

Ricci Flow from the Renormalization of Nonlinear Sigma Models in the Framework of Euclidean Algebraic Quantum Field Theory

Mauro Carfora^{1,2,3,a}, Claudio Dappiaggi^{1,2,3,b}, Nicolò Drago^{1,2,3,c} and Paolo Rinaldi^{1,d}

¹ Dipartimento di Fisica – Università di Pavia, Via Bassi 6, 27100 Pavia, Italy.

² INFN, Sezione di Pavia – Via Bassi 6, 27100 Pavia, Italy.

³ Istituto Nazionale di Alta Matematica – Sezione di Pavia, Via Ferrata, 5, 27100 Pavia, Italy.

^a mauro.carfora@pv.infn.it , ^b claudio.dappiaggi@unipv.it , ^c nicolo.drago@unipv.it , ^d paolo.rinaldi01@universitadipavia.it

May 21, 2019

Abstract

The perturbative approach to nonlinear Sigma models and the associated renormalization group flow are discussed within the framework of Euclidean algebraic quantum field theory and of the principle of general local covariance. In particular we show in an Euclidean setting how to define Wick ordered powers of the underlying quantum fields and we classify the freedom in such procedure by extending to this setting a recent construction of Khavkine, Melati and Moretti for vector valued free fields. As a by-product of such classification, we provide a mathematically rigorous proof that, at first order in perturbation theory, the renormalization group flow of the nonlinear Sigma model is the Ricci flow.

Keywords: locally covariant quantum field theory, algebraic quantum field theory, nonlinear Sigma models, Ricci flow.

MSC 2010: 81T20, 81T05

1 Introduction

In the realm of geometric analysis, there are several open avenues of research which have benefited from results and models arising from classical and quantum field theory. One, if not the most prominent example is the Ricci flow, which has come to the fore in the past few years thanks to Perelman proof [Per02, Per03] of the geometrization programme for three-dimensional manifolds due to Thurston [Thu97]. Introduced in the mathematical literature in the early eighties by Hamilton [Ham82], the Ricci flow has appeared independently in the context of quantum field theory mainly thanks to the early works of Friedan [Fri80, Fri85] within the analysis of nonlinear Sigma models over two-dimensional Riemannian manifolds as source and with a Riemannian manifold of arbitrary dimension as target space. Despite the apparent distance between the two settings in which the Ricci flow first appeared, the mutual influences have been manifold and the field theoretical approach has been of inspiration for some of the ground breaking results of Perelman and for several analyses of the structural properties of such flow, see *e.g.* [Car14].

From the viewpoint of nonlinear Sigma models, the Ricci flow arises when, by considering a perturbative approach to the underlying Euclidean field theory, one studies at first order the renormalization group flow, see *e.g.* [Car10, CM17] and also [Gaw99]. This is the main aspect on which we wish to focus in this paper and in particular we shall address the criticism towards such derivation of Ricci flow, which is often labelled not to be fully mathematically rigorous.

In order to tackle this problem, we shall work within the framework of algebraic quantum field theory, a mathematically rigorous approach which was first formulated by Haag and Kastler [HK63]. Especially in the

past few years it has been employed successfully to unveil and to characterize several structural properties of free and interacting quantum field theories, ranging from the formulation of the principle of general local covariance to a mathematically rigorous analysis of regularization and renormalization – see the recent reviews [BDFY15, Rej16]. Yet, the vast majority of the efforts have been addressed towards formulating and understanding the algebraic approach for field theories living on an underlying Lorentzian spacetime and thus many results and constructions are tied to such class of backgrounds. For example the quantization of a classical free field theory, the construction of an algebra of Wick polynomials or accounting interactions via a perturbative approach (including the ensuing renormalization procedure) are nowadays fully understood. Nonetheless a close scrutiny of all results unveils clearly that they rely on key structures which are tied to Lorentzian metrics. Notable instances of this statement are the realization of the canonical commutation relations in terms of advanced and retarded fundamental solutions associated to normally hyperbolic partial differential operators or the construction of Wick ordered quantum fields as a by-product of the existence of Hadamard states, see [BDFY15].

Yet there is no a priori obstruction to work within the algebraic framework while considering classical or quantum field theories which are living over a Riemannian manifold. Starting from the early seventies a few works in this direction have appeared in the literature [OS73, OS75] and, despite most of the efforts went towards formulating algebraic quantum field theory on Lorentzian backgrounds, it is clear that most of the ideas, of the technique and of the structural aspects admit a well-defined Euclidean counterpart. A notable example in this direction are the recent works by Keller on the formulation of Euclidean Epstein-Glaser renormalization [Kel09, Kel10], see also [Sch98, Wa79].

Hence, motivated and inspired by these works, we decide to opt for a bottom-up approach towards the analysis of the nonlinear Sigma models which are at the heart of the Ricci flow. At a classical level such models are realized considering as kinematic configurations arbitrary smooth maps ψ from a two-dimensional Riemannian manifold (Σ, γ) into a target Riemannian background (M, g) of arbitrary dimension. The dynamics is ruled by the stationary points of the so-called harmonic Lagrangian \mathcal{L}_H and considering its linearisation around an arbitrary configuration, we obtain a free field theory, which up to a source term, is governed by an elliptic operator E . This model is closely connected to string theory and its quantization from the algebraic viewpoint has been considered in [BRZ14].

As a starting point, we address the question of studying the quantization of the ensuing linearised theory. To this end, first we define the notion of an Euclidean locally covariant quantum field theory translating to a Riemannian framework the renown principle of general local covariance, formulated in a Lorentzian setting in [BFV03]. This leads us naturally to identifying an Euclidean quantum field theory as a functor between a suitable category of background data into that of unital $*$ -algebras which satisfies in addition a scaling hypothesis. Without entering into the technical details in the introduction, this requirement entails, that there exists an action of $\mathbb{R}_+ := (0, \infty)$ on the background data which, in turn, yields a corresponding isomorphism between the algebras of observables associated to each of the backgrounds constructed via such action. In order for the model of our interest to fit in this scheme, we need as second step to show how to associate a $*$ -algebra of locally covariant observables to the linear theory ruled by the operator E .

To this end, we work with the functional formalism, which has been successfully applied to the Lorentzian setting, see for example [BDF09, Rej16]. On the one hand we cannot follow slavishly these references, since we need to cope with several features which are tied to the Riemannian setting. On the other hand this approach has the net advantage that it allows to individuate in the ensuing algebra of observables a class of elements which could naturally be interpreted as Wick ordered powers of the underlying quantum field.

This observation leads to the second part of our paper in which we address the question of giving an abstract definition of Wick powers of an associated quantum field. This brings us to two relevant results. On the one hand, we characterize and classify the freedom which exists in constructing such polynomials, starting from the given definition. In tackling this problem, we extend to our framework the recent work of Khavkine, Melati and Moretti [KMM17], who have completely answered this question for vector valued Bosonic linear field theories, extending the seminal works of [HW01, HW02, HW05]. On the other hand we show that, since per assumption there exists an action of \mathbb{R}_+ on the background data, this induces a one-parameter family of Wick ordered powers of the underlying quantum field. It is worth mentioning that the formalism used in our work is strongly connected to the one of the recent monograph [Her19]. Yet, in this reference, perturbative quantum field theory is presented from a very general viewpoint highlighting the minimal set of

underlying assumptions which allow for the whole procedure to work. On the contrary we focus on a very specific scenario and in particular we follow a different approach to discuss the renormalization ambiguities of the underlying model.

Subsequently, following the standard rationale used in perturbative algebraic quantum field theory [HW03], it turns out that, to each coherent assignments of a one-parameter family of Wick polynomials, one associates a corresponding family of locally covariant Lagrangian densities, where parametric dependence is codified in the coupling constant, namely the metric of the target Riemannian manifold. As a last step, using the classification result of the ambiguities between two coherent assignment of Wick polynomials, we prove that such one-parameter family of metrics obeys the Ricci flow equation.

The paper is organized as follows: In the next subsection we fix the notation and we introduce all the geometric and analytic building blocks necessary for our investigation, in particular the nonlinear sigma models, we are interested in and their linearisation. Section 2 is devoted entirely to defining and to studying locally covariant Euclidean field theory. In particular in Subsection 2.1, first we introduce all the categories that we will be using and subsequently we give the formal definition of an Euclidean locally covariant theory, emphasizing in particular the so-called scaling hypothesis. In Subsection 2.2 we show instead that the linearisation on the nonlinear Sigma models, that we consider, fits in the framework formulated in Subsection 2.1. In Subsection 2.3 we still focus on the model of our interest, defining the notion of locally covariant observables and studying their behaviour under the action of the scaling which is intrinsic in the definition of locally covariant Euclidean field theory. Finally in Subsection 2.4, we generalize [KMM17] to our setting defining first what is a family of Wick powers and then classifying the ambiguities existing in giving such definition. In Section 3 we apply the results and the construction of Section 2 to give a rigorous derivation of the Ricci flow from the perturbative renormalization group of the nonlinear Sigma models introduced in Subsection 1.1.

1.1 General Setting

The goal of this section is to fix the notation and to introduce all the geometric and analytic building blocks necessary for our investigation.

To start with, we consider two connected, oriented, Riemannian manifolds (Σ, γ) and (M, g) where $\dim M = D$, while $\dim \Sigma = D'$. Later we will consider only the case $D' = 2$. In order to avoid confusion when dealing with the geometric structures associated to these backgrounds, we shall employ the convention that Greek (*resp.* Latin) indices are associated to quantities related to Σ (*resp.* to M). In addition, we denote with ∇^Σ, ∇^M the Levi-Civita connections defined respectively on $T\Sigma, TM$.

Remark 1: For future convenience we recall the definition of pull-back bundle and of pull-back connection, *cf.* [Hus94]. Let $B \xrightarrow{\pi_B} M$ be a vector bundle over M – typically $B = TM$ or $B = T^*M$ – and let $\psi \in C^\infty(\Sigma; M)$. The pull-back bundle ψ^*B is the vector bundle over Σ defined by

$$\psi^*B := \{(x, \xi) \in \Sigma \times B \mid \pi_B(\xi) := \psi(x)\} \quad \pi_{\psi^*B}(x, \xi) := x, \quad (1)$$

From the definition it follows that $\widehat{\psi}: \psi^*B \ni (x, \xi) \mapsto \xi \in B|_{\psi(x)}$ is an injective morphism of vector bundles which lifts ψ , namely $\widehat{\psi} \circ \pi_B = \psi \circ \pi_{\psi^*B}$. This leads to an injective morphism of vector spaces $\psi^\sharp: \Gamma(B) \rightarrow \Gamma(\psi^*B)$ defined by $\psi^\sharp s(x) := (\widehat{\psi}^{-1} \circ s \circ \psi)(x) = (x, s \circ \psi(x))$ for all $s \in \Gamma(B)$. Considering the Levi-Civita connection ∇^B on B the pull-back connection ∇^ψ on ψ^*B is defined as follows [MS76, App. C]. The push-forward $d\psi: T\Sigma \rightarrow TM$ induces an injective homomorphism of sections $d\psi^*: \Gamma(T^*M) \rightarrow \Gamma(T^*\Sigma)$. The pull-back connection $\nabla^\psi: \Gamma(B) \rightarrow \Gamma(T^*\Sigma \otimes B)$ is the unique one such that $\nabla^\psi \circ \psi^\sharp = (d\psi^* \otimes \psi^\sharp) \circ \nabla^M$. Furthermore, per definition $\nabla^\psi \alpha := \nabla^\Sigma \alpha$ for all $\alpha \in \Gamma(T^*\Sigma)$.

On top of Σ , we consider kinematic configurations $\psi \in C^\infty(\Sigma; M)$ while dynamics is ruled by the stationary points of the so-called harmonic Lagrangian density \mathcal{L}_H :

$$\mathcal{L}_H[\psi, \gamma, g] := \text{tr}_\gamma(\psi^*g)\mu_\gamma \stackrel{\text{loc.}}{=} g_{ab}\gamma^{\alpha\beta}(d\psi)_\alpha^a(d\psi)_\beta^b\mu_\gamma, \quad (2)$$

where μ_γ is the volume form induced by γ while $d\psi: T\Sigma \rightarrow TM$ is the push-forward along ψ . In this paper we shall not work directly with (2), rather we consider an expansion of \mathcal{L}_H up to the second order with

respect to an arbitrary, but fixed background kinematic configuration ψ . More precisely, for $\varphi \in \Gamma(\psi^*TM)$ let $\psi_\nu: \Sigma \rightarrow M$ be $\psi_\nu(x) := \exp_{\psi(x)}[\nu\varphi(x)]$, where $\exp_{\psi(x)}: T_{\psi(x)}M \rightarrow M$ is the exponential map at $\psi(x) \in M$, while $\nu \in I \subset \mathbb{R}$, where I is an open subset of \mathbb{R} which includes the origin – cf. remark 1. The Taylor expansion of $\mathcal{L}_H(\psi_\nu, \gamma, g)$ centred at $\nu = 0$ yields

$$\mathcal{L}(\psi_\nu, \gamma, g; \varphi) = \mathcal{L}_H(\psi, \gamma, g) + \left[\nu g(\varphi, Q(\psi)) - \frac{\nu^2}{2} \langle \varphi, E\varphi \rangle + \frac{\nu^2}{2} h(\text{Riem}(\varphi, d\psi)\varphi, d\psi) \right] \mu_\gamma + O(\nu^3), \quad (3)$$

where $\psi_0(x) \equiv \psi(x)$, $h|_x \stackrel{\text{loc.}}{=} g_{ab}|_{\psi(x)} \gamma^{\alpha\beta}|_x \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta} \otimes dy^a \otimes dy^b$, while the operator E is defined by

$$E: \Gamma(\psi^*TM) \rightarrow \Gamma(\psi^*T^*M) \quad E\varphi := \text{tr}_h(\nabla^\psi \circ \nabla^\psi \varphi), \quad (4)$$

where ∇^ψ stands for the pull-back connection associated with ∇^M, ∇^Σ on the pull-back bundle ψ^*TM over Σ – cf. remark 1. Finally the operator $Q: C^\infty(\Sigma, M) \rightarrow \Gamma(\psi^*T^*M)$ is a differential operator whose explicit form is inessential for what follows – see [Car10] for details. For our purposes the following property is of paramount relevance:

Lemma 2: The operator E is elliptic and its principal symbol coincides with that of $\widehat{E}: \Gamma(\psi^*TM) \rightarrow \Gamma(\psi^*T^*M)$, locally defined by $(\widehat{E}\varphi)_a(x) := g_{ab}(\psi(x))\Delta_\gamma(\varphi^b(x))$ for $\varphi \in \Gamma(\psi^*TM)$, where Δ_γ is the Laplace-Beltrami operator built out of γ .

Proof. For any point $x \in \Sigma$ and for any local trivialization of ψ^*TM centred at x , (4) reads

$$(E\varphi)_b \stackrel{\text{loc.}}{=} g_{ab}\Delta_\gamma(\varphi^a) + g_{ab}\Delta_\gamma(\psi^\ell)\Gamma_{\ell c}^a[g]\varphi^c + g_{ab}\gamma^{\alpha\beta} \left[(d\psi)_\alpha^\ell \frac{\partial}{\partial x^\beta} (\Gamma_{\ell c}^a[g]\varphi^c) + (d\psi)_\alpha^\ell \Gamma_{\ell c}^a[g] \frac{\partial \varphi^c}{\partial x^\beta} \right] + g_{ab}\gamma^{\alpha\beta} (d\psi)_\beta^\ell (d\psi)_\alpha^p \Gamma_{\ell c}^a[g] \Gamma_{pd}^c[g] \varphi^d.$$

where $\varphi \in \Gamma(\psi^*TM)$. Considering the definition of principal symbol,

$$\sigma_E(d\zeta)\varphi := \lim_{z \rightarrow +\infty} z^{-2} e^{-z\zeta} E(e^{z\zeta}\varphi), \quad \forall \zeta \in C^\infty(\Sigma), \varphi \in \Gamma(\psi^*TM), \quad (5)$$

the sought statement follows. \square

Remark 3: From now on the main object of our interest will be the expansion in (3) and therefore ψ, γ, g will be considered as parameters/background structures of the theory, whereas the role of kinematic configuration will be taken by $\varphi \in \Gamma(\psi^*TM)$. Observe that, in (3), $\langle \varphi, E\varphi \rangle \mu_\gamma$, plays the rôle of a kinetic term ruled by E , the elliptic operator (4) associated to the Lagrangian \mathcal{L} .

To conclude the section, we focus on the behaviour of the background structures under scaling and in particular we are interested in the *engineer dimension* of (ψ, φ, g) . The latter can be computed as follows: Consider the transformation

$$\gamma \rightarrow \gamma_\lambda, \quad (\gamma_\lambda)_{\alpha\beta} := \lambda^{-2} \gamma_{\alpha\beta}, \quad \lambda > 0. \quad (6)$$

The engineer dimensions $d_\psi, d_\varphi, d_g \in \mathbb{R}$ respectively of ψ, φ, g , appearing in (3), are the unique real numbers such that, if

$$\psi \rightarrow \psi_\lambda := \lambda^{d_\psi} \psi, \quad \varphi \rightarrow \varphi_\lambda := \lambda^{d_\varphi} \varphi, \quad g \rightarrow g_\lambda, \quad (g_\lambda)_{ab} := \lambda^{d_g} g_{ab}, \quad (7)$$

then the corresponding scaled Lagrangean density $\mathcal{L}(\psi_\lambda, \gamma_\lambda, g_\lambda; \varphi_\lambda)$ remains invariant, that is

$$\mathcal{L}(\psi_\lambda, \gamma_\lambda, g_\lambda; \varphi_\lambda) = \mathcal{L}(\psi, \gamma, g; \varphi).$$

Considering (3) and that $\mu_{\gamma_\lambda} = \lambda^{-D'} \mu_\gamma$, a straightforward computation leads to

$$d_\psi = d_\varphi = 0, \quad d_g = D' - 2. \quad (8)$$

2 Locally Covariant Euclidean Field Theories

In this Section, our goal is twofold. On the one hand we want to introduce locally covariant Euclidean quantum field theories using the language of categories and of functionals as first introduced in [BFV03] and [BDF09] respectively. Since, contrary to these seminal papers, we will also be interested in vector valued fields defined over Riemannian manifolds, we will also benefit greatly from [Kel09, Kel10] and from the recent works [KM16, KMM17]. At the same time we want to reinterpret and to analyse the model introduced in Section 1.1 within this more general conceptual framework.

