Thermal state with quadratic interaction

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Abstract

We consider the perturbative construction, proposed in [37], for a thermal state $\Omega_{\beta,\lambda V\{f\}}$ for the theory of a real scalar Klein-Gordon field ϕ with interacting potential $V\{f\}$. Here f is a spacetime cut-off of the interaction V and λ is a perturbative parameter. We assume that V is quadratic in the field ϕ and we compute the adiabatic limit $f \to 1$ of the state $\Omega_{\beta,\lambda V\{f\}}$. The limit is shown to exist, moreover, the perturbative series in λ sums up to the thermal state for the corresponding (free) theory with potential V. In addition, we exploit the same methods to address a similar computation for the non-equilibrium steady state (NESS) [59] recently constructed in [25].

1 Introduction

Algebraic quantum field theory (AQFT) is a mathematically rigorous approach to quantum field theory (QFT). Nowadays, AQFT is a well-established set-up to describe the propagation of quantum fields on curved spacetimes [15, 64].

The approach can be summarized as follows. To any physical system one associates a *-algebra \mathcal{A} , whose elements are interpreted as the observables of the system. Algebraic relations reproduce natural assumptions on the structural properties of the observables, while the whole construction is subjected to the requirement of covariance [17, 51], which ensures that the *-algebra \mathcal{A} is coherently constructed on any globally hyperbolic spacetime [8, 9]. The dynamics can be implemented algebraically through the time-slice axiom [19]. Once the algebra \mathcal{A} has been identified, the notion of state can be introduced [15, Chap.5]. The latter is, per definition, a linear, positive and normalized functional $\Omega: \mathcal{A} \to \mathbb{C}$. Yet, not all states are found to be physically relevant: a necessary constraint is the so-called Hadamard condition [36, 41, 40, 64] which has been recast in the framework of Microlocal Analysis [50] in the seminal works [56, 57].

The algebraic approach has been successfully applied and it is well-understood for free theories [15]. Interacting theories can be addressed with the same techniques, however, the underlying non-linearity of the equations of motion creates additional difficulties. To handle this problem, one usually switches to the perturbative approach, which can be described as follows. Since the non-linear dynamics can be read as a correction V of a linear dynamics, one can try to expand interacting observables as formal power series in a formal parameter λ for the interacting potential λV . From a technical point of view, the above mentioned expansion of interacting observables is realized through the so-called quantum Møller operator $\mathsf{R}_{\lambda V}$ [6, 39, 55]. This operator is defined through the famous Bogoliubov' formula, which requires the introduction of Wick polynomials and of the time-ordered product [16, 33, 47, 48, 49, 52, 53] see also [15, Chap.2]. Once an extension of the time-ordered product has been fixed, the algebra \mathcal{A}_V of interacting observables can be defined with the quantum Møller operator $\mathsf{R}_{\lambda V}$ as a *-subalgebra of the algebra $\mathcal{A}[[\lambda]]$ of formal power series in λ with values in \mathcal{A} . The resulting algebra \mathcal{A}_V satisfies the condition of covariance [17, 51] as well as the time-slice axiom [19]. The whole construction applies assuming that the perturbation V is itself an element of the algebra \mathcal{A} . The procedure of removing the compactness in the support of V is known under the name of adiabatic limit. The latter has been implemented algebraically [15, Chap.1-2] and it has recently been improved thanks to the results of [4, 46]. In particular in [4] the authors have shown convergence of the expectation value of the quantum Møller operator in the case of the Sine-Gordon Model.

The problem of identifying physically interesting states on the interacting algebra \mathcal{A}_V has been addressed recently in [37]. Therein, the authors successfully applied a construction proposed in [2, 12, 13] in the framework of C^* -algebras. The latter allows to construct a thermal equilibrium state $\Omega_{\beta,\lambda V}$ [11, 42, 44, 58, 61] for the interacting theory once a corresponding thermal state Ω_{β} for the free theory has been given. In [37] the construction of $\Omega_{\beta,\lambda V}$ has been achieved in terms of a formal power series in λ exploiting the time-slice axiom [19] of the algebra \mathcal{A}_V . Further results on this state can be found in [25, 26, 28, 45].

In this paper we analyse the state $\Omega_{\beta,\lambda V}$ for the case of a quadratic potential V. In this particular case the perturbed free theory leads to another free theory. This perturbation may model a variation in the mass term of the Klein-Gordon operator $\Box + m^2 \rightarrow \Box + m^2 + \lambda m_0^2 - \text{with } m^2 > 0, m^2 + \lambda m_0^2 > 0$ – though more general situations are allowed. The assumption on V allows to investigate the adiabatic limit of the resulting state $\Omega_{\beta,\lambda V}$, which is computed order-by-order. The series for the resulting state can be evaluated directly and it is shown to lead to the corresponding state associated with the perturbed free theory, see Theorem 3.

The convergence of the state $\Omega_{\beta,\lambda V}$ in the case of a quadratic potential V is expected – see for example [23, 24, 60] where non-bounded perturbations of KMS were considered – however, the results and the tools exploited in proving the main result are noteworthy for several reasons. First of all, this computation shows that perturbation theory is reliable: The adiabatic limit can be taken order-by-order, leading to a series which sums up to the correct result. This behaviour is expected but a priori not guaranteed and this result increases the chances of perturbation theory of being the correct approach to interacting theories.

The second remarkable point of this analysis is that the tools used in the proof of the main result can potentially be generalized to a generic non-linear potential V. In particular, the first bit of information exploited in the computation of the adiabatic limit is the possibility to interchange the quantum Møller operator $R_{\lambda V}$ with its classical counterpart $R_{\lambda V}^{cl}$ [55] – see equation (19). From a computational point of view, this leads to a great simplification, due to the results of [20, 27, 28]. From an abstract point of view, equation (19) can be understood as an effective resummation of the perturbative series and it should be compared to other approaches [1, 3, 54, 62, 63]. It would be extremely interesting to understand to which extent equation (19) can be generalized to non-linear potential V. Most likely, this would allow to interchange the quantum Møller operator $R_{\lambda V}$ with a classical one $R_{\lambda V_{eff}}^{cl}$, with V_{eff} being an effective potential built out of V [14, 65]. From this point of view, the results of this paper can be understood as a promising starting point for an "effective analysis" of perturbative AQFT (pAQFT).

Finally, this result points towards a non-perturbative version of pAQFT, whose first steps will be necessarily based on a systematic check of the convergence of the perturbative approach, in the spirit of [4].

The paper is organized as follows: In section 2 we briefly summarize the functional approach to perturbative algebraic quantum field theory (pAQFT) for the Klein-Gordon field on Minkowski spacetime as well as the construction proposed in [37]. Section 3 contains the main result of the paper, see Theorem

3, which is proved in section 3.2. Finally, in section 4, the techniques developed in the previous sections are applied to the non-equilibrium steady state (NESS) [59] constructed in [25].

2 Brief resumé of pAQFT

In this section we give a brief introduction to the quantization of the real scalar Klein-Gordon field in the framework of algebraic quantum field theory [6, 17, 31, 38, 39, 51], see also [15, Chap.2]. This approach applies on any globally hyperbolic spacetime [7, 8, 9], and it is covariant in the sense of a generally covariant local theory introduced in [17, 51] see also [47]. For practical purposes we focus our attention to Minkowski spacetime M, because the results of [37] were developed on this particular background. The main reference for this section is [15, Chap.2].

2.1 Free theory

In this section we outline the quantization of a *free* real scalar Klein-Gordon field ϕ , whose dynamics is ruled by the massive Klein-Gordon equation $\Box \phi + m^2 \phi = 0$, m > 0, $\Box = -\eta^{ab} \partial_a \partial_b$ where $\eta =$ diag(-1, 1, 1, 1) – we exploit natural units $\hbar = c = 1$. We will consider the functional approach [31], where the (off-shell) *-algebra of observables is identified as that of functionals over kinematic configurations $\phi \in C^{\infty}(M)$, namely $F: C^{\infty}(M) \to \mathbb{C}$. For the sake of simplicity, we focus on *polynomial* functionals $F \in \mathcal{P}$, which lead to some simplification without spoiling the full generality of this approach. Notice that a polynomial functional F is automatically smooth, that is, for all $\phi, \psi \in C^{\infty}(M)$ the function $\mathbb{R} \ni x \mapsto F(\phi + x\psi)$ is differentiable at x = 0 and, for all natural numbers $n \ge 1$, its *n*-th derivative at x = 0 defines a symmetric distribution, denoted $F^{(n)}[\phi]$ and called the *n*-th functional derivative of Fat ϕ . Explicitly $F(\phi + x\psi)^{(n)}|_{x=0} = F^{(n)}[\phi](\psi^{\otimes n})$. Unless stated otherwise, from now on all functionals will be implicitly considered to be polynomial.

Among all, local functionals will play an important rôle in the construction of the algebra of free observables. A functional $F: C^{\infty}(M) \to \mathbb{C}$ is said to be local if it satisfies the two following conditions: (i) F is compactly supported, that is $\operatorname{spt}(F) := \bigcup_{\phi \in C^{\infty}(M)} \operatorname{spt}(F^{(1)}[\phi])$ is compact; (ii) for all $n \ge 1$ and $\phi \in C^{\infty}(M)$, the *n*-th functional derivative of F at ϕ is supported on the full diagonal of M^n , that is $\operatorname{spt}(F^{(n)}[\phi]) \subseteq \{(x_1, \ldots, x_n) \in M^n | x_1 = \ldots = x_n\}$. The set of local functionals will be denoted by $\mathcal{P}_{\operatorname{loc}}$.

