathcal{E}_1$ is a ball produced by Step 1, case B.

Step 2, case A. If $\mathcal{H}(x, \varrho) \subset \mathcal{B}\left(V, \frac{1}{5}\varrho^2\right)$ for some $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$, then B is one of the balls that we want to keep in our final covering $\mathcal{F}$; we define

$$\mathcal{G}_1 \doteq \left\{B(x,\varrho) \in \mathcal{E}_1 \;\middle|\; \mathcal{H}(x,\varrho) \subset \mathcal{B}\left(V, \frac{1}{5}\varrho^2\right) \text{ for some } V \in \mathbf{H}^{k-1}(\mathbb{R}^m)\right\}. \tag{4.171}$$

Step 2, case B. If instead $B \notin \mathcal{G}_1$, this means that $\mathcal{H}(x,\varrho)$ spans $\frac{1}{5}\varrho^2$-effectively a k-space $L(x,\varrho)$, again by Lemma 4.54. Thus, applying Lemmas 4.71 to 4.73 with the *same constants* as in Step 1, Case B, we get:

1. $\mathcal{S} \cap B(x,\varrho) \subset \mathcal{B}\Big(L(x,\varrho), \frac{1}{5}\varrho^2\Big)$ for some $L(x,\varrho) \in \mathbf{H}^k(\mathbb{R}^m)$;
2. $\vartheta(z, \frac{1}{5}\varrho^2) > E - \gamma$ for all $z \in L(x,\varrho) \cap B(x,\varrho)$;
3. $z \in \mathcal{S}\Big[k; \frac{1}{2}\eta; \delta_0 \varrho^2\Big](u)$ for all $z \in L(x,\varrho) \cap B(x,\varrho)$.

Now we cover $\mathcal{S} \cap B(x,\varrho) \setminus \bigcup \mathcal{G}_1$ with balls of radius ϱ^2 such that for any pair $B \neq B'$ of such balls we have $\frac{1}{5}B \cap \frac{1}{5}B' = \varnothing$ and $\frac{1}{5}B \subset B(x,\varrho) \setminus \bigcup \mathcal{G}_1$; define $\mathcal{E}_{2,x}$ such a covering. Define

$$\mathcal{E}_2 \doteq \bigcup \{\mathcal{E}_{2,x} \mid B(x,\varrho) \in \mathcal{E}_1 \setminus \mathcal{G}_1\}. \tag{4.172}$$

This concludes Step 2.

After the j^{th} step, we have:

- j families of balls $\mathcal{G}_0, \dots, \mathcal{G}_{j-1}$, with the following properties: if $B \in \mathcal{G}_h$ then $B = B\Big(x, \varrho^h\Big)$ for some x, and $\mathcal{H}\Big(x, \varrho^h\Big)$ is contained in $\mathcal{B}\Big(V, \frac{1}{5}\varrho^{h+1}\Big)$ for some $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$;
- A family $\mathcal{E}_j$ of balls of radius ϱ^j.

If $j = \hat{\jmath}$, then we are done. Otherwise, we proceed in the same fashion. Let $B = B(x,\varrho^j) \in \mathcal{E}_j$.

Step $j+1$, case A. If $\mathcal{H}(x,\varrho^j) \subset \mathcal{B}\Big(V, \frac{1}{5}\varrho^{j+1}\Big)$ for some $V \in \mathbf{H}^{k-1}(\mathbb{R}^m)$, then B is one of the balls that we want to keep in our final covering $\mathcal{F}$; we define

$$\mathcal{G}_j \doteq \left\{ B\Big(x,\varrho^j\Big) \in \mathcal{E}_j \;\middle|\; \mathcal{H}(x,\varrho^j) \subset \mathcal{B}\Big(V, \frac{1}{5}\varrho^{j+1}\Big) \text{ for some } V \in \mathbf{H}^{k-1}(\mathbb{R}^m) \right\}. \tag{4.173}$$

Step $j+1$, case B. If instead $B \notin \mathcal{G}_j$, this means that $\mathcal{H}(x,\varrho^j)$ spans $\frac{1}{5}\varrho^{j+1}$-effectively a k-space $L(x,\varrho^j)$. Thus, applying Lemmas 4.71 to 4.73 with the *same constants* as in Case B of the previous steps, we get:

1. $\mathcal{S} \cap B \subset \mathcal{B}\Big(L(x,\varrho^j), \frac{1}{5}\varrho^{j+1}\Big)$ for some $L(x,\varrho^j) \in \mathbf{H}^k(\mathbb{R}^m)$;
2. $\vartheta(z, \frac{1}{5}\varrho^{j+1}) > E - \gamma$ for all $z \in L(x,\varrho^j) \cap B(x,\varrho^j)$;
3. $z \in \mathcal{S}\Big[k; \frac{1}{2}\eta; \delta_0 \varrho^{j+1}\Big](u)$ for all $z \in L(x,\varrho^j) \cap B(x,\varrho^j)$.

Now we cover $\mathcal{S} \cap B(x,\varrho^j) \setminus \bigcup_{h \leq j} \bigcup \mathcal{G}_h$ with balls of radius ϱ^{j+1} such that for any pair $B \neq B'$ of such balls we have $\frac{1}{5}B \cap \frac{1}{5}B' = \varnothing$ and $\frac{1}{5}B \subset B(x,\varrho) \setminus \bigcup_{h \leq j} \bigcup \mathcal{G}_h$; define $\mathcal{E}_{j+1,x}$ such a covering. Define

$$\mathcal{E}_{j+1} \doteq \bigcup \{\mathcal{E}_{j+1,x} \mid B(x,\varrho) \in \mathcal{E}_j \setminus \mathcal{G}_j\}. \tag{4.174}$$

Iterating the procedure until $\hat{\jmath}$, we obtain the desired construction.

Definition 4.78. If $\mathcal{F} = \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_{\hat{\jmath}-1} \cup \mathcal{E}_{\hat{\jmath}}$ is the covering just constructed, define the following sets of centers:

$$\mathcal{D}_h \doteq \Big\{x \in B(0,1) \,\Big|\, B\big(x, \varrho^h\big) \in \mathcal{G}_h\Big\}, \quad 0 \le h \le \hat{\jmath}-1 \tag{4.175}$$

$$\mathcal{D}_{\hat{\jmath}} \doteq \Big\{x \in B(0,1) \,\Big|\, B\big(x, \varrho^{\hat{\jmath}}\big) \in \mathcal{E}_{\hat{\jmath}}\Big\} \tag{4.176}$$

$$\mathcal{C} \doteq \mathcal{D}_0 \cup \cdots \cup \mathcal{D}_{\hat{\jmath}-1} \cup \mathcal{D}_{\hat{\jmath}} \tag{4.177}$$

$$\mathcal{C}_\ell \doteq \mathcal{D}_{\hat{\jmath}-\ell} \cup \cdots \cup \mathcal{D}_{\hat{\jmath}}, \quad 0 \le \ell \le \hat{\jmath}. \tag{4.178}$$

Moreover, if $x \in \mathcal{C}$, we will also denote by r_x the radius of the ball centered at x which is contained in $\mathcal{F}$. Notice that

$$\mathcal{C}_\ell = \Big\{x \in \mathcal{C} \,\Big|\, r_x \le \varrho^{\hat{\jmath}-\ell}\Big\}, \tag{4.179}$$

and $\mathcal{C} = \mathcal{C}_{\hat{\jmath}} \subset \mathcal{C}_{\hat{\jmath}-1} \subset \cdots \subset \mathcal{C}_1 \subset \mathcal{C}_0 = \mathcal{D}_{\hat{\jmath}}$. ◊

The next step is probably the most important of the whole construction: indeed, we show that *we have a control on the k^{th} powers of the radii of the balls in $\mathcal{F}$*. Here is where the refined techniques of Section 4.5.2 become involved: we use Reifenberg Theorem 4.63 to achieve the final estimate, and Theorem 4.66 to check Reifenberg's hypothesis. Unfortunately, the proof is a bit intricate: we split it in several subtheorems.

***Remark* 4.79.** From now on, we will assume without loss of generality that ϱ is of the form $5^{-\kappa}$ for some $\kappa \in \mathbb{N}$. This does not affect in any way the general procedure (in the proof of Proposition 4.82 we will choose ϱ as an arbitrary number smaller than a certain constant) and simplifies a bit some computations. ◊

Proposition 4.80 (Volume estimates). *Let $\mathcal{F} = \{B(x, r_x)\}_{x \in \mathcal{C}}$ be the covering constructed in the previous paragraph. Recall that $\varrho, \eta, \gamma, E > 0$ are fixed constants. If $\gamma > 0$ and $\varrho > 0$ are chosen small enough, there exists a constant $C_{\mathrm{I}} = C_{\mathrm{I}}(m, \varrho)$ such that*

$$\sum_{x \in \mathcal{C}} r_x^k \le C_{\mathrm{I}}. \tag{4.180}$$

By Reifenberg Theorem 4.63, the estimate (4.180) is achieved if the condition

$$\int_{B(w,\tau)} \int_0^\tau \beta_\mu^k(y,s)^2 \, \frac{ds}{s} \, d\mu(y) < \delta_R \tau^k \tag{4.181}$$

holds for any ball $B(w,\tau)$ with $w \in B(0,1)$ and $0 < \tau < 1$ (or $0 < \tau < \tau_{\max}$ for some $\tau_{\max}$, at the only price of worsening the constants involved). Here $\mu \doteq \sum_{x \in \mathcal{C}} r_x^k \delta_x$. For any $0 \le h \le \hat{\jmath}$, we now consider the measure μ_h associated to the set of centers $\mathcal{C}_h$ defined in Definition 4.78: $\mu_h \doteq \sum_{x \in \mathcal{C}_h} r_x^k \delta_x$. Clearly $\mu = \mu_{\hat{\jmath}}$; first of all, we state a very elementary "induction property" of the measures μ_h.

Subtheorem 4.80.1. *Let $h \in \{0, \ldots, \hat{\jmath}-1\}$. Let $C_{in} > 0$ be a constant. Assume without loss of generality $\varrho < \frac{1}{10}$. Assume that, for all $x \in B(0,1)$ and all $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, \varrho^{\hat{\jmath}-h}\right]$, it holds*

$$\mu_h(B(x,s)) \le C_{in} s^k. \tag{4.182}$$

Then there exists a constant C_f depending only on C_{in}, ϱ and m such that

$$\mu_{h+1}(B(x,s)) \le C_f s^k \tag{4.183}$$

whenever one of the following holds:

(i) $B(x,s) \cap (\mathcal{C}_{h+1} \setminus \mathcal{C}_h) = \varnothing$ *and* $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, 2\varrho^{\hat{\jmath}-(h+1)}\right]$.

(ii) $B(x,s)$ *contains points of* $\mathcal{C}_{h+1} \setminus \mathcal{C}_h$ *and* $s \in \left[2\varrho^{\hat{\jmath}-h}, 2\varrho^{\hat{\jmath}-(h+1)}\right]$.