2.1 General Local Covariance

Following [BFV03], the starting point of the principle of general local covariance consists of identifying a suitable set of categories which encode all necessary information of the underlying model. For the scopes of this paper the necessary ingredients are:

1. $\text{Bkg}_{D',D}$, the category of background geometries, such that

- $\text{Obj}(\text{Bkg}_{D',D})$ are pairs $(N; b)$ where $N \equiv (\Sigma, M)$ identifies a pair of smooth, connected, oriented manifolds, with $\dim \Sigma = D'$ and $\dim M = D$, while $b \equiv (\psi, \gamma, g)$ codifies the background data, that is $\psi \in C^\infty(\Sigma; M)$ while γ and g are smooth Riemannian metrics respectively on Σ and M .
- $\text{Ar}(\text{Bkg}_{D',D})$ are pairs (τ, t) where $\tau : \Sigma \rightarrow \tilde{\Sigma}$ and $t : M \rightarrow \tilde{M}$ are orientation preserving, isometric embeddings subject to the compatibility condition

$$\tilde{\psi} \circ \tau = t \circ \psi, \quad (9)$$

where $\tilde{\psi} \in C^\infty(\tilde{\Sigma}; \tilde{M})$. If $\dim \Sigma = D' = 2$, with a slight abuse of notation we write $\text{Bkg} \equiv \text{Bkg}_{2,D}$ as D plays no relevant role in our analysis

2. Alg is the category whose objects are unital $*$ -algebras, while the arrows are unit preserving, injective $*$ -homomorphisms.
3. Vec is the category whose objects are real vector spaces while the arrows are injective linear morphisms.

Remark 4: Observe that, in comparison with [KMM17, Def. 3.4], we adopt a slightly different definition of category of background geometries, adapted to the framework we consider. Nevertheless $\text{Bkg}_{D',D}$ still enjoys the notable property of being *dimensionful* in the sense of [KMM17]. In other words there exists an action of $\mathbb{R}_+ := (0, \infty)$ on $\text{Obj}(\text{Bkg}_{D',D})$

$$(N; b) = (\Sigma, M, \psi, \gamma, g) \rightarrow (N; b_\lambda) := (\Sigma, M; \psi_\lambda, \gamma_\lambda, g_\lambda) := (\Sigma, M; \psi, \lambda^{-2}\gamma, \lambda^{D'-2}g), \quad (10)$$

which is preserved by the arrows of $\text{Bkg}_{D',D}$ and whose definition is tied to the engineer dimension of ψ, g as per (8).

Definition 5: An *Euclidean locally covariant theory* is a covariant functor $\mathcal{A} : \text{Bkg}_{D',D} \rightarrow \text{Alg}$ which satisfies the scaling hypothesis: For all $\lambda > 0$, let $\mathcal{A}_\lambda : \text{Bkg}_{D,D'} \rightarrow \text{Alg}$ be the covariant functor $\mathcal{A} \circ \rho_\lambda$, where $\rho_\lambda : \text{Bkg}_{D,D'} \rightarrow \text{Bkg}_{D,D'}$ is the functor defined as the identity on morphisms while on objects it acts as per

(10). Then, for all $\lambda, \mu, \sigma > 0$, there exists a natural isomorphism $\mathcal{A}_\mu \xrightarrow{\varsigma_{\lambda,\mu}} \mathcal{A}_\lambda$ (with inverse $\mathcal{A}_\lambda \xrightarrow{\varsigma_{\lambda,\mu}^{-1}} \mathcal{A}_\mu$) such that

$$\varsigma_{\lambda,\mu}[N; b] = \varsigma_{\lambda,\sigma}[N; b] \circ \varsigma_{\sigma,\mu}[N; b_\lambda], \quad \varsigma_{\lambda,\lambda}[N; b] = \text{Id}_{\mathcal{A}[N; b]}, \quad (11)$$

for all $(N; b) \in \text{Obj}(\text{Bkg}_{D',D})$.

Remark 6: For notational convenience we shall adopt the convention

$$\varsigma_\lambda[N; b] \doteq \varsigma_{1,\lambda}[N; b]. \quad \forall [N; b] \in \text{Obj}(\text{Bkg})$$

Remark 7: The role of ς_λ is to ensure that the scaling $(N; b) \rightarrow (N; b_\lambda)$ is consistently implemented in the theory described by the functor \mathcal{A} . In turn \mathcal{A}_λ can be interpreted as the functor describing the theory \mathcal{A} at the scale λ , while the map $\varsigma_\lambda[N; b]: \mathcal{A}_\lambda[N; b] \rightarrow \mathcal{A}[N; b]$ codifies the rules needed to transform the same theory between different scales. This interpretation will have a significant rôle in our main result, see Theorem 47.

2.2 Linearised nonlinear Sigma models as a locally covariant theory

In this subsection we will show how to reformulate the model in Section 1.1 as an Euclidean locally covariant theory as per Definition 5. Therefore, henceforth $\dim \Sigma = 2$ and we will only be interested in the category of background geometries $\text{Bkg} \equiv \text{Bkg}_{2,D}$.

As starting point we focus on an arbitrary, but fixed, background geometry $(N; b) \in \text{Obj}(\text{Bkg})$ showing how to build the algebra $\mathcal{A}[N; b]$ associated with the Lagrangian (3), reformulating the whole construction in terms of categories only at a later stage.

Let thus $(N; b) = (\Sigma, M; \psi, \gamma, g) \in \text{Obj}(\text{Bkg})$ be a background geometry and let $E: \Gamma(\psi^*TM) \rightarrow \Gamma(\psi^*T^*M)$ be the elliptic differential operator (4). Since E is elliptic as per Lemma 2, it admits a parametrix $P: \Gamma_c(\psi^*T^*M) \rightarrow \Gamma(\psi^*TM)$, *c.f.* [Wel08, Th. 4.4], unique up to smoothing operators such that

$$PE - \text{Id}_{\Gamma_c(\psi^*T^*M)} \in \Gamma(\psi^*TM \boxtimes \psi^*T^*M), \quad EP - \text{Id}_{\Gamma_c(\psi^*T^*M)} \in \Gamma(\psi^*TM \boxtimes \psi^*T^*M). \quad (12)$$

Remark 8: Throughout this paper we shall employ the following notation. Given a vector bundle $B \xrightarrow{\pi_B} M$ and $k \in \mathbb{N}$ we denote with $S^{\otimes k}B \xrightarrow{\pi_{S^{\otimes k}B}} M$ the k -th symmetric tensor product of B . With $B^{\boxtimes n} \xrightarrow{\pi_{B^{\boxtimes n}}} M^n$ we identify the k -th exterior tensor product of B , that is, the vector bundle over M^k with fibre $\pi_{B^{\boxtimes n}}^{-1}(x_1, \dots, x_n) = \otimes_{\ell=1}^n \pi_B^{-1}(x_\ell)$. For a given $s \in \Gamma(B^{\boxtimes n})$ we denote with $[s] \in \Gamma(B^{\otimes n})$ the section obtained by considering the coinciding point limit of s , that is, the section obtained by pull-back of s along the inclusion of the total diagonal $D_n \rightarrow M^n$ where $D_n := \{(x_1, \dots, x_n) \in M^n \mid x_1 = \dots = x_n\}$. The notation $\text{SG}(B^{\boxtimes n})$ always refers to smooth sections over $B^{\boxtimes n}$ which are symmetrized with respect to the base points. Notice that if $s \in \text{SG}(B^{\boxtimes n})$ then $[s] \in \Gamma(S^{\otimes n}B)$. In particular, for $s \in \Gamma(B)$ we denote $s^{[\otimes]n} := [s^{\otimes n}] \in \Gamma(S^{\otimes n}B)$ where $s^{\otimes n} \in \text{SG}(B^{\boxtimes n})$.

Notice that the properties of the parametrices of being symmetric will play a distinguished rôle in the construction of a commutative algebra of observables – *cfr.* Definition 20, in sharp contrast with the outcome of the same procedure in a Lorentzian setting where the dynamics is ruled by symmetric hyperbolic partial differential operators.

Remark 9: Since E is formally self-adjoint, it follows that the formal adjoint of any parametrix P is again a parametrix for E . We can therefore consider formally self-adjoint parametrices, whose space will be denoted with $\text{Par}[N; b]$. Notice that, because of equation (12), $\text{Par}[N; b]$ is an affine space modelled over $\text{SG}(\psi^*TM^{\boxtimes 2})$.

Remark 10: The parametrix P admits locally a Hadamard representation which is constructed in detail in Appendix A – *cfr.* proposition 52. Here we recall the final result: Let (x, x') be a pair of points lying in a suitably constructed convex, geodesic neighbourhood $\mathcal{O} \hookrightarrow \Sigma$ centred at x . Then the integral kernel of P reads locally

$$P^{ab}(x, x') = H^{ab}(x, x') + W_P^{ab}(x, x'), \quad H^{ab}(x, x') = V^{ab} \log \frac{\sigma(x, x')}{\ell_H^2}, \quad (13)$$

where $\sigma(x, x')$ is the halved squared geodesic distance between x and x' , $V \in \text{SG}(\psi^*T\mathcal{O}^{\boxtimes 2})$ is a suitable symmetric tensor, while H codifies the singular part of the parametrix and $W_P \in \text{SG}(\psi^*T\mathcal{O}^{\boxtimes 2})$. It is important to keep in mind that, although (13) is meaningful only locally, one can use $[W_P] \in \Gamma(S^{\otimes 2}\psi^*T\mathcal{O})$ together with a partition of unity argument in order to identify a globally defined $[W_P] \in \Gamma(S^{\otimes 2}\psi^*TM)$ – which does not depend on the chosen partition of unity – where the subscript is used to highlight the dependence on the choice of the parametrix P – *cfr.* remark 1. The Hadamard representation and $[W_P]$ will be particularly important in the following construction as well as in the definition of locally covariant Wick powers – *cfr.* Example 43.

In the following we will indicate with $P \in \text{Par}[N; b]$ both the linear operator $P: \Gamma(\psi^*T^*M) \rightarrow \Gamma(\psi^*TM)$ and its associated distribution $P \in \text{SG}_c(\psi^*T^*M^{\boxtimes 2})'$ [Hö03, Thm.8.2.12], the subscript c indicating that we consider distributions over compactly supported test-sections. Recall that with Γ_c we are implicitly assuming that the sections are symmetrized also with respect to the base points.

Now we consider an arbitrary but fixed $P \in \text{Par}[N; b]$, using it first to define a suitable unital $*$ -algebra associated to the system whose dynamics is ruled by the operator E . Secondly we show how to built an algebra which is independent from the chosen P .

Definition 11: We denote with $\mathcal{P}_{\text{loc}}[N; b]$ the complex vector space of functionals $F: \Gamma(\psi^*TM) \rightarrow \mathbb{C}$ spanned by monomial functionals

$$F_{\omega_k}(\varphi) := \int_{\Sigma} \langle \varphi^{\boxtimes k}, \omega_k \rangle, \quad \omega_k \in \Gamma_c(\wedge^{\text{top}}T^*\Sigma \otimes S^{\otimes k}\psi^*T^*M), \quad k \in \mathbb{N} \cup \{0\}, \quad (14)$$

where $\varphi^{\boxtimes k} \in \Gamma(S^{\otimes k}\psi^*TM)$ denotes the coinciding point limit of the symmetric tensor product $S^{\otimes k}\varphi \in \text{SG}(\psi^*TM^{\boxtimes k})$ – cf. remark 1. Moreover, $\Gamma_c(\wedge^{\text{top}}T^*\Sigma \otimes S^{\otimes k}\psi^*T^*M)$ denotes the compactly supported sections of the vector bundle $\wedge^{\text{top}}T^*\Sigma \otimes S^{\otimes k}\psi^*T^*M$ – here $\wedge^{\text{top}}T^*\Sigma$ denotes the bundle of densities on Σ – while $\langle \varphi^k, \omega_k \rangle$ denotes the pairing $\varphi^{a_1}(x) \dots \varphi^{a_k}(x)\omega_{a_1 \dots a_k}(x)$. We refer to $\mathcal{P}_{\text{loc}}[N; b]$ as to the space of local polynomial functionals (with no derivatives of the configurations). For future convenience we set $\mathcal{P}[N; b] := \bigoplus_{n \geq 0} \mathcal{P}_{\text{loc}}^{\otimes n}[N; b]$ with $\mathcal{P}_{\text{loc}}^{\otimes 0} \equiv \mathbb{C}$.

Remark 12: Notice that any $F \in \mathcal{P}_{\text{loc}}[N; b]$ enjoys the following remarkable properties which will be exploited in the forthcoming discussion:

1. F is smooth, namely, for all $\varphi, \varphi_1, \dots, \varphi_n \in \Gamma(\psi^*TM)$, $n \geq 1$, the n -th functional derivative $F^{(n)}[\varphi]$, defined as

$$\langle F^{(n)}[\varphi], \varphi_1 \otimes \dots \otimes \varphi_n \rangle := \frac{\partial^n}{\partial s_1 \dots \partial s_n} F \left(\varphi + \sum_{i=1}^n s_i \varphi_i \right) \Big|_{s_1 = \dots = s_n = 0}, \quad (15)$$

identifies a symmetric, compactly supported distribution $F^{(n)}[\varphi] \in \text{SG}(\psi^*TM^{\boxtimes n})'$.

2. F is compactly supported, that is, $\overline{\bigcup_{\varphi} \text{supp}(F^{(1)}[\varphi])}$ is compact;
3. F is local, because for all $\varphi \in \Gamma(\psi^*TM)$, the n -th functional derivative $F^{(n)}[\varphi]$ is supported on the thin diagonal of $\Sigma^n = \underbrace{\Sigma \times \dots \times \Sigma}_n$, that is $\text{supp}(F^{(n)}[\varphi]) \subset D_n := \{(x_1, \dots, x_n) \in \Sigma^n \mid x_1 = \dots = x_n\}$. Moreover $\text{WF}(F^{(n)}[\varphi])$ is transversal $T^*D_n =$, where $\text{WF}(F^{(n)}[\varphi])$ stands for the wave front set of $F^{(n)}[\varphi]$, [Hö03, Def. 8.1.2].

For simplicity – cf. the proof of Proposition 13 and Remark 15 – our definition excludes local polynomial functionals which contain derivatives of the configuration φ ; the latter class will play no rôle in what follows.

Proposition 13: The vector space $\mathcal{P}[N; b]$ consisting of smooth, local, polynomial functionals is an associative and commutative $*$ -algebra if endowed with the product

$$(F \cdot_P G)(\varphi) = (\mathcal{M} \circ \exp(\Upsilon_P)(F \otimes G))(\varphi) := F(\varphi)G(\varphi) + \sum_{n \geq 1} \frac{1}{n!} \langle F^{(n)}[\varphi], P^{\otimes n}G^{(n)}[\varphi] \rangle, \quad (16)$$

where $P^{\otimes n}G^{(n)}[\varphi] \in \text{SG}_c(\psi^*T^*M^{\boxtimes n})'$ is the extension of $\underbrace{P \otimes \dots \otimes P}_n$ to $G^{(n)}[\varphi]$ according to [Hö03, Thm. 8.2.13], while

$$\mathcal{M}(F \otimes G)(\varphi) = F(\varphi)G(\varphi),$$

is the pointwise product. Here Υ_P is such that, for all $\varphi_1, \varphi_2 \in \Gamma(\psi^*TM)$,

$$\Upsilon_P(F \otimes G)(\varphi_1, \varphi_2) := \langle F^{(1)}[\varphi_1], PG^{(1)}[\varphi_2] \rangle.$$

The $*$ -involution is completely characterized on $\mathcal{P}[N; b]$ by requiring $F^*(\varphi) := \overline{F(\varphi)}$. We denote with $\mathcal{F}_P[N; b]$ the $*$ -algebra $(\mathcal{P}[N; b], \cdot_P, *)$.

Proof. The first step consists of observing that (16) is well-defined. Convergence of the sum is guaranteed since, the functionals being polynomial, only a finite set of terms contributes. The only problem might arise from $\langle F^{(n)}[\varphi], P^{\otimes n} G^{(n)}[\varphi] \rangle$. Yet, in the case at hand, the singular behaviour of $F^{(n)}[\varphi], G^{(n)}[\varphi]$ is known – cf. remark 12 – while that of P is that of the Hadamard parametrix H whose local behaviour is logarithmic in the halved squared geodesic distance σ , see equations (67). Hence, using that the scaling degree of H is smaller than 2 and that F, G do not contain derivatives of φ , we can use [BF99, Thm. 5.2] to infer that the contraction of $P^{\otimes n} \in \text{SG}_c(\psi^* T^* M^{\boxtimes 2n})'$ with $F^{(n)} \otimes G^{(n)} \in \text{SG}(\psi^* T^* M^{\boxtimes 2n})'$ yields a well-defined distribution with compact support in $C^\infty(\Sigma)'$, which can thus be integrated against the constant function. To conclude the proof, we observe that associativity is guaranteed per construction while commutativity is a by-product of the fact that the parametrix P is symmetric – cf. remark 9. \square

Remark 14: Equation (16) is well-defined as a consequence of [Hö03, Thm. 8.2.13] and of the extension of the parametrix P as a distribution on $\Gamma_c(\wedge^{\text{top}} T^* \Sigma \otimes \psi^* T^* M)$. This is defined setting $P\alpha := \mu_\gamma P(*_\gamma^{-1} \alpha)$ for $\alpha \in \Gamma_c(\wedge^{\text{top}} T^* \Sigma \otimes \psi^* T^* M)$ and

$$\langle \alpha, P\beta \rangle := \int_\Sigma \langle \alpha, *_\gamma P\alpha \rangle, \quad (17)$$

for $\alpha, \beta \in \Gamma_c(\wedge^{\text{top}} T^* \Sigma \otimes \psi^* T^* M)$ while $*_\gamma$ denotes the Hodge operator $*_\gamma: \Gamma(\wedge^\bullet T^* \Sigma) \rightarrow \Gamma(\wedge^{\dim \Sigma - \bullet} T^* \Sigma)$.

Remark 15: Notice that the previous Proposition strongly relies on the assumption $\dim \Sigma = 2$ as well as on Definition 11 of smooth polynomial local functionals without derivatives. As a matter of fact, for higher dimension or considering polynomial functionals including derivatives of the configuration φ , the contraction between $P^{\otimes n}$ and $F^{(n)} \otimes G^{(n)}$ would not be uniquely defined. In this case, different extensions exist as one can infer following [BF99] and thus one has to cope with *families* of well-defined products \cdot_P . An application of these ideas has already been studied in [FR12, FR13] in the context of gauge theories. The discussion of such scenario is behind the scopes of this paper and it is postponed to a future work [DDR19], see also [Kel09, Kel10] and [Da14].

