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Monotonicity Formulas in Nonlinear Potential Theory and their geometric applications

Author:
Luca BENATTI

Supervisor:
Prof. Lorenzo MAZZIERI

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Abstract

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In the setting of Riemannian manifolds with nonnegative Ricci curvature, we provide geometric inequalities as consequences of the Monotonicity Formulas holding along the flow of the level sets of the p-capacitary potential. The work is divided into three parts.

(i) In the first part, we describe the asymptotic behaviour of the p-capactitary potential in a natural class of Riemannian manifolds.

(ii) The second part is devoted to the proof of our Monotonicity-Rigidity Theorems.

(iii) In the last part, we apply the Monotonicity Theorems to obtain geometric inequalities, focusing on the Extended Minkowski Inequality.

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Introduction

Geometric Inequalities have always played a central role in mathematics. Their relevance is known to go far beyond the realm of mere geometric applications. For example, they have a clear influence on the study of shape optimisation problems. They are at the basis of some of the most fundamental functional inequalities, which are in turn frequently used in the analysis of partial differential equations. Geometric Inequalities also find a considerable place in mathematical physics and notably in the study of geometric aspects of General Relativity. To mention one, the Riemannian Penrose Inequality offers a most relevant example. They boast a long tradition. Ancient Greeks knew, for instance, the *Isoperimetric Inequality*, at least in its planar version, which reads as

$$\frac{|\mathbb{S}^{n-1}|^n}{|\mathbb{B}^n|^{n-1}} \leq \frac{|\partial\Omega|^n}{|\Omega|^{n-1}}$$

for every $\Omega \subseteq \mathbb{R}^n$ open bounded and convex. Despite this, we had to wait until the XIX$^{\text{th}}$ century for the mathematically rigorous proof, when Weierstrass completed the earlier argument by Steiner. In the first half of the XX$^{\text{th}}$ century, the problem received many contributions among others by Hurwitz [Hur02; Hur32], Caratheodory and Study [CS09] and Blaschke [Bla56]. Among the countless natural generalisations of this inequality produced in subsequent years, we emphasise the contributions of Minkowski [Min03], Aleksandrov [Ale37; Ale38] and Fenchel [Fen29]. In his work, Minkowski computed the volume expansion of the set $\{x \in \mathbb{R}^n \mid \mathrm{d}(x, \Omega) \leq t\}$ as

$$|\{x \in \mathbb{R}^n \mid \mathrm{d}(x, \Omega) \leq t\}| = \sum_{k=0}^{n} V_{n-k}(\Omega) t^k.$$

The coefficients $V_{n-k}(\Omega)$ of the above polynomial are also known in literature as *quermassintegrals*. It is easy to see that $V_n(\Omega)$ corresponds to the volume of Ω, while $V_{n-1}(\Omega)$ is its perimeter. In general, the quantity $V_{n-k}(\Omega)$ is related to the integral of a symmetric function of the principal curvatures of the boundary. Indeed, denoting by $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$ the principal curvatures of Ω, one has

$$V_{n-k}(\Omega) = \frac{1}{n-k} \int_{\partial\Omega} \mathfrak{S}_k(\kappa)\, \mathrm{d}\sigma \tag{1}$$

for all $k \geq 0$, where $\mathfrak{S}_k$ is the k-th elementary symmetric function, namely

$$\mathfrak{S}_k(t_1, \dots, t_{n-1}) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} t_{i_1} t_{i_2} \dots t_{i_k}.$$

A few years later, Aleksandrov and Fenchel proved that quermassintegrals are involved in a family of inequalities. Known as Aleksandrov-Fenchel Inequalities, they read as

$$V_{n-k}(\Omega) V_{n-k-2}(\Omega) \leq V_{n-k-1}(\Omega)^2.$$

A simple induction argument (see [CW11]) shows that this family of inequalities imply

$$\left(V_{n-k-1}(\Omega)\right)^{\frac{1}{n-k}} \leq \mathrm{C}_1(n,k) \left(V_{n-k}(\Omega)\right)^{\frac{1}{n-k-1}}. \tag{2}$$

One can recognise the Isoperimetric Inequality among them, choosing $k = 0$. Furthermore, in view of (1), (2) can be rewritten as

$$\left(\int_{\partial\Omega} \mathfrak{S}_{k-1}(\kappa) \, \mathrm{d}\sigma \right)^{\frac{1}{n-k}} \leq \mathrm{C}_2(n,k) \left(\int_{\partial\Omega} \mathfrak{S}_k(\kappa) \, \mathrm{d}\sigma \right)^{\frac{1}{n-k-1}}, \tag{3}$$

for all $k \geq 1$. In the second half of the XX$^{\text{th}}$ century, also the range of validity of the Isoperimetric Inequality has been improved to cover a more general class of subsets in various settings. De Giorgi [DeG58] proved that it holds for any subset with finite perimeter in the sense of Caccioppoli, giving a completely satisfying result in the Euclidean space. In the Riemannian setting, results were not slow in coming. Among others, we mention the version on compact Riemannian manifold with positive Ricci curvature obtained by Gromov [Gro80] after the work of Lévy [Lév22], and on noncompact Cartan-Hadamard manifolds by Kleiner [Kle92] and Croke [Cro84]. Recently, Agostiniani, Fogagnolo and Mazzieri [AFM20] proved it on 3-dimensional Riemannian manifolds with nonnegative Ricci curvature and Euclidean Volume Growth. Then, Brendle [Bre22] extended it with a different approach to all dimensions, showing that

$$\frac{\left|\mathbb{S}^{n-1}\right|^n}{\left|\mathbb{B}^n\right|^{n-1}} \mathrm{AVR}(g) \leq \frac{\left|\partial\Omega\right|^n}{\left|\Omega\right|^{n-1}} \tag{4}$$

for all bounded subset Ω with finite perimeter in a Riemannian Manifold (M, g), where $\mathrm{AVR}(g)$ is the Asymptotic Volume Ratio of (M, g) (see Notation and main settings below for the definition). It is also worth mentioning that Balogh and Kristàly [BK22] proved the validity of the Isoperimetric Inequality also in the nonsmooth setting. The stability of the Isoperimetric Inequality has also been investigated and produced *quantitative* version of this inequality, namely inequalities where a suitable distance of Ω from a ball of the same volume is controlled by the difference of its perimeter from the perimeter of the same ball (see [Fug89; HHW91; Hal92; FMP08; CL12]).

At this point, one may wonder whether the same program can be carried out for the other Aleksandrov-Fenchel Inequalities. The first in order, and the leading goal of this work, is (2) for $k = 1$. In this case $\mathfrak{S}_1(\kappa)$ is the total mean curvature H of $\partial\Omega$ and $\mathfrak{S}_0(\kappa) = 1$. Then, (3) reads as

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma. \tag{5}$$

Named after Minkowski, who firstly proved it in [Min03], it has attracted the attention of the mathematical community only recently. Castillon [Cas10] and Chang and Wang [CCW16; CW11] successfully employed Optimal Transport techniques to remove the convexity assumption in favour of milder constraints. A different approach recently used in this kind of problem is based on geometric flows. The Minkowski Inequality has a deep relation with the Inverse Mean Curvature Flow, which is a geometric flow ruled by

$$\frac{\partial \Psi_t}{\partial t}(q) = \frac{\nu_t(q)}{\mathrm{H}_t(q)} \qquad \text{for every } t \in [0, T) \text{ and } q \in \Sigma, \tag{6}$$

where $\Psi_t : \Sigma \to \mathbb{R}^n$ is a family of diffeomorphisms, ν_t and H_t are the outward pointing unit normal vector and the mean curvature of $\Sigma_t = \Psi_t(\Sigma)$ respectively. Observe that (6) makes sense as long as $\mathrm{H}_t > 0$. By direct computations, the quantity

$$\mathcal{Q}(t) = |\Sigma_t|^{-\frac{n-2}{n-1}} \int_{\Sigma_t} \mathrm{H}_t \,\mathrm{d}\sigma \tag{7}$$

is monotone nonincreasing along this flow. If the flow is immortal and converges smoothly enough to expanding spheres, one gets (5). Gerhardt [Ger90] and Urbas [Urb90] asserted that all star-shaped subsets with strictly mean-convex smooth boundary meet these requirements. A complete derivation of (5) for this larger class of subsets, together with other Aleksandrov-Fenchel inequalities, is contained in [GL09]. This approach has some topological obstructions. Indeed, the existence of an immortal flow obliges all the evolved hypersurfaces to be diffeomorphic to each other and since they must converge to spheres, the starting hypersurface must be topologically spherical. To avoid these constraints one can appeal to the weak formulation of the IMCF introduced by Huisken and Ilmanen in [HI01], which constituted the main step in their approach to prove the celebrated Riemannian Penrose Inequality. We briefly recall that the weak IMCF is a solution to the partial differential equation

$$\operatorname{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right) = |\mathrm{D}w| \tag{8}$$

in a precise nonstandard variational sense, obtained freezing the right-hand term (see Definition 2.4.1 below). This tool allows extending (5) to a larger class of subsets, which includes elements that are not topologically equivalent to spheres (see [FS14]). The IMCF has been successfully employed to obtain results in curved Riemannian manifolds as well in [LG16; Wei18; BHW16; McC17].

The new approach based on Nonlinear Potential Theory also falls in the category of geometric flows. The origin of this techniques is contained in two works by Agostiniani, Fogagnolo, Mazzieri and Pinamonti [AM20; FMP19; AFM22]. They considered for $p \in$

$(1, n)$, the solution to the boundary value problem

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } \mathbb{R}^n \smallsetminus \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{9}$$

and proved that the function $F : [1, +\infty) \to \mathbb{R}$ defined as

$$F_p(t) = t^{\frac{n-1}{n-p}} \int\limits_{\{u=1/t\}} |\mathrm{D}u|^p \, \mathrm{d}\sigma,$$

satisfies two *effective monotonicity inequalities*, which are

$$\lim_{t \to +\infty} F(t) \leq F(1) \qquad\qquad F'(1) \leq 0. \tag{10}$$

Coupling them with a well-known behaviour of the p-capacitary potential coming from [Col+15; KV86], they obtained an L^p-version of (5), that is

$$\widehat{\mathrm{C}}\mathrm{ap}(\partial\Omega)^{\frac{n-p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma, \tag{11}$$

where $\widehat{\mathrm{C}}\mathrm{ap}(\partial\Omega)$ denotes a normalised (setting $\widehat{\mathrm{C}}\mathrm{ap}(\mathbb{S}^{n-1}) = 1$) version of the p-capacity of $\partial\Omega$, that is denoted by $\mathrm{Cap}(\partial\Omega)$ (see Definition 1.3.1 below for the definition). Then the Minkowski-type inequality

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|} \right)^{\frac{n-2}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right| \mathrm{d}\sigma \tag{12}$$

follows from (11) letting $p \to 1^+$, taking into account that

$$\lim_{p \to 1^+} \widehat{\mathrm{C}}\mathrm{ap}(\partial\Omega) = \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}. \tag{13}$$

Here Ω^* denotes the strictly outward minimising hull of Ω, which is, briefly speaking, the maximal volume solution of the perimeter minimisation problem among all sets containing Ω (see Section 2.4.2 below for the precise definition).

This technique and the one using IMCF share the same idea of finding a monotone quantity holding along the evolution of the set and compare its starting value with the values assumed approaching infinity. However, the global monotonicity of the quantities provided via the Nonlinear Potential approach is not known, but only the monotonicity inequalities (10), anyway sufficient to prove (12), are ensured. On the contrary, here we gain the long time existence, without assuming any geometric constrain on the starting subset. Moreover, the flow is based on the level set of a more regular function then the weak solution of the IMCF would dictate (which is only merely Lipschitz).

It does not seem a false hope to reproduce the approach based on Nonlinear Potential Theory in the setting of complete noncompact Riemannian manifolds with nonnegative

Ricci curvature. Evidence to support our claim lies in the successful history of Linear Potential theoretic proof of the Willmore Inequality, firstly proved in [AM20] and then exported in [AFM20], to cover this class of manifolds. Moreover, the Minkowski Inequality

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \left(\frac{|\mathbb{S}^{n-1}|}{|\mathbb{B}^{n-1}|}\right)^{\frac{1}{n-1}} \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma \tag{14}$$

holds on Riemannian manifolds with nonnegative sectional curvature, being a consequence of the Michael-Simon Sobolev Inequality provided by Brendle in [Bre22]. This inequality has the merit of depending only on the perimeter of Ω and not on its strictly outward minimising hull. However, (14) is not sharp even in the flat Euclidean space (where $\mathrm{AVR}(g) = 1$), since $|\mathbb{S}^{n-1}|/|\mathbb{B}^{n-1}| > 1$.

One of the major difficulties in merely mimicking the lines pursued in the flat Euclidean case is to describe the asymptotic behaviour of the p-capacitary potential. For $p = 2$, the authors in [AFM20] carried it out in the great generality of complete Riemannian manifold with nonnegative Ricci curvature. However, their proof is built on [LTW97] or [CM97], who in turn used the representation formula and the Almgren frequency function respectively, neither of them being available for general p. There are two possibilities to avoid this problem. The first one consists in imposing some further constraints on the ambient manifold to ensure that the p-capacitary potential has a well-known behaviour at infinity. As we will see, this program can be completed in the particular but natural class of Asymptotically Conical Riemannian manifolds. On the other hand, one of the observations of our work is that the knowledge of the asymptotic behaviour can be avoided if one replaces the effective monotonicity inequalities (10) with the whole monotonicity of the function F_p.

The aim of this thesis is to give a well-rounded panoramic of problem (9) to better understand the approach via Nonlinear Potential Theory to the Minkowski Inequality and to export it to the Riemannian setting. In particular, we will show that

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma$$

holds for every Ω open bounded subset with smooth boundary in a complete Riemannian manifold with nonnegative Ricci curvature and Euclidean Volume Growth (that is $\mathrm{AVR}(g) > 0$). We will also characterise those sets that saturate the inequality proving the splitting of the manifold outside Ω into a truncated Riemannian cone.

In the following sections, we illustrate the content of each chapter, highlighting the difficulties and the main techniques used.

The p-capacitary potential and the weak IMCF

We take advantage of this work to collect some relevant facts from the classical Nonlinear Potential Theory. The reason for this is twofold. Firstly we aim at making this work as much as possible self-contained and make all the material consistent. Secondly, the theory is fragmented and developed at its very generality. This makes it frustrating to always ensure the conditions to apply the theory in particular cases. It is well-known that a solution to (9) exists if and only if the Riemannian manifold (M, g) is p-nonparabolic, that is if there exists a compact set of positive p-capacity. It turns out that this condition is ensured for all $1 < p < n$ if (M, g) is in either with nonnegative Ricci curvature and Euclidean Volume Growth or Asymptotically Conical with the Ricci tensor satisfying

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(\mathrm{d}(x,o)+1)^2}$$

for some $o \in M$, $\kappa \in \mathbb{R}$ and every $x \in M$ (see the Notation and main settings below for more details). Moreover, in these frameworks, a double side Li-Yau-type estimate and a Cheng-Yau-type estimate are provided, namely, there exists a positive constant $\mathrm{C} > 0$ such that

$$\mathrm{C}^{-1}\,\mathrm{d}(x,o)^{-\frac{n-p}{p-1}} \leq u_p(x) \leq \mathrm{C}\,\mathrm{d}(x,o)^{-\frac{n-p}{p-1}}, \qquad |\mathrm{D}\log u_p|(x) \leq \frac{\mathrm{C}}{\mathrm{d}(x,o)},$$

holds for every $x \in M$, for some fixed $o \in M$. Built on these inequalities, the main result obtained in this part concerns the asymptotic behaviour of the p-capacitary potential on Asymptotically Conical Riemannian manifolds. Based on the proof by [KV86], then exploited in [Col+15], we show that in a merely $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold

$$u_p = \mathrm{C}(p, \Omega, M)\rho^{-\frac{n-p}{p-1}}(1 + o(1)) \tag{15}$$

as $\rho \to +\infty$ characterising the constant in terms of the p-capacity of $\partial\Omega$ and the Asymptotic Volume Ratio of g, where ρ represents the radial coordinate on the asymptotic cone. This result extends the asymptotic analyses carried out in [AMO22; AMO21; HM20; MMT20] to the nonlinear setting, although without so refined estimates of the error terms. The identity (15) says that the level sets of the p-capacitary potential approximate the cross-section of the asymptotic cone, far away from Ω. Building on Schauder estimates, we also describe the behaviour of higher-order derivatives, requiring a faster rate of convergence for the metric and its derivatives. As aforementioned, this result was one of the two key ingredients in our first proof of the L^p-Minkowski Inequality. Indeed, coupled with the effective monotonicity inequalities (10), it directly gives (11) and in turn, letting $p \to 1^+$, (12) for Asymptotically Conical manifolds. This procedure has the disadvantage of being valid on a restricted class of manifolds, but it does not rely on the Isoperimetric Inequality, as the one based on the full monotonicity does (see the next section for further details).

The force secretly driving this approach to Minkowski Inequality dwells in the relation between the p-capacitary potential and the (weak) IMCF. In fact, a well-known result proved in the flat Euclidean case by Moser [Mos07; Mos08], subsequently extended to Riemannian manifolds by Kotschwar and Ni [KN09] and that inspire the results in [MRS19], says that if u_p is the p-capacitary potential associated with Ω the family of functions $w_p = -(p-1)\log u_p$ converges locally uniformly to a proper solution w_1 of (8) as $p \to 1^+$. This approach also provides a proof of the existence of the weak IMCF which is alternative to that of Huisken and Ilmanen [HI01]. Exactly as we did for the p-capaciatary potential, we spend some time rephrasing these results in our main settings. Indeed, while the results in [MRS19] aim at establishing the existence of the weak IMCF in the greatest possible generality, assuming a very weak curvature bound, our purpose is to derive more detailed information taking advantage of the specific features of our settings. We then employ the same proof leading to (15) to obtain the asymptotic behaviour of the IMCF, that is

$$w_1 = \mathrm{C}(\Omega, M) + (n-1)\log(\rho)(1+o(1)),$$

as $\rho \to +\infty$. This result actually extends the one in [HI01, Blowdown Lemma 7.1] from Asymptotically Flat to Asymptotically Conical Riemannian manifolds. Differently from Huisken and Ilmanen, here we are able to characterise the constant $\mathrm{C}(\Omega, M)$ in terms of $|\partial\Omega^*|$ and $\mathrm{AVR}(g)$. This slight improvement is a further evidence of the deep connection between the p-capacitary potential and the weak IMCF. Indeed, the constant $\mathrm{C}(\Omega, M)$ is the limit of $-(p-1)\log \mathrm{C}(p, \Omega, M)$ as $p \to 1^+$, hence, even if *a priori* the convergence of w_1 to w_p as $p \to 1^+$ is only locally uniform, this relation is preserved in the asymptotic behaviours.

These results apply in particular to Asymptotically Locally Euclidean *gravitational instantons*, that are noncompact hyperkhäler Ricci Flat 4-dimensional manifolds that have a role in Euclidean Quantum Gravity Theory, Gauge Theory and String Theory (see [HE73; EH79; Kro89a; Kro89b; Min09; Min10; Min11]). It is not difficult to see that these results can be applied even if (M, g) is not complete. The method is indeed blind to whatever happens inside Ω. It permits to include Asymptotically Flat Riemannian manifolds with compact boundary, relevant in General Relativity.

Full Monotonicity-Rigidity Theorems and their geometric consequences

The second result we propose aims to fully extend the monotonicity formulas discovered in [AM20; AFM20; Col12; CM14a] to the nonlinear case in general Riemannian manifold. Given for $1 < p < n$ a solution to the problem

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } M \smallsetminus \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } \mathrm{d}(o, x) \to +\infty, \end{cases} \tag{16}$$

we build the family of functions $F_p^\beta : [1, +\infty) \to \mathbb{R}$ defined as

$$F_p^\beta(t) = t^{\beta\frac{(n-1)(p-1)}{(n-p)}} \int\limits_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)} \, \mathrm{d}\sigma. \tag{17}$$

Differently from [FMP19; AFM20], we do not just prove the effective monotonicity inequalities but that each F_p^β admits a monotone and convex $\mathscr{C}^1$-representative for all $\beta > (n-p)/[(n-1)(p-1)]$. If $(F_p^\beta)'(t_0) = 0$ for some $t_0 \in [1, +\infty)$ we show that $\{u \leq 1/t_0\}$ is isometric to a truncated cone. We remark that in principle formula (17) does not grant a well-posed definition for free. Indeed, the mild $\mathscr{C}^{1,\beta}$ regularity of the solution to (16) is not enough to apply Sard's Theorem to control the critical set of u. In addition, nothing ensures that the monotonicity survives after a jump that might be of full measure. However, we managed to solve these problems following the approach in [GV21], where the authors were forced to face several technical problems due to the lack of regularity which is typical in the nonsmooth setting. We compute the first derivative of F_p^β using the full strength of the coarea formula. As it concerns the second derivative, we make a detour to the route traced in [GV21] digging up the cut-off argument in [AM20; AFM20] to obtain the higher regularity of our quantities.

Exploiting the full monotonicity, we obtain the main result of the thesis, that is the Extended Minkowski Inequality

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma, \tag{18}$$

which holds for any open bounded subset Ω with smooth boundary in a complete Riemannian manifold (M, g) with nonnegative Ricci curvature and Euclidean Volume Growth (that is $\mathrm{AVR}(g) > 0$). The above inequality is sharp and provides the optimal Minkowski Inequality for outward minimising subsets, that is when $|\partial\Omega| = |\partial\Omega^*|$, recovering the results originally sketched in [Hui]. We want to mention that, in (12) we characterise the constant in terms of the geometry of the ambient manifold.

Differently from [AFM20; FMP19], the proof is based on a contradiction argument. This new idea allows extending its validity from the class of Asymptotically Conical manifolds to the setting of Riemannian manifolds with nonnegative Ricci curvature and Euclidean Volume Growth, where we cannot ensure the pointwise asymptotic behaviour of the (derivative) of the p-capacitary potential. Indeed, the identity for the derivatives of the two sides of (15) implies that large level sets of the solution u_p to (16) have the same topology, which is in contrast with the existence of manifolds with nonnegative Ricci curvature and infinite topology (see [Men00; Khu+22]).

Heuristically, the proof can be implemented for the smooth IMCF flow as follows. By contradiction, assume that at some open subset $\Omega \subseteq \mathbb{R}^n$ with smooth boundary, the inequality

$$(n-1)\left|\mathbb{S}^{n-1}\right|^{\frac{1}{n-1}}\theta^{\frac{1}{n-1}} \geq |\partial\Omega|^{-\frac{n-2}{n-1}}\int_{\partial\Omega}\mathrm{H}\,\mathrm{d}\sigma$$

holds for some $\theta < 1$. Suppose that the solution to (6) starting at $\partial\Omega$ exists for all time and denote by Ω_t the set enclosed by $\Sigma_t = \Psi_t(\partial\Omega)$. Since $\mathcal{Q}$ in (7) decreases, the above inequality can only worsen along the flow. Thus, for every $t > 0$ it holds

$$(n-1)\left|\mathbb{S}^{n-1}\right|^{\frac{1}{n-1}}\theta^{\frac{1}{n-1}} \geq |\partial\Omega_t|^{-\frac{n-2}{n-1}}\int_{\partial\Omega_t}\mathrm{H}_t\,\mathrm{d}\sigma_t.$$

Applying the Hölder's Inequality to the right hand side, we obtain that

$$(n-1)\left|\mathbb{S}^{n-1}\right|^{\frac{1}{n-1}}\theta^{\frac{1}{n-1}}\int_{\partial\Omega_t}\frac{1}{\mathrm{H}_t}\,\mathrm{d}\sigma_t \geq |\partial\Omega_t|^{\frac{n}{n-1}}.$$

Integrating both sides for $t \in [0, T]$, using the coarea formula and evolution equations in [HP99], we end with

$$(n-1)\left|\mathbb{S}^{n-1}\right|^{\frac{1}{n-1}}\theta^{\frac{1}{n-1}}|\Omega_T| \geq \frac{n-1}{n}\left(|\partial\Omega_T|^{\frac{n}{n-1}} - |\Omega|^{\frac{n}{n-1}}\right)$$

which contradicts the Isoperimetric Inequality for T very large. This argument strongly relies on the long-time existence of IMCF, which is not ensured in general. One may try to avoid this problem by proving the monotonicity of the function $\mathcal{Q}$ in (7) along the flow of the level set of the weak IMCF. It is well-known that a solution to (8) can jump discontinuously across a region of positive volume. Therefore, the main question is under which conditions the monotonicity survives after these jumps. On the other hand, the L^p-Minkowski Inequality can be obtained with a similar argument considering the full monotonicity of the function F_β^p (for $\beta = 1/(p-1)$) and replacing the Isoperimetric Inequality with the sharp Iso-p-capacitary Inequality

$$\frac{\mathrm{Cap}_p(\mathbb{S}^{n-1})^n}{|\mathbb{B}^n|^{n-p}}\mathrm{AVR}(g)^p \leq \frac{\mathrm{Cap}_p(\partial K)^n}{|K|^{n-p}},$$

which follows from (4) using a classical argument inspired by [Jau12]. In conclusion, (18) follows as in [FMP19; AFM22] letting $p \to 1^+$ in the L^p-Minkowski Inequality and exploiting (13).

It is easy to see that cross-sections of Riemannian cones saturate inequality (18). Actually, the Extended Minkowski Inequality is explicitly seen to be sharp in $\mathscr{C}^0$-Asymptotically Conical Riemannian manifolds where

$$\inf\left\{ |\partial\Omega^*|^{-\frac{n-2}{n-1}} \int\limits_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma \;\middle|\; \Omega \subset M, \text{ with } \partial\Omega \text{ smooth} \right\} = \left(\left|\mathbb{S}^{n-1}\right|\mathrm{AVR}(g)\right)^{\frac{1}{n-1}}.$$

The main problem is characterising those sets that saturate inequality (18). The rigidity of the L^p-Minkowski Inequalities is a consequence of the rigidity statement of the function F_p^β in (17) that in turn follows from the vanishing of the nonnegative divergence of a vector field. This quantity degenerates as $p \to 1^+$ and its limit behaviour is far from being understood. This forces us to discuss the rigidity in a separate argument based on the IMCF. Evolving a strictly outward minimising subset with strictly-mean convex smooth boundary by IMCF, we obtain a foliation that remains totally umbilic and with vanishing normal component of the Ricci tensor for a while. The Codazzi-Mainardi equation implies that a totally umbilic surface with vanishing normal component of Ricci tensor in a nonnegative Ricci curvature ambient space is constant mean curvature. Coupling this with an extension of the Bishop-Gromov Theorem we are able to conclude that the equality is achieved in (18) at Ω only if $(M \smallsetminus \Omega, g)$ is isometric to a truncated cone on $\partial\Omega$.

We also provide a pinching condition theorem for the mean curvature of $\partial\Omega$ or the normal derivative of the function u_p at $\partial\Omega$ that forces Ω to be a Euclidean ball inside $\mathbb{R}^n$ extending the result in [BMM19] to the nonlinear setting in Riemannian manifolds. Such a program is carried on exploiting the Monotonicity-Rigidity theorem of the function

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u|.$$

The proof of this last result encounters the same critical issues as the proof of the L^p-Minkowski Inequality, on its way to be exported from the Asymptotically Conical setting to manifolds with nonnegative Ricci curvature. The contradiction argument above is quite flexible to deal with this case as well.

Further developments

The results collected here have given rise to collateral questions and suggested further problems that we did not face, but we intend to approach in future works.

Concerning the first part, we proved the asymptotic behaviour of the p-capacitary potential. In particular, we proved that given a solution u to (16) associated to an open bounded subset Ω with smooth boundary of an Asymptotically Conical Manifold (M, g) with asymptotically nonnegative Ricci curvature

$$u_p = \mathrm{C}(p, \Omega, M)\rho^{-\frac{n-p}{p-1}}(1 + o(1))$$

as $\rho \to +\infty$. One may aim at studying the error committed replacing the function u with $\mathrm{C}\,\rho^{-(n-p)/(p-1)}$. For the harmonic potential, on Asymptotically Flat Riemannian manifolds with order τ, it is known that the error is of order $\rho^{-(n-2)-\tau}$ approaching infinity (we address the reader to [AMO22; AMO21; HM20; MMT20] for the definition and the result). However, this improvement is based on the complete characterisation of the harmonic functions on $\mathbb{R}^n$ (that are known since the Laplacian operator is linear) and the representation formula. These refined asymptotics seems not available in literature for $p \neq 2$, not even in the flat Euclidean Space. Some insight in this direction were given in [Chr90], where the author proved that on Asymptotically Flat Riemannian manifolds with order τ, if the error is of order $\rho^{-(n-p)/(p-1)-\varepsilon}$ for some $\varepsilon < \tau$, then it is of order $\rho^{-(n-p)/(p-1)-\beta}$ for all $\beta < \min\{\tau, 1\}$. However, this result requires knowing *a priori* some estimates on the error which are not granted for free. These refined asymptotics would lead to a proof of the Riemannian Penrose Inequality as sketched at the end of [AMO21]. Moreover, it would give some ideas on how to define a mass-type invariant *à la* Biquard-Hein [BH19] on Asymptotically Locally Euclidean Riemannian manifolds.

Regarding the Monotonicity-Rigidity theorem, we mentioned that our proof inherits some techniques used in [GV21]. In this work, the authors proved the Monotonicity Formulas, holding along the level set flow of the harmonic potential, in the setting of RCD spaces. Moreover, in [Vio21], the author obtains a Willmore-type estimate in RCD employing the knowledge gained in [GV21] and the techniques in [AFM20]. Hence, also our results are likely to be exported to this framework. The main issue is to deal with the regularity of p-harmonic functions that is still unknown in this context, except for $p = 2$. This would lead to the L^p-Minkowski Inequalities thanks to the Iso-p-capacitary inequality, which can be obtained by coupling the recent Isoperimetric Inequality in [BK22] and the Polya-Szego Inequality in [NV21]. The problem of sending $p \to 1^+$ to obtain the Extended Minkowski Inequality seems to hide further challenges as one has to define the strictly outward minimising hull of a set and ensure all required properties to prove (13).

The Extended Minkowski Inequality also requires some additional work. Firstly, it is still open the question whether the strictly outward minimising hull can be replaced with the subset Ω in (18), at least when $\partial\Omega$ is strictly mean-convex. We recall that a lower bound of the total mean curvature in terms of $|\partial\Omega|$ actually holds in Riemannian manifolds with nonnegative sectional curvature thanks to (14), if one accepts to lessen the dimensional constant. However, the problem with the sharp constant is still open even in the flat Euclidean space. An attempt in this direction was made by Trudinger [Tru94] but the proof has been debunked in [Gua+10].

To conclude, both the Isoperimetric Inequality and the Minkowski Inequality belong to the family of Aleksandrov-Fenchel Inequalities. It is then reasonable that natural questions arising from the Isoperimetric Inequality can be redirected to the Minkowski Inequality. The first one is the Isoperimetric problem, namely finding subsets that minimise the perimeter among all those with the same volume. Although the literature on the Isoperimetric problem is quite vast ([Ant+22; AFP21; APP22; Rit01; Rit17; Nar14] to mention some of them), the analogue problem of minimising the total mean curvature among sets of fixed perimeter looks like it has not been faced yet. A second important question is the stability of the inequalities. As already said, quantitative versions of the Isoperimetric Inequality were already provided in literature. At the moment, our rigidity statement of the Extended Minkowski Inequality requires some further constraints on the subset Ω saturating it. In view of this difference with the Isoperimetric Inequality, some snares may be hidden along the way to obtain a quantitative Minkowski Inequality. In [AFM20], the authors proved that the following Willmore-type inequality

$$\left|\mathbb{S}^{n-1}\right| \mathrm{AVR}(g) \leq \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma \tag{19}$$

holds for any open bounded subset Ω with smooth boundary in a complete Riemannian manifold with nonnegative Ricci curvature and Euclidean Volume Growth. Moreover, the equality is achieved just on truncated cone over $\partial\Omega$. The inequality (19) is deduced from the monotonicity of the quantity $U : (0,1] \to \mathbb{R}$ which is defined as $U(t) = F_p^\beta(1/t)$, where F_p^β is given in (17) for $p = 2$ and $\beta = (n-2)$. Actually, the arguments contained in [AFM20] appears to be more quantitative, since it yields

$$\left|\mathbb{S}^{n-1}\right| \mathrm{AVR}(g) - \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^{n-1} \mathrm{d}\sigma \leq -\int_0^1 U'(t)\,\mathrm{d}t \leq 0. \tag{20}$$

The quantity $U'(t)$ is related to the L^2-norm of the trace-free hessian of a function, which identically vanishes on cones. These integral quantities are surprisingly similar to the ones considered in [CC96], where the authors showed that they control the Gromov-Hausdorff distance of the manifold from a cone. In [GV21], the authors find that the quantity $U'(t)$ actually controls the pointed measured Gromov-Hausdorff distance from a metric cone outside some compact set. Requiring some additional assumptions, one can imagine that this distance can be intended in a stronger topology. We obtained a partial result on Ricci-flat Asymptotically Conical Riemannian manifolds. In particular, if the left-hand side of (20) is close enough to 0 then $\partial\Omega$ is actually metrically close to the link of the asymptotic cone, provided some additional conditions on $\partial\Omega$ are fulfilled.

The achievement of this work were obtained in joint papers with L. Mazzieri and M. Fogagnolo. In particular, the results in Chapter 2 are contained in the forthcoming paper [BFM22], while the results in Chapters 3 and 4 are contained in [BFM21].

Notation and main settings

Riemannian manifolds

We will denote Riemannian manifolds by (M,g) and by $(V,(x^1,\dots,x^n))$ a local chart of M and by g_{ij}, $i,j=1,\dots,n$, the components of the metric with respect to that chart. The Levi-Civita connection will be denoted by D, the Riemannian curvature tensor by Riem and the Ricci curvature tensor by Ric. The Christoffel symbols associated with the Levi-Civita connection will be denoted by Γ_{ij}^k, $i,j,k=1,\dots,n$, with respect to the chart $(V,(x^1,\dots,x^n))$. The scalar product associated with the metric g on the tangent space will be usually denoted by $\langle\,\cdot\,|\,\cdot\,\rangle_g$ and the subscript g will be dropped if it is clear the metric we are referring to. Similarly, the norm associated with the scalar product will be denoted by $|\cdot|_g$, with the same convention as above. $\mathrm{d}_g(\,\cdot\,,\,\cdot\,)$ will denote the distance induced on (M,g) by the metric g. Given a smooth hypersurface N in M, we define $g^\top$ the metric induced by g on N. For a given function $f: M\to\mathbb{R}$, we denote

$$\mathrm{D}^\perp f=\langle \mathrm{D}f\,|\,\nu\rangle\nu \qquad \text{and} \qquad \mathrm{D}^\top f=\mathrm{D}f-\mathrm{D}^\perp f,$$

where ν is the unit normal vector field on N.

$\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ will always denote the n-dimensional ball and the $(n-1)$-dimensional sphere in the flat n-dimensional Euclidean space $\mathbb{R}^n$ respectively.

We will often indicate by μ_g the Lebesgue measure on M associated with the metric g. If N is an hypersurface of M, we will denote by σ_g the Lebesgue measure associated with $g^\top$ on N or in some occasion the $(n-1)$-dimensional Hausdorff measure on N. For every measurable set E, $|E|_g$ will denote the measure of a measurable subset accordingly to the most natural Hausdorff dimension, for example, if the general set in a family of measurable sets $(E_i)_{i\in I}$ is a hypersurface $|E_i|_g$ will denote the $(n-1)$-dimensional Hausdorff measure.

We refer the reader to [Pet06] to any other geometric notations we employ.

Function spaces and norm

We will denote L^p, $p\in[1,+\infty]$ the Lebesgue spaces and $W^{k,p}$, $p\in[1,+\infty]$ and $k\in\mathbb{N}$, the Sobolev space with respect to the measure μ associated with the metric g. $\mathscr{C}^{k,\alpha}$, $k\in\mathbb{N}$ and $\alpha\in[0,1]$ will denote the space of k^{th}-continuously differentiable functions with α-Hölder

k^{th}-derivative. 0-Hölder functions are intended to be continuous and we will write $\mathscr{C}^k$ in place of $\mathscr{C}^{k,0}$. We will prefer to denote Lip the space of Lipschitz continuous functions $\mathscr{C}^{0,1}$. We will also use AC to denote the space of absolutely continuous functions and BV for the space of functions with bounded variation. The subscript c will denote that the space is restricted to compactly supported functions. The norm of some function space will be indicated by $\|\cdot\|$ with the subscript space. The subscript loc will indicate that a function belongs to the given space for every compact subset. The subscript 0 will indicate the closure of compactly supported function with respect to the given norm.

Main settings

Throughout this work, we will mainly focus on two classes of Riemannian manifolds (M,g) that are the following.

(i) *Riemannian manifolds with nonnegative Ricci curvature*. With $\mathrm{AVR}(g)$ we will denote the Asymptotic Volume Ratio of g, defined as

$$\mathrm{AVR}(g) = \lim_{r\to+\infty} \frac{|B(o,r)|}{|\mathbb{B}^n| r^n} \left(= \lim_{r\to+\infty} \frac{|\partial B(o,r)|}{|\mathbb{S}^{n-1}| r^{n-1}} \right),$$

where $B(o,r)$ represents a geodesic ball of radius r centred at $o \in M$. The celebrated Bishop-Gromov Theorem [Bis64; Gro81] gives that for every $o \in M$ the ratio

$$[0,+\infty) \ni r \mapsto \frac{|B(o,r)|}{|\mathbb{B}^n| r^n} \tag{21}$$

is nonincreasing and thus it admits a limit as $r \to +\infty$, that actually does not depends on o. Moreover, $\mathrm{AVR}(g)$ belongs to $[0,1]$ and it is 1 only on the flat Euclidean space. We will often require that (M,g) has Euclidean Volume Growth, that is $\mathrm{AVR}(g) > 0$.

(ii) *$\mathscr{C}^{k,\alpha}$-Asymptotically Conical Riemannian manifolds*. According to [CEV17], a manifold is said to be $\mathscr{C}^{k,\alpha}$-Asymptotically Conical, for some $k \in \mathbb{N}$ and some $\alpha \in [0,1)$, if outside of a suitable compact set it is diffeomorphic to a truncated cone $[1,+\infty) \times L$, where (L, g_L) is a closed $(n-1)$-dimensional closed Riemannian manifold called the *link* of the cone, and the metric g is asymptotic to the cone metric $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ in the $\mathscr{C}^{k,\alpha}$-topology at infinity (see Definition 2.2.1). In some cases we will require an additional condition on the Ricci tensor, which is

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(\mathrm{d}(x,o)+1)^2}$$

for some $o \in M$, $\kappa \in \mathbb{R}$ and every $x \in M$. We will see that the quantity (21) admits a limit as $r \to +\infty$, even if it is not monotone. This limit will be denoted by $\mathrm{AVR}(g)$ as well and it coincides with $|L|/|\mathbb{S}^{n-1}|$. Asymptotically Conical Riemannian manifolds have Euclidean Volume Growth, but in general $\mathrm{AVR}(g)$ could exceed 1. $\mathrm{AVR}(g)$ could also be 1 even if (M,g) is not the flat Euclidean Space. This is for example the case of Asymptotically Flat spaces.

1

Preliminary results in nonlinear potential theory

1.1 Structure of the chapter

In this chapter, we recall, for ease of the reader, the classical theory of p-Laplace equation on Riemannian manifolds, which is one of the archetypal partial differential equations in Nonlinear Potential Theory. Almost all results in the next sections are essentially well-known but spread in different works, with different notations and with the aim of full generality. Here we seek to recollect the ideas and build a guidebook that can be used for the next chapters. In Section 1.2 we introduce the p-harmonic functions, the regularity results and some estimates related to them. Section 1.3 is devoted to the theory of the existence of positive p-Green's function on Riemannian manifolds with a focus on their asymptotic behaviour.

1.2 Main properties of p-harmonic functions

We start by rearranging the theory of p-harmonic functions on Riemannian manifolds, applying results contained in [LU68; Tol83; Lie88; Hol90; Hol99; Val13; Maz70; HK88; DiB83; Eva82; Ura68; Lew83; Lou08; Ser64; Tru67; FMP19]. We split the arguments into two parts. The first part concerns the definition of (weak) p-harmonic functions and the interior regularity estimates both at every point and where the gradient does not vanish. It is well-known that in the latter case the equation becomes uniformly elliptic, thus the classical theory for elliptic equations applies. For the sake of completeness, we also discuss the boundary regularity estimates. We conclude with the compactness of p-harmonic functions under locally uniform convergence both of the function and the underlying metric. The second part is focused on basic, though crucial, estimates. We recall here the most famous Comparison Principles, the Harnack's Inequality and the Harnack's Principle following from it. As concerns the first derivative, we recall the p-Cheng-Yau estimate proved in [WZ10]. The last part is reserved for the Kato-type Identity that will constitute the core of the proof in the rigidity part of our Monotonicity-Rigidity Theorem Theorems 3.2.1 and 3.2.2.

1.2.1 p-harmonic functions and their regularity

Given an open subset U in a complete Riemannian manifold (M, g) and $p > 1$, we say that $v \in W^{1,p}_{\mathrm{loc}}(U)$ is *p-harmonic* if

$$\int_U \left\langle |\mathrm{D}v|^{p-2}\mathrm{D}v \,|\, \mathrm{D}\psi \right\rangle \mathrm{d}\mu = 0 \tag{1.1}$$

for any test function $\psi \in \mathscr{C}^\infty_c(U)$. In this case we will write that

$$\Delta_g^{(p)} v = \mathrm{div}\left(|\mathrm{D}v|^{p-2}\mathrm{D}v\right) = 0 \qquad \text{on } U. \tag{1.2}$$

Given a subset $U \subset M$ and a chart $(V, (x^1, \ldots, x^n))$ the above equation can be written in coordinates on $V \cap U$ as

$$\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\left(g^{ks}\frac{\partial v}{\partial x^k}\frac{\partial v}{\partial x^s}\right)^{\frac{p-2}{2}} g^{ij}\frac{\partial v}{\partial x^j}\right) = 0. \tag{1.3}$$

Problem (1.3) is a 2nd order elliptic equation in divergence form that degenerates as $|\mathrm{D}v|$ vanishes. The regularity of the solutions to this operator has been investigated by many authors. We recollect here some results for ease of future reference.

Let U be an open subset of M. We can cover U with a countably family of bounded charts $(V_i, (x_i^1, \ldots, x_i^n))_{i\in\mathbb{N}}$. Suppose that for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$, there exist two constants $0 < \lambda^U \leq \Lambda^U_{k,\alpha} < +\infty$ such that

$$\begin{aligned} &g_{ij}(x)\xi^i\xi^j > \lambda^U \delta_{ij}\xi^i\xi^j && \text{for every } x \in U \cap V_s \text{ and } \xi \in \mathbb{R}^n, \\ &\|g_{ij}\|_{\mathscr{C}^{k,\alpha}(U\cap V_s)} \leq \Lambda^U_{k,\alpha} && \text{for every } s = 1, \ldots, N, \end{aligned} \tag{1.4}$$

where g_{ij} are the components of the metric g with respect to the coordinates $(x_s^1, \ldots, x_s^n)$. If $U \subset M$ is bounded the condition $0 < \lambda^U \leq \Lambda^U_{k,\alpha} < +\infty$ is satisfied. Differently from the flat Euclidean case, in a general noncompact complete Riemannian manifolds the metric g can degenerate approaching infinity. The above conditions are satisfied for unbounded subsets U when the metric g has some well-known behaviour at infinity, for example, if the manifold is asymptotically close in the $\mathscr{C}^{0,\alpha}$-topology to some given model.

Suppose that U is an open subset of M such that $0 < \lambda^U \leq \Lambda^U_{0,\alpha} < +\infty$, regularity results for p-harmonic functions (see [Tol83; DiB83; Eva82; Ura68; Lew83]) yield $v \in \mathscr{C}^{1,\beta}_{\mathrm{loc}}(U)$ for some $\beta \leq \alpha$ that depends on n, p, α, $\Lambda^U_{0,\alpha}$ and λ^U. Moreover, for any compact subset $K \subset U$ there exists a positive $\mathrm{C}_{1,\beta} > 0$ depending only on n, p, α, $\Lambda^U_{0,\alpha}$, λ^U and the distance of K from the boundary of U such that

$$\|v\|_{\mathscr{C}^{1,\beta}_{\mathrm{loc}}(K)} \leq \mathrm{C}$$

for every p-harmonic function v with $\|v\|_{L^\infty(U)} \leq 1$. The theorem below easily follows by a scaling argument, being that $v/\|v\|_{L^\infty(U)}$ is p-harmonic as well.

Theorem 1.2.1 (Schauder interior estimates)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold, $U \subset M$ be an open subset such that $0 < \lambda^U \leq \Lambda_{0,\alpha} < +\infty$ in (1.4), for some $\alpha > 0$. Let $p > 1$. Then, any bounded solution $v \in W^{1,p}_{\mathrm{loc}}(U)$ of the problem $\Delta^{(p)}_g v = 0$ on U belongs to $\mathscr{C}^{1,\beta}_{\mathrm{loc}}(U)$ for some positive $\beta = \beta(n, p, \alpha, \Lambda^U_{0,\alpha}, \lambda^U) \leq \alpha$. Moreover, for every compact $K \subset U$ the estimate*

$$\|v\|_{\mathscr{C}^{1,\beta}(K)} \leq C_{1,\beta} \, \|v\|_{L^\infty(U)},$$

holds for a positive constant $C_{1,\beta} = C_{1,\beta}(n, p, \alpha, \mathrm{d}(K, \partial U), \Lambda^U_{0,\alpha}, \lambda^U)$.

Actually, in [Tol83; LU68; DiB83] Sobolev regularity for first order derivative are implicitly stated. In particular, Proposition 1 in [Tol83] (basing the proof on [LU68]) shows that

$$\begin{aligned} &v \in W^{2,2}_{\mathrm{loc}}(U) \cap W^{1,\infty}_{\mathrm{loc}}(U) && \text{if } p \geq 2, \\ &v \in W^{2,p}_{\mathrm{loc}}(U) \cap W^{1,p+2}_{\mathrm{loc}}(U) && \text{if } p < 2 \end{aligned}$$

hold for a broader class of operators, but requiring a uniform ellipticity condition for $p \geq 2$. On the other hand [DiB83] stress out that $|\mathrm{D}v|^{(p-2)/2}\mathrm{D}^2 v$ belongs to L^2_{loc}, but the author prefers to assume that solution can be approximated weakly $W^{1,p}$ by classical solution of nondegenerate elliptic equations. We also want to highlight that the $W^{2,2}_{\mathrm{loc}}$ regularity is also in force for $p < 3$ as one can find in [Sci14, Theorem 1.1]. What we really need in the following is the next theorem, proved in [Lou08] (see Appendix C for the proof based on the p-Bochner formula in Appendix A)

Theorem 1.2.2 (Sobolev interior regularity)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold, $U \subset M$ be an open subset and $p > 1$. If $v \in W^{1,p}_{\mathrm{loc}}(U)$ is a bounded solution of the problem $\Delta^{(p)}_g v = 0$ on U, then*

$$|\mathrm{D}v|^{p-1} \in W^{1,2}_{\mathrm{loc}}(U).$$

Given a p-harmonic function $v \in W^{1,p}_{\mathrm{loc}}(U)$, expanding the divergence in (1.3) we obtain the following equation

$$\begin{aligned} &\left[g^{ij} + \frac{(p-2)g^{im}g^{jl}}{|\mathrm{D}v|^2} \frac{\partial v}{\partial x^m} \frac{\partial v}{\partial x^l} \right] \frac{\partial^2 v}{\partial x^m \partial x^l} \\ &\quad + \left[\frac{1}{\sqrt{\det g}} \frac{\partial(\sqrt{\det g} g^{ik})}{\partial x^k} - \frac{(p-2)g^{im}g^{jl}\Gamma^k_{ij}}{|\mathrm{D}v|^2} \frac{\partial v}{\partial x^m} \frac{\partial v}{\partial x^l} \right] \partial_k v = 0. \end{aligned} \tag{1.5}$$

If $|\mathrm{D}v| > 0$ at some point, problem (1.5) becomes a 2^{nd} order nondegenerate elliptic equation with $\mathscr{C}^{0,\beta}$ coefficient thanks to Theorem 1.2.1. Hence, the classical regularity theory for quasilinear nondegenerate elliptic equations ensures that Sobolev functions satisfying (1.1) are smooth around the points where the gradient does not vanish (see [LSU68, Chapter 4 Section 6]). Moreover, by a bootstrap argument we can infer the following $\mathscr{C}^{k,\beta}$ estimate for p-harmonic functions around points where the gradient does not vanish (see [GT15, Proposition 6.6]).

Theorem 1.2.3 (Higher-order Schauder estimates)**.** *Let (M,g) be a n-dimensional Riemannian manifold, $U \subset M$ be an open subset of M such that $0 < \lambda^U \leq \Lambda^U_{k,\alpha} < +\infty$ in (1.4), for some $k \geq 1$ and $\alpha > 0$. Let $p > 1$. Then, any bounded solution $v \in W^{1,p}_{\mathrm{loc}}(U)$ of the problem $\Delta^{(p)}_g v = 0$ on U such that $|\mathrm{D}v| > 0$ on U belongs to $\mathscr{C}^\infty(U)$. Moreover, for any $k \in \mathbb{N}$ and $\alpha \in (0,1)$ and any compact $K \subset U$ the estimate*

$$\|v\|_{\mathscr{C}^{k+1,\alpha}(K)} \leq \mathrm{C}_{k+1,\alpha}\, \|v\|_{L^\infty(U)}$$

holds for a positive constant $\mathrm{C}_{k+1,\alpha} = \mathrm{C}_{k+1,\alpha}(n,p,k,\alpha,\mathrm{d}(K,\partial U),\Lambda^U_{k,\alpha},\lambda^U,\inf_K|\mathrm{D}v|)$.

We then discuss the boundary regularity. Given $U \subset M$ open bounded, we say that $v \in W^{1,p}(U)$ solves the Dirichlet problem with boundary datum $\psi \in W^{1,p}(U)$ if

$$\begin{cases} \Delta^{(p)}_g v = 0 & \text{on } U, \\ v - \psi \in W^{1,p}_0(U). \end{cases} \tag{1.6}$$

An open bounded subset $U \subset M$ is said to be a *p-regular domain* (see [Hol90; Val13]) if the *Wiener criterion* is satisfied at every point of the boundary, that is

$$\int_0^1 \frac{1}{t}\left(\frac{\mathrm{Cap}_p(\overline{B(x,t)} \cap (M \smallsetminus U), B(x,2t))}{\mathrm{Cap}_p(\overline{B(x,t)}, B(x,2t))}\right)^{\frac{1}{p-1}} \mathrm{d}t = +\infty, \tag{1.7}$$

for every $x \in \partial U$ (see Definition 1.3.1 below for the definition of $\mathrm{Cap}_p(\cdot,\cdot)$). It is proved in [Maz70] (see also [Hol90; Hol99; Val13]) that if U is p-regular domain and $\psi \in W^{1,p}(U) \cap \mathscr{C}^0(\overline{U})$ then the solution v to (1.6) attains continuously the datum at the boundary. A major contribution of [Lie88] is that the Schauder estimates can be extended up to the boundary if both the boundary and the boundary datum are regular enough. We report here both the regularity theorems and the relative estimates for completeness.