***Remark* 4.81.** We could obviously state the same property with more general constants in front of the radii involved. This is however the form we will need: notice that the "upper bound" for the radius gains a factor 2. ◊

Proof. Fix $x \in B(0,1)$ and $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, 2\varrho^{\hat{\jmath}-(h+1)}\right]$. We can split $\mu_{h+1}(B(x,s))$ as

$$\begin{aligned} \mu_{h+1}(B(x,s)) &= \mu_h(B(x,s)) + \sum_{\substack{z \in \mathcal{C}_{h+1} \setminus \mathcal{C}_h \\ z \in B(x,s)}} \varrho^{k(\hat{\jmath}-h-1)} \\ &= \mu_h(B(x,s)) + \varrho^{k(\hat{\jmath}-h-1)} \operatorname{card}(B(x,s) \cap \mathcal{C}_{h+1} \setminus \mathcal{C}_h). \end{aligned} \tag{4.184}$$

Now:

- If $B(x,s) \cap (\mathcal{C}_{h+1} \setminus \mathcal{C}_h) = \varnothing$ and $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, \varrho^{\hat{\jmath}-h}\right]$, then the first term is smaller than or equal to $C_{in} s^k$ by assumption; the second term is trivially zero.
- If $s \in \left[\varrho^{\hat{\jmath}-h}, 2\varrho^{\hat{\jmath}-h-1}\right]$, then: we can cover $B(x,s) \cap \operatorname{spt}(\mu_h)$ with a controlled number $c_1(\varrho, m)$ of balls centered in $\mathcal{C}_h$ with radius $\varrho^{\hat{\jmath}-h}$, so that we obtain:

  $$\mu_h(B(x,s)) \le c_1 C_{in} \varrho^{k(\hat{\jmath}-h)} \le c_1 C_{in} s^k; \tag{4.185}$$

  moreover, the number $\operatorname{card}(B(x,s) \cap (\mathcal{C}_{h+1} \setminus \mathcal{C}_h))$ is also bounded by a constant $c_2(\varrho, m)$, because balls centered in $(\mathcal{C}_{h+1} \setminus \mathcal{C}_h)$ with radius $\frac{1}{5}\varrho^{\hat{\jmath}-h-1}$ do not contain points of $\mathcal{C}$ other then their center. Thus

  $$\varrho^{k(\hat{\jmath}-h-1)} \operatorname{card}(B(x,s) \cap \mathcal{C}_{h+1} \setminus \mathcal{C}_h) \le \frac{c_2}{\varrho^k} s^k. \tag{4.186}$$

By choosing $C_f \doteq \max\left\{C_{in}, c_1 C_{in} + \dfrac{c_2}{\varrho^k}\right\}$ we get the result. □

The next step is to prove that the estimate Equation (4.182) actually holds when $h = 0$.

Subtheorem 4.80.2. *There exists a constant $C_0(\varrho, m)$ such that: for any $x \in B(0,1)$ and $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, \varrho^{\hat{\jmath}}\right]$,*

$$\mu_0(B(x,s)) \le C_0 s^k. \tag{4.187}$$

Proof. An argument already used in Subtheorem 4.80.1: if $x \neq y \in \mathcal{C}_0$, then $B\left(x, \frac{1}{5}\varrho^{\hat{\jmath}}\right)$ and $B\left(y, \frac{1}{5}\varrho^{\hat{\jmath}}\right)$ are disjoint, thus the number of such centers contained in $B(x,s)$ is bounded by a constant (the same $c_2(\varrho, m)$ as in the previous proof). Thus

$$\mu_0(B(x,s)) \leq c_2(\varrho, m)\varrho^{\hat{\jmath}k} \leq 5^k c_2 s^k, \tag{4.188}$$

which is what we needed. □

It may seem that, having an inductive step and a base step, we could already get the volume estimate we need. The problem is that we are applying Subtheorem 4.80.1 with an initial constant C_{in} that keeps getting bigger at any step; instead, we would need in the end a universal constant that only depends on ϱ and m, since the number of steps is not fixed *a priori*, and we don't want our constants to depend on it. Here is where Reifenberg Theorem comes into play. The trick will be to prove that the estimate

$$\int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y,s)^2 \frac{ds}{s}\, d\mu_h(y) < \delta_R \tau^k \tag{4.189}$$

holds for any μ_h.

Subtheorem 4.80.3. *Let $\varrho > 0$ (small enough) and $C_f > 0$ be fixed constants; $\eta, E, \mathcal{S}$ as before. There exists a constant $\gamma = \gamma(\varrho, C_f, m, p, \eta)$ such that the following holds. Assume that $\mathcal{F}$ is the covering of $\mathcal{S}$ associated to the constant γ, and that μ_h verifies the conclusion of Subtheorem 4.80.1,* i.e.*: the estimate*

$$\mu_h(B(x,s)) \leq C_f s^k \tag{4.190}$$

holds whenever one of the following holds:

(i) $B(x,s) \cap (\mathcal{C}_h \setminus \mathcal{C}_{h-1}) = \varnothing$ and all $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, 2\varrho^{\hat{\jmath}-h}\right]$.

(ii) $B(x,s)$ contains a point of $\mathcal{C}_h \setminus \mathcal{C}_{h-1}$ and $s \in \left[2\varrho^{\hat{\jmath}-(h-1)}, 2\varrho^{\hat{\jmath}-h}\right]$.

Then the following estimate is also true:

$$\mu_h(B(x,s)) \leq C_R s^k \tag{4.191}$$

for all $x \in B(0,1)$ and all $s \in \left[\frac{1}{5}\varrho^{\hat{\jmath}}, \varrho^{\hat{\jmath}-h}\right]$, where C_R is the constant appearing in Reifenberg Theorem 4.63.

Proof. We proceed in several steps. Let $h \in \{0, \dots, \hat{\jmath}-1\}$ be fixed.

Step 1 *(Application of Reifenberg Theorem).* Clearly, if we are able to prove that

$$\int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y,s)^2 \frac{ds}{s}\, d\mu_h(y) < \delta_R \tau^k \tag{4.192}$$

holds for all $w \in B(0,1)$ and all $\tau < \varrho^{\hat{\jmath}-h}$, then we can exploit the rescaled version of Reifenberg Theorem (Corollary 4.64), and we get exactly the thesis.

Step 2 *(Application of the Estimates on β^k_μ).* Notice that the integral with respect to μ_h appearing in Equation (4.192) is actually a sum on $y \in \mathcal{C}_h \cap B(w,\tau)$. Let $y \in \mathcal{C}_h$, and consider $\beta^k_{\mu_h}(y,s)$. Then:

- If $s \leq \frac{1}{5} r_y$, then $\beta^k_{\mu_h}(y, s) = 0$, because y is the only point of $\mathcal{C}_h$ contained in $B(y, s)$ (and thus any k-plane through y is a best approximating plane for μ_h);
- If $s \geq \frac{1}{5} r_y$, by property 4 of the covering $\mathcal{F}$ (specifically Equation (4.169)), u is not $\left(\frac{1}{2}\eta, k+1\right)$-invariant in $B(y, 5\delta_0 s)$. Thus we can use Theorem 4.66 with $\bar{r} = 5\delta_0$ and $\sigma = 5$ (for example!) and obtain:

$$\beta^k_{\mu_h}(y, s)^2 \leq C_J s^{-k} \int_{B(y,s)} (\vartheta(z, 5s) - \vartheta(z, s))\, d\mu_h(z). \tag{4.193}$$

  where now C_J depends on m, p and η only.

More compactly, if we define the following function:

$$\begin{aligned} W(x, r) &\doteq [\vartheta(x, 5r) - \vartheta(x, r)]\chi_{\mathcal{C}_h}(x)\chi_{(r_x/5,\infty)}(r) = \\ &= \begin{cases} \vartheta(x, 5r) - \vartheta(x, r) & \text{if } x \in \mathcal{C}_h \text{ and } r \geq \frac{1}{5} r_x \\ 0 & \text{otherwise} \end{cases}, \end{aligned} \tag{4.194}$$

then for all $s > 0$ (smaller than a suitable constant) we have:

$$\beta^k_{\mu_h}(y, s)^2 \leq C_J s^{-k} \int_{B(y,s)} W(z, s)\, d\mu_h(z). \tag{4.195}$$

Step 3. By Step 2 and by Tonelli's Theorem, for a fixed $h \in \{0, \dots, \hat{\jmath} - 1\}$ we have:

$$\begin{aligned} \int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y, s)^2 \frac{ds}{s}\, d\mu_h(y) &\leq \int_{B(w,\tau)} \int_0^\tau C_J s^{-k} \int_{B(y,s)} W(z, s)\, d\mu_h(z) \frac{ds}{s}\, d\mu_h(y) \leq \\ &\leq \int_0^\tau C_J s^{-k} \int_{B(w,\tau)} \int_{B(y,s)} W(z, s)\, d\mu_h(z)\, d\mu_h(y) \frac{ds}{s}. \end{aligned} \tag{4.196}$$

Notice that by the triangle inequality

$$|z - w| \leq |z - y| + |y - w|, \tag{4.197}$$

so the set

$$\{(y, z) \in \mathbb{R}^m \times \mathbb{R}^m \mid y \in B(w, \tau), z \in B(y, s)\} \tag{4.198}$$

is contained in

$$\{(y, z) \in \mathbb{R}^m \times \mathbb{R}^m \mid z \in B(w, \tau + s), y \in B(z, s)\}. \tag{4.199}$$

Using again Tonelli Theorem, we also switch the two integrals in μ_h, thus getting:

$$\begin{aligned} \int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y, s)^2 \frac{ds}{s}\, d\mu_h(y) &\leq \int_0^\tau C_J s^{-k} \int_{B(w,\tau+s)} W(z, s) \left(\int_{B(z,s)} d\mu_h(y) \right) d\mu_h(z) \frac{ds}{s} \leq \\ &\leq \int_0^\tau C_J s^{-k} \int_{B(w,2\tau)} W(z, s)\, \mu_h(B(z, s))\, d\mu_h(z) \frac{ds}{s}. \end{aligned} \tag{4.200}$$

Now for all the relevant pairs (z,s) (for which W is not 0) the estimate (4.190) holds:

$$\int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y,s)^2 \frac{ds}{s}\, d\mu_h(y) \le C_J C_f \int_{B(w,2\tau)} \int_0^\tau W(z,s) \frac{ds}{s}\, d\mu_h(z). \tag{4.201}$$

STEP 4. Let $x \in \mathcal{D}_\ell$, so that $r_x = \varrho^\ell = 5^{-\kappa\ell}$ (see the Remark before the statement). The following estimate holds:

$$\begin{aligned} \int_{\frac{1}{5}r_x}^{\frac{1}{5}} (\vartheta(x,5s) - \vartheta(x,s)) \frac{ds}{s} &= \sum_{j=1}^{\kappa\ell} \int_{\left(\frac{1}{5}\right)^{j+1}}^{\left(\frac{1}{5}\right)^j} \frac{\vartheta(x,5s)-\vartheta(x,s)}{s}\, ds \le \\ &\le \sum_{j=1}^{\kappa\ell} \frac{\vartheta(x,5^{1-j}) - \vartheta(x,5^{-1-j})}{5^{-1-j}} \left(\frac{1}{5}\right)^j \left(1 - \left(\frac{1}{5}\right)\right) \le \\ &\le C\left[\vartheta(x,1) - \vartheta\left(x,5^{-\kappa\ell}\right) + \vartheta\left(x,\frac{1}{5}\right) - \vartheta\left(x,\frac{1}{5}5^{-\kappa\ell}\right)\right] \le \\ &\le C_{12}\gamma, \end{aligned} \tag{4.202}$$

where the last inequality is a consequence of property 4 of the covering $\mathcal{F}$ (specifically Equation (4.168)) and C_{12} depends on ϱ and η. Plugging this information into Equation (4.201) (provided that $\tau \le \frac{1}{5}$), we get

$$\int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y,s)^2 \frac{ds}{s}\, d\mu_h(y) \le C_J C_f C_{12}\gamma\, \mu_h(B(w,2\tau)). \tag{4.203}$$

The left hand side is 0 whenever $B(w,2\tau)$ contains a single point of $\mathcal{C}_h$; in all the other cases, the assumption (4.190) holds, thus

$$\int_{B(w,\tau)} \int_0^\tau \beta^k_{\mu_h}(y,s)^2 \frac{ds}{s}\, d\mu_h(y) \le 2^{-k} C_J C_f^2 C_{12}\gamma\tau^k. \tag{4.204}$$

Choosing $\gamma(\varrho, C_f, m, p, \eta) \le \frac{\delta_R(m)}{2^{-k}C_J C_f^2 C_{12}}$, we have the desired result. □

We can finally prove Proposition 4.80.

Proof of Proposition 4.80. The proof is now a simple induction: by Subtheorem 4.80.2 we have an estimate on μ_0 depending on a constant C_0; applying Subtheorem 4.80.1 the same estimate holds for μ_1 with $C_f = C_f(C_0, m, p, \varrho)$; but then we apply Subtheorem 4.80.3 to improve the constant: the estimate now holds for μ_1 with $C_R(m)$. So we can repeat the procedure: the final constant for each μ_h will still be $C_R(m)$. □

4.6.2 Second covering

The goal now is to refine the covering in order to find balls which satisfy a clean energy drop; that is, we get rid in some sense of the sets of type $\mathcal{H}(x,r)$, where the uniform energy drop does not happen, and which are already bound to lie in the fattening of $(k-1)$-dimensional planes.