Remark 16: It is worth observing that the algebra of local polynomial functionals $\mathcal{F}_P[N; b]$ already includes elements which can be interpreted as Wick powers of a field φ . As a concrete example, thought especially for a reader who is more familiar with the standard point splitting procedure, consider the functional $F_\omega(\varphi) = \int_\Sigma d\mu_\gamma \varphi^a(x) \varphi^b(x) \omega_{ab}(x)$ with $\omega \in \Gamma_c(S^{\otimes 2} \psi^* T^* M)$. One can in turn pick any sequence $(g_n)_a(x) (f_n)_b(x')$ with $f_n, g_n \in \Gamma_c(\psi^* T^* M)$ for all $n \in \mathbb{N}$ such that, in the weak topology, $\lim_{n \rightarrow \infty} (f_n)_a(x) (g_n)_b(x') = \omega_{ab}(x) \delta(x, x')$. As a consequence one can rewrite

$$F_\omega(\varphi) = \lim_{n \rightarrow \infty} F'_{f_n}(\varphi) F'_{g_n}(\varphi) = \lim_{n \rightarrow \infty} ((F'_{f_n} \cdot_P F'_{g_n})(\varphi) - P(f_n, g_n)),$$

where $F'_{f_n}(\varphi) = \int_\Sigma d\mu_\gamma f_n(x) \varphi(x)$. The right hand side of this last chain of equalities translates in the functional language the standard expression yielding the definition of a Wick ordered, squared field via a point splitting procedure.

Notice that $(N; b) \rightarrow \mathcal{F}_P[N; b]$ does *not* identify an Euclidean locally covariant theory as per Definition 5 due to the choice of an arbitrary $P \in \text{Par}[N; b]$. Our next goal is to overcome this hurdle and the first step in this direction consists of showing that, for a fixed object in Bkg , all choices of P are equivalent. The following Proposition generalizes to the case in hand a well-known property, see e.g. [HW01, Lemma 2.1] for the counterpart in a Lorentzian setting. Since the proof is identical, mutatis mutandis to that of [Lin13, Prop. 1.4.7], [Kel09, Prop. II.4], we omit it.

Proposition 17: Let $(N; b) \in \text{Obj}(\text{Bkg})$ be arbitrary but fixed and let $P, \tilde{P} \in \text{Par}[N; b]$. Then the algebras $\mathcal{F}_P[N; b]$ and $\mathcal{F}_{\tilde{P}}[N; b]$ are $*$ -isomorphic, the $*$ -isomorphism being realized by

$$\alpha_{\tilde{P}}: \mathcal{F}_{\tilde{P}}[N; b] \rightarrow \mathcal{F}_P[N; b], \quad (\alpha_{\tilde{P}} F)(\varphi) := \left[\exp[\Upsilon_{P-\tilde{P}}] F \right](\varphi), \quad (18)$$

where

$$\left[\exp[\Upsilon_{P-\tilde{P}}] F \right](\varphi) = \sum_{n=0}^{\infty} \frac{1}{2^{2n} n!} \langle (P - \tilde{P})^{\otimes n}, F^{(2n)}[\varphi] \rangle \quad (19)$$

and where $\Upsilon_{P-\tilde{P}}$ is such that

$$(\Upsilon_{P-\tilde{P}}F)(\varphi) := \frac{1}{2}\langle P - \tilde{P}, F^{(2)}[\varphi] \rangle.$$

In view of this last Proposition we can recollect all $*$ -algebras $\mathcal{F}_P[N; b]$ in a single object:

Definition 18: We call $\mathcal{E}[N; b]$ the bundle

$$\mathcal{E}[N; b] := \bigcup_P \mathcal{F}_P[N; b], \quad (20)$$

with base space $\text{Par}[N; b]$ and projection map $\pi_{\mathcal{E}[N; b]}(F_P) := P$.

Remark 19: Notice that the action $P \rightarrow P + W$ of $\text{S}\Gamma(\psi^*TM^{\boxtimes 2})$ on $\text{Par}[N; b]$ can be lifted to $\mathcal{E}[N; b]$ via the $*$ -isomorphism (18):

$$\alpha_W(F_P) := \alpha_{P+W}^P F_P, \quad \forall F_P \in \mathcal{E}[N; b]. \quad (21)$$

Definition 20: Let $\Gamma_{\text{eq}}(\mathcal{E}[N; b])$ be the the complex vector space of equivariant sections on $\mathcal{E}[N; b]$

$$\Gamma_{\text{eq}}(\mathcal{E}[N; b]) := \{F \in \Gamma(\mathcal{E}[N; b]) \mid F(P) = \alpha_P^{\tilde{P}} F(\tilde{P}) \quad \forall P, \tilde{P} \in \text{Par}[N; b]\}. \quad (22)$$

We denote with $\mathcal{A}[N; b] \equiv (\Gamma_{\text{eq}}(\mathcal{E}[N; b]), \cdot, *)$ the unital $*$ -algebra with the pointwise product \cdot_P as in (16) and with the fiberwise involution

$$(F \cdot G)(P) := F(P) \cdot_P G(P), \quad F^*(P) := F(P)^*, \quad (23)$$

for all $F, G \in \Gamma_{\text{eq}}(\mathcal{E}[N; b])$.

We can now prove that $\mathcal{A}[N; b]$ is the sought algebra.

Remark 21: The algebra $\mathcal{A}[N; b]$ can also be read as a concrete realization of the unique (up to $*$ -isomorphism) $*$ -algebra $\mathfrak{A}[N; b]$ for which there exists a family of $*$ -isomorphisms $\alpha_P: \mathfrak{A}[N; b] \rightarrow \mathcal{F}_P[N; b]$ for all $P \in \text{Par}[N; b]$ such that, $\alpha_P = \alpha_Q^P \circ \alpha_Q$ for all $P, Q \in \text{Par}[N; b]$. In addition we observe that the concrete algebras that we have constructed do not carry any topology. Following [BDLR18] one can bypass this limitation. Yet, working with topological $*$ -algebras would not change significantly the properties and the constructions in this paper. On the contrary it would play a key role whenever one looks for algebraic states on $\mathcal{A}[N; b]$ and for the associated GNS representation. Since this issue goes well beyond the scope of this work, we shall not further comment about it.

Remark 22: In the following we need to specify a few additional functor. We call

$$\begin{aligned} \Gamma: \text{Obj}(\text{Bkg}) &\rightarrow \text{Obj}(\text{Vec}) & \Gamma[N; b] &:= \Gamma(\psi^*TM), \\ \Gamma_c: \text{Obj}(\text{Bkg}) &\rightarrow \text{Obj}(\text{Vec}) & \Gamma_c[N; b] &:= \Gamma_c(\psi^*T^*M), \\ \text{Par}: \text{Obj}(\text{Bkg}) &\rightarrow \text{Obj}(\text{Vec}) & [N; b] &\mapsto \text{Par}[N; b]. \end{aligned}$$

Let $(\tau, t) \in \text{Ar}(\text{Bkg})$ be an arrow from $(N; b)$ to $(\tilde{N}; \tilde{b})$, that is, $\tau: \Sigma \rightarrow \tilde{\Sigma}$ and $t: M \rightarrow \tilde{M}$ are isometric, orientation preserving embeddings such that $\tilde{\psi} \circ \tau = t \circ \psi$. The map τ can be lifted to an isomorphism of vector bundles $\hat{\tau}: \tau^*\tilde{\psi}^*T\tilde{M} \rightarrow \psi^*T^*M|_{\tau(\Sigma)}$ by setting $\hat{\tau}(x, \xi) := (\tau(x), \xi)$ – cf. Remark 1. In addition the compatibility condition (9) implies

$$\tau^*\tilde{\psi}^*T\tilde{M} = (\tilde{\psi} \circ \tau)^*T\tilde{M} = (t \circ \psi)^*T\tilde{M} = \psi^*t^*T\tilde{M} = dt \circ \hat{\psi}(\psi^*TM),$$

where $dt: TM \rightarrow T\tilde{M}$ is the push-forward along t , while $\hat{\psi}: \psi^*TM \rightarrow TM|_{\psi(\Sigma)}$ has been defined in Remark 1. The composition $\hat{\tau}_t := \hat{\tau} \circ dt \circ \hat{\psi}: \psi^*TM \rightarrow \tilde{\psi}^*T\tilde{M}|_{\tau(\Sigma)}$ is thus an injective morphism of vector bundles and the same applies to $\hat{\tau}_{t,c}: \psi^*T^*M \rightarrow \tilde{\psi}^*T^*\tilde{M}|_{\tau(\Sigma)}$. Hence, we can consider

$$\begin{aligned} \Gamma[\tau, t]: \Gamma[\tilde{N}; \tilde{b}] &\rightarrow \Gamma[N; b] & \Gamma[\tau, t]\tilde{\varphi} &:= \hat{\tau}_t^{-1} \circ \tilde{\varphi} \circ \tau, \\ \Gamma_c[\tau, t]: \Gamma_c[N; b] &\rightarrow \Gamma_c[\tilde{N}; \tilde{b}] & \Gamma_c[\tau, t]\omega &:= \hat{\tau}_{t,c} \circ \omega \circ \tau^{-1}, \\ \text{Par}[\tau, t]: \text{Par}[\tilde{N}; \tilde{b}] &\rightarrow \text{Par}[N; b] & \text{Par}[\tau, t]\tilde{P} &:= \Gamma[\tau, t] \circ \tilde{P} \circ \Gamma_c[\tau, t]. \end{aligned}$$

Notice that $\Gamma_c[\tau, t]\omega$ is well-defined on account of the support properties of ω , in particular $\Gamma_c(\tau, t)\omega|_x = 0$ if $x \notin \tau(\Sigma)$. In other words Γ, Par (*resp.* Γ_c) are contravariant (*resp.* covariant) functors from Bkg to Vec .

Proposition 23: For all $(N; b) \in \text{Obj}(\text{Bkg})$, let $\mathcal{A}: \text{Bkg} \rightarrow \text{Alg}$ be such that, for all $(N; b) \in \text{Obj}(\text{Bkg})$, $\mathcal{A}[N; b]$ is the unital $*$ -algebra as per Definition 20, while, for every $(\tau, t) \in \text{Arr}(\text{Bkg})$, $\mathcal{A}[\tau, t] \in \text{Ar}(\text{Alg})$ as

$$\mathcal{A}[\tau, t]: \mathcal{A}[N; b] \rightarrow \mathcal{A}[\tilde{N}; \tilde{b}] \quad \mathcal{A}[\tau, t]F := \Gamma[\tau, t]^\# \circ F \circ \text{Par}[\tau, t],$$

where

$$\Gamma[\tau, t]^\#: \mathcal{F}_{\text{Par}[\tau, t]\tilde{P}}[N; b] \rightarrow \mathcal{F}_{\tilde{P}}[\tilde{N}, \tilde{b}] \quad \Gamma[\tau, t]^\#F := F \circ \Gamma[\tau, t].$$

Then \mathcal{A} is a covariant functor.

Proof. It suffices to observe that \mathcal{A} is well-defined when acting on objects since $\mathcal{A}[N; b] = \Gamma_{\text{eq}}[N; b]$ is per construction a unital $*$ -algebra, whereas the analysis in Remark 22 entails that $\mathcal{A}[\tau, t] \in \text{Ar}(\text{Alg})$ for all $(\tau, t) \in \text{Ar}(\text{Bkg})$. Since all structure used to define \mathcal{A} act covariantly, \mathcal{A} is a covariant functor. \square

In order to conclude that the functor \mathcal{A} introduced in Definition 20 identifies an Euclidean locally covariant theory as per Definition 5, the scaling property remains to be discussed. It is particularly important to stress the relation between such property and the local Hadamard representation of the parametrix – *cf.* remark 25.

Proposition 24: Let $(N; b) = (\Sigma, M; \psi, \gamma, g) \in \text{Obj}(\text{Bkg})$, $(N; b_\lambda) := (\Sigma, M; \psi, \lambda^{-2}\gamma, g) \in \text{Obj}(\text{Bkg})$, for $\lambda > 0$ – *cf.* Remark 4. Let $\mathcal{A}[N; b], \mathcal{A}[N; b_\lambda]$ be the associated $*$ -algebras as per definition 20. Then the map

$$s_\lambda: \text{Par}[N; b] \rightarrow \text{Par}[N; b_\lambda], \quad s_\lambda P := P_\lambda := \lambda^{-2}P, \quad (24)$$

is an isomorphism of affine spaces. Furthermore, the map

$$s_\lambda: \mathcal{A}[N; b_\lambda] \rightarrow \mathcal{A}[N; b], \quad (s_\lambda F)(P, \varphi) := \widehat{s}_\lambda F(P_\lambda, \varphi), \quad (25)$$

is an isomorphism of $*$ -algebras such that condition (11) holds true. Here $\widehat{s}_\lambda: \mathcal{E}[N; b_\lambda] \rightarrow \mathcal{E}[N; b]$ denotes the unique lift of s_λ to an isomorphism of vector bundles such that $\pi_{\mathcal{E}[N; b]} \circ \widehat{s}_\lambda = s_\lambda \circ \pi_{\mathcal{E}[N; b_\lambda]}$.

Proof. The first assertion is a direct consequence of the defining properties (12) for $P \in \text{Par}[N; b]$ and of the behaviour under the scaling $\gamma \rightarrow \lambda^{-2}\gamma$ of the operator E defined in (4), that is $E \rightarrow \lambda^2 E$. The associated map $\widehat{s}_\lambda: \mathcal{E}[N; b_\lambda] \rightarrow \mathcal{E}[N; b]$ is defined by $\widehat{s}_\lambda(P, F) := (P_\lambda, F)$. Notice that this guarantees that $s_\lambda: \Gamma_{\text{eq}}[N; b_\lambda] \rightarrow \Gamma_{\text{eq}}[N; b]$ is well-defined and it satisfies condition (11).

It remains to be shown that the map s_λ defined in (25) is a $*$ -isomorphism between $\mathcal{A}[N; b_\lambda]$ and $\mathcal{A}[N; b]$. For that it is enough to show that $s_\lambda(F \cdot G) = s_\lambda F \cdot s_\lambda G$ for all $F, G \in \mathcal{A}[N; b_\lambda]$. Let $P \in \text{Par}[N; b]$. A direct computation shows that

$$(s_\lambda F)[P]^{(n)} = \widehat{s}_\lambda F[P_\lambda]^{(n)}. \quad (26)$$

Moreover, we have that

$$\langle (s_\lambda F)[P]^{(n)}, P^{\otimes n} (s_\lambda G)[P]^{(n)} \rangle = \widehat{s}_\lambda \langle F^{(n)}[P_\lambda], P_\lambda^{\otimes n} G^{(n)}[P_\lambda] \rangle = \widehat{s}_\lambda \langle F^{(n)}[P_\lambda], P_\lambda^{\otimes n} G^{(n)}[P_\lambda] \rangle, \quad (27)$$

where in the first equality we used equation (26) and the fact that \widehat{s}_λ commutes with the contraction with P , while the second equality follows from the scaling properties of the Hodge operator $*_\gamma \rightarrow \lambda^2 *_\gamma$ – *cf.* remark 14. By inserting these results in the equations (16-23) for \cdot the equality $s_\lambda(F \cdot G)[P] = [s_\lambda(F) \cdot s_\lambda(G)][P]$ follows. \square

Remark 25: With reference to Remark 10, we compare the local Hadamard expansion of the integral kernels $P^{bc}(x, x'), P_\lambda^{bc}(x, x')$ of the parametrices $P, P_\lambda = \lambda^{-2}P$. Notice that, although $P_\lambda = \lambda^{-2}P$, at the level of integral kernels it holds $P_\lambda^{bc}(x, x') = P^{bc}(x, x')$ because of the presence of the different volume forms $\mu_\gamma, \mu_{\gamma_\lambda} = \lambda^2 \mu_\gamma$ – *cf.*, Remark 14. Yet the Hadamard parametrix H, H_λ , appearing in equation (13), and the

associated smooth remainders W_P, W_{P_λ} do change. More precisely, under the scaling $\gamma \rightarrow \lambda^{-2}\gamma$, in any geodesic neighbourhood $\mathcal{O} \subset \Sigma$, the singular part of the parametrix transforms as

$$H^{ab}(x, x') \rightarrow H_\lambda^{ab}(x, x') = V_\lambda^{ab}(x, x') \log \frac{\sigma_\lambda(x, x')}{\ell_H^2},$$

where $V_\lambda \in \text{SG}(\psi^*TM^{\otimes 2})$. As a consequence, whenever we choose a parametrix P , which decomposes as $P^{ab} = H^{ab} + W^{ab}$, the counterpart associated with the rescaled Hadamard parametrix H_λ reads

$$P^{ab}(x, x') = H_\lambda^{ab}(x, x') + W_\lambda^{ab}(x, x'), \quad (28)$$

As already highlighted in Remark 10, one can consider the coinciding point limit $x \rightarrow x'$ to construct $[W_P] \in \Gamma(\text{S}^{\otimes 2}\psi^*TM)$. On account of proposition 52 of Appendix A, under scaling the global section $[W_P]$ transforms as $[W_{P,\lambda}]^{bc} = [W_P]^{bc} - 2g^{bc} \log \lambda$.

Collecting definition (20) and propositions 17-23-24 we have the following result which concludes the construction of an Euclidean locally covariant theory as per Definition 5.

Proposition 26: The functor $\mathcal{A}: \text{Bkg} \rightarrow \text{Alg}$ identifies an Euclidean locally covariant theory as per Definition 5.

Remark 27: Notice that the whole construction of the functor \mathcal{A} profits from a simplification due to the dimensional restriction $D = 2$. Indeed, as pointed out in remark 15, for $D > 2$ the singularity behaviour of the parametrices P would spoil the possibility to define the product \cdot as per Definition 20 on the whole set of local polynomials $\mathcal{P}[N; b]$. In this latter case the product would have been defined on the subset $\mathcal{P}_{\text{reg}}[N; b] \subset \mathcal{P}[N; b]$ made of those elements $F \in \mathcal{P}[N; b]$ with smooth functional derivatives at any order. An extension procedure should be applied to define the product \cdot among local polynomial functionals in the same spirit of [BDF09, HW02, Kel09]. We will refrain from describing such a procedure here, see however [DDR19].