Theorem 1.2.4 (Global Schauder estimates)**.** *Let (M,g) be a complete n-dimensional Riemannian manifold and $p > 1$. Let U be an open subset of M such that $0 < \lambda^U \leq \Lambda_{0,\alpha} < +\infty$ in (1.4), for some $\alpha > 0$. Suppose that U has $\mathscr{C}^{1,\alpha}$ boundary and $\psi \in \mathscr{C}^{1,\alpha}(\overline{U})$. Let $p > 1$. Then, any solution $v \in W^{1,p}(U)$ be a solution of the problem (1.6) belongs to $\mathscr{C}^{1,\beta}(\overline{U})$ for some positive $\beta = \beta(n,p,\alpha,\Lambda^{\overline{U}}_{0,\alpha},\lambda^{\overline{U}}) \leq \alpha$. Moreover, the estimate*

$$\|v\|_{\mathscr{C}^{1,\beta}(\overline{U})} \leq \mathrm{C}_{1,\beta}\left(\|v\|_{L^\infty(U)} + \|\psi\|_{\mathscr{C}^{1,\alpha}(\partial U)}\right)$$

holds for a positive constant $\mathrm{C}_{1,\beta} = \mathrm{C}_{1,\beta}(n,p,\alpha,\Lambda^{\overline{U}}_{0,\alpha},\lambda^{\overline{U}},U)$.

When the gradient does not vanish and both the boundary and the boundary datum are more regular, we can infer the higher-order regularity, appealing again to the classical theory for elliptic equations.

Theorem 1.2.5 (Higher-order global Schauder estimates)**.** *Let (M,g) be a complete n-dimensional Riemannian manifold and $p > 1$. Let U be an open subset of M such that $0 < \lambda^U \leq \Lambda_{k,\alpha} < +\infty$ in (1.4), for some $k \geq 1$ and $\alpha > 0$. Suppose that U has $\mathscr{C}^{k,\alpha}$*

boundary and $\psi \in \mathscr{C}^{k,\alpha}(\overline{U})$*. Let* $p > 1$*. Let* $v \in W^{1,p}(U)$ *be a solution of problem* (1.6) *such that* $|\mathrm{D}v| > 0$ *in* $\overline{U}$*. Then* $v \in \mathscr{C}^{k+1,\alpha}(\overline{U})$*. Moreover, the estimate*

$$\|v\|_{\mathscr{C}^{k+1,\alpha}(\overline{U})} \leq \mathrm{C}_{k+1,\alpha} \left(\|v\|_{L^\infty(U)} + \|\psi\|_{\mathscr{C}^{k+1,\alpha}(\partial U)} \right),$$

holds for a positive constant $\mathrm{C}_{k+1,\alpha} = \mathrm{C}_{k+1,\alpha}(n, p, k, \alpha, \Lambda^{\overline{U}}_{k,\alpha}, \lambda^{\overline{U}}, \inf_{\overline{U}} |\mathrm{D}v|, U)$.

To conclude, we want to recall a compactness theorem that holds for p-harmonic functions. It is a natural question whether the limit of a sequence of p-harmonic functions is still p-harmonic. The weak formulation in (1.1) suggests that $\mathscr{C}^1$ convergence on compact subsets is enough to ensure that also the limit function is p-harmonic. The following theorem relaxes this hypothesis in favour of uniform convergence on compact subsets.

Theorem 1.2.6 (Compactness Theorem)**.** *Let* (M, g) *be complete* n*-dimensional Riemannian manifold,* $U \subset M$ *be an open and* $p > 1$*. Let* $U \subset M$ *be an open subset Let* $(v_n)_{n \in \mathbb{N}}$ *be a sequence of* p*-harmonic functions on* U *that converges uniformly to* v *on compact subsets of* U *as* $n \to +\infty$*. Then* $v \in W^{1,p}_{\mathrm{loc}}(U)$ *is* p*-harmonic on* U*.*

Proof. See [HK88, Theorem 3.2]. □

Remark 1.2.7. *Suppose that* $(U_n)_{n \in \mathbb{N}}$ *is a sequence of open subsets converging to* U *open subset as* $n \to +\infty$*. Let* g_n *be a metric on* U_n *for every* $n \in \mathbb{N}$ *that locally uniformly converges to some metric* g *on* U *as* $n \to +\infty$*. The above theorem still holds if* v_n *is* p*-harmonic with respect to the metric* g_n*. As a consequence, the limit function* v *is* p*-harmonic with respect to the metric* g*.*

1.2.2 Estimates for p-harmonic functions

We retrieve the Hopf's Maximum Principle [Tol83, Proposition 3.2.1] and the Comparison Principles [Val13, Proposition 2.1.4] by Valtorta and [Tol83, Lemma 3.1, Proposition 3.3.2] by Tolksdorf, specialised for our purposes.

Theorem 1.2.8 (Hopf's Maximum Principle)**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold,* $U \subset M$ *be an open bounded subset and* $p > 1$*. Let* $v \in \mathscr{C}^1(\overline{U})$ *be a* p*-harmonic function. Let* B *a ball contained in* U *such that* $v > 0$ *in* B *and* $v(x) = 0$ *at some* $x \in \partial B$*. Then* $\mathrm{D}v(x) \neq 0$*.*

Theorem 1.2.9 (Comparison Principles)**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold,* $U \subset M$ *be an open bounded subset,* $v_1, v_2 \in W^{1,p}(U)$ *be two* p*-harmonic functions and* $p > 1$*.*

- (Weak) Comparison Principle. *If* $\min\{v_1 - v_2, 0\} \in W^{1,p}_0(U)$ *then* $v_1 \geq v_2$ *almost everywhere on* U*.*
- Strong Comparison Principle. *Suppose in addition that* U *is connected,* $v_1 \in \mathscr{C}^1(\overline{U})$*,* $v_2 \in \mathscr{C}^2(\overline{U})$ *and* $|\nabla v_2| \geq \delta > 0$ *in* U*. If* $v_1 \leq v_2$ *(resp.* $v_1 \geq v_2$*) on* U*, then* $v_1 = v_2$ *or* $v_1 < v_2$ *(resp.* $v_1 > v_2$*) on* U*.*

In [Tol83], the Strong Comparison Principle is stated in a slightly different way, namely, for a larger class of operators but only in one of the two cases expressed above.

Our version can be deduced by applying the original theorem firstly to v_1 and v_2 and secondly to $-v_1$ and $-v_2$, that are also solutions to (1.1).

By the important contributions of [Tru67] and [Ser64] (see also [HK88]), Harnack's Inequality holds. Given an open subset $U \subset M$ and a compact connected $K \subset U$, there exists a constant $\mathrm{C}_H = \mathrm{C}_H(n, p, \Lambda^K_{0,0}, \lambda^K, \mathrm{diam}(K))$ such that

$$\sup_K v \leq \mathrm{C}_H \inf_K v, \tag{1.8}$$

for any nonnegative p-harmonic function v on U, where λ^K and $\Lambda^K_{0,0}$ are defined in (1.4). The consequence of Harnack's Inequality is the so celebrated Harnack's principle, whose proof can be found in [HK88, Theorem 3.3].

Theorem 1.2.10 (Harnack's Principle)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold, let U be an open subset of M and $p > 1$. Suppose that $(v_i)_{i\in\mathbb{N}}$ is an increasing sequence of p-harmonic functions on U. Then $v = \lim_i v_i$ is either identically $+\infty$ or a p-harmonic function in U. In the latter case the convergence of $v_i \to v$ is uniform on compact subsets of U.*

A way to prove Harnack's Inequality is by patching up a local Harnack's Inequality on geodesic balls inferred from the Cheng-Yau-type estimate provided in [WZ10].

Theorem 1.2.11 (Cheng-Yau-type estimate)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and $p > 1$. Let $v \in W^{1,p}_{\mathrm{loc}} B(o, 2R)$ be a positive p-harmonic function on a geodesic ball $B(o, 2R)$ for some $R > 0$ where $\mathrm{Ric} \geq -(n-1)\kappa^2$. Then there exists a constant $\mathrm{C} = \mathrm{C}(p, n)$ such that*

$$\sup_{B(o,R)} |\mathrm{D} \log v| \leq \mathrm{C} \left(\frac{1}{R} + \kappa \right). \tag{1.9}$$

Harnack's Inequality follows by integrating (1.9) on a path connecting two points $x, y \in B(o, R)$. Since the points are general, we end with

$$\sup_{B(o,R)} v \leq \mathrm{e}^{\mathrm{C}(1+R\kappa)} \inf_{B(o,R)} v,$$

for every nonnegative p-harmonic function v on $B(o, 2R)$.

To conclude, we recall a Kato-type Identity for p-harmonic function and consequently a refined version of the Kato-type Inequality for p-harmonic functions.

Proposition 1.2.12 (Kato-type Identity)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold, $U \subset M$ be an open subset of M and $p > 1$. Let $v \in W^{1,p}_{\mathrm{loc}}(U)$ be a p-harmonic function on U. Then, in an open neighbourhood of a point $x \in U$ where $|\mathrm{D}v|(x) > 0$, the following identity holds true*

$$\begin{aligned} |\mathrm{DD}v|^2 - \left(1 + \frac{(p-1)^2}{n-1}\right) |\mathrm{D}|\mathrm{D}v||^2 = &|\mathrm{D}v|^2 \left| \mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top \right|^2 \\ &+ \left(1 - \frac{(p-1)^2}{n-1}\right) \left|\mathrm{D}^\top |\mathrm{D}v|\right|^2, \end{aligned} \tag{1.10}$$

according to the orthogonal decomposition with respect to the level sets of v. Moreover, if $|\mathrm{D}v| > 0$ and the right hand side of (1.10) vanishes in $\{t_0 \leq v \leq t_1\}$ for some $t_0, t_1 \in \mathbb{R}$, $t_0 < t_1$ with t_1

possibly infinite, then the Riemannian manifold $(\{t_0 \leq v \leq t_1\}, g)$ *is isometric to the warped product* $([t_0, t_1] \times \{v = t_0\}, \mathrm{d}t \otimes \mathrm{d}t + \eta^2(t) g_{\{v=t_0\}})$*, where the relation between* v*,* η *and* t *is given by*

$$\eta(t) = \left(\frac{v'(t_0)}{v'(t)}\right)^{\frac{p-1}{n-1}}.$$

Proof. See [FMP19, Proposition 4.4]. □

Corollary 1.2.13 (Kato-type Inequalities for p-harmonic functions)**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold,* $U \subset M$ *be an open subset of* M *and* $p > 1$*. Let* $v \in W^{1,p}_{\mathrm{loc}}(U)$ *be a* p*-harmonic function on* U*.*

(i) If $(p-1)^2 > n-1$*, then, in a neighbourhood of any* $x \in U$ *such that* $|\mathrm{D}v|(x) > 0$

$$|\mathrm{DD}v|^2 \geq \left(1 + \frac{(p-1)^2}{n-1}\right) |\mathrm{D}|\mathrm{D}v||^2.$$

Moreover, if equality is achieved on $\{t_0 \leq v \leq t_1\}$ *and* $|\mathrm{D}v| > 0$ *in this region, then* $(\{t_0 \leq v \leq t_1\}, g)$ *has the same warped product structure as in the rigidity case of Proposition 1.2.12.*

(ii) If $(p-1)^2 > n-1$*, then, in a neighbourhood of any* $x \in U$ *such that* $|\mathrm{D}v|(x) > 0$*,*

$$|\mathrm{DD}v|^2 \geq 2|\mathrm{D}|\mathrm{D}v||^2. \tag{1.11}$$

Moreover, if the equality is achieved on $\{t_0 \leq v \leq t_1\}$ *and* $|\mathrm{D}v| > 0$ *in this region, then* $(\{t_0 \leq v \leq t_1\}, g)$ *splits as a Riemannian product* $([t_0, t_1] \times \{v = t_0\}, \mathrm{d}t^2 + g_{\{v=t_0\}})$ *and* v *is an affine function of* t*.*

(iii) If $(p-1)^2 = n-1$*, then, in a neighbourhood of any* $x \in U$ *such that* $|\mathrm{D}v|(x) > 0$ *it holds* (1.11)*. If the inequality holds in* (1.11) *at some point* x *with* $|\mathrm{D}v|(x) > 0$*, then* x *is an umbilical point of* $\{v = v(x)\}$*, that is a smooth hypersurface in a neighbourhood of* x*.*

Proof. See [Fog20, Corollary 2.6] and [FMP19, Corollary 4.6]. □

1.3 p-Green's functions on Riemannian manifolds

In this section, we mainly focus on characterising those Riemannian manifolds (M, g) that admit a positive p-Green's function with some vanishing property at infinity. We recall that a p-Green's function G is a distributional solution to the problem $\Delta_p G = \delta_o$, where δ_o is the Dirac delta at $o \in M$. We are seeing that the existence of positive G is related to the existence of a compact set of positive p-capacity. Mainly referring to [HKM18], we recall the theory of p-capacity for bounded and non-bounded condensers and all basic properties we think will be useful in the following dissertation. We then define p-nonparabolic (and consequently p-parabolic) Riemannian manifolds as those having a positive p-capacity at infinity, and consequently a positive p-Green's function. We conclude by studying the asymptotic behaviour of the p-Green's function at infinity, which is related to some geometric properties of the unbounded components of the manifold. This last part is inspired by the work of Holopainen [Hol99], with some insights coming from the subsequent [MRS19]. We also refer the reader to [LT87; LT92; LT95] for the harmonic analogue.

1.3.1 p-capacity on Riemannian manifolds

We find convenient to recall here the definition of *p-capacity of a condenser* (K, U) where U is an open subset of M and K is a compact subset of U.

Definition 1.3.1 (p-capacity and normalised p-capacity)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold, (K, U) a condenser and $p > 1$. The* p-capacity *of (K, U) is defined as*

$$\operatorname{Cap}_p(K, U) = \inf\left\{ \int_U |\mathrm{D}\varphi|^p \, \mathrm{d}\mu \;\middle|\; \varphi \in \mathscr{C}_c^\infty(U),\ \varphi \geq 1 \text{ on } K \right\}. \tag{1.12}$$

If $U = M$ we simply denote $\operatorname{Cap}_p(K) = \operatorname{Cap}_p(K, M)$. We also define the normalised p-capacity *as*

$$\widehat{\operatorname{Cap}}_p(K, U) = \left(\frac{p-1}{n-p}\right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \operatorname{Cap}_p(K, U). \tag{1.13}$$

Consider a condenser (K, U) with U bounded in M and $\psi \in \mathscr{C}_c^\infty(U)$ such that $\psi = 1$ in a neighbourhood of K. Then there exists a solution $u \in W^{1,p}(U \smallsetminus K)$ to the problem

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } U \smallsetminus K, \\ u - \psi \in W_0^{1,p}(U \smallsetminus K). \end{cases}$$

One can observe that such a function realises the minimum in (1.12). Indeed, since u can be approximated in $W^{1,p}(U \smallsetminus K)$ by functions $\varphi \in \mathscr{C}_c^\infty(U)$ with $\varphi \geq 1$ on K, then

$$\operatorname{Cap}_p(K, U) \leq \int_{U \smallsetminus K} |\mathrm{D}u|^p \, \mathrm{d}\mu.$$

On the other hand, the weak formulation in (1.1) can be relaxed in duality with functions in $W_0^{1,p}(U \smallsetminus K)$. Hence, taking any competitor $\varphi \in \mathscr{C}_c^\infty(U)$ with $\varphi \geq 1$ on K, we clearly have that $u - \varphi \in W_0^{1,p}(U \smallsetminus K)$ and by (1.1) we obtain

$$\int_{U \smallsetminus K} |\mathrm{D}u|^p \, \mathrm{d}\mu = \int_{U \smallsetminus K} \left\langle |\mathrm{D}u|^{p-2} \mathrm{D}u \;\middle|\; \mathrm{D}u \right\rangle \mathrm{d}\mu = \int_{U \smallsetminus K} \left\langle |\mathrm{D}u|^{p-2} \mathrm{D}u \;\middle|\; \mathrm{D}\varphi \right\rangle \mathrm{d}\mu.$$

Applying Hölder's Inequality to the right hand side, we are left with

$$\int_{U \smallsetminus K} |\mathrm{D}u|^p \, \mathrm{d}\mu \leq \int_{U \smallsetminus K} |\mathrm{D}\varphi|^p \, \mathrm{d}\mu$$

for every competitor φ in (1.12), proving in fact that

$$\operatorname{Cap}_p(K, U) = \int_{U \smallsetminus K} |\mathrm{D}u|^p \, \mathrm{d}\mu.$$

If U is not bounded M, we can consider and exhaustion $(U_i)_{i\in\mathbb{N}}$ for U with $K \subseteq U_1$. Since $U_i \subset U$ for every $i \in \mathbb{N}$, (1.12) one can easily see that

$$\mathrm{Cap}_p(K, U_i) \geq \mathrm{Cap}_p(K, U), \tag{1.14}$$

for every $i \in \mathbb{N}$. Conversely, for every $\varepsilon > 0$ let $\varphi_\varepsilon \in \mathscr{C}^\infty_c(U)$, $\varphi \geq 1$ on K such that

$$\int_U |\mathrm{D}\varphi|^p \,\mathrm{d}\mu - \varepsilon \leq \mathrm{Cap}_p(K, U).$$

Since U_i is an exhaustion for U there exists $I \in \mathbb{N}$ such that $\operatorname{supp}\varphi \subset U_i$ for every $i \geq I$. Hence,

$$\mathrm{Cap}_p(K, U_i) - \varepsilon \leq \int_U |\mathrm{D}\varphi|^p \,\mathrm{d}\mu - \varepsilon \leq \mathrm{Cap}_p(K, U). \tag{1.15}$$

Combining (1.14) and (1.15) we obtain that

$$\mathrm{Cap}_p(K, U) = \lim_{i\to+\infty} \mathrm{Cap}_p(K, U_i). \tag{1.16}$$

For each condenser (K, U_i) let u_i be its p-capacitary potential. The Comparison Principle in Theorem 1.2.9 gives that $(u_i)_{i\in\mathbb{N}}$ is increasing. By the Harnack's Principle Theorem 1.2.10, the limit function $u = \sup u_i$ is p-harmonic in $U \smallsetminus K$. The function u realises the p-capacity of the condenser (K, U) that is

$$\mathrm{Cap}_p(K, U) = \int_{U\smallsetminus K} |\mathrm{D}u|^p \,\mathrm{d}\mu, \tag{1.17}$$

and it will be called the *p-capacitary potential* associated with the condenser (K, U). Indeed, $(u_i)_{i\in\mathbb{N}}$ converges locally uniformly $|\mathrm{D}u_i|$ is bounded in $L^p(U \smallsetminus K)$, hence by lower semicontinuity

$$\int_U |\mathrm{D}u|^p \,\mathrm{d}\mu \leq \liminf_{i\to+\infty} \int_U |\mathrm{D}u_i|^p \,\mathrm{d}\mu = \mathrm{Cap}_p(K, U).$$

Conversely, by [HKM18, Lemma 1.33], $u \in L^p_{\mathrm{loc}}(U \smallsetminus K)$ and there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$, $\varphi_n \in \mathscr{C}^\infty_c(U)$ and $\varphi_n \geq 1$ on K such that $\mathrm{D}\varphi_n \to \mathrm{D}u$ in L^p as $n \to +\infty$. Hence, we have that

$$\mathrm{Cap}_p(K, U) \leq \int_{U\smallsetminus K} |\mathrm{D}u|^p \,\mathrm{d}\mu,$$

concluding the proof of (1.17). Observe that the function u may not belong to $W^{1,p}(U \smallsetminus K)$ in general. Indeed, if U is unbounded and $\mathrm{Cap}_p(K, U) = 0$ the p-capacitary potential u is identically 1, which is not in $L^p(U \smallsetminus K)$.

Observe that the very definition of p-capacity readily implies that

$$\mathrm{Cap}_p(K, U) = \mathrm{Cap}_p(\partial K, U),$$

whenever K is a compact subset of U.

We recall here some well-known, though useful, properties of the p-capacity. We refer the reader to [HKM18, Theorem 2.2] and [Hol90, Lemma 3.6, 3.7, 3.8] for the proofs.

Proposition 1.3.2. *Let (M, g) be a complete n-dimensional Riemannian manifold and $p > 1$.*

(i) If (K, U_1) and (K, U_2) are two condensers such that $U_1 \subset U_2$, then

$$\mathrm{Cap}_p(K, U_1) \geq \mathrm{Cap}_p(K, U_2). \tag{1.18}$$

(ii) If (K_1, U) and (K_2, U) are two condensers such that $K_1 \subset K_2$, then

$$\mathrm{Cap}_p(K_1, U) \leq \mathrm{Cap}_p(K_2, U). \tag{1.19}$$

(iii) Given a condenser (K, U) and $u \in W^{1,p}_{\mathrm{loc}}(U \smallsetminus K)$ its p-capacitary potential, if the set $(\{u \geq b\}, \{u > a\})$ is a condenser for some $0 \leq a < b \leq 1$, then

$$\mathrm{Cap}_p(\{u \geq b\}, \{u > a\}) = \frac{\mathrm{Cap}_p(K, U)}{(b-a)^{p-1}}. \tag{1.20}$$

Using the coarea formula (see Appendix B), we can characterise the p-capacity of a condenser as a weighted surface integral. We prove this result in the following statement.

Proposition 1.3.3. *Let (M, g) be a complete n-dimensional Riemannian manifold and $p > 1$. Consider a condenser (K, U) in M and let $u \in W^{1,p}_{\mathrm{loc}}(U \smallsetminus K)$ its p-capacitary potential. Suppose that $\{u \geq c\}$ is compact in U for every $c \in (0, 1)$, then*

$$\mathrm{Cap}_p(K, U) = \int\limits_{\{u=t\}} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma, \tag{1.21}$$

for almost every $t \in (0, 1)$, including any t regular for u. Moreover, if K has $\mathscr{C}^2$ boundary then

$$\mathrm{Cap}_p(K, U) = \int\limits_{\partial K} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma. \tag{1.22}$$

Proof. Observe that $\mathrm{Cap}_p(K, U) < +\infty$. Hence, the p-capacitary potential u of (K, U) is such that $|\mathrm{D}u| \in L^p(K \smallsetminus U)$. Moreover, by Theorem 1.2.1 u is locally Lipschitz. The coarea formula Proposition B.3 for $f = |\mathrm{D}u|^{p-1}$ yields

$$\mathrm{Cap}_p(K, U) = \int_0^1 \int\limits_{\{u=t\}} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}t. \tag{1.23}$$

Moreover, employing again the coarea formula Proposition B.3 with $f = \psi'(u)|\mathrm{D}u|^{p-1}$ and an integration by parts we have that

$$\begin{aligned}\int_0^1 \psi'(t) \int\limits_{\{u=t\}} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}t &= \int\limits_{M \smallsetminus \overline{\Omega}} \psi'(u)|\mathrm{D}u|^p \, \mathrm{d}\mu \\ &= -\int\limits_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u|^{p-2} \langle \mathrm{D}u \,|\, \mathrm{D}(\psi(u)) \rangle \, \mathrm{d}\mu \\ &= \int\limits_{M \smallsetminus \overline{\Omega}} \psi(u) \, \mathrm{div}(|\mathrm{D}u|^{p-2}\mathrm{D}u) \, \mathrm{d}\mu = 0,\end{aligned}$$

for every $\psi \in \mathscr{C}_c^\infty(0,1)$. The function

$$t \mapsto \int\limits_{\{u=t\}} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma \in L^1(0,1) \tag{1.24}$$

admits a constant representative, that coupled with (1.23) gives (1.21). If $t \in (0,1)$ is regular for u the function defined in (1.24) is smooth, concluding the proof of the first part of the statement.

Suppose that K has $\mathscr{C}^2$ boundary, then the p-capacitary potential is $\mathscr{C}^{1,\beta}$ up to ∂K. Moreover, by Hopf's Maximum Principle Theorem 1.2.8 the gradient does not vanishes at the boundary. Hence, the function (1.24) is continuous around $\{t = 1\} = \partial K$, proving also (1.22). □

It is also worth mentioning that one can define p-capacity for a condenser (F, U) where F is any subset of U by taking

$$\mathrm{Cap}_p(F, U) = \inf_{\substack{E \subset A \\ A \text{ open}}} \mathrm{Cap}_p^*(A, U),$$

where for any $A \subseteq U$

$$\mathrm{Cap}_p^*(A, U) = \sup_{\substack{K \subset A \\ K \text{ compact}}} \mathrm{Cap}_p(K, U).$$

Firstly, we observe that is not ambiguous having two definitions for compact subsets since they coincide. Moreover, the set function $\mathrm{Cap}_p(\cdot, U)$ is a Choquet capacity. Choquet's Theorem (see [Cho54]) asserts that all Borel (in fact, all analytic) subsets of U are capacitable, that is

$$\mathrm{Cap}_p(A, U) = \sup_{\substack{K \subset A \\ K \text{ compact}}} \mathrm{Cap}_p(K, U).$$

For a general F compactly contained in U it holds

$$\mathrm{Cap}_p(F, U) \leq \mathrm{Cap}_p(\overline{F}, U),$$

but the equality is achieved only for some F, for example, the ball (see [HKM18]). Indeed, consider a compact subset $K \subset \mathbb{R}^n$ and a countable dense family of points $(x_n)_{n \in \mathbb{N}}$ in K. For any point choose a ball B_n small enough such that

$$\mathrm{Cap}_p(B_n) = \mathrm{Cap}(\overline{B_n}) \leq 2^{-(n+2)} \, \mathrm{Cap}_p(K).$$

Let F be the union of such balls. Since $\overline{F} \supset K$ then $\mathrm{Cap}_p(\overline{F}) = \mathrm{Cap}_p(K)$, but the σ-subadditivity gives $\mathrm{Cap}_p(F) \leq \mathrm{Cap}_p(K)/2$. For a more detailed treatment of the theory of capacities we refer the reader to [Maz70; Hol90; HKM18; Cho54].

1.3.2 Existence of positive p-Green's function

We focus on the existence of a positive p-Green's function. Firstly, we define it for bounded p-regular subsets, which, as said before, are the subsets $U \subset M$ satisfying the Wiener criterion (1.7) at every $x \in \partial U$. Then we extend it to *p-nonparabolic* Riemannian manifolds.

Definition 1.3.4 (p-Green's function)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and $p > 1$. Let o be a point in a p-regular domain $U \subset M$. A* p-Green's function *is a positive $G(o, \cdot) \in W^{1,p}_{\mathrm{loc}}(U \smallsetminus \{o\}) \cap \mathscr{C}^0(U \smallsetminus \{o\})$ which satisfies $\lim_{x \to z} G(o, x) = 0$ for every $z \in \partial U$ and $\Delta_g^{(p)} G(o, \cdot) = -\delta_o$ where δ_o is the Dirac delta centred at o, that is*

$$\int_U \left\langle |\mathrm{D}G(o, \cdot)|^{p-2} \mathrm{D}G(o, \cdot) \,\middle|\, \mathrm{D}\psi \right\rangle \mathrm{d}\mu = \psi(o), \qquad \text{for any } \psi \in \mathscr{C}^\infty_c(U).$$

By [Hol90, Theorem 3.19], to each p-regular domain one can associate a p-Green's function. Observe that we do not require G to be symmetric. Actually, this is not true for $p \neq 2$.

Proposition 1.3.5. *Let (M, g) be a complete n-dimensional Riemannian manifold and $p > 1$. Let o be a point in a bounded open p-regular domain $U \subset M$. Then there exists a p-Green's function $G(o, \cdot)$ in U. Moreover, $G(o, x) \to +\infty$ as $\mathrm{d}(x, o) \to 0$ and it satisfies*

$$\mathrm{Cap}_p(\{G(o, \cdot) \geq b\}, \{G(o, \cdot) > a\}) = (b - a)^{-(p-1)} \tag{1.25}$$

for every $b > a \geq 0$.

According to [Hol90, Theorem 3.25], given en exhaustion $\{U_i\}_{i \in \mathbb{N}}$ of p-regular domains and $o \in U_1$, one can choose a family $\{G_i(o, \cdot)\}_{i \in \mathbb{N}}$ such that $G_i(o, \cdot)$ is the p-Green's function on U_i and $G_i(o, \cdot) \leq G_{i+1}(o, \cdot)$ on U_i, for every $i \in \mathbb{N}$. Then by the Harnack's Principle Theorem 1.2.10 the pointwise limit G of G_i as $i \to +\infty$ can be either identically $+\infty$ or p-harmonic in $M \smallsetminus \{o\}$. In the latter case G is said to be the p-Green's of M.

It is clear that not every Riemannian manifold admits a p-Green's function. This is not anecdotal, even if $p = 2$, since for example in $\mathbb{R}^2$ there is no positive harmonic Green's function. Suppose that a manifold (M, g) admits a p-Green's function and let $a > 0$ be large enough so that $\{G(o, \cdot) \geq a\}$ is compact, then by (1.16) and (1.25)

$$\mathrm{Cap}_p(\{G(o, \cdot) \geq a\}) = \lim_{b \to +\infty} \mathrm{Cap}_p(\{G(o, \cdot) \geq a\}, \{G(o, \cdot) > b\}) = a^{-(p-1)} > 0$$

Hence, we can give the following definition.

Definition 1.3.6 (p-parabolic and p-nonparabolic manifolds)**.** *Let (M, g) be a complete non-compact n-dimensional Riemannian manifold and $p > 1$. (M, g) is* p-nonparabolic *if there exists a compact $K \subset M$ such that $\mathrm{Cap}_p(K) > 0$. (M, g) is* p-parabolic *if it is not p-nonparabolic.*

We have already proved that if (M, g) admits a p-Green's function, then it is p-nonparabolic. Suppose now that (M, g) is p-nonparabolic, let $K \subset M$ be a positive p-capacity compact subset. Consider then a closed ball $\overline{B}$ containing K and $y \in B$. Let $\{U_i\}_{i \in \mathbb{N}}$ be an exhaustion for M of p-regular domains such that $B \subset U_1$ and $\{G_i(o, \cdot)\}_{i \in \mathbb{N}}$

the increasing sequence of p-Green's function associated with each domain. Define $m_i = \min\{G_i(o,x) \mid x \in \partial B\}$. By the Comparison Principle Theorem 1.2.9 it follows that

$$\overline{B} \subset \{G_i(o,\cdot) \geq m_i\}.$$

Employing (1.25) and (1.18), we have that

$$m_i \leq \mathrm{Cap}_p(\overline{B}, U_i)^{-\frac{1}{p-1}} \leq \mathrm{Cap}_p(\overline{B})^{-\frac{1}{p-1}}.$$

In particular, the p-Green's function obtained taking the limit of $\{G_i(y,\cdot)\}_{i\in\mathbb{N}}$ cannot be identically $+\infty$. We have proved (see [Hol90; Hol92]) the following statement.

Theorem 1.3.7. *Let (M,g) be a complete noncompact n-dimensional Riemannian manifold and $p>1$. (M,g) admits a p-Green's function if and only if it is p-nonparabolic.*

Let (M,g) be a noncompact Riemannian manifold, and consider a compact K in M. Let E be a connected component of $M \smallsetminus K$. In [IPS17], the authors extend the potential theory to Riemannian manifolds with smooth boundary also relying on the extension theorem in [PV20]. However, it is not in our interest to develop such a theory. We define a condenser in E as the triple $(C,U;E)$ such that $U \subset M$ is open, C is closed in U and $(C \cup \partial E, U)$ is a condenser in $\overline{E}$. We define the p-capacity of $(C,U;E)$ the quantity

$$\mathrm{Cap}_p(C,U;E) = \inf\left\{ \int\limits_{U\cap\overline{E}} |\mathrm{D}v|^p \,\mathrm{d}\mu \;\middle|\; v \in \mathscr{C}^\infty_c(U),\, v \geq 1 \text{ on } C \cup \partial E \right\}.$$

When $U = M$ we will simply denote $\mathrm{Cap}_p(C;E) = \mathrm{Cap}_p(C,M;E)$. Observe that the C does not need to be bounded, only the $C \cap \overline{E}$ has to be. One can define a p-capacitary potential, exactly in the same way we did in the setting of complete Riemannian manifold. In particular, given (C,U) with C containing K and $u \in W^{1,p}_{\mathrm{loc}}(U \smallsetminus C)$ its p-capacitary potential, then the restriction of u to E is the p-capacitary potential of $(C,U;E)$. Suppose now that $M \smallsetminus K$ has a finite number of connected components. Denote them $E_1,\dots,E_N$. It is clear that if one of them, namely E_i, is bounded then

$$\mathrm{Cap}_p(\partial E_i; E_i) = 0.$$

Hence, the capacity of the compact K is influenced only by the unbounded components of $M \smallsetminus K$.

Definition 1.3.8 (Ends of manifolds)**.** *Let (M,g) be a noncompact n-dimensional Riemannian manifold. An* end *of a noncompact Riemannian manifold (M,g) with respect to a bounded subset $K \subseteq M$ is an unbounded connected component of $M \smallsetminus \overline{K}$. We say that (M,g) has a finite number of ends if the number of ends with respect to any bounded subset $C \subseteq M$ is bounded by a number m independent of C. In this case, we say that (M,g) has m ends and they will be denoted by $E_1,\dots,E_N$.*

Since now we are going to consider only Riemannian manifolds with at most a finite number of ends. We are now introducing the definition of p-parabolicity for the ends.

Definition 1.3.9. *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold and $p > 1$. An end E is said to be p-nonparabolic if $\mathrm{Cap}_p(C; E) > 0$ for some condenser $(C; E)$ in E. An end is said to be p-parabolic if it is not p-nonparabolic.*

The previous definition readily distinguishes unbounded components of $M \smallsetminus K$ that do not contribute to the p-capacity of K from one that does. Hence, p-nonparabolicity of the manifold is deeply related to the p-nonparabolicity of its ends.

Proposition 1.3.10. *Let (M, g) be a complete noncompact Riemannian manifold. (M, g) is p-nonparabolic if and only if at least one of its ends is p-nonparabolic.*

Proof. Suppose that M is p-nonparabolic. Then there exists a compact $C \subset M$ of positive p-capacity. Let $E_1, \dots, E_N$ be the ends of M with respect to K. Let u be the p-capacitary potential associated with the condenser $(C \cup K, M)$ then

$$\begin{aligned}\mathrm{Cap}_p(C \cup K, M) &= \int\limits_{M \smallsetminus (C \cup K)} |\mathrm{D}u|^p \,\mathrm{d}\mu = \sum_{i=1}^{m} \int\limits_{E_i \smallsetminus (C \cup K)} |\mathrm{D}u|^p \,\mathrm{d}\mu \\ &= \sum_{i=1}^{m} \mathrm{Cap}_p(C \cup K; E_i)\end{aligned}$$

where the last identity is given by the fact that the restriction of u to $E_i \smallsetminus (C \cup K)$ is the p-capacitary potential associated with the condenser $(C; E_i)$. Since the p-capacity is nonnegative, and the p-capacity of $(C \cup K, M)$ is strictly positive we have the desired implication. Following the same procedure with C such that $\mathrm{Cap}_p(C; E_i) > 0$, we get the other way round. □

1.3.3 Asymptotic behaviour of the p-Green's function

Here we want to study the asymptotic behaviour of the p-Green's function. The Harnack's Inequality (1.8) ensures that for a given nonnegative function the infimum cannot be to much far from its supremum. In many cases, this implies the uniform convergence on compact sets, as in the Harnack's Principle Theorem 1.2.10. We are seeing that if the constant in Harnack's Inequality is stable for a sequence of compact sets escaping to infinity, this is enough to grant that the p-Green's function admits a limit at infinity.

Definition 1.3.11 (Harnack end)**.** *Let (M, g) be a complete Riemannian manifold and E an end of M. The end E is said to be Harnack if there exists a sequence of compact subsets $(K_i)_{i \in \mathbb{N}}$ such that ∂E belongs to a bounded component of $M \smallsetminus K_i$, for every compact K in M it holds that $K_i \subseteq E \smallsetminus K$ for every i large enough and there exists a constant $\mathrm{C}_H = \mathrm{C}_h(n, p)$ such that*

$$\sup_{K_i} v \leq \mathrm{C}_H \inf_{K_i} v$$

holds for every nonnegative p-harmonic function v in E.

The following result is proved in [Hol99, Lemma 3.2] (see also [Hol94, Lemma 3.23]),

Lemma 1.3.12. *Let E be a Harnack end of (M, g) and $o \in M$. Then for any nonnegative p-harmonic function v on E there exists a real $a \in [0, +\infty]$ such that*

$$\lim_{\substack{d(o,x) \to +\infty \\ x \in E}} v = a.$$

Let (M, g) be a complete Riemannian manifold and $o \in M$. Let E be an end of M with respect to some bounded $K \subset B(o, R)$. According to [Hol99], the end is said to be *p-large* (resp. *p-small*) if

$$\int_R^{+\infty} \left(\frac{t}{|B(o,t) \cap E|} \right)^{\frac{1}{p-1}} \mathrm{d}t < +\infty \qquad \text{(resp. } = +\infty\text{).} \tag{1.26}$$

The definition is clearly independent of the choice of o and R. Moreover, this definition is related to the notion of p-parabolic and p-nonparabolic manifold (see [Var81; Gri83; Gri87; Hol90; KZ96; CHS01]).

Proposition 1.3.13. *Let (M, g) be a complete n-dimensional Riemannian manifold and $p > 1$. Every p-small end is p-parabolic. In particular, M is p-parabolic if*

$$\int_R^{+\infty} \left(\frac{t}{|B(o,t)|} \right)^{\frac{1}{p-1}} \mathrm{d}t = +\infty.$$

The converse is not true in general. Inspired by [Var83], Holopainen [Hol99] presents as a counterexample a conformal version of $\mathbb{R}^n$ that is p-large for every $p > 1$, but it is n-parabolic. A class of ends for which being p-parabolic implies being p-small is the class of partially homogeneous ends. We report here the definition for ease of future references.

Definition 1.3.14 (Homogeneous ends)**.** *Let (M, g) be a complete Riemannian manifold, $o \in M$ and E an end of M. E is said to be* homogeneous *(resp.* partially homogeneous*) if*

(i) A weak $(1, p)$-Poincaré Inequality *is satisfied, namely, there exists a positive constant C_P such that for every $R > 0$ and $x \in E \setminus B(o, 2R)$ with $B(x, R) \subset E$ and for every $B(y, 2r) \subset B(x, R)$ we have that*

$$\fint_{B(y,r)} |v - \overline{v}| \, \mathrm{d}\mu \leq C_P \, r \left(\fint_{B(y,2r)} |\mathrm{D}v|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \qquad \textit{with } \overline{v} = \fint_{B(y,r)} v \, \mathrm{d}\mu,$$

holds whenever $v \in W^{1,p}(B(y, 2r))$.

(ii) A volume-doubling property *is satisfied, namely, there exists a constant C_D such that for every $R > 0$ and $x \in E \setminus B(o, 2R)$ with $B(x, R) \subset E$ and for every $B(y, 2r) \subset B(x, R)$*

$$|B(y, 2r)| \leq C_D \, |B(y, r)|.$$

(iii) *A* volume comparison *(resp.* partial volume comparison*) condition is satisfied, namely, there exists a constant* C_V *and* $R > 0$ *such that*

$$|B(o,r) \cap E| \leq C_V\,|B(x,r/8)|$$

holds for every $r \geq R$ *and* $x \in \partial B(o,r) \cap E$ *(resp.* $x \in \partial E(r)$ *where* $E(r)$ *is the unbounded component of* $E \smallsetminus \overline{B(o,r)}$*).*

The following theorem is a consequence of [Hol99, Theorem 4.7].

Proposition 1.3.15. *Let* (M,g) *be a* n*-dimensional Riemannian manifold,* $p > 1$ *and* E *an end of* M*. If* E *is partially homogeneous and* p*-parabolic then* E *is* p*-small.*

As a consequence, on a partially homogeneous end the integral in (1.26) is finite if and only if the end is p-nonparabolic. Since this condition is equivalent to the existence of a p-Green's function, it is not surprising that this two quantities are somehow related. In the following we analyse this relation.

Proposition 1.3.16. *Let* (M,g) *be a* n*-dimensional Riemannian manifold with a homogeneous* p*-large end* E *for some* $p > 1$*. Let* $o \in M \smallsetminus E$ *and* $G(o,\cdot) \in W^{1,p}_{\mathrm{loc}}(M \smallsetminus \{o\}) \cap \mathscr{C}^0(M \smallsetminus \{o\})$ *be the* p*-Green's function of* M*. Then there exists a constant* $C = C(p, C_D, C_P, C_V)$ *and* $R > 0$ *such that*

$$\sup_{x \in E(r)} G(o,x) \leq C \int_{2r}^{+\infty} \left(\frac{t}{|B(o,t) \cap E|}\right)^{\frac{1}{p-1}} \mathrm{d}t < +\infty \tag{1.27}$$

holds for every $r \geq R$*, where* $E(r)$ *is the unbounded component of* $E \smallsetminus \overline{B(o,r)}$*. If the end* E *is Harnack, then* $G(o,x) \to 0$ *as* $\mathrm{d}(o,x) \to +\infty$ *on* E*.*

Proof. The proof of (1.27) is contained in [Hol99, Proposition 5.7]. It follows that there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ in E such that $G(o,x_n) \to 0$ as $n \to +\infty$. Then Lemma 1.3.12 concludes the proof. □

We can also prove a lower bound for the p-Green's function. The proof is inspired to [Hol99, Proposition 5.9]. However, here we aim to a lower bound on each p-large homogeneous end, rather than on all of them.

Proposition 1.3.17. *Let* (M,g) *be a* n*-dimensional Riemannian manifold with a homogeneous* p*-large end* E *for some* $p > 1$*. Let* $o \in M \smallsetminus E$ *and* $G(o,\cdot) \in W^{1,p}_{\mathrm{loc}}(M \smallsetminus \{o\}) \cap \mathscr{C}^0(M \smallsetminus \{o\})$ *be a* p*-Green's function of* M*. Then, there exists a constant* C *depending on* p *and* E *and* $R > 0$ *such that*

$$\sup_{x \in \partial E(r)} G(o,x) \geq C^{\frac{1}{p-1}} \int_{2r}^{+\infty} \left(\frac{t}{|B(o,t) \cap E|}\right)^{\frac{1}{p-1}} \mathrm{d}t \tag{1.28}$$

for every $r \geq R$*, where* $E(r)$ *is the unbounded component of* $E \smallsetminus \overline{B(o,r)}$*. Moreover, the constant can be chosen as*

$$C = a\ \mathrm{Cap}_p(\partial E; E), \tag{1.29}$$

where $a > 0$ *is such that* $\left\{G(o,\cdot) \geq a^{1/(p-1)}\right\}$*.*

Proof. Denote $\mathrm{Cap}_p(\{G(o,x) \geq d\}; E) = c^{-(p-1)} > 0$, where $d > 0$ is such that the set $\{G(o,x) \geq d\}$ contains ∂E. Define $\overline{G} = cd^{-1}G$. Employing (1.25), we obtain that

$$\begin{aligned}\mathrm{Cap}_p(\{\overline{G}(o,x) \geq b\}; E) &= \mathrm{Cap}_p(\left\{G(o,x) \geq dc^{-1}b\right\}; E) \\ &= \left(\frac{c}{b}\right)^{p-1} \mathrm{Cap}_p(\{G(o,x) \geq d\}; E) = b^{-(p-1)}\end{aligned}$$

holds for every $b > 0$. Define $M_r = \max\{\overline{G}(o,x) \,|\, x \in \partial E(r)\}$. For r large enough we can assume that both $\{\overline{G}(o,\cdot\,) \geq M_r\}$ and $\overline{B(o,r)}$ are closed and contain ∂E. By the Comparison Principle Theorem 1.2.9 yields $\{\overline{G}(o,\cdot\,) \geq M_r\} \cap E \subset \overline{B(o,r)} \cap E$. The monotonicity of the p-capacity and (1.25) give

$$\mathrm{Cap}_p(\overline{B(o,r)}; E) = \mathrm{Cap}_p(E \setminus E(r); E) \geq \mathrm{Cap}_p(\{\overline{G}(o,\cdot\,) \geq M_r\}; E) = M_r^{-(p-1)}.$$

Since for every $s > t > 0$ the function $(\mathrm{d}(o,\cdot) - s)/(t-s)$ is Lipschitz (in particular belongs to $W^{1,p}_{\mathrm{loc}}(M)$ for $p > 1$), approximating it with test functions in (1.12), one can prove the following estimate

$$\mathrm{Cap}_p(\overline{B(o,t)}, B(o,s); E) \leq \frac{|B(o,s) \cap E|}{t^p}$$

whenever $\overline{B(o,t)} \supset \partial E$. Using [HKM18, Theorem 5.32] we immediately get that for every $k > 0$

$$\begin{aligned} M_r &\geq \left(\mathrm{Cap}_p(\overline{B(o,r)}, B(o,2^k r)\right)^{-\frac{1}{p-1}} \geq \sum_{j=0}^{k-1} \left(\mathrm{Cap}_p(\overline{B(o,2^j r)}, B(o,2^{j+1}r))\right)^{-\frac{1}{p-1}} \\ &\geq \sum_{j=0}^{k-1} \left(\frac{(2^j r)^p}{|B(o,2^{j+1}r) \cap E|}\right)^{\frac{1}{p-1}} \geq \sum_{j=0}^{k-1} \int_{2^{j+1}r}^{2^{j+2}r} \left(\frac{t}{|B(o,t) \cap E|}\right)^{\frac{1}{p-1}} \mathrm{d}t \\ &= \int_{2r}^{2^{k+1}r} \left(\frac{t}{|B(o,t) \cap E|}\right)^{\frac{1}{p-1}} \mathrm{d}t. \end{aligned}$$

Taking the limit as $k \to +\infty$ and recalling the relation between G and $\overline{G}$, we get (1.28) with the constant $\mathrm{C} = dc^{-1}$. Choosing $d = a^{1/(p-1)}$ and using the monotonicity of the p-capacity by inclusion we get the constant in (1.29). □

Observe that, if a Harnack's Inequality holds on $\partial E(r)$ and the constant is independent of r, it is possible to deduce a lower bound on the p-Green's function from (1.28). In general Propositions 1.3.16 and 1.3.17 cannot be extended to a bound on $\partial B(o,r)$, since, even if (M,g) has only one end, infinitely many bounded components may appear approaching infinity.

2

Asymptotic behaviour of the p-capacitary potential and the IMCF

2.1 Structure of the chapter

This chapter is devoted to describing the asymptotic behaviour of the p-capacitary potential and the IMCF, which are the main results in [BFM22]. In Section 2.2, we report the precise definition of Asymptotically Conical Riemannian manifolds and all properties that will be useful in the subsequent sections. In Section 2.3, we study the existence and estimates of a p-capacitary potential associated to some open bounded subset Ω with smooth boundary. We then specialise these results to the setting of Riemannian manifolds with nonnegative Ricci curvature and Asymptotically Conical Riemannian manifolds. This section also contains the asymptotic behaviour of the p-capacitary potential and its derivatives on Asymptotically Conical Riemannian manifolds. The last section highlights the relation between the Nonlinear Potential Theory and the Inverse Mean Curvature Flow. For the sake of completeness, we re-prove here the existence of the (weak) IMCF on Riemannian manifolds with nonnegative Ricci curvature. We conclude by specialising and improving the results in [MRS19] in our Asymptotically Conical Riemannian setting, inferring also the asymptotic behaviour of the (weak) IMCF, characterising the limit in [HI01, Lemma 7.1].

2.2 Asymptotically Conical Riemannian manifolds

We give here the precise definition of Asymptotically Conical Riemannian manifolds according to [CEV17]. For better comprehension, we recall the definition of the Hölder seminorm of a tensor field. A tensor field $T \in \mathcal{T}^q_s(M)$ is α-Hölder continuous at x for some $\alpha \in [0,1]$ if there exists a geodesically convex open neighbourhood U_x centred at x such that

$$\sup_{y \in U_x \smallsetminus \{x\}} \frac{\left|T(x) - T(y)\right|_g}{\left(\mathrm{d}(x,y)\right)^{\alpha}} < +\infty$$

is finite, where, to compute the difference between $T(x)$ and $T(y)$, we parallel transport $T(y)$ onto x. The tensor field T is said to be α-Hölder continuous on $U \subset M$ if it is α-

Hölder continuous at every $x \in U$. We sometimes omit the subscript g if it is clear the metric we are referring to.

Consider a *cone with link* L, namely $((0,+\infty) \times L, \hat{g})$ where $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$. In this case, let $s > 0$ be such that $B_s(x)$ is geodesically convex in $((0,+\infty) \times L, \hat{g})$ for every $x \in \{1\} \times L$. Then, for every $x \in (0,+\infty) \times L$ the ball of radius $s\rho(x)$ centred at x is still geodesically convex, where $\rho : (0,+\infty) \times L \to (0,+\infty)$ is the projection onto the first coordinate. Given an α-Hölder continuous tensor field T, we define the α-Hölder seminorm of T at x as

$$[T]^{(s)}_{\alpha,\hat{g}}(x) = \sup_{y \in B_{s\rho(x)}(x) \smallsetminus \{x\}} \frac{|T(x) - T(y)|_{\hat{g}}}{(\mathrm{d}(x,y))^{\alpha}}.$$

Observe that, if T is bounded (with respect to $|\cdot|_{\hat{g}}$) and $s, t > 0$ satisfy the above assumptions, $[T]^{(s)}_{\alpha,\hat{g}}(x) = [T]^{(t)}_{\alpha,\hat{g}}(x)$ for any $x \in (R,+\infty) \times L$ provided R is large enough. Then, the following definition is well-posed and we can drop the superscript (s).

Definition 2.2.1 ($\mathscr{C}^{k,\alpha}$-Asymptotically Conical Riemannian manifolds)**.** *Let (M,g) be a Riemannian manifold, $k \in \mathbb{N}$ and $\alpha \in [0,1]$. M is said to be $\mathscr{C}^{k,\alpha}$*-Asymptotically Conical *if there exists an open bounded subset $K \subseteq M$, a closed smooth hypersurface L and a diffeomorphism $\hat{\pi} : M \smallsetminus K \to [1,+\infty) \times L$ such that*

$$\sum_{i=0}^{k} \rho^i \left| \mathrm{D}^{(i)}_{\hat{g}}(\hat{\pi}_* g - \hat{g}) \right|_{\hat{g}} + \rho^{k+\alpha} \left[\mathrm{D}^{(k)}_{\hat{g}}(\hat{\pi}_* g - \hat{g}) \right]_{\alpha,\hat{g}} = o(1) \qquad \text{as } \rho \to +\infty, \tag{2.1}$$

where $\rho : [1,+\infty) \times L \to [1,+\infty)$ is the projection map onto the first component and $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ is the cone metric. In the case $\alpha = 0$, we use the notation $\mathscr{C}^k$-Asymptotically Conical instead of $\mathscr{C}^{k,0}$-Asymptotically Conical.