Construction of the second covering Consider again the "first covering" $\mathcal{F}$ for $\mathcal{S}$. It is split in $\mathcal{F} = \mathcal{G} \cup \mathcal{E}$, where $\mathcal{F}, \mathcal{E}$ and $\mathcal{G}$ have the following properties:

1. Balls in $\mathcal{E} \doteq \mathcal{E}^{(0)} \doteq \mathcal{E}_{\hat{\jmath}}$ have radius equal to $r = \varrho^{\hat{\jmath}}$;
2. Balls in $\mathcal{G} \doteq \mathcal{G}_0 \cup \cdots \cup \mathcal{G}_{\hat{\jmath}-1}$ have radius ϱ^h with $h < \hat{\jmath}$; if $B = B(x, r) \in \mathcal{G}$, it satisfies the condition

$$\mathcal{H}_B = \left\{ y \in B \cap \mathcal{S} \;\middle|\; \vartheta\Big(y, \frac{1}{5}\varrho r\Big) > E - \delta \right\} \subset \mathcal{B}\Big(V_B, \frac{1}{5}\varrho r\Big) \tag{4.205}$$

   for some $(k-1)$-affine subspace $V_B \in \mathbf{H}^{k-1}(\mathbb{R}^m)$.
3. The estimate $\sum_{B \in \mathcal{F}} r_B^k \le C_{\mathrm{I}}(m)$ holds, where r_B is the radius of B.

We now refine $\mathcal{F}$ inductively, applying at each step a rescaled version of the procedure from Section 4.6.1.

Step 1. Consider $B \in \mathcal{G}$ and the associated $(k-1)$-plane V_B. We cover $B \cap \mathcal{S}$ with balls of radius ϱr_B, divided in three subcoverings: $\mathcal{E}_B$ (with radius *equal* to r), $\mathcal{D}_B$ (satisfying an *energy drop* condition), $\mathcal{W}_B$ (*wild* balls on which we have no control).

- If $r_B = \varrho^{\hat{\jmath}-1}$, simply cover $B \cap \mathcal{S}$ with at most $C_{13}(m, \varrho)$ balls of radius $\varrho^{\hat{\jmath}}$. Call this covering $\mathcal{E}_B$; set $\mathcal{D}_B = \mathcal{W}_B = \varnothing$. (Actually $C_{13}(m, \varrho) = C(m)\varrho^{-m}$, but it's irrelevant.)
- If $r_B > \varrho^{\hat{\jmath}-1}$, we cover $\mathcal{B}\Big(\mathcal{H}_B, \frac{1}{5}\varrho r_B\Big)$ with at most $C_{14}(m)\varrho^{-(k-1)}$ balls of radius ϱr_B; call this covering $\mathcal{W}_B$. This is possible since $\mathcal{H}_B \subset \mathcal{B}\Big(V_B, \frac{1}{5}\varrho r_B\Big)$; notice that the case $\mathcal{H}_B = \varnothing$ is included. Cover $(B \cap \mathcal{S}) \setminus \mathcal{B}\Big(\mathcal{H}_B, \frac{1}{5}\varrho r_B\Big)$ with at most $C_{13}(m, \varrho)$ balls of radius ϱr_B; call this covering $\mathcal{D}_B$. Set $\mathcal{E}_B = \varnothing$. Notice that if $\tilde{B} \in \mathcal{D}_B$ then it satisfies the uniform energy drop condition.

At this point we have a covering of $\mathcal{S}$ of this type:

$$\mathcal{F}^{(1)} = \mathcal{E}^{(1)} \cup \mathcal{D}^{(1)} \cup \mathcal{W}^{(1)}, \tag{4.206}$$

where

$$\mathcal{E}^{(1)} \doteq \mathcal{E}^{(0)} \cup \bigcup_{B \in \mathcal{G}} \mathcal{E}_B, \qquad \mathcal{D}^{(1)} \doteq \bigcup_{B \in \mathcal{G}} \mathcal{D}_B, \qquad \mathcal{W}^{(1)} \doteq \bigcup_{B \in \mathcal{G}} \mathcal{W}_B, \tag{4.207}$$

and

$$\sum_{B \in \mathcal{E}^{(0)}} r_B^k \le C_{\mathrm{I}}(m), \qquad \sum_{B \in \mathcal{E}^{(1)} \setminus \mathcal{E}^{(0)}} r_B^k \le \varrho^k C_{13}(m, \varrho) C_{\mathrm{I}}(m) \tag{4.208}$$

$$\sum_{B \in \mathcal{D}^{(1)}} r_B^k \le \varrho^k C_{13}(m, \varrho) C_{\mathrm{I}}(m), \qquad \sum_{B \in \mathcal{W}^{(1)}} r_B^k \le C_{14}(m) C_{\mathrm{I}}(m) \varrho^k \varrho^{-k+1}. \tag{4.209}$$

Introduce the constants

$$K_1(\varrho, m) = C_{\mathrm{I}}(m)(1 + 2\varrho^k C_{13}) \tag{4.210}$$

$$K_2(m) = C_{\mathrm{I}}(m) C_{14}(m). \tag{4.211}$$

so that

$$\sum_{\mathcal{E}^{(1)} \cup \mathcal{D}^{(1)}} r_B^k \le K_1, \qquad \sum_{\mathcal{W}^{(1)}} r_B^k \le K_2 \varrho. \tag{4.212}$$

Step $h+1$**.** Assume that, for some $h \le \hat{\jmath}-1$, we have a covering of $\mathcal{S}$ of the form $\mathcal{F}^{(h)} = \mathcal{E}^{(h)} \cup \mathcal{D}^{(h)} \cup \mathcal{W}^{(h)}$ with the following properties:

1. If $B \in \mathcal{E}^{(h)}$, then $r_B = r = \varrho^{\hat{\jmath}}$;
2. If $B \in \mathcal{D}^{(h)}$, then the energy drop condition holds in B;
3. If $B \in \mathcal{W}^{(h)}$, then $\varrho^{\hat{\jmath}} < r_B \le \varrho^h$;
4. The estimates

$$\sum_{B\in\mathcal{E}^{(h)}\cup\mathcal{D}^{(h)}} r_B^k \le K_1 \sum_{j=0}^{h-1} (K_2\varrho)^j, \qquad \sum_{B\in\mathcal{W}^{(h)}} r_B^k \le (K_2\varrho)^h \tag{4.213}$$

hold true.

Consider a ball $B^\star \in \mathcal{W}^{(h)}$. Applying a rescaled version of the first construction (and of Proposition 4.80) we first find a covering $\mathcal{F}_{B^\star}$ for $\mathcal{S}\cap B^\star$ of the type

$$\mathcal{F}_{B^\star} = \mathcal{G}_{B^\star;h} \cup \cdots \cup \mathcal{G}_{B^\star;\hat{\jmath}-1} \cup \mathcal{E}_{B^\star;\hat{\jmath}} = \mathcal{G}_{B^\star} \cup \mathcal{E}_{B^\star}, \tag{4.214}$$

where $\mathcal{G}_{B^\star}$ are balls on which the energy drop condition is verified up to a neighborhood of a $(k-1)$-plane, $\mathcal{E}_{B^\star}$ are balls of radius $r = \varrho^{\hat{\jmath}}$, and $\sum_{B\in\mathcal{F}_{B^\star}} r_B^k \le C_{\mathrm{I}}(m) r_{B^\star}^k$. Secondly, re-cover each ball of $\mathcal{G}_{B^\star}$ with a rescaled version of Step 1, thus obtaining

$$\mathcal{F}_{B^\star}^{(h+1)} = \mathcal{E}_{B^\star}^{(h+1)} \cup \mathcal{D}_{B^\star}^{(h+1)} \cup \mathcal{W}_{B^\star}^{(h+1)}, \tag{4.215}$$

where balls of $\mathcal{E}_{B^\star}^{(h+1)}$ have radius r, balls of $\mathcal{D}_{B^\star}^{(h+1)}$ satisfy the energy drop condition, balls of $\mathcal{W}_{B^\star}^{(h+1)}$ have radius $r < r_B \le \varrho^{h+1}$, and

$$\sum_{B\in\mathcal{E}_{B^\star}^{(h+1)}\cup\mathcal{D}_{B^\star}^{(h+1)}} r_B^k \le K_1 r_{B^\star}^k, \qquad \sum_{B\in\mathcal{W}_{B^\star}^{(h+1)}} r_B^k \le K_2\varrho r_{B^\star}^k. \tag{4.216}$$

Then define

$$\mathcal{E}^{(h+1)} \doteq \mathcal{E}^{(h)} \cup \bigcup_{B^\star\in\mathcal{W}^{(h)}} \mathcal{E}_{B^\star}^{(h+1)}, \qquad \mathcal{D}^{(h+1)} \doteq \mathcal{D}^{(h)} \cup \bigcup_{B^\star\in\mathcal{W}^{(h)}} \mathcal{D}_{B^\star}^{(h+1)}, \tag{4.217}$$

$$\mathcal{W}^{(h+1)} \doteq \bigcup_{B^\star\in\mathcal{W}^{(h)}} \mathcal{W}_{B^\star}^{(h+1)}, \tag{4.218}$$

$$\mathcal{F}^{(h+1)} = \mathcal{E}^{(h+1)} \cup \mathcal{D}^{(h+1)} \cup \mathcal{W}^{(h+1)}. \tag{4.219}$$

All the conditions 1 to 3 are satisfied with $h+1$ instead of h; as for the estimates (4.213), we have

$$\begin{aligned} \sum_{B\in\mathcal{E}^{(h+1)}\cup\mathcal{D}^{(h+1)}} r_B^k &= \sum_{B\in\mathcal{E}^{(h)}\cup\mathcal{D}^{(h)}} r_B^k + \sum_{B^\star\in\mathcal{W}^{(h)}} \sum_{B\in\mathcal{E}_{B^\star}^{(h+1)}\cup\mathcal{D}_{B^\star}^{(h+1)}} r_B^k \le \\ &\le K_1 \sum_{j=0}^{h-1} (K_2\varrho)^j + K_1 \sum_{B^\star\in\mathcal{W}^{(h)}} r_{B^\star}^k \le K_1 \sum_{j=0}^{h} (K_2\varrho)^j \end{aligned} \tag{4.220}$$

and

$$\begin{aligned}\sum_{B\in\mathcal{W}^{(h+1)}} r_B^k = \sum_{B^\star\in\mathcal{W}^{(h)}} \sum_{B\in\mathcal{W}^{(h+1)}_{B^\star}} r_B^k \le \\ \le K_2\varrho \sum_{B^\star\in\mathcal{W}^{(h)}} r_{B^\star}^k \le (K_2\varrho)^{h+1}.\end{aligned} \tag{4.221}$$

Thus, as a consequence of this procedure, we have the following.

Proposition 4.82. *Let $u \in W^{1,p}(\Omega,\mathcal{N})$ be a map with energy bounded by Λ that satisfies the p-minimality or p-stationariety conditions of (4.70 | SC). Let $\eta > 0$ be a constant and $1 \le k \le m$. Assume that $E \le \Lambda$ is such that $\vartheta(y,1) \le E$ for all $y \in B(0,1)\cap\mathcal{S}$. Let $\mathcal{S} \subset \mathcal{S}[k;\eta;\delta_0 r](u)$ for some $r > 0$. There exists a finite covering $\mathcal{F}^\star$ of $\mathcal{S}$ with the following properties:*

(i) All the radii satisfy $r_B \ge \bar\varrho r$, where $\bar\varrho \in (0,1)$ is a constant only depending on m;

(ii) The k^{th} powers of the radii are controlled by $\sum_{B\in\mathcal{F}^\star} r_B^k \le C_{\mathrm{II}}$, where C_{II} depends only on m.

(iii) $\mathcal{F}^\star = \mathcal{L}^\star \cup \mathcal{D}^\star$, where:

(A) For all $B \in \mathcal{L}^\star$, $r_B \le r$ (that is, r_B is lower *than or equal to the needed radius);*

(B) Every ball $B \in \mathcal{D}^\star$ satisfies a uniform energy drop *condition:*

$$\vartheta\left(y, \frac{1}{5} r_B\right) < E - \delta \qquad \text{for all } y \in \mathcal{S}\cap B. \tag{4.222}$$

Here both δ_0 and δ are constants that depend on $m, p, \mathcal{N}, \Lambda, \eta$ (and nothing else).