For the rest of this paper we will consider the local covariant theory \mathcal{A} introduced in Definition 20.

2.3 Local covariance of observables and of quantum fields

In this section we will be especially interested in identifying a distinguished class of elements of $\mathcal{A}[N; b]$ yielding notion of locally covariant observable. For future convenience we first introduce the following functors.

Definition 28: Let $(N; b) = (\Sigma, M; \psi, \gamma, g), (\tilde{N}, \tilde{b}) = (\tilde{\Sigma}, \tilde{M}; \tilde{\psi}, \tilde{\gamma}, \tilde{g}) \in \text{Obj}(\text{Bkg})$ and let $(\tau, t) \in \text{Ar}(\text{Bkg})$ be an arrow from $(N; b)$ to $(\tilde{N}; \tilde{b})$. We call $C_c^\infty: \text{Bkg} \rightarrow \text{Alg}$ the covariant functor

$$C_c^\infty[N; b] := C_c^\infty(\Sigma) \quad C_c^\infty[\tau, t]f := f \circ \tau^{-1} \quad \forall f \in C_c^\infty[N; b]. \quad (29)$$

Notice that $C_c^\infty[\tau, t]f$ is well-defined on account of the support properties of f , in particular $C_c^\infty[\tau, t]f(x) = 0$ whenever $x \notin \tau(\Sigma)$. Let $\wedge^{\text{top}}T^*\Sigma$ denotes the bundle of densities on Σ . Then we can

Definition 29: Let $k \in \mathbb{N}$. We call $\text{SG}_c^k: \text{Bkg} \rightarrow \text{Alg}$, the covariant functor such that, for all $(N; b) \in \text{Obj}(\text{Bkg})$

$$\text{SG}_c^k[N; b] := \bigoplus_{m=0}^{\infty} \text{SG}_c((\wedge^{\text{top}}T^*\Sigma \otimes \text{S}^{\otimes k}\psi^*T^*M)^{\boxtimes m}), \quad \text{SG}_c((\wedge^{\text{top}}T^*\Sigma \otimes \text{S}^{\otimes k}\psi^*T^*M)^{\boxtimes 0}) \equiv \mathbb{C} \quad (30)$$

where $\text{SG}_c((\wedge^{\text{top}}T^*\Sigma \otimes \text{S}^{\otimes k}\psi^*T^*M)^{\boxtimes m})$ denotes the compactly supported symmetric sections of the m -th exterior tensor product of the vector bundle $\wedge^{\text{top}}T^*\Sigma \otimes \text{S}^{\otimes k}\psi^*T^*M$ – cf. Remark 8. In addition, for all $(\tau, t) \in \text{Arr}(\text{Bkg})$,

$$\text{SG}_c^k[\tau, t]: \text{SG}_c^k[N; b] \rightarrow \text{SG}_c^k[\tilde{N}; \tilde{b}],$$

where $\text{SG}_c^k[\tau, t]\omega := \hat{\tau}_{t,c}^{m,k} \circ \omega \circ \tau^{-1}$ for all $\omega \in \text{SG}_c^k((\wedge^{\text{top}}T^*\Sigma \otimes \text{S}^{\otimes k}\psi^*T^*M)^{\boxtimes m})$. Here $\hat{\tau}_{t,c}: \psi^*T^*M \rightarrow \tilde{\psi}^*T^*\tilde{M}|_{\tau(\Sigma)}$ is an injective morphism of vector bundles – cf. proof of proposition 23, which extends to a map $\hat{\tau}_{t,c}^{m,k}: (\wedge^{\text{top}}T^*\Sigma \otimes \text{S}^{\otimes k}\psi^*T^*M)^{\boxtimes m} \rightarrow (\wedge^{\text{top}}T^*\tilde{\Sigma} \otimes \text{S}^{\otimes k}\tilde{\psi}^*T^*\tilde{M})^{\boxtimes m}|_{\tau(\Sigma)}$, $m \in \mathbb{N} \cup \{0\}$, by considering a suitable symmetrized tensor product and pull-back for top-densities.

Observe that similarly we define the contravariant functor $\Gamma^k: \text{Bkg} \rightarrow \text{Alg}$ as

$$\text{S}\Gamma^k[N; b] := \bigoplus_{m=0}^{\infty} \text{S}\Gamma_c((\text{S}^{\otimes k} \psi^* T M)^{\boxtimes m}), \quad (31)$$

The associated arrow $\Gamma^k[\tau, t]: \Gamma[\tilde{N}; \tilde{b}] \rightarrow \Gamma^k[N; b]$ is obtained by considering the injective morphism of vector bundle $\widehat{\tau}_t: \psi^* T M \rightarrow \widetilde{\psi}^* T \widetilde{M}|_{\tau(\Sigma)}$ – cf. proof of Proposition 23 – and its extension to $\widehat{\tau}_t^{m,k}: (\text{S}^{\otimes k} \psi^* T M)^{\boxtimes m} \rightarrow (\text{S}^{\otimes k} \widetilde{\psi}^* T \widetilde{M})^{\boxtimes m}|_{\tau(\Sigma)}$ for all $m \in \mathbb{N} \cup \{0\}$. The arrow $\text{S}\Gamma^k[\tau, t]: \text{S}\Gamma^k[\tilde{N}; \tilde{b}] \rightarrow \text{S}\Gamma^k[N; b]$ is then defined as $\text{S}\Gamma^k[\tau, t]\widetilde{C} := (\widehat{\tau}_t^{m,k})^{-1} \circ \widetilde{C} \circ \tau$ for all $\widetilde{C} \in \text{S}\Gamma((\text{S}^{\otimes k} \widetilde{\psi}^* T \widetilde{M})^{\boxtimes m})$.

Remark 30: Since, from time to time, it is convenient to focus on a fixed m -th symmetric, exterior tensor product of $\wedge^{\text{top}} T^* \Sigma \otimes \text{S}^{\otimes k} \psi^* T^* M$, we can work also with the covariant functors $\text{S}\Gamma_c^{k,m}: \text{Bkg} \rightarrow \text{Vec}$, $m \in \mathbb{N} \cup \{0\}$, such that for all $(N; b) \in \text{Obj}(\text{Bkg})$ and for all $(\tau, t) \in \text{Ar}(\text{Bkg})$,

$$\text{S}\Gamma_c^{k,m}[N; b] := \text{S}\Gamma_c((\wedge^{\text{top}} T^* \Sigma \otimes \text{S}^{\otimes k} \psi^* T^* M)^{\boxtimes m}), \quad \text{S}\Gamma_c^{k,m}[\tau, t] := \text{S}\Gamma_c^k[\tau, t]|_{\text{S}\Gamma_c^{k,m}[N; b]}. \quad (32)$$

In the following we will be mainly interested in functionals which are constructed out of compactly supported sections of an arbitrary $(\wedge^{\text{top}} T^* \Sigma \otimes \text{S}^{\otimes k} \psi^* T^* M)^{\boxtimes m}$. We stress that our analysis is tied to the functor \mathcal{A} introduced in Definition 20, though the procedure can be extended to any Euclidean locally covariant theory as per Definition 5. As a first step we need to relate the functors \mathcal{A} and $\text{S}\Gamma_c^k$. Inspired by [BFV03], we introduce

Definition 31: Let $k \in \mathbb{N}$ and let $\mathcal{A}: \text{Bkg} \rightarrow \text{Alg}$ and $\text{S}\Gamma_c^k: \text{Bkg} \rightarrow \text{Alg}$ be the functors as per Definition 20 and 29 respectively. We call *locally covariant observable of degree k* a natural transformation $\mathcal{O}_k: \text{S}\Gamma_c^k \rightarrow \mathcal{A}$ i.e., for every $(N; b) \in \text{Obj}(\text{Bkg})$, $\mathcal{O}_k[N; b]: \text{S}\Gamma_c^k[N; b] \rightarrow \mathcal{A}[N; b]$ is an arrow in Alg such that, for every $(\tau, t) \in \text{Ar}(\text{Bkg})$ mapping $(N; b)$ to $(\tilde{N}; \tilde{b})$, it holds that

$$\mathcal{O}_k[\tilde{N}; \tilde{b}] \circ \text{S}\Gamma_c^k[\tau, t] = \mathcal{A}[\tau, t] \circ \mathcal{O}_k[N; b]. \quad (33)$$

For concreteness, we underline that, for all $(N; b) \in \text{Obj}(\text{Bkg})$, $\mathcal{O}_k[N; b]$ can be read as an algebra-valued distribution, that is, for all $P \in \text{Par}[N; b]$, for all $\varphi \in \Gamma(\psi^* T M)$ and for all $m \in \mathbb{N}$,

$$\mathcal{O}_k[N; b](\bullet, P, \varphi): \text{S}\Gamma_c((\wedge^{\text{top}} T^* \Sigma \otimes \text{S}^{\otimes k} \psi^* T^* M)^{\boxtimes m}) \ni \omega_m \mapsto \mathcal{O}_k[N; b](\omega_m, P, \varphi) \in \mathbb{C}, \quad (34)$$

defines a distribution $\mathcal{O}_k[N; b](\bullet, P, \varphi) \in \text{S}\Gamma_c((\wedge^{\text{top}} T^* \Sigma \otimes \text{S}^{\otimes k} \psi^* T^* M)^{\boxtimes m})'$. Henceforth we will follow (34) writing for notational simplicity $\mathcal{O}_k[N; b](\omega, P, \varphi)$ in place $[\mathcal{O}_k[N; b](\omega)](P, \varphi)$.

Remark 32: The previous definition can be generalized by substituting the functor $\text{S}\Gamma_c^k$ with an arbitrary functor $F: \text{Bkg} \rightarrow \text{Alg}$. In this case we still call local and covariant observable any natural transformation $\mathcal{O}: F \rightarrow \mathcal{A}$ – cf. equation (57) in section 3.

Remark 33: Notice that, since $\mathcal{O}_k[N; b] \in \text{Ar}(\text{Alg})$, for all $m \in \mathbb{N}$ and $\omega_1 \otimes \dots \otimes \omega_m \in \text{S}\Gamma_c^{k,m}[N; b]$ with $\omega_j \in \text{S}\Gamma_c^{k,1}[N; b]$ for all $j = 1, \dots, m$ it holds

$$\mathcal{O}_k[N; b](\omega_1 \otimes \dots \otimes \omega_m) = \mathcal{O}_k[N; b](\omega_1) \cdots \mathcal{O}_k[N; b](\omega_m).$$

This property implies that a locally covariant observable as per Definition 31 is known once it is known its value on degree $m \in \{0, 1\}$. In this sense, a locally covariant observable consists of a locally covariant polynomial in the field – $\mathcal{O}_k[N; b]$ at degree $m = 1$ – together with its powers according to the product \cdot of $\mathcal{A}[N; b]$ – see Definition 20. As already stressed in Remark 27 these observations depend crucially on the dimensional restriction $D = 2$. For generic D , the identification of a locally covariant observable \mathcal{O}_k can be interpreted as: (a) the identification of a local and covariant polynomial functional in the field configuration φ , namely of $\mathcal{O}_k[N; b]$ at degree $m = 1$; (b) the identification of an extension of the product \cdot , which allows to define the product between $\mathcal{O}_k[N; b]$ with itself.

Example 34: Consider $(N; b) \in \text{Obj}(\text{Bkg})$ and let $\Phi[N; b]: \text{S}\Gamma_c^1[N; b] \rightarrow \mathcal{A}[N; b]$ be defined as follows. If $\omega_1 \in \text{S}\Gamma_c^{1,1}[N; b]$ then $\Phi[N; b](\omega_1)$ is the linear functional such that, for all $(P, \varphi) \in \text{Par}[N; b] \times \Gamma(\psi^* T M)$,

$$\Phi[N; b](\omega_1, P, \varphi) := \int_{\Sigma} \langle \omega_1, \varphi \rangle. \quad (35)$$

As pointed out in Remark 33, this fixes completely $\Phi[N; b]$ on the whole $\text{ST}_c^{1,1}[N; b]$. Let now $(\tau, t) \in \text{Ar}(\text{Bkg})$ be a mapping from $(N; b)$ to $(\tilde{N}, \tilde{b}) \in \text{Obj}(\text{Bkg})$. To conclude that Φ is a natural transformation, we need to show that $[\mathcal{A}(\tau, t)] \circ \Phi[N; b] = \Phi[\tilde{N}, \tilde{b}] \circ \text{ST}_c^1(\tau, t)$. This is a direct consequence of the definition, as one can readily infer, since, for every $\omega_1 \in \text{ST}_c^{1,1}[N; b]$ and $\tilde{P} \in \text{Par}(\tilde{N}; \tilde{b})$, $\tilde{\varphi} \in \Gamma(\tilde{\psi}^* T\tilde{M})$

$$\begin{aligned} \left[\mathcal{A}[\tau, t] \Phi[N; b](\omega_1) \right] (\tilde{P}, \tilde{\varphi}) &= \Phi[N; b](\omega_1, \text{Par}[\tau, t] \tilde{P}, \text{ST}_c^1[\tau, t] \tilde{\varphi}) = \int_{\Sigma} \langle \tilde{\varphi}, \text{ST}_c^1[\tau, t] \omega_1 \rangle \\ &= \Phi[\tilde{N}, \tilde{b}](\text{ST}_c^1(\tau, t) \omega_1, \tilde{P}, \tilde{\varphi}), \end{aligned}$$

where $\text{Par}[\tau, t] \tilde{P} \in \text{Par}[\tilde{N}; \tilde{b}]$ has been defined in the proof of proposition 23. In view of its definition and of its properties $\Phi[N; b]$ identifies a locally covariant observable of degree 1 to which we refer as a *locally covariant quantum field*.

Example 35: In order to define powers of a locally covariant quantum field, which could be interpreted also as locally covariant observables, the starting point is Remark 16. Here a candidate for a well-defined Wick ordered, squared field is introduced, but the definition depends on the choice of a parametrix P , a procedure which is intrinsically non locally covariant. In order to bypass this hurdle, constructing at the same time an equivariant section of $\mathcal{E}[N; b]$, we need to rely on the Hadamard representation of any parametrix P as in Equation (13). As outlined in Example 10, we can use such representation to identify from each parametrix P , $[W_P] \in \Gamma(\text{S}^{\otimes 2} \psi^* TM)$. Bearing in mind this information, consider $(N; b) \in \text{Obj}(\text{Bkg})$ and let $\Phi^2[N; b]: \text{ST}_c^2[N; b] \rightarrow \mathcal{A}[N; b]$ be defined as follows. If $\omega_1 \in \text{ST}_c^{2,1}[N; b]$ then $\Phi^2[N; b](\omega_1)$ is the linear functional such that, for all $(P, \varphi) \in \text{Par}[N; b] \times \Gamma(\psi^* TM)$,

$$\Phi^2[N; b](\omega_1, P, \varphi) := \int_{\Sigma} \langle \varphi^{[\otimes 2]} + [W_P], \omega_1 \rangle, \quad (36)$$

where $\varphi^{[\otimes 2]} \in \Gamma(\text{S}^{\otimes 2} \psi^* TM)$ – cf. remark 8. In order to realize that Φ^2 identifies a locally covariant observable of degree 1, it suffices to proceed as in Example 34 and thus we shall not dwell into the details. It is important to observe that (36) is a possible realization of a Wick power of Φ , but it is not the unique one. We will discuss this issue in detail in the next section.

For later convenience we introduce a notion which intertwines locally covariant observables with scaling yielding as a by-product an abstract notion of engineer dimension which matches the one discussed at the end of Section 1.1.

Definition 36: Let $k \in \mathbb{N}$, and let $\mathcal{O}_k: \text{ST}^k \rightarrow \mathcal{A}$ be a locally covariant observable of degree k as per Definition 31. For any $[N; b] \in \text{Obj}(\text{Bkg})$ we call *rescaled locally covariant observable* at scale $\lambda > 0$, $S_\lambda \mathcal{O}_k$ the locally covariant observable defined by

$$(S_\lambda \mathcal{O}_k)[N; b] := \mathcal{O}_k[N; b_\lambda], \quad (37)$$

where $[N; b_\lambda]$ is defined in (10). In addition we say that $\mathcal{O}_k[N; b]$ has engineering dimension $d_{\mathcal{O}_k} \in \mathbb{R}$ if

$$(S_\lambda \mathcal{O}_k)[N; b](\omega_m) = \lambda^{d_{\mathcal{O}_k} m} \mathcal{O}_k[N; b](\omega_m), \quad (38)$$

holds for all $[N; b] \in \text{Obj}(\text{Bkg})$ and $\omega_m \in \text{ST}_c^{k,m}[N; b]$. On the contrary we say that \mathcal{O}_k scales *almost homogeneously* with dimension $\kappa \in \mathbb{R}$ and order $\ell \in \mathbb{N}$ if

$$S_\lambda \mathcal{O}_k[N; b](\omega_m) = \lambda^{\kappa m} \mathcal{O}_k[N; b](\omega_m) + \lambda^{\kappa m} \sum_{j=0}^{\ell} \log(\lambda)^j \mathcal{O}_j[N; b](\omega_m), \quad (39)$$

holds for all $[N; b] \in \text{Obj}(\text{Bkg})$ and $\omega_m \in \text{ST}_c^{k,m}[N; b]$ where, for all $j \in \{0, \dots, \ell\}$, \mathcal{O}_j is a locally covariant observables which scales almost homogeneously with degree κ and order $\ell - j$.

Remark 37: We stress, that while our analysis could be slavishly applied to models for which the dimension of Σ is arbitrary, Definition 36 relies on $\dim \Sigma = 2$. In the general case, (37) should be modified as follows

$$(S_\lambda \mathcal{O})[N; b] := \varsigma_\lambda[N; b] \circ \mathcal{O}[N; b_\lambda],$$

where $\dim \Sigma = D$ while $\varsigma_\lambda[N; b]: \mathcal{A}[N; b_\lambda] \rightarrow \mathcal{A}[N; b]$ is the scaling transformation introduced in Definition 5.

The engineering dimension can be computed explicitly in many notable instances:

Example 38: Consider the locally covariant observables defined in Example 34 via the natural transformations Φ . Putting together (35) and (10), one can compute that $S_\lambda \Phi = \Phi$ that is the engineering dimension of Φ is 0.