Once equipped with the pointwise product, the set \mathcal{P}_{loc} generates the algebra $\mathcal{P}_{\text{mloc}}$ of *multilocal* functionals. Together with the *-involution defined by the complex conjugation $F^*(\phi) := \overline{F(\phi)}$, one obtains a commutative *-algebra, identified with that of classical observables for the Klein-Gordon field. In order to introduce its quantum counterpart, one needs to deform the pointwise product of $\mathcal{P}_{\text{mloc}}$. This is realized by choosing a so-called Hadamard distribution ω [56, 57], which is defined as a positive distribution $\omega \in C_c^{\infty}(M^2)'$ which satisfies the canonical commutation relations (CCR), that is $\omega(f,g) - \omega(g,f) = i\mathbf{G}(f,g)$. Here G denotes the causal propagator [7] associated to $\Box + m^2$. Moreover, the Wave Front Set [50] of the distribution ω is required to satisfy the microlocal spectrum condition [56, 57, 64] – see also equation (2). The latter requirement ensures that the singular behaviour of ω is the same as that of the Minkowski vacuum.

Once an Hadamard distribution has been chosen one may define an associative, non-commutative, \star -product on $\mathcal{P}_{\text{mloc}}$ as follows [6, 21, 22, 31]: for all $F, G \in \mathcal{P}_{\text{mloc}}$ one sets

$$\left(F\star_{\omega} G\right)(\phi) := F(\phi)G(\phi) + \sum_{n\geq 1} \frac{1}{n!} \omega^{\otimes n} \left(F^{(n)}[\phi], G^{(n)}[\phi]\right).$$

$$\tag{1}$$

Notice that the series is convergent because F, G are assumed to be polynomial functionals. The \ast -algebra \mathcal{A}_{ω} obtained by equipping $\mathcal{P}_{\text{mloc}}$ with the \star -product (1) and the \ast -involution given by complex

conjugation is called the algebra of ω -renormalized quantum observables. Different choices of ω lead to *-isomorphic algebras: This is a consequence of the fact that, if ω, ω' are Hadamard distributions, then $\omega - \omega' \in C^{\infty}(M^2)$ [56, 57].

States on \mathcal{A}_{ω} are defined as linear, positive and normalized functionals $\Omega: \mathcal{A}_{\omega} \to \mathbb{C}$. Among all the possible choices, we will mainly consider the one obtained considering the evaluation functional $\Omega(F) := F(0)$. This defines a so-called quasi-free state [15, Chap.5], namely a state entirely determined by the distribution $C_c^{\infty}(M)^2 \ni (f_1, f_2) \mapsto \Omega(F_{f_1} \star_{\omega} F_{f_2})$, where $F_{f_k}(\phi) = \int_M f_k \phi$. The latter distribution is called the two-point function associated to Ω and coincides with ω . In general, the two-point function ω of a Poincaré invariant Hadamard state Ω can be Fourier expanded as follows: for all $f, g \in C_c^{\infty}(M)$

$$\omega(f,g) = \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon} \sum_{\pm} c_{\pm}(k) \widehat{f}(\pm\epsilon,k) \widehat{g}(\mp\epsilon,-k), \qquad \epsilon = \epsilon(k) := \sqrt{|k|^2 + m^2}. \tag{2}$$

The functions c_{\pm} identify completely the state Ω . Actually ω is an Hadamard distribution if and only if $c_{+} + c_{-} \geq 0$, $c_{+} - c_{-} = 1$ and c_{-} is smooth and rapidly decreasing.

The algebra \mathcal{A}_{ω} is an algebra of *off-shell* functionals, namely functionals which are not constrained by any dynamical requirement. The *on-shell* algebra of quantum observables $\mathcal{A}_{\omega,\text{on}}$ is identified with the quotient of $\mathcal{A}_{\omega}/\mathfrak{I}_{\omega}$ with respect to the *-ideal \mathfrak{I}_{ω} which contains "dynamically trivial" functionals. For the case of an Hadamard distribution ω which is a weak bisolution of $\Box + m^2$ the ideal \mathfrak{I}_{ω} consists of functionals vanishing on solutions of the Klein-Gordon equation $\Box \phi + m^2 \phi = 0$.

The on-shell algebra enjoys the remarkable property of the time-slice axiom [19], which is described as follows. Let $\mathcal{O} \subseteq M$ be a region of M such that $J(\mathcal{O}) := J^{\uparrow}(\mathcal{O}) \cup J^{\downarrow}(\mathcal{O}) = M$, where $J^{\uparrow}(\mathcal{O})$ (resp. $J^{\downarrow}(\mathcal{O})$) denotes the causal future (resp. past) of \mathcal{O} [5, 7]. Let $\mathcal{A}_{\omega,\mathrm{on}}(\mathcal{O})$ be the on-shell algebra generated by $F \in \mathcal{P}_{\mathrm{loc}}$ with $\mathrm{spt}(F) \subseteq \mathcal{O}$. This algebra is clearly embedded in the whole algebra $\mathcal{A}_{\omega,\mathrm{on}}$: the time-slice axiom ensures that this embedding is in fact a *-isomorphism. In the following, we will mostly deal with the off-shell algebra.

2.2 Interacting theory

Interactions for a real scalar Klein-Gordon field are non-linear corrections to the linear operator $\Box + m^2$ which are described by a self-adjoint element of the algebra $V \in \mathcal{A}_{\omega}$ [15, Chap.2]. This amounts to assume that the dynamics of the interacting field is ruled by the operator $\Box + m^2 + \lambda V^{(1)}[\cdot]$, where λ is the coupling of the interaction. Notice that V has compact support so that a perturbative approach is justified: The interacting observables are then expanded in formal power series of λ – which is regarded as a formal parameter – leading to elements in $\mathcal{P}_{mloc}[[\lambda]]$.

Once this step has been accomplished, it remains to discuss the so-called adiabatic limit, where a suitable limit $\operatorname{spt}(V) \to M$ is considered. In the algebraic setting, this is a two-steps procedure. On the one hand, the adiabatic limit can be performed at the level of algebras, the so-called algebraic adiabatic limit, leading a *-algebra $\mathcal{A}_{V,\operatorname{ad}}$. On the other hand, the adiabatic limit $\operatorname{spt}(V) \to 1$ can also be considered on family of functionals $\{\Omega_f\}_f$ such that, for each test function f, Ω_f defines a state for the interacting algebra \mathcal{A}_V with $\operatorname{spt}(V) = \operatorname{spt}(f)$ – see for example the family of states identified by (12). The distributional limit $f \to 1$ is defined in an appropriately sense – cf. Section 3 – and its analysis is ultimately a case-by-case study. The purpose of this paper is to show the convergence of a particular sequence of states for the algebra obtained with a quadratic interaction V.

In the following we briefly sketch the construction of the algebras of interacting observables associated with a perturbation $V\{f\}$, where the notation stresses the dependence of V on the cut-off $f \in C_c^{\infty}(M)$, that is $\operatorname{spt}(V\{f\}) = \operatorname{spt}(f)$. We will not discuss the construction in full details, referring instead to the vast literature on the topic [6, 15, 16, 19, 32, 33, 39, 47, 48, 49]. In this section we assume that a choice for an Hadamard distribution ω has been made and denote with $\mathcal{A} := \mathcal{A}_{\omega}$ the corresponding algebra.

2.2.1 Quantum Møller operator

Following [15, Chap.2], the *-algebra of interacting observables of $\mathcal{A}_{V\{f\}}$ is introduced as a *-subalgebra of $\mathcal{A}[[\lambda]]$. This *-subalgebra is defined through the so-called quantum Møller operator $\mathsf{R}_{\lambda V\{f\}}$ [6], which can be defined as a map $\mathsf{R}_{\lambda V\{f\}}$: $\mathcal{P}_{\mathrm{loc}} \to \mathcal{A}[[\lambda]]$ by the well-known Bogoliubov formula – see equation (3). The definition of this latter maps requires the introduction of the time-ordered product \cdot_T [15, Chap.2] [48]. This is an associative and commutative product on the *-subalgebra $\mathcal{P}_{\mathrm{mreg}} \subset \mathcal{P}_{\mathrm{mloc}}$ made of polynomial functionals with smooth functional derivatives of all orders. The time-ordered product can be extended to the whole $\mathcal{P}_{\mathrm{mloc}}$ with a non-unique extension procedure [33], where the ambiguities in the extension are controlled by the so-called renormalization freedoms [47, 48, 49]. Once an extension of the time-ordered product has been identified, the quantum Møller operator is defined through the Bogoliubov formula

$$\mathsf{R}_{\lambda V\{f\}}(F) := \exp_T \left[i\lambda V \right]^{-1} \star_\omega \left(\exp_T \left[i\lambda V \right] \cdot_T F \right) \in \mathcal{A}[[\lambda]], \tag{3}$$

where $F \in \mathcal{P}_{\text{loc}}$ and \exp_T denotes the exponential computed with the time-ordered product while $\exp_T [iV]^{-1}$ is the inverse of $\exp_T [iV]$ with respect to \star_{ω} . For the sake of simplicity we just summarize the construction as a definition:

Definition 1: Let $V\{f\} \in \mathcal{P}_{\text{loc}}$. The *-algebra of interacting observables for the real scalar Klein-Gordon theory associated with the perturbation $V\{f\}$ is the *-subalgebra $\mathcal{A}_{V\{f\}} \subset \mathcal{A}[[\lambda]]$ generated by $\mathsf{R}_{\lambda V\{f\}}(\mathcal{P}_{\text{loc}})$.