Proof. Let $\bar\varrho = \bar\varrho(m) \le \frac{1}{2}K_2(m)^{-1}$, where K_2 is the constant introduced in (4.211). Once $\bar\varrho(m)$ is chosen, also a constant $\bar\gamma(m,p,\eta)$ is fixed by Subtheorem 4.80.3; as a consequence, $\delta_0(m,p,\eta)$ gets determined by Lemma 4.71 and the constant $\delta(m,p,\eta)$ is fixed as well by Lemma 4.71 and Lemma 4.72. Assume that $\bar\varrho^{\hat\jmath} \le r < \bar\varrho^{\hat\jmath-1}$ for some $\hat\jmath \in \mathbb{N}$. Perform the construction (of the first covering and then) of the second covering until Step $\hat\jmath$. Then:

- $\mathcal{W}^{(\hat\jmath)} = \varnothing$ (by the bounds on the radii), so $\mathcal{F}^{(\hat\jmath)} = \mathcal{E}^{(\hat\jmath)} \cup \mathcal{D}^{(\hat\jmath)}$;
- The balls in $\mathcal{E}^{(\hat\jmath)}$ have radius $\bar\varrho^{\hat\jmath} \le r$;
- The balls in $\mathcal{D}^{(\hat\jmath)}$ satisfy the energy drop condition;
- Each ball in $\mathcal{F}^{(\hat\jmath)}$ has radius greater than or equal to $\bar\varrho^{\hat\jmath} \ge \bar\varrho r$.

Moreover, by Equation (4.213),

$$\sum_{B\in\mathcal{F}^{(\hat\jmath)}} r_B^k \le K_1(m,\bar\varrho(m)) \sum_{j=0}^{\hat\jmath-1} (K_2\varrho)^j \le K_3(m) \sum_{h=0}^{\infty} \left(\frac{1}{2}\right)^h \le 2K_3(m). \tag{4.223}$$

This proves the proposition, by setting $C_{\mathrm{II}}(m) = 2K_3(m)$, $\mathcal{L}^\star = \mathcal{E}^{(\hat\jmath)}$, $\mathcal{D}^\star = \mathcal{D}^{(\hat\jmath)}$. □

4.7 Proof of the main theorems

Finally, we can prove the result stated in Theorem 4.56. We split the proof in the following two paragraphs.

4.7.1 Volume estimate

Theorem 4.83. *Let $u \in W^{1,p}(\Omega, \mathcal{N})$ be a map with energy bounded by Λ. Let $\eta > 0$ and $1 \le k \le m$. Assume that $\mathcal{N}$ and u satisfy the condition (4.70 | SC) for the strong compactness theorem. There exists a constant $C_{15} = C_{15}(m, \mathcal{N}, p, \Lambda, \eta)$ such that for any $r > 0$*

$$\mathscr{L}^m\Big(\mathcal{B}\Big(\mathcal{S}^k_{\eta,\delta_0 r}(u), r\Big) \cap B(0,1)\Big) \le C_{15} r^{m-k}. \tag{4.224}$$

The proof is a straightforward consequence of the following lemma:

Lemma 4.84. *Let m, p, Λ, η, k be constants, u a map and $r > 0$ as in Theorem 4.83. For any number $i \in \mathbb{N}$ there exists a covering $\mathcal{F}^\star_i$ of the set $\mathcal{S} \doteq \mathcal{S}[k; \eta; \delta_0 r](u)$ consisting of open balls, satisfying the following properties:*

(i) The radii r_B satisfy

$$\sum_{B \in \mathcal{F}^\star_i} r_B^k \le (c_3(m) C_{\mathrm{II}}(m))^i \tag{4.225}$$

for some new dimensional constant $c_3(m)$ and the old constant $C_{\mathrm{II}}(m)$ coming from Proposition 4.82;

(ii) $\mathcal{F}^\star_i = \mathcal{L}^\star_i \cup \mathcal{D}^\star_i$, where:

(A) For all $B \in \mathcal{L}^\star_i$, $\bar\varrho r \le r_B \le r$, where $\bar\varrho = \bar\varrho(m) \in (0,1)$ is the constant introduced in Proposition 4.82;

(B) For all $B \in \mathcal{D}^\star_i$ and all $y \in \mathcal{S} \cap B$, we have

$$\vartheta\Big(y, \frac{1}{5} r_B\Big) \le \Lambda - i\delta,$$

where $\delta = \delta(m, p, \Lambda, \mathcal{N}, \eta)$ is the constant determined by Proposition 4.82.

Proof. We proceed by induction on $i \in \mathbb{N}$. For $i = 0$, we can simply take $\mathcal{F}^\star_0 = \mathcal{D}^\star_0 = \{B(0,1)\}$. Assume then the lemma is true for some $i \ge 0$. Consider a ball $B_0 \in \mathcal{D}^\star_i$, and cover it with $c_3(m)$ balls of radius $\frac{1}{5} r_{B_0}$ (call $\mathcal{D}_{i,B_0}$ this covering); for each of these balls B consider the rescaling of B (and u) through the transformation that maps it into the unit ball. Applying Proposition 4.82 with $E = \Lambda - i\delta$ and $\mathcal{S} \setminus \bigcup_{j=1}^i \mathcal{L}_j$, and scaling back to the original B, we find a covering $\mathcal{F}^\star_B = \mathcal{L}^\star_B \cup \mathcal{D}^\star_B$ satisfying:

1. If $\tilde B \in \mathcal{L}^\star_B$, then $\bar\varrho r \le r_{\tilde B} \le r$;
2. If $\tilde B \in \mathcal{D}^\star_B$, then $\vartheta\Big(y, \frac{1}{5} r_{\tilde B}\Big) \le \Lambda - i\delta - \delta$ for all $y \in \tilde B \cap \mathcal{S}$;
3. $\sum_{\tilde B \in \mathcal{F}^\star_B} r_{\tilde B}^k \le C_{\mathrm{II}} r_B^k$.

Thus, by defining

$$\mathcal{D}^\star_{i+1} \doteq \bigcup_{B_0\in\mathcal{D}^\star_i} \bigcup_{B\in\mathcal{D}_{B_0,i}} \mathcal{D}^\star_B \tag{4.226}$$

$$\mathcal{L}^\star_{i+1} \doteq \mathcal{L}^\star_i \cup \bigcup_{B_0\in\mathcal{D}^\star_i} \bigcup_{B\in\mathcal{D}_{B_0,i}} \mathcal{L}^\star_B, \tag{4.227}$$

we get the needed result. □

Proof of Theorem 4.83. We apply Lemma 4.84 assuming the integer i to be $\hat{\imath}(m,p,\Lambda,\mathcal{N},\eta) \doteq \left\lfloor \frac{\Lambda}{\delta} \right\rfloor + 1$, where $\delta = \delta(m,p,\Lambda,\mathcal{N},\eta)$ is the constant determined by Proposition 4.82. Consider the covering $\mathcal{F}^\dagger \doteq \mathcal{F}^\star_{\hat{\imath}}$ of $\mathcal{S}[k;\eta;\delta_0 r](u) \cap B(0,1)$: the energy drop condition $\vartheta\left(y, \frac{1}{5}r_B\right) \le \Lambda - \hat{\imath}\delta < 0$ can never hold, since by definition the normalized energy ϑ is non-negative. Thus $\mathcal{F}^\dagger$ consists of a collection of balls satisfying

$$\sum_{B\in\mathcal{F}^\dagger} r_B^k \le (c_3(m)C_{\mathrm{II}}(m))^{\hat{\imath}} \doteq C_{\mathrm{III}}(m,p,\Lambda,\mathcal{N},\eta), \tag{4.228}$$

$$\bar{\varrho} r \le r_B \le r \text{ for any } B \in \mathcal{F}^\dagger. \tag{4.229}$$

By elementary geometric properties, the collection of balls $\left\{B(x,2r) \,\middle|\, B(x,r_B) \in \mathcal{F}^\dagger\right\}$ covers the tubular neighborhood $\mathcal{B}(\mathcal{S}[k;\eta;\delta_0 r](u), r) \cap B(0,1)$. In particular, by Equations (4.228) and (4.229):

$$\begin{aligned}\mathscr{L}^m(\mathcal{B}(\mathcal{S}[k;\eta;\delta_0 r](u), r) \cap B(0,1)) &\le \sum_{B\in\mathcal{F}^\dagger} \omega_m (2r)^m \le \omega_m 2^m r^{m-k} \sum_{B\in\mathcal{F}^\dagger} \left(\frac{r_B}{\bar{\varrho}}\right)^k \le \\ &\le \frac{\omega_m 2^m C_{\mathrm{III}}}{\bar{\varrho}^k} r^{m-k},\end{aligned} \tag{4.230}$$

where $\bar{\varrho} = \bar{\varrho}(m)$ and $C_{\mathrm{III}} = C_{\mathrm{III}}(m,p,\Lambda,\mathcal{N},\eta)$. By defining

$$C_{15}(m,\mathcal{N},p,\Lambda,\eta) \doteq \frac{\omega_m 2^m C_{\mathrm{III}}}{\bar{\varrho}^k}, \tag{4.231}$$

the result is proved. □

***Remark* 4.85** (Bound on the Hausdorff measure)**.** Notice that, thanks to the properties (4.228) and (4.229) of the covering $\mathcal{F}^\dagger$ one already obtains an explicit estimate on the k-dimensional Hausdorff measure of the stratum $\mathcal{S}[k;\eta;\delta_0 r](u)$, without going through Lemma 4.24 and the (yet to prove) rectifiability. In particular, at this stage we already know that any stratum $\mathcal{S}^k_{\eta,\delta_0 r}(u)$ (and $\mathcal{S}^k_\eta$, as a consequence) has a finite k-dimensional Hausdorff measure. ♢

4.7.2 Rectifiability

We now tackle the problem of the rectifiability of the strata of type $\mathcal{S}^k_\eta$. It is clear that we will need to use the second part of Theorem 4.63; the technique is basically the same we used for the volume estimates, even with some simplifications.

Theorem 4.86. *Let $u \in W^{1,p}(\Omega,\mathcal{N})$, and assume that $\mathcal{N}$ and u satisfy the condition (4.70 | SC) for the strong compactness theorem. For any $\eta > 0$ and any $0 \le k \le m$, the stratum $\mathcal{S}^k_\eta(u)$ is k-rectifiable.*

As we will see shortly, the result follows easily from this lemma.

Lemma 4.87. *Let m, p, Λ, η be fixed. There exist a universal constant $\kappa(m, p, \Lambda, \mathcal{N}, \eta)$ with $0 < \kappa < 1$ such that the following holds. Let u satisfy the condition (4.70 | SC), and let $\mathcal{S} \subset \mathcal{S}^k_\eta(u)$ be a $\mathscr{H}^k$-measurable subset. There exists a $\mathscr{H}^k$-measurable subset $\mathcal{R} \subset \mathcal{S}$ with the following properties:*

1. *$\mathscr{H}^k(\mathcal{R}) \leq \kappa \mathscr{H}^k(\mathcal{S})$;*
2. *The set $\mathcal{S} \setminus \mathcal{R}$ is k-rectifiable.*

Before proving this lemma, which requires some effort, we show how it is applied to prove Theorem 4.86.

Proof of Theorem 4.86. Notice that by Theorem 4.83 and Remark 4.85 we already know that $\mathscr{H}^k\Big(\mathcal{S}^k_\eta(u)\Big)$ is finite. By induction, for any $j \in \mathbb{N}$ there exists a $\mathscr{H}^k$-measurable set $\mathcal{R}_j \subset \mathcal{S}^k_\eta(u)$ such that:

- $\mathscr{H}^k(\mathcal{R}_j) \leq \kappa^j \mathscr{H}^k\Big(\mathcal{S}^k_\eta(u)\Big) < \infty$;
- The set $(\mathcal{S}^k_\eta(u)) \setminus \mathcal{R}_j$ is k-rectifiable.