At the same time, if we consider the locally covariant observable Φ^2 as per Example 35, in order to evaluate its behaviour under scaling we need to take into account Remark 25 according to which $[W_{P,\lambda}]^{bc} = [W_P]^{bc} - 2g^{bc} \log \lambda$. Hence, for all $\lambda > 0$

$$(S_\lambda \Phi^2) = \Phi^2 + \mathcal{V} \log \lambda,$$

where \mathcal{V} is the locally covariant observable of degree 0 such that

$$\mathcal{V}[N; b](\omega_1, P, \varphi) = -2 \int_{\Sigma} \langle g^\sharp, \omega_1 \rangle, \quad \omega_1 \in \text{ST}_c^{2,1}[N; b].$$

In other words Φ^2 scales almost homogeneously with dimension 0 and order 1.

2.4 Wick ordered powers of quantum fields

Following our previous analysis, in this section we address the issue of Wick ordering in order to construct, for any $[N; b] \in \text{Obj}(\text{Bkg})$, well-defined algebra valued distributions, which can be read as locally covariant powers of the underlying, locally covariant quantum field Φ as the one introduced in Example 34.

Although, in the Lorentzian framework, this is an overkilled topic starting from the seminal work [HW01], here we will be mainly interested in the Euclidean setting and in vector-valued fields. For this reason we shall follow mainly the rationale used in [KMM17]. In particular, tackling the problem of Wick ordering can be divided in two separate issues, the first concerning the existence of a well-defined ordering scheme, the second addressing the question of classifying the possible ambiguities in the construction of Wick ordered observables, while keeping track of local covariance.

In the following we give an abstract definition of Wick ordered powers of a quantum field adapting to the case in hand [KMM17, Def. 5.2].

Definition 39: Let Φ be a locally covariant observable defined in Example 34. A family of Wick powers associated to Φ is a family of natural transformations $\Phi^\bullet = \{\Phi^k\}_{k \in \mathbb{N}}$ with $\Phi^k: \text{ST}_c^k \rightarrow \mathcal{A}$ such that it holds

1. For all $k \in \mathbb{N} \cup \{0\}$, Φ^k is a locally covariant observable which scales almost homogeneously with dimension $\kappa = 0$.
2. If $k = 1$, $\Phi^1 = \Phi$ while, if $k = 0$ $\Phi^0 := 1_{\mathcal{A}}$, where for all $(N; b) \in \text{Obj}(\text{Bkg})$ and for all $z \in \mathbb{C}$, $1_{\mathcal{A}}[N; b](z) := z 1_{\mathcal{A}[N; b]}$, the right hand side of this equality being the identity element of $\mathcal{A}[N; b]$.
3. For all $k \in \mathbb{N} \cup \{0\}$, it holds, that, for all $(N; b) \in \text{Obj}(\text{Bkg})$, $\omega_1 \in \text{ST}_c^{k,1}[N; b]$, $P \in \text{Par}[N; b]$ and $\varphi_1, \varphi_2 \in \Gamma(\psi^* TM)$, then,

$$\langle \Phi^k[N; b](\omega_1, P)^{(1)}[\varphi_1], \varphi_2 \rangle = k \Phi^{k-1}[N; b](\varphi_2 \lrcorner \omega_1, P, \varphi_1), \quad (40)$$

where $\varphi_2 \lrcorner \omega_1 \in \text{ST}_c^{k-1,1}[N; b]$ denotes the section which reads locally $\varphi_2^{a_1}(\omega_1)_{a_1 \dots a_k}$, while the superscript (1) refers to the functional derivative as per Definition 11.

4. Let $d \in \mathbb{N}$ and let $(N; b_s) \in \text{Obj}(\text{Bkg})$ be such that $\{b_s = (\psi, \gamma_s, g_s)\}_{s \in \mathbb{R}^d}$ is a smooth, compactly supported d -dimensional family of variations of $b = (\psi, \gamma, g)$ as per Definition 55. For all smooth family $\{P_s\}_{s \in \mathbb{R}^d}$ where $P_s \in \text{Par}(N, b_s)$ for all $s \in \mathbb{R}^d$, let $\mathcal{U}_k \in \Gamma_c(\pi_d^* \text{S}^{\otimes k} \psi^* T^* M)'$ be the distribution on the pullback bundle $\pi_d^* \text{S}^{\otimes k} \psi^* T^* M$ over the base space $\mathbb{R}^d \times \Sigma$ – here $\pi_d: \mathbb{R}^d \times \Sigma \rightarrow \Sigma$ denotes the canonical projection – defined by

$$\mathcal{U}_k(\chi \otimes \omega_1) := \int_{\mathbb{R}^d} ds \Phi^k[N; b_s](\omega_1, P_s, 0) \chi(s), \quad \omega_1 \in \text{ST}_c^{k,1}[N; b], \chi \in C_c^\infty(\mathbb{R}^d). \quad (41)$$

We require that, for all $k \in \mathbb{N} \cup \{0\}$.

$$\text{WF}(\mathcal{U}_k) = \emptyset, \quad (42)$$

where $\text{WF}(\mathcal{U}_k)$ denotes the wavefront set of \mathcal{U}_k , [H603, Def. 8.1.2].

Remark 40: Notice that condition 4 in Definition 39 exploits a smooth, compactly supported d -dimensional family of variations (γ_s, g_s) of the metrics γ and g while the background configuration ψ has been fixed. The choice of a smooth family of parametrices $\{P_s\}_{s \in \mathbb{R}^d}$ should be compared with the smooth class of states $\omega \circ \tau_s^{-1}$ introduced in [KMM17, Def. 5.2]. In particular $\{P_s\}$ is associated with a unique $P \in \text{SG}_c(\pi_d^* \psi^* T^* M^{\boxtimes 2})' - \text{cf.}$ Definition 55 in Appendix B. The existence of the family $\{P_s\}_{s \in \mathbb{R}^d}$ is a consequence of the smoothness in the parameter $s \in \mathbb{R}^d$ of the elliptic operator E_s associated to the background data $b_s - \text{cf.}$ equation (4) – and of the construction of P_s as a pseudodifferential operator [Shu87, Thm. 5.1]. Notice that, given $\{P_s\}_{s \in \mathbb{R}^d}$, any other family of parametrices is of the form $\{P_s + W_s\}_{s \in \mathbb{R}^d}$ where $\{W_s\}_{s \in \mathbb{R}^d}$ is a smooth family such that $W_s \in \text{SG}(\psi^* T M^{\boxtimes 2}) - \text{that is,}$ $\{W_s\}_{s \in \mathbb{R}^d}$ is associated with a unique $W \in \text{SG}(\pi_d^* \psi^* T M^{\boxtimes 2})$.

Remark 41: Observe that, if condition (42) holds true for Φ^ℓ with $\ell \leq k$ then, to verify it for Φ^k , it suffices to check it for any, but fixed choice of the family of $\{P_s\}_{s \in \mathbb{R}^d}$.

The proof of this statement goes by induction: Condition (42) holds true for $k = 0, 1$ independently from $\{P_s\}_{s \in \mathbb{R}^d}$ since both $\Phi^0[N; b]$ and $\Phi^1[N; b] = \Phi[N; b]$ identify per construction constant sections over $\mathcal{E}[N; b]$. Let now assume that condition (42) holds true for \mathcal{U}_ℓ for all $\ell \leq k$ and for all smooth family $\{P_s\}_{s \in \mathbb{R}^d}$. We now show that, if (42) holds true for \mathcal{U}_k built out of a particular smooth family $\{P_s\}_{s \in \mathbb{R}^d}$, then it holds true for all distributions $\tilde{\mathcal{U}}_k$ associated with any other smooth family $\{\tilde{P}_s\}_{s \in \mathbb{R}^d}$. From the equivariance condition – see (22) – it descends

$$\tilde{\mathcal{U}}_k(\chi \otimes \omega) = \int_{\mathbb{R}^d} ds \Phi^k[N; b_s](\omega, \tilde{P}_s, 0) \chi(s) = \int_{\mathbb{R}^d} ds \alpha_{\tilde{P}_s}^{P_s} [\Phi^k[N; b_s](\omega, P_s)](0) \chi(s).$$

Definition (18) entails that $\alpha_{\tilde{P}_s}^{P_s} [\Phi^k[N; b_s](\omega, P_s)]$ is a linear combination of functional derivatives with respect to $\varphi \in \Gamma(\psi^* T M)$ of $\Phi^k[N; b_s](\omega, P_s)$ evaluated at $\varphi = 0$. By condition (40) each of such derivatives which is non trivial can be reduced to a Wick power Φ^ℓ with $\ell \leq k$. Explicitly it holds

$$\alpha_{\tilde{P}_s}^{P_s} [\Phi^k[N; b_s](\omega, P_s)](0) = \Phi^k[N; b_s](\omega, P_s, 0) + \sum_{\substack{\ell=2 \\ 2\ell \leq k}}^k \frac{k!!}{(2\ell)!!(k-2\ell)!!} \Phi^{k-2\ell}[N; b_s](\tilde{P}_s - P_s)^{\otimes \ell} \lrcorner \omega, P_s, 0),$$

where locally $(\tilde{P}_s - P_s)^{\otimes \ell} \lrcorner \omega_k(x) = (\tilde{P}_s - P_s)^{a_1 a_2}(x, x) \dots (\tilde{P}_s - P_s)^{a_{\ell-1} a_\ell}(x, x) (\omega_k)_{a_1 \dots a_k}(x)$. Observe that $\tilde{P}_s - P_s \in \text{SG}(\psi^* T M^{\boxtimes 2})$ is smooth in $s \in \mathbb{R}^d$ and moreover the last expression contains only terms of the form \mathcal{U}_ℓ , $\ell \leq k$, where (42) holds true by the inductive hypothesis.

Remark 42: Notice that, if Φ^\bullet identifies a family of Wick powers as per Definition 39, then for all $\lambda > 0$ we can construct another family of Wick powers Φ_λ^\bullet via scaling – cf. Definition 36. Actually we set

$$\Phi_\lambda^k[N; b] := (S_\lambda \Phi^k)[N; b] := \Phi^k[N; b_\lambda], \quad (43)$$

where $(N; b_\lambda)$ is defined in (10). This fact will play a crucial rôle in Section 3.

Example 43: We provide a constructive scheme yielding a natural candidate to play the role of a family of Wick powers. Let $k \in \mathbb{N}$ and let $(N; b) \in \text{Obj}(\text{Bkg})$ and $\omega \in \text{SG}_c^{k,1}[N; b]$ be arbitrary. Define $\phi^k[N; b] \in \mathcal{P}_{\text{loc}}[N; b] - \text{see}$ Definition 11 – to be

$$\phi^k[N; b](\omega, \varphi) := \int_{\Sigma} \langle \varphi^{[\otimes k]}, \omega \rangle, \quad (44)$$

where $\varphi^{[\otimes k]} \in \Gamma(\text{S}^{\otimes k} \psi^* T M)$ has been defined in remark 8, while locally $\langle \varphi^{[\otimes k]}, \omega \rangle = \varphi^{a_1} \dots \varphi^{a_k} \omega_{a_1 \dots a_k}$. Since we are interested in functionals which are equivariant with respect to the choice of $P \in \text{Par}[N; b]$, we set for all $\omega \in \text{SG}_c^{k,1}[N; b]$ and $\varphi \in \Gamma(\psi^* T M)$,

$$:\Phi^k : [N; b](\omega, P, \varphi) := \left[\exp \left[\Upsilon_{[W_P]} \right] \phi^k[N; b](\omega) \right] (\varphi), \quad (45)$$

where $\Gamma_{[W_P]}$ is defined as in (19) where we also exploited the support property of the functional derivatives of any local functionals – see Definition 11. Moreover $[W_P] \in \Gamma(S^{\otimes 2}\psi^*TM)$ is defined as in Remark 10. Notice that such a local decomposition depends on the chosen background geometry $(N; b)$ out of which H is identified.

By extending $\Phi^k[N; b]$ to a locally covariant observable – see Remark 33 – the collection of all Φ^k defines a family of Wick powers as per Definition 39. Indeed observe that, adapting (43) to Φ^k , this scales almost homogeneously with dimension $\kappa = 0$ while the second and third condition in Definition 39 follow per construction. Finally (42) holds true since

$$:\Phi^{2\ell+1}: [N; b](\omega, P, 0) = 0, \quad :\Phi^{2\ell}: [N; b](\omega, P, 0) = \int_{\Sigma} \langle [W_P]^{[\otimes \ell]}, \omega \rangle,$$

where $[W_P]^{[\otimes \ell]} := [[W_P]^{\otimes \ell}]$ – cf. remark 8. The smoothness of the associated distribution \mathcal{U}_k – see equation (41) – follows.

Our next step consists of addressing the question concerning the characterization and the classification of the freedom in the construction of a family of Wick powers. In the Lorentzian setting this question has already been thoroughly investigated for a large class of field theories, see [HW01, KMM17, KM16], while here we tackle the same problem for the model in hand, introduced in Section 1.1.

In the same spirit of [KMM17, Thm. 5.2, Thm. 6.2] the result is divided in two parts – see Theorems 45 and 46. In the first we prove a general formula – Equation (50) – which starts from two families of Wick powers, say $\widehat{\Phi}^\bullet$ and Φ^\bullet , relating each $\widehat{\Phi}^k$ to a linear combination of $\{\Phi^\ell\}_{\ell \leq k}$ whose coefficients are a collection of locally covariant observables $\{C_\ell\}_{1 \leq \ell \leq k-2}$. This result profits of the Peetre-Slovák theorem which we briefly recall in Appendix B. In the second part, we prove additional structural properties of the coefficients C_ℓ , recasting in this framework [KMM17, Thm. 6.2].

Before stating the key results of this section, we prove a key lemma.

Lemma 44: Let $k \in \mathbb{N}$ and for all $(N; b) \in \text{Obj}(\text{Bkg})$ let $c_k[N; b] \in \text{S}\Gamma^{k,1}[N; b]$ be such that

$$c_k[\widetilde{N}; \widetilde{b}] = \tau^* c_k[N, b],$$

for all $[\tau, t] \in \text{Ar}(\text{Bkg})$ between (N, b) and $(\widetilde{N}, \widetilde{b})$. Then, for any $(N; b) = (\Sigma, M; \psi, \gamma, g) \in \text{Obj}(\text{Bkg})$, there exists a map $D_{k,\Sigma,M}: \Gamma(S^{\otimes 2}T^*\Sigma \otimes S^{\otimes 2}\psi^*T^*M) \rightarrow \Gamma^{k,1}[N; b]$ such that

$$c_k[N; b] = c_k[\Sigma, M; \psi, \gamma, g] = D_{k,\Sigma,M}(\gamma, \psi^*g). \quad (46)$$

Proof. Per hypothesis, for every pair $[N; b] = (\Sigma, M; \psi, \gamma, g)$, $[\widetilde{N}, \widetilde{b}] = (\widetilde{\Sigma}, \widetilde{M}; \widetilde{\psi}, \widetilde{\gamma}, \widetilde{g}) \in \text{Obj}(\text{Bkg})$ such that there exists $(\tau, t) \in \text{Ar}(\text{Bkg})$ from $[N; b]$ to $[\widetilde{N}, \widetilde{b}]$, it holds

$$c_k(\widetilde{\Sigma}, \widetilde{M}; \widetilde{\psi}, \widetilde{\gamma}, \widetilde{g}) = \tau^* c_k(\Sigma, M; \psi, \gamma, g). \quad (47)$$

Consider the special case where $[\widetilde{N}, \widetilde{b}]$ and (τ, t) are such that $\widetilde{\Sigma} = \Sigma$, $\widetilde{M} = M$, $\tau = \text{Id}_\Sigma$ while $t: M \rightarrow M$ is any diffeomorphism in M such that $t|_{\psi(\Sigma)} = \text{Id}|_\Sigma$. Condition (9) entails $\widetilde{\psi} = \psi \circ \tau = \psi$, while $g = t^*\widetilde{g}$, $\widetilde{\gamma} = \gamma$. Equation (47) implies

$$c_k[N; b] = c_k(\Sigma, M; \psi, \gamma, \widetilde{g}) = c_k(\Sigma, M; \psi, \gamma, t^*\widetilde{g}).$$

It follows that $c_k(\Sigma, M; \psi, \gamma, g)$ depends on g only via ψ^*g , that is

$$c_k(\Sigma, M; \psi, \gamma, g) =: d_k(\Sigma, M; \psi, \gamma, \psi^*g).$$

We prove that $c_k[N; b]$ depends on ψ only via ψ^*g . As above let $\tau = \text{Id}_\Sigma$ and let $t: M \rightarrow M$ be any diffeomorphism: by condition (46) we find

$$\begin{aligned} d_k(\Sigma, M; \widetilde{\psi}, \widetilde{\gamma}, \widetilde{\psi}^*\widetilde{g}) &= c_k(\Sigma, M; \widetilde{\psi}, \widetilde{\gamma}, \widetilde{g}) = c_k(\Sigma, M; \psi, \gamma, t^*\widetilde{g}) \\ &= d_k(\Sigma, M; \psi, \gamma, \psi^*t^*\widetilde{g}) = d_k(\Sigma, M; \psi, \gamma, \widetilde{\psi}^*\widetilde{g}), \end{aligned}$$

where we exploited equation (9) so that $\psi^*t^*\widetilde{g} = (t \circ \psi)^*\widetilde{g} = \widetilde{\psi}^*\widetilde{g}$. This implies that

$$c_k(\Sigma, M; \psi, \gamma, g) =: D_{k,\Sigma,M}(\gamma, \psi^*g),$$

which entails the sought result. \square

Theorem 45: Let $\widehat{\Phi}^\bullet$ and Φ^\bullet be two families of Wick powers associated to Φ as per Definition 39. Then, for all integers $k > 2$, there exists a collection $\{C_\ell\}_{2 \leq \ell \leq k}$ of locally covariant observables $C_\ell: \text{SF}_c^\ell \rightarrow \mathcal{A}$, each of which scales almost homogeneously with dimension $\kappa = 0$ so that, for all $(N, b) \in \text{Obj}(\text{Bkg})$ and for all $\omega_1 \in \text{SF}_c^{k,1}[N; b]$

$$C_\ell[N; b](\omega_1) = \int_\Sigma \langle c_\ell[N; b], \omega_1 \rangle 1_{\mathcal{A}[N; b]}, \quad (48)$$

where $c_\ell[N; b] \in \text{SF}_c^{\ell,1}[N; b]$ for all $(N; b) \in \text{Obj}(\text{Bkg})$. Furthermore, if $(N; b) = (\Sigma, M; \psi, \gamma, g)$, then

$$c_\ell(\Sigma, M; \psi, \gamma, g) = D_{\ell, \Sigma, M}(\gamma, \psi^* g), \quad (49)$$

where $D_{\ell, \Sigma, M}: \Gamma(\text{S}^{\otimes 2} T^* \Sigma \otimes \text{S}^{\otimes 2} \psi^* T^* M) \rightarrow \Gamma(\text{S}^{\otimes \ell} \psi^* T M)$ is a differential operator of locally bounded order in the sense of Definition 53 in Appendix B. In addition, for all $(N; b) \in \text{Obj}(\text{Bkg})$ and for all $\omega_1 \in \text{SF}_c^{k,1}[N; b]$,

$$\widehat{\Phi}^k[N; b](\omega_1) = \Phi^k[N; b](\omega_1) + \sum_{\ell=0}^{k-2} \binom{k}{\ell} \Phi^\ell[N; b](c_{k-\ell}[N; b] \lrcorner \omega_1), \quad (50)$$

where $c_{k-\ell}[N; b] \lrcorner \omega_1 \in \text{SF}_c^{\ell,1}[N; b]$ reads locally $(c_{k-\ell}[N; b] \lrcorner \omega_1)_{a_1 \dots a_\ell} = c_{k-\ell}^{a_{\ell+1} \dots a_k}[N; b](\omega_1)_{a_1 \dots a_\ell}$.