The previous definition says that in a $\mathscr{C}^{k,\alpha}$-Asymptotically Conical Manifold the metric g approaches the metric $\hat{g}$ of a truncated cone with link L with respect to a scale-invariant $\mathscr{C}^{k,\alpha}$-norm. The diffeomorphism $\hat{\pi} : M \smallsetminus K \to L \times [1,+\infty)$ identifies the boundary of K with the link L. With abuse of notation, $\hat{\pi}_* \rho : M \smallsetminus K \to [1,+\infty)$ will be denoted by ρ and $\hat{\pi}_* \hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ by $\hat{g}$. Moreover, by convention $\rho < 1$ on K and accordingly $\{\rho \leq r\} = M \smallsetminus \{\rho > r\}$ and $\{1 \leq \rho \leq r\} = M \smallsetminus (\{\rho > r\} \cup K)$. Given any coordinate system $(\vartheta^1, \dots, \vartheta^{n-1})$ on an open subset U of L, $(\rho, \vartheta^1, \dots, \vartheta^{n-1})$ are coordinates on $(1,+\infty) \times U \subset M \smallsetminus \overline{K}$. The condition $|g - \hat{g}|_{\hat{g}} = o(1)$ as $\rho \to +\infty$ is equivalent to a condition on the coordinates that can be read as

$$g_{\rho\rho} = 1 + o(1), \qquad g_{\rho j} = o(\rho), \qquad g_{ij} = \rho^2 g^L_{ij} + o(\rho^2),$$

for every $i, j = 1, \dots, n-1$ as $\rho \to +\infty$. By using Cramer's rule to solve the system and the Laplace expansion to compute determinants, we obtain

$$g^{\rho\rho} = 1 + o(1), \qquad g^{\rho j} = o(\rho^{-1}), \qquad g^{ij} = \rho^{-2} g_L^{ij} + o(\rho^{-2}).$$

The $\mathscr{C}^{0,\alpha}$-Asymptotically Conical condition for $\alpha > 0$ gives, in addition, information on the Hölder seminorm of the components. Indeed, arguing as before we get that

$$[g_{\rho\rho} - 1]_{\alpha,\hat{g}} = o(\rho^{-\alpha}), \qquad [g_{\rho j}]_{\alpha,\hat{g}} = o(\rho^{1-\alpha}), \qquad \left[g_{ij} - \rho^2 g^L_{ij} \right]_{\alpha,\hat{g}} = o(\rho^{2-\alpha}),$$

for every $i, j = 1, \dots, n-1$ as $\rho \to +\infty$. Increasing k in the $\mathscr{C}^{k,\alpha}$-Asymptotically Conical assumption we gain knowledge about the k-th derivative of the components of g.

Consider for every $s > 0$ the family of diffeomorphism on $(0, +\infty) \times L$ defined as

$$\begin{aligned} \omega_s : \quad (0,+\infty) \times L &\longrightarrow (0,+\infty) \times L \\ (\rho, \vartheta^1, \dots, \vartheta^{n-1}) &\longmapsto (s\rho, \vartheta^1, \dots, \vartheta^{n-1}), \end{aligned} \tag{2.2}$$

With abuse of language, we will also denote by ω_s any restriction of it to some truncated cone. Since ω_s induces a family of diffeomorphisms from $(1/s, +\infty) \times L$ onto $\{\rho \geq 1\} \subset M$ through the composition with π in Definition 2.2.1, we will also denote by ω_s such map. Condition (2.1) can be also interpreted as the convergence of the family of metrics on the cone $(0, +\infty) \times L$, built for every $s \geq 1$ by pulling the metric g back through the diffeomorphism ω_s and properly rescaling them. This is the content of the following lemma.

Lemma 2.2.2. *Let (M, g) be a complete n-dimensional Riemannian manifold. Then, (M, g) is $\mathscr{C}^{k,\alpha}$-Asymptotically Conical if and only if the metric $g_{(s)} = s^{-2}\omega_s^* g$ satisfies*

$$\sum_{i=0}^{k} \rho^i \left| \mathrm{D}_{\hat{g}}^{(i)}(g_{(s)} - \hat{g}) \right|_{\hat{g}} + \rho^{k+\alpha} \left[\mathrm{D}_{\hat{g}}^{(k)}(g_{(s)} - \hat{g}) \right]_{\alpha, \hat{g}} = o(1) \qquad \text{as } s \to +\infty,$$

uniformly on $[R, +\infty) \times L$ for every $R > 0$.

Proof. Since $\omega_s^* \mathrm{d}\rho = r\mathrm{d}\rho$ and $\omega_s^* \mathrm{d}\vartheta^i = \mathrm{d}\vartheta^i$, is clear that $s^{-2}\omega_s^* \hat{g} = \hat{g}$. Thus the the case of $\mathscr{C}^0$-Asymptotically Conical manifold follows from algebra operations on tensors. The result for $k \in \mathbb{N}$ and $\alpha \in [0, 1]$ follows in the same way from the fact that $\mathrm{D}_{\hat{g}} s^{-2}\omega_s^* g = s^{-2}\omega_s^*(\mathrm{D}_{\hat{g}} g)$ and $\mathrm{d}_{\hat{g}}(x, y) = \mathrm{d}_{\hat{g}}(\omega_s(x), \omega_s(y))/s$ for every $x, y \in (1/s, +\infty) \times L$. □

We highlight the relation between the coordinate ρ and the distance induced by g on M.

Lemma 2.2.3. *Let (M, g) be a n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold and $o \in M$. Then*

$$\lim_{\mathrm{d}(o,x) \to +\infty} \frac{\mathrm{d}(o, x)}{\rho(x)} = 1. \tag{2.3}$$

Since K is compact, there exists a $R > 0$ such that $\mathrm{d}(o, x) > R$ implies $x \in M \smallsetminus K$, hence (2.3) makes sense. Since π is a diffeomorphism, taking the limit for $\mathrm{d}(o, x) \to +\infty$ is the same of taking it for $\rho(x) \to +\infty$.

Proof. Since $|\mathrm{D}\rho|_g = 1 + o(1)$, for every $\varepsilon > 0$ we can find $R_\varepsilon > 1$ such that $1 - \varepsilon \leq |\mathrm{D}\rho|_g \leq 1 + \varepsilon$ on $\{\rho \geq R_\varepsilon\}$. Pick $x \in \{\rho \geq R_\varepsilon\}$ and a curve $\gamma : [R_\varepsilon, \rho(x)] \to M$ which is the solution to the problem

$$\begin{cases} \dot{\gamma}(s) = \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|^2}(\gamma(s)), \\ \gamma(\rho(x)) = x. \end{cases}$$

Computing the length of γ we get

$$L(\gamma) = \int_{\rho(x)}^{R_\varepsilon} |\dot{\gamma}(s)| \, \mathrm{d}s = \int_{\rho(x)}^{R_\varepsilon} \frac{1}{|\mathrm{D}\rho|_g}(\gamma(s)) \, \mathrm{d}s \leq \frac{\rho(x) - R_\varepsilon}{1-\varepsilon},$$

which ensures that

$$\limsup_{\rho(x)\to+\infty} \frac{\mathrm{d}(o,x)}{\rho(x)} \leq \limsup_{\rho(x)\to+\infty} \frac{L(\gamma) + 2\,\mathrm{diam}(\{\rho \leq R_\varepsilon\})}{\rho(x)} \leq \frac{1}{1-\varepsilon}.$$

Conversely, consider any geodesic $\sigma : [0, L] \to M$, parametrised by arc length, joining $\sigma(0) \in \{\rho = R_\varepsilon\}$ and $\sigma(L) = x$. Then we obtain

$$\rho(x) - R_\varepsilon = \int_0^L \langle \mathrm{D}\rho \,|\, \dot{\sigma}(s)\rangle \, \mathrm{d}s \leq \int_0^L |\mathrm{D}\rho|_g(\sigma(s)) \, \mathrm{d}s \leq (1+\varepsilon)L,$$

which yields

$$\begin{aligned}\liminf_{\rho(x)\to+\infty} \frac{\mathrm{d}(x,o)}{\rho(x)} &\geq \liminf_{\rho(x)\to+\infty} \frac{L - 2\,\mathrm{diam}(\{\rho \leq R_\varepsilon\})}{\rho(x)} \\ &\geq \liminf_{\rho(x)\to+\infty} \frac{\rho(x) - R_\varepsilon - 2\,\mathrm{diam}(\{\rho \leq R_\varepsilon\})}{(1+\varepsilon)\rho(x)} = \frac{1}{1+\varepsilon}.\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ we can conclude. □

In Riemannian manifolds with nonnegative Ricci curvature, in virtue of the Bishop-Gromov Theorem, one can define an Asymptotic Volume Ratio since

$$\mathrm{AVR}(g) = \lim_{r\to+\infty} \frac{|B(o,R)|}{|\mathbb{B}^n| R^n}$$

exists and does not depend on $o \in M$. Here we relaxed the condition on Ricci curvature so we cannot apply Bishop-Gromov Theorem, but on the other side we require an asymptotic behaviour for the metric, hence we can hope in defining an Asymptotic Volume Ratio as well.

Lemma 2.2.4. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Then*

$$\frac{|L|}{|\mathbb{S}^{n-1}|} = \lim_{R\to+\infty} \frac{|\{\rho \leq R\}|}{|\mathbb{B}^n| R^n} = \lim_{R\to+\infty} \frac{|\{\rho = R\}|}{|\mathbb{S}^{n-1}| R^{n-1}}, \tag{2.4}$$

where L is the link of the cone (M, g) is asymptotic to.

Proof. One can easily show that $\det(g) = \det(\hat{g})(1 + o(1)) = \rho^{2(n-1)} \det(g_L)(1 + o(1))$ as $\rho \to +\infty$. Hence, for every $\varepsilon > 0$ there exists $R_\varepsilon \geq 1$ such that

$$|L|(1-\varepsilon)\frac{(R^n - R_\varepsilon^n)}{n} \leq |\{R_\varepsilon \leq \rho \leq R\}| \leq |L|(1+\varepsilon)\frac{(R^n - R_\varepsilon^n)}{n}.$$

Dividing each term by $|\mathbb{B}^n|R^n$ one gets

$$|L|(1-\varepsilon)\frac{(R^n-R_\varepsilon^n)}{n|\mathbb{B}^n|R^n} \leq \frac{|\{R_\varepsilon \leq \rho \leq R\}|}{|\mathbb{B}^n|R^n} \leq |L|(1+\varepsilon)\frac{(R^n-R_\varepsilon^n)}{n|\mathbb{B}^n|R^n}.$$

Since $|\{1 \leq \rho \leq R_\varepsilon\}|/(|\mathbb{B}^n|R^n)$ vanishes as $R \to +\infty$ we obtain

$$(1-\varepsilon)\frac{|L|}{n|\mathbb{B}^n|} \leq \lim_{R\to+\infty} \frac{|\{1 \leq \rho \leq R\}|}{|\mathbb{B}^n|R^n} \leq (1+\varepsilon)\frac{|L|}{n|\mathbb{B}^n|},$$

which in turn gives the first identity in (2.4) by arbitrariness of ε. We now turn to prove the second identity. Since De L'Hôpital rule gives

$$\lim_{R\to+\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}R}|\{1 \leq \rho \leq R\}|}{|\mathbb{S}^{n-1}|R^{n-1}} = \lim_{R\to+\infty} \frac{|\{1 \leq \rho \leq R\}|}{|\mathbb{B}^n|R^n} = \frac{|L|}{|\mathbb{S}^{n-1}|}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}R}|\{1 \leq \rho \leq R\}| = \frac{\mathrm{d}}{\mathrm{d}R}\int_1^R \int_{\{\rho=s\}} \frac{1}{|\mathrm{D}\rho|}\,\mathrm{d}\sigma\mathrm{d}s = |\{\rho = R\}|(1+o(1)),$$

we conclude the proof. □

Coupling this result with Lemma 2.2.3 one gets that

$$\lim_{R\to+\infty} \frac{|B(o,R)|}{R^n|\mathbb{B}^n|} = \lim_{R\to+\infty} \frac{|\{\rho \leq R\}|}{|\mathbb{B}^n|R^n} = \frac{|L|}{|\mathbb{S}^{n-1}|},$$

for every $o \in M$. Hence the left hand side limit exists and does not depend on the point $o \in M$. We can lastly give the following definition.

Definition 2.2.5. *Let (M,g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. The* Asymptotic Volume Ratio *of (M,g) is defined as*

$$\mathrm{AVR}(g) = \frac{|L|}{|\mathbb{S}^{n-1}|},$$

where L is the link of the cone (M,g) is asymptotic to.

In this case $\mathrm{AVR}(g) > 0$, but in general $\mathrm{AVR}(g)$ could exceed 1 and $\mathrm{AVR}(g) = 1$ does not imply that (M,g) is isometric to the flat Euclidean space.

Given the conical metric $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ and the change of coordinates $\rho = \mathrm{e}^t$, we have $\hat{g} = \mathrm{e}^{2t}\,\mathrm{d}t^2 + \mathrm{e}^{2t}\,g_L = \mathrm{e}^{2t}(\mathrm{d}t^2 + g_L)$. Hence, every conical metric $\hat{g}$ is conformally equivalent to the cylindrical metric $\overline{g} = \mathrm{d}t^2 + g_L$. It is reasonable to define the cylindrical counterpart of Definition 2.2.1. To this purpose, we have to define the α-Hölder seminorm onto cylinders. Consider a *cylinder on L*, namely $(\mathbb{R} \times L, \overline{g})$. In this case, let $s > 0$ be such that $B_s(x)$ is geodesically convex in $(\mathbb{R} \times L, \overline{g})$ for every $x \in \{0\} \times L$. Then, for every $x \in \mathbb{R} \times L$ the ball of radius s centred at x is still geodesically convex. We define the α-Hölder seminorm of T at x as

$$[T]^{(s)}_{\alpha,\overline{g}}(x) = \sup_{y \in B_{s(x)}(x)\setminus\{x\}} \frac{|T(x) - T(y)|_{\overline{g}}}{(\mathrm{d}(x,y))^\alpha}.$$

Observe that, if T vanishes as $t \to +\infty$ and $s, t > 0$ satisfy the above assumptions, $[T]^{(s)}_{\alpha,\overline{g}}(x) = [T]^{(t)}_{\alpha,\overline{g}}(x)$ for any $x \in (R, +\infty) \times L$ provided R is large enough. Then, the following definition is well-posed and we can drop the superscript (s).

Definition 2.2.6 ($\mathscr{C}^{k,\alpha}$-Asymptotically Cylindrical Riemannian manifolds)**.** *Let (M, g) be a Riemannian manifold, $k \in \mathbb{N}$ and $\alpha \in [0, 1]$. M is said to be $\mathscr{C}^{k,\alpha}$-*Asymptotically Cylindrical *if there exists an open bounded subset $K \subseteq M$, a closed smooth hypersurface L and a diffeomorphism $\overline{\pi} : M \smallsetminus K \to [0, +\infty) \times L$ such that*

$$\sum_{i=0}^{k} \left| \mathrm{D}^{(i)}_{\overline{g}} (\overline{\pi}_* g - \overline{g}) \right|_{\overline{g}} + \left[\mathrm{D}^{(k)}_{\overline{g}} (\overline{\pi}_* g - \overline{g}) \right]_{\alpha,\overline{g}} = o(1) \qquad \textit{as } t \to +\infty,$$

where $t : [0, +\infty) \times L \to [0, +\infty)$ is the projection map onto the first component and $\overline{g} = \mathrm{d}t^2 + g_L$ is the cone metric. In the case $\alpha = 0$, we use the notation $\mathscr{C}^k$-Asymptotically Cylindrical instead of $\mathscr{C}^{k,0}$-Asymptotically Cylindrical.

As before, with an abuse of notation, $\overline{\pi}_* t : M \smallsetminus K \to [\mathrm{e}, +\infty)$ will be denoted by t and $\overline{\pi}_* \overline{g} = \mathrm{d}t^2 + g_L$ by $\overline{g}$. As before, the condition $|g - \overline{g}|_{\overline{g}} = o(1)$ is equivalent to a condition on the coordinates that can be read as

$$g_{tt} = 1 + o(1), \qquad g_{tj} = o(1), \qquad g_{ij} = g^L_{ij} + o(1).$$

Suppose (M, g) is a $\mathscr{C}^{k,\alpha}$-Asymptotically Conical Riemannian maniofold. Composing the change of coordinates $\rho = \mathrm{e}^t$ and the diffeomorphism $\hat{\pi}$ of Definition 2.2.1 we obtain the map $\overline{\pi}$ of Definition 2.2.6. Moreover,

$$\mathrm{e}^{-2t} g_{tt} = 1 + o(1), \qquad \mathrm{e}^{-2t} g_{tj} = o(1), \qquad \mathrm{e}^{-2t} g_{ij} = g^L_{ij} + o(1),$$

and the same behaviour occurs for the derivatives. Hence, the following computational lemma holds true.

Lemma 2.2.7. *A complete Riemannian manifold (M, g) is $\mathscr{C}^{k,\alpha}$-Asymptotically Conical if and only if it is conformally equivalent to a $\mathscr{C}^{k,\alpha}$-Asymptotically Cylindrical Riemannian manifold.*

Lemma 2.2.2 can be rephrased in this context, where ω_s in (2.2) is suitably replaced with a family of translations and accordingly no rescaling is needed. Clearly Lemma 2.2.3 and Lemma 2.2.4 hold as well with e^t in the place of ρ.

As already mentioned, we will often require the following curvature constraint

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(\mathrm{d}(x, o) + 1)^2} \tag{2.5}$$

for every $x \in M$, where we fixed $o \in M$. Since the right hand side of (2.5) converges to 0 as $\mathrm{d}(x, o) \to +\infty$, one may wonder whether the limit cone has nonnegative Ricci curvature. Trivially if the link of the limit cone L satisfies $\mathrm{Ric}_L \geq (n-2) g_L$, then the limit con has nonnegative Ricci curvature. If the geometry of the link is not known, one can assume a better speed rate of convergence for the metric in order to pass to the limit (2.5). Since the Ricci curvature depends on the second order derivatives of the metric, $\mathscr{C}^2$-convergence is enough. In the next proposition, we reduce this assumption in favour of $\mathscr{C}^0$-convergence (see Appendix A for a smooth approach when the convergence is in the $\mathscr{C}^1$-topology).

Lemma 2.2.8. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Suppose that* $\mathrm{Ric} \geq -f(\mathrm{d}(x, o))$ *for some nonnegative smooth function $f(t) = o(1)$ as $t \to +\infty$, for some $o \in M$. Then $\mathrm{Ric}_{\hat{g}} \geq 0$, where $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ is the asymptotic conic metric of g and L is the link of the limit cone.*

Proof. By Lemma 2.2.2 we can assume that f is a function of ρ. Let ω_s be as in (2.2), we denote by $g_{(s)}$ the metric $s^{-2}\omega_s^* g$ on $[1/s, +\infty) \times L$. It is easy to prove that $([1/s, +\infty) \times L, \mathrm{d}_{g_{(s)}}, x)$ converges in the pointed-Gromov-Hausdorff topology to $([0, +\infty) \times L, \mathrm{d}_{\hat{g}}, x)$ for some $x \in \{\rho = 2\}$. Since

$$\lim_{s \to +\infty} |B(x, 1)|_{g_{(s)}} = |B(x, 1)|_{\hat{g}},$$

by [DG18, Theorem 1.2] $([1/s, +\infty) \times L, \mathrm{d}_{g_{(s)}}, \mu_{g_{(s)}}, x)$ converges to $([0, +\infty) \times L, \mathrm{d}_{\hat{g}}, \mu_{\hat{g}}, x)$ in the pointed-measured-Gromov-Hausdorff topology. By [GMS15, Theorem 7.2] $\mathrm{Ric}_{\hat{g}} \geq -f(s)$ for every s, hence $\mathrm{Ric}_{\hat{g}} \geq 0$. □

A direct consequence of the previous lemma is that a Euclidean-like Isoperimetric Inequality with positive constant holds on Asymptotically Conical manifolds such that $\mathrm{Ric} \geq -f(\mathrm{d}(x, o))$, namely there exists a positive constant $\mathrm{C}_I > 0$ such that

$$\mathrm{C}_I \leq \frac{|\partial K|^n}{|K|^{n-1}}$$

for every compact domain K. Indeed, we know that in manifolds with nonnegative Ricci curvature and Euclidean volume growth this inequality is true, as observed by [Var85] and proved very recently with a sharp constant by [Bre22] (see also [Car94] and [Heb99, Theorem 8.4]). Being this inequality true on the asymptotic cone, it is plausible that it can be transferred to our setting.

It is well-known that the existence of a positive Isoperimetric constant is equivalent to the existence of a finite constant for a global L^1-Sobolev Inequality, which is $\mathrm{C}_S < +\infty$ depending only on the geometry of the manifold such that

$$\left(\int_M |\varphi|^{\frac{n}{n-1}} \, \mathrm{d}\mu \right)^{\frac{n-1}{n}} \leq \mathrm{C}_S \int_M |\mathrm{D}\varphi| \, \mathrm{d}\mu \tag{2.6}$$

for every $\varphi \in \mathrm{Lip}_c(M)$ (see [FF60, Remark 6.6] or [SY94, pp. 89-90]). Applying the above inequality to the function $f^{p\frac{n-1}{n-p}}$ and using Hölder's Inequality one obtains the L^p-Sobolev Inequality

$$\left(\int_M |\varphi|^{\frac{np}{n-p}} \, \mathrm{d}\mu \right)^{\frac{n-p}{np}} \leq \mathrm{C}_{S,p} \left(\int_M |\mathrm{D}\varphi|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

for every $\varphi \in \mathrm{Lip}_c(M)$, where $\mathrm{C}_{S,p}$ depends only C_S, n and p. In the following proposition we prove that all these inequalities are in force in the above considered setting.

Proposition 2.2.9. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Suppose that $\mathrm{Ric}_g \geq -f(\mathrm{d}(x, o))$ for some nonnegative smooth function $f(t) = o(1)$ as $t \to +\infty$ and some $o \in M$. Then (M, g) admits a global L^1-Sobolev Inequality (2.6) for some finite constant C_S or equivalently it has a positive Isoperimetric constant $\mathrm{C}_I > 0$.*

Proof. In virtue of [PST14, Theorem 3.2] it is enough to prove that a L^1-Soboelv Inequality is satisfied outside some compact set. By Lemma 2.3.11, (M, g) has only a finite number of ends each of them corresponding to one connected component of the link L. Thus, we can assume that (M, g) has a single Asymptotically Conical end E. By Lemma 2.2.8, E asymptotically behaves as a cone with nonnegative Ricci curvature, that satisfies the L^1-Sobolev Inequality for some constant C_S. Suppose by contradiction that for every compact set $K \subset E$, the L^1-Sobolev Inequality is not satisfied on $E \smallsetminus K$. Since the metric g converges to the metric $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$, for every $\varepsilon > 0$ there exists a compact set K_ε such that

$$\left| \int_M |\varphi|^{\frac{n}{n-1}} \,\mathrm{d}\mu_g - \int_M |\varphi|^{\frac{n}{n-1}} \,\mathrm{d}\mu_{\hat{g}} \right| \leq \varepsilon \int_M |\varphi|^{\frac{n}{n-1}} \,\mathrm{d}\mu_{\hat{g}}$$

and

$$\left| \int_M |\mathrm{D}\varphi|_g \,\mathrm{d}\mu_g - \int_M |\mathrm{D}\varphi|_{\hat{g}} \,\mathrm{d}\mu_{\hat{g}} \right| \leq \varepsilon \int_M |\mathrm{D}\varphi|_{\hat{g}} \,\mathrm{d}\mu_{\hat{g}}$$

for every $\varphi \in \mathrm{Lip}_c(E \smallsetminus K_\varepsilon)$. Moreover, for every C there exists a function $\varphi \in \mathrm{Lip}_c(E \smallsetminus K_\varepsilon)$ such that

$$\left(\int_M |\varphi|_g^{\frac{n}{n-1}} \,\mathrm{d}\mu_g \right)^{\frac{n-1}{n}} > \mathrm{C} \int_M |\mathrm{D}\varphi|_g \,\mathrm{d}\mu_g$$

Then for every $\varepsilon < 1$ and C we have $\varphi \in \mathrm{Lip}_c(E \smallsetminus K_\varepsilon)$ that satisfies

$$\begin{aligned}\left(\int_M |\varphi|_{\hat{g}}^{\frac{n}{n-1}} \,\mathrm{d}\mu_{\hat{g}} \right)^{\frac{n-1}{n}} &\geq \frac{1}{(1+\varepsilon)^{\frac{n-1}{n}}} \left(\int_M |\varphi|_g^{\frac{n}{n-1}} \,\mathrm{d}\mu_g \right)^{\frac{n-1}{n}} > \frac{\mathrm{C}}{(1+\varepsilon)^{\frac{n-1}{n}}} \int_M |\mathrm{D}\varphi|_g \,\mathrm{d}\mu_g \\ &\geq \mathrm{C}\, \frac{(1-\varepsilon)}{(1+\varepsilon)^{\frac{n-1}{n}}} \int_M |\mathrm{D}\varphi|_{\hat{g}} \,\mathrm{d}\mu_{\hat{g}}.\end{aligned}$$

It is enough to choose $\varepsilon < 1$ and C such that

$$\mathrm{C}\, \frac{(1-\varepsilon)}{(1+\varepsilon)^{\frac{n-1}{n}}} > \mathrm{C}_S$$

to obtain a contradiction to the L^1-Sobolev Inequality on the asymptotic cone. ☐

2.3 The p-capacitary potential

This section is devoted to the study of the *p-capacitary potential* on complete noncompact Riemannian manifolds (M, g). By the previous discussions, we know that given Ω an open bounded subset with smooth boundary there exists a function $u \in W^{1,p}_{\mathrm{loc}}(M \setminus \overline{\Omega})$ that realises $\mathrm{Cap}_p(\partial\Omega)$. However, this function u can be identically 1. We now want to furnish some conditions so that u is not constant. Therefore, from now on, with the locution p-capacitary potential we intend a solution $u \in W^{1,p}_{\mathrm{loc}}(M \setminus \overline{\Omega})$ to the problem

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } M \setminus \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } \mathrm{d}(o,x) \to +\infty. \end{cases} \tag{2.7}$$

The regularity results previously discussed ensure that u belongs to $\mathscr{C}^{1,\beta}_{loc}(M \setminus \Omega)$ and it is smooth near the points where the gradient does not vanish. In particular, by Hopf's Maximum Principle in Theorem 1.2.8 the datum on $\partial\Omega$ is attained smoothly. It turns out that a solution to (2.7) exists and is unique for every Ω provided the manifold admits a positive p-Green's function vanishing at infinity. Moreover, since both the p-Green's function and u are solutions to the same partial differential equation, by the Comparison Principle we can put forward all estimates in Section 1.3 for to the p-capacitary potential. We spend the first part of the section restating all previous theorems in the special cases of manifolds with nonnegative Ricci curvature, which will be useful in the subsequent chapters, dedicated to the proof of our monotonicity formulas and geometric inequalities.

The remaining part contains one of the main results of this chapter. As already mentioned in the Introduction, we are proving that on Asymptotically Conical manifold the p-capacitary potential is vanishing at infinity with a well-known asymptotic behaviour. In particular, it can be compared with the fundamental radial solution on the asymptotic cone, namely $\rho^{-(n-p)/(p-1)}$, where ρ is the radial coordinate on the cone.

2.3.1 Existence of the p-capacitary potential

We firstly prove that the existence of a vanishing p-Green's function implies that a solution to (2.7) exists for any $\Omega \subset M$ open bounded subset with smooth boundary.

Theorem 2.3.1 (Existence of p-capacitary potential)**.** *Let (M,g) be a complete p-nonparabolic n-dimensional Riemannian manifold, for $p > 1$. Assume also that the p-Green's function G satisfies $G(o,x) \to 0$ as $\mathrm{d}(o,x) \to +\infty$ for some $o \in M$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then there exists a unique solution u to* (2.7) *and it attains smoothly the boundary value on $\partial\Omega$.*

Proof. Without loss of generality, we can suppose that $o \in \Omega$. Consider an exhaustion of p-regular domains $(U_i)_{i\in\mathbb{N}}$ for M, such that $\overline{\Omega} \subseteq U_i$ for every $i \in \mathbb{N}$. Let $(u_i)_{i\in\mathbb{N}}$ be the p-capacitary potential associated with the condenser $(\overline{\Omega}, U_i)$. There exists a constant C not depending on $i \in \mathbb{N}$ such that $u_i \leq \mathrm{C}\, G(o,\cdot)$ on $\partial\Omega$. Since U_i is p-regular, $u_i = 0$ on ∂U_i, then $u_i \leq \mathrm{C}\, G(o,\cdot)$ on $U_i \setminus \overline{\Omega}$ by the Comparison Principle Theorem 1.2.9. Since $\sup u_i \leq \mathrm{C}\, G(o,\cdot)$, by the Harnack's Principle Theorem 1.2.10 the function $u = \sup u_i$ is a solution to (2.7). As it may concern the uniqueness, let v be any other solution to (2.7) and let $k \in \mathbb{N}$. By the vanishing of u at infinity there exists an open subset $U_k \subset M$ such that $u \leq v + 1/k$ on ∂U_k. Applying the Boundary Comparison Principle Theorem 1.2.9 one gets that $u \leq v + 1/k\ U_k \setminus \overline{\Omega}$. Since we can always assume that $(U_k)_{k\in\mathbb{N}}$ is an increasing sequence, we obtain $u \leq v$ on $M \smallsetminus \overline{\Omega}$ letting $k \to +\infty$. Exchanging u and v gives the opposite inequality, showing the uniqueness. □

As a byproduct of the previous theorem we also obtain an upper bound of the p-capacitary potential in terms of the p-Green's function. In the following proposition we improve it to a double sides bound.

Proposition 2.3.2. *Let (M, g) be a complete p-nonparabolic n-dimensional Riemannian manifold, for $p > 1$. Assume also that the p-Green's function G satisfies $G(o, x) \to 0$ as $\mathrm{d}(o, x) \to +\infty$ for some $o \in M$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then the solution u to (2.7) satisfies*

$$\frac{G(o, \cdot)}{\sup_{\partial\Omega} G} \leq u \leq \frac{G(o, \cdot)}{\inf_{\partial\Omega} G}$$

on $M \smallsetminus \Omega$.

Proof. We only prove the lower bound, since the upper bound was provided in the previous theorem and the strategy is the same. Let $\mathrm{C} = 1/\sup_{\partial\Omega} G$, then $\mathrm{C}\, G(o, \cdot) \leq u$ on $\partial\Omega$. Since both u and G vanishes at infinity, for any $\delta > 0$ we have $\mathrm{C}^{-1}\, G(o, \cdot) \leq u + \delta$ on ∂U_δ for any U_δ big enough. The Comparison Principle Theorem 1.2.9 gives that $\mathrm{C}\, G(o, \cdot) \leq u + \delta$ on $U_\delta \smallsetminus \overline{\Omega}$. Assuming that $(U_\delta)_{\delta > 0}$ is increasing as δ goes to 0^+, one obtains that $\mathrm{C}\, G(o, \cdot) \leq u$ on $M \smallsetminus \Omega$ letting $\delta \to 0^+$. □

To conclude, we apply the Cheng-Yau-type estimate Theorem 1.2.11 to obtain a gradient bound on u.

Proposition 2.3.3. *Let (M, g) be a p-nonparabolic Riemannian manifold of dimensione n, for $p > 1$. Let $\Omega \subset M$ be open bounded with smooth boundary. Assume that the Ricci tensor satisfies the condition*

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(1 + \mathrm{d}(x, o))^2}$$

for some $o \in \Omega$ and $\kappa \in \mathbb{R}$ and that the p-Green's function satisfies $G(o, x) \to 0$ as $\mathrm{d}(o, x) \to +\infty$. Let u the p-capacitary potential associated with Ω. Then, there exists a positive constant $\mathrm{C} > 0$ such that

$$|\mathrm{D} \log u| \leq \frac{\mathrm{C}}{\mathrm{d}(x, o)} \tag{2.8}$$

holds on the whole $M \smallsetminus \Omega$.

Proof. By the $\mathscr{C}^1$-regularity of u, it clearly suffices to show that (2.8) holds true outside some compact set containing Ω. Let then $o \in \Omega$ and $R > 0$ be such that $\Omega \subset B(o, R)$, and let $x \in M \smallsetminus \overline{B(o, 2R)}$. With this choice, we have $B(x, \mathrm{d}(o, x) - R) \subset M \smallsetminus \overline{B(o, R)}$. Thus, applying inequality (1.9) to the function u in the ball $B(x, \mathrm{d}(o, x) - R)$ we get

$$\frac{|\mathrm{D}u|}{u} \leq 2\mathrm{C}\left(\frac{1}{\mathrm{d}(o, x) - R} + \frac{\kappa}{\mathrm{d}(o, x) + 1}\right) \leq 2\mathrm{C}\,\frac{(2 + \kappa)}{\mathrm{d}(o, x)}$$

concluding the proof. □

2.3.2 Properties of the p-capacitary potential in Riemannian manifolds with nonnegative Ricci curvature

Here we specialise all results obtained in this chapter in the case of Riemannian manifold (M, g) with nonnegative Ricci curvature. First of all, as a consequence of the Cheeger-Gromoll Splitting Theorem [CG71] (see [AFM20, Proposition 2.10] for a detailed proof), we have the following characterisation of the number of ends.

Proposition 2.3.4. *Let (M, g) be a complete noncompact Riemannian manifold with $\mathrm{Ric} \geq 0$. If (M, g) is not a Riemannian cylinder, then it has just one end.*

If (M, g) is a Riemannian manifold with nonnegative Ricci curvature. By [Hol99] (see Examples 3.1 and 4.2) every end of M is Harnack (see also [AG90]) and homogeneous. The following proposition is then a consequence of Proposition 1.3.15.

Proposition 2.3.5. *Let (M, g) be a complete noncompact Riemannian manifold with nonnegative Ricci curvature. Then, M is p-nonparabolic if and only if*

$$\int_R^{+\infty} \left(\frac{t}{|B(o,t)|} \right)^{\frac{1}{p-1}} \mathrm{d}t < +\infty. \tag{2.9}$$

for some $o \in M$ and $R > 0$. In particular, if (M, g) has Euclidean Volume Growth, then it is p-nonparabolic for every $1 < p < n$.

Proof. Assume that (M, g) has Euclidean Volume Growth. Then,

$$\int_R^{+\infty} \left(\frac{t}{|B(o,t)|} \right)^{\frac{1}{p-1}} \mathrm{d}t \leq \mathrm{C} \int_R^{+\infty} t^{-\frac{n-1}{p-1}} \,\mathrm{d}t = \mathrm{C}\,\frac{p-1}{n-p} R^{-\frac{n-p}{p-1}} < +\infty,$$

concluding the proof. □

Observe that the integral in (2.9) is not finite on cylinders, thus a p-nonparabolic Riemannian manifold (M, g) with nonnegative Ricci has one end. As a consequence of the previous proposition, the p-Green's function of (M, g) satisfies the upper bound in Proposition 1.3.16 and, since the end is Harnack, it vanishes at infinity. In virtue of Theorem 2.3.1, this implies the existence of a solution to the problem (2.7) for every $\Omega \subseteq M$ open bounded with smooth boundary.

Proposition 2.3.6. *Let (M, g) be a complete p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$, for $p > 1$. Then, for every $\Omega \subset M$ open bounded subset with smooth boundary there exists a unique solution $u : M \smallsetminus \Omega \to (0, 1]$ to*

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } M \smallsetminus \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } \mathrm{d}(o, x) \to +\infty. \end{cases}$$

Moreover, it attains smoothly the datum at the boundary.

Moreover, the following sharp lower bound for the p-Green's function holds.

Proposition 2.3.7. *Let (M, g) be a complete p-nonparabolic n-dimensional Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and $o \in M$. Then,*

$$\mathrm{d}(o, x)^{-\frac{n-p}{p-1}} \leq G(o, x) \tag{2.10}$$

holds for every $x \in M \smallsetminus \{o\}$.

Proof. Let $r(x) = \mathrm{d}(o, x)$. We first show that $\Delta_p r^{-(n-p)/(p-1)} \geq 0$ holds in the weak sense, that is

$$\int_M \left\langle \left| \mathrm{D} r^{-\frac{n-p}{p-1}} \right|^{p-2} \mathrm{D} r^{-\frac{n-p}{p-1}} \,\middle|\, \mathrm{D}\psi \right\rangle \mathrm{d}\mu \leq 0$$

for any $\psi \in \mathscr{C}_c^\infty(M)$. In fact, we have

$$\begin{aligned} \int_M \left\langle \left| \mathrm{D} r^{-\frac{n-p}{p-1}} \right|^{p-2} \mathrm{D} r^{-\frac{n-p}{p-1}} \,\middle|\, \mathrm{D}\psi \right\rangle \mathrm{d}\mu &= -\left(\frac{n-p}{p-1}\right)^{p-1} \int_M r^{1-n} \langle \mathrm{D} r \,|\, \mathrm{D}\psi \rangle \, \mathrm{d}\mu \\ &= \frac{1}{n-2} \left(\frac{n-p}{p-1}\right)^{p-1} \int_M \langle \mathrm{D} r^{2-n} \,|\, \mathrm{D}\psi \rangle \, \mathrm{d}\mu \leq 0, \end{aligned}$$

where the last inequality is the Laplacian Comparison Theorem, that is $\Delta_g r^{2-n} \geq 0$ in the sense of distributions. Fix $\delta > 0$. Since both $r^{-(n-p)/(p-1)}$ and G vanish at infinity, we have $r^{-(n-p)/(p-1)} \leq G + \delta$ on $\partial B(o, R)$ for any $R > 0$ big enough. On the other hand, the general result [Ser64, Theorem 12] ensures that $G_p(o, x)$ is asymptotic to $r(x)^{-(n-p)/(p-1)}$ as $r(x) \to 0^+$, and thus we also get $r^{-(n-p)/(p-1)} \leq G + \delta$ on $\partial B(o, \varepsilon)$ for any $\varepsilon > 0$ small enough. Thus, applying the Comparison Principle [HKM18, p. 3.7] (see also [GLM86, Lemma 2.3]) to the subsolution $r^{-(n-p)/(p-1)}$ and to the solution $G + \delta$ (with respect to the p-Laplacian), in the annulus $B(o, R) \smallsetminus \overline{B(o, \varepsilon)}$, we get $r^{-(n-p)/(p-1)} \leq G + \delta$ on such annulus. Letting $\varepsilon \to 0^+$ and $R \to +\infty$, we deduce that the same holds on the whole $M \smallsetminus \{o\}$. Finally, letting $\delta \to 0^+$, we are left with (2.10). □

We point out that (2.10) is sharp, since the p-Green's function of $\mathbb{R}^n$ is exactly given by the formula $G(x, y) = \mathrm{d}(x, y)^{-(n-p)/(p-1)}$ for any $x \neq y$.

For what it concerns the upper bound for the p-Green's function, we observe that in [MRS19, Theorem 3.8] it is shown, building on [Hol99, Proposition 5.10], that if (M, g) in addition to the assumptions of Proposition 2.3.7 has Euclidean Volume Growth, then we also have

$$G(o, x) \leq \mathrm{C}\, \mathrm{d}(o, x)^{-\frac{n-p}{p-1}} \tag{2.11}$$

for some constant C with well understood dependencies. In virtue of Proposition 2.3.2, this behaviour can be translated in terms of the p-capacitary potential.

Proposition 2.3.8 (Li-Yau-type estimates)**.** *Let (M, g) be a complete p-nonparabolic n-dimensional Riemannian manifold with* $\mathrm{Ric} \geq 0$*, for $p > 1$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary, $u : M \smallsetminus \Omega \to (0, 1]$ be the solution to* (2.7) *and $o \in \Omega$. Then, there exists a constant $\mathrm{C}_1(n, p, \Omega, M) > 0$ such that*

$$\mathrm{C}_1^{-\frac{1}{p-1}}\, \mathrm{d}(o, x)^{-\frac{n-p}{p-1}} \leq u(x)$$

holds true for every $x \in M \smallsetminus \Omega$. *If in addition* (M, g) *has Euclidean Volume Growth, then there also exists another positive constant* $C_2(n, p, \Omega, M) > 0$ *such that*

$$u(x) \leq C_2^{\frac{1}{p-1}} \, d(o, x)^{-\frac{n-p}{p-1}},$$

for every $x \in M \smallsetminus \Omega$. *Moreover, in this last case* C_1 *and* C_2 *are bounded as* $p \to 1^+$.

Proof. It only remains to prove that C_1 and C_2 are bounded as $p \to 1^+$. In virtue of [MRS19, Theorem 3.6], the constant in (2.11) can be chosen so that $C^{(p-1)}$ is bounded as $p \to 1^+$. Indeed, the condition of the theorem are satisfied since a Sobolev Inequality is in force as a consequence of the Isoperimetric Inequality [Bre22] (see [FF60, Remark 6.6] or [SY94, pp. 89-90]). By Propositions 2.3.2 and 2.3.7

$$u(x) \geq \frac{G(o, x)}{\sup_{\partial\Omega} G(o, x)} \geq \frac{1}{C} \left(\frac{d(o, x)}{d(o, \partial\Omega)} \right)^{-\frac{n-p}{p-1}},$$

then it is enough to choose $C_1 = C^{(p-1)} / d(o, \partial\Omega)^{n-p}$. Conversely, employing again Propositions 2.3.2 and 2.3.7

$$u \leq \frac{G(o, x)}{\inf_{\partial\Omega} G(o, x)} \leq C \left(\frac{d(x, o)}{\operatorname{diam}(\Omega)} \right)^{-\frac{n-p}{p-1}}$$

then it is enough to choose $C_2 = C^{(p-1)} \operatorname{diam}(\Omega)^{n-p}$. □

Moreover, on p-nonparabolic Riemannian manifolds with nonnegative Ricci curvature the statement of Proposition 2.3.3 can be simplified as follows.

Proposition 2.3.9 (Cheng-Yau-type estimate)**.** *Let* (M, g) *be a complete* p*-nonparabolic* n*-dimensional Riemannian manifold with* $\mathrm{Ric} \geq 0$, *for* $p > 1$. *Let* $\Omega \subset M$ *open bounded with smooth boundary and let* u *be its* p*-capacitary potential and* $o \in \Omega$. *Then, there exits a positive constant* C *such that*

$$|D \log u| \leq \frac{C}{d(x, o)} \tag{2.12}$$

holds on the whole $M \smallsetminus \Omega$.

Proof. By Proposition 1.3.16, the assumptions of Proposition 2.3.3 are satisfied and (2.12) is a consequence of (2.8). □

To conclude, we provide the sharp Iso-p-capacitary Inequality in complete noncompact Riemannian manifolds with nonnegative Ricci curvature and Euclidean volume growth. As for the standard Iso-p-capacitary Inequality in Euclidean setting, the proof fully relies on the Isoperimetric Inequality combined with a Pólya-Szegö Principle. In particular, the sharpness of the inequality that follows is a direct consequence of the sharp Isoperimetric constant in this setting, that has been found first in dimension 3 in [AFM20] and later extended to all dimensions in [Bre22]. See also [FM20; BK22; Joh21] for related results. The proof below is classical, and it is inspired by [Jau12], where it is illustrated for the capacity in $\mathbb{R}^n$.

Theorem 2.3.10 (Iso-p-capacitary Inequality)**.** *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean Volume Growth. Let $K \subset M$ be a compact domain with boundary ∂K. Then*

$$\frac{\mathrm{Cap}_p(\mathbb{S}^{n-1})^n}{|\mathbb{B}^n|^{n-p}}\mathrm{AVR}(g)^p \leq \frac{\mathrm{Cap}_p(\partial K)^n}{|K|^{n-p}} \tag{2.13}$$

Moreover, if K has smooth boundary and satisfies the equality in (2.13) then (M, g) is isometric to the Euclidean Space and K is a ball.

Proof. Let $u \in W^{1,p}_{\mathrm{loc}}(M \smallsetminus K)$ be the p-capacitary potential associated with K. By (1.22) and the coarea formula in Proposition B.3 for $f = |\mathrm{D}u|^{p-1}$ we have that

$$\mathrm{Cap}_p(K) = \int\limits_{M \smallsetminus K} |\mathrm{D}u|^p \,\mathrm{d}\mu = \int\limits_0^1 \int\limits_{\{u=\tau\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma\mathrm{d}\tau. \tag{2.14}$$

The Hölder's Inequality with exponents $a = p$ and $b = p/(p-1)$ gives

$$|\{u = \tau\}|^p \leq \left(\int\limits_{\{u=\tau\}} |\mathrm{D}u|^{p-1} \,\mathrm{d}\sigma \right) \left(\int\limits_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|} \,\mathrm{d}\sigma \right)^{p-1} \tag{2.15}$$

for almost every $\tau \in (0, 1]$. Define $V' : (0, 1] \to \mathbb{R}$ to be

$$V'(\tau) = -\int\limits_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|} \,\mathrm{d}\sigma \tag{2.16}$$

Moreover, let $V : (0, 1] \to \mathbb{R}$ be the primitive of $V'(\tau)$ chosen as

$$V(\tau) = |K| - \int\limits_\tau^1 V'(s) \,\mathrm{d}s = |\{u \geq \tau\} \cup K \smallsetminus \mathrm{Crit}\, u|,$$

where the second identity is obtained coupling (2.16) with the coarea formula Proposition B.2 applied with $f = (1 - \chi_{\mathrm{Crit}\, u})|\mathrm{D}u|^{-1}$.

By the Isoperimetric Inequality in [Bre22, Corollary 1.3] we have that

$$\begin{aligned} |\{u = \tau\}| \geq |\partial(\{u \geq \tau\} \cup K)| &\geq |\{u \geq \tau\} \cup K|^{\frac{n-1}{n}} \mathrm{AVR}(g)^{\frac{1}{n}} n |\mathbb{B}^n|^{\frac{1}{n}} \\ &\geq V(\tau)^{\frac{n-1}{n}} \mathrm{AVR}(g)^{\frac{1}{n}} n |\mathbb{B}^n|^{\frac{1}{n}}. \end{aligned} \tag{2.17}$$

Let $R(\tau)$ be the radius of the ball in $\mathbb{R}^n$ which has volume $V(\tau)$, then $V(\tau) = |\mathbb{B}^n| R(\tau)^n$ and $V'(\tau) = |\mathbb{S}^{n-1}| R(\tau)^{n-1} R'(\tau)$. Coupling (2.17) with (2.14), (2.15) and (2.16) we obtain

$$\begin{aligned} \mathrm{Cap}_p(K) &\geq \int\limits_0^1 \frac{|\{u = \tau\}|^p}{[-V'(\tau)]^{p-1}} \,\mathrm{d}\tau \geq n^p \left(|\mathbb{B}^n| \mathrm{AVR}(g)\right)^{\frac{p}{n}} \int\limits_0^1 \frac{V(\tau)^{\frac{p(n-1)}{n}}}{[-V'(\tau)]^{p-1}} \,\mathrm{d}\tau \\ &= |\mathbb{S}^{n-1}| \mathrm{AVR}(g)^{\frac{p}{n}} \int\limits_0^1 \frac{R(\tau)^{n-1}}{[-R'(\tau)]^{p-1}} \,\mathrm{d}\tau. \end{aligned}$$

Let now $v : \mathbb{R}^n \smallsetminus B(0, R(1)) \subset \mathbb{R}^n \to (0,1]$ be the function which is τ on $\partial B(0, R(\tau))$. By (2.17) and Propositions 2.3.8 and 2.3.9 there exists a positive constant $\mathrm{C} = \mathrm{C}(p,n)$ such that

$$- V'(\tau) = \int\limits_{\{u=\tau\}} \frac{1}{|\mathrm{D}u|} \, \mathrm{d}\sigma \geq \mathrm{C} \, |K|^{\frac{n-1}{n}} \tau^{\frac{n-p}{p-1}}.$$

Seeing as $|\mathrm{D}v| = -1/R'(\tau) = -|\mathbb{S}^{n-1}| R^{n-1}(\tau)/V'(\tau)$ the function v is locally Lipschitz. Since $|\mathbb{S}^{n-1}| R(\tau)^{n-1} = |\partial B(0, R(\tau))| = |\{v = \tau\}|$ by the coarea formula Proposition B.2 applied with $f = |\mathrm{D}v|^{p-1}$ we have

$$\begin{aligned}
\left|\mathbb{S}^{n-1}\right| \mathrm{AVR}(g)^{\frac{p}{n}} \int\limits_0^1 \frac{R(\tau)^{n-1}}{[-R'(\tau)]^{p-1}} \, \mathrm{d}\tau &= \mathrm{AVR}(g)^{\frac{p}{n}} \int\limits_0^1 \int\limits_{\{v=\tau\}} |\mathrm{D}v|^{p-1} \, \mathrm{d}\sigma \, \mathrm{d}\tau \\
&= \mathrm{AVR}(g)^{\frac{p}{n}} \int\limits_{\{|x| \geq R(1)\}} |\mathrm{D}v|^p \, \mathrm{d}x \geq \mathrm{AVR}(g)^{\frac{p}{n}} \operatorname{Cap}_p \left(\partial B(0, R(1)), \mathbb{R}^n\right),
\end{aligned}$$

where the last identity follows by the definition of the p-capacity (1.12) in flat $\mathbb{R}^n$. Using (1.23) and the fact that $|\{|x| \leq R(1)\}| = V(1) \geq |K|$, we finally obtain

$$\begin{aligned}
\mathrm{AVR}(g)^{\frac{p}{n}} \operatorname{Cap}_p \left(\partial B(0, R(1))\right) &= \mathrm{AVR}(g)^{\frac{p}{n}} \operatorname{Cap}_p \left(\mathbb{S}^{n-1}\right) R(1)^{n-p} \\
&\geq \mathrm{AVR}(g)^{\frac{p}{n}} \frac{\operatorname{Cap}_p(\mathbb{S}^{n-1})}{|\mathbb{B}^n|^{\frac{n-p}{n}}} |K|^{\frac{n-p}{n}},
\end{aligned}$$

and consequently (2.13).

Clearly, if equality holds true in (2.13) and ∂K is smooth then also equality holds in the use of the Isoperimetric Inequality and [Bre22, Theorem 1.2] forces the rigidity both of the ambient manifold and K. □

2.3.3 Properties of the p-capacitary potential in Asymptotically Conical Riemannian manifolds

As we did in the setting of Riemannian manifolds with nonnegative Ricci curvature, we specialise the results obtained in this chapter in the framework of Asymptotically Conical Riemannian manifolds. Following this analogy, we proceed characterising the number of ends.