This is easily proved: the step $j = 1$ comes from the application of Lemma 4.87 to the stratum $\mathcal{S}^k_\eta(u)$, while the $(j+1)^{\text{th}}$ step descends from the application of the same lemma to $\mathcal{R}_j$. Now we can define

$$\tilde{\mathcal{R}} \doteq \bigcap_{j \in \mathbb{N}} \mathcal{R}_j \tag{4.232}$$

$$\tilde{\mathcal{S}} \doteq \mathcal{S}^k_\eta(u) \setminus \tilde{\mathcal{R}} = \bigcup_{j \in \mathbb{N}} \Big(\mathcal{S}^k_\eta(u) \setminus \mathcal{R}_j\Big). \tag{4.233}$$

Here $\tilde{\mathcal{R}}$ has $\mathscr{H}^k$-measure zero; and $\tilde{\mathcal{S}}$ is the countable union of sets, each of which is countable union of Lipschitz k-graphs; therefore $\tilde{\mathcal{S}}$ itself is a countable union of Lipschitz k-graphs. This means precisely that $\mathcal{S}^k_\eta(u)$ is k-rectifiable. $\square$

Now we turn to prove Lemma 4.87.

Proof. We can assume that $\mathscr{H}^k(\mathcal{S}) > 0$, otherwise the statement is trivial.

Step 1. Consider the following map: for $x \in B(0, 1)$ and $r > 0$ (small enough),

$$f_r(x) \doteq \vartheta(x, r) - \vartheta(x, 0), \tag{4.234}$$

where $\vartheta(x, 0) \doteq \lim_{s \to 0} \vartheta(x, s)$. As r tends to 0, the map f_r converges pointwise and decreasingly to the constant function $f_0 \equiv 0$; moreover, all the maps f_r are bounded by the constant map Λ, which is integrable with respect to the measure $\mathscr{H}^k \llcorner \mathcal{S}$ by Theorem 4.83. Now fix a $\delta > 0$. By the Dominated Convergence Theorem, there exists a $\bar{r} > 0$ depending on δ such that

$$\int_{\mathcal{S}} f_{5\bar{r}}(x)\, d\mathscr{H}^k(x) \leq \delta^2 \mathscr{H}^k(\mathcal{S}). \tag{4.235}$$

Consider the following sets:

$$F_\delta \doteq \left\{ x \in \mathcal{S} \,\middle|\, f_{5\bar{r}(\delta)}(x) > \delta \right\} \tag{4.236}$$

$$G_\delta \doteq \left\{ x \in \mathcal{S} \,\middle|\, f_{5\bar{r}(\delta)}(x) \le \delta \right\} = \mathcal{S} \setminus F_\delta; \tag{4.237}$$

observe that, since $f_{5\bar{r}}$ is nonnegative, we have:

$$\int_{\mathcal{S}} f_{5\bar{r}}(x)\, d\mathscr{H}^k(x) \ge \int_{F_\delta} f_{5\bar{r}}(x)\, d\mathscr{H}^k(x) \ge \delta \mathscr{H}^k(F_\delta); \tag{4.238}$$

this, combined with Equation (4.235), gives

$$\mathscr{H}^k(F_\delta) \le \delta \mathscr{H}^k(\mathcal{S}). \tag{4.239}$$

We claim that, for δ sufficiently small, the set G_δ is k-rectifiable; if we manage to show this, then the lemma is proved. In order to prove this claim, we consider a *finite* covering $\{B(x_i, \bar{r})\}_{i=1}^L$ of G_δ made with balls of the fixed radius $\bar{r}(\delta)$. It is sufficient to show that for δ small $G_\delta \cap B(x_i, \bar{r}(\delta))$ is rectifiable for any i: our main aim will be now to check the applicability of the second Reifenberg Theorem (Theorem 4.63, part (ii)), that gives exactly that result.

Step 2. Fix a ball $B(x_i, \bar{r}(\delta))$, with $i \in \{1, \dots, L\}$, and apply the usual transformation $\lambda_{x_i,\bar{r}}^{-1}$. We set

$$\tilde{u} = T_{x_i,\bar{r}} u, \qquad \tilde{G}_\delta = \lambda^{\leftarrow}_{x_i,\bar{r}}(G_\delta) \cap B(0,1). \tag{4.240}$$

Also, we define μ_δ to be the measure $\mathscr{H}^k \llcorner \tilde{G}_\delta$ on the unit ball $B(0,1)$. Notice that for any $x \in \tilde{G}_\delta$ we have:

$$\vartheta^{\tilde{u}}(x,5) - \vartheta^{\tilde{u}}(x,0) \le \delta, \tag{4.241}$$

by the definition of G_δ and the usual scale invariance properties of ϑ. Now the original G_δ was a subset of $\mathcal{S}^k_\eta(u)$, hence u was not $(\eta, k+1)$-invariant in $B(x, \bar{r}s)$ for any point $x \in G_\delta$ and for any $s > 0$; consequently, for any point x in the transformed set $\tilde{G}_\delta$ and for any $s > 0$, $\tilde{u}$ is not $(\eta, k+1)$-invariant in $B(x,s)$. This is what we need to apply Theorem 4.66 on any ball $B(x,s)$; and we apply it to the finite measure $\mu_\delta = \mathscr{H}^k \llcorner \tilde{G}_\delta$. We obtain that, for any $x \in \tilde{G}_\delta$ and any $0 < s \le 1$,

$$\beta^k_{\tilde{G}_\delta}(x,s)^2 \le C_{16}(m,p,\eta) s^{-k} \int_{B(x,s)} \vartheta(y,5s) - \vartheta(y,s)\, d\mu_\delta(y). \tag{4.242}$$

This goes in the direction we need, since we are trying to check if the Reifenberg condition (4.122 | k-Reif) is satisfied. Following what we did in the proof of Proposition 4.80, we first fix $w \in B(0,1)$ and $r \le 1$; for all $0 < s \le r$ we compute:

$$\int_{B(w,r)} \beta^k_{\tilde{G}_\delta}(x,s)^2\, d\mu_\delta(x) \le C_{16} s^{-k} \int_{B(w,r)} \left(\int_{B(x,s)} \left[\vartheta^{\tilde{u}}(y,5s) - \vartheta^{\tilde{u}}(y,s) \right] d\mu_\delta(y) \right) d\mu_\delta(x) \tag{4.243}$$

Observe that we are allowed to do this since μ_δ is supported in $\tilde{G}_\delta$. As we have already noticed in Proposition 4.80, if $|x-w| < r$ and $|y-x| < s$, then $|y-w| < r+s$: thus we can estimate

(4.244)

$$\begin{aligned}\int_{B(w,r)} \beta^k_{\tilde{G}_\delta}(x,s)^2\, d\mu_\delta(x) &\le C_{16} s^{-k} \int_{B(w,r+s)} \int_{B(y,s)} \left[\vartheta^{\tilde{u}}(y,5s) - \vartheta^{\tilde{u}}(y,s) \right] d\mu_\delta(x)\, d\mu_\delta(y) \le \\ &\le C_{16} s^{-k} \int_{B(w,r+s)} \left[\vartheta^{\tilde{u}}(y,5s) - \vartheta^{\tilde{u}}(y,s) \right] \mathscr{H}^k\left(\tilde{G}_\delta \cap B(y,s) \right) d\mu_\delta(y).\end{aligned}$$

But now we can exploit the uniform volume estimates given by Theorem 4.83 (appropriately rescaled); we get the following uniform *a priori* upper bound:

$$\mathscr{H}^k\Big(\lambda^{\leftarrow}_{x_i,\bar{r}}\Big(\mathcal{S}^k_\eta(u)\Big)\cap B(y,s)\Big)\le C_{15}(m,p,\mathcal{N},\Lambda,\eta)s^k; \tag{4.245}$$

notice that thanks to this *a priori* estimate it is not necessary to reproduce the induction argument of Proposition 4.80. Plugging this information in the previous inequality we get:

$$\int_{B(w,r)}\beta^k_{\tilde{G}_\delta}(x,s)^2\,d\mu_\delta(x)\le C_{16}C_{15}\int_{B(w,r+s)}\Big[\vartheta^{\tilde{u}}(y,5s)-\vartheta^{\tilde{u}}(y,s)\Big]\,d\mu_\delta(y). \tag{4.246}$$

In order to check the validity of Equation (4.122 | k-Reif), we now consider the left hand side of that inequality: applying Tonelli Theorem (twice), we find:

$$\begin{aligned}\int_{B(w,r)}\left(\int_0^r\beta^k_{\tilde{G}_\delta}(x,s)^2\frac{ds}{s}\right)d\mu_\delta(x)&=\int_0^r\left(\int_{B(w,r)}\beta^k_{\tilde{G}_\delta}(x,s)^2\,d\mu_\delta(x)\right)\frac{ds}{s}\le\\&\le C_{16}C_{15}\int_0^r\left(\int_{B(w,2r)}\Big[\vartheta^{\tilde{u}}(y,5s)-\vartheta^{\tilde{u}}(y,s)\Big]\,d\mu_\delta(y)\right)\frac{ds}{s}=\\&=C_{16}C_{15}\int_{B(w,2r)}\left(\int_0^r\Big[\vartheta^{\tilde{u}}(y,5s)-\vartheta^{\tilde{u}}(y,s)\Big]\frac{ds}{s}\right)d\mu_\delta(y).\end{aligned} \tag{4.247}$$

Consider for a moment the inner integral; r can simply be bounded by 1. We use basically the same trick we exploited in Proposition 4.80:

$$\begin{aligned}\int_0^1\Big[\vartheta^{\tilde{u}}(y,5s)-\vartheta^{\tilde{u}}(y,s)\Big]\frac{ds}{s}&=\sum_{j=0}^\infty\int_{5^{-(j+1)}}^{5^{-j}}\frac{\vartheta^{\tilde{u}}(y,5s)-\vartheta^{\tilde{u}}(y,s)}{s}ds\le\\&\le\sum_{j=0}^\infty\int_{5^{-(j+1)}}^{5^{-j}}\frac{\vartheta^{\tilde{u}}(y,5^{-j+1})-\vartheta^{\tilde{u}}(y,5^{-j-1})}{5^{-j-1}}ds\le\\&\le C_{17}\sum_{j=0}^\infty\Big[\vartheta^{\tilde{u}}(y,5^{-j+1})-\vartheta^{\tilde{u}}(y,5^{-j-1})\Big]\le\\&\le C_{17}\Big[\Big(\vartheta^{\tilde{u}}(y,5)-\vartheta^{\tilde{u}}(y,0)\Big)+\Big(\vartheta^{\tilde{u}}(y,1)-\vartheta^{\tilde{u}}(y,0)\Big)\Big]\le\\&\le 2C_{17}\delta.\end{aligned} \tag{4.248}$$

Therefore we can insert this piece of information in the previous integral; using again the upper bound (4.245) on the measure of the singular stratum, we find, for a new constant $C_{19}(m,p,\mathcal{N},\Lambda,\eta)$:

$$\begin{aligned}\int_{B(w,r)}\left(\int_0^r\beta^k_{\tilde{G}_\delta}(x,s)^2\frac{ds}{s}\right)d\mu_\delta(x)&\le C_{18}\mu_\delta(B(w,2r))\delta\le\\&\le C_{19}\delta r^k.\end{aligned} \tag{4.249}$$

Taking

$$\delta<\frac{\delta_{\mathrm{R}}(m)}{C_{19}(m,p,\mathcal{N},\Lambda,\eta)}, \tag{4.250}$$

we get exactly the hypothesis needed for the second part of Reifenberg Theorem: thus $\tilde{G}_\delta$ is k-rectifiable, and tracing back the steps of the proof this proves the k-rectifiability of G_δ. $\square$