The algebraic adiabatic limit is related to the following properties of the quantum Møller operator [15, Chap.1-2]. Let $f_1, f_2 \in C_c^{\infty}(M)$ and $F \in \mathcal{P}_{\text{loc}}$, then

$$\mathsf{R}_{\lambda V\{f_1\}}(F) = F, \qquad \text{if } J^{\downarrow}(\operatorname{spt}(F)) \cap J^{\uparrow}(\operatorname{spt}(V\{f_1\})) = \emptyset.$$
(4)

Similarly, if $J^{\downarrow}(\operatorname{spt}(V\{f_1 - f_2\})) \cap J^{\uparrow}(\operatorname{spt}(F)) = \emptyset$ then there exists a formal unitary $U_{f_1, f_2} \in \mathcal{A}[[\lambda]]$ such that

$$\mathsf{R}_{\lambda V\{f_1\}} = U_{f_1, f_2}^{-1} \star \mathsf{R}_{\lambda V\{f_2\}}(F) \star U_{f_1, f_2} \,. \tag{5}$$

Out of properties (4-5) the algebraic adiabatic limit can be performed, leading to a *-algebra $\mathcal{A}_{V,\mathrm{ad}}$, independent from the cut-off of V [15, Chap.2]. Actually one considers a net of algebras $\mathcal{O} \mapsto \mathcal{A}_{V\{f\}}(\mathcal{O})$ where \mathcal{O} is any double cone of M, that is, there exists $x, y \in M$ such that $\mathcal{O} = J^{\uparrow}\{x\} \cap J^{\downarrow}\{y\}$. For each of these algebra one considers the cut-off f to be in the class $1_{\mathcal{O}}$ of functions $g \in C_c^{\infty}(M)$ such that $g|_{\mathcal{O}} = 1$. Thanks to property (5), for all $f, g \in 1_{\mathcal{O}}$ the algebras $\mathcal{A}_{V\{f\}}(\mathcal{O})$ and $\mathcal{A}_{V\{g\}}(\mathcal{O})$ are unitary equivalent. This allows to identify, for each double cone \mathcal{O} , the algebra $\mathcal{A}_{V,\mathrm{ad}}(\mathcal{O})$ as a direct limit, leading to a net of *-algebras in the sense of Haag and Kastler [43]. The global algebra $\mathcal{A}_{V,\mathrm{ad}}$ can then be identified with the direct limit of this net.

Finally, the interacting *-algebra $\mathcal{A}_{V\{f\}} \subseteq \mathcal{A}[[\lambda]]$ can be projected on its on-shell version $\mathcal{A}_{V\{f\},on} := \mathcal{A}_{V\{f\}}/\mathcal{I}_{V\{f\}}$ where $\mathcal{I}_{V\{f\}} := \mathcal{A}_{V\{f\}} \cap \mathcal{I}_{\omega}$. As for \mathcal{A} , the time-slice axiom holds true for $\mathcal{A}_{V\{f\},on}$ as well as for $\mathcal{A}_{V,ad,on}$ [19]. Once again, the whole construction can be shown to be covariant in the sense of a generally covariant local theory introduced in [17, 51]

In what follows, we will exploit the time-slice axiom and the covariance of the construction. Indeed, we will focus on the off-shell algebra $\mathcal{A}_{V\{f\}}(J^{\uparrow}(\Sigma))$, with Σ being a Cauchy surface for M [7, 8, 9]. If not

stated otherwise, in the following we will leave the Σ -dependence of $\mathcal{A}_{V\{f\}}(J^{\uparrow}(\Sigma))$ implicit. In particular, exploiting an arbitrary but fixed inertial frame for which $\Sigma = t^{-1}\{0\}$, we will choose the cut-off f as a product $h\chi$, where $h \in C_c^{\infty}(\mathbb{R}^3)$ and $\chi \in C_c^{\infty}(\mathbb{R})$ with $\operatorname{spt}(\chi) \subseteq (-1, +\infty)$. Moreover, due to property (4) it is not restrictive to assume $\chi \in C^{\infty}(\mathbb{R})$ be such that $\chi(t) = 1$ for $t \ge 0$.

2.2.2 Interacting thermal states

In this section we summarize the construction proposed in [37]. The latter aims to define an interacting thermal state out of an arbitrary chosen thermal state for the free theory. This construction is inspired by analogy to the one proposed in [2] in the framework of C^* -algebras.

Thermal equilibrium states are identified by the so-called Kubo-Martin-Schwinger (KMS) condition [11, 12, 13, 42, 44, 58, 61]. In the algebraic approach the latter requires the identification of a one-parameter group of automorphism on the algebra of interest, which is interpreted as the group of time translations. In the case of the free algebra \mathcal{A} this can be defined as follows. For all $t \in \mathbb{R}$ and $\phi \in C^{\infty}(M)$ let ϕ_t be the time translation of ϕ by t (we implicitly fixed an inertial frame). Then, for all $F \in \mathcal{A}$ the time translation is defined as

$$F \mapsto \tau_t(F) := F_t \,, \quad F_t(\phi) := F(\phi_t) \,. \tag{6}$$

As for the interacting algebra the one-parameter group is defined on each generator of $\mathcal{A}_{V\{f\}}$ as [37]

$$\mathsf{R}_{\lambda V\{f\}}(F) \mapsto \tau_{V\{f\},t} \big(\mathsf{R}_{\lambda V\{f\}}(F) \big) := \mathsf{R}_{\lambda V\{f\}}(F_t) \,. \tag{7}$$

Once a one-parameter group of *-automorphism has been fixed one may introduce KMS states as follows [12, 13, 61].

Definition 2: Let A be a topological *-algebra and let $\alpha \in \hom(\mathbb{R}, \operatorname{Aut}(A))$ be a one-parameter group of *-automorphism of A. A state Ω over A is called a (β, α) -KMS state at inverse temperature $\beta > 0$ if, for all $a, b \in A$, the function $t \mapsto \Omega(a\alpha_t(b))$ admits an analytic continuation – denoted with $\Omega(a\alpha_z(b))$ – in the complex strip $\mathcal{S}_{\beta} := \{z \in \mathbb{C} \mid 0 < \Im z < \beta\}$ which is continuous on the closure $\overline{\mathcal{S}_{\beta}}$ and such that

$$\Omega(a\alpha_z(b))|_{z=i\beta} = \Omega(ba).$$
(8)

In the case of the free algebra \mathcal{A} , for all $\beta > 0$ there is a unique KMS state Ω_{β} which is a quasi-free state whose two-point function is given by, *cf.* expression (2),

$$\omega_{\beta}(f,g) := \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon} \sum_{\pm} b_{\pm}(\beta,\epsilon) \widehat{f}(\pm\epsilon,k) \widehat{g}(\mp\epsilon,-k) \,, \qquad b_{\pm}(\beta,\epsilon) := \frac{\mp 1}{e^{\mp\beta\epsilon} - 1} \,. \tag{9}$$

The identity $b_{-}(\beta,\epsilon) = e^{-\beta\epsilon}b_{+}(\beta,\epsilon)$ ensures the KMS condition (8) as well as the Hadamard property.

Let now $V\{h\chi\} \in \mathcal{P}_{\text{loc}}$. Following a previous construction in the context of C^* -algebras [2], in [37] the causality properties (4-5) were exploited to built an intertwiner between the free time evolution τ and the interacting time evolution $\tau_{V\{h\chi\}}$. Actually, for all $F \in \mathcal{A}_{V\{h\chi\}}$ there exists a unitary cocycle $U_{V\{h\chi\}}(t) \in \mathcal{A}[[\lambda]]$ such that

$$\tau_{V\{h\chi\},t} \big[\mathsf{R}_{\lambda V\{h\chi\}}(F) \big] = U_{V\{h\chi\}}(t)^{-1} \star_{\beta} \tau_t \big[\mathsf{R}_{\lambda V\{h\chi\}}(F) \big] \star_{\beta} U_{V\{h\chi\}}(t) \,. \tag{10}$$

Here we have implicitly identified the free algebra \mathcal{A} with the ω_{β} -renormalized algebra $\mathcal{A}_{\omega_{\beta}}$ and $\star_{\beta} := \star_{\omega_{\beta}}$ is a short notation. The cocycle $U_{V\{h_{\chi}\}}(t)$ satisfies the cocycle condition

$$U_{V\{h\chi\}}(t+s) = U_{V\{h\chi\}}(t) \star_{\beta} \tau_t \left[U_{V\{h\chi\}}(s) \right],$$
(11)

which has a cohomological interpretation [18]. Property (11) implies that the state

$$\Omega_{\beta,\lambda V\{h\chi\}}(A) := \frac{\Omega_{\beta}(A \star_{\beta} U_{V\{h\chi\}}(t))}{\Omega_{\beta}(U_{V\{h\chi\}}(t))} \bigg|_{t=i\beta} \quad \forall A \in \mathcal{A}_{V\{h\chi\}},$$
(12)

is a well-defined $(\beta, \tau_{V\{h\chi\}})$ -KMS state for the interacting algebra $\mathcal{A}_{V\{h\chi\}}$ [37]. Notice that the evaluation at $t = i\beta$ is justified at each order in λ by the analytic properties of Ω_{β} .