Lemma 2.3.11. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Then (M, g) has a finite number of $\mathscr{C}^0$-Asymptotically Conical ends with connected boundary each one diffeomorphic to one connected componnt of L.*

Proof. Consider the compact set K given by Definition 2.2.1. Then ∂K is a compact embedded smooth hypersurface, hence it has a finite number of connected components. Each end with respect to K is therefore diffeomorphic to a cone on a connected component of the link L and it is $\mathscr{C}^0$-Asymptotically Conical by the very definition. □

We denote by $E_1, \dots, E_N$ the ends of (M, g). Since they are $\mathscr{C}^0$-Asymptotically Conical by Lemma 2.2.4 we can define an Asymptotic Volume Ratio of E_i as

$$\mathrm{AVR}(g; E_i) = \lim_{r \to +\infty} \frac{|B(o,r) \cap E_i|}{r^n |\mathbb{B}^n|} = \frac{|L_i|}{|\mathbb{S}^{n-1}|}, \tag{2.18}$$

for $i = 1, \dots, N$. It follows that

$$\mathrm{AVR}(g) = \sum_{i=1}^{m} \mathrm{AVR}(g; E_i).$$

Each end $E_i \subset M$ also satisfies the volume comparison condition given in Definition 1.3.14 (3). Indeed, by (2.4) there exists R_ε such that

$$|B(x,r) \cap E_i|_g \leq |\{\rho \leq 2r\} \cap E_i|_g \leq \frac{2^n}{n}(1+\varepsilon) r^n |L_i|_{\hat{g}}$$

holds for every $r > R_\varepsilon$ and $x \in \partial B(o,r) \cap E_i$. Conversely, by the convergence of the metric, we can choose R_ε so that

$$|B(x,r/8)|_g \geq (1-\varepsilon)|B(x,r/8)|_{\hat{g}} \geq \frac{17^n - 15^n}{n 16^n}(1-\varepsilon) r^n \inf_{y \in L_i} \left\{ |B(y, 1/16)|_{g_L} \right\},$$

which is positive since L_i is compact and smooth. Hence if R_ε is big enough to satisfy $\varepsilon < \mu(L_i)/2$, Definition 1.3.14 (3) is ensured for every $r \geq R_\varepsilon$ and $x \in \partial B(o,r) \cap E_i$.

Assume now that (M, g) is an Asymptotically Conical Riemannian manifold the curvature constraint

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(\mathrm{d}(x,o)+1)^2} \tag{2.19}$$

holds for every $x \in M$, where we fixed $o \in M$. By [Hol99] (see Examples 2.20, 3.1 and 4.2) every end is Harnack and it is homogeneous.

Remark 2.3.12. *In [Hol99, Example 2.20], the author actually derives from the Buser's Isoperimetric Inequality [Bus82] a weak $(1,1)$-Poincaré Inequality, namely,*

$$\fint_{B(y,r)} |\varphi - \overline{\varphi}| \,\mathrm{d}\mu \leq \mathrm{C}_P\, r \fint_{B(y,2r)} |\mathrm{D}\varphi| \,\mathrm{d}\mu \qquad \text{with } \overline{\varphi} = \fint_{B(y,r)} \varphi \,\mathrm{d}\mu,$$

for every Lipschitz function $\varphi \in \mathrm{Lip}_c(B(x,4r))$ with $B(x,4r) \subset \{\rho \geq R\}$ for some R big enough.

Moreover, since $\mathrm{AVR}(g; E_i) > 0$ we can find R_ε such that

$$\int_{R_\varepsilon}^{+\infty} \left(\frac{t}{|B(o,t) \cap E_i|} \right)^{\frac{1}{p-1}} \mathrm{d}t \leq \left(\frac{2}{\mathrm{AVR}(g; E_i)} \right)^{\frac{1}{p-1}} \left(\frac{p-1}{n-p} \right) R_\varepsilon^{-\frac{n-p}{p-1}} < +\infty.$$

Therefore, for every $1 < p < n$ every end is p-large and p-nonparabolic, in virtue of Proposition 1.3.15. Since every end is also Harnack, $G(o,x)$ vanishes at infinity by Proposition 1.3.16. This condition coupled with Theorem 2.3.1 ensures the existence and uniqueness of a solution to (2.7) for every $\Omega \subset M$ open, bounded with smooth boundary.

Proposition 2.3.13. *Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying* (2.19)*. Let $1 < p < n$. Then, for every $\Omega \subset M$ open bounded subset with smooth boundary there exists a unique solution $u : M \smallsetminus \Omega \to (0, 1]$ to*

$$\begin{cases} \Delta_g^{(p)} u = 0 & \text{on } M \smallsetminus \Omega, \\ u = 1 & \text{on } \partial\Omega, \\ u \to 0 & \text{as } \mathrm{d}(o, x) \to +\infty. \end{cases}$$

Moreover, it attains smoothly the datum at the boundary.

Consider $\Omega \subseteq M$ some open subset with smooth boundary and $u : M \smallsetminus \Omega \to \mathbb{R}$ the p-capacitary potential associated with Ω. There exists a $T \in [1, +\infty)$ such that $\{u > 1/t\}$ contains K for every $t \geq T$. We define the *p-capacity of $\partial\Omega$ with respect to the end E_i* the quantity

$$\mathrm{Cap}_p^{(i)}(\partial\Omega) = t^{p-1} \int_{\{u=1/t\} \cap E_i} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma \tag{2.20}$$

for some $t \geq T$, and accordingly the *normalised p-capacity of Ω with respect to the end E_i* the quantity

$$\widehat{\mathrm{C}}\mathrm{ap}_p^{(i)}(\partial\Omega) = \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \mathrm{Cap}_p^{(i)}(\partial\Omega).$$

Observe that the two quantities are well defined by (1.20). Moreover, it is readily checked that, the p-capacity of Ω splits as

$$\mathrm{Cap}_p(\partial\Omega) = \sum_{i=1}^{m} \mathrm{Cap}_p^{(i)}(\partial\Omega).$$

$\mathrm{Cap}_p^{(i)}(\partial\Omega)$ represents the portion of $\partial\Omega$ that contributes to its p-capacity under the influence of the end E_i. Actually, if Ω already contains K we have $\mathrm{Cap}_p^{(i)}(\partial\Omega) = \mathrm{Cap}_p(\partial\Omega \cap E_i; E_i)$.

On cones, the p-capacity of the cross section $\{\rho = r\}$ can be easily computed since the function $u = (\rho/r)^{-(n-p)/(p-1)}$ is the p-capacitary potential associated with these sets. In Asymptotically Conical Riemannian manifolds one might expect that the p-capacity of $\{\rho = r\}$ approaches the model one for large r. Despite the definition of the p-capacity involves the first order derivatives of the p-capacitary potential, the convergence is also true even if the metric converges only in the $\mathscr{C}^0$-topology.

Lemma 2.3.14. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Let ρ be the radial coordinate on M. Let $E_1, \ldots, E_N$ be the ends of M with respect to the bounded K in Definition* 2.2.1*. Then,*

$$\lim_{r \to +\infty} \frac{\mathrm{Cap}_p^{(i)}(\{\rho = r\})}{r^{n-p} |\mathbb{S}^{n-1}|} = \left(\frac{n-p}{p-1} \right)^{p-1} \mathrm{AVR}(g; E_i) \tag{2.21}$$

holds for every $i = 1, \ldots, N$.

Proof. We prove the statement in the presence of one single end, being the general case an easy consequence. We then drop the index i in the following lines. Since the metric g converges to the metric $\hat{g}$, for every $\varepsilon > 0$ there exists a $R_\varepsilon > 0$ such that for every $r \geq R_\varepsilon$

$$\left| \int_M |D\varphi|_g^p \, d\mu_g - \int_M |D\varphi|_{\hat{g}}^p \, d\mu_{\hat{g}} \right| \leq \varepsilon \int_M |D\varphi|_{\hat{g}}^p \, d\mu_{\hat{g}}$$

holds for every function $\varphi \in \mathscr{C}_c^\infty(\{\rho \geq r\})$ such that $\varphi = 1$ on $\{\rho = r\}$. In particular, we have that

$$(1-\varepsilon) \int_M |D\varphi|_{\hat{g}}^p \, d\mu_{\hat{g}} \leq \int_M |D\varphi|_g^p \, d\mu_g \leq (1+\varepsilon) \int_M |D\varphi|_{\hat{g}}^p \, d\mu_{\hat{g}}.$$

The set $\{\rho \geq r\}$ is diffeomorphic to $[r, +\infty) \times L$ where L is the cross section of the cone (M, g) is asymptotic to. Hence, the family of φ's considered above are in one-to-one correspondence with the competitors for the p-capacity of $\{\rho = r\}$ in the Riemannian cone $[r, +\infty) \times L$. Dividing each side by $|\mathbb{S}^{n-1}|$, recalling the characterisation of $\mathrm{AVR}(g)$ in Definition 2.2.5 and taking the infimum on each side of the previous chain of inequalities we are left with

$$(1-\varepsilon) r^{n-p} \mathrm{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1} \leq \frac{\mathrm{Cap}_p(\{\rho = r\})}{|\mathbb{S}^{n-1}|} \leq (1+\varepsilon) r^{n-p} \mathrm{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1}.$$

dividing each term by r^{n-p} and letting $r \to +\infty$ we have that

$$(1-\varepsilon) \mathrm{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1} \leq \lim_{r \to +\infty} \frac{\mathrm{Cap}_p(\{\rho = r\})}{r^{n-p} |\mathbb{S}^{n-1}|} \leq (1+\varepsilon) \mathrm{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1},$$

which in turns gives (2.21) by arbitrariness of ε. □

We next provide Li-Yau-type estimates. As we did in the framework of Riemannian manifolds with nonnegative Ricci curvature, they are obtained combining the estimates on the p-Green's function with Proposition 2.3.2. Proposition 1.3.16 and Proposition 1.3.17 do not give the desired bound, since $E \smallsetminus \overline{B(o, r_k)}$ could have bounded components for a divergent sequence $\{r_k\}_{k \in \mathbb{N}}$. We use the Asymptotically Conical structure to foliate levels of the distance by cross-sections of the asymptotic cone for witch the following uniform Harnack's Inequality holds.

Proposition 2.3.15 (Harnack's Inequality). *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying* (2.19)*. There exists a constant $\mathrm{C} > 0$ that depends only on the dimension, p and κ, but not on r, such that*

$$\sup_{\{\rho = r\} \cap E} u \leq \mathrm{C}_H^{\frac{1}{p-1}} \inf_{\{\rho = r\} \cap E} u \tag{2.22}$$

for every positive p-harmonic function u on E. Moreover, the constant C_H is bounded as $p \to 1^+$.

Proof. We only provide here the existence of C_H. As it may concerns its boundedness, the constant obtained in the following lines is not sharp. The proof is based on the Cheng-Yau-type estimate showed in [WZ10], hence the constant we are providing satisfies $(p -$

1) $C \to +\infty$ as $p \to 1^+$. The result can be improved using [MRS19, Theorem 3.4], whose assumptions are fulfilled thanks to Proposition 2.2.9 and Remark 2.3.12.

The main tool we are employing is the Cheng-Yau-type estimate Theorem 1.2.11. Fixing $o \in M$, let $R > 0$ be such that $\{\rho = r\} \subset E \setminus \overline{B(o, 3r/4)}$ for every $r \geq R$. Applying (1.9) in $B(y, r/4)$ for some $y \in \{\rho = r\}$, we get

$$\sup_{B(y,r/4)} \frac{|\mathrm{D}u|}{u} \leq \mathrm{C}_1 \left(\frac{1}{r} + \frac{2\kappa}{2 + r} \right) \leq \frac{\mathrm{C}_2}{r} \tag{2.23}$$

where C_2 does not depend on r but only on again depending only on n and p. Here, we employed $B(y, r/4) \subset E \setminus B(o, r/4)$ and the lower bound on the Ricci curvature given by (2.19). As the diameter of $\{\rho = r\}$, which is connected, grows with r by the assumed asymptotic conicality, we deduce from (2.23) that

$$\log(u(x)) - \log(u(y)) \leq \sup_{\{\rho = r\} \cap E} |\mathrm{D} \log u| \operatorname{diam}\{\rho = r\} \leq \mathrm{C}$$

for some new constant $\mathrm{C}_H^{1/(p-1)}$ not depending on r. □

Proposition 2.3.16. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with* Ric *satisfying (2.19). Let $p > 1$. Let E be given G the p-Green's function on M, then there exist two positive finite constants $\mathrm{C}_L, \mathrm{C}_U > 0$ such that*

$$\mathrm{C}_L^{\frac{1}{p-1}} \, \mathrm{d}(o, x)^{-\frac{n-p}{p-1}} \leq G(o, x) \leq \mathrm{C}_U^{\frac{1}{p-1}} \, \mathrm{d}(o, x)^{-\frac{n-p}{p-1}}. \tag{2.24}$$

for every $x \in M \setminus \{o\}$. Moreover, the constant C_U is bounded as $p \to 1^+$ and there exists $R > 0$ not depending on p such that the lower bound in (2.24) holds with constant $\mathrm{C}_L = \mathrm{C}\ a\ \mathrm{Cap}_p(\overline{K}; E)$ on $M \setminus B(o, R)$, where $a > 0$ is such that $G(x, o) \geq a^{1/(p-1)}$ on $M \setminus K$, C is bounded as $p \to 1^+$ and K is the bounded set in Definition 2.2.1.

Proof. The upper bound in (2.24) with C_U bounded as $p \to 1^+$ follows from [MRS19, Theorem 3.6]. Indeed, the assumptions are satisfied since a weighted p-Sobolev Inequality by Proposition 2.2.9 and the arguments therein.

For what it concerns the lower bound, we are in position to apply Proposition 1.3.17. The main issue is that you do not have control on the bounded components of $E \setminus \overline{B(o, R)}$. Consider the function $R : [1, +\infty) \to \mathbb{R}$ defined as

$$R(t) = \max\{\mathrm{d}(o, x) \,|\, x \in \{\rho = t\}\}.$$

Observe that $\{\rho \geq t\} \supset E(2R(t))$. Then applying the Harnack's Inequality (2.22), Comparison Principle Theorem 1.2.9 and Proposition 1.3.17 one gets that

$$\begin{aligned} \inf_{\{\rho = t\} \cap E} G_p(o, x) &\geq \mathrm{C}_H \sup_{\{\rho = t\} \cap E} G_p(o, x) \geq \mathrm{C}_H \sup_{\partial E(2R(t))} G_p(o, x) \\ &\geq \mathrm{C}_1 \int_{4R(t)}^{+\infty} \left(\frac{r}{|B(o, r) \cap E|} \right)^{\frac{1}{p-1}} \mathrm{d}r \geq \mathrm{C}_2 \, R(t)^{-\frac{n-p}{p-1}}, \end{aligned}$$

for every $t \geq T$, for some T large enough C_1 is the constant given by (1.29) in Proposition 1.3.17 and $\mathrm{C}_2 = \mathrm{C}\,\mathrm{C}_1$ where C_1 is bounded as $p \to 1^+$. By Lemma 2.2.3 there exists $R_1 \geq R(T)$ such that $\partial B(o,r) \subset \{\rho \leq 2r\}$ and $R(2r) \leq 4r$ holds for every $r \geq R_1$. Then

$$\inf_{\partial B(o,r)} G(o,x) \geq \inf_{\{\rho \leq 2r\}} G(o,x) \geq \mathrm{C}_2\, R(2t)^{-\frac{n-p}{p-1}} \geq \left(\frac{\mathrm{C}_2}{4^{n-p}}\right)^{\frac{1}{p-1}} r^{-\frac{n-p}{p-1}},$$

holds for every $t \geq R_1$, concluding the proof. The global lower bound, follows since itis satisfied near the pole o and the two functions are continuous in the remaining annulus. However, the new constant C_L might go to 0 as $p \to 1^+$. □

The lower bound observed in Proposition 2.3.16 improves the one given by a direct application of [MRS19, Corollary 2.8]. In [MRS19] the authors take only into account the behaviour of the Ricci tensor considering as the model a manifold for which the lower bound (2.19) is actually achieved. In Proposition 1.3.17 we actually used as the model the manifold itself. The comparison can be performed in virtue of the asymptotic behaviour of the volume ensured by the asymptotically conical property.

Proposition 2.3.17 (Li-Yau-type estimates). *Let (M,g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical manifold with* Ric *satisfying* (2.19). *Let $\Omega \subset M$ be open bounded with smooth boundary, $u : M \smallsetminus \Omega \to (0,1]$ the solution to* (2.7) *and $o \in \Omega$. Then, there exists a constant $\mathrm{C} > 0$ such that*

$$\mathrm{C}^{-1}\, \mathrm{d}(o,x)^{-\frac{n-p}{p-1}} \leq u(x) \leq \mathrm{C}\, \mathrm{d}(o,x)^{-\frac{n-p}{p-1}}$$

hold true for every $x \in M \smallsetminus \Omega$.

Proof. By Proposition 2.3.16, the assumptions of Proposition 2.3.2 are satisfied. □

Proposition 2.3.18 (Cheng-yau-type estimate). *Let (M,g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical manifold with* Ric *satisfying* (2.19). *Let $\Omega \subset M$ open bounded with smooth boundary and let u be its p-capacitary potential and $o \in \Omega$. Then, there exists a constant $\mathrm{C} > 0$ such that*

$$|\mathrm{D}u| \leq \frac{\mathrm{C}}{\mathrm{d}(x,o)} \tag{2.25}$$

holds on the whole $M \smallsetminus \Omega$.

Proof. By Proposition 2.3.16, the assumptions of Proposition 2.3.3 are satisfied and (2.25) is a consequence of (2.8). □

Remark 2.3.19. *A direct consequence of Proposition 2.2.9 is a Iso-p-capacitary Inequality in Asymptotically Conical Riemannian manifolds as in* (2.13)*, where* $\mathrm{AVR}(g)$ *is replaced with the Isoperimetric constant.*

2.3.4 Asymptotic behaviour of the p-capacitary potential

We conclude this section proving the asymptotic behaviour of the p-capacitary potential associated u with an open bounded subset with smooth boundary by the problem (2.7). We prove that in Asymptotically Conical Riemannian manifolds u is equivalent to $\gamma\rho^{-(n-p)/(p-1)}$ at a large distance, for some constant γ that is well characterised in terms of Ω and the geometric of the ambient manifold.

Theorem 2.3.20 (Asymptotic behaviour of the p-capacitary potential)**.** *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with* Ric *satisfying*

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(1+\mathrm{d}(x,o))^2},$$

for some fixed $o \in M$, $\kappa \in \mathbb{R}$ and for every $x \in M$. Let $E_1, \dots, E_N$ be the (finitely many) ends of M with respect to the bounded K in Definition 2.2.1. Consider $\Omega \subset M$ be an open bounded subset with smooth boundary and $u : M \smallsetminus \Omega \to \mathbb{R}$ the solution to the problem (2.7). Then

$$u(x) = \left(\frac{\widehat{\mathrm{Cap}}_p^{(i)}(\partial\Omega)}{\mathrm{AVR}(g;E_i)}\right)^{\frac{1}{p-1}} \rho(x)^{-\frac{n-p}{p-1}} + o\left(\rho(x)^{-\frac{n-p}{p-1}}\right) \tag{2.26}$$

on E_i as $\mathrm{d}(o,x) \to +\infty$ for every $i = 1, \dots, N$, where $\widehat{\mathrm{Cap}}_p^{(i)}(\partial\Omega)$ and $\mathrm{AVR}(g;E_i)$ are defined respectively in (2.20) and (2.18).

In fact, we prove a more general statement that provides information also about the asymptotic behaviour of the derivatives of u, if the asymptotic structure of the underlying metric is suitably reinforced. Indeed, taking advantage of the Schauder estimates Theorem 1.2.5, we can deduce the following result.

Theorem 2.3.21 (Asymptotic behaviour of the p-capacitary potential)**.** *Let (M, g) be a complete $\mathscr{C}^{k,\alpha}$-Asymptotically Conical Riemannian manifold for some $\alpha > 0$ and $k \in \mathbb{N}$ with* Ric *satisfying*

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(1+\mathrm{d}(x,o))^2},$$

for every $x \in M$ and for some fixed $o \in M$. Let $E_1, \dots, E_N$ be the (finitely many) ends of M with respect to the compact K in Definition 2.2.1. Consider $\Omega \subset M$ be an open bounded subset with smooth boundary and $u : M \smallsetminus \Omega \to \mathbb{R}$ a solution of the problem (2.7). Then

$$\left| \mathrm{D}^\ell u - \left(\frac{\widehat{\mathrm{Cap}}_p^{(i)}(\partial\Omega)}{\mathrm{AVR}(g;E_i)}\right)^{\frac{1}{p-1}} \mathrm{D}^\ell \rho^{-\frac{n-p}{p-1}} \right|_{\hat{g}} = o\left(\rho^{-\frac{n-p}{p-1}-\ell}\right) \tag{2.27}$$

on E_i as $\mathrm{d}(o,x) \to +\infty$ for every $i = 1, \dots, N$ and $\ell \leq k+1$, where $\widehat{\mathrm{Cap}}_p^{(i)}(\partial\Omega)$ and $\mathrm{AVR}(g;E_i)$ are defined respectively in (2.20) and (2.18).

In the case $\mathrm{Ric} \geq 0$ and $p = 2$ this result is carried out in the greater generality of Euclidean Volume Growth in the two works [LTW97] and [CM97] (see also [Din02] and [AFM20]), where the asymptotic $\mathscr{C}^0$-expansion of the harmonic Green's function is respectively deduced using the existence of a representation formula and the monotonicity of Almgren's frequency function. This result also extend to the nonlinear setting the asymptotic analyses carried out in [AMO22, Theorem 2.2], [AMO21, Lemma 2.2], [HM20, Lemma 4.1] and [MMT20, Lemma A.2], although without so refined estimates of the error terms. Up to our knowledge, these tools are not known in the general case of p-harmonic functions, and appreciable results for the p-capacitary potential have indeed been established only in $\mathbb{R}^n$ in [KV86; Col+15]. The key property of Euclidean spaces employed in these contributions is the dilation invariant structure of $\mathbb{R}^n$. A careful reading of these works suggest that such a strategy could be successful also on Riemannian cones.

Along with the proof, we extend Lemma 2.3.14 showing the the p-capacity that p-capacitary potential behaves like the p-capacity of the cross sections approaching infinity.

Proposition 2.3.22 (Asymptotic behaviour of the p-capacity of level sets)**.** *In the same assumptions and notations of Theorem 2.3.20, set, for $i = 1, \dots, N$,*

$$v_i = \left(\frac{\widehat{\mathrm{Cap}}_p^{(i)}(\partial\Omega)}{\mathrm{AVR}(g; E_i)} \right)^{\frac{1}{n-p}} u^{-\frac{p-1}{n-p}}.$$

Then, we have

$$\lim_{s \to +\infty} \frac{\mathrm{Cap}_p(\{v_i = s\} \cap E_i; E_i)}{s^{n-p} |\mathbb{S}^{n-p}|} = \left(\frac{n-p}{p-1} \right)^{p-1} \mathrm{AVR}(g; E_i).$$

Moreover, as a byproduct, we obtain the following uniqueness result.

Proposition 2.3.23. *Let $((0, +\infty) \times L, \hat{g})$ be a n-dimensional Riemannian cone with* $\mathrm{Ric} \geq 0$, *where L is a closed connected smooth hypersurface. Let u be nonnegative p-harmonic function on $(0, +\infty) \times L$ satisfying $u(x) \leq \mathrm{C}\, \rho(x)^{-(n-p)/(p-1)}$ for every $x \in (0, +\infty) \times L$ for some constant* $\mathrm{C} \geq 0$. *Then, there exists a nonnegative $\gamma \in \mathbb{R}$ such that*

$$u(x) = \gamma \rho(x)^{-\frac{n-p}{p-1}}$$

holds on $(0, +\infty) \times L$.

In [KV86], a stronger result is proved in $\mathbb{R}^n \smallsetminus \{0\}$, where the upper bound and accordingly the function u are shifted by a constant. The analogue can be obtained in the same way in our setting. The proof of this proposition is inspired to the one of [HI01, Proposition 7.2]. The techniques used underline the relation between the p-capacitary potential and (weak) IMCF we will see in the next section, where the following proof will be rephrased in this latter geometric flavour.

Proof of Theorems 2.3.20 and 2.3.21 and Propositions 2.3.22 and 2.3.23. It is enough to prove the theorems in the case M has only one end. The proof of the general case then follows applying the result to each end. We will denote by $g_{(s)}$ the metric $s^{-2}\omega_s^* g$ on $[1/s, +\infty) \times$

L, being ω_s the family of diffeomorphisms defined in (2.2). We find it convenient to organise the proof in four steps. The first three steps are devoted to prove Theorem 2.3.20. In particular, the second and the third ones contain the proofs of Propositions 2.3.22 and 2.3.23 respectively. In the last, we complete the proof of the asymptotic behaviour of higher-order derivatives, mainly using Schauder estimates for p-harmonic functions.

Step 1. Suppose that (M, g) is $\mathscr{C}^0$-Asymptotically Conical. Define the family of functions $u_s : [1/s, +\infty) \times L \to \mathbb{R}$ as

$$u_s(x) = s^{\frac{n-p}{p-1}} u \circ \omega_s(x),$$

where ω_s is the map defined in (2.2). The aim of this step is to prove compactness of $(u_s)_{s\geq 1}$ with respect to local uniform convergence on $(0, +\infty) \times L$. In particular, by Theorem 1.2.6 (see Remark 1.2.7) any limit point w of the sequence $(u_s)_{s\geq 1}$ is p-harmonic with respect to the metric $\hat{g}$ on $(0, +\infty) \times L$. Moreover, there exists a positive constant C such that

$$\mathrm{C}^{-1} \rho(x)^{-\frac{n-p}{p-1}} \leq w(x) \leq \mathrm{C}\, \rho(x)^{-\frac{n-p}{p-1}} \tag{2.28}$$

is satisfied for every $x \in (0, +\infty) \times L$.

By Proposition 2.3.17 we have that

$$\mathrm{C}_1^{-1} \left(\mathrm{d}(o, x)\right)^{-\frac{n-p}{p-1}} \leq u(x) \leq \mathrm{C}_1 \left(\mathrm{d}(o, x)\right)^{-\frac{n-p}{p-1}}$$

holds on $M \smallsetminus \Omega$. In particular, since by Lemma 2.2.3 the distance function from o behaves asymptotically as the coordinate ρ, we deduce that there exist $S_2, \mathrm{C}_2 > 0$ such that

$$\mathrm{C}_2^{-1} \rho(x)^{-\frac{n-p}{p-1}} \leq u_s(x) \leq \mathrm{C}_2\, \rho(x)^{-\frac{n-p}{p-1}} \tag{2.29}$$

holds on $[1/s, +\infty) \times L$ for every $s \geq S_2$. Then, $(u_s)_{s\geq 1}$ is equibounded. By the gradient estimate Propositions 2.3.17 and 2.3.18

$$|\mathrm{D}u|(x) \leq \mathrm{C}_3\, u(x)^{\frac{n-1}{n-p}} \leq \mathrm{C}_4 \left(\mathrm{d}(o, x)\right)^{-\frac{n-1}{p-1}},$$

for some positive constants $\mathrm{C}_3, \mathrm{C}_4$. Hence, employing again Lemma 2.2.3 there exist $S_5, \mathrm{C}_5 > 0$ such that

$$|\mathrm{D}u_s|_{g_{(s)}}(x) \leq \mathrm{C}_5\, \rho(x)^{-\frac{n-1}{p-1}} \tag{2.30}$$

holds on $[1/s, +\infty) \times L$ for every $s \geq S_5$. By Lemma 2.2.2 we have that for some $\varepsilon > 0$ there is $S_6 > 0$ such that

$$|\mathrm{D}u_s|_{\hat{g}} \leq (1 + \varepsilon) |\mathrm{D}u_s|_{g_{(s)}}$$

holds for every $s \geq S_6$. Combining it with (2.30) we obtain that the family $(u_s)_{s\geq 1}$ is equicontinuous. By the Arzelà-Ascoli Theorem we conclude that $(u_s)_{s\geq 1}$ is precompact with respect to the local uniform convergence. (2.28) follows from (2.29).

Step 2. Here we prove that any limit point v of the family $(u_s)_{s\geq 1}$ has the form

$$v(x) = \gamma \rho(x)^{-\frac{n-p}{p-1}}, \tag{2.31}$$

for some nonnegative $\gamma \in \mathbb{R}$, proving also Proposition 2.3.23. Let $v : (0, +\infty) \times L \to \mathbb{R}$ be a nonnegative p-harmonic function satisfying the bound $v(x) \leq \mathrm{C}\,\rho(x)^{-\frac{n-p}{p-1}}$ on $(0, +\infty) \times L$.

Define the function $\mathfrak{e}_v : (0, +\infty) \to \mathbb{R}$ as

$$\mathfrak{e}_v(t) = \frac{R(t)}{r(t)},$$

where $[r(t), R(t)] \times L$ is the smallest annulus containing $\{v = 1/t\}$ for every $t \in (0, +\infty)$. Observe that, $\mathfrak{e}_v(t) \geq 1$ and $\mathfrak{e}_v(t) = 1$ if and only if $\{v = 1/t\}$ is a cross-section of the cone. By the Comparison Principle Theorem 1.2.9, using the potentials of $\{\rho = r(t)\}$ and $\{\rho = R(t)\}$ as barriers, we have that

$$r(t)\left(\frac{t}{T}\right)^{\frac{n-p}{p-1}} \leq \rho(x) \leq R(t)\left(\frac{t}{T}\right)^{\frac{n-p}{p-1}} \tag{2.32}$$

holds for every $x \in \{v = 1/T\}$ for every $T \geq t$. Hence, $\mathfrak{e}_v$ is nonincreasing. Moreover, since $(0, +\infty) \times L$ is connected, $\rho(x)^{-\frac{n-p}{p-1}}$ is p-harmonic and $\mathscr{C}^2((0, +\infty) \times L)$ and $|\mathrm{D}\rho^{-\frac{n-p}{p-1}}| \geq \frac{n-p}{p-1} R^{-\frac{n-1}{p-1}}$ holds on $(0, R) \times L$ for every $R > 0$, by the Strong Comparison Principle Theorem 1.2.9 the inequalities in (2.32) are strict unless $\{v = t\}$ is a cross-section. It is not hard to see that $\mathfrak{e}_v$ is scale invariant

Consider for $s \leq 1$ the family $v_s : [1, +\infty) \times L \to \mathbb{R}$ defined as

$$v_s(x) = s^{\frac{n-p}{p-1}} v \circ \omega_s(x)$$

where ω_s is defined in (2.2). Using the same argument of Step 1 we have that

$$v_s(x) \leq \mathrm{C}\,\rho(x)^{-\frac{n-p}{p-1}} \qquad \text{and} \qquad |\mathrm{D}v_s|(x) \leq \mathrm{C}\,\rho(x)^{-\frac{n-1}{p-1}}$$

holds on $(0, +\infty) \times L$ for some constant $\mathrm{C} > 0$. Hence, appealing to the Arzelà-Ascoli Theorem, $(v_s)_{s\leq 1}$ is precompact with respect to the local uniform convergence. Let w be a limit point for $(v_s)_{s\leq 1}$. Since $\mathfrak{e}$ is scale invariant, $\mathfrak{e}_{v_s}(t) = \mathfrak{e}_v(t/s)$. Then, $\mathfrak{e}_w(t)$ is constant equal to some $\mathfrak{e}_w \in [1, +\infty)$ that by monotonicity satisfies $\mathfrak{e}_w = \sup_t \mathfrak{e}_v(t) \in [1, +\infty)$. Suppose by contradiction that $\mathfrak{e}_w > 1$. Then the level $\{w = 1\} \subset [r(1), \mathfrak{e}_w r(1)] \times L$ and $\{w = 1\}$ touches both the cross-sections $\{\rho = r(1)\}$ and $\{\rho = \mathfrak{e}_w r(1)\}$ without being equal to either one. By (2.32) and the Strong Comparison Principl Theorem 1.2.9

$$r(1) t^{\frac{p-1}{n-p}} < \rho(x) < \mathfrak{e}_w r(1) t^{\frac{p-1}{n-p}}$$

holds for every $x \in \{w = 1/t\}$ for every $t > 1$. We therefore have that $\mathfrak{e}_w(t) < \mathfrak{e}_w$ which is a contradiction. In conclusion $\mathfrak{e}_w$ must be 1 and since $1 \leq \mathfrak{e}_v(t) \leq \mathfrak{e}_w = 1$, v is as in (2.31).

Step 3. By Step 2, any limit point w of the family $(u_s)_{s\geq 1}$ given by Step 1 has the form $\gamma\rho^{-\frac{n-p}{p-1}}$, where $\gamma > 0$ by (2.28). We are now going to prove that

$$\gamma = \widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega)^{\frac{1}{p-1}} \mathrm{AVR}(g)^{-\frac{1}{p-1}}. \tag{2.33}$$

The characterisation (2.33) ensures that any converging subsequence admits the same limit, proving that the whole family $(u_s)_{s\geq 1}$ locally uniformly converges to $\gamma\rho^{-\frac{n-p}{p-1}}$ as $s \to +\infty$. In particular, for every $\varepsilon > 0$ there exists a $S \geq 1$ such that

$$\sup_{\{\rho=s\}} s^{\frac{n-p}{p-1}} \left| u - \left(\frac{\widehat{\mathrm{Cap}}_p(\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{p-1}} \rho^{-\frac{n-p}{p-1}} \right| = \sup_{\{\rho=1\}} \left| u_s - \left(\frac{\widehat{\mathrm{Cap}}_p(\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{p-1}} \rho^{-\frac{n-p}{p-1}} \right| \leq \varepsilon$$

for every $s \geq S$ proving Theorem 2.3.20 and Proposition 2.3.22.

Observe that $\gamma > 0$ by Proposition 2.3.17. In order to prove (2.33), we find convenient to work with the auxiliary function

$$v = \left(\frac{u}{\gamma}\right)^{-\frac{p-1}{n-p}}.$$

Since w is a limit point for the family $(u_s)_{s\geq 1}$, there is a subsequence $(u_{s_k})_{k\in\mathbb{N}}$, s_k increasing and divergent as $k \to +\infty$, such that $u_{s_k} \to w = \gamma\rho^{-\frac{n-p}{p-1}}$ locally uniformly on $(0, +\infty) \times L$ as $k \to +\infty$. Then for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\} \subset \{v \leq s_k\} \subset \left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}$$

holds for every $k \geq k_\varepsilon$. The monotonicity of the p-capacity with respect to the inclusion (1.19) yields

$$\mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\}\right) \leq \mathrm{Cap}_p\left(\{v \leq s_k\}\right) \leq \mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}\right)$$

By (1.20) we can compute the p-capacity of level sets of v in terms of the capacity of $\partial\Omega$, that is

$$\mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\}\right) \leq \gamma^{-(p-1)} s_k^{n-p}\, \mathrm{Cap}_p(\partial\Omega) \leq \mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}\right)$$

Dividing each side by $|\mathbb{S}^{n-1}| s_k^{n-p}$, letting $k \to +\infty$ and using Lemma 2.3.14 we infer that

$$\left(\frac{n-p}{p-1}\right)^{p-1} \frac{\mathrm{AVR}(g)}{(1+\varepsilon)^{n-p}} \leq \gamma^{-(p-1)} \frac{\mathrm{Cap}_p(\partial\Omega)}{|\mathbb{S}^{n-1}|} \leq \left(\frac{n-p}{p-1}\right)^{p-1} \frac{\mathrm{AVR}(g)}{(1-\varepsilon)^{n-p}}$$

Then (2.33) follows by arbitrariness of $\varepsilon > 0$, keeping in mind the characterisation of $\mathrm{AVR}(g)$ in Definition 2.2.5 and the relation between the p-capacity and the normalised p-capacity (1.13).

Step 4. Suppose now (M, g) is $\mathscr{C}^{0,\alpha}$-Asymptotically Conical for $\alpha > 0$. By Theorem 1.2.1 $u_s \in \mathscr{C}^{1,\beta}_{\mathrm{loc}}((1/s, +\infty) \times L)$ for some $\beta \in (0, \alpha)$ and for every $K \subset (1/s, +\infty) \times L$ there exists constant $\mathrm{C} > 0$ such that

$$\|u_s\|_{\mathscr{C}^{1,\beta}(K)} \leq \mathrm{C}\, \|u_s\|_{\mathscr{C}^0((1/s,+\infty)\times L)}. \tag{2.34}$$

Since the metric $g_{(s)}$ locally $\mathscr{C}^{0,\alpha}$ -converges to $\hat{g}$ on $(0,+\infty)\times L$ by Lemma 2.2.2, the constant in (2.34) can be chosen not depending on s. Hence, by Arzelà-Ascoli Theorem, the family $(u_s)_{s\geq 1}$ $\mathscr{C}^1$-locally converges on $(0,+\infty)\times L$ to the function

$$\left(\frac{\widehat{\mathrm{Cap}}_p(\partial\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{p-1}}\rho^{-\frac{n-p}{p-1}} \tag{2.35}$$

as $s\to+\infty$. This proves Theorem 2.3.21 for $k=0$ and $\ell\leq 1$.

If (M,g) is $\mathscr{C}^{k,\alpha}$-Asymptotically Conical for $k\geq 1$ and $\alpha>0$, we already proved that (2.27) holds for $\ell\leq 1$. In particular, for every R there exists $S>0$ such that $|\mathrm{D}u_s|>0$ holds on every compact $K\subset(R,+\infty)\times L$ for every $s\geq S$. Applying Theorem 1.2.3, $u_s\in\mathscr{C}^\infty((R,+\infty)\times L)$ for every $s\geq S$. Moreover, for every $K\subset(R,+\infty)\times L)$ there exists a constant $\mathrm{C}>0$ such that

$$\|u_s\|_{\mathscr{C}^{k+1,\alpha}(K)}\leq \mathrm{C}\,\|u_s\|_{\mathscr{C}^0((1/s,+\infty)\times L)}. \tag{2.36}$$

Since $g_{(s)}$ locally $\mathscr{C}^{k+1,\alpha}$-converges to $\hat{g}$ on $(0,+\infty)\times L$, the constant in (2.36) can be chosen not depending on s. Since R is arbitrary, $(u_s)_{s\geq 1}$ is precompact with respect to the local $\mathscr{C}^{k+1}$-topology. Hence, $(u_s)_{s\geq 1}$ converges on compact subsets of $(0,+\infty)\times L$ to the function defined in (2.35) as $s\to+\infty$ up to its $(k+1)$-th derivative, concluding the proof of Theorem 2.3.21. □

As a consequence of Theorem 2.3.20, we extend Lemma 2.2.4 showing that the volume of level sets of a suitable function of the p-capacitary potential behaves like the volume of geodesic spheres approaching infinity.

Proposition 2.3.24 (Asymptotic behaviour of the area of level sets)**.** *Under the same assumptions and notations of Theorem 2.3.20, set, for $i=1,\dots,N$,*

$$v_i=\left(\frac{\widehat{\mathrm{Cap}}_p^{(i)}(\partial\Omega)}{\mathrm{AVR}(g;E_i)}\right)^{\frac{1}{n-p}}u^{-\frac{p-1}{n-p}}.$$

Then, we have

$$\mathrm{AVR}(g;E_i)=\lim_{s\to+\infty}\frac{|\{v_i\leq s\}\cap E_i|}{s^n|\mathbb{B}^n|}. \tag{2.37}$$

Moreover, if the assumptions of Theorem 2.3.21 are satisfied, then

$$\mathrm{AVR}(g;E_i)=\lim_{s\to+\infty}\frac{|\{v_i=s\}\cap E_i|}{s^{n-1}|\mathbb{S}^{n-1}|}. \tag{2.38}$$

Proof. We prove the statement in the case M has only one end, being the general case a direct consequence. We drop the index i in the following lines. By Theorem 2.3.20, for any $\varepsilon>0$ there exists $R_\varepsilon>0$ such that

$$(1-\varepsilon)\rho\leq v\leq(1+\varepsilon)\rho$$

holds on $\{\rho \geq R_\varepsilon\}$. Thus we have that

$$\left\{\rho \leq \frac{s}{1+\varepsilon}\right\} \subset \{v \leq s\} \subset \left\{\rho \leq \frac{s}{1-\varepsilon}\right\}.$$

By the monotonicty of the volume we have that

$$\left|\left\{\rho \leq \frac{s}{1+\varepsilon}\right\}\right| \leq |\{v \leq s\}| \leq \left|\left\{\rho \leq \frac{s}{1-\varepsilon}\right\}\right|.$$

Dividing each side by $s^n|\mathbb{B}^n|$ and passing to the limit as $s \to +\infty$ we can conclude that

$$\frac{\mathrm{AVR}(g)}{(1+\varepsilon)^n} \leq \lim_{s\to+\infty} \frac{|\{v \leq s\}|}{s^n|\mathbb{B}^n|} \leq \frac{\mathrm{AVR}(g)}{(1-\varepsilon)^n}.$$

Since $\varepsilon > 0$ is arbitrary, we conclude the proof of the first identity in (2.37). A straightforward computation relying on the identity

$$|\mathrm{D}v| = \left(\frac{\widehat{\mathrm{Cap}}_p(\partial\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{n-p}} \left(\frac{p-1}{n-p}\right) u^{-\frac{n-1}{p-1}}|\mathrm{D}u|$$

permits to write

$$\mathrm{AVR}(g) = \frac{1}{|\mathbb{S}^{n-1}|s^{n-1}} \int\limits_{\{v=s\}} |\mathrm{D}v|^{p-1}\,\mathrm{d}\sigma.$$

If the assumptions of Theorem 2.3.21 are satisfied, then $|\mathrm{D}v|$ approaches 1 at infinity, hence we have

$$\mathrm{AVR}(g) = \lim_{s\to+\infty} \frac{1}{|\mathbb{S}^{n-1}|s^{n-1}} \int\limits_{\{v=s\}} |\mathrm{D}v|^{p-1}\,\mathrm{d}\sigma = \lim_{s\to+\infty} \frac{|\{v=s\}|}{s^{n-1}|\mathbb{S}^{n-1}|},$$

which concludes the proof of (2.38). □

2.4 The (weak) Inverse Mean Curvature Flow

We conclude the chapter highlighting the relation between the p-capacitary potential and the (weak) Inverse Mean Curvature Flow, introduced by Huisken and Ilmanen in [HI01] to resolve the Riemannian Penrose Conjecture. Such relation was pointed out by Moser in [Mos07; Mos08] who showed that, after a suitable transformation, the p-capacitary potential converges to the weak IMCF as $p \to 1^+$. This result has been extended to cover a larger class of Riemannian manifolds in [KN09; MRS19].

We firstly recall here the definition of the weak IMCF, which is a solution to the equation

$$\mathrm{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right) = |\mathrm{D}w|$$

in a weak sense. Without any attempt to be as general as in [MRS19], we prove the existence of the weak IMCF on Riemannian manifolds with nonnegative Ricci curvature.

We follow here the original argument by Moser, specifying those passages that were unclear in the original paper [Mos07]. In the Asymptotically Conical setting, we instead prefer to assume the existence provided in [MRS19, Theorem 1.7], but we leverage all the work done in this section to improve the Li-Yau-type estimates on the solution to the weak IMCF problem, obtaining an upper bound with a sharp coefficient. We conclude by showing the asymptotic behaviour at infinity on Asymptotically Conical Riemannian manifolds, which is the analogue of Theorem 2.3.20 for the (weak) IMCF.

2.4.1 The level set formulation of the IMCF

Let (M, g) be a complete Riemannian manifold and Σ be a closed complete smooth strictly mean-convex hypersurface in (M, g). A classical (smooth) solution of the Inverse Mean Curvature Flow starting at Σ is a family of a diffeomorphisms $\Psi_t : \Sigma \to M$ with $t \in [0, T)$, $T > 0$, such that $\Psi_0(\Sigma) = \Sigma$ and

$$\frac{\partial \Psi_t}{\partial t}(q) = \frac{\nu_t(q)}{\mathrm{H}_t(q)} \qquad \text{for every } t \in [0, T) \text{ and } q \in \Sigma, \tag{2.39}$$

where $\nu_t(q)$ and $\mathrm{H}_t(q)$ are the unit normal vector and the mean curvature of $\Sigma_t = \Psi_t(\Sigma)$ at q. The mean curvature of Σ_t may tend to zero at some point as t approaches T. In this case the flow develops singularities a time T, which is called an extinguish time for the smooth IMCF. In the case $T = +\infty$ the flow is said immortal.

On the other hand, one can introduce a level-set formulation of the IMCF, where the evolved hypersurfaces are the level sets of a smooth function $w : M \to \mathbb{R}$ such that $|\mathrm{D}w| > 0$. One can see that (2.39) is equivalent to

$$\Delta_g^{(1)} w_1 = \mathrm{div}\left(\frac{\mathrm{D}w_1}{|\mathrm{D}w_1|}\right) = |\mathrm{D}w|, \tag{2.40}$$

since the left hand side represents the mean curvature of the level set $\{w = t\}$ and the right hand side is the inverse of the speed of the level set flow. This is a second order degenerate elliptic partial differential equation. One can temporarily abandon the strict constraint of dealing with diffeomorphisms as in (2.39) in favour of a lower regularity but a more flexible tool, looking for weak solutions of (2.40). We give the following definition of weak solutions to (2.40).

Definition 2.4.1 (Weak IMCF). *Let (M, g) be a complete n-dimensional Rieamannian manifold and $U \subseteq M$ an open subset. A locally Lipschitz function $w_1 \in \mathrm{Lip}_{\mathrm{loc}}(U)$ is a weak solution of* (2.40) *in U if for every $\psi \in \mathrm{Lip}_{\mathrm{loc}}(U)$ and any compact $K \subset U$ with $\{w_1 \neq \psi\} \subset K$,*

$$J^K_{w_1}(w_1) \leq J^K_{w_1}(\psi) \tag{2.41}$$

where

$$J^K_{w_1}(\psi) = \int_K |\mathrm{D}\psi| + \psi |\mathrm{D}w_1| \,\mathrm{d}\mu. \tag{2.42}$$

Given an open bounded subset $\Omega \subset M$ with smooth boundary, the weak IMCF starting at Ω is a weak solution $w_1 \in \mathrm{Lip}_{\mathrm{loc}}(M \smallsetminus \Omega)$ to

$$\begin{cases} \Delta_g^{(1)} w_1 = |\mathrm{D}w_1| & \text{on } M \smallsetminus \Omega, \\ w_1 = 0 & \text{on } \partial\Omega, \\ w_1 \to +\infty & \text{as } \mathrm{d}(x,o) \to +\infty, \end{cases} \tag{2.43}$$

for some $o \in \Omega$.

The reason to introduce the functional (2.42) is that (2.40) is neither a gradient flow nor the Eulero-Lagrange equation of some energy functional. $J^K_{w_1}(\,\cdot\,)$ is then obtained freezing the right-hand side.

In [HI01, Theorem 3.1] the authors proved that a solution to (2.43) exists provided the manifold satisfies some additional assumptions that are in force, for example, on asymptotically flat manifolds. This result has been achieved with a classical technique for elliptic partial differential equations, using the approximation by the regularised problem

$$\mathrm{div}\left(\frac{\mathrm{D}w_1^\varepsilon}{\sqrt{|\mathrm{D}w_1^\varepsilon|^2+\varepsilon}}\right) = \sqrt{|\mathrm{D}w_1^\varepsilon|^2+\varepsilon}.$$

Moreover, in [HI01, Thoerem 2.2 (iii)] the authors show that the properness of the solution guarantees its uniqueness without further requirements on the ambient manifold. We recall that a function $w : M \to \mathbb{R}$ is proper if its sublevel sets $\{w \leq t\}$ are compact for each $t \in \mathbb{R}$.

A different idea was thereafter proposed by Moser in [Mos07; Mos08] in the flat Euclidean case, then extended by Kotschwar and Ni [KN09] and Mari, Rigoli and Setti [MRS19] in the Riemannian framework. He adopted the p-capacitary potential (2.7) u_p associated with Ω to approximate the solution of (2.40). Indeed, it is simple to realise that the family of functions function $w_p = -(p-1)\log u_p$ solves the initial value problem

$$\begin{cases} \Delta_g^{(p)} w_p = |\mathrm{D}w_p|^p & \text{on } M \smallsetminus \Omega, \\ w_p = 0 & \text{on } \partial\Omega, \\ w_p \to +\infty & \text{as } \mathrm{d}(x,o) \to +\infty, \end{cases} \tag{2.44}$$

for some $o \in \Omega$. Moreover, (2.44) formally approximate (2.43) as $p \to 1^+$. This convergence is not only formal, but w_p converges to a proper $\mathrm{Lip}_{\mathrm{loc}}(M \smallsetminus \Omega)$ weak solution of (2.40) locally uniformly.

2.4.2 Properties of the IMCF on Riemannian manifolds

We recall here some results that follow from the analysis of the IMCF in this section and that will be useful both to prove its asymptotic behaviour and in the derivation of the Minkowski Inequality. We mainly refer to [FM20] and [HI01]. We are denoting with $\partial^* E$ the reduced boundary of a finite perimeter set E.

Definition 2.4.2 (Outward minimising and strictly outward minimising sets)**.** *Let (M,g) be a complete n-dimensional Riemannian manifold. Let $E \subset M$ be a bounded measurable set with* *finite perimeter. E is* outward minimising *if for any $F \supseteq E$ we have $|\partial^* E| \leq |\partial^* F|$, where by $\partial^* F$ we denote the reduced boundary of a set F. E is* strictly outward minimising *if it is outward minimising and whenever $|\partial^* E| = |\partial^* F|$ for some $F \supseteq E$ we have that $|F \setminus E| = 0$.*

We can define the *strictly outward minimising hull* Ω^* of an open bounded subset Ω with smooth boundary as

$$\Omega^* = \operatorname{Int} E \qquad \begin{array}{l} \text{for some bounded } E \text{ containing } \Omega \\ \text{such that } |E| = \inf\limits_{F \in \mathrm{SOMBE}(\Omega)} |F|, \end{array} \tag{2.45}$$

where by $\mathrm{SOMBE}(\Omega)$ we denote the family of all bounded strictly outward minimising sets containing Ω and $\operatorname{Int} E$ is the measure theoretic interior of E. As a consequence of [FM20, Theorem 1.1], [Bre22] and Proposition 2.2.9 we have the following result.

Theorem 2.4.3. *Let (M, g) a complete n-dimensional Riemannian manifold satisfying one of the following two conditions:*

- *(i) (M, g) has nonnegative Ricci curvature and Euclidean Volume Growth;*
- *(ii) (M, g) is $\mathscr{C}^0$-Asympotically Conical and the link L of the asymptotic cone satisfies* $\mathrm{Ric} \geq -f(\mathrm{d}(o, x))$ *for some nonnegative smooth function $f(t) = o(1)$ as $t \to +\infty$ and some $o \in M$.*

Then, every bounded open set $\Omega \subset M$ with finite perimeter admits a strictly outward minimising hull $\Omega^* \subset M$ in the sense of (2.45). Moreover, the set Ω^* is an open bounded maximal volume solution to the least area problem with obstacle Ω, that is $|\Omega^*| = \max |E|$, where E solves the problem

$$|\partial^* E| = \inf\{|\partial^* F| \mid F \text{ is bounded with finite perimeter and } \Omega \subseteq F\}.$$

Outward minimising sets can be characterised as those satisfying

$$|\partial \Omega| = |\partial \Omega^*|.$$

A set Ω is strictly outward minimising if and only if it coincides with Ω^*.

Moreover, we recall the following regularity result for the strictly outward minimising hull (see [HI01, Regularity Theorem 1.3], [FM20, Theorem 2.18] and in the references therein).