Notations

$B_r(x)$	ball centered at x with radius r
$B(x,r)$	ball centered at x with radius r – used when cumbersome notations are involved
$\mathcal{B}_r(S)$	set of points that dist less than r from the set S
$\mathcal{B}(x,r)$	set of points that dist less than r from the set S – used when cumbersome notations are involved
$\mathscr{B}(\mathsf{X})$	Borel σ-algebra on (X,d)
β_μ^k	k-dimensional Jones' number of the measure μ (Definition 4.57)
$\mathbf{C}^1_{\mathbb{H}}$	space of horizontally $\mathbf{C}^1$ functions from subdomains of $\mathbb{H}^n$ (Definition 1.95)
$\mathbf{C}^1_{\mathbb{W}}$	space of intrinsically $\mathbf{C}^1$ functions (Definition 3.10)
C_{db}	doubling constant of a metric measure space (Definition 1.25)
Ch_p	Cheeger energy (Definition 1.12)
C_{PI}	(local) Poincaré constant of a metric measure space (Definition 1.28)
$\lvert\nabla(\cdot)\rvert$	slope of a function (Definition 1.3)
$\lvert\nabla(\cdot)\rvert_{\mathrm{w}}$	minimal weak upper gradient of a function (Definition 1.10 and Remark 1.15)
$\lvert\mathrm{D}(\cdot)\rvert_*$	total variation of a function (def:bv-tv)
$\nabla_{\mathbb{H}}$	horizontal gradient in $\mathbb{H}^n$ (Definition 1.95)
∇^f	intrinsic gradient associated to f (Definition 3.7)
d_{cc}	Carnot-Carathéodory distance (Equation (1.99))
d_{eu}	Euclidean distance
$\mathsf{d}_{\mathrm{H}}, \mathsf{d}_{\mathrm{GH}}, \mathsf{d}_{\mathrm{mGH}}$	Hausdorff, Gromov-Hausdorff, measured Gromov-Hausdorff distance respectively (Section 1.1.4)
$\mathrm{Geo}(\mathsf{X})$	space of geodesics on the metric space (X,d) (Definition 1.5)
$\mathbf{G}^k(\mathbb{R}^m)$	family of k-dimensional linear subspaces of $\mathbb{R}^m$ Notation 4.36
$\mathbb{H}^n$	Heisenberg group (Section 1.4.1)
$\mathscr{H}^k$	k-dimensional Hausdorff measure
$\mathbf{H}^k(\mathbb{R}^m)$	family of k-dimensional affine subspaces of $\mathbb{R}^m$ Notation 4.36

$H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$	Sobolev space through approximation by Lipschitz functions (Definition 1.7, see also $N^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$)
$H^{1,p}(\Omega,\mathsf{d},\mathfrak{m})$	local Sobolev space (Definition 1.19)
$H^{1,p}_0(\Omega,\mathsf{d},\mathfrak{m})$	Sobolev functions with zero boundary values, defined as closure of Lip_c in $H^{1,p}$ (Definition 1.18)
$\mathcal{I}_{(\mathsf{X},\mathsf{d},\mathfrak{m})}$	isoperimetric profile of $(\mathsf{X},\mathsf{d},\mathfrak{m})$ (Definition 1.36)
$(J_{K,N},\mathsf{d}_{\mathrm{eu}},\mathfrak{m}_{K,N})$	model space with curvature $K>0$ and dimension $N>1$
$\mathcal{K}_{\mathrm{ob}}(\Omega,\mathsf{d},\mathfrak{m})$	family of (a.e.) non-negative maps in $H^{1,2}_0(\Omega,\mathsf{d},\mathfrak{m})$ (Equation (2.66))
$\mathscr{L}^n$	n-dimensional Lebesgue measure
$\mathrm{Lip}(\mathsf{X},\mathsf{d})$	Lipschitz functions from (X,d) to $\mathbb{R}$
$\mathrm{Lip}_c(\mathsf{X},\mathsf{d})$	Lipschitz functions from (X,d) to $\mathbb{R}$ with compact support
$\mathrm{Lip}_b(\mathsf{X},\mathsf{d})$	bounded Lipschitz functions from (X,d) to $\mathbb{R}$
$\mathrm{Lip}_{\mathrm{loc}}(\mathsf{X},\mathsf{d})$	functions from (X,d) to $\mathbb{R}$ such that for any point x there exists $B_r(x)$ on which the function is Lipschitz
$\mathscr{M}^k$	k-dimensional Minkowski content (Definition 4.18)
$N^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$	Newtonian space (Remark 1.15, see also $H^{1,p}(\mathsf{X},\mathsf{d},\mathfrak{m})$)
$\mathscr{P}(\mathsf{Y})$	probability measures on Y (Notation 1.21)
$\mathscr{P}_2(\mathsf{Y})$	probability measures on Y with finite second moment (Notation 1.21)
$\mathrm{Per}_{\mathbb{H}}$	intrinsic perimeter in the Heisenberg group (Definition 1.86 and Notation 1.92)
$\mathrm{RS}(f)$	relaxed slope of the function f (Definition 1.7)
$\mathcal{S}(u)$	singular set of the function u (Definition 4.2)
$\mathcal{S}^k_{\eta,r}(u)$	singular k^{th} stratum of u, with scale parameter r and closeness parameter η (Definition 4.46)
$u^\sharp$	one-dimensional decreasing rearrangement of u (Definition 1.40)
$u^{\mathrm{sg}\sharp}$	signed one-dimensional decreasing rearrangement of u (Definition 2.16)
$u^\star$	equimeasurable decreasing rearrangement of u (Definition 1.55 and Definition 1.75)
$u^{\mathrm{sg}\star}$	signed equimeasurable decreasing rearrangement of u (Definition 2.17)
W_2	Wasserstein distance (Definition 1.22)
$W^{1,p}(\Omega)$	classical Sobolev spaces through integration by parts (Notation 1.20, see also $H^{1,p}$ and $N^{1,p}$)
$W^{1,p}_{\mathbb{X}}(\Omega)$	sub-Riemannian Sobolev space (Equation (1.101))

Bibliography

[ABB20] A. Agrachev, D. Barilari, and U. Boscain. *A comprehensive introduction to sub-Riemannian geometry*. Vol. 181. Cambridge Studies in Advanced Mathematics. From the Hamiltonian viewpoint, With an appendix by Igor Zelenko. Cambridge University Press, Cambridge, 2020, pp. xviii+745.

[ACD15] L. Ambrosio, M. Colombo, and S. Di Marino. "Sobolev spaces in metric measure spaces: reflexivity and lower semicontinuity of slope". In: *Variational methods for evolving objects*. Vol. 67. Adv. Stud. Pure Math. Math. Soc. Japan, [Tokyo], 2015, pp. 1–58.

[AD14] L. Ambrosio and S. Di Marino. "Equivalent definitions of BV space and of total variation on metric measure spaces". In: *J. Funct. Anal.* 266.7 (2014), pp. 4150–4188.

[AFP00] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.

[AG13] L. Ambrosio and N. Gigli. "A user's guide to optimal transport". In: *Modelling and optimisation of flows on networks*. Vol. 2062. Lecture Notes in Math. Springer, Heidelberg, 2013, pp. 1–155.

[AG16] L. Ambrosio and R. Ghezzi. "Sobolev and bounded variation functions on metric measure spaces". In: *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. II*. EMS Ser. Lect. Math. Eur. Math. Soc., Zürich, 2016, pp. 211–273.

[AGM15] L. Ambrosio, R. Ghezzi, and V. Magnani. "BV functions and sets of finite perimeter in sub-Riemannian manifolds". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32.3 (2015), pp. 489–517.

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Second. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008, pp. x+334.

[AGS13] L. Ambrosio, N. Gigli, and G. Savaré. "Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces". In: *Rev. Mat. Iberoam.* 29.3 (2013), pp. 969–996.

[AGS14a] L. Ambrosio, N. Gigli, and G. Savaré. "Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below". In: *Invent. Math.* 195.2 (2014), pp. 289–391.

[AGS14b] L. Ambrosio, N. Gigli, and G. Savaré. “Metric measure spaces with Riemannian Ricci curvature bounded from below”. In: *Duke Math. J.* 163.7 (2014), pp. 1405–1490.

[AH18] L. Ambrosio and S. Honda. “Local spectral convergence in $\mathrm{RCD}^*(K,N)$ spaces”. In: *Nonlinear Anal.* 177.part A (2018), pp. 1–23.

[AL88] M. Avellaneda and F.-H. Lin. “Fonctions quasi affines et minimisation de $\int |\nabla \mathbf{u}|^p$”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 306.8 (1988), pp. 355–358.

[Alm00] F. J. Almgren Jr. *Almgren's big regularity paper*. Vol. 1. World Scientific Monograph Series in Mathematics. Q-valued functions minimizing Dirichlet's integral and the regularity of area-minimizing rectifiable currents up to codimension 2, With a preface by Jean E. Taylor and Vladimir Scheffer. World Scientific Publishing Co., Inc., River Edge, NJ, 2000, pp. xvi+955.

[Amb01] L. Ambrosio. “Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces”. In: *Adv. Math.* 159.1 (2001), pp. 51–67.

[Amb02] L. Ambrosio. “Fine properties of sets of finite perimeter in doubling metric measure spaces”. In: vol. 10. 2-3. Calculus of variations, nonsmooth analysis and related topics. 2002, pp. 111–128.

[Amb18] L. Ambrosio. “Calculus, heat flow and curvature-dimension bounds in metric measure spaces”. In: *Proceedings of the International Congress of Mathematicians (ICM 2018)*. 2018, pp. 301–340. eprint: https://www.worldscientific.com/doi/pdf/10.1142/9789813272880_0015.

[ANS] L. Ambrosio, S. Nicolussi Golo, and F. Serra Cassano. “Classical flows of vector fields with exponential or sub-exponential summability”. Forthcoming.

[APS19] L. Ambrosio, A. Pinamonti, and G. Speight. “Weighted Sobolev spaces on metric measure spaces”. In: *J. Reine Angew. Math.* 746 (2019), pp. 39–65.

[AR04] L. Ambrosio and S. Rigot. “Optimal mass transportation in the Heisenberg group”. In: *J. Funct. Anal.* 208.2 (2004), pp. 261–301.

[ASV06] L. Ambrosio, F. Serra Cassano, and D. Vittone. “Intrinsic regular hypersurfaces in Heisenberg groups”. In: *J. Geom. Anal.* 16.2 (2006), pp. 187–232.

[AT15] J. Azzam and X. Tolsa. “Characterization of n-rectifiability in terms of Jones' square function: Part II”. In: *Geom. Funct. Anal.* 25.5 (2015), pp. 1371–1412.

[Bae19] A. Baernstein II. *Symmetrization in analysis*. Vol. 36. New Mathematical Monographs. With David Drasin and Richard S. Laugesen, With a foreword by Walter Hayman. Cambridge University Press, Cambridge, 2019, pp. xviii+473.

[Ban76] C. Bandle. “Bounds for the solutions of boundary value problems”. In: *J. Math. Anal. Appl.* 54.3 (1976), pp. 706–716.

[Ban80] C. Bandle. *Isoperimetric inequalities and applications*. Vol. 7. Monographs and Studies in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1980, pp. x+228.

[BB11] A. Björn and J. Björn. *Nonlinear potential theory on metric spaces.* Vol. 17. EMS Tracts in Mathematics. European Mathematical Society (EMS), Zürich, 2011, pp. xii+403.

[BBI01] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry.* Vol. 33. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2001, pp. xiv+415.

[BBK06] J. Björn, S. Buckley, and S. Keith. “Admissible measures in one dimension”. In: *Proc. Amer. Math. Soc.* 134.3 (2006), pp. 703–705.

[BBS20] A. Björn, J. Björn, and N. Shanmugalingam. “Locally p-admissible measures on $\mathbb{R}$”. In: *J. Funct. Anal.* 278.4 (2020), pp. 108344, 17.

[BCS15] F. Bigolin, L. Caravenna, and F. Serra Cassano. “Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation”. In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32.5 (2015), pp. 925–963.

[BDG69] E. Bombieri, E. De Giorgi, and E. Giusti. “Minimal cones and the Bernstein problem”. In: *Invent. Math.* 7 (1969), pp. 243–268.

[Bel96] A. Bellaïche. “The tangent space in sub-Riemannian geometry”. In: *Sub-Riemannian geometry.* Vol. 144. Progr. Math. Birkhäuser, Basel, 1996, pp. 1–78.

[Ber27] S. Bernstein. “Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differentialgleichungen vom elliptischen Typus”. In: *Math. Z.* 26.1 (1927), pp. 551–558.

[Ber51] L. Bers. “Isolated singularities of minimal surfaces”. In: *Ann. of Math. (2)* 53 (1951), pp. 364–386.

[BKS18] Z. M. Balogh, A. Kristály, and K. Sipos. “Geometric inequalities on Heisenberg groups”. In: *Calc. Var. Partial Differential Equations* 57.2 (2018), Paper No. 61, 41.