The state $\Omega_{\beta,\lambda V\{h_{\chi}\}}$ enjoys the following expansion [37, Prop. 3]

$$\Omega_{\beta,\lambda V\{h\chi\}}(A) = \Omega_{\beta}(A) + \sum_{n\geq 1} (-1)^n \int_{\beta S_n} \mathrm{d}U \,\Omega_{\beta}^c \left[A \otimes \bigotimes_{\ell=1}^n K_{iu_{\ell}} \right], \qquad \forall A \in \mathcal{A}_{V\{h\chi\}}.$$
(13)

Here, $S_n := \{U := (u_1, \ldots, u_n) \in \mathbb{R}^n | 0 \le u_1 \le \ldots \le u_n \le 1\}$ is the canonical *n*-dimensional simplex while $K := \frac{\mathrm{d}}{i\mathrm{d}t} U_{V\{h\chi\}}(t)|_{t=0} = \mathsf{R}_{\lambda V\{h\chi\}}(\lambda V\{h\chi\})$, the dot being time derivative. Moreover, Ω_{β}^c denotes the connected part of Ω_{β} which is defined by

$$\Omega^{\beta}(A_1 \star \ldots \star A_n) = \sum_{P \in \mathsf{P}\{1,\ldots,n\}} \prod_{I \in P} \Omega^{c}_{\beta} \left(\bigotimes_{\ell \in I} A_{\ell}\right), \qquad \forall A_1,\ldots,A_n \in \mathcal{A}, \forall n \in \mathbb{Z}_+,$$
(14)

together with the condition $\Omega_{\beta}^{c}(1_{\mathcal{A}}) = 0$ – here $\mathsf{P}\{1,\ldots,n\}$ denotes the set of partition of $\{1,\ldots,n\}$ in non-empty subsets.

In [37] the dependence of the state $\Omega_{\beta,\lambda V\{h\chi\}}$ on the cut-off present in V was studied. In the massive case, the clustering properties of the state Ω_{β} guarantees that the limit $h \to 1$ can be performed in the sense of van Hove [37, Def. 2], leading to a KMS state $\Omega_{\beta,\lambda V\{\chi\}}$. In [28] similar conclusions were drawn also in the massless case, where the lack of clustering properties for Ω_{β} can be treated by exploiting the so-called principle of perturbative agreement (PPA) [49]. In [25], the long time behaviour of the state $\Omega_{\beta,\lambda V\{\chi\}}$ has been investigated, leading to one of the first example of non-equilibrium steady state (NESS) [59] in the context of Quantum Field Theory – see also [45]. In particular this state is defined as the weak limit

$$\Omega_{\text{NESS}}(A) := \lim_{t \to \infty} \frac{1}{t} \int_0^t \mathrm{d}s \,\Omega_{\beta,\lambda V\{\chi\}}[\tau_s(A)], \qquad \forall A \in \mathcal{A}_{V\{\chi\}}.$$
(15)

The thermodynamical properties of Ω_{NESS} have been discussed in [26].

3 Main result

The goal of this paper is to study the adiabatic limit of the interacting thermal state $\Omega_{\beta,\lambda Q\{h\chi\}}$ constructed in [37] in the case of a perturbation given by a quadratic interaction Q:

$$Q\{h\chi\}(\phi) := \frac{m_0^2}{2} \int_M h\chi \phi^2 \,, \tag{16}$$

where $m_0^2 > 0$ has the dimension of a squared mass. The precise definition of the adiabatic limit is the following: After performing the van Hove limit $h \to 1$ of (13) we obtain a state $\Omega_{\beta,\lambda Q\{\chi\}}$, which depends on χ [37]. We then consider $\chi \in C^{\infty}(\mathbb{R})$ be such that

$$\operatorname{spt}(\chi) \subseteq (-1, +\infty), \qquad \chi(t) = 1 \quad \text{for } t \ge 0,$$
(17)

and we set $\chi_{\mu}(t) := \chi(t/\mu)$ for all $\mu > 0$. We denote with $Q\{\chi_{\mu}\}$ and $\Omega_{\beta,\lambda Q\{\chi_{\mu}\}}$ the quadratic perturbation (16) and the corresponding state (13) where χ has been substituted with χ_{μ} . The adiabatic limit of $\Omega_{\beta,\lambda Q\{h\chi\}}$ is, per definition, the weak limit of the sequence of states $\Omega_{\beta,\lambda Q\{\chi_{\mu}\}}$ as $\mu \to +\infty$. Notice that this prescription for the adiabatic limit consists in removing the space cut-off h – which is the "thermodynamical limit" for the interaction $\lambda Q\{h\chi\}$ – and then in addressing the limit for the time cut-off χ . Exchanging the limits $h \to 1, \chi \to 1$ described above would probably lead to a rather trivial result because for fixed h observables F with $\operatorname{spt}(F) \cap J(\operatorname{spt}(h)) = \emptyset$ remain unaffected by the action of the Møller operator $\mathsf{R}_{\lambda Q\{h\chi\}} - cf$. equation (4).

Thanks to the structural assumption on $Q\{\chi_{\mu}\}$ – see in particular equation (19) – we will be able to compute the limit $\mu \to +\infty$ of each term in the series (13) for $\Omega_{\beta,\lambda Q\{\chi_{\mu}\}}$. Moreover, we will be also able to give a closed form for the series itself: Then the resulting state is compared to the KMS state on the algebra of the Klein-Gordon theory with mass $m^2 + \lambda m_0^2$. For the convenience of the reader we state here the main result:

Theorem 3: For a quadratic perturbation $Q\{\chi_{\mu}\}$ as in (16), the state Ω_{AD} defined as the weak limit of the sequence $\Omega_{\beta,\lambda Q\{\chi_{\mu}\}}$ for $\mu \to +\infty$ is the quasi-free state whose two-point function reads

$$\omega_{\rm AD}(f,g) = \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} b_{\pm}(\beta,\epsilon_{\lambda}) \widehat{f}(\pm\epsilon_{\lambda},k) \widehat{g}(\mp\epsilon_{\lambda},-k) \,, \qquad \epsilon_{\lambda} = \epsilon_{\lambda}(k) := \sqrt{|k|^2 + m^2 + \lambda m_0^2} \,, \tag{18}$$

In other words, in the adiabatic limit, $\Omega_{\beta,\lambda Q\{\chi_{\mu}\}}$ converges to the KMS state for the Klein-Gordon theory with mass $m^2 + \lambda m_0^2$.

Remark 4: (i) We stress that we do not make any claims about the case of a tachyonic (imaginary) mass $m^2 + \lambda m_0^2 < 0$. (ii) Notice that the general form of a purely quadratic local functional would contain first derivatives of the field, *i.e.*, terms proportional to $\partial_a \phi \partial_b \phi$, $\phi \partial_a \phi$. While the latter term would not be a great deal and may model the presence of an external heat flux (described by an interaction $\sim \phi Q^a \partial_a \phi$), the former one would spoil some of the result of [28] which we will need in the following sections.

3.1 Preliminary observations

3.1.1 Quantum Møller operator

In this section we describe how the simple structure of the interaction potential $Q\{\chi_{\mu}\}$ given in (16) allows to simplify the expression involving the quantum Møller operator.

Indeed, with reference to [28], we recall that for any quadratic local functional $Q\{\chi_{\mu}\}$ it holds

$$\mathsf{R}_{\lambda Q\{\chi_{\mu}\}} = \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}} \circ \gamma_{\lambda Q\{\chi_{\mu}\}}, \qquad (19)$$

where $\mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$ is the classical Møller operator [55] while $\gamma_{\lambda Q\{\chi_{\mu}\}}$ is a contraction map between local functionals – see [28] for details. The classical Møller operator $\mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$: $\mathcal{P}_{\mathrm{loc}} \to \mathcal{A}[[\lambda]]$ can be thought as the classical limit of $\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}$ [20, 28, 30, 32]. Actually, it is an *exact*, *i.e.* non-perturbative, *isomorphism $\mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$: $\mathcal{A}_{\lambda Q\{\chi_{\mu}\}} \to \mathcal{A}$ between the algebra $\mathcal{A}_{\lambda Q\{\chi_{\mu}\}}$ of quantum observables associated to the free Klein-Gordon field whose dynamics is ruled by the operator $\Box + m^2 + \lambda m_0^2 \chi_{\mu}$ and the algebra \mathcal{A} . Its pull-back action on states has been studied in [20, 29] and will be exploited in the following – *cf.* equation (20).