Proof. The proof proceeds per induction with respect to k . First of all we prove Equation (50) for $k = 2$. Hence, we set $C_2 := \widehat{\Phi}^2 - \Phi^2$, showing that it is of the wanted form. Let $(N; b) \in \text{Obj}(\text{Bkg})$ and let $\omega_1 \in \text{SF}_c^{2,1}[N; b]$, while $\varphi_1, \varphi_2 \in \Gamma(\psi^* T M)$ and $P \in \text{Par}[N; b]$. Equation (40) entails

$$\langle C_2[N; b](\omega_1, P)^{(1)}[\varphi_1], \varphi_2 \rangle = 2(\Phi - \widehat{\Phi})[N; b](\varphi_2 \lrcorner \omega_1, P, \varphi_1) = 0.$$

It follows that, as an element of $\mathcal{A}[N; b]$, $C_2[N; b](\omega_1)$ does not depend on (P, φ) , that is, it is a multiple of the identity element:

$$C_2[N; b](\omega_1, P, \varphi) = \int_\Sigma \langle c_2[N; b], \omega_1 \rangle 1_{\mathcal{A}[N; b]},$$

where c_2 is an assignment to $(N; b) \in \text{Obj}(\text{Bkg})$ of an element in $\text{SF}_c^{2,1}[N; b]$ on account of the regularity condition (42). Moreover C_2 inherits from Φ^2 and $\widehat{\Phi}^2$ the property of scaling almost homogeneously with degree $\kappa = 0$. Since the arrows of Bkg act on $c_2[N; b]$ via pull-back, the hypotheses of Lemma 44 are met and we can conclude that $c_2(\Sigma, M; \psi, \gamma, g) = D_{2, \Sigma, M}(\gamma, \psi^* g)$ for all $(N; b) = (\Sigma, M; \psi, \gamma, g)$. It descends that, for all $x \in \Sigma$, $D_{2, \Sigma, M}(\gamma, \psi^* g)(x)$ depends only on the germ of $\gamma, \psi^* g$ at x . Furthermore, condition (42) ensures that $(\gamma, \psi^* g) \mapsto D_{2, \Sigma, M}(\gamma, \psi^* g)$ is weakly regular as per Definition 56. By the Peetre-Slovák Theorem – see Appendix B – it follows that $D_{2, \Sigma, M}: \Gamma(\text{S}^{\otimes 2} T^* \Sigma \otimes \text{S}^{\otimes 2} \psi^* T^* M) \rightarrow \Gamma(\text{S}^{\otimes 2} \psi^* T M)$ is a differential operator of locally bounded order. This concludes the proof of the theorem for $k = 2$.

Let us assume that, for all $2 \leq p \leq k$, $(N, b) \in \text{Obj}(\text{Bkg})$ and for all $\omega_1 \in \text{SF}_c^{p,1}[N; b]$

$$\widehat{\Phi}^p[N; b](\omega_1) = \Phi^p[N; b](\omega_1) + \sum_{\ell=0}^{p-2} \binom{p}{\ell} \Phi^\ell[N; b](c_{p-\ell}[N; b] \lrcorner \omega_1). \quad (51)$$

Here for all $q \in \{1, \dots, p-2\}$, C_q is a locally covariant observable which scales almost homogeneously with dimension $\kappa = 0$, so that

$$C_q[N; b](\omega_1) = \int_\Sigma \langle c_q[N; b], \omega_1 \rangle 1_{\mathcal{A}[N; b]} \quad \forall \omega_1 \in \text{SF}_c^{q,1}[N; b],$$

where $c_q[N; b] = c_q(\Sigma, M; \psi, \gamma, g) = D_{q, \Sigma, M}(\gamma, \psi^* g) \in \text{SF}_c^{q,1}[N; b]$, being $D_{q, \Sigma, M}$ a differential operator of locally bounded order. We prove the inductive step, namely that equation (51) holds true for $p = k+1$. As for the case $k = 2$, let C_{k+1} be defined as

$$C_{k+1}[N; b](\omega_1) := \widehat{\Phi}^{k+1}[N; b](\omega_1) - \Phi^{k+1}[N; b](\omega_1) - \sum_{\ell=0}^{k-1} \binom{k+1}{\ell} \Phi^\ell[N; b](c_{k+1-\ell}[N; b] \lrcorner \omega_1), \quad (52)$$

for all $\omega_1 \in \text{S}\Gamma_c^{k+1,1}[N; b]$. Equation (40) and the inductive hypothesis (51) entail that $C_{k+1}[N; b](\omega_1)$ is an element of $\mathcal{A}[N; b]$ that does not depend on the choice of (P, φ) . Hence there exist an assignment to $(N; b) \in \text{Obj}(\text{Bkg})$ of an element $c_{k+1}[N; b] \in \text{S}\Gamma_c^{k+1,1}[N; b]$ such that

$$C_{k+1}[N; b](\omega_1) = \int_{\Sigma} \langle c_{k+1}[N; b], \omega_1 \rangle 1_{\mathcal{A}[N; b]}.$$

where we used the regularity condition (42). In addition, still on account of the inductive hypothesis (51), C_{k+1} scales almost homogeneously with degree $\kappa = 0$. This implies that $c_{k+1}[N; b]$ satisfies the hypothesis of Lemma 44 and, thus, it follows that $c_{k+1}(\Sigma, M; \psi, \gamma, g) = D_{k+1, \Sigma, M}(\gamma, \psi^*g)$. The regularity condition (42) ensures that $D_{k+1, \Sigma, M}: \Gamma(\text{S}^{\otimes 2}T^*\Sigma \otimes \text{S}^{\otimes 2}\psi^*T^*M) \rightarrow \Gamma(\text{S}^{\otimes k+1}\psi^*TM)$ is weakly regular and that, for all $x \in \Sigma$, $D_{k+1, \Sigma, M}(\gamma, \psi^*g)(x)$ depends on γ, ψ^*g only via their germs at x . By the Peetre-Slovák Theorem $D_{k+1, \Sigma, M}$ is a differential operator of locally bounded order. This completes the proof. \square

To conclude we state the last result of this section.

Theorem 46: Under the same assumptions of Theorem 45, it holds that, for each $k \in \mathbb{N}$ and for each $(N; b) = (\Sigma, M; \psi, \gamma, g) \in \text{Obj}(\text{Bkg})$, the map $D_{k, \Sigma, M}$ defined in equation (49) enjoys the following properties:

1. $D_{k, \Sigma, M}: \Gamma(\text{S}^{\otimes 2}T^*\Sigma \otimes \text{S}^{\otimes 2}\psi^*T^*M) \rightarrow \Gamma^{k,1}[N; b]$ is a differential operator of globally bounded order – see Definition 53 of Appendix B ;
2. for all $x \in \Sigma$, $\gamma \in \Gamma(\text{S}^{\otimes 2}T^*\Sigma)$ and $\psi^*g \in \Gamma(\text{S}^{\otimes 2}\psi^*T^*M)$ it holds

$$D_{k, \Sigma, M}(\gamma, \psi^*g)(x) = D_k \left(\gamma^{\alpha\beta}(x), \epsilon^{\alpha\beta}(x), R_{\alpha\beta\mu\nu}[\gamma](x), \dots, \nabla_{\alpha_1}^{\Sigma} \dots \nabla_{\alpha_p}^{\Sigma} R_{\alpha\beta\mu\nu}[\gamma](x), \dots, g^{ab}(\psi(x)), R_{abcd}[g](\psi(x)), \dots, \nabla_{a_1}^M \dots \nabla_{a_r}^M R_{abcd}[g](\psi(x)) \right), \quad (53)$$

where D_k is a tensor, covariantly constructed from its arguments, where the symbol R in the above expression indicates the Riemann tensor while $\epsilon^{\alpha\beta}$ is the totally antisymmetric Levi-Civita tensor.

3. Each D_k is an homogeneous of degree $\kappa = 0$, linear combination of finitely many covariantly constructed tensors. These are polynomials in all the arguments on which D_k depends in (53) and the functional form does not depend on the choice of $[N; b] \in \text{Obj}(\text{Bkg})$.

Proof. On account of Lemma 44 the proof can follow almost slavishly that of [KMM17, Thm. 6.2]. For this reason we omit it. \square

3 Renormalization and Ricci flow

Our main goal is to apply the results of Section 2.4 giving a rigorous derivation of the Ricci flow from the renormalization of (the linearisation of) the non-linear Sigma-model introduced in Section 1.1 – see [Car10] and also [Gaw99].

Perturbative Euclidean statistical field theory. In the framework of Euclidean algebraic quantum field theory the expectation value of a (locally covariant) observable \mathcal{O} is typically built out of a Lagrangian density \mathcal{L} , as the one introduced in equation (3), which is regarded as the covariance of an infinite dimensional Gaussian measure. However, except for some rather special cases, this approach brings several difficulties in dealing with non-linearities and thus one must resort to a perturbative approach. Fixing \mathcal{L} to be the one of (3), following the discussion and the notation at the beginning of Section 1.1, we split \mathcal{L} in two contributions

$$\mathcal{L}(\psi, \gamma, g; \varphi) := \mathcal{L}_{\text{free}}(\psi, \gamma, g; \varphi) + \mathcal{L}_{\text{int}}(\psi, \gamma, g; \varphi), \quad (54)$$

$$\mathcal{L}_{\text{free}}(\psi, \gamma, g; \varphi) := -\frac{\nu^2}{2} \langle \varphi, E\varphi \rangle \mu_{\gamma}, \quad (55)$$

$$\mathcal{L}_{\text{int}}(\psi, \gamma, g, \varphi) := \mathcal{L}_{\text{H}}(\psi, \gamma, g) + \left[\nu g(\varphi, Q(\psi)) + \frac{\nu^2}{2} h(\text{Riem}(\varphi, d\psi)\varphi, d\psi) \right] \mu_{\gamma}. \quad (56)$$

As the notation suggests, we interpret $\mathcal{L}_{\text{free}}$ as the Lagrangian density of a free field theory while \mathcal{L}_{int} is interpreted as an interacting part.

One has to keep in mind that such subdivision is arbitrary and our choice is dictated by the fact that the dynamics encoded in $\mathcal{L}_{\text{free}}$ is ruled by the elliptic operator E . According to Proposition 26, we can associate to it an Euclidean locally covariant theory $\mathcal{A} : \text{Bkg} \rightarrow \text{Alg}$. At the same time, to \mathcal{L}_{int} we can associate a locally covariant observable as per Definition 31 with the following procedure.

Consider a family of Wick powers Φ^\bullet as per Definition 39 and, starting from \mathcal{L}_{int} , define the following natural transformation which we indicate for simplicity as $\mathcal{L}_{\text{int}}[\Phi^\bullet] : C_c^\infty \rightarrow \mathcal{A}$:

$$\mathcal{L}_{\text{int}}[\Phi^\bullet][N; b](f) := \mathcal{L}_{\text{H}}[N; b](f) 1_{\mathcal{A}[N; b]} + \nu \Phi[N; b](f Q(\psi) \mu_\gamma) + \frac{\nu^2}{2} \Phi^2[N; b](f \theta[N; b] \mu_\gamma), \quad (57)$$

where $(N; b) \in \text{Obj}(\text{Bkg})$, $f \in C_c^\infty(\Sigma)$. The covariant functor $C_c^\infty : \text{Bkg} \rightarrow \text{Alg}$ has been defined in definition 28. In addition $\theta[N; b] \in \Gamma(S^{\otimes 2} \psi^* TM)$ is locally defined by

$$\theta[N; b]_{cd}(x) := \gamma^{\alpha\beta}(x) g_{\ell b}(\psi(x)) R_{cad}{}^\ell[g](d\psi)_\alpha^a (d\psi)_\beta^b, \quad (58)$$

while $\mathcal{L}_{\text{H}}[N; b](f) := \int_\Sigma d\mu_\gamma f \mathcal{L}_{\text{H}}(\psi, \gamma, g)$, \mathcal{L}_{H} being the Lagrangian in (2).

Within the perturbative approach to Euclidean field theory one defines the (generating function) partition function as the natural transformation $\mathcal{Z}[\Phi^\bullet] : C_c^\infty \rightarrow \mathcal{A}$

$$\mathcal{Z}[\Phi^\bullet][N; b](f) := \exp_{\mathcal{A}} \left[z \mathcal{L}_{\text{int}}[\Phi^k][N; b](f) \right] := \sum_{n \geq 0} \frac{z^n}{n!} \mathcal{L}_{\text{int}}[\Phi^k][N; b](f)^n \in \mathcal{A}[N; b][[z]], \quad (59)$$

where the exponential series is considered as a formal power series in the formal parameter z and the product is the one defined by $\mathcal{A}[N; b]$, for all $[N; b] \in \text{Obj}(\text{Bkg})$. Out of $\mathcal{Z}[\Phi^\bullet]$ one can build perturbatively the above mentioned expectation value of any locally covariant observable \mathcal{O} . A complete discussion of the structural properties of the perturbative approach to Euclidean algebraic field theories is beyond the scope of this paper and we postpone it to a forthcoming work [DDR19].

Application to Ricci flow. As explained at the beginning of this section, once it has been fixed a family of Wick powers Φ^\bullet – cf. Definition 39 – we may define a corresponding locally covariant Lagrangian density $\mathcal{L}_{\text{int}}[\Phi^\bullet]$ – see equation (57) – and the associated partition function $\mathcal{Z}[\Phi^\bullet]$ as per equation (59). The key point consists in realizing that different choices of Φ^\bullet yield different explicit forms for $\mathcal{L}_{\text{int}}[\Phi^\bullet]$, which, on account of theorems 45 and 46, differ only by a linear combination of terms proportional to certain locally covariant quantum fields – see equation 50.

In the framework of the renormalization group approach such ambiguity is studied by choosing, for each real $\lambda > 0$, $\widehat{\Phi}^\bullet := S_\lambda \Phi^\bullet$, see Remark 42, in particular Equation (43). As a consequence we consider an interacting Lagrangean density $\mathcal{L}_{\text{int}}[S_\lambda \Phi^\bullet]$ which, by Theorem 45 can be written as $\mathcal{L}_{\text{int}}[S_\lambda \Phi^\bullet] = \mathcal{L}_{\text{int}}[\Phi^\bullet] + \mathcal{R}_\lambda[\Phi^\bullet]$, where $\mathcal{R}_\lambda[\Phi^\bullet]$ is a suitable remainder. The main idea behind the renormalization group approach is that $\mathcal{R}_\lambda[\Phi^\bullet]$ can be reabsorbed in the full Lagrangian density, namely, for every $\lambda > 0$, there exists a natural transformation, dubbed renormalized Lagrangian at the scale λ , $\mathcal{L}_{\text{int}, \lambda}[\Phi^\bullet] : C_c^\infty \rightarrow \mathcal{A}$ such that

$$\mathcal{L}_{\text{int}}[S_\lambda \Phi^\bullet] = \mathcal{L}_{\text{int}}[\Phi^\bullet] + \mathcal{R}_\lambda[\Phi^\bullet] =: \mathcal{L}_{\text{int}, \lambda}[\Phi^\bullet]. \quad (60)$$

In what follows we will compute explicitly the renormalized Lagrangian density at scale $\lambda > 0$, $\mathcal{L}_\lambda[\Phi^\bullet] := \mathcal{L}_{\text{free}}[\Phi^\bullet] + \mathcal{L}_{\text{int}, \lambda}[\Phi^\bullet]$ – see Theorem 47. For concreteness we will work with the family of Wick powers defined in Example 43, though any well-defined, different choice can be made without affecting the final result. Eventually we comment how the result of Theorem 47 are linked to the derivation of the Ricci flow [Car10] – see Lemma 49.

Theorem 47: Let Φ^\bullet be the family of Wick powers as per Example 43 and let $\mathcal{L}_{\text{int}}[\Phi^\bullet]$ be the locally covariant interacting Lagrangian density as per Equation (57). For all $\lambda > 0$, let $\mathcal{L}_{\text{int}}[S_\lambda \Phi^\bullet]$ be the counterpart of $\mathcal{L}_{\text{int}}[\Phi^\bullet]$ – defined in (60) – constructed out of the rescaled natural transformation $S_\lambda \Phi^\bullet$. Then it holds

$$\mathcal{L}_{\text{int}, \lambda}[\Phi^\bullet][N; b](f) = \mathcal{L}_{\text{H}, \lambda}[N; b](f) 1_{\mathcal{A}[N; b]} + \nu \Phi[N; b](f Q(\psi) \mu_\gamma) + \frac{\nu^2}{2} \Phi^2[N; b](f \theta[N; b] \mu_\gamma), \quad (61)$$

where $f \in C_c^\infty(\Sigma)$ and $[N; b] = (\Sigma, M; \psi, \gamma, g)$ while

$$\mathcal{L}_{H,\lambda}[N; b](f) := \int_{\Sigma} f \operatorname{tr}_{\gamma}(\psi^* g_{\log \lambda}) \mu_{\gamma}, \quad (g_{\log \lambda})_{ab}(x) = g_{ab}(x) - \nu^2 \log(\lambda) R_{ab}[g](x). \quad (62)$$

Proof. Let $[N; b] = (\Sigma, M; \psi, \gamma, g) \in \operatorname{Obj}(\mathbf{Bkg})$, $f \in C_c^\infty(\Sigma, M)$, $P \in \operatorname{Par}[N; b]$ and $\varphi \in \Gamma(\psi^* TM)$. Recalling Definition 36 as well as equation (45) – see Example 43 – it holds

$$\mathcal{L}_{\text{int}}[S_{\lambda} \Phi^{\bullet}][N; b] = \mathcal{L}_{\text{int}}[\Phi^{\bullet}][N; b_{\lambda}]. \quad (63)$$

Recalling Proposition 52 and Remark 25, we can use the local Hadamard representation of the parametrix $P = H + W$ for E to realize that, in a geodesic neighbourhood of any point $x \in \Sigma$, $P = H_{\lambda} + W_{\lambda}$ where $[W_{P,\lambda}]^{ab}(x) := W_{\lambda}^{ab}(x, x) = [W_P]^{ab}(x) - 2 \log(\lambda) g^{ab}(\psi(x))$.