Theorem 2.4.4. *Let (M, g) be a complete n-dimensional Riemannian manifold satisfying one of the two conditions in Theorem 2.4.3. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then there exists a (possibily empty) subset* $\mathrm{Sing} \subset \partial\Omega^* \setminus \partial\Omega$ *of Hausdorff dimension $n - 8$,* $\mathrm{Sing} = \varnothing$ *if $n \leq 7$, such that*

- *(i) $\partial\Omega^*$ is smooth around each point in the interior of $\partial\Omega \cap \partial\Omega^*$ with respect to the induced topology on $\partial\Omega^*$;*
- *(ii) $\partial\Omega^*$ is smooth around each point of $\partial\Omega^* \setminus (\partial\Omega \cup \mathrm{Sing})$;*
- *(iii) $\partial\Omega^*$ is a $\mathscr{C}^{1,1}$-hypersurface around each point of $\partial\Omega^* \setminus \mathrm{Sing}$.*

The $\mathscr{C}^{1,1}$-regularity is essentially optimal (see [SZW91; SWZ93]). A direct consequence is that Ω^* is the unique open bounded maximal volume solution to the least area problem with obstacle Ω, in the sense that any other solution E satisfies $\operatorname{Int} E = \Omega^*$ (see [FM20, Theorem 2.19].

A simple variational argument shows that outward minimising subsets has nonnegative mean curvature. It is possible to give a notion of weak mean curvature for a subset which is a $\mathscr{C}^{1,1}$-hypersurface outside a singular set as the one naturally defined in the natural almost everywhere sense (we refer the reader to [HI01, Section 1] for a precise definition). The same variational argument implies that $\partial\Omega^*$ is weakly mean-convex.

As highlighted in [HI01], the notions of strictly outward minimising sets and weak IMCF are deeply related. Let $\Omega \subset M$ be an open bounded with smooth boundary and w the solution to (2.43). The following facts hold along the weak IMCF (see [HI01, Minimizing Hull Property 1.4] or [FM20, Section 3.2]):

(i) for every $t > 0$, the set $\{w < t\}$ is outward minimising;

(ii) for every $t \geq 0$, the set $\operatorname{Int}\{w \leq t\}$ is strictly outward minimising;

(iii) for every $t \geq 0$, the strictly outward minimising hull of $\{w < t\}$ coincides with $\operatorname{Int}\{w \leq t\}$;

(iv) for every $t \geq 0$ the set $\{w \leq t\}$ satisfies

$$J_w^K(\{w \leq t\}) \leq J_w^K(F)$$

for every $F \subset M$ with locally finite perimeter such that $\operatorname{Int}\{w \leq t\} \triangle F$ is compactly contained in $M \smallsetminus \overline{\Omega}$ and every compact K containing $\operatorname{Int}\{w \leq t\} \triangle F$, where

$$J_w^K(F) = |\partial^* F \cap K| - \int_{F \cap K} |\mathrm{D}w| \, \mathrm{d}\mu. \tag{2.46}$$

In particular, the weak IMCF starting at Ω suddenly jumps to its strictly outward minimising hull and it remains strictly outward minimising until $\{w < t\}$ is only outward minimising. At that moment, it jumps to its strictly outward minimising hull and continues as before. Differently from the first jump, at which the area can drop to a strictly less value, $\{w < t\}$ is still outward minimising for every $t > 0$, then $|\partial\{w < t\}| = |\partial\{w \leq t\}|$. Moreover, the area grows exponentially as provided in [HI01, Exponential Growth Lemma 1.6].

Lemma 2.4.5. *Let (M, g) be a complete n-dimensional Riemannian manifold satisfying one of the two conditions in Theorem 2.4.3. Let Ω be an open bounded subset with smooth boundary and w the solution to (2.43) starting at Ω. Then*

$$|\partial\{w \leq t\}| = |\partial\{w < t\}| = \mathrm{e}^t \, |\partial\Omega^*|$$

holds for every $t > 0$.

Beyond this, minimising properties are at the basis of the overlapping of the two notions of IMCF. It is actually easy to see that the classical definition of the IMCF may not coincide with its weak version, even if both exist. For example, the classical IMCF starting at some open bounded subset with strictly mean-convex boundary Ω foliate the manifold in the tubular neighbourhood of $\partial\Omega$ where it is defined. While, if Ω is not strictly outward minimising, the weak IMCF suddenly jumps to Ω^*. The reason is that the hypersurfaces obtained through the smooth IMCF only minimises the functional (2.42) only in the neighbourhood they foliate. This is the content of [HI01, Smooth Flow Lemma 2.3], we recall here (see also [HI01, Lemma 1.1]).

Lemma 2.4.6 (Smooth Flow Lemma)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold. Let $\Sigma \subset M$ be a closed complete smooth strictly mean-convex hypersurfce. For every $t \in [0, T)$, let $\Psi_t : \Sigma \to M$ be a classical solution to (2.39) starting at Σ and $\Sigma_t = \Psi_t(\Sigma)$. Let w a function such that $\{w = t\} = \Sigma_t$ and $\{w < t\}$ is the region bounded by Σ_t. Then $\{w < t\}$ minimises the functional $J_w^K(\cdot)$ in (2.46) for every K compactly contained in $\{0 \leq w < T\}$ and for every $t \in [0, T)$. In particular, w is a solution to the weak IMCF in $\{0 \leq w < T\}$, according to Definition 2.4.1.*

Observe that the solution provided in the previous lemma is only local. If it exists, the global weak solution in $M \smallsetminus \overline{\Omega}$ may not coincide with it. In the particular case the smooth flow is immortal, the function w is the global weak IMCF starting at the set enclosed by Σ and in particular each Σ_t is strictly outward minimising. Another case when the two definitions agrees is when the weak IMCF starts at some Ω which is both strictly outward minimising and has strictly mean-convex boundary, as proved in [HI01, Smooth Start Lemma 2.4].

Lemma 2.4.7 (Smooth Start Lemma)**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $\Omega \subset M$ be an open bounded strictly outward minimising subset with smooth strictly mean-convex boundary. Suppose there exists a solution w of (2.43) with initial condition Ω. Then $\{w = t\}$ coincides with the smooth classical solution for a short time.*

The (weak) IMCF problem shares the same compactness property of p-harmonic functions with respect to the local uniform convergence, thanks to [HI01, Compactness Theorem 2.1].

Theorem 2.4.8 (Compactness Theorem)**.** *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of solutions to (2.41) on U that converges locally uniformly to some function w with locally uniformly equibounded gradient. Then, $w \in \mathrm{Lip}_{\mathrm{loc}}(U)$ solves (2.41).*

Remark 2.4.9. *Suppose that $(U_n)_{n\in\mathbb{N}}$ is a sequence of open subsets converging to U open subset as $n \to +\infty$. Let g_n be a metric on U_n for every $n \in \mathbb{N}$ that locally uniformly converges to some metric g on U as $n \to +\infty$. The above theorem still holds if w_n is p-harmonic with respect to the metric g_n and w is p-harmonic on U with respect to g.*

To conclude, we stress another relation between the weak IMCF and the p-capacitary potential. Such a result is contained in the far more general [FM20, Theorem 1.2], having in mind the relation between the p-capacity and the normalised p-capacity given in Definition 1.3.1.

Theorem 2.4.10. *Let (M, g) be a complete n-dimensional Riemannian manifold satisfying one of the two properties in Theorem 2.4.3. Let Ω be an open bounded subset with smooth boundary. Then*

$$\lim_{p\to 1^+} \mathrm{Cap}_p(\partial\Omega) = |\partial\Omega^*|.$$

2.4.3 Existence of the weak IMCF

Here we prove the existence of the (weak) Inverse Mean Curvature Flow starting at $\Omega \subset M$ open bounded with smooth boundary in some natural classes of Riemannian manifolds.

First of all, we observe that a weak solution of

$$\Delta_g^{(p)} w_p = |\mathrm{D} w_p|^p \tag{2.47}$$

is also a minimiser of the functional

$$J^{p,K}_{w_p}(\psi) = \int_K \frac{1}{p}|\mathrm{D}\psi|^p + \psi|\mathrm{D} w_p|^p \,\mathrm{d}\mu, \tag{2.48}$$

which means that for every compact $K \subset M \setminus \overline{\Omega}$ and all functions $\psi \in W^{1,p}_{\mathrm{loc}}(M \setminus \overline{\Omega})$ such that $\operatorname{supp}(w_p - \psi) \subset K$ it holds

$$J^{p,K}_{w_p}(w_p) \leq J^{p,K}_{w_p}(\psi). \tag{2.49}$$

Indeed, consider $\psi \in W^{1,p}_{\mathrm{loc}}(M \setminus \Omega)$ that satisfies $\operatorname{supp}(w_p - \psi) \subset K \subset M \setminus \overline{\Omega}$, by (2.47) we obtain

$$\int_{M\setminus\overline{\Omega}} (w_p - \psi)|\mathrm{D} w_p|^p \,\mathrm{d}\mu = \int_{M\setminus\overline{\Omega}} |\mathrm{D} w_p|^{p-2}\big\langle \mathrm{D} w_p \,\big|\, \mathrm{D}\psi - \mathrm{D} w_p\big\rangle \,\mathrm{d}\mu.$$

Using Young's Inequality and since $w_p = \psi$ almost everywhere outside K it holds

$$\begin{aligned}\int_K \frac{1}{p}|\mathrm{D} w_p|^p + w_p|\mathrm{D} w_p|^p \,\mathrm{d}\mu &= \int_K \frac{1-p}{p}|\mathrm{D} w_p|^p + \psi|\mathrm{D} w_p|^p + |\mathrm{D} w_p|^{p-1}|\mathrm{D}\psi| \,\mathrm{d}\mu \\ &\leq \int_K \frac{1}{p}|\mathrm{D}\psi|^p + \psi|\mathrm{D} w_p|^p \,\mathrm{d}\mu,\end{aligned}$$

proving the validity of (2.49).

We now want to pass to the limit as $p \to 1^+$ in the functional (2.48). The main issue is that the functional depends on p also through the solution w_p to (2.47). For this reason we first need to control the sequence $(w_p)_{p>1}$. We recall that by Proposition 2.3.8 there exist a constant $\mathrm{C} > 0$

$$(n-p)\log\big(\mathrm{d}(x,o)\big) - \mathrm{C} \leq w_p(x) \leq (n-p)\log\big(\mathrm{d}(x,o)\big) + \mathrm{C}\,. \tag{2.50}$$

and the constant C is bounded as $p \to 1^+$. Moreover, by [MRS19, Theorem 2.24] we also have the following uniform gradient bound, which triggers the Arzelà-Ascoli Theorem and grants the local uniform convergence of $(w_p)_{p\geq 1}$.

Proposition 2.4.11. *Let (M, g) a complete p-nonparabolic n-dimensional Riemannian manifold with $\mathrm{Ric} \geq 0$, for $p > 1$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. The solution w_p to (2.44) satisfies*

$$|\mathrm{D} w_p| \leq \frac{n-p}{R}, \tag{2.51}$$

where R is the supremum of all $r > 0$ such that for each $x \in \partial\Omega$ there exists a geodesic ball $B_r \subset \Omega$ such that $x \in \partial B_r$.

We now have all the tools required to prove the following theorem.

Theorem 2.4.12. *Let (M, g) be a Riemannian Manifold with $\mathrm{Ric} \geq 0$ and Euclidean Volume Growth. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. For every $1 < p < n$, let w_p be the solution to (2.44), then $w_p \to w_1$ locally uniformly as $p \to 1^+$ and $w_1 \in \mathrm{Lip}(M \smallsetminus \overline{\Omega})$ solves the weak IMCF (2.43) starting at Ω. Moreover, given $o \in M$, the function w_1 satisfies*

$$(n-1)\log\big(\mathrm{d}(x,o)\big) - \mathrm{C} \leq w_1 \leq (n-1)\log\big(\mathrm{d}(x,o)\big) + \mathrm{C} \tag{2.52}$$

where $\mathrm{C} = \mathrm{C}(\Omega, M, n) > 0$, and

$$|\mathrm{D}w_1| \leq \max_{\partial\Omega}\{\mathrm{H}, 0\}, \tag{2.53}$$

where H is the mean curvature of $\partial\Omega$.

Proof. By (2.50) and (2.51) there exists a subsequence $(p_k)_{k\in\mathbb{N}}$, $p_k \to 1^+$ as $k \to +\infty$ and $w_1 \in \mathrm{Lip}_{\mathrm{loc}}(M \smallsetminus \Omega)$ satisfying (2.52) and (2.53), such that $w_{p_k} \to w_1$ locally uniformly as $k \to +\infty$. In particular, w_1 is proper. If we prove that w_1 satisfies the minimisation problem (2.41) we conclude that w_1 does not depend on the chosen subsequence and consequently the theorem.

Consider $K \subset M \smallsetminus \overline{\Omega}$ compact and $\psi \in \mathrm{Lip}_{\mathrm{loc}}(M \smallsetminus \Omega)$ with $\mathrm{supp}(w_1 - \psi) \subset K$. Choose $\eta \in \mathscr{C}^\infty_c(M \smallsetminus \overline{\Omega})$ with $0 \leq \eta \leq 1$ and $\eta = 1$ on K. Replacing ψ with $\eta\psi + (1-\eta)w_p$ and K with $\mathrm{supp}\,\eta$ in (2.49), we obtain

$$\begin{aligned}\int\limits_{\mathrm{supp}\,\eta} \frac{1}{p}|\mathrm{D}w_p|^p + w_p|\mathrm{D}w_p|^p \,\mathrm{d}\mu \leq \int\limits_{\mathrm{supp}\,\eta} &\frac{1}{p}|\eta\mathrm{D}\psi + (1-\eta)\mathrm{D}w_p + (\psi - w_p)\mathrm{D}\eta|^p \\ &+ (\eta\psi + (1-\eta)w_p)|\mathrm{D}w_p|^p \,\mathrm{d}\mu.\end{aligned}$$

Using Hölder's Inequality and reorganising the terms, we get

$$\begin{aligned}\int\limits_{\mathrm{supp}\,\eta} \frac{1}{p}|\mathrm{D}w_p|^p + \eta(w_p - \psi)|\mathrm{D}w_p|^p \,\mathrm{d}\mu \leq \frac{3^{p-1}}{p}\int\limits_{\mathrm{supp}\,\eta} &\eta^p|\mathrm{D}\psi|^p + (1-\eta)^p|\mathrm{D}w_p|^p \\ &+ |\psi - w_p|^p|\mathrm{D}\eta|^p \,\mathrm{d}\mu.\end{aligned} \tag{2.54}$$

Choose $p = p_k$ and replace ψ with w_1. Since $|\mathrm{D}w_{p_k}|^{p_k}$ is uniformly bounded by (2.51) and $w_{p_k} \to w_1$ locally uniformly as $k \to +\infty$ we have that

$$\limsup_{k\to+\infty} \frac{1}{p_k}\int\limits_{M\smallsetminus\overline{\Omega}} |\mathrm{D}w_{p_k}|^{p_k}(1 - 3^{p_k-1}(1-\eta)^{p_k})\,\mathrm{d}\mu = \limsup_{k\to+\infty}\int\limits_{M\smallsetminus\overline{\Omega}} \eta|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu,$$

while clearly

$$\limsup_{k\to+\infty}\int\limits_{M\smallsetminus\overline{\Omega}} \eta^{p_k}|\mathrm{D}w_1|^{p_k} - \eta(w_{p_k} - w_1)|\mathrm{D}w_{p_k}|^{p_k} + |\psi - w_{p_k}|^{p_k}|\mathrm{D}\eta|^{p_k}\,\mathrm{d}\mu = \int\limits_{M\smallsetminus\overline{\Omega}} \eta|\mathrm{D}w_1|\,\mathrm{d}\mu,$$

since $|\psi - w_1||\mathrm{D}\eta| = 0$ almost everywhere by construction. Plugging the two above limits into (2.54) we finally obtain

$$\limsup_{k\to+\infty} \int_{M\smallsetminus\overline{\Omega}} \eta|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu \leq \int_{M\smallsetminus\overline{\Omega}} \eta|\mathrm{D}w_1|\,\mathrm{d}\mu. \tag{2.55}$$

Moreover, since for every compact $K \subseteq M \smallsetminus \overline{\Omega}$ we have that

$$\sup_{k\in\mathbb{N}} \int_K |w_{p_k}| + |\mathrm{D}w_{p_k}|\,\mathrm{d}\mu < +\infty,$$

up to a not relabeled further subsequence, $\mathrm{D}w_{p_k}$ locally weakly* converges to $\mathrm{D}w_1$. Hence, employing Jensen's Inequality, we have

$$\begin{aligned}\liminf_{k\to+\infty} \fint_{\operatorname{supp}\eta} \eta|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu &\geq \liminf_{k\to+\infty} \fint_{\operatorname{supp}\eta} \eta^{p_k}|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu \\ &\geq \liminf_{k\to+\infty}\left(\fint_{\operatorname{supp}\eta} \eta|\mathrm{D}w_{p_k}|\,\mathrm{d}\mu\right)^{p_k} \geq \fint_{\operatorname{supp}\eta} \eta|\mathrm{D}w_1|\,\mathrm{d}\mu.\end{aligned} \tag{2.56}$$

for every $\eta \in \mathscr{C}^\infty_c(M \smallsetminus \overline{\Omega})$ with $0 \leq \eta \leq 1$. Combining it with (2.55), we get

$$\lim_{k\to+\infty} \int_{M\smallsetminus\overline{\Omega}} \eta|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu = \int_{M\smallsetminus\overline{\Omega}} \eta|\mathrm{D}w_1|\,\mathrm{d}\mu.$$

To conclude, replace p with p_k in (2.54). By the lower semicontinuity (2.56), we obtain

$$\liminf_{k\to+\infty} \int_{\operatorname{supp}\eta} \left(\frac{1}{p_k} - \frac{3^{p_k-1}}{p_k}(1-\eta)^{p_k}\right)|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu \geq \int_{\operatorname{supp}\eta} \eta|\mathrm{D}w_1|\,\mathrm{d}\mu.$$

Coupled with the uniform gradient bound, the local uniform convergence of $(w_{p_k})_{k\in\mathbb{N}}$ gives

$$\lim_{k\to+\infty} \int_{\operatorname{supp}\eta} \eta(w_{p_k} - \psi)|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu = \int_{\operatorname{supp}\eta} \eta(w_1 - \psi)|\mathrm{D}w_1|\,\mathrm{d}\mu,$$

$$\lim_{k\to+\infty} \int_{\operatorname{supp}\eta} \eta^{p_k}|\mathrm{D}\psi|^{p_k} + (1-\eta)^{p_k}|\mathrm{D}w_{p_k}|^{p_k}\,\mathrm{d}\mu = \int_{\operatorname{supp}\eta} \eta|\mathrm{D}\psi| + (1-\eta)|\mathrm{D}w_1|\,\mathrm{d}\mu$$

and

$$\lim_{k\to+\infty} \int_{\operatorname{supp}\eta} |w_{p_k} - \psi|^{p_k}|\mathrm{D}\eta|^{p_k}\,\mathrm{d}\mu = \int_{\operatorname{supp}\eta} |w_1 - \psi||\mathrm{D}\eta|\,\mathrm{d}\mu = 0,$$

since $w_1 = \psi$ outside K and $\mathrm{D}\eta = 0$ on K. Passing to the limit as $k \to +\infty$, we conclude that

$$\int_{M\smallsetminus\overline{\Omega}} \eta(|\mathrm{D}w_1| + w_1|\mathrm{D}w_1|)\,\mathrm{d}\mu \leq \int_{M\smallsetminus\overline{\Omega}} \eta(|\mathrm{D}\psi| + \psi|\mathrm{D}w_1|)\,\mathrm{d}\mu$$

holds for every $\eta \in \mathscr{C}^\infty_c(M \smallsetminus \overline{\Omega})$ with $0 \leq \eta \leq 1$. Since $\mathrm{supp}(w_1 - \psi) \subset \{\eta = 1\}$, w_1 minimises (2.41). □

Actually, a more precise gradient bound can be obtained in this case

Proposition 2.4.13 (Cheng-Yau-type estimate for the IMCF)**.** *Under the same assumptions of Theorem 2.4.12 there exists a positive constant* $\mathrm{C} = \mathrm{C}(n, M, \Omega) > 0$ *such that*

$$|\mathrm{D}w_1|(x) \leq \frac{\mathrm{C}}{\mathrm{d}(x, o)}$$

holds for almost every $x \in M \smallsetminus \Omega$.

Proof. It follows from Remark 4.9 and Theorem 1.7 in [MRS19]. □

In [MRS19, Theorem 1.7] the authors prove that existence of the (weak) IMCF is guaranteed whenever the Ricci curvature satisfies a negative and nondecreasing lower bound and a global L^1-Sobolev Inequality. A direct consequence of Proposition 2.2.9 is that we can apply [MRS19, Theorem 1.7] in the class of $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold satisfying

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(\mathrm{d}(x,o)+1)^2} \tag{2.57}$$

for some $o \in M$, $\kappa \in \mathbb{R}$ and every $x \in M$.

Theorem 2.4.14. *Let* (M, g) *be a complete* n*-dimensional* $\mathscr{C}^0$*-Asymptotically Conical Riemannian manifold satisfying the Ricci curvature bound* (2.57)*. Let* $\Omega \subset M$ *be an open bounded subset with smooth boundary. Let* w_p *be the solution to* (2.44)*, then* $w_p \to w_1$ *locally uniformly as* $p \to 1^+$ *and* $w_1 \in \mathrm{Lip}(M \smallsetminus \overline{\Omega})$ *solves the weak IMCF* (2.43) *starting at* Ω*. Moreover, given* $o \in \Omega$*, the function* w_1 *satisfies*

$$(n-1)\log\big(\mathrm{d}(x,o)\big) - \mathrm{C} \leq w_1 \leq (n-1)\log\big(\mathrm{d}(x,o)\big) + \mathrm{C} \tag{2.58}$$

and $|\mathrm{D}w_1| \leq \mathrm{C}$*, where* $\mathrm{C} = \mathrm{C}(\Omega, M, n)$*.*

Proof. The existence is guaranteed by [MRS19, Theorem 1.7], whose assumptions are satisfied in virtue of Proposition 2.2.9 and (2.57). The lower bound is the consequence of [MRS19, Theorem 1.7 and 1.3]. Let R be such that

$$-(p-1)\log G_p(x,o) \leq (n-p)\log \mathrm{d}(x,o) - \log \mathrm{C}_L \tag{2.59}$$

on $M \smallsetminus B(o,R)$ with the constant $\mathrm{C}_L = \mathrm{C}\; a\; \mathrm{Cap}_p(\overline{K}; E)$ described in the statement of Proposition 2.3.17. By [MRS19], $-(p-1)\log G_p(x,o)$ locally uniformly converges as $p \to 1^+$. Then we can choose a in the constant C_L independent of p so that $G_p(o,x) \geq a^{1/(p-1)}$

holds on $M \smallsetminus K$ where K is the bounded set in Definition 2.2.1. Moreover, by Theorem 2.4.10, since ∂K is smooth, $\mathrm{Cap}_p(\overline{K}; E)$ is bounded as $p \to 1^+$. Hence the constant C_L in (2.59) does not depends on p. Passing to the limit as $p \to 1^+$, in virtue of the upper bound in [MRS19, Theorem 1.7], we obtain

$$w \leq (n-1) \log \mathrm{d}(x,o) + \mathrm{C}$$

outside some $B(o,R)$. Since both the left and side and the right hand side are continuous, the bound can be extended to $M \smallsetminus \Omega$. □

We refer the reader to [MRS19, Theorem 1.7] for the precise gradient bound in the previous theorem. For what follows, we need to improve it to include a decay term depending on the distance. In [MRS19, Remark 4.9] the authors obtained a gradient bound that reads as

$$|\mathrm{D} w_1| \leq \frac{\mathrm{C}}{\mathrm{d}(x,o)^{1/\kappa'}}, \qquad \text{where } \kappa' = \frac{1+\sqrt{1+4\kappa^2}}{2} \geq 1,$$

for some constant $\mathrm{C} > 0$ depending only on Ω, the dimension n and the geometry of the ambient manifold. By [GW79] (see also [MRS19, Remark 4.5]) The exponent κ' can be chosen equal to 1 if the lower bound on the Ricci curvature is of the kind $\mathrm{Ric} \geq -(n-1) f(\mathrm{d}(x,o))$ for some smooth nonnegative function $f(t)$, such that

$$\int_0^{+\infty} t f(t) \, \mathrm{d}t < +\infty. \tag{2.60}$$

The [HI01, Weak Existence Theorem 3.1] the function w_1 satisfies

$$|\mathrm{D} w_1|(x) \leq \sup_{\partial\Omega \cap B(x,r)} \mathrm{H}^+ + \frac{\mathrm{C}}{r} \tag{2.61}$$

for almost every $x \in M \smallsetminus \Omega$ and for every r for which there exists a function $\psi \in \mathscr{C}^2(B(x,r))$ such that $\psi \geq \mathrm{d}(x,\cdot)^2$, $\psi(x) = 0$, $|\mathrm{D}\psi| \leq 3\mathrm{d}(x,\cdot)$, $\mathrm{D}^2\psi \leq 3g$ and $\mathrm{Ric} \geq -\mathrm{C}/r^2$ in $B(x,r)$. The existence of ψ is guaranteed if a sectional curvature lower bound is ensured. Otherwise, one can require an higher rate of convergence of the metric.

Proposition 2.4.15. *Let (M,g) be a complete n-dimensional $\mathscr{C}^1$-Asymptotically Conical Riemannian manifold satisfying the Ricci curvature bound (2.57). Let $\Omega \subset M$ be an open bounded subset with smooth boundary, $o \in M$. There exists a positive constant $\mathrm{C} = \mathrm{C}(n, M, \Omega) > 0$, such that the solution $w_1 \in \mathrm{Lip}(M \smallsetminus \Omega)$ of the weak IMCF starting at Ω, given by Theorem 2.4.14 satisfies*

$$|\mathrm{D} w_1|(x) \leq \frac{\mathrm{C}}{\mathrm{d}(x,o)} \tag{2.62}$$

for almost every $x \in M \smallsetminus \Omega$.

Proof. By [HI01, Weak Existence Theorem 3.1] the function w_1 satisfies (2.61) for almost every $x \in M \smallsetminus \Omega$. In virtue of the discussion in [HI01, Definition 3.3] (see also the proof of [HI01, Blowdown Lemma 7.1]) there exists a constant $\mathrm{C} > 0$ and $R > 0$ such that $r \geq \mathrm{C}\,\mathrm{d}(x,o)$ in (2.61) for every $x \in M \smallsetminus B(o,R)$. Then (2.62) follows taking r so that $\partial\Omega \cap B(x,r) = \varnothing$. □

Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold defined in Definition 2.2.1 and denote by $E_1, \dots, E_N$ the (finitely many) ends. Consider $\Omega \subset M$ open bounded subset with smooth boundary and $w : M \smallsetminus \Omega \to [0, +\infty)$ the weak IMCF w starting at Ω. As we did for the p-capacity we can define the area of the strictly outward minimising hull of Ω with respect to one end E_i. Indeed, there exists a time T such that $\{w \leq t\}$ contains the compact K defined in Definition 2.2.1 for every $t \geq T$. We then define the area of $\partial\Omega^*$ with respect to E_i as

$$|\partial\Omega^*|^{(i)} = \frac{|\partial\{w \leq t\} \cap E_i|}{\mathrm{e}^t} \tag{2.63}$$

for some $t \geq T$. Observe that such a definition is well posed by Lemma 2.4.5. Moreover, it is readily checked that $|\partial\Omega^*|$ splits as

$$|\partial\Omega^*| = \sum_{i=1}^{m} |\partial\Omega^*|^{(i)}.$$

Actually, if $K \subset \Omega$ then $|\partial\Omega^*|^{(i)} = |\partial\Omega^* \cap E_i|$. The relation between $\mathrm{Cap}_p^{(i)}(\partial\Omega)$ and $|\partial\Omega^*|^{(i)}$ is the same as the one in Theorem 2.4.10, as we are showing in the next lemma.

Lemma 2.4.16. *Let (M, g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying (2.57) and $E_1, \dots, E_N$ its ends with respect to the bounded K in Definition 2.2.1. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then,*

$$\lim_{p \to 1^+} \mathrm{Cap}_p^{(i)}(\partial\Omega) = |\partial\Omega^*|^{(i)}$$

holds for every $i = 1, \dots, N$

Proof. Let w_1 be the solution to the weak IMCF starting at Ω and T large enough so that $\{w_1 \leq T\}$ contains K in Definition 2.2.1. By Theorem 2.4.14 we have that w_p converges locally uniformly to w as $p \to 1^+$. In particular, for every $t \geq T$ there exists $p_t \in (1, n)$ such that $\{w_1 \leq T\} \subseteq \{w_p \leq t\}$ holds for every $p < p_t$. Arguing as in [FM20, Theorem 1.2], since an Isoperimetric Inequality is in force by Proposition 2.2.9, we can prove that $|\partial\{w \leq T\} \cap E_i| \leq \mathrm{C}_{n,p} \mathrm{Cap}_p(\partial\{w \leq T\}; E_i)$, for some constant $\mathrm{C}_{n,p}$ such that $\mathrm{C}_{n,p} \to 1$ as $p \to 1^+$. In particular, by the monotonicity of the p-capacity (1.19) and (1.20) we have that

$$\begin{aligned}|\partial\Omega^*|^{(i)} &\leq \mathrm{C}_{n,p}\, \mathrm{e}^{-T} \mathrm{Cap}_p\left(\{w_1 \leq T\} \cap E_i; E_i\right) \leq \mathrm{C}_{n,p}\, \mathrm{e}^{-T} \mathrm{Cap}_p\left(\left\{w_p \leq t\right\} \cap E_i; E_i\right) \\ &= \mathrm{C}_{n,p}\, \mathrm{e}^{t-2T} \mathrm{Cap}_p\left(\left\{w_p \leq T\right\} \cap E_i; E_i\right) \leq \mathrm{C}_{n,p}\, \mathrm{e}^{t-T} \mathrm{Cap}_p^{(i)}(\partial\Omega)\end{aligned}$$

letting $p \to 1^+$ and then $t \to T^+$, we have that

$$|\partial\Omega^*|^{(i)} \leq \lim_{p \to 1^+} \mathrm{Cap}_p^{(i)}(\partial\Omega).$$

If for some $i = 1, \dots, N$ the inequality is strict, then

$$|\partial\Omega^*| = \sum_{i=1}^{N} |\partial\Omega^*|^{(i)} < \sum_{i=1}^{N} \lim_{p \to 1^+} \mathrm{Cap}_p^{(i)}(\partial\Omega) = \lim_{p \to 1^+} \mathrm{Cap}_p(\partial\Omega) = |\partial\Omega^*|$$

which is a contradiction. □

On Riemannian cones, it is easy to see that cross-sections are strictly outward minimising since the function

$$w(x) = (n-1)\log(\rho(x)) \qquad \text{with } x \in (0,+\infty)\times L \tag{2.64}$$

is a solution to (2.41), where L is a closed $(n-1)$-dimensional manifold. The most natural question arising is whether cross-sections are still strictly outward minimising on Asymptotically Conical Riemannian manifolds. In the following lemma, we show that it is true for all cross-sections large enough, using a calibration argument that is somewhat inspired by Huisken-Ilmanen's weak formulation of the IMCF [HI01] and by arguments employed in [Rit17].

Lemma 2.4.17. *Let (M,g) be a complete n-dimensional $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Then $\{\rho \leq r\}$ is strictly outward minimising for r large enough, where ρ is the radial coordinate on the cone.*

Proof. Consider any $\varphi \in \mathscr{C}^\infty_c(\{\rho \geq r\})$, then

$$\int_{\{\rho\geq r\}} \operatorname{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\varphi\,\mathrm{d}\mu = -\int_{\{\rho\geq r\}} \left\langle \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\,\middle|\,\mathrm{D}\varphi\right\rangle \mathrm{d}\mu - \int_{\{\rho=r\}} \varphi\,\mathrm{d}\sigma_g.$$

Observe that the right hand side of the previous identity depends only on the coefficient of the metric and not on their derivatives. Since the metric g converges to the metric $\hat{g}$, for every $\varepsilon > 0$ there exists R_ε such that for every $r \geq R_\varepsilon$

$$\left| \int_M \operatorname{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\varphi\,\mathrm{d}\mu_g - \int_M \frac{n-1}{\rho}\varphi\,\mathrm{d}\mu_{\hat{g}}\right| \leq \varepsilon \int_M \frac{n-1}{\rho}\varphi\,\mathrm{d}\mu_{\hat{g}} \tag{2.65}$$

holds for every $\varphi \in \mathscr{C}^\infty_c(\{\rho \geq r\})$ and

$$\left||E|_g - |E|_{\hat{g}}\right| \leq \varepsilon |E|_{\hat{g}}$$

for every measurable $E \subset \{\rho \geq r\}$. By (2.65) and the density of compactly supported smooth functions, for every $E \subset \{\rho \geq r\}$ we have that

$$\int_E \operatorname{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\mathrm{d}\mu_g \geq (1-\varepsilon)\frac{n-1}{\sup_E \rho}|E|_{\hat{g}}.$$

Let F be a subset of finite perimeter containing $\{\rho < r\}$, then

$$\begin{aligned}\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\left(\frac{n-1}{\sup_F \rho}\right)|F \smallsetminus \{\rho < r\}|_g \leq (1-\varepsilon)\frac{n-1}{\sup_F\rho}|F\smallsetminus\{\rho\leq r\}|_{\hat{g}} &\leq \int_M \operatorname{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\mathrm{d}\mu_g \\ &\leq \int_{\partial^* F}\left\langle \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\,\middle|\,\nu_{\partial^*F}\right\rangle \mathrm{d}\sigma_g - \int_{\{\rho=r\}}\left\langle \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\,\middle|\,\nu_{\{\rho=r\}}\right\rangle\mathrm{d}\sigma_g \\ &\leq |\partial^*F|_g - |\{\rho=r\}|_g.\end{aligned}$$

This proves that $|\{\rho=r\}|_g \leq |\partial^*F|_g$. Moreover, $|\{\rho=r\}|_g = |\partial^*F|_g$ if and only if $|F\smallsetminus\{\rho<r\}|_g = 0$, which gives that $\{\rho<r\}$ is strictly outward minimising. □

2.4.4 Asymptotic behaviour of the IMCF

We are now ready to prove the analogue of Theorem 2.3.20 for the IMCF. To our knowledge, the following result with the explicit constant was known only in the flat case of $\mathbb{R}^n$. In this setting, the level sets of the weak IMCF become starshaped (and thus smooth) after a sufficiently long time as a consequence of [HI08, Theorem 2.7]. At this point, the constant could be easily deduced by classical results [Ger90; Urb90] for the smooth IMCF. It is worth pointing out that the arguments we employ got an important inspiration also from those in the proof of [HI01, Blowdown Lemma 7.1], that actually helped also in establishing Theorem 2.3.20. The following result actually simplifies [HI01, Blowdown Lemma 7.1] extending it from Asymptotically Flat to Asymptotically Conical Riemannian manifold and giving an explicit constant which depends on the geometry of the ambient manifold and the geometry of $\partial\Omega$.

Theorem 2.4.18 (Asymptotic behaviour of the Inverse Mean Curvature Flow)**.** *Let (M, g) be a complete n-dimensional $\mathscr{C}^1$-Asymptotically Conical Riemannian manifold with* Ric *satisfying*

$$\mathrm{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(1+\mathrm{d}(x,o))^2},$$

for some fixed $o \in M$, $\kappa \in \mathbb{R}$ and for every $x \in M$. Let $E_1, \dots, E_N$ be the (finitely many) ends of M with respect to the bounded K in Definition 2.2.1. Consider $\Omega \subset M$ be an open bounded subset with smooth boundary and $w : M \smallsetminus \Omega \to \mathbb{R}$ the solution to the problem (2.43)*. Then*

$$w(x) = (n-1)\log(\rho(x)) - \log\left(\frac{|\partial\Omega^*|^{(i)}}{|\mathbb{S}^{n-1}|\mathrm{AVR}(g;E_i)}\right) + o\,(1) \tag{2.66}$$

on E_i as $\mathrm{d}(o,x) \to +\infty$ for every $i = 1, \dots, N$, where $|\partial\Omega^*|^{(i)}$ and $\mathrm{AVR}(g;E_i)$ are defined respectively in (2.63) *and* (2.18)*.*

Clearly, we also obtain the analogue of Proposition 2.3.22.

Proposition 2.4.19 (Asymptotic behaviour of the area of level sets)**.** *Under the same assumptions and notations of Theorem 2.4.18, set, for $i = 1, \dots, N$,*

$$v_i = \left(\frac{|\partial\Omega_i^*|}{|\mathbb{S}^{n-1}|\mathrm{AVR}(g;E_i)}\right)^{\frac{1}{n-1}} \mathrm{e}^{\frac{w}{n-1}}.$$

Then, we have

$$\lim_{s\to+\infty} \frac{|\{v_i = s\} \cap E_i|}{s^{n-1}|\mathbb{S}^{n-1}|} = \mathrm{AVR}(g).$$

Observe that a similar result for the p-capacitary potential was obtained in (2.38). In that case a first order asymptotic behaviour for the p-capacitary potential was required. The reason is that the area is linked to the level sets of IMCF in the same way the p-capacity is linked to the level set of p-capacitary potential. A simple $\mathscr{C}^0$-convergence of the function is therefore enough.

As a byproduct we also obtain the counterpart of Proposition 2.3.23 proving that (2.64) is the unique solution on $(0, +\infty) \times L$ up to a constant. A refined result on the flat Euclidean space is the content of [HI01, Proposition 7.2].

Proposition 2.4.20. *Let $((0,+\infty)\times L, \hat{g})$ be a n-dimensional Riemannian cone with* $\mathrm{Ric}\geq 0$, *where L is a closed connected smooth hypersurface. Let w be a solution to (2.41) on $(0,+\infty)\times L$ satisfying $w(x)\geq (n-1)\log\rho(x)+\mathrm{C}$ for every $x\in(0,+\infty)\times L$ for some constant $\mathrm{C}\geq 0$. Then, there exists a $\gamma\in\mathbb{R}$ such that*

$$w(x)=(n-1)\log(\rho(x))+\gamma \qquad\qquad \text{with } x\in(0,+\infty)\times L,$$

holds on $(0,+\infty)\times L$.

Proof of Theorem 2.4.18 and Propositions 2.4.19 and 2.4.20. The proof follows the same lines of Theorem 2.3.20. We prove the theorem in the case M has only one end, since the general case follows applying the result to each end. We denote by $g_{(s)}$ the metric $s^{-2}\omega_s^* g$ on $[1/s,+\infty)\times L$, being ω_s the family of diffeomorphism defined in (2.2). We divide the proof in three steps. The second and the third ones contain the proofs of Propositions 2.4.19 and 2.4.20 respectively.

Step 1. Define for every $s\geq 1$ the family of functions $w_s:[1/s,+\infty)\times L\to\mathbb{R}$ as

$$w_s = w\circ\omega_s-(n-1)\log(s),$$

where ω_s is the differmorphism in (2.2). Employing Li-Yau-type estimates (2.58) in Theorem 2.4.14 and Proposition 2.4.15 as in the proof of Theorem 2.3.20, it is easy to show that $(w_s)_{s\geq 1}$ is equibounded and equi-Lipschitz. By the Arzelà-Ascoli Theorem, $(w_s)_{s\geq 1}$ is precompact with respect to the local uniform convergence on $(0,+\infty)\times L$. Moreover, by Theorem 2.4.8 every limit point u is a solution to the (weak) IMCF on $(0,+\infty)\times L$ and by (2.58) there exists a positive constant $\mathrm{C}>0$ such that

$$(n-1)\log(\rho(x))-\mathrm{C}\leq u(x)\leq (n-1)\log(\rho(x))+\mathrm{C}$$

is satisfied on $(0,+\infty)\times L$.

Step 2. Here we prove Proposition 2.4.20, inferring in particular that any limit point v of $(w_s)_{s\geq 1}$ satisfies

$$v(x)=(n-1)\log\rho(x)+\gamma$$

on $(0,+\infty)\times L$ for some $\gamma\in\mathbb{R}$. Let $\mathfrak{e}_v:\mathbb{R}\to\mathbb{R}$ be defined as

$$\mathfrak{e}_v(t)=\frac{R(t)}{r(t)},$$

where, for every $t\in\mathbb{R}$, $[r(t),R(t)]\times L$ is the smallest annulus containing $\{v=t\}$. Arguing as in Step 2 of Theorem 2.3.20, starting from any weak IMCF v on $(0,+\infty)\times L$ we can produce a function $u:(0,+\infty)\times L\to\mathbb{R}$ such that $\mathfrak{e}_u(t)$ is constant and is equal to $\mathfrak{e}_u=\sup_t\mathfrak{e}_v(t)\in[1,+\infty)$. Suppose by contradiction that $\mathfrak{e}_u>1$. Then the level $\{u=0\}\subset[r(0),\mathfrak{e}_u r(0)]\times L$ and touches both the cross-sections $\{\rho=r(0)\}$ and $\{\rho=\mathfrak{e}_u r(0)\}$ without being equal to neither. To compare the weak solution with the strong solution we have to place something in between of them. Perturb $\{\rho\leq r(0)\}$ outward and $\{\rho\leq\mathfrak{e}_u r(0)\}$ inward to obtain D^- and D^+ respectively with the following properties:

- $\{\rho \leq r(0)\} \subset D^- \subset \{u \leq 0\}$ and $\{u \leq 0\} \subset D^+ \subset \{\rho \leq \mathfrak{c}_u r(0)\}$;
- D^- and D^+ are starshaped with smooth strictly mean-convex boundary.

Then, the smooth IMCF starting at D^+ and D^- exists for all time by [Zho18, Theorem 3.1] and by Lemma 2.4.6 it coincides with the weak notion of the IMCF. Denote by $(D_t^-)_{t\geq 0}$ and $(D_t^+)_{t\geq 0}$ the sublevel sets of the two weak (and smooth) IMCF starting at D^- and D^+ respectively. By the Strong Comparison Principle for smooth flows we have that

$$\rho(x) > r(0)\, \mathrm{e}^{\frac{t}{n-1}} \text{ for } x \in \partial D_t^- \qquad \text{and} \qquad \rho(x) < \mathfrak{c}_u r(0)\, \mathrm{e}^{\frac{t}{n-1}} \text{ for } x \in \partial D_t^+. \tag{2.67}$$

On the other hand, by the Weak Comparison Theorem [HI01, Theorem 2.2(ii)], we get

$$D_t^- \subset \{u \leq t\} \subset D_t^+. \tag{2.68}$$

Coupling (2.67) and (2.68), we conclude that $\mathfrak{c}_u(t) < \mathfrak{c}_u$, which is the desired contradiction. Then $\mathfrak{c}_u = 1$ that completes the proof of Proposition 2.4.20, as in Step 2 of Theorem 2.3.20.

Step 3. Let $u = (n-1)\log\rho + \gamma$ be a limit point of the family $(w_s)_{s\geq 1}$. We are now going to prove that

$$\gamma = \log\left(\frac{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\right). \tag{2.69}$$

The characterisation proves Proposition 2.4.19 and implies that the limit point is unique, concluding the proof. We work with the auxiliary function

$$v = \mathrm{e}^{\frac{w-\gamma}{n-1}}.$$

Since u is a limit point for the family $(w_s)_{s\geq 1}$, there exists a subsequence $(w_{s_k})_{k\in\mathbb{N}}$, s_k increasing and divergent as $k \to +\infty$, such that $w_{s_k} \to u = (n-1)\log\rho + \gamma$ locally uniformly on $(0, +\infty) \times L$ as $k \to +\infty$. Then, for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\} \subset \{v \leq s_k\} \subset \left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}$$

holds for every $k \geq k_\varepsilon$. By Lemma 2.4.17 we can assume k_ε large enough so that both the left most and the right most sets are strictly outward minimising for any $k \geq k_\varepsilon$. Then the perimeter is monotone by inclusion and by Lemma 2.4.5 we have

$$\left|\left\{\rho = \frac{s_k}{1+\varepsilon}\right\}\right| \leq \mathrm{e}^{\gamma}\, s_k^{n-1}|\partial\Omega^*| \leq \left|\left\{\rho = \frac{s_k}{1-\varepsilon}\right\}\right|$$

Dividing both sides by $|\mathbb{S}^{n-1}|s_k^{n-1}$, letting $k \to +\infty$ and using Lemma 2.2.4 we infer that

$$\frac{\mathrm{AVR}(g)}{(1+\varepsilon)^{n-1}} \leq \mathrm{e}^{\gamma}\,\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|} \leq \frac{\mathrm{AVR}(g)}{(1-\varepsilon)^{n-1}}$$

Then, (2.69) follows by arbitrariness of $\varepsilon > 0$. □

Firstly, observe that we do not have the analogue of Theorem 2.3.21 for the IMCF. The asymptotic behaviour of higher-order derivatives of the p-capacitary potential is indeed a consequence of the higher regularity of the functions ruling the flow, given by Schauder estimates. The solution to the weak IMCF is only locally Lipschitz, so we cannot infer that the gradient stops vanishing approaching infinity, that would be the starting point to use the classic regularity theory for elliptic partial differential equations.

The result above is to be compared with [HI01, Lemma 7.1]. We obtain here an explicit characterisation of the constants c_λ that is

$$c_\lambda = -(n-1)\log\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\lambda\right).$$

The constants appearing in (2.66) satisfy

$$\log\left(\frac{\mathrm{AVR}(g;E_i)|\mathbb{S}^{n-1}|}{|\partial\Omega^*|^{(i)}}\right) = \lim_{p\to 1^+} -(p-1)\log\left[\left(\frac{\mathrm{Cap}_p^{(i)}(\partial\Omega)}{\mathrm{AVR}(g;E_i)}\right)^{\frac{1}{p-1}}\right]$$

thanks to Lemma 2.4.16, where the quantities in the limit are the ones in (2.26), transformed in accordance with $w_p = -(p-1)\log u_p$. Hence, even if by Theorem 2.4.14 $w_p \to w_1$ only locally uniformly as $p \to 1^+$, the asymptotic behaviour of w_1 is anyway effected by this procedure.

To conclude, we need $\mathscr{C}^1$-convergence of the metric in order to apply Proposition 2.4.15. This requirement can be weakened in favour of a mild $\mathscr{C}^0$-convergence if, for example the Ricci curvature satisfies a bound $\mathrm{Ric} \geq -(n-1)f(\mathrm{d}(x,o))$ for some nonnegative function $f(t)$ that satisfies (2.60). A Cheng-Yau-type estimate as in Theorem 1.2.11 with a constant $\mathrm{C} > 0$ such that $(p-1)\,\mathrm{C}$ remains bounded as $p \to 1^+$ would be a sufficient way to avoid the requirement and also an interesting tool on its own, with important consequences in the study of the weak IMCF.

3

Monotonicity formulas on p-nonparabolic Riemannian manifolds

3.1 Strucutre of the chapter

In this chapter, we are going to prove our Monotonicity Formulas in the p-nonparabolic setting. The results we present here are the natural extensions of the ones shown in [AM20; AFM20] as well as of the ones obtained in [FMP19; AFM22]. In the first two mentioned papers the authors established the monotonicity along the level set of the harmonic capacitary potential, respectively in $\mathbb{R}^n$ and in a general nonparabolic manifold with nonnegative Ricci curvature, whereas in the second two papers an analogous theory has been developed in the case of the p-capacitary potential in the Euclidean setting. More precisely, in [FMP19], the authors worked out the smooth computations and took advantage of the fact that the p-capacitary potential associated with a convex domain is smooth and has no critical points (see [Col+15; Lew77]), whereas the main technical achievement in [AFM22] is the treatment of the general case when the critical points are present and even possibly arranged in sets of full measure. On the other hand, the approach presented in [AFM22] only produces *effective monotonicity inequalities* (10), which are anyway sufficient to prove the Extended Minkowski Inequality in the flat setting, as mentioned in the Introduction. In Section 3.2 we are introducing the quantities F_p^β and F_p^∞, built on the level sets of a solution u to (2.7) for $1 < p < n$ and $\beta \in [0, +\infty)$, and the precise statement of their Monotonicity-Rigidity in the range $\beta \geq -(n-p)/[(n-1)(p-1)]$. Section 3.3 is devoted to reformulating the results in a conformally related Riemannian manifold. Even if one can perform all computations in the original Riemannian manifold, the structure of the conformal change simplifies a lot the calculation and produces Monotonicity results for the related problem. We conclude the section by showing the relation between the Monotonicity-Rigidity theorems in the two different settings. The last section contains the proof of the results in the conformal setting. The main difficulty amounts to ensuring that the monotonicity survives the singular values of u, which, as far as we know, could even form a set of positive measure. Inspired by the analysis in [GV21], where the authors were forced to face severe technical problems caused by the typical low regularity of the nonsmooth setting, we compute the derivative of our integral quantities (3.1) in the distributional sense, appealing to the full strength of the coarea formula (see Appendix B) and exploiting the Sobolev regularity of the gradient of a p-harmonic function (see Appendix C for a self-contained proof of this result).

From now on we assume that (M, g) is a complete Riemannian manifold of dimension $n \geq 3$ and $1 < p < n$.

3.2 Statement of the Monotonicity-Rigidity Theorems

As just said we now introduce the Monotone quantities we are going to study. Consider a complete Riemannian manifold (M, g) of dimension $n \geq 3$. Let $u : M \setminus \Omega \to \mathbb{R}$ be a solution of (2.7). For $\beta \in [0, +\infty)$ we consider the function

$$F_p^\beta(t) = t^{\beta\frac{(n-1)(p-1)}{(n-p)}} \int\limits_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)}\, \mathrm{d}\sigma \tag{3.1}$$

defined for every $t \geq 1$ such that $|\{u = 1/t\} \cap \mathrm{Crit}(u)| = 0$, which is fulfilled for almost every $t \in [1, +\infty)$ by Lemma B.1. We also set

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u|, \tag{3.2}$$

that is defined on the whole $[1, +\infty)$. If $1/t$ is a regular value for u, then F_p^β is differentiable at t for every $\beta \in [0, +\infty)$ and its derivative is

$$(F_p^\beta)'(t) = -\beta t^{\beta\frac{(n-1)(p-1)}{(n-p)}-2} \int\limits_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-1}\Big(\mathrm{H} - \tfrac{(n-1)(p-1)}{(n-p)}|\mathrm{D}\log u|\Big)\, \mathrm{d}\sigma. \tag{3.3}$$

As said before, the aim of this section is to prove Monotonicity-Rigidity Theorems for $t \mapsto F_p^\beta(t)$ and $t \mapsto F_p^\infty(t)$. We start with the statement for $(n-p)/[(p-1)(n-1)] < \beta < +\infty$.