[Bob96] S. Bobkov. “Extremal properties of half-spaces for log-concave distributions”. In: *Ann. Probab.* 24.1 (1996), pp. 35–48.

[Bou06] J.-C. Bourgoin. “The minimality of the map $\frac{x}{\|x\|}$ for weighted energy”. In: *Calc. Var. Partial Differential Equations* 25.4 (2006), pp. 469–489.

[Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599.

[Bre87] Y. Brenier. “Décomposition polaire et réarrangement monotone des champs de vecteurs”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 305.19 (1987), pp. 805–808.

[Bre91] Y. Brenier. “Polar factorization and monotone rearrangement of vector-valued functions”. In: *Comm. Pure Appl. Math.* 44.4 (1991), pp. 375–417.

[BS10] F. Bigolin and F. Serra Cassano. “Distributional solutions of Burgers’ equation and intrinsic regular graphs in Heisenberg groups”. In: *J. Math. Anal. Appl.* 366.2 (2010), pp. 561–568.

[BSV07] V. Barone Adesi, F. Serra Cassano, and D. Vittone. “The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations”. In: *Calc. Var. Partial Differential Equations* 30.1 (2007), pp. 17–49.

[BTV17] Z. M. Balogh, J. T. Tyson, and E. Vecchi. "Intrinsic curvature of curves and surfaces and a Gauss-Bonnet theorem in the Heisenberg group". In: *Math. Z.* 287.1-2 (2017), pp. 1–38.

[BTV20] Z. M. Balogh, J. T. Tyson, and E. Vecchi. "Correction to: Intrinsic curvature of curves and surfaces and a Gauss-Bonnet theorem in the Heisenberg group". In: *Math. Z.* 296.1-2 (2020), pp. 875–876.

[Cap+07] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* Vol. 259. Progress in Mathematics. Birkhäuser Verlag, Basel, 2007, pp. xvi+223.

[Car92] M. do Carmo. *Riemannian geometry.* Mathematics: Theory & Applications. Translated from the second Portuguese edition by Francis Flaherty. Birkhäuser Boston, Inc., Boston, MA, 1992, pp. xiv+300.

[CC00a] J. Cheeger and T. H. Colding. "On the structure of spaces with Ricci curvature bounded below. II". In: *J. Differential Geom.* 54.1 (2000), pp. 13–35.

[CC00b] J. Cheeger and T. H. Colding. "On the structure of spaces with Ricci curvature bounded below. III". In: *J. Differential Geom.* 54.1 (2000), pp. 37–74.

[CC97] J. Cheeger and T. H. Colding. "On the structure of spaces with Ricci curvature bounded below. I". In: *J. Differential Geom.* 46.3 (1997), pp. 406–480.

[CCT02] J. Cheeger, T. H. Colding, and G. Tian. "On the singularities of spaces with bounded Ricci curvature". In: *Geom. Funct. Anal.* 12.5 (2002), pp. 873–914.

[CG89] J.-M. Coron and R. Gulliver. "Minimizing p-harmonic maps into spheres". In: *J. Reine Angew. Math.* 401 (1989), pp. 82–100.

[Che+05] J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang. "Minimal surfaces in pseudohermitian geometry". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 4.1 (2005), pp. 129–177.

[Che99] J. Cheeger. "Differentiability of Lipschitz functions on metric measure spaces". In: *Geom. Funct. Anal.* 9.3 (1999), pp. 428–517.

[CM17] F. Cavalletti and A. Mondino. "Sharp and rigid isoperimetric inequalities in metric-measure spaces with lower Ricci curvature bounds". In: *Invent. Math.* 208.3 (2017), pp. 803–849.

[CM18] F. Cavalletti and A. Mondino. "Isoperimetric inequalities for finite perimeter sets under lower Ricci curvature bounds". In: *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 29.3 (2018), pp. 413–430.

[CM21] F. Cavalletti and E. Milman. "The globalization theorem for the curvature-dimension condition". In: *Invent. Math.* 226.1 (2021), pp. 1–137.

[CMS01] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger. "A Riemannian interpolation inequality à la Borell, Brascamp and Lieb". In: *Invent. Math.* 146.2 (2001), pp. 219–257.

[CN13a] J. Cheeger and A. Naber. "Lower bounds on Ricci curvature and quantitative behavior of singular sets". In: *Invent. Math.* 191.2 (2013), pp. 321–339.

[CN13b] J. Cheeger and A. Naber. “Quantitative stratification and the regularity of harmonic maps and minimal currents”. In: *Comm. Pure Appl. Math.* 66.6 (2013), pp. 965–990.

[De 65] E. De Giorgi. “Una estensione del teorema di Bernstein”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 19 (1965), pp. 79–85.

[DGN08] D. Danielli, N. Garofalo, and D. M. Nhieu. “A notable family of entire intrinsic minimal graphs in the Heisenberg group which are not perimeter minimizing”. In: *Amer. J. Math.* 130.2 (2008), pp. 317–339.

[DT12] G. David and T. Toro. “Reifenberg parameterizations for sets with holes”. In: *Mem. Amer. Math. Soc.* 215.1012 (2012), pp. vi+102.

[EG15] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions.* Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299.

[ENV16] N. Edelen, A. Naber, and D. Valtorta. “Quantitative Reifenberg theorem for measures”. In: *arXiv preprint arXiv:1612.08052* (2016).

[ENV19] N. Edelen, A. Naber, and D. Valtorta. “Effective Reifenberg theorems in Hilbert and Banach spaces”. In: *Math. Ann.* 374.3-4 (2019), pp. 1139–1218.

[Eva10] L. C. Evans. *Partial differential equations. Second edition.* Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, 2010.

[Eva82] L. C. Evans. “A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic p.d.e”. In: *J. Differential Equations* 45.3 (1982), pp. 356–373.

[Fab23] G. Faber. “Beweiss dass unter allen homogenen Membranen von gleicher Fláche und gleicher Spannung die kreisförmgige den leifsten Grundton gibt”. In: *Sitz. Bayer Acad. Wiss.* (1923), pp. 169–172.

[Fed69] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969, pp. xiv+676.

[FF60] H. Federer and W. H. Fleming. “Normal and integral currents”. In: *Ann. of Math. (2)* 72 (1960), pp. 458–520.

[Fle62] W. H. Fleming. “On the oriented Plateau problem”. In: *Rend. Circ. Mat. Palermo (2)* 11 (1962), pp. 69–90.

[Fol99] G. B. Folland. *Real analysis.* Second. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xvi+386.

[FSS01] B. Franchi, R. Serapioni, and F. Serra Cassano. “Rectifiability and perimeter in the Heisenberg group”. In: *Math. Ann.* 321.3 (2001), pp. 479–531.

[Fuk87] K. Fukaya. “Collapsing of Riemannian manifolds and eigenvalues of Laplace operator”. In: *Invent. Math.* 87.3 (1987), pp. 517–547.

[GH16] N. Gigli and B.-X. Han. “Independence on p of weak upper gradients on RCD spaces”. In: *J. Funct. Anal.* 271.1 (2016), pp. 1–11.

[Gig18] N. Gigli. “Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below”. In: *Mem. Amer. Math. Soc.* 251.1196 (2018), pp. v+161.

[Giu84] E. Giusti. *Minimal surfaces and functions of bounded variation.* Vol. 80. Monographs in Mathematics. Birkhäuser Verlag, Basel, 1984, pp. xii+240.

[GMS15] N. Gigli, A. Mondino, and G. Savaré. "Convergence of pointed non-compact metric measure spaces and stability of Ricci curvature bounds and heat flows". In: *Proc. Lond. Math. Soc. (3)* 111.5 (2015), pp. 1071–1129.

[Gol18] S. Golo. "Some remarks on contact variations in the first Heisenberg group". In: *Ann. Acad. Sci. Fenn. Math.* 43.1 (2018), pp. 311–335.

[GP02] N. Garofalo and S. D. Pauls. "The Bernstein Problem in the Heisenberg Group". In: *arXiv Mathematics e-prints* (Sept. 2002).

[GR15] M. Galli and M. Ritoré. "Area-stationary and stable surfaces of class C^1 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$". In: *Adv. Math.* 285 (2015), pp. 737–765.

[Gro07] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces.* English. Modern Birkhäuser Classics. Based on the 1981 French original, With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates. Birkhäuser Boston, Inc., Boston, MA, 2007, pp. xx+585.

[Gro81a] M. Gromov. "Groups of polynomial growth and expanding maps". In: *Inst. Hautes Études Sci. Publ. Math.* 53 (1981), pp. 53–73.

[Gro81b] M. Gromov. *Structures métriques pour les variétés riemanniennes.* Vol. 1. Textes Mathématiques [Mathematical Texts]. Edited by J. Lafontaine and P. Pansu. CEDIC, Paris, 1981, pp. iv+152.

[GŚ98] W. Gangbo and A. Święch. "Optimal maps for the multidimensional Monge-Kantorovich problem". In: *Comm. Pure Appl. Math.* 51.1 (1998), pp. 23–45.

[Haj96] P. Hajłasz. "Sobolev spaces on an arbitrary metric space". In: *Potential Anal.* 5.4 (1996), pp. 403–415.

[Hal03] B. C. Hall. *Lie groups, Lie algebras, and representations.* Vol. 222. Graduate Texts in Mathematics. An elementary introduction. Springer-Verlag, New York, 2003, pp. xiv+351.

[Hal13] B. C. Hall. *Quantum theory for mathematicians.* Vol. 267. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+554.

[Har02] P. Hartman. *Ordinary differential equations.* Vol. 38. Classics in Applied Mathematics. Corrected reprint of the second (1982) edition [Birkhäuser, Boston, MA; MR0658490 (83e:34002)], With a foreword by Peter Bates. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002, pp. xx+612.

[Hél02] F. Hélein. *Harmonic maps, conservation laws and moving frames.* Second. Vol. 150. Cambridge Tracts in Mathematics. Translated from the 1996 French original, With a foreword by James Eells. Cambridge University Press, Cambridge, 2002, pp. xxvi+264.

[Hél90] F. Hélein. "Régularité des applications faiblement harmoniques entre une surface et une sphère". In: *C. R. Acad. Sci. Paris Sér. I Math.* 311.9 (1990), pp. 519–524.

[HK00] P. Hajłasz and P. Koskela. "Sobolev met Poincaré". In: *Mem. Amer. Math. Soc.* 145.688 (2000), pp. x+101.

[HK14] S. Hencl and P. Koskela. *Lectures on mappings of finite distortion.* Vol. 2096. Lecture Notes in Mathematics. Springer, Cham, 2014, pp. xii+176.

[HK95] P. Hajłasz and P. Koskela. “Sobolev meets Poincaré”. In: *C. R. Acad. Sci. Paris Sér. I Math.* 320.10 (1995), pp. 1211–1215.

[HKL86] R. Hardt, D. Kinderlehrer, and F.-H. Lin. “Existence and partial regularity of static liquid crystal configurations”. In: *Comm. Math. Phys.* 105.4 (1986), pp. 547–570.

[HKM06] J. Heinonen, T. Kilpeläinen, and O. Martio. *Nonlinear potential theory of degenerate elliptic equations.* Unabridged republication of the 1993 original. Dover Publications, Inc., Mineola, NY, 2006, pp. xii+404.

[HKST15] J. Heinonen, P. Koskela, N. Shanmugalingam, and J. T. Tyson. *Sobolev spaces on metric measure spaces.* Vol. 27. New Mathematical Monographs. An approach based on upper gradients. Cambridge University Press, Cambridge, 2015, pp. xii+434.

[HL87] R. Hardt and F.-H. Lin. “Mappings minimizing the L^p norm of the gradient”. In: *Comm. Pure Appl. Math.* 40.5 (1987), pp. 555–588.

[HLW98] R. Hardt, F. Lin, and C. Wang. “The p-energy minimality of $x/|x|$”. In: *Comm. Anal. Geom.* 6.1 (1998), pp. 141–152.

[Hon01] M.-C. Hong. “On the minimality of the p-harmonic map $\frac{x}{|x|} : B^n \to S^{n-1}$”. In: *Calc. Var. Partial Differential Equations* 13.4 (2001), pp. 459–468.