The main feature of $\gamma_{\lambda Q\{\chi_{\mu}\}}$ is that it does not increase the number of fields present in each observable. Hence, when applied on linear or quadratic functionals, $\gamma_{\lambda Q\{\chi_{\mu}\}}$ is the identity up to constant, actually

$$\gamma_{\lambda Q\{\chi_{\mu}\}}(F) = F, \qquad \gamma_{\lambda Q\{\chi_{\mu}\}}(Q') = Q' + c,$$

where $F(\phi) := \int_M f \phi$ and Q' is any quadratic functional. The conclusion is that, on linear and quadratic functionals, the quantum Møller operator $\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}^{cl}$ and the classical Møller operator $\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}^{cl}$ coincide up to constant. The latter will play no rôle in the subsequent discussion due to the presence of the connected part Ω_{β}^c of Ω_{β} .

This observation allows to rewrite the connected part Ω_{β}^{c} of Ω_{β} appearing in (13) as

$$\Omega_{\beta}^{c} \left[\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(F_{1}) \star_{\beta} \ldots \star_{\beta} \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(F_{n}) \otimes \bigotimes_{\ell=1}^{n} \left[\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}Q\{\dot{\chi}_{\mu}\}\right]_{iu_{\ell}} \right]$$
$$= \Omega_{\beta}^{c} \left[\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}^{cl}(F_{1}) \star_{\beta} \cdots \star_{\beta} \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}^{cl}(F_{n}) \otimes \bigotimes_{\ell=1}^{n} \left[\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}^{cl}Q\{\dot{\chi}_{\mu}\}\right]_{iu_{\ell}} \right]$$
$$= \left[\Omega_{\beta} \circ \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}^{cl} \right]^{c} \left[F_{1} \star_{\lambda Q\{\chi_{\mu}\}} \cdots \star_{\lambda Q\{\chi_{\mu}\}} F_{n} \otimes \bigotimes_{\ell=1}^{n} Q\{\dot{\chi}_{\mu}\}_{iu_{\ell}} \right],$$

where we exploited the fact $\Omega_{\beta}^{c}(1_{\mathcal{A}}) = 0$. The *-product $\star_{\lambda Q\{\chi_{\mu}\}}$ is the one induced by the *-isomorphism $\mathsf{R}^{cl}_{\lambda Q\{\chi_{\mu}\}}$, *i.e.* it is induced by $\omega_{\beta} \circ \mathsf{R}^{cl}_{\lambda Q\{\chi_{\mu}\}}$. This computation shows that the χ_{μ} -dependent part of (13) is either in the appearance of $Q\{\dot{\chi}_{\mu}\}$ -terms in the connected function or in the presence of the pull-back state $\Omega_{\beta} \circ \mathsf{R}^{cl}_{\lambda Q\{\chi_{\mu}\}}$.

In the following we will compute the limit $\mu \to +\infty$ of each term of the series (13), exploiting the exact knowledge provided by $\mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$. In a sense, equation (19) allows to switch to an effective description where the classical part of the perturbative series in λ has already been summed, leaving untouched the pure quantum contributions. This simplification is expected to hold for a more general potential V, though it would probably appear as $\mathsf{R}_{\lambda V\{f\}} = \mathsf{R}^{\mathrm{cl}}_{\lambda V_{\mathrm{eff}}\{f\}} \circ \gamma_{\lambda V\{f\}}$, with $V_{\mathrm{eff}}\{f\}$ being an effective potential, perturbatively built out of $V\{f\}$ [14, 65].

As explained in [20, 28] the state $\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$ is a quasi-free state whose two-point function $\omega_{\lambda V\{\chi_{\mu}\}}$ is given by

$$\omega_{\lambda V\{\chi_{\mu}\}}(f,g) = \int_{\mathbb{R}} \mathrm{d}t \int_{\mathbb{R}} \mathrm{d}t' \int_{\mathbb{R}^3} \mathrm{d}k \, \widetilde{f}(t,k) \widetilde{g}(t',k) \left[b_+(\beta,\epsilon) T_{k,\mu}(t) \overline{T_{k,\mu}(t')} + b_-(\beta,\epsilon) \overline{T_{k,\mu}(t)} T_{k,\mu}(t') \right],$$
(20)

where $f, g \in C_c^{\infty}(\mathbb{R}^4)$ and \tilde{f}, \tilde{g} denotes the Fourier transform of f, g in three momentum $k \in \mathbb{R}^3$. The modes $T_{k,\mu}(t)$ are solutions of the following differential equation

$$\ddot{T}_{k,\mu}(t) + \epsilon_{\mu}(k,t)^2 T_{k,\mu}(t) = 0, \qquad T_{k,\mu}(t) = \frac{e^{-i\epsilon t}}{\sqrt{2\epsilon}} \qquad \text{for } t \notin \operatorname{spt}(\chi),$$
(21)

where $\epsilon_{\mu}(k,\tau) := \sqrt{\epsilon(k)^2 + (\epsilon_{\lambda}(k)^2 - \epsilon(k)^2)\chi_{\mu}(t)}$, subject to the Wronskian condition

$$\overline{\dot{T}}_{k,\mu}(t)T_{k,\mu}(t) - \overline{T_{k,\mu}(t)}\overline{\dot{T}}_{k,\mu}(t) = i.$$
(22)

The limit of this latter state as $\mu \to +\infty$ was investigated in [20, 27, 28]:

Lemma 5 ([27]): The limit $\mu \to +\infty$ of the sequence $\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$ exists and defines a quasi-free state $\Omega_{\mathrm{AD,cl}}$ whose two point function $\omega_{\mathrm{AD,cl}}$ reads

$$\omega_{\rm AD,cl}(f,g) := \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} b_{\pm}(\beta,\epsilon) \widehat{f}(\pm\epsilon_{\lambda},k) \widehat{g}(\mp\epsilon_{\lambda},-k) \,. \tag{23}$$

Remark 6: As remarked in [27] the limit for $\mu \to +\infty$ of a thermal state Ω_{β} under the action of the classical Møller operator $\mathsf{R}^{cl}_{\lambda Q\{\chi_{\mu}\}}$ fails to be the corresponding thermal state for the theory with mass $m^2 + \lambda m_0^2$. The reason can be traced back to the fact that, while the ϵ -modes of the states are correctly changed into the ϵ_{λ} ones -i.e. $T_{k,\mu}(t)\overline{T_{k,\mu}(t')} \to (2\epsilon_{\lambda})^{-1}\exp[-i\epsilon_{\lambda}(t-t')]$ – the thermal coefficients $b_{\pm}(\beta,\epsilon)$ remain untouched. Theorem 3 shows that the terms needed to restore the KMS property are exactly those provided by the perturbative series (13).

For future convenience we state the following lemma.

Lemma 7: Let $\chi \in C^{\infty}(\mathbb{R})$ with the property (17) and set $\chi_{\mu}(t) := \chi(t/\mu)$ for $\mu > 0$. For any $k \in \mathbb{R}^3$, let $T_{k,\mu}(t)$ be the modes defined as in (21). Then

$$\lim_{\mu \to +\infty} \int_{\mathbb{R}} \mathrm{d}t \ T_{k,\mu}(t)^2 \frac{\mathrm{d}}{\mathrm{d}t} \chi_{\mu}(t) = 0, \qquad \lim_{\mu \to +\infty} \int_{\mathbb{R}} \mathrm{d}t \ \left| T_{k,\mu}(t) \right|^2 \frac{\mathrm{d}}{\mathrm{d}t} \chi_{\mu}(t) = \frac{1}{\epsilon_{\lambda} + \epsilon}.$$
(24)

Proof. We recall a few facts from [20, Lemma 5.1], [28, Appendix D]. First of all we set, for all $k \in \mathbb{R}^3$,

$$T_{\mathbf{a},k,\mu}(t) := \frac{1}{\sqrt{2\epsilon_{\mu}(k,t)}} \exp\left[-i \int_{t_0}^t \mathrm{d}s \; \epsilon_{\mu}(k,s)\right], \qquad \epsilon_{\mu}(k,t) = \sqrt{\epsilon^2(k) + (\epsilon_{\lambda}^2(k) - \epsilon^2(k))\chi_{\mu}(t)}, \quad (25)$$

where $t_0 \notin \operatorname{spt}(\chi_{\mu})$ is arbitrary but fixed. Following [20, 28] one finds that, for all $k \in \mathbb{R}^3$ and $t \in \mathbb{R}$, $T_{\mathbf{a},k,\mu}(t)\overline{T_{\mathbf{a},k,\mu}(t')} \to (2\epsilon_{\lambda}(k))^{-1} \exp[-i\epsilon_{\lambda}(k)(t-t')]$ as $\mu \to +\infty$. Notice however that $T_{\mathbf{a},k,\mu}(t)$ has no limit for $\mu \to +\infty$. Finally, $|T_{k,\mu} - T_{\mathbf{a},k,\mu}| \to 0$ as $\mu \to +\infty$ uniformly in t, see the proof of [20, Lemma 5.1] or [28, Appendix D].