Theorem 3.2.1 (Monotonicity-Rigidity Theorem for F_p^β)**.** *Let (M, g) be a p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subset M$ be a bounded open subset with smooth boundary. Let F_p^β be the function defined in (3.1) with $(n-p)/[(n-1)(p-1)] < \beta < +\infty$. Then $F_p^\beta \in W^{2,1}(1, +\infty)$ and*

$$\begin{aligned}(F_p^\beta)'(t) = -\beta\left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} &\int\limits_{\{u\leq 1/t\}\setminus \mathrm{Crit}\, u} u^{2-\beta\frac{(p-1)(n-1)}{(n-p)}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \\ &\left\{\left[\beta - \frac{(n-p)}{(n-1)(p-1)}\right]\left[\mathrm{H} - \left[\frac{(n-1)(p-1)}{(n-p)}\right]|\mathrm{D}\log u|\right]^2\right. \\ &+ (p-1)\left[\beta + \frac{p-2}{p-1}\right]\frac{|\mathrm{D}^\top|\mathrm{D}u||^2}{|\mathrm{D}u|^2} \\ &\left.+ \left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 + \mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|}, \frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\right\}\mathrm{d}\mu\end{aligned} \tag{3.4}$$

and

$$
\begin{aligned}
(F_p^\beta)''(t) = \beta \left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} t^{\beta\frac{(n-1)(p-1)}{(n-p)}-4} \int\limits_{\{u=1/t\}} |\mathrm{D}u|^{(\beta+1)(p-1)-2} \\
\Bigg\{ \left[\beta - \frac{(n-p)}{(n-1)(p-1)}\right]\left[\mathrm{H} - \left[\frac{(n-1)(p-1)}{(n-p)}\right]|\mathrm{D}\log u|\right]^2 \\
+ (p-1)\left[\beta + \frac{p-2}{p-1}\right]\frac{\left|\mathrm{D}^\top|\mathrm{D}u|\right|^2}{|\mathrm{D}u|^2} \\
+ \left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 + \mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|}, \frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\Bigg\}\, \mathrm{d}\mu
\end{aligned} \tag{3.5}
$$

hold for almost every $t \in [1, +\infty)$. In particular, F_p^β admits a convex and monotone nonincreasing $\mathscr{C}^1$-representative. Moreover, $(F_p^\beta)'(t_0) = 0$ at some $t_0 \geq 1$ such that $1/t_0$ regular value for u if and only if $(\{u \leq 1/t_0\}, g)$ is isometric to

$$
\left([\tau_0, +\infty) \times \{u = 1/t_0\}, \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}}\right),
$$

$$
\textit{with } \tau_0 = \left(\frac{|\{u = 1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.
$$

In this case $\{u = 1/t_0\}$ is a connected totally umbilical hypersurface with constant mean curvature in $(M \smallsetminus \Omega, g)$.

The threshold case $\beta = (n-p)/[(n-1)(p-1)]$ is a little more delicate since we cannot easily pass to the limit as in [AM20]. The first term (3.5) cannot be handled via Monotone Convergence Theorem or Dominated Convergence Theorem. What we can actually prove is a weaker version of Theorem 3.2.1.

Theorem 3.2.2 (Monotonicity-Rigidity Theorem for the threshold case)**.** *Let (M, g) be a p-nonparabolic Riemannian manifold with* Ric ≥ 0*. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Let F_p^β be the function defined in (3.1) with $\beta = (n-p)/[(n-1)(p-1)]$. Then, $F_p^\beta \in W^{1,1}(1, +\infty)$, $(F_p^\beta)' \in \mathrm{BV}(1, +\infty)$ and*

$$
\begin{aligned}
(F_p^\beta)'(t) \leq -\beta \left(\frac{(n-2)(p-1)}{(n-p)}\right)^{(\beta+1)(p-1)} \int\limits_{\{u \leq 1/t\} \smallsetminus \mathrm{Crit}\, u} u^{2-\beta\frac{(p-1)(n-1)}{(n-p)}} |\mathrm{D}u|^{(\beta+1)(p-1)-1} \\
\Bigg\{ (p-1)\left[\beta + \frac{p-2}{p-1}\right]\frac{\left|\mathrm{D}^\top|\mathrm{D}u|\right|^2}{|\mathrm{D}u|^2} \\
+ \left|\mathrm{h} - \frac{\mathrm{H}}{n-1}g^\top\right|^2 + \mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|}, \frac{\mathrm{D}u}{|\mathrm{D}u|}\right)\Bigg\}\, \mathrm{d}\mu
\end{aligned} \tag{3.6}
$$

hold for almost every $t \in [1, +\infty)$. *In particular,* F_p^β *admits a monotone nonincreasing* $\mathrm{AC}(1, +\infty)$*-representative. Moreover,* $(F_p^\beta)'(t_0) = 0$ *at some* $t_0 \geq 1$ *such that* $1/t_0$ *regular value for* u *if and only if* $(\{u \leq 1/t_0\}, g)$ *is isometric to*

$$\left([\tau_0, +\infty) \times \{u = 1/t_0\}, \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}} \right),$$

$$\text{with } \tau_0 = \left(\frac{|\{u = 1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

In this case $\{u = 1/t_0\}$ *is a connected totally umbilical hypersurface with constant mean curvature in* $(M \smallsetminus \Omega, g)$.

We observe that the rigidity statement is expressed in terms of the derivative. However, if $F_p^\beta(t) = F_p^\beta(T)$ for $1 \leq t < T < +\infty$ such that $1/t$ and $1/T$ are regular values for u, the rigidity statement still triggers. Indeed, since the set of regular values is open, monotonicity ensures the existence of a decreasing sequence $(t_j)_{j \in \mathbb{N}}$ such that $t_j \to t$ as $j \to +\infty$, $1/t_j$ is regular for u and $(F_p^\beta)'(t_j) = 0$. Since $t \mapsto F_p^\beta(t)$ is smooth in a neighbourhood of t, this implies that $(F_p^\beta)'(t) = 0$ hence the splitting of $\{u \leq 1/t\}$ follows.

We conclude dealing with the limit case $\beta = +\infty$.

Theorem 3.2.3 (Monotonicity-Rigidity theorem for F_p^∞)**.** *Let* (M, g) *be a* p*-nonparabolic Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let* $\Omega \subseteq M$ *be an open bounded subset with smooth boundary. Let* F_p^∞ *be the function defined in* (3.2). *Then,* F_p^∞ *is a continuous monotone nonincreasing function. Furthermore, we have*

$$\left[\mathrm{H}_g - \frac{(n-1)(p-1)}{(n-p)} |\mathrm{D} \log u|_g \right](x_t) = -(p-1) \frac{\partial}{\partial \nu_t} \log \frac{|\mathrm{D}u|_g}{u^{\frac{n-1}{n-p}}}(x_t) \geq 0 \tag{3.7}$$

where $x_t \in \{u = 1/t\}$ *is the point that realises* $\sup_{\{u=1/t\}} |\mathrm{D}u|_g / u^{(n-1)/(n-p)}$ *and* $\nu_t = -\mathrm{D}u/|\mathrm{D}u|_g$ *is the unit normal to* $\{u = 1/t\}$. *Moreover,* $F_p^\infty(t_0) = F_p^\infty(T)$ *for some* $t_0 < T$ *or the equality holds in* (3.7) *for some* t_0 *such that* $1/t_0$ *is regular for* u *if and only if* $(\{u \leq 1/t_0\}, g)$ *is isometric to*

$$\left([\tau_0, +\infty) \times \{u = 1/t_0\}, \mathrm{d}\tau \otimes \mathrm{d}\tau + \left(\frac{\tau}{\tau_0}\right)^2 g_{\{u=1/t_0\}} \right),$$

$$\text{with } \tau_0 = \left(\frac{|\{u = 1/t_0\}|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

In this case $\{u = 1/t_0\}$ *is a connected totally umbilical hypersurface with constant mean curvature in* $(M \smallsetminus \Omega, g)$.

A direct consequence of the monotonicity of F_p^∞ is the following regularity theorem for the p-capacitary potential.

Corollary 3.2.4. *The function* F_p^∞ *is strictly positive and continuous. In particular, every level of* u *has at least one regular point.*

We want also to emphasise that these theorems can be applied in particular in $\mathbb{R}^n$ for every Ω open bounded with smooth boundary, where they naturally extend the Monotonicity-Rigidity Theorems in [FMP19; AFM22].

3.3 Conformal formulation of the Monotonicity-Rigidity Theorems

This section is essentially divided into two parts. The first one contains the preparatory material rewriting the geometric quantities involved in the theorem of the previous section. We essentially take the computations from [AFM22; FMP19] and extend them to the Riemannian case, where the Ricci tensor is allowed to be non zero. We then prove a splitting principle that will be the core of the Rigidity statement of the Monotonicity theorems. The second part contains the statements of the conformal versions of Theorems 3.2.1 to 3.2.3. We conclude by showing that the two formulations are equivalent.

3.3.1 The conformal setting

Let $u : M \smallsetminus \Omega \to \mathbb{R}$ be the solution of the problem (2.7). As shown first in [AM20], it is easier to work in the conformally related Riemannian manifold $(M \smallsetminus \Omega, \tilde{g})$, where $\tilde{g}$ is given by

$$\tilde{g} = u^{2\left(\frac{p-1}{n-p}\right)} g. \tag{3.8}$$

It is also convenient to consider the new variable

$$\varphi = -\frac{(p-1)(n-2)}{(n-p)} \log u, \tag{3.9}$$

so that the metric $\tilde{g}$ can be equivalently rewritten as

$$\tilde{g} = \mathrm{e}^{-\frac{2\varphi}{n-2}} g.$$

Observe that, in light of the optimal $\mathscr{C}^{1,\alpha}$-regularity of u, the metric $\tilde{g}$ is not a smooth Riemannian metric, at least at points where the gradient of u vanishes. The following computations make sense on the whole $M \smallsetminus \Omega$ if they involve only the first derivative, while if higher derivatives are required they have to be intended as carried out outside the critical set of u.

Fix local coordinates $(x^1, \dots, x^n)$ in M. Using standard formulas (see [Bes08; HE73] and [FMP19; Fog20] for the same computations in the Euclidean case) we obtain

$$\tilde{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} - \frac{1}{n-2}\left(\delta^{\gamma}_{\alpha}\partial_{\beta}\varphi + \delta^{\gamma}_{\beta}\partial_{\alpha}\varphi - g_{\alpha\beta}g^{\gamma\eta}\partial_{\eta}\varphi\right),$$

where $\tilde{\Gamma}^{\gamma}_{\alpha\beta}$ and $\Gamma^{\gamma}_{\alpha\beta}$ are the Christoffel symbols associated with the metric $\tilde{g}$ and g respectively. We denote by ∇ the covariant derivative with respect to the metric $\tilde{g}$. For any function $f \in \mathscr{C}^2$, the Hessian and the Laplacian of f are respectively

$$\nabla\nabla f = \mathrm{DD} f + \frac{1}{n-2}\left(\mathrm{d}f \otimes \mathrm{d}\varphi + \mathrm{d}\varphi \otimes \mathrm{d}f - \langle \mathrm{D}f \,|\, \mathrm{D}\varphi\rangle_g g\right), \tag{3.10}$$

$$\Delta_{\tilde{g}} f = \mathrm{e}^{\frac{2\varphi}{n-2}}\left(\Delta_g f - \langle \mathrm{D}f \,|\, \mathrm{D}\varphi\rangle_g\right). \tag{3.11}$$

Applying this relations to u, recalling that $\Delta_g^{(p)} u = 0$, we can compute its p-Laplacian with respect to the conformally related Riemannian metric

$$\begin{aligned}\Delta_{\tilde g}^{(p)} u = \operatorname{div}_{\tilde g}\left(|\nabla u|_{\tilde g}^{p-2}\nabla u\right) &= |\nabla u|_{\tilde g}^{p-2}\Delta_{\tilde g} u + (p-2)|\nabla u|_{\tilde g}^{p-4}\nabla^2 u(\nabla u, \nabla u)\\ &= (p-1)\frac{|\nabla u|_{\tilde g}^{p}}{u},\end{aligned}$$

where we used the fact that by definition of $\tilde g$ we have

$$|\mathrm{D}u|_g = u^{\frac{p-1}{n-p}}|\nabla u|_{\tilde g}. \tag{3.12}$$

As a consequence we have that φ is p-harmonic with the respect the conformal p-Laplacian.

Lemma 3.3.1. *Let u be a solution to (2.7), let φ defined in (3.9) and $\tilde g$ the metric obtained through (3.8). Then,*

$$\Delta_{\tilde g}^{(p)}\varphi = 0$$

on $M \smallsetminus (\overline{\Omega} \cup \operatorname{Crit} u)$.

Proof. The above computations can be performed on $M \smallsetminus (\overline{\Omega} \cup \operatorname{Crit} u)$. By definition (3.9) it is enough to prove that $\log u$ is p-harmonic with respect to the metric $\tilde g$. Thus,

$$\begin{aligned}\Delta_{\tilde g}^{(p)} \log u &= \operatorname{div}_{\tilde g}\left(u^{-(p-1)}|\nabla u|_{\tilde g}^{p-2}\nabla u\right)\\ &= -(p-1)u^{-p}|\nabla u|_{\tilde g}^{p} + u^{-(p-1)}\Delta_{\tilde g}^{(p)} u,\end{aligned}$$

that vanishes by (3.12). □

We now want to describe the Ricci curvature of $\tilde g$ in order to have a complete reformulation of problem (2.7).

Lemma 3.3.2. *Let u be a solution to (2.7), let φ defined in (3.9) and $\tilde g$ the metric obtained through (3.8). Then,*

$$\operatorname{Ric}_{\tilde g} - \nabla\nabla\varphi + \frac{\mathrm{d}\varphi\otimes\mathrm{d}\varphi}{n-2} = \operatorname{Ric}_g + \left(\frac{|\nabla\varphi|_{\tilde g}^2}{n-2} - \left(\frac{p-2}{n-2}\right)\frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_{\tilde g}^2}\right)\tilde g$$

on $M \smallsetminus (\overline{\Omega} \cup \operatorname{Crit} u)$.

Proof. Recall that the Ricci curvature tensor can be computed as

$$\operatorname{Ric}_{\tilde g} = \operatorname{Ric}_g + \mathrm{DD}\varphi + \frac{\mathrm{d}\varphi\otimes\mathrm{d}\varphi}{n-2} + \left(\frac{\Delta_g\varphi - |\mathrm{D}\varphi|_g^2}{n-2}\right) g.$$

By (3.10), (3.11) and the definition of $\tilde g$ we get that

$$\operatorname{Ric}_{\tilde g} = \operatorname{Ric}_g + \nabla\nabla\varphi - \frac{\mathrm{d}\varphi\otimes\mathrm{d}\varphi}{n-2} + \left(\frac{\Delta_{\tilde g}\varphi + |\nabla\varphi|_{\tilde g}^2}{n-2}\right)\tilde g. \tag{3.13}$$

Using Lemma 3.3.1 we have that

$$\Delta_{\tilde{g}}\varphi = -(p-2)\frac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2_{\tilde{g}}},$$

that coupled with (3.13) concludes the proof. □

Problem (2.7) can be translated in terms of $\tilde{g}$ and φ as

$$\begin{cases} \Delta^{(p)}_{\tilde{g}}\varphi = 0 & \text{in } M \setminus (\overline{\Omega} \setminus \operatorname{Crit}\varphi), \\ \operatorname{Ric}_{\tilde{g}} - \nabla\nabla\varphi + \dfrac{\mathrm{d}\varphi\otimes\mathrm{d}\varphi}{n-2} = \operatorname{Ric}_g & \\ \qquad + \left(\dfrac{|\nabla\varphi|^2_{\tilde{g}}}{n-2} - \left(\dfrac{p-2}{n-2}\right)\dfrac{\nabla\nabla\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2_{\tilde{g}}}\right)\tilde{g} & \text{in } M \setminus (\overline{\Omega} \setminus \operatorname{Crit}\varphi), \\ \varphi = 0 & \text{on } \partial\Omega, \\ \varphi(x) \to +\infty & \text{as } \mathrm{d}(x,o) \to +\infty. \end{cases} \tag{3.14}$$

Problem (3.14) coincides with the one studied in [AM20] for $p = 2$, whereas for general p and for M the flat Euclidean space one can recognise the problem in [FMP19; AFM22].

We now recall the relations between the geometric quantities on level sets of u and φ. The following identities are exactly the same as in [FMP19], since the curvature of the ambient space does not play any role in it. First of all, observe that

$$|\nabla\varphi|_{\tilde{g}} = \frac{(n-2)(p-1)}{(n-p)}\frac{|\mathrm{D}u|_g}{u^{\frac{n-1}{n-p}}}. \tag{3.15}$$

In particular, $\{\varphi = s\}$ is a regular level set if and only if $\left\{u = e^{-(n-p)s/[(p-1)(n-2)]}\right\}$ is a regular level set. Moreover, we recognise from the above expression and the estimate (2.12) the fundamental property of $|\nabla\varphi|$ to be uniformly bounded.

Lemma 3.3.3. *Let (M, g) be a p-nonparabolic Riemannian manifolds with* $\operatorname{Ric} \geq 0$*. Let $\Omega \subset M$ be an open bounded subset with smooth boundary and u be the solution to (2.7). Let φ and $\tilde{g}$ associated with u and g through (3.9) and (3.8). Then, there exists a constant* C *such that*

$$|\nabla\varphi|_{\tilde{g}} \leq \mathrm{C} \tag{3.16}$$

on the whole $M \setminus \Omega$.

Consider the g-unit vector field defined as

$$\nu_g = -\frac{\mathrm{D}u}{|\mathrm{D}u|_g} = \frac{\mathrm{D}\varphi}{|\mathrm{D}\varphi|_g}$$

and the $\tilde{g}$-unit vector field defined as

$$\nu_{\tilde{g}} = -\frac{\nabla u}{|\nabla u|_{\tilde{g}}} = \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}}.$$

We can now compute the second fundamental form h_g and $\mathrm{h}_{\tilde{g}}$ of the level sets of u and φ with respect the metric g and the conformally related metric $\tilde{g}$ obtaining

$$\mathrm{h}_g = -\frac{\mathrm{DD}u}{|\mathrm{D}u|_g} = \frac{\mathrm{DD}\varphi}{|\mathrm{D}\varphi|_g} \qquad \text{and} \qquad \mathrm{h}_{\tilde{g}} = -\frac{\nabla\nabla u}{|\nabla u|_{\tilde{g}}} = \frac{\nabla\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}}$$

respectively. Tracing the above expressions with respect to the induced metric on the level sets we obtain the mean curvature in the two settings

$$\mathrm{H}_g = -\frac{\Delta_g u}{|\mathrm{D}u|_g} + \frac{\mathrm{DD}u(\mathrm{D}u, \mathrm{D}u)}{|\mathrm{D}u|_g} \qquad \text{and} \qquad \mathrm{H}_{\tilde{g}} = \frac{\Delta_{\tilde{g}}\varphi}{|\nabla\varphi|_{\tilde{g}}} - \frac{\nabla\nabla\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^3_{\tilde{g}}}.$$

Recalling that $\Delta^{(p)}_g u = 0$ and $\Delta^{(p)}_{\tilde{g}}\varphi = 0$ we can rewrite the above quantities as

$$\mathrm{H}_g = \frac{p-1}{p}\frac{\langle \mathrm{D}|\mathrm{D}u|^p_g \,|\, \mathrm{D}u\rangle_g}{|\mathrm{D}u|^{p+1}} = (p-1)\frac{\mathrm{DD}u(\mathrm{D}u, \mathrm{D}u)}{|\mathrm{D}u|^3_g}$$

and

$$\mathrm{H}_{\tilde{g}} = -\frac{p-1}{p}\frac{\langle \nabla|\nabla\varphi|^p_{\tilde{g}} \,|\, \nabla\varphi\rangle_{\tilde{g}}}{|\nabla\varphi|^{p+1}_{\tilde{g}}} = (p-1)\frac{\nabla\nabla\varphi(\nabla\varphi, \nabla\varphi)}{|\nabla\varphi|^3_{\tilde{g}}}. \tag{3.17}$$

The second fundamental forms h_g and $\mathrm{h}_{\tilde{g}}$ are then related by the following formula

$$\mathrm{h}_{\tilde{g}} = u^{\frac{p-1}{n-p}}\left(\mathrm{h}_g - \frac{p-1}{n-p}\frac{|\mathrm{D}u|_g}{u} g^{\top}\right),$$

where $g^{\top}$ is the metric induced on the level set of u. Tracing the above identity with respect to $\tilde{g}^{\top}$ we obtain the relation between the mean curvatures H_g and $\mathrm{H}_{\tilde{g}}$

$$\mathrm{H}_{\tilde{g}} = u^{-\frac{p-1}{n-p}}\left(\mathrm{H}_g - \frac{(n-1)(p-1)}{(n-p)}\frac{|\mathrm{D}u|_g}{u}\right). \tag{3.18}$$

Finally, we recall the relation between the Lebesgue measures $\mathrm{d}\mu_g$ and $\mathrm{d}\mu_{\tilde{g}}$ on M

$$\mathrm{d}\mu_{\tilde{g}} = u^{\frac{n(p-1)}{(n-p)}}\mathrm{d}\mu_g \tag{3.19}$$

and the relation between the induced measure on the level sets

$$\mathrm{d}\sigma_{\tilde{g}} = u^{\frac{(n-1)(p-1)}{(n-p)}}\mathrm{d}\sigma_g.$$

To conclude we rewrite the p-Bochner formula for the function φ. The following proposition is obtained replacing the Ricci tensor with respect to the metric $\tilde{g}$ in the second equation of (3.14) with the expression coming from p-Bochner formula and plugging in $\Delta^{(p)}_{\tilde{g}}\varphi = 0$. Since the details are straightforward, we do not report them.

Proposition 3.3.4. *Let (M, g) be a p-nonparabolic manifold with* Ric ≥ 0 *and $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Let φ be a solution to (3.14). Then, in a neighbourhood of each point where $|\nabla\varphi| > 0$ we have*

$$\begin{aligned}\Delta_{\tilde g}|\nabla\varphi|^p_{\tilde g} + (p-2)\frac{\nabla\nabla|\nabla\varphi|^p_{\tilde g}(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^2_{\tilde g}} - \frac{n-p}{n-2}\left\langle \nabla|\nabla\varphi|^p_{\tilde g} \,\middle|\, \nabla\varphi\right\rangle_{\tilde g} \\ = p|\nabla\varphi|^{p-2}_{\tilde g}\left(|\nabla\nabla\varphi|^2_{\tilde g} + p(p-2)\left\langle \nabla|\nabla\varphi|_{\tilde g} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde g}}\right\rangle^2_{\tilde g} + \mathrm{Ric}(\nabla\varphi,\nabla\varphi)\right).\end{aligned} \tag{3.20}$$

Formula (3.20) is at the base of the proof of Monotonicity Theorems. For F_p^∞ it permits to show that $|\nabla\varphi|^p$ satisfies a maximum principle, while for F_p^β leads to the proof of its convexity. As a consequence, we have the following lemma that is the key of the rigidity part of the Montonicity theorems.

Lemma 3.3.5 (Splitting principle)**.** *Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Let φ be a solution to (3.14). Assume that $|\nabla\varphi|_{\tilde g} > 0$ on $\{s_0 \leq \varphi \leq s_1\}$ for some $s_0, s_1 \in [0, +\infty)$, $s_0 < s_1$ with s_1 possibly infinite, and that $\big|\nabla|\nabla\varphi|_{\tilde g}\big|_{\tilde g} = 0$ on this region. Then, the Riemannian manifold $(\{s_0 \leq \varphi \leq s_1\}, \tilde g)$ is isometric to the Riemannian product $([s_0, s_1] \times \{\varphi = s_0\}, \mathrm{d}s^2 + \tilde g_{\{\varphi = s_0\}})$ and φ is an affine function of s. In this case $\{\varphi = s_0\}$ is connected and totally geodesic in $(M \smallsetminus \Omega, \tilde g)$.*

Proof. If $\big|\nabla|\nabla\varphi|_{\tilde g}\big|_{\tilde g} = 0$, by (3.20) we have that $|\nabla\nabla\varphi|_{\tilde g} = 0$ on $\{s_0 \leq \varphi \leq s_1\}$. In particular φ is harmonic with respect to the metric $\tilde g$ and by Kato-type Identity Proposition 1.2.12 for $p = 2$ we obtain

$$\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\tilde g^\top\right|^2_{\tilde g^\top} = 0 \qquad \text{and} \qquad \left|\nabla^\top|\nabla\varphi|_{\tilde g}\right|_{\tilde g} = 0$$

on $\{s_0 \leq \varphi \leq s_1\}$. Hence the rigidity part of Proposition 1.2.12 applies giving that $\{s_0 \leq \varphi \leq s_1\}$ splits into the warped product $([s_0, s_1] \times \{\varphi = s_0\}, \mathrm{d}s^2 + \eta^2(s)\tilde g_{\{\varphi = s_0\}})$ where

$$\eta(s) = \left(\frac{\varphi'(s_0)}{\varphi'(s)}\right)^{\frac{1}{n-1}}.$$

Deriving it one more time and taking into account that $\varphi''(s) = 0$, we have that $\eta'(s) = 0$ and since $\eta(s_0) = 1$, we have that η is constantly one. Moreover, $\varphi'(s) = \varphi'(s_0)$ which, by integrating it, gives that φ is an affine function of s. □

3.3.2 Conformal formulation of the Monotonicity-Rigidity Theorems

The remaining part of this section is devoted to rephrasing the Monotonicity-Rigidity Theorems 3.2.1 and 3.2.3 in the conformally related setting and proving their equivalence.

From now on, given a p-nonparabolic manifold (M, g) with Ric ≥ 0 *and u a solution to (2.7), φ and $\tilde g$ will indicate the solutions of (3.14) obtained from u and g trough (3.8) and (3.9).*

Using the previous relations, the family of functions F_p^β for $\beta \in [0,+\infty)$ and F_p^∞ defined in (3.1) and (3.2) can be rewritten in terms of $\tilde{g}$ and φ. Then, we define $\Phi_p^\beta : [0,+\infty) \to \mathbb{R}$ by

$$\Phi_p^\beta(s) = \int\limits_{\{\varphi = s\}} |\nabla \varphi|_{\tilde{g}}^{(\beta+1)(p-1)} \, \mathrm{d}\sigma_{\tilde{g}}, \tag{3.21}$$

for every $0 \leq \beta < +\infty$ and $\Phi_p^\infty : [0,+\infty) \to \mathbb{R}$ by

$$\Phi_p^\infty(s) = \sup_{\{\varphi = s\}} |\nabla \varphi|_{\tilde{g}}.$$

The function Φ_p^β can be obtained from F_p^β through a change of variable that is

$$\Phi_p^\beta(s) = F_p^\beta\left(\mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)} s}\right).$$

For $\beta < +\infty$ it holds that

$$(\Phi_p^\beta)'(s) = \frac{(n-p)}{(p-1)(n-2)} \, \mathrm{e}^{\frac{n-p}{(p-1)(n-2)} s} (F_p^\beta)'\left(\mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)} s}\right), \tag{3.22}$$

whenever one side of the identity makes sense.

We are now ready to reformulate Theorems 3.2.1, 3.2.2 and 3.2.3 in the conformal setting.

Theorem 3.3.6 (Monotonicity-Rigidity theorem for Φ_p^β)**.** *Let (M,g) be a p-nonparabolic Riemannian manifold with* Ric ≥ 0. *Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Let Φ_p^β be the function defined in* (3.1) *with $(n-p)/[(n-1)(p-1)] < \beta < +\infty$. Then, $\Phi_p^\beta \in W^{2,1}_{\mathrm{loc}}(0,+\infty)$ and*

$$\begin{aligned}(\Phi_p^\beta)'(s) = -\beta \, \mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)} s} \int\limits_{\{\varphi \geq s\} \smallsetminus \mathrm{Crit}\,\varphi} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)} \varphi} \, |\nabla \varphi|_{\tilde{g}}^{(\beta+1)(p-1)-1} \\ \left\{ \left| \mathrm{h}_{\tilde{g}} - \frac{\mathrm{H}_{\tilde{g}}}{n-1} \tilde{g}^\top \right|^2_{\tilde{g}^\top} + (p-1)^2 \left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right] \frac{\left|\nabla^\perp |\nabla \varphi|_{\tilde{g}}\right|^2_{\tilde{g}}}{|\nabla \varphi|^2_{\tilde{g}}} \right. \\ \left. + (p-1)\left[\beta + \frac{p-2}{p-1}\right] \frac{\left|\nabla^\top |\nabla \varphi|_{\tilde{g}}\right|^2_{\tilde{g}}}{|\nabla \varphi|^2_{\tilde{g}}} + \mathrm{Ric}\left(\frac{\nabla \varphi}{|\nabla \varphi|_{\tilde{g}}}, \frac{\nabla \varphi}{|\nabla \varphi|_{\tilde{g}}}\right) \right\} \mathrm{d}\mu_{\tilde{g}},\end{aligned} \tag{3.23}$$

holds for almost every $s \in [0,+\infty)$. In particular, Φ_p^β admits a monotone nonincreasing $\mathscr{C}^1$-representative. Moreover, $(\Phi_p^\beta)'(s_0) = 0$ at some $s_0 \geq 0$ regular for φ if and only if $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to

$$\left([s_0,+\infty) \times \{\varphi = s_0\}, \mathrm{d}s \otimes \mathrm{d}s + \tilde{g}_{\{\varphi = s_0\}}\right).$$

In this case $\{\varphi = s_0\}$ is a connected totally geodesic hypersurface in $(M \smallsetminus \Omega, \tilde{g})$.

Theorem 3.3.7 (Monotonicity-Rigidity theorem for the threshold case)**.** *Let (M, g) be a p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Let Φ_p^β be the function defined in (3.1) with $\beta = (n-p)/[(n-1)(p-1)]$. Then, $\Phi_p^\beta \in W^{1,1}_{\mathrm{loc}}(0, +\infty)$, $(\Phi_p^\beta)' \in \mathrm{BV}_{\mathrm{loc}}(0, +\infty)$ and*

$$\begin{aligned}(\Phi_p^\beta)'(s) \leq -\beta\, \mathrm{e}^{\frac{(n-p)}{(p-1)(n-2)}s} \int\limits_{\{\varphi \geq s\} \smallsetminus \mathrm{Crit}\,\varphi} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi} |\nabla \varphi|_{\tilde{g}}^{(\beta+1)(p-1)-2} \Bigg\{ \left| \mathrm{h}_{\tilde{g}} - \frac{\mathrm{H}_{\tilde{g}}}{n-1} \tilde{g}^\top \right|^2_{\tilde{g}^\top} \\ + (p-1)\left[\beta + \frac{p-2}{p-1}\right] \frac{|\nabla^\top |\nabla\varphi|_{\tilde{g}}|^2_{\tilde{g}}}{|\nabla\varphi|^2_{\tilde{g}}} + \mathrm{Ric}\left(\frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}}, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}}\right) \Bigg\}\, \mathrm{d}\mu_{\tilde{g}},\end{aligned} \tag{3.24}$$

holds for almost every $s \in [0, +\infty)$. In particular, Φ_p^β admits a monotone nonincreasing $\mathrm{AC}_{\mathrm{loc}}$-representative. Moreover, $(\Phi_p^\beta)'(s_0) = 0$ at some $s_0 \geq 0$ regular for φ if and only if $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to

$$\left([s_0, +\infty) \times \{\varphi = s_0\}, \mathrm{d}s \otimes \mathrm{d}s + \tilde{g}_{\{\varphi = s_0\}}\right).$$

In this case $\{\varphi = s_0\}$ is a connected totally geodesic hypersurface in $(M \smallsetminus \Omega, \tilde{g})$.

Theorem 3.3.8 (Monotonicity-Rigidity theorem for Φ_p^∞)**.** *Let (M, g) be a p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Let Φ_p^∞ be the function defined in (3.21). Then, Φ_p^β is a continuous monotone nonincreasing function. Furthermore, we have*

$$\mathrm{H}_{\tilde{g}}(x_s) = -(p-1)\frac{\partial}{\partial \nu_s} \log |\nabla\varphi|_{\tilde{g}}(x_s) \geq 0 \tag{3.25}$$

where $x_s \in \{\varphi = s\}$ is the point where $\sup_{\{\varphi = s\}} |\nabla\varphi|_{\tilde{g}}$ is achieved and ν_s is the unit normal to the level set of φ. Moreover, $\Phi_p^\infty(s_0) = \Phi_p^\infty(S)$ for some $s_0 < S$ with s_0 regular value for φ or the equality holds in (3.25) for some s_0 regular value for φ if and only if $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to

$$\left([s_0, +\infty) \times \{\varphi = s_0\}, \mathrm{d}s \otimes \mathrm{d}s + \tilde{g}_{\{\varphi = s_0\}}\right).$$

In this case $\{\varphi = s_0\}$ is a connected totally geodesic hypersurface in $(M \smallsetminus \Omega, \tilde{g})$.

The proof of these theorems is postponed to the next section. We are now going to deduce the Monotonicity Formulas (3.7) and (3.4), as well as the rigidity statements for F_p^β and F_p^∞, from their conformal versions. Differently from Theorem 3.2.1, in Theorem 3.3.6 we do not prove any identity for the second derivative of Φ_p^β because it does not seem to show interesting convexity properties of such a function. On the other hand, (3.5) follows from (3.4) simply by the coarea formula.

Proof of Theorems 3.2.1 to 3.2.3 after Theorems 3.3.6 to 3.3.8. To obtain both the identity (3.4) and the estimate (3.6) for F_β^p, we are only left to rewrite (3.23) in terms of u and apply (3.22).

In (3.9), (3.15) and (3.19) we have already highlighted the relation between u and φ, $|\mathrm{D}u|_g$ and $|\nabla\varphi|_{\tilde{g}}$ and $\mathrm{d}\mu_g$ and $\mathrm{d}\mu_{\tilde{g}}$ respectively. The other quantities are related as follows

$$\begin{aligned}\left|\mathrm{h}_{\tilde{g}} - \frac{\mathrm{H}_{\tilde{g}}}{n-1}\tilde{g}^{\top}\right|^2_{\tilde{g}^{\top}} &= u^{-2\frac{p-1}{n-p}}\left|\mathrm{h}_g - \frac{\mathrm{H}_g}{n-1}g^{\top}\right|^2_{g^{\top}},\\ \frac{\left|\nabla^{\top}|\nabla\varphi|_{\tilde{g}}\right|^2_{\tilde{g}}}{|\nabla\varphi|^2_{\tilde{g}}} &= u^{-2\frac{p-1}{n-p}}\frac{\left|\mathrm{D}^{\top}|\mathrm{D}u|_g\right|^2_g}{|\mathrm{D}u|^2_g},\\ \frac{\left|\nabla^{\perp}|\nabla\varphi|_{\tilde{g}}\right|^2_{\tilde{g}}}{|\nabla\varphi|^2_{\tilde{g}}} &= \frac{u^{-2\frac{p-1}{n-p}}}{(p-1)^2}\left[\mathrm{H}_g - \frac{(n-1)(p-1)}{(n-p)}\frac{|\mathrm{D}u|_g}{u}\right]^2\end{aligned}$$

and

$$\mathrm{Ric}\left(\frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}}, \frac{\nabla\varphi}{|\nabla\varphi|_{\tilde{g}}}\right) = u^{-2\frac{p-1}{n-p}}\,\mathrm{Ric}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|_g}, \frac{\mathrm{D}u}{|\mathrm{D}u|_g}\right).$$

As already mentioned (3.5) follows from (3.4) by the coarea formula.

We are thus left to show that the cylindrical splitting of $\tilde{g}$ implies the conical splitting for the metric g. We know that φ is an affine function of the coordinate s. Hence, we can write $\varphi = as + b$, with $a \neq 0$, since φ is not constant. By (3.8) we have

$$g = \mathrm{e}^{\frac{2(as+b)}{n-2}}\left(\mathrm{d}s \otimes \mathrm{d}s + \tilde{g}_{\{\varphi = s_0\}}\right).$$

Defining the new coordinate ρ as

$$\mathrm{d}\rho = \mathrm{e}^{\frac{as+b}{n-2}}\,\mathrm{d}s,$$

we get

$$g = \mathrm{d}\rho \otimes \mathrm{d}\rho + \frac{\mathrm{e}^{2\frac{as+b}{n-2}}}{\mathrm{e}^{2\frac{s_0}{n-2}}}g_{\{u=1/t_0\}} = \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\{u=1/t_0\}},$$

where ρ_0 is such that $\{u \leq 1/t_0\} = \{\rho \geq \rho_0\}$. Observe that, by the conical splitting, the measure of the level sets of ρ satisfy

$$|\{\rho = R\}| = \left(\frac{R}{\rho_0}\right)^{n-1}|\{u = 1/t_0\}|.$$

We can then easily compute the claimed value of ρ_0 by using Lemma 2.2.4 we have

$$\mathrm{AVR}(g) = \lim_{R\to+\infty}\frac{|\{\rho = R\}|}{R^{n-1}|\mathbb{S}^{n-1}|} = \frac{|\{u = t_0\}|}{\rho_0^{n-1}|\mathbb{S}^{n-1}|}.$$

The proof of (3.7) as well as its rigidity statement, follows in the same way from (3.25), (3.15) and (3.18). □

3.4 Proof of the Monotonicity-Rigidity Theorems

In this section we are proving Theorems 3.3.6 to 3.3.8. The first part of the section is devoted to proving the Monotonicity-Rigidity Theorem for Φ_p^β. It turns out that the weak derivative of Φ_p^β is related to a vector field X (see (3.26)) with nonnegative divergence. Taking advantage of the higher regularity of the function φ outside the critical set, we will prove that the integral of $\operatorname{div}(X)$ is related to the second derivative of Φ_p^β, showing at the same time the regularity and the monotonicity of the function. As it may concern Φ_p^∞, its monotonicity follows from a Maximum Principle applied to an elliptic operator which has $|\mathrm{D}\varphi|^p$ as a subsolution. The last part of the section contains the proof of this limit case.

From now on, we will drop the subscript $\tilde{g}$ whenever it is clear which metric we are referring to.

3.4.1 Monotonicity-Rigidity Theorems for Φ_p^β

In this section we prove Theorems 3.3.6 and 3.3.7. A basic property we will need is the essential uniform boundedness of Φ_p^β defined in (3.21).

Lemma 3.4.1. *Let (M, g) be a p-nonparabolic Riemannian manifold. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. For every $\beta \in [0, +\infty)$, Φ_p^β is essentially uniformly bounded, namely, $\Phi_p^\beta(s) \leq \mathrm{C}$ for almost every $s \in [0, +\infty)$, including any s that is regular for φ.*

Proof. It suffices to write Φ_p^β as

$$\begin{aligned}\Phi_p^\beta(s) &= \int\limits_{\{\varphi=s\}} |\nabla\varphi|^{(\beta+1)(p-1)}\,\mathrm{d}\sigma \leq \mathrm{C}^{\beta(p-1)} \int\limits_{\{\varphi=s\}} |\nabla\varphi|^{p-1}\,\mathrm{d}\sigma \\ &= \mathrm{C}^{\beta(p-1)}\left[\frac{(n-2)(p-1)}{(n-p)}\right]^{p-1} \int\limits_{\{u=1/t\}} |\mathrm{D}u|_g^{p-1}\,\mathrm{d}\sigma_g,\end{aligned}$$

where C is the constant appearing in Lemma 3.3.3, the last identity is due to (3.15) and (3.9) taking $s = -[(p-1)(n-2)/(n-p)]\log t$. By Proposition 1.3.3 we have that the integral on the rightmost hand side coincides with $\mathrm{Cap}_p(\partial\Omega)$ for almost any t, including any of those such that $1/t$ is a regular value for u. This settles the boundedness of Φ_p^β for $\beta \in [0, +\infty)$. □

Consider now the vector field

$$X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}\left(\nabla|\nabla\varphi|^{\beta(p-1)} + (p-2)\nabla^{\perp}|\nabla\varphi|^{\beta(p-1)}\right), \tag{3.26}$$

defined in a neighbourhood of each point such that $|\nabla\varphi| \neq 0$. Observe a first crucial property of the vector field X.

Lemma 3.4.2 (Integrability of X). *Let (M, g) be a p-nonparabolic manifold and X be the vector field defined in (3.26). Then*

$$|\langle X \,|\, \nabla\varphi\rangle| \leq \beta(p-1)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{\beta(p-1)}\big|\nabla|\nabla\varphi|^{p-1}\big| \in L^2_{\mathrm{loc}}(M \smallsetminus \overline{\Omega}).$$

Proof. $\langle X \,|\, \nabla\varphi\rangle$ makes sense also on the critical set of φ, since $\nabla|\nabla\varphi|^{p-1} = 0$ almost everywhere on $\mathrm{Crit}\,\varphi$ by Theorem 1.2.2. By easy computations we have

$$\begin{aligned}\langle X \,|\, \nabla\varphi\rangle &= (p-1)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{p-2}\Big\langle \nabla|\nabla\varphi|^{\beta(p-1)} \,\Big|\, \nabla\varphi\Big\rangle \\ &= \beta(p-1)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^{\beta(p-1)-1}\Big\langle \nabla|\nabla\varphi|^{p-1} \,\Big|\, \nabla\varphi\Big\rangle\end{aligned}$$

thus, we conclude in virtue of (3.16) and Theorem 1.2.2. □

The vector field X is related to derivative of Φ^β_p, due to the following identity.

Proposition 3.4.3. *Let (M, g) be a p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. For every $\beta \in [0, +\infty)$, the function $s \mapsto \Phi^\beta_p(s)$ defined in (3.21) belongs to $W^{1,1}_{\mathrm{loc}}(0, +\infty)$ and its derivative is given by*

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s) = \frac{1}{p-1}\int\limits_{\{\varphi=s\}} \left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma. \tag{3.27}$$

for almost every $s \in [0, +\infty)$, where X is the vector field defined in (3.26).

Proof. By the definition of X, it is easy to check that

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle = \frac{1}{p-1}\left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle$$

holds around each point such that $|\nabla\varphi| \neq 0$. Hence, it remains only to prove that $\Phi^\beta_p(s) \in W^{1,1}_{\mathrm{loc}}(0 + \infty)$ and that

$$(\Phi^\beta_p)'(s) = \int\limits_{\{\varphi=s\}} \left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma$$

holds for almost any $s \in (0, \infty)$. Let $\eta \in \mathscr{C}^\infty_c(0, +\infty)$. Since $|\nabla\varphi|$ is bounded by Lemma 3.3.3, applying the coarea formula Proposition B.2 with $f = |\nabla\varphi|^{(\beta+1)(p-1)}$ and the chain rule we obtain that

$$\begin{aligned}\int_0^{+\infty} \eta'(s)\Phi^\beta_p(s)\,\mathrm{d}s &= \int_0^{+\infty} \eta'(s)\int\limits_{\{\varphi=s\}} |\nabla\varphi|^{(\beta+1)(p-1)}\,\mathrm{d}\sigma\,\mathrm{d}s \\ &= \int\limits_{M\smallsetminus\overline{\Omega}} \eta'(s)\langle\nabla\varphi \,|\, \nabla\varphi\rangle|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\mu \\ &= \int\limits_{M\smallsetminus\overline{\Omega}} \langle\nabla(\eta(\varphi)) \,|\, \nabla\varphi\rangle|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\mu.\end{aligned}$$

Integrating by parts the right hand side, $\Delta^{(p)}\varphi = 0$ yields

$$\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s = -\int_{M\setminus\overline{\Omega}} \eta(\varphi)\left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \nabla\varphi\right\rangle \mathrm{d}\mu.$$

Thanks to Lemma 3.4.2, we are in position to apply the coarea formula in Proposition B.3 with $f = \eta(\varphi)\langle|\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \mid \nabla\varphi\rangle/|\nabla\varphi|$, to get

$$\int_0^{+\infty} \eta'(s)\Phi_p^\beta(s)\,\mathrm{d}s = -\int_0^1 \eta(s)\int_{\{\varphi=s\}} \left\langle |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)} \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma\,\mathrm{d}s,$$

which ensures both that $(\Phi_p^\beta)' \in W^{1,1}_{\mathrm{loc}}(0,+\infty)$ and (3.27). □

Combining (3.27) and (3.17), one can actually show that

$$(\Phi_p^\beta)'(s) = -\beta\int_{\{\varphi=s\}} \mathrm{H}\,|\nabla\varphi|^{(\beta+1)(p-1)-1}\,\mathrm{d}\sigma \tag{3.28}$$

for almost every $s \in [0,+\infty)$.

The nonnegative divergence of X is what substantially rules the monotonicity of Φ_p^β, and this is true when β ranges in a suitable set of parameters.

Lemma 3.4.4 (Divergence of X)**.** *Let (M,g) be a p-nonparabolic manifold and X be the vector field defined in (3.26). Then*

$$\operatorname{div} X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,Q, \tag{3.29}$$

holds at any point such that $|\nabla\varphi| > 0$, with

$$\begin{aligned} Q = \beta(p-1)|\nabla\varphi|^{(\beta+1)(p-1)-1}\Bigg\{&\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\tilde{g}^\top\right|^2 \\ &+ (p-1)^2\left[\beta - \frac{(n-p)}{(p-1)(n-1)}\right]\frac{|\nabla^\perp|\nabla\varphi||^2}{|\nabla\varphi|^2} \\ &+ (p-1)\left[\beta + \frac{p-2}{p-1}\right]\frac{|\nabla^\top|\nabla\varphi||^2}{|\nabla\varphi|^2} + \mathrm{Ric}_g\left(\frac{\nabla\varphi}{|\nabla\varphi|}, \frac{\nabla\varphi}{|\nabla\varphi|}\right)\Bigg\}, \end{aligned} \tag{3.30}$$

according to the orthogonal decomposition with respect to the level sets of φ, where h *and* H *are respectively the second fundamental form and the mean curvature of the level sets of φ with respect to the unit normal $\nabla\varphi/|\nabla\varphi|$ and* Ric_g *denotes the Ricci tensor of the background metric. In particular,* $\operatorname{div}(X) \geq 0$ *for $(n-p)/[(n-1)(p-1)] \leq \beta < +\infty$.*

Proof. The proof follows the same lines of [AFM22, Lemma 4.1] and we report all computations here for the sake of completeness. We can write X as the sum of two vector fields

$$X = \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}(W+Z),$$

where

$$W = |\nabla\varphi|^{p-2}\nabla|\nabla\varphi|^{\beta(p-1)}$$

and

$$Z = (p-2)|\nabla\varphi|^{p-2}\nabla^{\perp}|\nabla\varphi|^{\beta(p-1)}.$$

We proceed computing the divergence of the two vector fields separately.

Step 1. Divergence of W. By the chain rule we have that

$$W = \frac{\beta(p-1)}{p}|\nabla\varphi|^{\beta(p-1)-2}\nabla|\nabla\varphi|^{p}, \tag{3.31}$$

hence its divergence is

$$\operatorname{div} W = \frac{\beta(p-1)}{p}|\nabla\varphi|^{\beta(p-1)-2}\Delta|\nabla\varphi|^{p} + \beta(p-1)(\beta(p-1)-2)|\nabla\varphi|^{\beta(p-1)+p-4}|\nabla|\nabla\varphi||^{2}.$$

Plugging the p-Bochner formula for φ in Proposition 3.3.4 into the previous identity taking into account the definition (3.31) we have that

$$\begin{aligned}\operatorname{div} W - \frac{n-p}{n-2}\langle W \,|\, \nabla\varphi\rangle = \beta(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\Bigg[&|\nabla\nabla\varphi|^{2} \\ &+ p(p-2)\big|\nabla^{\perp}|\nabla\varphi|\big|^{2} - \frac{p-2}{p}\frac{\nabla\nabla|\nabla\varphi|^{p}(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|^{p}} \\ &+ (\beta(p-1)-2)|\nabla|\nabla\varphi||^{2} + \mathrm{Ric}_{g}(\nabla\varphi,\nabla\varphi)\Bigg]\end{aligned}$$

and computing the Hessian of $|\nabla\varphi|^{p}$ one has

$$\begin{aligned}\operatorname{div} W - \frac{n-p}{n-2}\langle W \,|\, \nabla\varphi\rangle = \beta(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\Bigg[&|\nabla\nabla\varphi|^{2} \\ &+ (p-2)\left(\big|\nabla^{\perp}|\nabla\varphi|\big|^{2} - \frac{\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|}\right) \\ &+ (\beta(p-1)-2)|\nabla|\nabla\varphi||^{2} + \mathrm{Ric}_{g}(\nabla\varphi,\nabla\varphi)\Bigg].\end{aligned}$$

Using the orthogonal decomposition

$$|\nabla|\nabla\varphi||^{2} = \big|\nabla^{\perp}|\nabla\varphi|\big|^{2} + \big|\nabla^{\top}|\nabla\varphi|\big|^{2},$$

on level sets of φ and using the Kato-type Identity (1.10), we conclude obtaining

$$\begin{aligned}\operatorname{div} W - \frac{n-p}{n-2}\langle W \,|\, \nabla\varphi\rangle = \beta(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}\Bigg[&\beta(p-1)\big|\nabla^{\top}|\nabla\varphi|\big|^{2} \\ &+ |\nabla\varphi|^{2}\left|\mathrm{h} - \frac{\mathrm{H}}{n-1}\tilde{g}^{\top}\right|^{2} - (p-2)\frac{\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|} \\ &+ \left((\beta(p-1) + \frac{(p-1)^{2}}{n-1} + p - 3\right)\big|\nabla^{\perp}|\nabla\varphi|\big|^{2} \\ &+ \mathrm{Ric}(\nabla\varphi,\nabla\varphi)\Bigg].\end{aligned}$$

Step 2. The divergence of Z. Since φ is p-harmonic, we get

$$\operatorname{div} Z = (p-2)|\nabla\varphi|^{p-2}\left\langle \nabla\left(\frac{|\nabla^{\perp}|\nabla\varphi|^{\beta(p-1)}|}{|\nabla\varphi|}\right) \,\middle|\, \nabla\varphi \right\rangle.$$

Since

$$\frac{|\nabla^{\perp}|\nabla\varphi|^{\beta(p-1)}|}{|\nabla\varphi|} = \beta(p-1)|\nabla\varphi|^{\beta(p-1)-2}|\nabla^{\perp}|\nabla\varphi||,$$

using the decomposition on the level sets of φ and the general fact that

$$\frac{\nabla\nabla\varphi(\nabla|\nabla\varphi|,\nabla\varphi)}{|\nabla\varphi|} = |\nabla|\nabla\varphi||^2,$$

the divergence of Z reads as

$$\begin{aligned}\operatorname{div} Z = \beta(p-2)(p-1)|\nabla\varphi|^{\beta(p-1)+p-4}&\left[\frac{\nabla\nabla|\nabla\varphi|(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|}\right.\\ &\left.+\left(\beta(p-1)^2-\beta(p-1)-2(p-2)\right)\big|\nabla^{\perp}|\nabla\varphi|\big|^2+(p-2)\big|\nabla^{\top}|\nabla\varphi|\big|^2\right].\end{aligned}$$

Step 3. Conclusions. Observe that $(p-1)\langle W \mid \nabla\varphi\rangle = \langle W+Z \mid \nabla\varphi\rangle$. Hence by straightforward computations

$$\operatorname{div}(W+Z) - \frac{(n-p)}{(n-2)(p-1)}\langle W+Z \mid \nabla\varphi\rangle = \operatorname{div} W - \frac{n-p}{n-2}\langle W \mid \nabla\varphi\rangle + \operatorname{div} Z = Q$$

which is equivalent to (3.29). □

Suppose that $|\nabla\varphi| \neq 0$ everywhere. We can apply the Divergence Theorem in the domain $\{s < \varphi < S\}$ to obtain

$$\int\limits_{\{\varphi=S\}}\left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma - \int\limits_{\{\varphi=s\}}\left\langle X \,\middle|\, \frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle \mathrm{d}\sigma = \int\limits_{\{s<\varphi<S\}} \operatorname{div} X \,\mathrm{d}\mu \geq 0. \tag{3.32}$$

Using (3.27) we deduce that

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^{\beta})'(s) \leq \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S}(\Phi_p^{\beta})'(S).$$

This almost concludes the prove of the Monotonicity Theorem for Φ_p^{β} with $(n-p)/[(n-1)(p-1)] \leq \beta < +\infty$ assuming the absence of critical points. Indeed, by integrating it, monotonicity will follow as in [FMP19, Theorem 3.4]. This case lies in the same trail blazed in [AM20] since if $|\nabla \varphi| \neq 0$ the p-Laplace operator is elliptic non degenerate, and thus the techniques used for harmonic functions fit perfectly.