[Hop50] E. Hopf. “On S. Bernstein’s theorem on surfaces $z(x, y)$ of nonpositive curvature”. In: *Proc. Amer. Math. Soc.* 1 (1950), pp. 80–85.

[HRR10] A. Hurtado, M. Ritoré, and C. Rosales. “The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$”. In: *Adv. Math.* 224.2 (2010), pp. 561–600.

[Jer86] D. Jerison. “The Poincaré inequality for vector fields satisfying Hörmander’s condition”. In: *Duke Math. J.* 53.2 (1986), pp. 503–523.

[Jon90] P. W. Jones. “Rectifiable sets and the traveling salesman problem”. In: *Inventiones Mathematicae* 102.1 (1990), pp. 1–15.

[Jui09] N. Juillet. “Geometric inequalities and generalized Ricci bounds in the Heisenberg group”. In: *Int. Math. Res. Not. IMRN* 13 (2009), pp. 2347–2373.

[Kes06] S. Kesavan. *Symmetrization & applications.* Vol. 3. Series in Analysis. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006, pp. xii+148.

[Kil94] T. Kilpeläinen. “Weighted Sobolev spaces and capacity”. In: *Ann. Acad. Sci. Fenn. Ser. A I Math.* 19.1 (1994), pp. 95–113.

[KP15] Y.-H. Kim and B. Pass. “Multi-marginal optimal transport on Riemannian manifolds”. In: *Amer. J. Math.* 137.4 (2015), pp. 1045–1060.

[Kra25] E. Krahn. “Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises”. In: *Mat. Ann.* 94 (1925), pp. 97–100.

[KS80] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications.* Vol. 88. Pure and Applied Mathematics. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980, pp. xiv+313.

[Lin87] F.-H. Lin. “Une remarque sur l’application $x/|x|$”. In: *CR Acad. Sci. Paris* 305.1 (1987), pp. 529–531.

[Lin99] F.-H. Lin. “Gradient estimates and blow-up analysis for stationary harmonic maps”. In: *Ann. of Math. (2)* 149.3 (1999), pp. 785–829.

[Lio79] P. L. Lions. “Quelques remarques sur la symétrisation de Schwarz”. In: *Nonlinear partial differential equations and their applications. Collège de France Seminar* 1 (1978/1979), pp. 308–319.

[LMP06] V. Latvala, N. Marola, and M. Pere. “Harnack’s inequality for a nonlinear eigenvalue problem on metric spaces”. In: *J. Math. Anal. Appl.* 321.2 (2006), pp. 793–810.

[Luc88] S. Luckhaus. “Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold”. In: *Indiana Univ. Math. J.* 37.2 (1988), pp. 349–367.

[LV09] J. Lott and C. Villani. “Ricci curvature for metric-measure spaces via optimal transport”. In: *Ann. of Math. (2)* 169.3 (2009), pp. 903–991.

[LY86] P. Li and S.-T. Yau. “On the parabolic kernel of the Schrödinger operator”. In: *Acta Math.* 156.3-4 (1986), pp. 153–201.

[Mag12] F. Maggi. *Sets of finite perimeter and geometric variational problems.* Vol. 135. Cambridge Studies in Advanced Mathematics. An introduction to geometric measure theory. Cambridge University Press, Cambridge, 2012, pp. xx+454.

[Mat95] P. Mattila. *Geometry of sets and measures in Euclidean spaces.* Vol. 44. Cambridge Studies in Advanced Mathematics. Fractals and rectifiability. Cambridge University Press, Cambridge, 1995, pp. xii+343.

[McC01] R. J. McCann. “Polar factorization of maps on Riemannian manifolds”. In: *Geom. Funct. Anal.* 11.3 (2001), pp. 589–608.

[McC94] R. J. McCann. *A convexity theory for interacting gases and equilibrium crystals.* Thesis (Ph.D.)–Princeton University. ProQuest LLC, Ann Arbor, MI, 1994, p. 163.

[Mi18] M. Miśkiewicz. “Discrete Reifenberg-type theorem”. In: *Ann. Acad. Sci. Fenn. Math.* 43.1 (2018), pp. 3–19.

[Mic50] E. J. Mickle. “A remark on a theorem of Serge Bernstein”. In: *Proc. Amer. Math. Soc.* 1 (1950), pp. 86–89.

[Mir03] M. Miranda Jr. “Functions of bounded variation on “good” metric spaces”. In: *J. Math. Pures Appl. (9)* 82.8 (2003), pp. 975–1004.

[MN19] A. Mondino and A. Naber. “Structure theory of metric measure spaces with lower Ricci curvature bounds”. In: *J. Eur. Math. Soc. (JEMS)* 21.6 (2019), pp. 1809–1854.

[Mos05] R. Moser. *Partial regularity for harmonic maps and related problems.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005, pp. viii+184.

[MS19] A. Mondino and D. Semola. “Polya-Szego inequality and Dirichlet p-spectral gap for non-smooth spaces with Ricci curvature bounded below”. In: *Journal de Mathématiques Pures et Appliquées* (2019).

[MSV08] R. Monti, F. Serra Cassano, and D. Vittone. “A negative answer to the Bernstein problem for intrinsic graphs in the Heisenberg group”. In: *Boll. Unione Mat. Ital. (9)* 1.3 (2008), pp. 709–727.

[MV21] A. Mondino and M. Vedovato. “A Talenti-type comparison theorem for $\mathrm{RCD}(K,N)$ spaces and applications”. In: *Calc. Var. Partial Differential Equations* 60.4 (2021), Paper No. 157, 43.

[Nas54] J. Nash. “C^1 isometric imbeddings”. In: *Ann. of Math. (2)* 60 (1954), pp. 383–396.

[Nit57] J. C. C. Nitsche. “Elementary proof of Bernstein’s theorem on minimal surfaces”. In: *Ann. of Math. (2)* 66 (1957), pp. 543–544.

[NS19] S. Nicolussi Golo and F. Serra Cassano. “The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group”. In: *Calc. Var. Partial Differential Equations* 58.4 (2019), Art. 141, 28.

[NSW85] A. Nagel, E. M. Stein, and S. Wainger. “Balls and metrics defined by vector fields. I. Basic properties”. In: *Acta Math.* 155.1-2 (1985), pp. 103–147.

[NV17] A. Naber and D. Valtorta. “Rectifiable-Reifenberg and the regularity of stationary and minimizing harmonic maps”. In: *Ann. of Math. (2)* 185.1 (2017), pp. 131–227.

[NV18] A. Naber and D. Valtorta. “Stratification for the singular set of approximate harmonic maps”. In: *Math. Z.* 290.3-4 (2018), pp. 1415–1455.

[NVV19] A. Naber, D. Valtorta, and G. Veronelli. “Quantitative regularity for p-harmonic maps”. In: *Comm. Anal. Geom.* 27.1 (2019), pp. 111–159.

[OV00] F. Otto and C. Villani. “Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality”. In: *J. Funct. Anal.* 173.2 (2000), pp. 361–400.

[Paj02] H. Pajot. *Analytic capacity, rectifiability, Menger curvature and the Cauchy integral.* Vol. 1799. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2002, pp. xii+119.

[Pay62] L. E. Payne. “Some isoperimetric inequalities in the torsion problem for multiply connected regions”. In: *Studies in mathematical analysis and related topics.* Stanford Univ. Press, Stanford, Calif., 1962, pp. 270–280.

[Pet16] P. Petersen. *Riemannian geometry.* Third. Vol. 171. Graduate Texts in Mathematics. Springer, Cham, 2016, pp. xviii+499.

[PPV21] B. Pass, A. Pinamonti, and M. Vedovato. “Multi-marginal optimal transport on the Heisenberg group”. In: *Methods Appl. Anal.* 28.1 (2021), pp. 61–75.

[PS51] G. Pólya and G. Szegő. *Isoperimetric Inequalities in Mathematical Physics.* Vol. 27. Princeton University Press, 1951.

[PW50] G. Polya and A. Weinstein. “On the torsional rigidity of multiply connected cross-sections”. In: *Ann. of Math. (2)* 52 (1950), pp. 154–163.

[Raj12] T. Rajala. “Local Poincaré inequalities from stable curvature conditions on metric spaces”. In: *Calc. Var. Partial Differential Equations* 44.3-4 (2012), pp. 477–494.

[Rit09] M. Ritoré. “Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity”. In: *Calc. Var. Partial Differential Equations* 34.2 (2009), pp. 179–192.

[Riv95] T. Rivière. “Everywhere discontinuous harmonic maps into spheres”. In: *Acta Math.* 175.2 (1995), pp. 197–226.

[RS05] M.-K. von Renesse and K.-T. Sturm. "Transport inequalities, gradient estimates, entropy, and Ricci curvature". In: *Comm. Pure Appl. Math.* 58.7 (2005), pp. 923–940.

[San15] F. Santambrogio. *Optimal transport for applied mathematicians.* Vol. 87. Progress in Nonlinear Differential Equations and their Applications. Calculus of variations, PDEs, and modeling. BirkhäuserSpringer, Cham, 2015, pp. xxvii+353.

[Sha00] N. Shanmugalingam. "Newtonian spaces: an extension of Sobolev spaces to metric measure spaces". In: *Rev. Mat. Iberoamericana* 16.2 (2000), pp. 243–279.

[Sim96] L. Simon. *Theorems on regularity and singularity of energy minimizing maps.* Lectures in Mathematics ETH Zürich. Based on lecture notes by Norbert Hungerbühler. Birkhäuser Verlag, Basel, 1996, pp. viii+152.

[Ste93] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* Vol. 43. Princeton Mathematical Series. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993, pp. xiv+695.

[Stu06a] K.-T. Sturm. "On the geometry of metric measure spaces. I". In: *Acta Math.* 196.1 (2006), pp. 65–131.

[Stu06b] K.-T. Sturm. "On the geometry of metric measure spaces. II". In: *Acta Math.* 196.1 (2006), pp. 133–177.

[SU82] R. Schoen and K. Uhlenbeck. "A regularity theory for harmonic maps". In: *J. Differential Geom.* 17.2 (1982), pp. 307–335.

[SV20] F. Serra Cassano and M. Vedovato. "The Bernstein problem in Heisenberg groups". In: *Matematiche (Catania)* 75.1 (2020), pp. 377–403.

[SV69] J. Serrin and D. E. Varberg. "A general chain rule for derivatives and the change of variables formula for the Lebesgue integral". In: *Amer. Math. Monthly* 76 (1969), pp. 514–520.

[Sze30] G. Szegö. "Über einige Extremalaufgaben der Potentialtheorie". In: *Math. Z.* 31.1 (1930), pp. 583–593.

[Tal76] G. Talenti. "Elliptic equations and rearrangements". In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 3.4 (1976), pp. 697–718.

[Tor95] T. Toro. "Geometric conditions and existence of bi-Lipschitz parameterizations". In: *Duke Math. J.* 77.1 (1995), pp. 193–227.

[TW95] T. Toro and C. Wang. "Compactness properties of weakly p-harmonic maps into homogeneous spaces". In: *Indiana Univ. Math. J.* 44.1 (1995), pp. 87–113.

[Uhl77] K. Uhlenbeck. "Regularity for a class of non-linear elliptic systems". In: *Acta Math.* 138.3-4 (1977), pp. 219–240.

[Ura68] N. N. Ural′ceva. "Degenerate quasilinear elliptic systems". In: *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 7 (1968), pp. 184–222.

[Ved21] M. Vedovato. "Quantitative regularity for p-minimizing maps through a Reifenberg theorem". In: *J. Geom. Anal.* 31.8 (2021), pp. 8271–8317.

[Vil03] C. Villani. *Topics in optimal transportation*. Vol. 58. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2003, pp. xvi+370.

[Vil09] C. Villani. *Optimal transport*. Vol. 338. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Old and new. Springer-Verlag, Berlin, 2009, pp. xxii+973.

[Wan98] C. Wang. "Minimality and perturbation of singularities for certain p-harmonic maps". In: *Indiana Univ. Math. J.* 47.2 (1998), pp. 725–740.

[Wei62] H. Weinberger. "Symmetrization in uniformly elliptic problems". In: *Studies in mathematical analysis and related topics* (1962), pp. 424–428.

[You21] R. Young. *Area-minimizing ruled graphs and the Bernstein problem in the Heisenberg group*. 2021. arXiv: 2105.08890 [math.CA].