We now address the second limit in (24):

$$\lim_{\mu \to +\infty} \int_{\mathbb{R}} \frac{\mathrm{d}t}{\mu} \left| T_{k,\mu}(t) \right|^2 \dot{\chi}\left(\frac{t}{\mu}\right) = \lim_{\mu \to +\infty} \int_{\mathbb{R}} \mathrm{d}t \left| T_{\mathrm{a},k,\mu}(\mu t) \right|^2 \dot{\chi}(t) = \lim_{\mu \to +\infty} \int_{\mathbb{R}} \mathrm{d}t \, \frac{\dot{\chi}(t)}{2\epsilon_{\mu}(k,\mu t)} = \frac{1}{\epsilon_{\lambda}(k) + \epsilon(k)}$$

The first limit in (24) is addressed similarly. Indeed, since $t_0 \leq 0$, one finds, for all $t \in \mathbb{R}$,

$$\mu \int_{t_0/\mu}^t \epsilon_\mu(k,\mu s) \mathrm{d}s - \mu \int_0^t \epsilon_\mu(k,\mu s) \mathrm{d}s \longrightarrow_{\mu \to +\infty} t_0 \epsilon_\lambda(k) \, ds$$

Therefore, as $\mu \to +\infty$, $T_{a,k,\mu}(\mu t)^2 \simeq (2\epsilon_{\mu}(k,\mu t))^{-1} \exp(i\mu\psi(t))$, with $\dot{\psi}(t) \neq 0$ and the thesis follows by Riemann's Lemma.

3.1.2 β -expansion of the Bose-Einstein factor

For later convenience we provide a formula for the β -derivatives of the "thermal coefficients" $b_{\pm}(\beta, \epsilon)$, which will be useful in the proof of Theorem 3. We recall that b_{\pm} are defined by

$$b_{\pm}(\beta,\epsilon) = \frac{\mp 1}{e^{\mp\beta\epsilon} - 1}.$$
(26)

In particular, b_{-} is the Bose-Einstein factor. Thanks to the relations $b_{\pm}(\beta, \epsilon) = e^{\pm\beta\epsilon}b_{\mp}(\beta, \epsilon)$ and $b_{+}-b_{-} = 1$ one may compute

$$\partial_{\beta}b_{\pm}(\beta,\epsilon) = -\epsilon b_{+}(\beta,\epsilon)b_{-}(\beta,\epsilon).$$
⁽²⁷⁾

Iterating relation (27) we find

$$\partial_{\beta}^{n}b_{\pm} = (-\epsilon)^{n} \sum_{k=1}^{n} c_{n,k}b_{\pm}^{n+1-k}b_{\pm}^{k}, \qquad (28)$$

where the coefficients $(c_{n,k})_{n>1,1\leq k\leq n}$ satisfy the following recursion relations:

$$c_{n,k} = \begin{cases} 1 & \text{for } k \in \{1,n\} \\ kc_{n-1,k} + (n+1-k)c_{n-1,k-1} & \text{for } 2 \le k \le n-1 . \end{cases}, \qquad c_{n,k} = c_{n,n+1-k} . \tag{29}$$

From equation (29) it follows that the coefficients $c_{n,k}$ coincide with the Eulerian numbers A(n,k) [10, Thm. 1.7], that is, $c_{n,k}$ is the number of *n*-permutations with k-1 descents. We recall that, for a given *n*-permutation σ of $\{1, \ldots, n\}$, an index $i \in \{1, \ldots, n-1\}$ is called descent of σ if $\sigma_i > \sigma_{i+1}$. In what follows we denote with \wp_n the set of *n*-permutations and with $\wp_{n,k} \subset \wp_n$ the subset of *n*-permutation with k-1 descents – so that $c_{n,k} = |\wp_{n,k}|$. See [34, 35] for similar applications of these structures in the QFT framework.

3.2 Proof of Theorem 3

We are now ready to prove Theorem 3.

Proof. (Thm.3) We recall that, $Q\{\chi_{\mu}\}$ is the quadratic functional (16) with cut-off χ_{μ} , where $\chi \in C^{\infty}(\mathbb{R})$ satisfies (17) while $\chi_{\mu}(t) := \chi(t/\mu)$.

At first, we focus on the case $A = \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(F) \star \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(G)$, where $F, G \in \mathcal{A}$ are two linear functionals defined by $F(\phi) := \int_{M} f\phi$, $G(\phi) := \int_{M} g\phi$ for $f, g \in C_{c}^{\infty}(M)$. The general case will be outlined at the end of the proof.

By exploiting the results of section 3.1.1 – see equation (19) – we may write the state (13) as

$$\Omega_{\beta,\lambda Q\{\chi_{\mu}\}} \left[\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(F) \star_{\beta} \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(G) \right] = \omega_{\lambda Q\{\chi_{\mu}\}}(f,g)$$

$$+ \sum_{n \ge 1} (-1)^{n} \int_{\beta S_{n}} \mathrm{d}U \left(\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}} \right)^{c} \left[FG \otimes \bigotimes_{\ell=1}^{n} (\lambda Q\{\dot{\chi}_{\mu}\})_{iu_{\ell}} \right].$$

$$(30)$$

Notice that, by Lemma 5, the first term on the right-hand side of (30) tends to $\omega_{AD,Cl}(f,g)$. In what follows, we will consider each term of the series appearing in (30) and compute its limit as $\mu \to +\infty$. We will then be able to sum the series. Notice that the *n*-th term in the series (30) is not the *n*-th order in perturbation theory of $\Omega_{\beta,\lambda Q}\{\chi_{\mu}\}$. Indeed the state $\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q}\{\chi_{\mu}\}$ depends on the formal parameter λ . The main advantage coming from the results recalled in section 3.1.1 is that $\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q}\{\chi_{\mu}\}$ can be considered as an exact – *i.e.* non perturbative – quantity, and what has to be computed is only the *n*-th order term of the series in (30). In other words, we are exploiting a partial summation, in which the contribution in λ coming from $\mathsf{R}^{\mathrm{cl}}_{\lambda Q}\{\chi_{\mu}\}$ are recollected.

With this in mind, we now focus our attention on the n-th term

$$\left(\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}\right)^{c} \left[FG \otimes \bigotimes_{\ell=1}^{n} (\lambda Q\{\dot{\chi}_{\mu}\})_{iu_{\ell}} \right].$$
(31)

Since the state $\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi_{\mu}\}}$ is a quasi-free state with two-point function given by (20), the term (31) can be expanded graphically as a sum of connected graphs, whose edges are associated with $\omega_{\lambda V\{\chi_{\mu}\}}$. Since FG and $Q\{\chi_{\mu}\}$ are quadratic functionals, we can write (31) equivalently as a sum over *n*-permutations. Indeed for each $\sigma \in \wp_n$, the corresponding contribution to (31) is

$$\frac{(\lambda m_0^2)^n}{2} \left[\int_{\mathbb{R}^{4(n+2)}} \mathrm{d}Z \left[f(z_0)g(z_{n+1}) + f(z_{n+1})g(z_0) \right] \omega_{\lambda Q\{\chi_\mu\}}(z_0, z_{\sigma_1}) \omega_{\lambda Q\{\chi_\mu\}}(z_{n+1}, z_{\sigma_n}) \cdot \prod_{i=1}^{n-1} \omega_{\lambda Q\{\chi_\mu\}}(z_{\sigma_j}, z_{\sigma_{j+1}}) \prod_{\ell=1}^n (\dot{\chi}_\mu)_{iu_\ell}(z_\ell^0) \right],$$

Here $z_0, \ldots, z_{n+1} \in M$ are arbitrary but fixed points of M and z_ℓ^0 denotes the time component of z_ℓ for $\ell \in \{1, \ldots, n\}$. Notice that the symmetric term in f, g gives the same contribution and cancels the factor 1/2 which pops out from formula (1).

For the sake of simplicity, we now provide the explicit computation in the case n = 1, which will be generalized later. For n = 1 the unique contribution in (31) is given by

$$\lambda m_0^2 \int_{\mathbb{R}^{12}} \mathrm{d}z_0 \mathrm{d}z_1 \mathrm{d}z_2 f(z_0) g(z_2) \omega_{\lambda V\{\chi_\mu\}}(z_0, z_1 + iue^0) \omega_{\lambda V\{\chi_\mu\}}(z_2, z_1 + iue^0) \dot{\chi}_\mu(z_1^0) , \qquad (32)$$

Here $y + iue^0$ denotes the complex time translation of the point $y = (\underline{y}, y^0)$ by $iue^0 - e^0$ being the unit time vector field – which we recall is well-defined once exploiting the analytic properties of ω_{β} .