If we want to pursue the previous path, even when the critical set of φ is not empty, we are first committed to provide a version of (3.32) that holds even in presence of critical values. The main issue is that $\operatorname{div}(X)$ does not belong to L^1_{loc} *a priori*. Following the same lines of [GV21, Proposition 4.6], testing $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)$ against nonnegative functions $\eta \in \mathscr{C}^\infty_c(0,+\infty)$ and using the coarea formula Proposition B.3 for $f = \langle X \,|\, \nabla \varphi \rangle$ one gets

$$(p-1)\int_0^{+\infty} \eta'(s)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)\,\mathrm{d}s = \int_{M\smallsetminus \mathrm{Crit}(\varphi)} \langle X \,|\, \nabla[\eta(\varphi)]\rangle\,\mathrm{d}\mu.$$

We now would like to integrate by parts and use the nonnegativitiy of $\operatorname{div}(X)$ outside the critical set of φ. In doing this, we are hampered by the fact that $\operatorname{div}(\chi_{M\smallsetminus \mathrm{Crit}\,\varphi}X)$ is actually a measure that is possibly not absolutely continuous. Hence we can aim at proving that $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(0,+\infty)$, but not the absolute continuity. Differently from the nonsmooth case, we can here employ the higher regularity of φ outside its critical set to refine the result.

Proposition 3.4.5. *Let (M,g) be a p-nonparabolic Riemannian manifold with $\mathrm{Ric} \geq 0$. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. For every $(n-p)/[(n-1)(p-1)] < \beta < +\infty$, the function $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)$ defined in (3.27) belongs to $W^{1,1}_{\mathrm{loc}}(0,+\infty)$ and its derivative is given by*

$$\left(\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)\right)' = \frac{1}{p-1}\int_{\{\varphi=s\}\smallsetminus \mathrm{Crit}(\varphi)} \frac{\operatorname{div} X}{|\nabla \varphi|}\,\mathrm{d}\sigma, \tag{3.33}$$

for almost every $s \in [0,+\infty)$, where X is the vector field defined in (3.26).

Proof. Proposition 3.4.5 follows if we prove that $\operatorname{div}(X)(1-\chi_{\mathrm{Crit}(\varphi)})$ belongs to $L^1_{\mathrm{loc}}(M \smallsetminus \overline{\Omega})$ and

$$(p-1)\int_0^{+\infty} \eta'(s)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)\,\mathrm{d}s = -\int_{M\smallsetminus \mathrm{Crit}\,\varphi} \eta(\varphi)\operatorname{div} X\,\mathrm{d}\mu \tag{3.34}$$

holds for every $\eta \in \mathscr{C}^\infty_c(0,+\infty)$. Indeed, by the coarea formula Proposition B.3 with $f = \operatorname{div}(X)(1-\chi_{\mathrm{Crit}\,\varphi})$ we would get

$$(p-1)\int_0^{+\infty} \eta'(s)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi^\beta_p)'(s)\,\mathrm{d}s = -\int_0^{+\infty} \eta(s)\int_{\{\varphi=s\}\smallsetminus \mathrm{Crit}\,\varphi} \frac{\operatorname{div} X}{|\nabla \varphi|}\,\mathrm{d}\sigma\,\mathrm{d}s,$$

which implies both that $\mathrm{e}^{-(n-p)s/(n-2)(p-1)}(\Phi^\beta_p)' \in W^{1,1}_{\mathrm{loc}}(0,+\infty)$ and (3.33).

Step 1. Proof for nonnegative η. Let $\eta \in \mathscr{C}^\infty_c(0,+\infty)$ be nonnegative. For every $\varepsilon > 0$, consider a smooth nonnegative cut-off function $\chi_\varepsilon : [0,+\infty) \to \mathbb{R}$ such that

$$\begin{cases} \chi_\varepsilon(t) = 0 & \text{in } t < \frac{1}{2}\varepsilon, \\ 0 < \chi_\varepsilon'(t) \leq \frac{2}{\varepsilon} & \text{in } \frac{1}{2}\varepsilon \leq t \leq \frac{3}{2}\varepsilon, \\ \chi_\varepsilon(t) = 1 & \text{in } t > \frac{3}{2}\varepsilon. \end{cases}$$

Define accordingly the vector field $X_\varepsilon = \chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})X$, where X is the vector field given in (3.26). Notice that $|\langle X_\varepsilon \,|\, \nabla\varphi\rangle| \leq |\langle X \,|\, \nabla\varphi\rangle|$. Hence Proposition 3.4.3, the coarea formula Proposition B.3 with $f = \eta'(\varphi)\langle X \,|\, \nabla\varphi/|\nabla\varphi|\rangle$ and the Dominated Convergence Theorem, whose assumptions are fulfilled in virtue of Lemma 3.4.2, imply

$$\begin{aligned}\int_0^{+\infty} \eta'(s)\, \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s} (\Phi_p^\beta)'(s)\, \mathrm{d}s &= \frac{1}{p-1}\int_M \eta'(\varphi)\langle X \,|\, \nabla\varphi\rangle\, \mathrm{d}\mu \\ &= \lim_{\varepsilon\to 0^+} \frac{1}{p-1}\int_M \eta'(\varphi)\langle X_\varepsilon \,|\, \nabla\varphi\rangle\, \mathrm{d}\mu.\end{aligned}$$

Integrating by parts we obtain that

$$\begin{aligned}\int_{M\smallsetminus\overline{\Omega}} \eta'(\varphi)\langle X_\varepsilon \,|\, \nabla\varphi\rangle\, \mathrm{d}\mu &= -\int_{M\smallsetminus\overline{\Omega}} \mathrm{div}(X_\varepsilon)\eta(\varphi)\, \mathrm{d}\mu \\ &= -\int_{M\smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})\, \mathrm{div}\, X\, \mathrm{d}\mu \\ &\qquad -\int_{N_{3\varepsilon/2}\smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon'(|\nabla\varphi|^{\beta(p-1)})\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)}\right\rangle \mathrm{d}\mu,\end{aligned}$$

where $N_\delta = \left\{|\nabla\varphi|^{\beta(p-1)} < \delta\right\}$ for every $\delta > 0$. By the Monotone Convergence Theorem, the first integral gives

$$\lim_{\varepsilon\to 0^+} \int_{M\smallsetminus N_{\varepsilon/2}} \eta(\varphi)\chi_\varepsilon(|\nabla\varphi|^{\beta(p-1)})\, \mathrm{div}\, X\, \mathrm{d}\mu = \int_{M\smallsetminus \mathrm{Crit}\,\varphi} \eta(\varphi)\, \mathrm{div}\, X\, \mathrm{d}\mu \geq 0.$$

We now aim at proving that the second integral vanishes as $\varepsilon \to 0^+$. Observe that $|\nabla\varphi|^{\beta(p-1)} \geq \varepsilon/2$ on $N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}$, then φ is smooth in that domain. By the definition of X in (3.26), we obtain that

$$\begin{aligned}\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)}\right\rangle &= \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\, |\nabla\varphi|^{p-2}\left(\left|\nabla|\nabla\varphi|^{\beta(p-1)}\right|^2 + (p-2)\left|\nabla^\perp|\nabla\varphi|^{\beta(p-1)}\right|^2\right) \\ &= \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\, |\nabla\varphi|^{p-2}\left(\left|\nabla^\top|\nabla\varphi|^{\beta(p-1)}\right|^2 + (p-1)\left|\nabla^\perp|\nabla\varphi|^{\beta(p-1)}\right|^2\right)\end{aligned}$$

which is nonnegative. By the coarea formula we obtain that

$$\left| \int_{N_{3\varepsilon/2} \smallsetminus N_{\varepsilon/2}} \eta(\varphi) \mathcal{X}'_{\varepsilon}(|\nabla\varphi|^{\beta(p-1)}) \left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle \mathrm{d}\mu \right| \\ \leq \frac{2}{\varepsilon} \|\eta\|_{L^\infty} \int_{\varepsilon/2}^{3\varepsilon/2} \int_{\partial N_s} \frac{\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|} \mathrm{d}\sigma \, \mathrm{d}s.$$

Let $R > 0$ and $\mathcal{H}$ be defined as

$$\mathcal{H}(r) = \int_{\partial N_r} \frac{\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|} \mathrm{d}\sigma,$$

for every regular value $r \in (0, R)$ of $|\nabla\varphi|$, hence for almost every $r \in (0, R)$ thanks to Sard's Theorem. By the Mean Value Theorem, showing that $\mathcal{H}(r)$ vanishes as $r \to 0^+$ is enough to conclude the proof.

Let $0 < t < r < R$ be two regular values for $|\nabla\varphi|$, applying the Divergence Theorem to the smooth vector field X on $N_r \smallsetminus N_t$ we get

$$\begin{aligned} \mathcal{H}(r) - \mathcal{H}(t) &= \int_{\partial N_r} \frac{\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|} \mathrm{d}\sigma - \int_{\partial N_t} \frac{\left\langle X \,\middle|\, \nabla|\nabla\varphi|^{\beta(p-1)} \right\rangle}{|\nabla|\nabla\varphi|^{\beta(p-1)}|} \mathrm{d}\sigma \\ &= \int_{N_r \smallsetminus N_t} \mathrm{div}(X) \, \mathrm{d}\mu = \int_t^r \int_{\partial N_s} \frac{\mathrm{div}(X)}{|\nabla|\nabla\varphi|^{\beta(p-1)}|} \mathrm{d}\sigma \, \mathrm{d}s. \end{aligned} \tag{3.35}$$

where the last identity is an application of the coarea formula. Since the integrand in the rightmost side is nonnegative and $\mathcal{H}$ is almost everywhere finite, $\mathcal{H}$ is locally absolutely continuous.

Since $\mathrm{Ric} \geq 0$ and $|\nabla\varphi|^2 |\mathrm{h} - \frac{\mathrm{H}}{n-1} g^\top|^2 \geq 0$, by (3.30) we have that

$$\begin{aligned} \mathrm{div}\, X \geq{}& \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)} S} \beta(p-1) |\nabla\varphi|^{\beta(p-1)+p-4} \left((p-1) \left[\beta + \frac{p-2}{p-1} \right] \left|\nabla^\top |\nabla\varphi|\right|^2 \right. \\ & \left. + (p-1)^2 \left[\beta - \frac{(n-p)}{(p-1)(n-1)} \right] \left| \nabla^\perp |\nabla\varphi| \right|^2 \right) \\ \geq{}& \beta^2 (p-1)^2 \, \mathrm{C} \, |\nabla\varphi|^{\beta(p-1)+p-4} \left(\left|\nabla^\perp|\nabla\varphi|\right|^2 + \left|\nabla^\top |\nabla\varphi|\right|^2 \right) \\ \geq{}& \mathrm{C} \, |\nabla\varphi|^{-\beta(p-1)+p-2} \left|\nabla |\nabla\varphi|^{\beta(p-1)}\right|^2, \end{aligned}$$

where

$$\mathrm{C} = \frac{1}{\beta} \mathrm{e}^{-\frac{(n-p)}{(p-1)(n-2)} S} \min \left\{ \left[\beta + \frac{p-2}{p-1} \right], (p-1) \left[\beta - \frac{(n-p)}{(p-1)(n-1)} \right] \right\} > 0.$$

Taking derivatives in (3.35) it holds that

$$\mathcal{H}'(r) = \int\limits_{\partial N_s} \frac{\operatorname{div}(X)}{|\nabla|\nabla\varphi|^{\beta(p-1)}|}\,\mathrm{d}\sigma \geq \mathrm{C} \int\limits_{\partial N_r} |\nabla\varphi|^{-\beta(p-1)+p-2}|\nabla|\nabla\varphi|^{\beta(p-1)}|\,\mathrm{d}\sigma = \mathrm{C}\,\frac{\mathcal{H}(r)}{r},$$

for almost any $r > 0$. Integrating for $R > r$, we obtain that

$$\frac{\mathcal{H}(r)}{r^{\mathrm{C}}} \leq \frac{\mathcal{H}(R)}{R^{\mathrm{C}}}.$$

We thus deduce that $\mathcal{H}(r) \to 0$ as $r \to 0^+$.

Step 2. Conclusion. Consider $K \subset M \smallsetminus \overline{\Omega}$, there exists a $\eta_K \in \mathscr{C}_c^\infty(0,+\infty)$, $\eta_K \geq 0$, such that $\eta_K(\varphi) \geq 1$ on K. Lemma 3.4.4 yields

$$\begin{aligned}\int\limits_K \operatorname{div}(X)(1-\chi_{\operatorname{Crit}\varphi})\,\mathrm{d}\mu &\leq \int\limits_{M\smallsetminus\operatorname{Crit}\varphi} \eta_K(\varphi)\operatorname{div}(X)\,\mathrm{d}\mu \\ &= -(p-1)\int\limits_0^{+\infty} \eta_K'(s)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s,\end{aligned}$$

which is finite by Proposition 3.4.3. This ensures that $\operatorname{div}(X)(1-\chi_{\operatorname{Crit}\varphi})$ belongs to $L^1_{\mathrm{loc}}(M\smallsetminus\overline{\Omega})$. In particular, (3.34) holds for every $\eta \in \mathscr{C}_c^\infty(0,+\infty)$. Employing the coarea formula in Proposition B.3 for $f = \eta(\varphi)\operatorname{div}(X)(1-\chi_{\operatorname{Crit}\varphi})/|\nabla\varphi|$, we get that

$$\begin{aligned}(p-1)\int\limits_0^{+\infty} \eta'(s)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s &= -\int\limits_{M\smallsetminus\operatorname{Crit}\varphi} \eta(\varphi)\operatorname{div}(X)\,\mathrm{d}\mu \\ &= -\int\limits_0^{+\infty} \eta(s) \int\limits_{\{\varphi=s\}} \frac{\operatorname{div}(X)}{|\nabla\varphi|}\,\mathrm{d}\sigma\,\mathrm{d}t.\end{aligned}$$

This ensures both that $e^{-\frac{(n-p)}{(n-2)(p-1)}s}\Phi_p^\beta \in W^{1,1}_{\mathrm{loc}}$ and (3.33). □

Corollary 3.4.6. *Let (M,g) be a p-nonparabolic Riemannian manifold with* $\operatorname{Ric} \geq 0$. *Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then, the function $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)$ defined in (3.27) for $\beta = (n-p)/[(n-1)(p-1)]$ belongs to $\mathrm{BV}_{\mathrm{loc}}(0,+\infty)$ and its derivative satisfies*

$$\left(\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\right)' \geq \frac{1}{p-1}\left(\int\limits_{\{\varphi=s\}\smallsetminus\operatorname{Crit}(\varphi)} \frac{\operatorname{div} X}{|\nabla\varphi|}\,\mathrm{d}\sigma\right)\mathcal{L}^1, \tag{3.36}$$

for almost every $s \in [0,+\infty)$, where X is the vector field defined in (3.26) and $\mathcal{L}^1$ is the Lebesgue measure on $[0,+\infty)$.

Proof. Denote

$$\begin{aligned}\operatorname{div}^+ X =&\beta(p-1)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|_{\tilde g}^{(\beta+1)(p-1)-1}\left\{\left|\mathrm{h}_{\tilde g}-\frac{\mathrm{H}_{\tilde g}}{n-1}\tilde g^{\top}\right|^2_{\tilde g^{\top}}\right.\\ &\left.+\,(p-1)\left[\beta+\frac{p-2}{p-1}\right]\frac{|\nabla^{\top}|\nabla\varphi|_{\tilde g}|^2_{\tilde g}}{|\nabla\varphi|^2_{\tilde g}}+\mathrm{Ric}\left(\frac{\nabla\varphi}{|\nabla\varphi|_{\tilde g}},\frac{\nabla\varphi}{|\nabla\varphi|_{\tilde g}}\right)\right\}.\end{aligned}$$

Since we have neglected a nonnegative term, $\operatorname{div}^+ X \le \operatorname{div} X$ holds for every $\beta > (n-p)/[(n-1)(p-1)] = \overline{\beta}$ and $\operatorname{div}^+ X = \operatorname{div} X$ for $\beta = \overline{\beta}$. In particular, by Proposition 3.4.5 and the coarea formula Proposition B.3 for $f = \eta(\varphi)\operatorname{div}^+(X)(1-\chi_{\operatorname{Crit}\varphi})/|\nabla\varphi|$

$$\int_0^{+\infty}\eta(s)\left(\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\right)'\,\mathrm{d}s \ge \frac{1}{p-1}\int_{M\smallsetminus\operatorname{Crit}\varphi}\eta(\varphi)\operatorname{div}^+ X\,\mathrm{d}\mu \tag{3.37}$$

holds for every $\beta > \overline{\beta}$ and every nonnegative $\eta \in \mathscr{C}^\infty_c(0,+\infty)$. We want to pass the limit as $\beta \to \overline{\beta}^+$. The right hand side can be split into two parts, the one where $|\nabla\varphi| \le 1$ and $|\nabla\varphi| > 1$. On $\{|\nabla\varphi| \le 1\}$ the term $|\nabla\varphi|^{\beta(p-1)}$ is increasing, we can thus employ the Monotone Convergence Theorem. Conversely, on $\{|\nabla\varphi| > 1\}$ we are far away from the critical set of φ. Hence, we can control second derivatives and use Dominated Convergence Theorem. As it concerns the left hand side, Proposition 3.4.3 and the coarea formula for $f = \eta(\varphi)\langle X \,|\, \nabla\varphi/|\nabla\varphi|\rangle$ give that

$$\int_0^{+\infty}\eta'(s)\,\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)\,\mathrm{d}s = \frac{1}{p-1}\int_M \eta'(\varphi)\langle X\,|\,\nabla\varphi\rangle\,\mathrm{d}\mu.$$

By Lemma 3.4.2 we can use the Dominated Convergence Theorem.

Then (3.37) holds also for $\beta = \overline{\beta}$. The derivative of $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)$ is a nonnegative distribution and by Riesz Representation Theorem, is a nonnegative Radon measure. This shows both that $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)$ belongs to $\mathrm{BV}_{\mathrm{loc}}(0,+\infty)$ and (3.36). □

Finally, all the tools required for the proof of Monotonicity are at our disposal.

Proof of Theorems 3.3.6 and 3.3.7. We use an argument due to Colding and Minicozzi in [CM14b]. If $\beta > (n-p)/[(p-1)(n-1)]$, by Propositions 3.4.3 and 3.4.5 Φ_p^β is of class $W^{2,1}_{\mathrm{loc}}(0,+\infty)$. By (3.33) $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)$ is nondecreasing. For every $0 \le s < S < +\infty$ we have

$$\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}(S-s)}(\Phi_p^\beta)'(s) \le (\Phi_p^\beta)'(S).$$

Integrating the above inequality, we get

$$\frac{(n-1)(p-1)}{(n-p)}\left(\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}(S-s)}-1\right)(\Phi_p^\beta)'(s) \le \Phi_p^\beta(S)-\Phi_p^\beta(s) \tag{3.38}$$

for every $0 \leq s < S < +\infty$. Suppose, by contradiction, that $(\Phi_p^\beta)'(s) > 0$ for some $s \in [0, +\infty)$. Passing to the limit as $S \to +\infty$ in (3.38) we would get that $\Phi_p^\beta(S) \to +\infty$ against the boundedness property ensured by Lemma 3.4.1. Hence, $(\Phi_p^\beta)'(s) \leq 0$ and in particular $s \mapsto \Phi_p^\beta(s)$ is nonincreasing. Notice that Φ_p^β is a bounded, nonincreasing $\mathscr{C}^1(0, +\infty)$ function, then $(\Phi_p^\beta)'(s) \to 0$ as $s \to +\infty$. Coupling Proposition 3.4.5 with the coarea formula in Proposition B.3 for $f = \mathrm{div}(X)(1 - \chi_{\mathrm{Crit}\,\varphi})/|\nabla\varphi|$ one gets that

$$\begin{aligned}
\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s) &= \lim_{S\to+\infty} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s) - \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S}(\Phi_p^\beta)'(S) \\
&= \lim_{S\to+\infty} -\int\limits_{\{s\leq\varphi\leq S\}\smallsetminus\mathrm{Crit}\,\varphi} \mathrm{div}\, X \,\mathrm{d}\mu = -\int\limits_{\{\varphi\geq s\}\smallsetminus\mathrm{Crit}\,\varphi} \mathrm{div}\, X \,\mathrm{d}\mu,
\end{aligned}$$

which also ensures that $\mathrm{div}\, X \in L^1(M \smallsetminus (\overline{\Omega} \cup \mathrm{Crit}(\varphi)))$.

Conversely, if $\beta = (n-p)/[(p-1)(n-1)]$, by Proposition 3.4.3 and Corollary 3.4.6 $s \mapsto \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s)$ admits a nondecreasing right-continuous representative. Hence, (3.38) holds also in this case for every $0 \leq s < S < +\infty$. Arguing as above, $(\Phi_p^\beta)'(s) \leq 0$, $s \mapsto \Phi_p^\beta(s)$ and (the right-continuous representative of) $(\Phi_p^\beta)'(s) \to 0$ as $s \to +\infty$. Coupling Corollary 3.4.6 with the coarea formula in (B.1) for $f = \mathrm{div}(X)(1 - \chi_{\mathrm{Crit}\,\varphi})/|\nabla\varphi|$ one gets that

$$\begin{aligned}
\mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s) &= \lim_{S\to+\infty} \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}s}(\Phi_p^\beta)'(s) - \mathrm{e}^{-\frac{(n-p)}{(n-2)(p-1)}S}(\Phi_p^\beta)'(S) \\
&\geq \lim_{S\to+\infty} -\int\limits_{\{s\leq\varphi\leq S\}\smallsetminus\mathrm{Crit}\,\varphi} \mathrm{div}\, X \,\mathrm{d}\mu = -\int\limits_{\{\varphi\geq s\}\smallsetminus\mathrm{Crit}\,\varphi} \mathrm{div}\, X \,\mathrm{d}\mu,
\end{aligned}$$

which ensures that $\mathrm{div}\, X \in L^1(M \smallsetminus (\overline{\Omega} \cup \mathrm{Crit}(\varphi)))$ also in the case $\beta = (n-p)/[(p-1)(n-1)]$ and (3.24).

For the rigidity statement, suppose that $(\Phi_p^\beta)'(s_0) = 0$ for some $s_0 \in [0, +\infty)$ regular for φ. Then (3.23) or (3.24) yields $\mathrm{div}\, X = 0$ on the region $\{\varphi \geq s_0\} \smallsetminus \mathrm{Crit}\,\varphi$. We can consider two cases. If $\beta > (n-p)/[(n-1)(p-1)]$, then by (3.30) both $|\nabla^\top|\nabla\varphi||$ and $|\nabla^\perp|\nabla\varphi||$ vanish and in turn $|\nabla|\nabla\varphi|| = 0$ near $\{\varphi = s_0\}$. In particular, $|\nabla\varphi|$ is positively constant on a small tubular neighbourhood of $\{\varphi = s_0\}$ and, since φ is at least $\mathscr{C}^1$, it remains positively constant on $\{\varphi \geq s_0\}$. Therefore, this implies that there are no critical values $s \geq s_0$ and $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to the cylinder $([s_0, +\infty) \times \{\varphi = s_0\}, \mathrm{d}s^2 + \tilde{g}_{\{\varphi = s_0\}})$ by Lemma 3.3.5.

The case $\beta = (n-p)/[(n-1)(p-1)]$ is a little more delicate. By (3.30) we have that $|\nabla^\top|\nabla\varphi|| = 0$ and $|\mathrm{h} - \frac{\mathrm{H}}{n-1}\tilde{g}^\top| = 0$. The rigidity part of Proposition 1.2.12 implies that $(\{\varphi \geq s_0\}, \tilde{g})$ splits to a warped product at least near the level set $\{\varphi = s_0\}$. In particular, the mean curvature of $\{\varphi = s\}$ depends only on s, for $s > s_0$ sufficiently close to s_0, and consequently (3.28) implies it is zero. Finally, (3.17) yields $|\nabla^\perp|\nabla\varphi|| = 0$. Since we have already observed that $|\nabla^\top|\nabla\varphi|| = 0$, we get that $|\nabla\varphi|$ is positively constant on a small tubular neighbourhood of $\{\varphi = s_0\}$. Arguing as above we get the same cylindrical splitting. □

3.4.2 Monotonicity-Rigidity Theorem for Φ_p^∞

As said before, the Monotonicity of Φ_p^∞ almost corresponds to a Maximum Principle for the function $|\nabla\varphi|$. To show this we adapt an argument used by Colding in [Col12, Theorem 3.1] to prove the sharp estimate on the harmonic Green's function on manifolds with nonnegative Ricci curvature. Differently from [FMP19], a small observation in applying the maximum principle is the key to infer the full Monotonicity of Φ_p^∞.

We firstly observe the crucial property of Φ_p^∞ to be uniformly bounded, which is the direct consequence of Lemma 3.3.3.

Lemma 3.4.7. *Let (M,g) be a p-nonparabolic Riemannian manifold. Let $\Omega\subset M$ be an open bounded subset with smooth boundary. For every $\beta\in[0,+\infty)$, Φ_p^∞ is uniformly bounded*

Consider the operator $\mathscr{L}$, acting on smooth functions f, defined as

$$\mathscr{L}(f)=\Delta f+(p-2)\nabla\nabla f\left(\frac{\nabla\varphi}{|\nabla\varphi|},\frac{\nabla\varphi}{|\nabla\varphi|}\right)-\frac{n-p}{n-2}\langle\nabla f\,|\,\nabla\varphi\rangle, \tag{3.39}$$

at each point where $|\nabla\varphi|>0$. The function $|\nabla\varphi|^p$ is a subsolution of the equation $\mathscr{L}f=0$. The proof follows the same line of [FMP19, Lemma 5.1], where the only difference is that the curvature term that appear when Bochner formula for p-harmonic functions is applied can be controlled by $\mathrm{Ric}\geq 0$.

Lemma 3.4.8. *Let (M,g) be a p-nonparabolic manifold with nonnegative Ricci curvature. Let $\mathscr{L}$ be the operator defined in (3.39). Then,*

$$\mathscr{L}(|\nabla\varphi|^p)\geq 0$$

at each point where $|\nabla\varphi|>0$.

Proof. By Bochner's Formula (A.2) for p-harmonic function we have that

$$\mathscr{L}(|\nabla\varphi|^p)=p|\nabla\varphi|^{p-2}\left(|\nabla\nabla\varphi|^2+p(p-2)\left\langle\nabla|\nabla\varphi|\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2+\mathrm{Ric}(\nabla\varphi,\nabla\varphi)\right).$$

Since $\mathrm{Ric}\geq 0$ and the standard Kato inequality, we obtain

$$\begin{aligned}\mathscr{L}(|\nabla\varphi|^p)&\geq p|\nabla\varphi|^{p-2}\left(|\nabla|\nabla\varphi||^2+p(p-2)\left\langle\nabla|\nabla\varphi|\,\middle|\,\frac{\nabla\varphi}{|\nabla\varphi|}\right\rangle^2\right)\\&\geq p|\nabla\varphi|^{p-2}\left(\left|\nabla^\top|\nabla\varphi|\right|^2+(p-1)^2\left|\nabla^\perp|\nabla\varphi|\right|^2\right)\geq 0,\end{aligned}$$

concluding the proof. □

The following lemma provides a function lying in the kernel of $\mathscr{L}$ in (3.39).

Lemma 3.4.9. *Let (M,g) be a p-nonparabolic manifold with nonnegative Ricci curvature. Let $\mathscr{L}$ be the operator defined in (3.39) then*

$$\mathscr{L}\left(\mathrm{e}^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\right)=0 \tag{3.40}$$

at each point where $|\nabla\varphi|>0$.

Proof. We just compute each addendum of $\mathscr{L}$. First of all we have that

$$\begin{aligned}\Delta\left(e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\right) &= \frac{(n-p)}{(n-2)(p-1)}\, e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\left[\frac{(n-p}{(n-2)(p-1)}|\nabla\varphi|^2+\Delta\varphi\right]\\ &= \frac{(n-p)}{(n-2)(p-1)}\, e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\left[\frac{(n-p}{(n-2)(p-1)}|\nabla\varphi|^2\right.\\ &\qquad \left.-(p-2)\nabla\nabla\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|},\frac{\nabla\varphi}{|\nabla\varphi|}\right)\right),\end{aligned}$$

where in the last identity we used the p-harmonicity of φ. The second term is equal to

$$\begin{aligned}\nabla\nabla\left(e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\right) &= \frac{(n-p)}{(n-2)(p-1)}\, e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\left[\frac{(n-p}{(n-2)(p-1)}|\nabla\varphi|^2\right.\\ &\qquad \left.+\nabla\nabla\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|},\frac{\nabla\varphi}{|\nabla\varphi|}\right)\right].\end{aligned}$$

The last one is

$$\left\langle \nabla\, e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\,\middle|\,\nabla\varphi\right\rangle = \frac{(n-p)}{(n-2)(p-1)}\, e^{\frac{(n-p)}{(n-2)(p-1)}\varphi}\,|\nabla\varphi|^2.$$

Combining the three expression above we finally get (3.40). □

We now have at our disposal all tools required for the proof of the Monotonicity-Rigidity theorem for Φ_p^∞.

Proof of Theorem 3.3.8. We firstly claim that

$$|\nabla\varphi|(x) \leq \sup_{\{\varphi=s\}}|\nabla\varphi| \tag{3.41}$$

for every $s\in[0,+\infty)$ and $x\in\{\varphi\geq s\}$, which trivially implies the monotonicity of $\Phi_p^\infty(s)$. Firstly suppose that $\Phi_p^\infty(s)>0$ and let $0<\delta<\Phi_p^\infty(s)$. By Lemma 3.4.7 $|\nabla\varphi|\leq C$ uniformly in $M\smallsetminus\Omega$. Fix $S>s$ and consider the function

$$w = |\nabla\varphi|^p - \sup_{\{\varphi=s\}}|\nabla\varphi|^p - C^p\, e^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)}$$

defined on $\{s\leq\varphi\leq S\}\smallsetminus N_\delta$ where $N_\delta=\{|\nabla\varphi|<\delta\}$. Since $w\leq 0$ on the boundary of $\{s\leq\varphi\leq S\}\smallsetminus N_\delta$ and $\mathscr{L}(w)\geq 0$ in its interior, by the Maximum Principle (see [GT15, Theorem 3.3]) we have that

$$|\nabla\varphi|^p \leq \sup_{\{\varphi=s\}}|\nabla\varphi|^p + C^p\, e^{\frac{n-p}{(n-2)(p-1)}(\varphi-S)} \tag{3.42}$$

on $\{s\leq\varphi\leq S\}\smallsetminus N_\delta$. Moreover, since $|\nabla\varphi|<\delta$ on N_δ, (3.42) is thus satisfied on the whole $\{s\leq\varphi\leq S\}$. Passing to the limit as $S\to+\infty$, (3.41) is proved for $s\in[0,+\infty)$ such that $\Phi_p^\infty(s)>0$.

To conclude the proof (3.41) it remains to show that $\Phi^{\beta}_p(s) > 0$ for every $s \in [0, +\infty)$, which is in particular the content of Corollary 3.2.4 in virtue of (3.15). Suppose by contradiction that $\Phi^{\infty}_p(s) = 0$ for some $s \in [0, +\infty)$. By Lemma B.1 there exists a sequence of $(s_j)_{j\in\mathbb{N}}$, $s_j \to s$ as $j \to +\infty$ and $\Phi^{\infty}_p(s_j) > 0$. If, up to a subsequence, we can assume that $\Phi^{\infty}_p(s_j) \to 0$ as $j \to +\infty$, then we can conclude. Indeed, $\Phi^{\infty}_p(s_j) \geq |\nabla\varphi|(x)$ for every $x \in \{\varphi \geq s\}$ and $\Phi^{\infty}_p(s_j) \to 0$ as $j \to +\infty$. Hence, by (3.41), $|\nabla\varphi| = 0$ on $\{\varphi \geq s\}$, contradicting the unboundedness of φ. Suppose now by contradiction that every subsequence of $\Phi^{\infty}_p(s_j)$ does not vanish, then there would be $\delta > 0$ and $J \in \mathbb{N}$ such that $\Phi^{\infty}_p(s_j) > \delta$ for every $j \geq J$. Since level sets of φ are compact, $\Phi^{\beta}_p(s_j) = |\nabla\varphi|(x_{s_j})$ at some point $x_{s_j} \in \{\varphi = s_j\}$. Moreover, $(x_{s_j})_{j\in\mathbb{N}}$ is bounded, since it is contained in $\{\varphi \leq s\}$. Hence, we can assume that there exists $x \in \{\varphi \leq s\}$ such that $x_{s_j} \to x$ as $j \to +\infty$. Since φ is $\mathscr{C}^1$, we obtain that $\varphi(x) = s$ and $|\nabla\varphi|(x) \geq \delta$, contradicting the fact that $\Phi^{\infty}_p(s) = 0$.

Using a similar argument we can infer that $s \mapsto \Phi^{\beta}_p(s)$ is left continuous. Indeed, by contradiction there would be $\delta > 0$ such that $\Phi^{\infty}_p(s) \geq \Phi^{\infty}_p(s_0) + \delta$ for any $s < s_0$. Let $x_s \in \{\varphi = s\}$ such that $\Phi^{\infty}_p(s) = |\nabla\varphi|(x_s)$. By the compactness of $\{\varphi \leq s_0\}$, there exists a sequence $(s_j)_{j\in\mathbb{N}}$ and a point $x \in \{\varphi \leq s_0\}$ such that $s_j < s_0$, $s_j \to s_0$ and $x_{s_j} \to x$. Since $\varphi \in \mathscr{C}^1$, $\varphi(x) = s_0$ and $|\nabla\varphi|(x) \geq \Phi^{\infty}_p(s_0) + \delta$, contradicting the definition of Φ^{∞}_p. To prove the right continuity it is enough to prove that $s \mapsto \Phi^{\infty}_p(s)$ is lower semicontinuous. Since $\Phi^{\infty}_p > 0$, the maximum of $|\nabla\varphi|$ on $\{\varphi = s\}$ is achieved at a regular point x. Let $(s_j)_{j\in\mathbb{N}}$ be a sequence such that $s_j \to s$ as $j \to +\infty$. Since $|\nabla\varphi|$ is continuous, there exists a sequence of points $(x_{s_j})_{j\in\mathbb{N}}$ such that $x_{s_j} \in \{\varphi = s_j\}$ and $x_{s_j} \to x$ as $j \to +\infty$. Since $|\nabla\varphi|(x_{s_j}) \leq \Phi^{\infty}_p(s_j)$ for every $j \in \mathbb{N}$, yielding the lower semicontinuity.

Let $x_s \in \{\varphi = s\}$ be the point where the maximum of $|\nabla\varphi|$ in $\{\varphi = s\}$ is achieved. By the monotonicity it is also the maximum in $\{\varphi \geq s\}$ and in particular

$$\frac{\partial}{\partial\nu}|\nabla\varphi|^p(x_s) \leq 0 \tag{3.43}$$

by the Hopf's Maximum Principle, since x_s is a regular point for φ. (3.25) follows then from (3.17).

It remains only to show the rigidity part of the theorem. Assume that $\Phi^{\infty}_p(s_0) = \Phi^{\infty}_p(S)$ for some $S > s_0$ with s_0 regular. By the monotonicity of Φ^{∞}_p, we can assume that S is so close to s_0 that $|\nabla\varphi|$ never vanishes on $\{s_0 \leq \varphi \leq S\}$. Consider $x_S \in \{\varphi = S\}$ such that $\Phi^{\infty}_p(s) = |\nabla\varphi|(x_S)$. Let $\delta > 0$ be small enough so that $|\nabla\varphi| > 0$ on $\{s_0 \leq \varphi \leq S + \delta\}$, then

$$\sup_{\{\varphi=S\}} |\nabla\varphi| = \sup_{\{\varphi=s_0\}} |\nabla\varphi| = \sup_{\{\varphi=s_0\}\cup\{\varphi=S+\delta\}} |\nabla\varphi|.$$

Since x_S is an interior point of $\{s_0 \leq \varphi \leq S + \delta\}$, by the Maximum Principle we have that $|\nabla\varphi|$ is positively constant on $\{s_0 \leq \varphi \leq S + \delta\}$. The continuity of $|\nabla\varphi|$ implies that no singular values bigger then s_0 can occur. In particular, δ can be taken arbitrarily big, showing that $|\nabla\varphi|$ is constant on $\{\varphi \geq s_0\}$ and thus that $(\{\varphi \geq s_0\}, \tilde{g})$ is isometric to the Riemannian product $([s_0, +\infty) \times \{\varphi = s_0\}, ds^2 + g_{\{\varphi=s_0\}})$ by Lemma 3.3.5.

Finally, suppose that equality holds in (3.25). In this case the normal derivative in (3.43) vanishes. Since x_s is a global maximum for $|\nabla\varphi|$ on $\{s_0 \leq \varphi \leq S\}$ for every $S > s_0$ and $|\nabla\varphi|^p$ is a subsolution of $\mathscr{L}(f) = 0$ provided it never vanishes on $\{s_0 \leq \varphi \leq S\}$, Hopf's Maximum principle implies that $|\nabla\varphi|^p$ is positively constant on this region. Arguing as above, we infer that no singular value can be bigger than s_0, $|\nabla\varphi|$ is constant on $\{\varphi \geq s_0\}$ and the same cylindrical splitting in virtue of Lemma 3.3.5. □

4

Geometric consequences of the monotonicity formulas

4.1 Structure of the chapter

In this chapter, we prove the geometric implications of the Monotonicity-Rigidity theorems, which are the Minkowski Inequalities, a rigidity result under pinching conditions and a sphere theorem. In Section 4.2 we show the family of $L^p_{(\beta)}$-Minkowski Inequalities. Among them, choosing particular β's, we can find the L^p-Minkowski Inequality holding on Riemannian manifolds with nonnegative Ricci curvature and Euclidean Volume Growth and also the Willmore-type Inequality that was already proved in [AFM20] with a different technique. Coupling these results with the Isoperimetric Inequality in [Bre22] we obtain their volumetric versions. All the inequalities presented are sharp and the equality is satisfied only on Riemannian cones. Letting $p \to 1^+$, we finally obtain the Extended Minkowski Inequality (18) which extends the result in [AFM22; FMP19]. This inequality turns out to be sharp on Asymptotically Conical Riemannian manifolds with nonnegative Ricci curvature and we characterise the equality case for strictly outward minimising subsets with smooth strictly mean-convex boundary using the Inverse Mean Curvature flow. Section 4.3 extends to the nonlinear setting the results in [BMM19; Fog20] which are two rigidity results under a pinching condition on the mean curvature and the normal derivative of the p-capacitary potential associated with the set Ω.

4.2 Minkowski-type Inequalities

The first geometric consequence we present is the sharp Minkowski Inequality on Riemannian manifolds with nonnegative Ricci curvature and Euclidean Volume Growth. As already said, the inequality is obtained by approximation, letting $p \to 1^+$ in the L^p-Minkowski Inequality. This family of inequalities follows by a contradiction argument that involves the Iso-p-capacitary Inequality Theorem 2.3.10 and the monotonicity of F_p^β for $\beta = 1/(p-1)$. For this choice of β, the function F_p^β is can be rewritten (up to a constant) as

$$t \mapsto \left(\widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega_t)\right)^{-\frac{n-p-1}{n-p}} \int_{\partial\Omega_t} |\mathrm{D}\log u|^p \,\mathrm{d}\sigma \tag{4.1}$$

for $t \in [1, +\infty)$, where $\partial\Omega_t$ is the level set of the p-capacitary potential u associated with Ω through (2.7). One can recognise that the quantity in (4.1) formally converges to the monotone quantity that is classically used to prove the Minkowski Inequality using the IMCF (see (7)). Even using Theorem 2.4.12 and Theorem 2.4.10, we cannot pass directly to the limit.

4.2.1 L^p-Minkowski Inequality

We are now ready to prove the $L^p_{(\beta)}$-Minkowski Inequality.

Theorem 4.2.1 ($L^p_{(\beta)}$-Minkowski Inequality)**.** *Let (M, g) be a complete Riemannian manifold with* Ric ≥ 0 *and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then, for every $1 < p < n$ and $\beta \geq (n-p)/[(p-1)(n-1)]$, the following inequality holds*

$$\widehat{\mathrm{Cap}}_p(\partial\Omega)^{1-\beta\frac{(p-1)}{(n-p)}} \mathrm{AVR}(g)^{\beta\frac{p-1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{(\beta+1)(p-1)} \mathrm{d}\sigma \tag{4.2}$$

Moreover, equality holds in (4.2) if and only if $(M \setminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega} \right), \qquad \text{with } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

Proof. We prove it for $\beta = 1/(p-1)$, the general case being obtained similarly. We first show that

$$\widehat{\mathrm{Cap}}_p(\partial\Omega)^{\frac{n-p-1}{n-p}} \mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^p \int\limits_{\partial\Omega} |\mathrm{D} \log u|^p \, \mathrm{d}\sigma \tag{4.3}$$

holds true for any open bounded subset $\Omega \subseteq M$ with smooth boundary. Let then $\theta < \mathrm{AVR}(g)$ and suppose by contradiction that there exists an open bounded subset $\Omega \subseteq M$ with smooth boundary, such that

$$\widehat{\mathrm{Cap}}_p(\partial\Omega)^{\frac{n-p-1}{n-p}} \theta^{\frac{1}{n-p}} \geq \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p} \right)^p \int\limits_{\partial\Omega} |\mathrm{D} \log u|^p \, \mathrm{d}\sigma.$$

Define $\tau = 1/t \in (0, 1]$ and $\Omega_\tau = \{u > \tau\} \cup \Omega$. By Theorem 3.2.1, the function $\tau \mapsto F_p^\beta(\tau)$ is nondecreasing for $\tau \in (0, 1]$. Exploiting this monotonicity we have

$$\begin{aligned} \left(\frac{n-p}{p-1} \right)^p |\mathbb{S}^{n-1}| \theta^{\frac{1}{n-p}} &\geq \widehat{\mathrm{Cap}}_p(\partial\Omega)^{-\frac{n-p-1}{n-p}} \int\limits_{\partial\Omega} |\mathrm{D} \log u|^p \, \mathrm{d}\sigma \\ &\geq \widehat{\mathrm{Cap}}_p(\partial\Omega_\tau)^{-\frac{n-p-1}{n-p}} \int\limits_{\{u=\tau\}} |\mathrm{D} \log u|^p \, \mathrm{d}\sigma. \end{aligned} \tag{4.4}$$

The Hölder's Inequality with conjugate exponents $a = (p+1)/p$ and $b = p+1$, yields

$$\mathrm{Cap}_p(\partial\Omega_\tau)^{\frac{p+1}{p}} \leq \left(\int\limits_{\{u=\tau\}} |\mathrm{D}\log u|^p \,\mathrm{d}\sigma \right) \left(\int\limits_{\{u=\tau\}} \frac{1}{|\mathrm{D}\log u|} \,\mathrm{d}\sigma \right)^{\frac{1}{p}}.$$

Therefore, plugging it into (4.4), we get

$$\left|\mathbb{S}^{n-1}\right| \widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega_\tau)^{\frac{n}{n-p}} \leq \left(\frac{n-p}{p-1}\right) \theta^{\frac{p}{n-p}} \int\limits_{\{u=\tau\}} \frac{1}{|\mathrm{D}\log u|} \,\mathrm{d}\sigma.$$

Using (1.20) and integrating both sides we obtain

$$\left|\mathbb{S}^{n-1}\right| \widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega)^{\frac{n}{n-p}} \int\limits_\tau^1 s^{-\frac{n(p-1)}{n-p}-1} \,\mathrm{d}s \leq \left(\frac{n-p}{p-1}\right) \theta^{\frac{p}{n-p}} \int_\tau^1 \int\limits_{\{u=s\}} \frac{1}{|\mathrm{D}u|} \,\mathrm{d}\sigma \,\mathrm{d}s,$$

that, together with the coarea formula Proposition B.3 with $f = (1-\chi_{\mathrm{Crit}\,u})|\mathrm{D}u|^{-1}$, leaves us with

$$\frac{|\mathbb{S}^{n-1}|}{n} \left(\widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega_\tau)^{\frac{n}{n-p}} - \widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega)^{\frac{n}{n-p}} \right) \leq \theta^{\frac{p}{n-p}} |\Omega_\tau \smallsetminus (\Omega \cup \mathrm{Crit}\, u)|,$$

for every $\tau \in [0,1)$. Applying the sharp Iso-p-capacitary inequality (2.13) to the left hand side we obtain

$$\mathrm{AVR}(g)^{\frac{p}{n-p}} \left(|\Omega_\tau| - \widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega)^{\frac{n}{n-p}} \right) \leq \theta^{\frac{p}{n-p}} |\Omega_\tau|.$$

Dividing both sides by $|\Omega_\tau|$ and passing to the limit as $\tau \to 0$, we get a contradiction with $\theta < \mathrm{AVR}(g)$, proving that for any $\theta < \mathrm{AVR}(g)$

$$\widehat{\mathrm{C}}\mathrm{ap}_p(\partial\Omega)^{\frac{n-p-1}{n-p}} \theta^{\frac{1}{n-p}} < \frac{1}{|\mathbb{S}^{n-1}|} \left(\frac{p-1}{n-p}\right)^p \int\limits_{\partial\Omega} |\mathrm{D}\log u|^p \,\mathrm{d}\sigma$$

holds true for every any bounded open $\Omega \subset M$ with smooth boundary. Letting $\theta \to \mathrm{AVR}(g)^-$ yields (4.3).

To conclude observe that Theorem 3.2.1 implies $(F_p)'(1) \leq 0$ and thus, thanks to (3.3), we have

$$\int\limits_{\partial\Omega} \left(\frac{p-1}{n-p}\right) |\mathrm{D}u|^p \,\mathrm{d}\sigma \leq \int\limits_{\partial\Omega} |\mathrm{D}u|^{p-1} \frac{\mathrm{H}}{n-1} \,\mathrm{d}\sigma.$$

By Hölder's Inequality with conjugate exponents $a = p/(p-1)$ and $b = p$, we get

$$\int\limits_{\partial\Omega} |\mathrm{D}\log u|^p \,\mathrm{d}\sigma \leq \left(\frac{n-p}{p-1}\right)^p \int\limits_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|^p \mathrm{d}\sigma, \tag{4.5}$$

that coupled with (4.3) concludes the proof of (4.2).

If we now assume that equality holds in (4.2), then the two sides of (4.5) are identical too. In particular, by (3.3), $F_p'(1) = 0$ and the rigidity statement in Theorem 3.2.1 applies. □

Two direct corollaries of this results are the L^p-Minkowski Inequalities [BFM21, Theorem 4.3] and the Willmore-type Inequality [AFM20, Theorem 1.1], respectively obtained from (4.2) for $\beta = 1/(p-1)$ and $\beta = (n-p)/(p-1)$.

Corollary 4.2.2 (L^p-Minkowski Inequalities)**.** *Let (M, g) be a complete Riemannian manifold with* Ric ≥ 0 *and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then, for every $1 < p < n$, the following inequality holds*

$$\widehat{\mathrm{Cap}}_p(\partial\Omega)^{\frac{n-p-1}{n-p}} \mathrm{AVR}(g)^{\frac{1}{n-p}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^p \mathrm{d}\sigma. \tag{4.6}$$

Moreover, equality holds in (4.6) if and only if $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega} \right), \qquad \textit{with } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

Corollary 4.2.3 (Willmore-type Inequality)**.** *Let (M, g) be a complete Riemannian manifold with* Ric ≥ 0 *and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then, the following inequality holds*

$$\left|\mathbb{S}^{n-1}\right| \mathrm{AVR}(g) \leq \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{n-1} \mathrm{d}\sigma. \tag{4.7}$$

Moreover, equality holds in (4.7) if and only if $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega} \right), \qquad \textit{with } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|} \right)^{\frac{1}{n-1}}.$$

Combining Theorem 4.2.1 with the sharp Iso-p-capacitary Inequality Theorem 2.3.10 we obtain a Volumetric $L^p_{(\beta)}$-Minkowski Inequality. The rigidity statement follows in this case from the rigidity of the Iso-p-capacitary Inequality.

Theorem 4.2.4 (Volumetric $L^p_{(\beta)}$-Minkowski Inequality)**.** *Let (M, g) be complete Riemannian manifold with* Ric ≥ 0 *and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then for every $1 < p < n$, the following inequality holds*

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|} \right)^{\frac{n-p}{n} - \beta\frac{p-1}{n}} \mathrm{AVR}(g)^{\beta\frac{p-1}{n} + \frac{p}{n}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left| \frac{\mathrm{H}}{n-1} \right|^{(\beta+1)(p-1)} \mathrm{d}\sigma.$$

Moreover, equality holds in (4.2) if and only if M is the Euclidean Space and Ω is a ball.

4.2.2 Extended Minkowski Inequality

Letting $p \to 1^+$ in the L^p-Minkowski Inequality (4.2) and employing the Dominated Convergence Theorem complete the proof of the Extended Minkowski Inequality.

Theorem 4.2.5 (Extended Minkowski Inequality)**.** *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded set with smooth boundary. Then*

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma. \tag{4.8}$$

Combining Theorem 4.2.5 with the sharp Isoperimetric Inequality for manifolds with nonnegative Ricci curvature [Bre22, Corollary 1.3], reading

$$\frac{|\mathbb{S}^{n-1}|^n}{|\mathbb{B}^n|^{n-1}} \mathrm{AVR}(g) \leq \frac{|\partial\Omega^*|^n}{|\Omega^*|^{n-1}},$$

we get the following sharp volumetric version of the Minkowski Inequality.

Theorem 4.2.6 (Volumetric Minkowski inequality)**.** *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean Volume Growth. Then,*

$$\left(\frac{|\Omega|}{|\mathbb{B}^n|}\right)^{\frac{n-2}{n}} \mathrm{AVR}(g)^{\frac{2}{n}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma.$$

As a corollary, the Minkowski Inequality can be simplified in class of outward minimising sets for which $|\partial\Omega| = |\partial\Omega^*|$ and $\mathrm{H} \geq 0$ (see Section 2.4.2).

Corollary 4.2.7 (Minkowski Inequality for outward minimising sets)**.** *Let (M, g) be a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$ *and Euclidean Volume Growth. Let $\Omega \subseteq M$ be a bounded outward minimising subset with smooth boundary, then*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} \leq \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} \frac{\mathrm{H}}{n-1} \mathrm{d}\sigma.$$

To conclude, we remark that the Extended Minkowski Inequality (4.8) can be also deduced from its version for outward minimising subsets up to dimension $n \leq 7$, using an approximation argument via Mean Curvature Flow (see [HI01; HI08]). In higher dimension this argument fails due to lower regularity of minimal surfaces involved in the definition of the strictly outward minimising hull.