Using (45), the first two terms in \mathcal{L}_{int} in (57) remain unchanged because they are respectively constant and linear in φ . On the contrary, the third term yields

$$\begin{aligned} \frac{\nu^2}{2} \Phi^2[N; b_{\lambda}][f \theta[N; b_{\lambda}] \mu_{\gamma_{\lambda}}, P, \varphi] &= \frac{\nu^2}{2} \Phi^2[N; b][f \theta[N; b] \mu_{\gamma}, P, \varphi] \\ &\quad - \nu^2 \log(\lambda) \int_{\Sigma} \gamma^{\alpha\beta} R_{ab}[g] (d\psi)_{\alpha}^a (d\psi)_{\beta}^b f \mu_{\gamma}, \end{aligned} \quad (64)$$

where we used (58) for $\theta[N; b]$. Inserting Equation (64) in (63), the sought result descends. \square

Remark 48: Notice that the results of Theorem 47 depend on the particular choice for Φ^{\bullet} . As a matter of fact Theorem 45 entails that any other choice, say Φ^{\bullet} would be so that $\Phi^2 =: \Phi^2 + C_2$, being C_2 a locally covariant quantum field which scales almost homogeneously with degree zero – cf. Theorem 46. As a by product, Equation (64) holds true also for Φ^2 if and only if C_2 scales *exactly* homogeneously with degree zero – that is, if and only if C_2 is invariant under scaling. Nevertheless, theorems 45-46 allow us to control the variation of equation (64) with respect to the chosen family of Wick powers Φ^{\bullet} .

Lemma 49 (Ricci flow): Under the assumptions of Theorem 47, setting $\lambda := e^{2\tau}$, the corresponding metric $g(\tau) := g_{2\tau}$ as per Equation (62) satisfies

$$\frac{d}{d\tau} g(\tau) = -2\nu^2 \operatorname{Ric}[g] = -2\nu^2 \operatorname{Ric}[g(\tau)] + O(\nu^3). \quad (65)$$

Proof. Considering Equation (62) and recalling the approximation made in Section 1.1,

$$g_{\log \lambda}^{ab} = g^{ab} - \nu^2 \log(\lambda) R^{ab}[g] + O(\nu^3), \quad \nu^2 \log(\lambda) R_{ab}[g_{\log \lambda}] = \nu^2 \log(\lambda) R_{ab}[g] + O(\nu^3).$$

Neglecting $O(\nu^3)$ -contributions the previous equation leads to the wanted Ricci flow equation for the renormalized metric $g(\tau)$. \square

Remark 50: It appears clear that the above derivation of the Ricci flow equation (65) is linked to the expansion in powers of ν made in the previous Section 1.1. It is also possible to consider an higher order expansion for the Lagrangian density (3), which leads to a corresponding improved Ricci flow equation. As an example, the expansion up to $o(\nu^4)$ leads to the so-called Ricci flow equation at two loops, also known as $RG - 2$ flow [CG18]. It is noteworthy that, thanks to Theorem 45, the present framework can be used to obtain an analogous of Theorem 47 from which the higher order corrections to the Ricci flow can be explicitly computed. Yet, in this work, we refrain from providing a detailed computation, which follows the lines of the proof of Theorem 47.

Remark 51: We stress that, in our derivation of the Ricci flow equation – see Remark 49 – as well as in the proof of all the results of the previous Sections, we only assume that $\psi \in C^\infty(\Sigma; M)$. In particular, we do not require ψ to be harmonic and the results of Theorem 47 and Remark 49 do not depend on ψ . Stated differently, the results of this paper hold true also considering *off-shell* background configuration ψ , rather than on-shell (harmonic) background configurations.

A Hadamard expansion for the parametrix of E

Goal of this appendix is to give a finer description of the local structure of a parametrix P associated with the elliptic operator E , introduced in Equation (4). Let $[N; b] = (\Sigma, M; \psi, \gamma, g) \in \text{Obj}(\text{Bkg})$ be arbitrary but fixed. In the following, we will be considering convex, geodesic neighbourhoods of Σ , but at the same time we will be concerned about their image under the action of ψ which is smooth, but not necessarily proper. Hence, whenever we consider convex, geodesic neighbourhoods of a point, we are implicitly constructing them as follows: For any $x \in \Sigma$, consider $\psi(x) \in M$ and any convex, geodesic neighbourhood $U \subset M$ centred at this point. Being ψ smooth, $\psi^{-1}(U)$ is an open subset of Σ centred at x . If this is not a convex, geodesic neighbourhood, then consider an open subset, which we identify with \mathcal{O} , which has this property. In addition $\psi(\mathcal{O})$ is a subset of U and hence any two points therein are connected by a unique geodesic of (M, g) .

We summarize our results in the following proposition:

Proposition 52: Let $(N; b) = (\Sigma, M, \gamma, g, \psi) \in \text{Obj}(\text{Bkg})$ and let $E: \Gamma(\psi^*TM) \rightarrow \Gamma(\psi^*T^*M)$ be the elliptic operator defined in (4). For $\lambda > 0$ let $H, H_\lambda \in \text{S}\Gamma_c(\psi^*T^*M^{\otimes 2})'$ be the Hadamard parametrices associated with background data $(N; b)$ and $(N; b_\lambda)$ respectively – cf. Remark 10 and Definition 4. It holds

$$H_\lambda^{bc}(x) - H^{bc}(x) = -2 \log(\lambda) V^{bc}(x), \quad (66)$$

where $V \in \Gamma(\text{S}^{\otimes 2}\psi^*T\mathcal{O})$ is constructed out the background data (ψ, γ, g) – cf. equation (69) – and it satisfies $[V]^{bc}(x) := g^{bc}(\psi(x))$.

Proof. Let \mathcal{O} be a geodesically convex neighbourhood of Σ . We begin by recalling the construction of the so-called Hadamard parametrix associated to the restriction to \mathcal{O} of E on the background data $(N; b)$. This is defined as the bi-distribution $H \in \text{S}\Gamma_c(\psi^*T^*\mathcal{O}^{\otimes 2})'$ whose integral kernel reads [G98, Mor99a, Mor99b]

$$H^{bc}(x, x') := V^{bc}(x, x') \log \frac{\sigma(x, x')}{\ell_H^2} := \sum_{n \geq 0} V_n^{bc}(x, x') \sigma(x, x')^n \log \frac{\sigma(x, x')}{\ell_H^2}, \quad (67)$$

where $\sigma(x, x')$ denotes the halved squared geodesic distance between $x, x' \in \mathcal{O}$, while $\ell_H \in \mathbb{R}$ is an arbitrary reference length, which will play no rôle in the proof. Before focusing on the tensor coefficients $V_n^{bc}(x, x')$, observe that Equation (67) defines H in terms of the so-called Hadamard expansion which is a formal power series in σ . Hence, with a slight abuse of notation, we have left implicit both the existence of a suitable cut-off which ensures convergence of (67) and the necessity of replacing σ with a regularized counterpart $\sigma + i\epsilon$, which controls the singularity as $x = x'$. Neither the cut-off nor the regularization will play a rôle in our analysis.

We focus now on the remaining unknowns, the tensor coefficients V_n^{bc} of (67). Recalling that both HE and EH coincide with the identity operator up to smooth terms, it holds locally that

$$\begin{aligned} (EH)_a^c &= \sum_{n \geq 0} E(V_n)^c_a \sigma^n \log \frac{\sigma}{\ell_H^2} \\ &+ \sum_{n \geq 0} \left[n g_{ab} V_n^{bc} (\Delta_\gamma \sigma + 2(n-1)) + 2n g_{ab} \gamma^{\alpha\beta} (\nabla^\psi V_n)^{bc}_\alpha (d\sigma)_\beta \right] \sigma^{n-1} \log \frac{\sigma}{\ell_H^2} \\ &+ \sum_{n \geq 0} \left[2g_{ab} \gamma^{\alpha\beta} (\nabla^\psi V_n)^{bc}_\alpha (d\sigma)_\beta + g_{ab} V_n^{bc} (\Delta_\gamma \sigma - 2 + 4n) \right] \sigma^{n-1}, \end{aligned} \quad (68)$$

where we omitted for notational simplicity the explicit dependence on (x, x') and where we exploited the identity $\gamma^{\alpha\beta} (d\sigma)_\alpha (d\sigma)_\beta = 2\sigma$, see e.g. [PPV11]. To ensure that $EH - \text{Id}_{\Gamma_c(\psi^*T^*M)} \in \Gamma(\psi^*TM \boxtimes \psi^*T^*M)$, the coefficients multiplying $\log \sigma$ and σ^{-1} ought to vanish. This leads to the following hierarchy of equations for V_n^{bc} :

$$2g_{ab} \gamma^{\alpha\beta} (\nabla^\psi V_0)^{bc}_\alpha (d\sigma)_\beta + g_{ab} V_0^{bc} (\Delta_\gamma \sigma - 2) = 0 \quad (69a)$$

$$E(V_{n-1})^c_a + 2n g_{ab} \gamma^{\alpha\beta} (\nabla^\psi V_n)^{bc}_\alpha (d\sigma)_\beta + n g_{ab} V_n^{bc} (\Delta_\gamma \sigma + 2(n-1)) = 0. \quad (69b)$$

Notice that the latter is a system of transport equations which can be solved recursively once we provide initial conditions for the tensors V_n^{bc} . The customary choice for the initial data is to consider the limit $x \rightarrow x'$ of equation (69). Denoting with $[A](x) := \lim_{x \rightarrow x'} A(x, x')$ the coinciding point limit of a generic smooth bi-tensor – cf. remark 8 – we get

$$[E(V_0)^c_a] + 2[g_{ab}V_1^{bc}] = 0, \quad [E(V_{n-1})^c_a] + 2n^2[g_{ab}V_n^{bc}] = 0, \quad (70)$$

where we used the identities

$$[\sigma] = 0, \quad [(d\sigma)_\alpha] = 0, \quad [(\nabla^\Sigma \circ \nabla^\Sigma \sigma)_{\alpha\beta}] = \gamma_{\alpha\beta}. \quad (71)$$

Notice that the equations in (70) specify initial data for V_n^{bc} for all $n \geq 1$, leaving us only with an arbitrariness in the choice of the initial datum for V_0 . In order for $EP - \text{Id}, PE - \text{Id} \in \Gamma(\psi^*TM \boxtimes \psi^*T^*M)$, we fix

$$[V_0^{bc}] = g^{bc}. \quad (72)$$

We now consider the Hadamard parametrix H_λ associated with E and background data $(N; b_\lambda)$. Once again we have

$$H_\lambda^{bc}(x, x') = \sum_{n \geq 0} V_{\lambda, n}^{bc}(x, x') \sigma_\lambda(x, x')^n \log \sigma_\lambda(x, x'),$$

where σ_λ is the halved squared geodesic distance built out of the metric $\lambda^{-2}\gamma_{\alpha\beta}$. The smooth tensors $V_{\lambda, n}^{bc}$ satisfy the system (69) with background data $(N; b_\lambda)$. Observe that equation (69a) is invariant under scaling $\gamma_{\alpha\beta} \rightarrow \lambda^{-2}\gamma_{\alpha\beta}$ because $\sigma_\lambda = \lambda^{-2}\sigma$. Together with the initial conditions $[V_0]^{bc} = [V_{\lambda, 0}]^{bc} = g^{bc}$ this entails $V_{\lambda, 0} = V_0$. By induction it easily follows from the scaling behaviour of equations (69b) that $V_{\lambda, n} = \lambda^{2n}V_n$. Therefore

$$H_\lambda^{bc} - H^{bc} = \sum_{n \geq 0} V_{\lambda, n}^{bc} \lambda^{-2n} \sigma^n (\log \sigma - 2 \log \lambda) - \sum_{n \geq 0} V_n^{bc} \sigma^n \log \sigma = -2 \log(\lambda) V^{bc}.$$

Using the initial condition 72, equation (66) follows. \square

B The Peetre-Slovák theorem

In this section we recall succinctly the Peetre-Slovák theorem as well as all ancillary definitions. For more details, we refer to [NS14] and especially to [KM16, Appendix A], to which this appendix is inspired. In the following $E \xrightarrow{\pi_E} B, F \xrightarrow{\pi_F} B$ are smooth bundles over a smooth manifold B , while $J^r E$ denotes the r -jet bundle over B for $r \in \mathbb{N}$ – refer to [KMS93] for definitions and properties.

Definition 53: A map $D: \Gamma(E) \rightarrow \Gamma(F)$ is called a *differential operator* of globally bounded order $r \in \mathbb{N}$ if there exists a smooth map $d: J^r E \rightarrow F$ such that $\pi_F \circ d = \pi_{J^r E}$ and

$$D(\varepsilon) = d(j^r \varepsilon) \quad \forall \varepsilon \in \Gamma(E), \quad (73)$$

where $j^r \varepsilon \in \Gamma(J^r E)$ denotes the r -jet extension of ε .

Definition 54: A map $D: \Gamma(E) \rightarrow \Gamma(F)$ is called a *differential operator of locally bounded order* if for all $x_0 \in B$ and for all $\varepsilon_0 \in \Gamma(E)$, there exists

1. an open subset $U \subseteq B$ containing x_0 and with compact closure,
2. an integer $r \in \mathbb{N}$, as well as a neighbourhood $Z^r \subseteq J^r E$ of $j^r \varepsilon_0(U)$ such that $\pi_{J^r E} Z^r = U$,
3. a smooth map $d: Z^r \rightarrow F$ such that $\pi_F \circ d = \pi_{J^r E}$

so that

$$D(\varepsilon)(x) = d(j^r \varepsilon)(x), \quad (74)$$

for all $x \in U$ and $\varepsilon \in \Gamma(E)$ with $j^r \varepsilon(U) \subseteq Z^r$.

The Peetre-Slovák's Theorem gives a sufficient condition for a map $D: \Gamma(E) \rightarrow \Gamma(F)$ to be a differential operator of locally bounded order.

In addition recall that, denoting with $\pi_d: B \times \mathbb{R}^d \rightarrow B$ the canonical projection to B , the pull-back bundle $\pi_d^*E \xrightarrow{\pi_{\pi_d^*E}} B \times \mathbb{R}^d$ is the smooth bundle defined by

$$\pi^*E := \{(s, x, e) \in \mathbb{R}^d \times B \times E \mid \pi_E(e) = \pi_d(s, x)\} \simeq \mathbb{R}^d \times E. \quad (75)$$

Denoting with $\pi_{d,E}$ the projection $\pi_{d,E}: \pi_d^*E \rightarrow E$, each smooth section $\zeta \in \Gamma(\pi_d^*E)$ induces a smooth family of sections $\{\zeta_s\}_{s \in \mathbb{R}^d}$ in $\Gamma(E)$ defined by $\zeta_s(x) := \pi_{d,E}\zeta((s, x))$ which, in turn, depends smoothly on the parameter $s \in \mathbb{R}^d$.

Definition 55: Let $d \in \mathbb{N}$ and let $\{\zeta_s\}_{s \in \mathbb{R}^d}$ be a smooth family of sections in $\Gamma(E)$ induced by a smooth section $\zeta \in \Gamma(\pi_d^*E)$. We say that $\{\zeta_s\}_{s \in \mathbb{R}^d}$ is a smooth compactly supported d -dimensional family of variations if there exists a compact $K \subseteq B$ such that $\zeta(s, x) = \zeta(s', x)$ for all $x \notin K$ and for all $s, s' \in \mathbb{R}^d$.

Definition 56: A map $D: \Gamma(E) \rightarrow \Gamma(F)$ is called weakly-regular if, for all $d \in \mathbb{N}$ and for all smooth compactly supported d -dimensional families of variations $\{\zeta_s\}_{s \in \mathbb{R}^d}$ – see Definition 55 – $\psi_s := D\zeta_s$ is a smooth compactly supported d -dimensional family of variations.

Theorem 57 (Peetre-Slovák): Let $D: \Gamma(E) \rightarrow \Gamma(F)$ be a smooth map such that

- for all $\varepsilon \in \Gamma(E)$ and for all $x \in B$, $D\varepsilon(x)$ depends only on the germ of ε at $x \in B$, i.e. $(D\varepsilon)(x) = (D\tilde{\varepsilon})(x)$ for all $\tilde{\varepsilon} \in \Gamma(E)$ which coincides with ε in a neighbourhood of x ;
- D is weakly regular as per Definition 56.

Then D is a differential operator of locally bounded order as per Definition 53.

C Fulfilment of the perturbative agreement

In this section we comment on the principle of perturbative agreement (PPA for short) for the model we have introduced in Definition 20.

The PPA has been introduced in [HW05] as a further constraint on the structure of Wick powers – see also [DHP16, Za15]. Loosely speaking, it requires that a theory associated with a quadratic perturbation E_s of the elliptic operator E introduced in equation (4) should yield to an algebra $\mathcal{A}_s[N; b]$ compatible with the unperturbed algebra $\mathcal{A}[N; b]$. Here $E_s - E \in \Gamma_c(S^{\otimes 2}\psi^*T^*M)$ is a smooth and compactly supported (1-dimensional) family of variation. The compatibility between \mathcal{A}_s and \mathcal{A} is in the sense of formal power series in s – cf. definition 61.

As pointed out in [Za15] the PPA is important in our setting because, among other things, it ensures that the renormalization group flow technique we applied in Section 3 does not depend on the splitting $\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$. A complete discussion of the PPA is out not within the scopes of this paper – for a complete discussion in the Riemannian setting see [DDR19]. In the present appendix we provide a brief resumé of the content of the PPA, proving that there exists a family of Wick powers as per definition 39 which fulfils it – cf. Proposition 63.