Exploiting the explicit form of the two-point function $\omega_{\lambda V\{\chi_{\mu}\}} - cf$. equation (20) – and Lemma 7 one finds that, in the limit $\mu \to +\infty$,

$$(32) \to \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \widehat{f}(\pm\epsilon_{\lambda}, k) \widehat{g}(\mp\epsilon_{\lambda}, -k) \frac{\lambda m_0^2}{\epsilon_{\lambda} + \epsilon} b_+(\beta, \epsilon) b_-(\beta, \epsilon) \,,$$

The final step is to recall (27) so that the limit for $\mu \to +\infty$ of the term n = 1 in (30) becomes

$$n = 1 \text{ term of } (30) \to_{\mu \to +\infty} \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \widehat{f}(\pm\epsilon_{\lambda}, k) \widehat{g}(\mp\epsilon_{\lambda}, -k) \frac{\beta \lambda m_0^2}{(\epsilon_{\lambda} + \epsilon)\epsilon} \partial_{\beta} b_{\pm}(\beta, \epsilon) \,.$$

The claim is that this formula can be generalized for all $n \ge 1$. Indeed, let consider the contribution to (31) at order $n \ge 1$. The combinatorial expansion gives

$$(31) = \sum_{\sigma \in \wp_n} (\lambda m_0^2)^n \left[\int_{\mathbb{R}^{4(n+2)}} dZ f(z_0) g(z_{n+1}) \omega_{\lambda Q\{\chi_\mu\}}(z_0, z_{\sigma_1}) \omega_{\lambda Q\{\chi_\mu\}}(z_{n+1}, z_{\sigma_n}) \cdots \right]_{j=1}^{n-1} \omega_{\lambda V\{\chi_\mu\}}(z_{\sigma_j}, z_{\sigma_{i+1}}) \prod_{\ell=1}^n (\dot{\chi}_\mu)_{iu_\ell}(z_\ell^0) \right],$$
(33)

Let us focus on the contribution to (33) from an arbitrary but fixed permutation $\sigma \in \wp_n$. Once the explicit expression (20) of $\omega_{\lambda V\{\chi_{\mu}\}}$ has been inserted into (33), one finds a sum of products of factors $b_{\pm}(\beta,\epsilon)$ with the corresponding modes. Notice that, since the limit $h \to 1$ has already been taken, there is a single integration over three momentum $k \in \mathbb{R}^3$. Invoking Lemma 7 several terms in the sum above disappear. Actually, the non-vanishing contributions are those which contain, for each $\ell \in \{1, \ldots, n\}$, the factor $|T_k(z_{\ell}^0)|^2$ – the other products would contain either a factor $\overline{T_k(z_{\ell}^0)^2}$ or a factor $T_k(z_{\ell}^0)^2$.

These non-trivial terms can be computed as follows. Let be j the number of descents of σ , that is, let $j \in \{1, \ldots, n\}$ be such that $\sigma \in \wp_{n,j}$. Then, in the adiabatic limit $\mu \to +\infty$, the non-trivial contribution to (33) obtained from σ is

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \left[\frac{\lambda m_0^2}{(\epsilon_{\lambda} + \epsilon)} \right]^n b_{\pm}(\beta, \epsilon)^{n+1-j} b_{\mp}(\beta, \epsilon)^j \widehat{f}(\pm \epsilon_{\lambda}, k) \widehat{g}(\mp \epsilon_{\lambda}, -k) \,. \tag{34}$$

Thus, the contribution as $\mu \to +\infty$ arising from a *n*-permutation $\sigma \in \wp_n$ depends uniquely on its subclass $\wp_{n,j}$, *i.e.* on the number of descends of σ . With this in mind we may rewrite the limit for $\mu \to +\infty$ of (33) as follows:

$$(33) \stackrel{\mu \to +\infty}{\to} \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \widehat{f}(\pm\epsilon_{\lambda}, k) \widehat{g}(\mp\epsilon_{\lambda}, -k) \left[\frac{\lambda m_0^2}{(\epsilon_{\lambda} + \epsilon)}\right]^n \sum_{j=1}^n c_{n,j} b_{\pm}(\beta, \epsilon)^{n+1-j} b_{\mp}(\beta, \epsilon)^j = \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \widehat{f}(\pm\epsilon_{\lambda}, k) \widehat{g}(\mp\epsilon_{\lambda}, -k) \left[\frac{-\lambda m_0^2}{(\epsilon_{\lambda} + \epsilon)\epsilon}\right]^n \partial_{\beta}^n b_{\pm}(\beta, \epsilon), \qquad (35)$$

where we exploited the equality $|\wp_{n,k}| = c_{n,k}$ and equation (28) as well as the symmetry property $c_{n,j} = c_{n,n+1-j}$.

Summing up, we have computed the limit as $\mu \to +\infty$ of the *n*-order term appearing in (30). Indeed, since (35) does not depends on $U = (u_1, \ldots, u_n)$, the integral over the simplex βS_n would simply provide a factor $\beta^n (n!)^{-1}$. We thus find

$$\begin{split} \lim_{\mu \to +\infty} \Omega_{\beta,\lambda Q\{\chi_{\mu}\}} \left[\mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(F) \star_{\beta} \mathsf{R}_{\lambda Q\{\chi_{\mu}\}}(G) \right] &= \omega_{\text{AD,cl}}(f,g) \\ &+ \sum_{n \geq 1} \int_{\mathbb{R}^{3}} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \widehat{f}(\pm\epsilon_{\lambda},k) \widehat{g}(\mp\epsilon_{\lambda},-k) \frac{1}{n!} \left[\frac{\beta\lambda m_{0}^{2}}{(\epsilon_{\lambda}+\epsilon)\epsilon} \right]^{n} \partial_{\beta}^{n} b_{\pm}(\beta,\epsilon) \\ &= \int_{\mathbb{R}^{3}} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} \widehat{f}(\pm\epsilon_{\lambda},k) \widehat{g}(\mp\epsilon_{\lambda},-k) b_{\pm} \left[\beta + \frac{\beta\lambda m_{0}^{2}}{(\epsilon_{\lambda}+\epsilon)\epsilon},\epsilon \right], \end{split}$$

where we used the explicit form (23) of $\omega_{\text{AD,cl}}(f,g)$. It is then a simple computation to check that $b_{\pm}(\beta + \beta\lambda m_0^2[(\epsilon_{\lambda} + \epsilon)\epsilon]^{-1}, \epsilon) = b_{\pm}(\beta, \epsilon_{\lambda}).$

The general case for $F, G \in \mathcal{P}_{loc}$ is treated analogously. Using relation (19) one reduces to the state $\Omega_{\beta} \circ \mathsf{R}^{cl}_{\lambda Q\{\chi_{\mu}\}}$ applied on local observables $\gamma_{\lambda Q\{\chi_{\mu}\}}(F), \gamma_{\lambda Q\{\chi_{\mu}\}}(G)$. Notice that $\gamma_{\lambda Q\{\chi_{\mu}\}}(F) = F + F'$, where F' is a local functional with less fields than F. The combinatorial expansion exploited above still applies and the combinatorics reproduces the usual Wick formula for a quasi-free state. The thesis follows.

Remark 8: One may wonder about the massless case m = 0. In this case the proof of Theorem 3 is affected by several infrared divergences, that is the integral over three momentum $k \in \mathbb{R}^3$ is divergent due to the singular behaviour of the Bose-Einstein factor at k = 0. For example, the contribution (34) is divergent due to the presence of the product $b_{\pm}(\beta, \epsilon)^{n+1-j}b_{\mp}(\beta, \epsilon)^j \simeq |k|^{-n-1}$. This singular behaviour can be understood by observing that the expansion $b_{\pm}(\beta, \epsilon_{\lambda}) = b_{\pm}(\beta + \beta\lambda m_0^2 [\epsilon(\epsilon + \epsilon_{\lambda})]^{-1}, \epsilon)$ becomes singular in the massless case.

4 Non-equilibrium steady state

In this section we compare the result obtained in Theorem 3 with those obtained in [25] where a nonequilibrium steady state (NESS) [59] Ω_{NESS} was built out of $\Omega_{\beta,\lambda Q\{\chi\}}$ with an ergodic mean – *cf.* equation (15). In the case of a quadratic perturbation Q as in (16), the steps of the proof of Theorem 3 can be carried out also in this latter case, leading to some simplification. However, such simplification would not suffice to sum the perturbative series.

As a preliminary result we state the following lemma.

Lemma 9: Let $\chi \in C^{\infty}(\mathbb{R})$ be such that (17) holds true. For any $k \in \mathbb{R}^3$, let $T_k(t) := T_{k,\mu=1}(t)$ be the modes defined as in (21). Then, for all $k \in \mathbb{R}^3$, there exist $A_{\pm} = A_{\pm}(k) \in \mathbb{C}$ such that, for all $t_1, t_2 \in \mathbb{R}$,

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{d}\tau \, T_k(t_1 + \tau) T_k(t_2 + \tau) = \frac{A_+ A_-}{2\epsilon_\lambda} \sum_{\pm} e^{\pm i\epsilon_\lambda(t_1 - t_2)} \,, \tag{36a}$$

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t \mathrm{d}\tau \, T_k(t_1 + \tau) \overline{T_k(t_2 + \tau)} = \frac{1}{2\epsilon_\lambda} \sum_{\pm} |A_{\pm}|^2 e^{\mp i\epsilon_\lambda(t_1 - t_2)} \,. \tag{36b}$$

Proof. First, we recall from [20, Lemma 5.1], see also [28, Appendix D] that the modes T_k defined in (21) are uniformly bounded, namely

$$|T_k(t)| \le \frac{1}{2\epsilon_{\lambda}} = \frac{1}{2\sqrt{|k|^2 + m^2 + \lambda m_0^2}}.$$
(37)