4.2.3 Sharpness of the Minkowski Inequality

We show that on a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with nonnegative Ricci curvature there is a sequence of strictly outward minimising sets that are arbitrarily close to saturate the Minkowski Inequality, inferring its sharpness. The sets $\{\rho = R\}$ are the natural candidates, since they satisfy the equality in (4.8) on cones. We show that, in fact, they are uniformly close to the model one for large enough R.

Proposition 4.2.8. *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with nonnegative Ricci curvature. Then,*

$$\lim_{r\to+\infty} |\{\rho = r\}|^{-\frac{n-2}{n-1}} \int_{\{\rho=r\}} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma = \Big(\mathrm{AVR}(g)|\mathbb{S}^{n-1}|\Big)^{\frac{1}{n-1}}. \tag{4.9}$$

In particular,

$$\inf\left\{ |\partial\Omega^*|^{-\frac{n-2}{n-1}} \int_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right| \mathrm{d}\sigma \;\middle|\; \Omega \subset M,\ \text{with } \partial\Omega \text{ smooth} \right\} = \Big(|\mathbb{S}^{n-1}|\mathrm{AVR}(g)\Big)^{\frac{1}{n-1}}.$$

Proof. $\{\rho \leq r\}$ is strictly outward minimising for large r by Lemma 2.4.17. Suppose by contradiction that (4.9) does not hold. Then, there is a constant $\theta > (\mathrm{AVR}(g)|\mathbb{S}^{n-1}|)^{1/(n-1)}$ such that

$$\int_{\{\rho=s\}} \frac{\mathrm{H}}{n-1}\,\mathrm{d}\sigma \geq \theta|\{\rho = s\}|^{\frac{n-2}{n-1}}$$

on $\{\rho \geq s\}$ for s large enough. Let $\varphi \in \mathscr{C}^\infty_c(1, +\infty)$ be nonnegative. The function $\psi_r(s) = \varphi(s/r)$ is nonnegative and $\psi_r \in \mathscr{C}^\infty_c(r, +\infty)$. Taking s large enough, we can assume that $|\mathrm{D}\rho| \geq \delta$ holds on $\{\rho \geq s\}$ for some $\delta > 0$. Multiplying by ψ_r, dividing by $|\mathrm{D}\rho|$ and integrating both sides we get

$$\int_r^{+\infty} \psi_r(s) \int_{\{\rho=s\}} \frac{\mathrm{H}}{n-1} |\mathrm{D}\rho|^{-1}\,\mathrm{d}\sigma\,\mathrm{d}s \geq \theta \int_r^{+\infty} \psi_r(s)|\{\rho = s\}|^{\frac{n-2}{n-1}} |\mathrm{D}\rho|^{-1}\,\mathrm{d}s.$$

By the coarea formula in Proposition B.3 with $f = \varphi(\rho)|\mathrm{D}\rho|^{-1}\,\mathrm{H}\,/(n-1)$ we obtain

$$\int_{\{\rho\geq r\}} \frac{\psi_r(s)}{n-1} \operatorname{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right) \mathrm{d}\mu \geq \theta \int_r^{+\infty} \psi_r(s)|\{\rho = s\}|^{\frac{n-2}{n-1}} |\mathrm{D}\rho|^{-1}\,\mathrm{d}s.$$

Integrating by parts the right hand side we have

$$-\int_{\{\rho\geq r\}} \frac{\psi_r'(s)}{(n-1)} |\mathrm{D}\rho|\,\mathrm{d}\mu \geq \theta \int_r^{+\infty} \psi_r(s)|\{\rho = s\}|^{\frac{n-2}{n-1}} |\mathrm{D}\rho|^{-1}\,\mathrm{d}s. \tag{4.10}$$

Observe that each side depends only on the coefficients of the metric and not on their derivatives. Hence for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\int_{\{\rho\geq r\}} \psi_r'(s)|\mathrm{D}\rho|\,\mathrm{d}\mu \geq (1+\varepsilon) \int_{\{\rho\geq r\}} \psi_r'(s)\,\mathrm{d}\mu_{\hat{g}} \tag{4.11}$$

and

$$\int_r^{+\infty} \psi_r(s)|\{\rho = s\}|^{\frac{n-2}{n-1}} |\mathrm{D}\rho|^{-1}\,\mathrm{d}s \geq (1-\varepsilon) \int_r^{+\infty} \psi_r(s) s^{n-2} |L|^{\frac{n-2}{n-1}}\,\mathrm{d}s. \tag{4.12}$$

Putting (4.11) and (4.12) into (4.10) we obtain

$$-\frac{(1+\varepsilon)}{(1-\varepsilon)}\int_{\{\rho\geq r\}}\frac{\psi_r'(s)}{(n-1)}\,\mathrm{d}\mu_{\hat{g}}\geq\theta\int_r^{+\infty}\psi_r(s)s^{n-2}|L|^{\frac{n-2}{n-1}}\,\mathrm{d}s.$$

Appealing again to the coarea formula on the left hand side, it leaves us with

$$-\frac{(1+\varepsilon)}{(1-\varepsilon)}\int_r^{+\infty}\frac{\psi_r'(s)}{(n-1)}|L|s^{n-1}\,\mathrm{d}s\geq\theta\int_r^{+\infty}\psi_r(s)s^{n-2}|L|^{\frac{n-2}{n-1}}\,\mathrm{d}s.$$

Integrating by parts the left hand side we conclude that

$$\begin{aligned}\frac{(1+\varepsilon)}{(1-\varepsilon)}|L|\int_r^{+\infty}\psi_r(s)s^{n-2}\,\mathrm{d}s\geq\theta|L|^{\frac{n-2}{n-1}}\int_r^{+\infty}\psi_r(s)s^{n-2}\,\mathrm{d}s&\\ >\Big(\mathrm{AVR}(g)|\mathbb{S}^{n-1}|\Big)^{\frac{1}{n-1}}|L|^{\frac{n-2}{n-1}}\int_r^{+\infty}\psi_r(s)s^{n-2}\,\mathrm{d}s&\\ =|L|\int_r^{+\infty}\psi_r(s)s^{n-2}\,\mathrm{d}s,&\end{aligned}$$

which yields the desired contradiction by the arbitrariness of $\varepsilon>0$. □

4.2.4 Rigidity of the Minkowski Inequality

We now prove the Rigidity statement of the Minkowski Inequality. Observe once again that the rigidity statement does not follow from the rigidity of the L^p-Minkowski Inequalities. Indeed, in that case, it was a consequence of the Monotonicity-Rigidity Theorem 3.2.1 that in turn follows by the vanishing of a nonnegative quantity. This quantity degenerates as $p\to 1^+$ and the correct understanding of limit behaviour seems to require a more delicate analysis. This forces us to discuss the rigidity statement in a separate argument, involving the study of the IMCF starting at boundaries of domains that saturate the Minkowski Inequality Theorem 4.2.5. Going into more detail, consider a Riemannian manifold (M,g) with nonnegative Ricci curvature and Euclidean Volume Growth. Let $\Omega\subset M$ be an open bounded subset with smooth strictly mean-convex boundary. We evolve $\partial\Omega$ by smooth IMCF (see (2.39)). We prove that the evolved hypersurfaces, in an outer neighbourhood of $\partial\Omega$, constitute the cross-sections of a truncated cone with the same volume ratio of (M,g). The conclusion then follows from a generalisation of the Bishop-Gromov Theorem. The conical splitting we aim to is inspired by an argument contained in [HI01, Section 8]. A first step consists in the following fundamental Lemma.

Lemma 4.2.9. *Let (M,g) be a complete Riemannian manifold with $\mathrm{Ric}\geq 0$ and $\Sigma\subseteq M$ a totally umbilical closed hypersuface such that $\mathrm{Ric}(\nu,\nu)=0$ where ν is the normal unit vector field to Σ. Then Σ has* constant *mean curvature.*

Proof. The (traced) Codazzi-Mainardi equations and the totally umbilicity yields

$$\mathrm{Ric}_{j\nu} = \mathrm{D}_i \,\mathrm{h}_{ij} - \mathrm{D}_j \,\mathrm{H} = -\frac{n-2}{n-1} \mathrm{D}_j \,\mathrm{H}$$

for any $j = 1, \dots, n-1$. Consider, at a fixed point on Σ, the vector $\eta_\lambda = \lambda \,\mathrm{D}^\top \mathrm{H} + \nu$, with $\lambda \in \mathbb{R}$. Since $\mathrm{Ric}(\nu, \nu) = 0$, we have

$$0 \leq \mathrm{Ric}(\eta, \eta) = 2 \,\mathrm{Ric}_{j\nu} \,\eta^j \eta^\nu + \mathrm{Ric}_{ij} \,\eta^i \eta^j = -2\lambda \frac{n-2}{n-1} \left|\mathrm{D}^\top \mathrm{H}\right|^2 + \lambda^2 \,\mathrm{Ric}_{ij} \,\mathrm{D}^i \,\mathrm{H} \,\mathrm{D}^j \,\mathrm{H}$$

for every $\lambda \in \mathbb{R}$. This can happen only if $|\mathrm{D}^\top \mathrm{H}| = 0$, hence H is constant on Σ. □

The following straightforward but very important consequence of the Bishop-Gromov monotonicity ensures in particular that if an outer neighbourhood of a bounded open set with smooth boundary $K \subset M$ is isometric to a truncated cone, then the whole complement of K is isometric to a truncated cone based at ∂K.

Lemma 4.2.10. *Let (M, g) be a complete noncompact Riemannian manifold with* $\mathrm{Ric} \geq 0$. *Let $K \subset M$ be a bounded open set. Suppose there exists a outer neighbourhood $A \subset M \smallsetminus K$ of K such that (A, g) is isometric to*

$$\left([\rho_0, \rho_1] \times \partial K, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right)$$

for $0 < \rho_0 < \rho_1$. Then

$$|\partial K| \geq \rho_0^{n-1} \left|\mathbb{S}^{n-1}\right| \mathrm{AVR}(g), \tag{4.13}$$

and equality holds if and only $(M \smallsetminus K, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial K, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right).$$

Proof. Consider the cone $(C, \hat{g})$ given by

$$\left((0, \rho_1) \times \partial K, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial K} \right),$$

and the Riemannian manifold, with a conical singularity, obtained by gluing $(C, \hat{g})$ with $(M \setminus (K \cup A), g)$ along $\{\rho = \rho_1\}$. By our assumptions, such manifold is well-defined with nonnegative Ricci curvature outside of the tip o of C, and coincides with (M, g) in the complement of K. In C, the geodesic distance from o is given by ρ, and in particular, by Bishop-Gromov's monotonicity,

$$\frac{|\{\rho = r\}|}{r^{n-1} |\mathbb{S}^{n-1}|} \geq \mathrm{AVR}(g),$$

for any $r \in (0, \rho_1)$. Since $|\{\rho = \rho_0\}| = |\partial K|$, setting $r = \rho_0$ proves (4.13). If equality holds, then, by the rigidity statement in Bishop-Gromov's Theorem for manifolds with a conical singularity, the whole manifold we constructed is isometric to a cone, and in particular $(M \setminus K, g)$ splits as claimed. This well-known, slightly enhanced version of the Bishop-Gromov rigidity statement can be readily deduced from its classic proof, or seen as a very special case of its version for nonsmooth metric spaces [DG16]. □

Theorem 4.2.11 (Rigidity statement)**.** *Let (M, g) be a Riemannian manifold with nonnegative Ricci curvature. Let $\Omega \subset M$ be a bounded open strictly outward minimising subset with smooth strictly mean-convex boundary, such that*

$$\left(\frac{|\partial\Omega|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\partial\Omega} \frac{\mathrm{H}}{n-1}\, \mathrm{d}\sigma. \tag{4.14}$$

Then $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \qquad \textit{with } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}. \tag{4.15}$$

In the Ricci-flat case a stronger rigidity result is in force. Indeed, the equation $\mathrm{Ric} = 0$ implies that g is analytic [DK81, Theorem 5.2] (see also [Bes08, Theorem 5.26]), and thus the conical structure (4.15) can be extended also inside Ω. Since the only smooth cone is the flat Euclidean space and Ω is totally umibilical we have the following corollary.

Corollary 4.2.12. *Let (M, g) be a Ricci-flat Riemannian manifold. Let $\Omega \subset M$ be a bounded open strictly outward minimising subset with smooth strictly mean-convex boundary that satisfies* (4.14)*. Then (M, g) is isometric to the flat Euclidean space and Ω is a ball.*

Proof of Theorem 4.2.11. Since $\partial\Omega$ is by assumption strictly mean-convex, we can evolve it by (smooth) IMCF $\partial\Omega_t$ defined in (2.39) for $t \in [0, T)$. By the Smooth Start Lemma 2.4.7, up to shortening the time interval, we can assume that Ω_t is strictly outward minimising for any $t \in [0, T)$. Indeed, since Ω is strictly outward minimising, the flow coincides for a short time with the weak notion of IMCF, that exists in our setting by Theorem 2.4.12. The sublevel sets of the weak IMCF being strictly outward minimising is a basic and fundamental property illustrated in [HI01, Minimizing Hull Property 1.4] (see otherwise Section 2.4.2). Consider then the function $\mathcal{Q} : [0, T) \to \mathbb{R}$ defined by

$$\mathcal{Q}(t) \;=\; |\partial\Omega_t|^{-\frac{n-2}{n-1}} \int\limits_{\partial\Omega_t} \mathrm{H}_t \;\mathrm{d}\sigma\,.$$

A straightforward computation, direct consequence of the evolution equations for curvature flows derived for example in [HP99, Theorem 3.2] shows that

$$\mathcal{Q}'(t) = -|\partial\Omega_t|^{-\frac{n-2}{n-1}} \int\limits_{\partial\Omega_t} \frac{|\mathring{\mathrm{h}}_t|^2 + \mathrm{Ric}(\nu_t, \nu_t)}{\mathrm{H}_t}\, \mathrm{d}\sigma \leq 0\,,$$

where by $\mathring{\mathrm{h}}_t$ we denote the trace-free part of the second fundamental form h_t of $\partial\Omega_t$. On the other hand, the strict inequality for some $t \in [0, T)$ would result in a contradiction to the Minkowski Inequality. Thus $\mathcal{Q}'(t)$ vanishes for any $t \in [0, T)$ and, in particular $\partial\Omega_t$ satisfies (4.14) for any $t \in [0, T)$. Hence, $\partial\Omega_t$ is totally umbilical and satisfies $\mathrm{Ric}(\nu_t, \nu_t) = 0$ in for every $t \in [0, T)$. By Lemma 4.2.9 $\partial\Omega_t$ has constant mean curvature for every $t \in [0, T)$.

Define the function w such that $\{w = t\} = \partial\Omega_t$. Since $\{w = t\}$ evolves by smooth IMCF for $t \in [0, T)$ it satisfies the relation

$$\mathrm{H}_t = \operatorname{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|}\right)(x) = |\mathrm{D}w|(x)$$

at any point $x \in \{w = t\}$, where H_t is the mean curvature of $\{w = t\}$. Hence, since $\mathrm{H}_t > 0$, a well-known extension of Gauss' Lemma yields

$$g = \frac{\mathrm{d}w \otimes \mathrm{d}w}{|\mathrm{D}w|^2} + g_{\partial\Omega_t} = \frac{\mathrm{d}t \otimes \mathrm{d}t}{\mathrm{H}(t,x)^2} + g_{\partial\Omega_t}. \tag{4.16}$$

The evolution equation (see [HP99, Theorem 3.2 (i)]) satisfied by $g_{\partial\Omega_t}$ is

$$\frac{\partial}{\partial t} g_{\partial\Omega_t} = 2\frac{\mathrm{h}_t}{\mathrm{H}_t} = \frac{2}{n-1} g_{\partial\Omega_t},$$

where the last identity is due to the total umbilicity of $\partial\Omega_t$. Integrating such an equation we deduce

$$g_{\partial\Omega_t} = \mathrm{e}^{\frac{2t}{(n-1)}} g_{\partial\Omega}, \tag{4.17}$$

On the other hand, the evolution equation for the mean curvature along the IMCF (see [HP99, Theorem 3.2 (v)]) declaims

$$\frac{\partial}{\partial t}\mathrm{H}_t = -\Delta_{\partial\Omega_t}\left(\frac{1}{\mathrm{H}_t}\right) - \frac{1}{\mathrm{H}_t}\left[|\mathrm{h}_t|^2 + \mathrm{Ric}(\nu_t, \nu_t)\right] = -\frac{\mathrm{H}_t}{n-1},$$

where the last identity is due to the fact that $\partial\Omega_t$ is totally umbilical, $\mathrm{Ric}(\nu_t, \nu_t) = 0$ and the mean curvature H_t of $\partial\Omega_t$ depends only on t. Integrating it we obtain that

$$\mathrm{H}_t = \mathrm{e}^{-\frac{t}{n-1}}\,\mathrm{H}_0, \tag{4.18}$$

where H_0 is the mean curvature of $\partial\Omega$.

Plugging (4.17) and (4.18) into (4.16), we deduce that $(\{0 \leq w < T\}, g)$ is isometric to

$$\left([0, T) \times \partial\Omega,\ \mathrm{e}^{\frac{2t}{n-1}}\,\frac{\mathrm{d}t \otimes \mathrm{d}t}{\mathrm{H}_0^2} + \mathrm{e}^{\frac{2t}{n-1}}\,g_{\partial\Omega}\right).$$

Performing the change of variables

$$\rho = \frac{(n-1)}{\mathrm{H}_0}\mathrm{e}^{\frac{t}{(n-1)}},$$

the metric can be written as

$$\left([\rho_0, \rho(T)) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right) \qquad \text{where } \rho_0 = \frac{(n-1)}{\mathrm{H}_0}.$$

On the other hand, since by assumption $\partial\Omega$ saturates the Minkowski Inequality, that is (4.14) holds true, we immediately get

$$\rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

We conclude by the rigidity statement in Lemma 4.2.10, that the whole $M \setminus \Omega$ is isometric to a truncated cone. □

In dimension $3 \leq n \leq 7$, an open bounded subset Ω with smooth strictly mean-convex boundary satisfying

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \frac{\mathrm{H}}{n-1} \, \mathrm{d}\sigma.$$

is *a priori* strictly outward minimising, and thus, in this case, such assumption can be dropped from Theorem 4.2.11. Indeed, by approximating Ω via Mean Curvature Flow with smooth strictly outward minimising domains, as described in [HI01, Lemma 5.6], we deduce that (4.8) holds also for $\mathscr{C}^{1,1}$-hypersurfaces. In particular, the Minkowski Inequality holds also for the strictly outward minimising hull of Ω (see Theorem 2.4.4) for every Ω with smooth boundary, provided the dimensional bound holds true. We can then argue by contradiction. Suppose that Ω^* does not coincide with Ω, then

$$\left(\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}} \mathrm{AVR}(g)^{\frac{1}{n-1}} = \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega} \frac{\mathrm{H}}{n-1} \, \mathrm{d}\sigma > \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\partial\Omega^*} \frac{\mathrm{H}}{n-1} \, \mathrm{d}\sigma$$

where the last inequality is due to the fact that $\mathrm{H} = 0$ on $\partial\Omega^* \smallsetminus \partial\Omega$. But this contradicts the Minkowski Inequality for Ω^*, hence $\Omega = \Omega^*$.

4.3 A pinching condition and a sphere theorem

In this section, we exploit the monotonicity of the function $t \mapsto F_p^\infty(t)$ defined in (3.2) to prove a couple of rigidity statements involving a pinching condition on the mean curvature of $\partial\Omega$ and an *a priori* bound on the gradient of the p-capacitary potential associated with Ω. These results without any convexity assumption are new also in $\mathbb{R}^n$, and they constitute the complete nonlinear generalisation of [BMM19, Corollary 1.4 and 1.9]. For convex subsets of the Euclidean space, they are the content of [FMP19, Corollary 2.16 and 2.17].

Theorem 4.3.1. *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean Volume Growth. If there exists an open bounded subset $\Omega \subseteq M$ with smooth boundary satisfying*

$$-\left[\frac{\mathrm{AVR}(g)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)}\right]^{\frac{1}{n-p}} \leq \frac{\mathrm{H}}{n-1} \leq \left[\frac{\mathrm{AVR}(g)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)}\right]^{\frac{1}{n-p}} \tag{4.19}$$

at every point of $\partial\Omega$, then $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right), \qquad \textit{with } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

In this case $\partial\Omega$ is a connected totally umbilical hypersurface with constant mean curvature in $(M \smallsetminus \Omega, g)$.

Proof. We can argue by contradiction as in Theorem 4.2.1 to prove that

$$\left(\frac{n-p}{p-1}\right)\left[\frac{\mathrm{AVR}(g)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega} |\mathrm{D}u|.$$

Indeed, we can follow the same lines replacing the consequence of the monotonicity of F_p^β with the corresponding of F_p^∞, that thanks to (1.20) can be rewritten as

$$F_p^\infty(t) = t^{\frac{n-1}{n-p}} \sup_{\{u=1/t\}} |\mathrm{D}u| = \left(\frac{\widehat{\mathrm{Cap}}_p(\partial\Omega_\tau)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)}\right)^{\frac{1}{n-p}} \sup_{\{u=\tau\}} |\mathrm{D}\log u|$$

where $\tau = 1/t \in (0,1]$ and $\Omega_\tau = \{u > \tau\} \cup \Omega$. Accordingly, we employ the Hölder's Inequality with conjugate exponents $a = +\infty$ and $b = 1$, that is

$$\mathrm{Cap}_p(\partial\Omega_\tau)^{\frac{1}{p}} \leq \sup_{\{u=\tau\}} |\mathrm{D}\log u| \left(\int\limits_{\{u=\tau\}} \frac{1}{|\mathrm{D}\log u|}\, \mathrm{d}\sigma \right)^{\frac{1}{p}}.$$

In the end, by Theorem 3.2.3 we get

$$\sup_{\partial\Omega} |\mathrm{D}u| \leq \frac{(n-p)}{(p-1)(n-1)} \sup_{\partial\Omega} |\mathrm{H}|,$$

and the equality holds if and only if $(M \smallsetminus \Omega, g)$ splits as in the statement. Condition (4.19) easily implies the equality. □

The above result is a rigidity theorem under a pinching condition on the mean curvature of $\partial\Omega$ with respect to its p-capacity. From the proof above we can also get that

$$\left(\frac{n-p}{p-1}\right)\left[\frac{\mathrm{AVR}(g)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega} |\mathrm{D}u| \tag{4.20}$$

and the equality is satisfied only on metric cones, by Theorem 3.2.3. Letting $p \to 1^+$ in (4.20), we get the following completely geometric consequence.

Proposition 4.3.2. *Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq 0$ and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then, the following inequality holds*

$$\left[\frac{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\right]^{\frac{1}{n-1}} \leq \sup_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|. \tag{4.21}$$

In particular, it holds

$$\left[\frac{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right]^{\frac{1}{n-1}} \leq \sup_{\partial\Omega} \left|\frac{\mathrm{H}}{n-1}\right|. \tag{4.22}$$

If the equality in (4.21) is achieved on a strictly outward minimising set with strictly mean-convex boundary, then $(M \smallsetminus \Omega, g)$ is isometric to

$$\left([\rho_0, +\infty) \times \partial\Omega, \mathrm{d}\rho \otimes \mathrm{d}\rho + \left(\frac{\rho}{\rho_0}\right)^2 g_{\partial\Omega}\right) \qquad \text{where } \rho_0 = \left(\frac{|\partial\Omega|}{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}\right)^{\frac{1}{n-1}}.$$

Proof. By (4.20) and Theorem 3.2.3 we have

$$\left[\frac{\mathrm{AVR}(g)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)}\right]^{\frac{1}{n-p}} \leq \sup_{\partial\Omega}\left|\frac{\mathrm{H}}{n-1}\right|.$$

Letting $p \to 1^+$ and using Theorem 2.4.10 we get, coupling it with $|\partial\Omega^*| \leq |\partial\Omega|$ we get both (4.21) and (4.22). In the case of outward minimising sets, (4.21) follows from the Minkowski Inequality Corollary 4.2.7. The rigidity follows from the rigidity of the Minkowski Inequality Theorem 4.2.11. □

The inequality (4.20) also gives a lower bound on the gradient of u on $\partial\Omega$ in terms of the p-capacity of Ω that, when attained, forces (M, g) to be (isometric to) $\mathbb{R}^n$ with Ω a Euclidean ball.

Theorem 4.3.3. *Let (M, g) be a complete Riemannian manifold with* Ric ≥ 0 *curvature and Euclidean Volume Growth. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary and u the p-capacitary potential associated with Ω and assume that*

$$\sup_{\partial\Omega}|\mathrm{D}u| \leq \left(\frac{n-p}{p-1}\right)\mathrm{AVR}(g)^{\frac{1}{p-1}}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{1}{n-1}}. \tag{4.23}$$

Then (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric and Ω is ball.

Proof. Under assumption (4.23), we get

$$\widehat{\mathrm{Cap}}_p(\partial\Omega) = \left(\frac{p-1}{n-p}\right)^{p-1}\frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\Omega}|\mathrm{D}u|^{p-1}\,\mathrm{d}\sigma \leq \mathrm{AVR}(g)\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{-\frac{n-p}{n-1}},$$

that yields

$$\begin{aligned}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-p}{n-1}} &\leq \frac{\mathrm{AVR}(g)}{\widehat{\mathrm{Cap}}_p(\partial\Omega)} \leq (p-1)^{n-p}\sup_{\partial\Omega}\left|\frac{\mathrm{D}u}{n-p}\right|^{n-p} \\ &\leq \mathrm{AVR}(g)^{\frac{n-p}{p-1}}\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega|}\right)^{\frac{n-p}{n-1}},\end{aligned} \tag{4.24}$$

where we used (4.20) together with condition (4.23). Thus, we obtain that $\mathrm{AVR}(g) = 1$, and hence, by Bishop-Gromov's Theorem, that (M, g) is isometric to $\mathbb{R}^n$ with the Euclidean metric. Since all inequalities in (4.24) become equalities, by the second we have the equality in (4.20). Hence, one we can apply the rigidity statement in Theorem 3.2.3 which ensures that $\partial\Omega$ is a compact connected and totally umbilical hypersurface of $\mathbb{R}^n$, that is, Ω is a ball. □

A

THE p-BOCHNER FORMULA

One of the most powerful tools in Riemannian geometry is the classical Bochner formula, which reads as

$$\frac{1}{2}\Delta|\mathrm{D}f|^2 = \langle \mathrm{D}\Delta f \,|\, \mathrm{D}f\rangle + |\mathrm{DD}f|^2 + \mathrm{Ric}(\mathrm{D}f, \mathrm{D}f)$$

for every function $f \in \mathscr{C}^\infty(M)$. This formula embodies the property of a Riemannian manifold to have Ricci curvature bounded from below, thanks to the following lemma.

Lemma A.1. *Let (M, g) a complete Riemannian manifold.* $\mathrm{Ric} \geq \kappa$ *for some $\kappa \in \mathbb{R}$ if and only if for every $f \in \mathscr{C}^\infty(M)$ it holds that*

$$\frac{1}{2}\Delta|\mathrm{D}f|^2 \geq \langle \mathrm{D}\Delta f \,|\, \mathrm{D}f\rangle + \kappa|\mathrm{D}f|^2 \tag{A.1}$$

Proof. If $\mathrm{Ric} \geq \kappa$ then (A.1) follows directly from the Bochner formula. Conversely, suppose (A.1) holds for every $f \in \mathscr{C}^\infty(M)$ and there exists a point $p \in M$ and a vector $v \in T_pM$ such that $\mathrm{Ric}(v, v) < \kappa|v|^2$. We can build a function $f \in \mathscr{C}^\infty(M)$ such that $\mathrm{D}f = v$ and $\mathrm{DD}f = 0$ at p. Then, by the Bochner formula

$$\frac{1}{2}\Delta|\mathrm{D}f|^2 < \langle \mathrm{D}\Delta f \,|\, \mathrm{D}f\rangle + \kappa|\mathrm{D}f|^2$$

holds at p, contradicting (A.1). □

The Bochner formula can be generalised to agree with the p-Laplacian also for $p \neq 2$, as in [Val13, Proposition 3..1.2] and [KN09, Lemma 2.1]. We recall that the p-Laplacian is the operator defined in (1.2) as

$$\Delta_g^{(p)} f = \mathrm{div}\left(|\mathrm{D}f|^{p-2}\mathrm{D}f\right).$$

One can see that, even for a smooth function $f \in \mathscr{C}^\infty(M)$ the p-Laplacian of f is not defined at points where $|\mathrm{D}f| = 0$. To overcome it, one can proceed in two different ways. The first one is to state the identity outside the critical set of f. The second one is to consider the ε-regularised version of the p-Laplacian, that is

$$\Delta_g^{(p,\varepsilon)} f = \mathrm{div}\left((|\mathrm{D}f|^2 + \varepsilon)^{\frac{p-2}{2}}\mathrm{D}f\right)$$

for every $f \in \mathscr{C}^\infty(M)$. This operator is no more degenerate, hence it is well defined at every point. In this spirit, we propose a version of the p-Bochner formula (ε-regularised) where the four terms involved are the exact generalisation of the four terms in the Bochner formula.

Proposition A.2 (p-Bochner formula)**.** *Let (M,g) a complete Riemannian manifold and $f \in \mathscr{C}^\infty(M)$, $\varepsilon \geq 0$ and $p > 1$. Then,*

$$\begin{aligned} \frac{1}{p}\operatorname{div}\left(|\mathrm{D}f|_\varepsilon^{p-2}\mathcal{A}(\mathrm{D}|\mathrm{D}f|_\varepsilon^{p})\right) = \left\langle \mathrm{D}\Delta_g^{(p,\varepsilon)}f \,\middle|\, |\mathrm{D}f|_\varepsilon^{p-2}\mathrm{D}f\right\rangle \\ + |\mathrm{D}f|_\varepsilon^{2p-4}|\mathrm{DD}f|^2_{\mathcal{A}} + |\mathrm{D}f|_\varepsilon^{2p-4}\operatorname{Ric}(\mathrm{D}f,\mathrm{D}f) \end{aligned} \tag{A.2}$$

holds at any point such that $|\mathrm{D}f|_\varepsilon > 0$, where

$$\mathcal{A} = g + (p-2)\frac{\mathrm{d}f\otimes\mathrm{d}f}{|\mathrm{D}f|_\varepsilon^2}, \qquad |\cdot|_\varepsilon = (|\cdot|^2+\varepsilon)^{\frac{1}{2}}$$

and $\Delta_g^{(p,\varepsilon)}$ is the ε-regularised p-Laplacian.

Proof. We prove it for $\varepsilon = 0$ only, the general case being obtained similarly. For $\varepsilon = 0$, the condition $|\mathrm{D}f|_\varepsilon > 0$ reads as $|\mathrm{D}f| > 0$. Coupling

$$\left\langle \mathrm{D}\Delta_p f \,\middle|\, |\mathrm{D}f|^{p-2}\mathrm{D}f\right\rangle = \left\langle \mathrm{D}\left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle + \mathrm{D}\left(|\mathrm{D}f|^{p-2}\Delta f\right)\,\middle|\,|\mathrm{D}f|^{p-2}\mathrm{D}f\right\rangle$$

and

$$\begin{aligned} \operatorname{div}\left(|\mathrm{D}f|^{p-2}\left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle\mathrm{D}f\right) = \left\langle \mathrm{D}\left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle\,\middle|\,|\mathrm{D}f|^{p-2}\mathrm{D}f\right\rangle \\ + \left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\Delta f\,\middle|\,|\nabla f|^{p-2}\mathrm{D}f\right\rangle \\ + \left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle^2, \end{aligned}$$

one obtains

$$\begin{aligned} \left\langle \mathrm{D}\Delta_p f\,\middle|\,|\mathrm{D}f|^{p-2}\mathrm{D}f\right\rangle = \operatorname{div}\left(|\mathrm{D}f|^{p-2}\left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle\mathrm{D}f\right) - \left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle^2 \\ + |\mathrm{D}f|^{2p-4}\langle \mathrm{D}\Delta f \mid \mathrm{D}f\rangle. \end{aligned}$$

By applying the classical Bochner formula to the last term we get

$$\begin{aligned} \left\langle \mathrm{D}\Delta_p f\,\middle|\,|\mathrm{D}f|^{p-2}\mathrm{D}f\right\rangle = |\mathrm{D}f|^{2p-4}\left(\frac{1}{2}\Delta|\mathrm{D}f|^2 - |\mathrm{DD}f|^2 - \operatorname{Ric}(\mathrm{D}f,\mathrm{D}f)\right) \\ + \operatorname{div}\left(|\mathrm{D}f|^{p-2}\left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle\mathrm{D}f\right) \\ - \left\langle \mathrm{D}|\mathrm{D}f|^{p-2}\,\middle|\,\mathrm{D}f\right\rangle^2. \end{aligned} \tag{A.3}$$

Plugging

$$\frac{1}{2}|\mathrm{D}f|^{2p-4}\Delta|\mathrm{D}f|^2 = \frac{1}{p}\operatorname{div}\left(|\mathrm{D}f|^{p-2}\mathrm{D}|\mathrm{D}f|^p\right) - 2(p-2)|\mathrm{D}f|^{2p-4}|\mathrm{D}|\mathrm{D}f||^2$$

and

$$\operatorname{div}\left(|Df|^{p-2}\left\langle D|Df|^{p-2}\,\middle|\,Df\right\rangle Df\right) = \frac{1}{p}\operatorname{div}\left((p-2)|Df|^{p-2}\left\langle D|Df|^{p}\,\middle|\,\frac{Df}{|Df|}\right\rangle\frac{Df}{|Df|}\right)$$

into (A.3) we have

$$\begin{aligned}\left\langle D\Delta_p f\,\middle|\,|Df|^{p-2}Df\right\rangle = \frac{1}{p}\operatorname{div}\left(|Df|^{p-2}\mathcal{A}(D|Df|^{p})\right)&\\ -(p-2)^2|Df|^{2p-6}\langle D|Df|\,|\,Df\rangle^2 - |Df|^{2p-4}|DDf|^2&\\ -2(p-2)|Df|^{2p-4}|D|Df||^2 - |Df|^{2p-4}\operatorname{Ric}(Df,Df).&\end{aligned}$$

That is equivalent to (A.2), since

$$|DDf|^2_{\mathcal{A}} = (p-2)^2\left\langle D|Df|\,\middle|\,\frac{Df}{|Df|}\right\rangle^2 + 2(p-2)|D|Df||^2 + |DDf|^2. \qquad \square$$

Observe that the operator $\mathcal{A}$ is exactly the metric g for $p = 2$. One can see that the metric induced by $\mathcal{A}$ is compatible with the metric induced by g, since

$$\min(1, p-1)g \leq \mathcal{A} \leq \max(1, p-1)g. \tag{A.4}$$

B

COAREA FORMULA

We recall here some classical results from [Mag12; Eva18] (see also [Mir03]) about the coarea formula. In its standard version, the coarea formula says that for every Lipschitz function $v : M \to \mathbb{R}$ and $E \subset M$ measurable it holds

$$\int_E |\mathrm{D}v| \,\mathrm{d}\mu = \int_{-\infty}^{+\infty} |\{v = t\} \cap E| \,\mathrm{d}t.$$

Clearly the formula can be extended also for functions v that are only locally Lipschitz. A direct consequence of the formula is the following lemma that can be seen as a weak version of the Morse-Sard Lemma. We refer the reader to [Mag12, Lemma 18.5] for the proof.

Lemma B.1. *Let (M, g) be a complete Riemannian manifold. Let $v : U \to \mathbb{R}$ be a locally Lipschitz function defined on an open subset $U \subset M$. Then,*

$$|\{v = t\} \cap \mathrm{Crit}(v)| = 0$$

for almost every $t \in \mathbb{R}$.

By a classical approximation argument involving simple functions, one can prove the following generalisation of the coarea formula.

Proposition B.2 (Coarea formula - measurable functions)**.** *Let (M, g) be a complete Riemannian manifold. Let $v : U \to \mathbb{R}$ be a locally Lipschitz function defined on an open subset $U \subset M$. Let $f \in L^0(U)$ be a nonnegative function. Then,*

$$\int_E f|\mathrm{D}v| \,\mathrm{d}\mu = \int_{-\infty}^{+\infty} \int_{\{v=t\}\cap E} f \,\mathrm{d}\sigma \,\mathrm{d}t$$

holds for evey measurable subset $E \subset U$.

Splitting the positive and negative parts of an integrable function we can prove the following proposition.

Proposition B.3 (Coarea formula - integrable functions)**.** *Let (M, g) be a complete Riemannian manifold. Let $v : U \to \mathbb{R}$ be a locally Lipschitz function defined on an open subset $U \subset M$. For every measurable function f such that $f|\mathrm{D}v| \in L^1(U)$*

$$\int_E f|\mathrm{D}v| \,\mathrm{d}\mu = \int_{-\infty}^{+\infty} \int_{\{v=t\}\cap E} f \,\mathrm{d}\sigma \,\mathrm{d}t \tag{B.1}$$

holds for every measurable subset $E \subset U$. Moreover, the function

$$t \mapsto \int_{\{v=t\}} f \,\mathrm{d}\sigma$$

belongs to $L^1(\mathbb{R})$ and its equivalence class does not depends on the representative of f.

Some remarks are mandatory here. Consider a locally Lipschitz function $v : U \subset M \to \mathbb{R}$ and let $f \in L^1(U)$ be such that $f = 0$ almost everywhere on $\mathrm{Crit}\, v$. Then, the function $g : U \to \mathbb{R}$ defined as

$$g = \begin{cases} f|\mathrm{D}v|^{-1} & \text{on } U \setminus \mathrm{Crit}\, v, \\ 0 & \text{on } \mathrm{Crit}\, v, \end{cases}$$

satisfies the hypothesis of Proposition B.3. Hence, employing also Lemma B.1, we have

$$\int_E f \,\mathrm{d}\mu = \int_E g|\mathrm{D}v| \,\mathrm{d}\mu = \int_{-\infty}^{+\infty} \int_{E\cap\{v=t\}} g \,\mathrm{d}\sigma \,\mathrm{d}t = \int_{-\infty}^{+\infty} \int_{E\cap\{v=t\}} \frac{f}{|\mathrm{D}v|} \,\mathrm{d}\sigma \,\mathrm{d}t.$$

We also highlight that given a function $\psi \in L^\infty(\mathbb{R})$, $\psi(v)$ belongs to $L^\infty(U)$. If f is such that $f|\mathrm{D}v| \in L^1(U)$, then $f\psi(v)|\mathrm{D}v|$ belongs to $L^1(U)$ as well. It follows that

$$\int_U \psi(v) f|\mathrm{D}v| \,\mathrm{d}\mu = \int_{-\infty}^{+\infty} \psi(t) \int_{\{v=t\}} f \,\mathrm{d}\sigma \,\mathrm{d}t.$$

To conclude, it is clear that we can relax the hypotheses on f, taking it in $L^1_{\mathrm{loc}}(U)$. In this case, the coarea formula (B.1) holds for every E measurable an compactly contained in U. If we denote by I the image of U through v and we suppose that $\{a \leq v \leq b\}$ is compact in U for every $[a, b] \subset I$, then the function

$$t \mapsto \int_{\{v=t\}} f \,\mathrm{d}\sigma$$

belongs to $L^1_{\mathrm{loc}}(I)$ and its equivalence class does not depend on the representative of f. Moreover, if $\psi \in L^\infty(I)$ has compact support in I, we have that

$$\int_U \psi(v) f|\mathrm{D}v| \,\mathrm{d}\mu = \int_I \psi(t) \int_{\{v=t\}} f \,\mathrm{d}\sigma \,\mathrm{d}t$$

holds as well.

C

Sobolev regularity of the gradient of p-harmonic functions

We give here a proof of Theorem 1.2.2 that is based on the p-Bochner formula (A.2). The reason leading this choice is that recently Monotonicity Formulas for harmonic functions have been made available in the nonsmooth setting by Gigli and Violo in [GV21] and [Vio21]. What we actually need in the proof of Theorem 3.3.6 is that

$$|\mathrm{D}u|^{p-1} \in W^{1,2}_{\mathrm{loc}}(M). \tag{C.1}$$

and the Cheng-Yau-type estimate Theorem 1.2.11 for the p-capacitary potential u associated with $\Omega \subset M$ open, bounded with smooth boundary. Since the Cheng-Yau-type estimate has been proved by Wang and Zhang [WZ10] exploiting the Sobolev regularity of the gradient of p-harmoinc functions, proving (C.1) in the nonsmooth setting would be a crucial step to obtain the generalisation of results in [GV21; Vio21] for $p \neq 2$ as well as a regularity result for p-harmonic functions, that are far away to be comprehended.

We recall some notations here for the ease of the reader. We define the ε-regularised p-Laplace operator $\Delta_g^{(p,\varepsilon)}$ as

$$\Delta_g^{(p,\varepsilon)} f = \operatorname{div}\left(|\mathrm{D}f|_\varepsilon^{p-2}\mathrm{D}f\right)$$

for a given function f, where

$$|\cdot|_\varepsilon = (|\cdot|^2+\varepsilon)^{\frac{1}{2}}.$$

Moreover, for a given function $f \in \mathscr{C}^\infty(M)$ we define the tensor $\mathcal{A}$ given

$$\mathcal{A} = g + (p-2)\frac{\mathrm{d}f \otimes \mathrm{d}f}{|\mathrm{D}f|_\varepsilon^2}.$$

As said in Appendix A, for $\varepsilon = 0$, $\Delta_g^{(p,\varepsilon)} = \Delta_g^{(p)}$ is the p-Laplacian. For $p = 2$, we have that $\mathcal{A} = g$ while in general $\mathrm{C}^{-1} g \leq \mathcal{A} \leq \mathrm{C}\, g$ for some finite positive constant C.

Proof of Theorem 1.2.2. Let v be a bounded p-harmonic function for $p > 1$ on some open bounded subset U. We aim at proving that $|\mathrm{D}v|^{p-1} \in W^{1,2}_{\mathrm{loc}}(B)$, for every ball $B \subset U$. For every $\varepsilon > 0$ there exists a unique function $v_\varepsilon \in W^{1,p}_{\mathrm{loc}}(B)$ solving

$$\begin{cases} \Delta_g^{(p,\varepsilon)} v_\varepsilon = 0 & \text{on } B, \\ v_\varepsilon - v \in W^{1,p}_0(B). \end{cases}$$

It is well-known (see [CCW16, Appendix]) that $v_\varepsilon \to v$ strongly in $W^{1,p}(B)$ and

$$\int_B |\mathrm{D}v_\varepsilon|_\varepsilon^p \, \mathrm{d}\mu \le \mathrm{C}_1, \tag{C.2}$$

where the constant C_1 does not depend on $\varepsilon > 0$. By De Giorgi estimate the family $(v_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(B)$ (see [Ser64, Theorem 1]).

Fix $\varepsilon > 0$ and $\eta \in \mathscr{C}_c^\infty(B)$ with $0 \le \eta \le 1$. Integrating the p-Bochner inequality (A.2) for $f = v_\varepsilon$ against η^2 one obtains that

$$\int_B \eta^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{DD}v_\varepsilon|_{\mathcal{A}}^2 \, \mathrm{d}\mu = -\frac{1}{p}\int_B |\mathrm{D}v_\varepsilon|_\varepsilon^{p-2} \langle \mathcal{A}(\mathrm{D}|\mathrm{D}v_\varepsilon|_\varepsilon^p) \,|\, \mathrm{D}\eta^2 \rangle \, \mathrm{d}\mu \\ - \int_B \eta^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} \,\mathrm{Ric}(\mathrm{D}v_\varepsilon, \mathrm{D}v_\varepsilon) \, \mathrm{d}\mu.$$

Recalling that $\mathcal{A}$ is equivalent to g (see (A.4)), there exists a constant C_2 independent of ε such that

$$|\mathrm{DD}v_\varepsilon|_{\mathcal{A}}^2 \ge \mathrm{C}_2 \, |\mathrm{DD}v_\varepsilon|^2.$$

The second integral can be controlled using the same approach. Using the Young's Inequality, the Kato Inequality and $\mathrm{D}|\mathrm{D}v_\varepsilon|_\varepsilon^2 = \mathrm{D}|\mathrm{D}v_\varepsilon|^2$, we obtain

$$\left| \frac{1}{p}\int_B |\mathrm{D}v_\varepsilon|_\varepsilon^{p-2} \langle \mathcal{A}(\mathrm{D}|\mathrm{D}v_\varepsilon|_\varepsilon^p) \,|\, \mathrm{D}\eta^2 \rangle \, \mathrm{d}\mu \right|, \le \delta^2 \,\mathrm{C}_3 \int_B \eta^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{DD}v_\varepsilon|^2 \, \mathrm{d}\mu \\ + \frac{1}{\delta^2} \int_B |\mathrm{D}\eta|^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{D}v_\varepsilon|^2 \, \mathrm{d}\mu$$

for some C_3 independent of ε. For the last integral, observe that on the ball the Ricci tensor is bounded from below by $-\kappa^2$, where κ depends on the ball. We thus get

$$-\int_B |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} \,\mathrm{Ric}(\mathrm{D}v_\varepsilon, \mathrm{D}v_\varepsilon) \eta^2 \, \mathrm{d}\mu \le \kappa^2 \int_B \eta^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{D}v_\varepsilon|^2 \, \mathrm{d}\mu.$$

Choosing δ so that $\delta^2 \,\mathrm{C}_3 \le \mathrm{C}_2$, taking into account (C.2) and $|\mathrm{D}v_\varepsilon| \le |\mathrm{D}v_\varepsilon|_\varepsilon$, we obtain a further constants C_4 such that

$$\int_B \eta^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{DD}v_\varepsilon|^2 \, \mathrm{d}\mu \le \mathrm{C}_4 \int_B |\mathrm{D}\eta|^2 |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{D}v_\varepsilon|^2 \, \mathrm{d}\mu + \mathrm{C}_4 \tag{C.3}$$

(C.3) is the same Caccioppoli-type estimate used by Lou in [Lou08, Lemma 2.1]. The proof follows now the same lines.

We claim that for any $K \subset B$ compactly contained

$$\int_K |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-4} |\mathrm{DD}v_\varepsilon|^2 \, \mathrm{d}\mu \le \mathrm{C}(K), \tag{C.4}$$

$$\int_K |\mathrm{D}v_\varepsilon|_\varepsilon^{2p-2} \, \mathrm{d}\mu \le \mathrm{C}(K) \tag{C.5}$$

hold for some constant $C(K) > 0$. Suppose that $\eta \geq 1$ on K.

If $p \leq 2$, since $p-1 \leq p/2$, Jensen's Inequality yields

$$\int_B |Dv_\varepsilon|_\varepsilon^{2p-2} \,d\mu \leq |B|^{\frac{2-p}{p}} \left(\int_B |Dv_\varepsilon|_\varepsilon^p \,d\mu \right)^{\frac{2p-2}{p}},$$

thus (C.5) follows from (C.2). (C.3), $|Dv_\varepsilon| \leq |Dv_\varepsilon|_\varepsilon$ and (C.5) then imply (C.4).

If $2 \leq p \leq 4$, replacing η by η^2 in (C.3) we have

$$\int_B \eta^4 |Dv_\varepsilon|_\varepsilon^{2p-4} |DDv_\varepsilon|^2 \,d\mu \leq C_5 \int_B \eta^2 |Dv_\varepsilon|_\varepsilon^{2p-4} |Dv_\varepsilon|^2 \,d\mu + C_5, \tag{C.6}$$

where we used the upper bound on $|D\eta|$ to remove the gradient term in the right hand side. Integrating by part, since v_ε solves the equation $\Delta^{(p,\varepsilon)} v_\varepsilon = 0$, we obtain that

$$\begin{aligned}\int_B \eta^2 |Dv_\varepsilon|_\varepsilon^{2p-4} |Dv_\varepsilon|^2 \,d\mu \leq & -\int_B (p-2)\eta^2 v_\varepsilon |Dv_\varepsilon|^{2p-5} \langle D|Dv_\varepsilon|_\varepsilon \,|\, Dv_\varepsilon \rangle \,d\mu \\ & - \int_B 2\eta v_\varepsilon \langle D\eta \,|\, Dv_\varepsilon \rangle |Dv_\varepsilon|_\varepsilon^{2p-4} \,d\mu.\end{aligned}$$

Using the Kato Inequality, the uniform upper bound on $\|v_\varepsilon\|_{L^\infty(B)}$ and Young's Inequality on both integrals we get

$$\begin{aligned}\int_B \eta^2 |Dv_\varepsilon|_\varepsilon^{2p-4} |Dv_\varepsilon|^2 \,d\mu \leq \delta^2 \int_B \eta^4 |Dv_\varepsilon|_\varepsilon^{2p-4} |DDv_\varepsilon|^2 \,d\mu + \frac{C_6}{\delta^2} \int_B |Dv_\varepsilon|_\varepsilon^{2p-4} \,d\mu \\ + \delta^2 \int_B \eta^2 |Dv_\varepsilon|_\varepsilon^{2p-4} |Dv_\varepsilon|^2 \,d\mu.\end{aligned}$$

Since $0 < p-2 \leq p/2$ using (C.2) and choosing $\delta^2 < 1/2$ we are left with

$$\int_B \eta^2 |Dv_\varepsilon|_\varepsilon^{2p-4} |Dv_\varepsilon|^2 \,d\mu \leq 2\delta^2 \int_B \eta^4 |Dv_\varepsilon|_\varepsilon^{2p-4} |DDv_\varepsilon|^2 \,d\mu + \frac{C_7}{\delta^2}. \tag{C.7}$$

Coupling it with (C.6) for sufficiently small δ we obtain

$$\int_B \eta^4 |Dv_\varepsilon|_\varepsilon^{2p-4} |DDv_\varepsilon|^2 \,d\mu \leq C_8$$

proving (C.4). Plugging this inequality into (C.7) and using

$$\int_B |Dv_\varepsilon|_\varepsilon^{2p-4} \,d\mu \leq |B|^{\frac{4-p}{p}} \left(\int_B |Dv_\varepsilon|_\varepsilon^p \,d\mu \right)^{\frac{2p-4}{p}},$$

we get (C.5) also in this case. A similar discussion triggers the proof for all $p \in (1, +\infty)$. By Kato's Inequality, (C.5) and (C.4) we have

$$\left\| |\mathrm{D} v_\varepsilon|_\varepsilon^{p-1} \right\|_{W^{1,2}(K)} \leq \mathrm{C}(K)$$

uniformly in ε. Since v_ε converges strongly in $W^{1,p}(B)$ to v we obtain the desired regularity of (C.1). □

We mention that in [Lou08, Lemma 2.1] the result is proved for the equation

$$\Delta_g^{(p)} v = f \in L^q(U)$$

with $q > n/p$, $q \geq 2$. A stronger result has been made available recently by [ACF21]. In this paper the authors obtain a better regularity, relaxing assumptions on the source term and using a technique that treats all p's in the same way.

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