For $\tau \ge 0$, $T_k(\tau)$ satisfies equation (21), where $\epsilon_{\mu=1}(k,\tau) = \epsilon_{\lambda}(k)$. Hence we may write $T_k(\tau)$ as

$$T_k(\tau) = \frac{1}{\sqrt{2\epsilon_\lambda}} \sum_{\pm} A_{\pm} e^{\pm i\epsilon_\lambda \tau}, \qquad A_{\pm} \in \mathbb{C}.$$
(38)

Let $t_1, t_2 \in \mathbb{R}$ be arbitrary but fixed and set

$$\tau_{\min} := \inf\{\tau \in \mathbb{R} | t_{\ell} + \tau \ge 0 \ \forall \ell \in \{1, 2\}\}.$$

$$(39)$$

For $\tau \geq \tau_{\min}$ we may compute

$$T_k(t_1 + \tau)\overline{T_k(t_2 + \tau)} = \frac{1}{2\epsilon_{\lambda}} \sum_{\pm} \left[|A_{\pm}|^2 e^{\pm i\epsilon_{\lambda}(t_1 - t_2)} \right] + 2\Re \left(A_{\pm} \overline{A_{-}} e^{-i\epsilon_{\lambda}(2\tau + t_1 + t_2)} \right)$$
(40a)

$$T_k(t_1 + \tau)T_k(t_2 + \tau) = \frac{1}{2\epsilon_\lambda} \sum_{\pm} \left[A_{\pm}^2 e^{\mp i\epsilon_\lambda(2\tau + t_1 + t_2)} + A_{\pm}A_{-}e^{\mp i\epsilon_\lambda(t_1 - t_2)} \right].$$
 (40b)

The limits (36) can be computed exploiting (37): considering (36b) we have

$$\lim_{t \to +\infty} \int_0^t \frac{\mathrm{d}\tau}{t} T_k(t_1 + \tau) \overline{T_k(t_2 + \tau)} = \lim_{t \to +\infty} \int_{\tau_{\min}}^t \frac{\mathrm{d}\tau}{t} T_k(t_1 + \tau) \overline{T_k(t_2 + \tau)} = \frac{1}{2\epsilon_\lambda} \sum_{\pm} A_{\pm}^2 e^{\mp i\epsilon_\lambda(t_1 - t_2)} A_{\pm}^2 e^{\pm i\epsilon_\lambda(t_1 - t_2)} A_{\pm}^2 e^{$$

where in the second equality we used (40a). A similar computation can be carried out for (36a). The thesis follows. $\hfill \Box$

Remark 10: (i) The coefficients A_{\pm} are subjected to the condition $|A_{+}|^{2} - |A_{-}|^{2} = 1$ which ensures the Wronskian condition (22) for the modes T_{k} .

(*ii*) The proof of Lemma (9) still holds true if one replaces the ergodic mean over $\tau \in (0, +\infty)$ with an ergodic mean over the real axis. In this latter case equation (36) acquires additional terms, proportional to the modes exp $[\pm i\epsilon(t_1 - t_2)]$.

As an immediate consequence of Lemma 9 we compute the ergodic limit of the state $\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi\}}$.

Corollary 11: The limit $t \to +\infty$ of the sequence of states defined by $A \mapsto t^{-1} \int_0^t \mathrm{d}s \left(\Omega \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi\}}\right) [\alpha_s(A)]$ exists and defines a quasi-free state $\Omega_{\mathrm{NESS,cl}}$ whose two-point function is given by

$$\omega_{\text{NESS,cl}}(f,g) := \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_{\lambda}} \sum_{\pm} c_{\pm}(\beta,k) \widehat{f}(\pm\epsilon_{\lambda},k) \widehat{g}(\mp\epsilon_{\lambda},-k) \,, \tag{41}$$

where

$$c_{+}(\beta,k) := \sum_{\pm} b_{\pm}(\beta,\epsilon) |A_{\pm}(k)|^{2}, \qquad c_{-}(\beta,k) := \sum_{\pm} b_{\pm}(\beta,\epsilon) |A_{\mp}(k)|^{2}.$$
(42)

Remark 12: Notice that the CCR relations for the state $\omega_{\text{NESS,cl}}$ are a direct consequence of the relations $b_+ - b_- = 1 = |A_+|^2 - |A_-|^2$.

We now follows the steps of the proof of Theorem 3 in the case of Ω_{NESS} . In particular, let $F, G \in \mathcal{A}$ be linear functionals, namely $F(\phi) := \int_M f\phi$ and $G(\phi) := \int_M g\phi$ for $f, g \in C_c^{\infty}(M)$. We shall compute

$$\Omega_{\text{NESS}}\left(\mathsf{R}_{\lambda Q\{\chi\}}(F)\star_{\beta}\mathsf{R}_{\lambda Q\{\chi\}}(G)\right) := \lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} \mathrm{d}s \ \Omega_{\beta,\lambda Q\{\chi\}}\left[\tau_{s}\left(\mathsf{R}_{\lambda Q\{\chi\}}(F)\right)\star_{\beta}\tau_{s}\left(\mathsf{R}_{\lambda Q\{\chi\}}(G)\right)\right].$$
(43)

The existence of the limit $t \to +\infty$ in the sense of formal power series in λ has already been proved in [25]. As in (30), we exploit the perturbative series (13) in order to compute the integrand

$$\Omega_{\beta,\lambda Q\{\chi\}} \left[\tau_s \left(\mathsf{R}_{\lambda Q\{\chi\}}(F) \right) \star \tau_s \left(\mathsf{R}_{\lambda Q\{\chi\}}(G) \right) \right] = \omega_{\lambda Q\{\chi\}} (f_s, g_s) + \sum_{n \ge 1} (-1)^n \int_{\beta S_n} \left[\Omega_\beta \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi\}} \right]^c \left[F_s G_s \otimes \bigotimes_{\ell=1}^n Q\{\dot{\chi}\}_{iu_\ell} \right].$$

$$(44)$$

Thanks to Corollary 11 one finds that, once integrated in s, the first term in the right-hand side of (44) converges to $\omega_{\text{NESS,cl}}(f,g)$. Similarly to the proof of Theorem 3 we focus on the expansion of the n-th term

$$\left[\Omega_{\beta} \circ \mathsf{R}^{\mathrm{cl}}_{\lambda Q\{\chi\}}\right]^{c} \left(F_{s}G_{s} \otimes \bigotimes_{\ell=1}^{n} Q\{\dot{\chi}\}_{iu_{\ell}}\right).$$

$$(45)$$

Once again, this contribution is the sum over n-permutations, in particular

$$(45) = \sum_{\sigma \in \wp_n} (\lambda m_0^2)^n \int_{\mathbb{R}^{4(n+2)}} \mathrm{d}Z f(z_0) g(z_{n+1}) \omega_{\lambda Q\{\chi\}}(z_0 + se^0, z_{\sigma_1}) \omega_{\lambda Q\{\chi\}}(z_{n+1} + se^0, z_{\sigma_n}) \Xi_{\sigma}(\hat{Z}, U) ,$$

$$(46)$$

$$\Xi_{\sigma}(\hat{Z}, U) := \prod_{j=1}^{n-1} \omega_{\lambda Q\{\chi\}}(z_{\sigma_j}, z_{\sigma_{j+1}}) \prod_{\ell=1}^n \dot{\chi}_{iu_\ell}(z_\ell^0), \qquad \hat{Z} := (z_1, \dots, z_n).$$
(47)

Exploiting equation (20) (for $\mu = 1$) one reduces the previous expression (46) to an integral in three momentum $k \in \mathbb{R}^3$. Considering the ergodic mean of the contribution to (46) of a single *n*-permutation $\sigma \in \wp_n$ and applying Lemma 9, one finds

ergodic mean of
the
$$\sigma$$
-contribution to (46) $\rightarrow_{t \to +\infty} (\lambda m_0^2)^n \int_{\mathbb{R}^3} \frac{\mathrm{d}k}{2\epsilon_\lambda} \left[\sum_{\pm} a_{\pm,\sigma}(k,U) \widehat{f}(\pm\epsilon_\lambda,k) \widehat{g}(\mp\epsilon_\lambda,-k) \right],$ (48)

where we defined

$$a_{\pm,\sigma}(k,U) := \int_{\mathbb{R}^{4n}} \mathrm{d}\hat{Z} \bigg[b_{+}(\epsilon)^{2} A_{+} A_{-} \overline{T_{k}(z_{\sigma_{1}})T_{k}(z_{\sigma_{n}})} + b_{-}(\epsilon)^{2} \overline{A_{+}A_{-}} T_{k}(z_{\sigma_{1}})T_{k}(z_{\sigma_{n}}) + b_{+}(\epsilon) b_{-}(\epsilon) \bigg(|A_{\pm}|^{2} \overline{T_{k}(z_{\sigma_{1}})}T_{k}(z_{\sigma_{n}}) + |A_{\mp}|^{2} T_{k}(z_{\sigma_{1}}) \overline{T_{k}(z_{\sigma_{n}})} \bigg) \bigg] \Xi_{\sigma}(\hat{Z},U) \,.$$

Hence, in spite of the fact that the ergodic limit of each term present in (46) can be computed, the resulting limit appears to depend in a quite complicate way on the chosen graph $\sigma \in \wp_n$. This fact spoils the chance to infer a closed form for the ergodic limit of the series (43).

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