In what follows E_s will always denote a smooth and compactly supported (1-dimensional) family of variation – cf. Definition 55 – of the elliptic operator E defined as per equation (4). In particular E_s is elliptic for all s . Notice that, for the sake of simplicity, we are assuming that $E_s - E \in \Gamma_c(S^{\otimes 2}\psi^*T^*M)$ is a differential operator of order at most 1. This is actually enough for our setting see however [DDR19, HW05] for completeness.

Formulation of the PPA. In order to formulate the PPA a few preliminary definitions are in due order. First of all we need a linear isomorphism $R_s: \text{Par}[N; b] \ni P \rightarrow P_s \in \text{Par}_s[N; b]$ between the space of parametrices $\text{Par}_s[N; b]$ associated with E_s and those of E . The construction of this map is rather standard, see [DDR19] for further details and [DD16, DHP16, HW05, Za15] for the corresponding map in the Lorentzian

setting. For what concerns the PPA, we just need the perturbative expansion of R_s as a formal power series in s up to a smooth remainder. Let $P \in \text{Par}[N; b]$; since $Q_s := E_s - E$ is compactly supported,

$$E_s = E + Q_s = E(\text{Id} + PQ_s) - SQ_s,$$

where $S \in \Gamma(\psi^*TM \boxtimes \psi^*T^*M)$ is such that $PE - \text{Id}_{\Gamma_c(\psi^*T^*M)} = S - cf.$ equation (12). We consider the map $R_{[[s]]}: \Gamma_c(\psi^*T^*M) \rightarrow \Gamma(\psi^*TM)[[s]]$ defined by

$$R_{[[s]]}\omega := \sum_{n \geq 0} (-PQ_s)^n P\omega. \quad (76)$$

This map can be interpreted as a perturbative expansion (up to a smooth remainder) of a well-defined isomorphism $R_s: \text{Par}[N; b] \rightarrow \text{Par}_s[N; b] - cf.$ [DD16, DDR19, DHP16, HW05].

The second ingredient we need is a $*$ -isomorphism $\beta_s: \mathcal{A}_{\text{reg}}[N; b] \rightarrow \mathcal{A}_{s, \text{reg}}[N; b]$. Here the subscript reg denotes the algebra generated by regular local functional, namely those with smooth functional derivatives of all orders. We will not enter into the details of this construction, however, we give the explicit form for β_s :

$$(\beta_s F)(P_s) := \exp[\Upsilon_{P_s - P}] F[P] \quad \forall F \in \mathcal{A}[N; b]. \quad (77)$$

This can be extended to a map $\beta_{[[s]]}: \mathcal{A}[N; b] \rightarrow \Gamma(\mathcal{E}[N; b])[s]$ with values in the algebra of formal power series in s with coefficients in $\Gamma(\mathcal{E}[N; b]) - cf.$ Definitions 18-20. The expansion is possible since, on account of equation (76), $P_s - P = P_{[[s]]} - P + R = \sum_{n \geq 1} (-PG_s)^n P + R$ has a well-defined coinciding point limit (here R is a smooth remainder). Therefore β_s is well-defined at each order in s . As explained in [DDR19, DHP16] the map $\beta_{[[s]]}$ can be interpreted as an extension of the expansion in formal power series of β_s .

We focus on Wick powers, strengthening the smoothness requirement of Definition 39 by allowing also variations of the elliptic operator E_s .

Definition 58: Let $d, n \in \mathbb{N}$ and let $(N; b_s) \in \text{Obj}(\text{Bkg})$ be such that $\{b_s = (\psi, \gamma_s, g_s)\}_{s \in \mathbb{R}^d}$ is a smooth, compactly supported d -dimensional family of variations of $b = (\psi, \gamma, g)$ as per Definition 55. Moreover let $E_{t,s}$ be a smooth and compactly supported n -dimensional family of variations of the elliptic operator E_s constructed out of the background data b_s as per equation (4). For all smooth families $\{P_{t,s}\}_{s \in \mathbb{R}^d}$ where $P_{t,s} \in \text{Par}_t(N, b_s)$ is a parametrix for $E_{t,s}$ for all $s \in \mathbb{R}^d$ and $t \in \mathbb{N}^n$, let $\mathcal{U}_k \in \Gamma_c(\pi_{d+n}^* S^{\otimes k} \psi^* T^* M)'$ be the distribution defined by

$$\mathcal{U}_k(\chi \otimes \omega_1) := \int_{\mathbb{R}^{d+n}} ds dt \Phi_t^k[N; b_s](\omega_1, P_{t,s}, 0) \chi(s, t), \quad (78)$$

where $\omega_1 \in \text{S}\Gamma_c^{k,1}[N; b]$, $\chi \in C_c^\infty(\mathbb{R}^{d+n})$. Here $\Phi_t^k[N; b_s]$ denotes the (k -th) Wick power associated with the background data $(N; b_s)$ and with the elliptic operator $E_{s,t}$. If $\text{WF}(\mathcal{U}_k) = \emptyset$, we call the family of Wick powers Φ^\bullet smooth.

Remark 59: Loosely speaking Definition 58 requires a suitable smoothness of Φ^\bullet with respect both to the background data b and to the variation of the elliptic operator E . For certain models – like the scalar field *cf.* [DDR19] – the variations of the background data exhaust all possible variations of the associated elliptic operator E . In this situation the smoothness as per Definition 58 coincides with the one required in Definition 39.

Remark 60: A smooth family of parametrices $P_{t,s} \in \text{Par}_t[N; b_s]$ can be constructed by setting $P_{t,s} := R_t P_s$ where $P_s \in \text{Par}[N; b_s]$ is a smooth family of parametrices for $E_s - cf.$ Remark 40.

From now on Φ^\bullet will denote a family of Wick powers as per Definition 39 satisfying the smoothness requirement of Definition 58. Notice that the family $:\Phi^\bullet:$ defined in Example 45 satisfies such smoothness requirement.

Definition 61: Let E_s denote a smooth and compactly supported (1-dimensional) family of variation – *cf.* definition 55 – of the elliptic operator E defined as per equation 4. We say that the family of Wick powers

Φ^\bullet satisfies the principle of perturbative agreement (PPA) if for all $k \geq 2$, $n \in \mathbb{N} \cup \{0\}$, $P \in \text{Par}[N; b]$, $\omega_m \in \text{SI}_c^{k,m}[N; b]$ it holds

$$\left. \frac{d^n}{ds^n} \Phi_s^k[N; b](\omega_m, P_s) \right|_{s=0} = \left. \frac{d^n}{ds^n} \beta_s(\Phi^k[N; b])(\omega_m, P_s) \right|_{s=0}. \quad (79)$$

Remark 62: A direct computation shows that the PPA is satisfied if and only if equation (79) holds true for $n = 1$ – cf. [DDR19]. Moreover, on account of the lack of renormalization ambiguities for $m \geq 2$ – cf. Remarks 15-27 – the PPA is fulfilled whenever it holds for $m = 1$.

We state the main result of this appendix.

Proposition 63: The family $:\Phi^\bullet:$ defined in equation (43) satisfies the PPA as per Definition 61 with respect to a family of variations E_s of the elliptic operator E such that $E_s - E \in \Gamma_c(S^{\otimes 2}\psi^*T^*M)$ is a differential operator of order at most 1.

Proof. Observe that on account of Theorems 45-46 we may write for all $k \geq 2$ and $\omega \in \text{SI}_c^{k,1}[N; b]$

$$\Phi_s^k[N; b](\omega_1) = :\Phi^k:_{s,1}[N; b](\omega_1) + \sum_{\ell=0}^{k-2} :\Phi^\ell:_{s,1}(c_{s,k-\ell}[N; b] \lrcorner \omega_1),$$

where $:\Phi^k:$ is defined as in Example 43 while $c_{s,\ell} \in \Gamma^{\ell,1}[N; b]$ satisfies the hypothesis of theorem 45-46. Moreover, $c_{s,\ell}$ is a smooth and compactly supported family of variation and we set $c_\ell := c_{s,\ell}|_{s=0}$.

Our aim is to show that $:\Phi^\bullet:$ satisfies equation (79). In particular we shall impose equation (79) for a generic family Φ^\bullet of Wick powers. This will constraint the coefficient $c_{s,\ell}$ defined above, in particular we shall prove that equation (79) implies that we can choose $c_{s,\ell} = 0$, that is, $\Phi^\bullet = :\Phi^\bullet:$. We first consider the case $k = 2$. Setting $\delta := \left. \frac{d}{ds} \right|_{s=0}$, by direct inspection it holds

$$\begin{aligned} \delta[\Phi^2[N; b](\omega_1, P)] &= \delta[:\Phi^2: [N; b](\omega_1, P) + C_{s,2}[N; b](\omega_1)] \\ &= \Upsilon_{\delta(W_P)} : \Phi^2 : [N; b](\omega_1, P) + \delta[C_2[N; b](\omega_1)], \end{aligned}$$

where $C_2[N; b](\omega_1) := \int_\Sigma \langle c_2[N; b], \omega_1 \rangle$. Similarly the first order in s in the right hand side of equation (79) reads

$$\left. \frac{d}{ds} \beta_s \Phi^k[N; b](\omega_1, P_s) \right|_{s=0} = \Upsilon_{\delta(P)} \Phi^2[N; b](\omega_1, P) = \Upsilon_{\delta(P)} : \Phi^2 : [N; b](\omega_1, P),$$

where we exploited equations (76)-(77). Equation (79) entails

$$\delta[C_2[N; b](\omega_1)] = -\Upsilon_{\delta(W_P) + \delta(P)} : \Phi^2 : [N; b](\omega_1, P) = \langle [\delta(H)], \omega_1 \rangle,$$

where we used equation (40) and Remark 10. Therefore the PPA for $k = 2$ can be fulfilled if

$$\delta(c_2) = -[\delta(H)]. \quad (80)$$

Since $\delta(H)$ is local and covariant the above equation can be considered as a definition of the coefficient c_2 – indeed, it respects all requirement of Theorems 45-46. In particular $c_{2,s}$ is the solution to the ODE $\delta(c_{2,s}) = -[\delta(H_s)]$.

For the general case we compute once again the right and the left hand side of equation (79) using equation (50). The final result is

$$\begin{aligned} 0 &= \sum_{\ell=0}^{k-2} :\Phi^\ell: [N; b](\delta(c_{k-\ell}[N; b]) \lrcorner \omega_1, P) \\ &\quad - \Upsilon_{[\delta(H)]} \left[:\Phi^k: [N; b](\omega_1, P) + \sum_{\ell=0}^{k-2} :\Phi^\ell: [N; b](c_{k-\ell}[N; b] \lrcorner \omega_1, P) \right], \end{aligned}$$

Equation (40) leads to the following generalization of equation (80)

$$\delta(c_{k-\ell}) = -(\ell + 2)(\ell + 1)[\delta(H)][\otimes]c_{k-2-\ell} \quad 2 \leq \ell \leq k - 3.$$

Once again this can be used to define inductively the coefficients c_ℓ .

We now prove that $[\delta(H)] = 0$, which implies that the choice $c_{s,\ell} = 0$ leads to a family of Wick powers satisfying the PPA. We recall that we are considering families of variations E_s such that $E_s - E \in \Gamma_c(S^{\otimes 2}\psi^*T^*M)$ is a differential operator of order at most 1. For $\varphi \in \Gamma(\psi^*TM)$ we can write locally

$$(E_s\varphi - E\varphi)_\alpha = (A_s)^\alpha_{ab}(\nabla^\psi\varphi)_\alpha^b + (T_s)_{ab}\varphi^b, \quad (81)$$

where $(A_s)^\alpha_{ab}, (T_s)_{ab}$ are suitable smooth tensors. This implies that the Hadamard parametrix H_s associated with E_s has the form

$$H_s = \sum_{n \geq 0} V_{s,n} \sigma^n \log \sigma,$$

where σ does not depend on s since so it does the principal symbol of E_s [G98]. The tensors $V_{s,n} \in \text{SI}(\psi^*TM^{\boxtimes 2})$ satisfies a hierarchy of transport equations analogous to system (69). In particular $V_{s,0}$ satisfies

$$2g_{ab}\gamma^{\alpha\beta}(\nabla^\psi V_{s,0})^{bc}_\alpha (d\sigma)_\beta + g_{ab}V_{s,0}^{bc}(\Delta_\gamma\sigma - 2) + (A_s)^\alpha_{ab}V_{s,0}^{bc}(d\sigma)_\alpha = 0 \quad [V_{s,0}]^{bc} = g^{bc}. \quad (82)$$

It then follows that $\delta(V_0)$ satisfies a transport equation with initial condition $[\delta(V_0)] = 0$. This implies that $\delta(V_0) = 0$ and therefore

$$\delta(H) = \sum_{n \geq 1} \delta(V_n) \sigma^n \log \sigma = O(\sigma). \quad (83)$$

□

Acknowledgments

The work of C. D. was supported by the University of Pavia, while that of N. D. was supported in part by a research fellowship of the University of Pavia. We are grateful to Federico Faldino, Igor Khavkine, Alexander Schenkel and Jochen Zahn for the useful discussions. We are especially grateful to Klaus Fredenhagen for the enlightening discussions on the rôle of the algebra of functionals. This work is based partly on the MSC thesis of P. R. .

References

- [BRZ14] D. Bahns, D., K. Rejzner, J. Zahn, “*The effective theory of strings*” Commun. Math. Phys. (2014) **327**, 779.
- [BDLR18] C. Brouder, N. V. Dang, C. Laurent-Gengoux and K. Rejzner, “*Properties of field functionals and characterization of local functionals,*” J. Math. Phys. **59** (2018) no.2, 023508 [[arXiv:1705.01937](https://arxiv.org/abs/1705.01937) [math-ph]].
- [BDFY15] R. Brunetti, C. Dappiaggi, K. Fredenhagen and J. Yngvason, “*Advances in algebraic quantum field theory,*” Math. Phys. Stud. (2015), Springer 453p.
- [BDF09] R. Brunetti, M. Duetsch, K. Fredenhagen, “*Perturbative Algebraic Quantum Field Theory and the Renormalization Groups,*”, Adv. Theor. Math. Phys. **13** (2009) no.5, 1541 [[arXiv:0901.2038](https://arxiv.org/abs/0901.2038) [math-ph]].

- [BF99] R. Brunetti and K. Fredenhagen, “*Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds*,” Commun. Math. Phys. **208** (2000) 623 [[math-ph/9903028](#)].
- [BFV03] R. Brunetti, K. Fredenhagen and R. Verch, “*The Generally covariant locality principle: A New paradigm for local quantum field theory*,” Commun. Math. Phys. **237** (2003) 31 doi:10.1007/s00220-003-0815-7 [[math-ph/0112041](#)].
- [Car10] M. Carfora, “*Renormalization Group and the Ricci flow*”, Milan J. of Math. **78** (2010) 319 [[arXiv:1001.3595](#)] [hep-th].
- [Car14] M. Carfora, “*The Wasserstein geometry of nonlinear σ models and the Hamilton-Perelman Ricci flow*,” Rev. Math. Phys. **29** (2016) no.01, 1750001 [[arXiv:1405.0827](#)] [math-ph].
- [CG18] M. Carfora, C. Guenther, “*Scaling and entropy for the RG-2 flow*”, [[arXiv:1805.09773v1](#)] [math.DG].
- [CM17] M. Carfora and A. Marzuoli, “*Quantum Triangulations*,” Lect. Notes Phys. **942** (2017), Springer 392p.
- [Da14] N. V. Dang, “*Extension of distributions, scalings and renormalization of QFT on Riemannian manifolds*”, [[arXiv:1411.3670](#)] [math-ph].
- [DD16] C. Dappiaggi, N. Drago, “*Constructing Hadamard States via an Extended Miller Operator*”, Lett. Math. Phys. **106** (2016) no.11, 1587 [[arXiv:1506.09122](#)] [math-ph].
- [DDR19] C. Dappiaggi, N. Drago, P. Rinaldi, “*The algebra of Wick polynomials of a scalar field on a Riemannian manifold*”, [[arXiv:1903.01258](#)] [math-ph].
- [DHP16] N. Drago, T-P. Hack, N. Pinamonti, “*The generalized principle of Perturbative Agreement and the thermal mass*”, Ann. Henri Poinc. **18** (2017) no.3, 807 [[arXiv:1502.02705](#)] [math-ph].
- [FR12] K. Fredenhagen and K. Rejzner, “*Batalin-Vilkovisky formalism in the functional approach to classical field theory*,” Commun. Math. Phys. **314** (2012) 93 [[arXiv:1101.5112](#)] [math-ph].
- [FR13] K. Fredenhagen and K. Rejzner, “*Batalin-Vilkovisky formalism in perturbative algebraic quantum field theory*,” Commun. Math. Phys. **317** (2013) 697, [[arXiv:1110.5232](#)] [math-ph].
- [Fri80] D. Friedan, “*Nonlinear Models in Two Epsilon Dimensions*,” Phys. Rev. Lett. **45** (1980) 1057.
- [Fri85] D. H. Friedan, “*Nonlinear Models in Two + Epsilon Dimensions*,” Annals Phys. **163** (1985) 318.
- [G98] P.R. Garabedian, “*Partial Differential Equations*”, (1998) Chelsea Pub Co, 672 p.
- [Gaw99] K. Gawedzki, Conformal Field Theory, In: P. Deligne, P. Etingof, D.D. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, E. Witten, (eds.) *Quantum Fields and Strings: A Course for Mathematicians*, Institute for Advanced Study, Vol. 2, AMS (1999)
- [HK63] R. Haag and D. Kastler, “*An Algebraic approach to quantum field theory*,” J. Math. Phys. **5** (1964) 848.
- [Her19] E. Herscovich, “*Renormalization in Quantum Field Theory (after R. Borchers)*” to appear in Astérisque, <https://www-fourier.ujf-grenoble.fr/~eherscov/Articles/Renormalization>
- [Ham82] R. S. Hamilton, “*Three-manifolds with positive Ricci curvature*”, J. Diff. Geom. **17** (1982), 255.
- [HW01] S. Hollands and R. M. Wald, “*Local Wick polynomials and time ordered products of quantum fields in curved space-time*,” Commun. Math. Phys. **223** (2001) 289 [[gr-qc/0103074](#)].

- [HW02] S. Hollands and R. M. Wald, “*Existence of local covariant time ordered products of quantum fields in curved space-time*,” Commun. Math. Phys. **231** (2002) 309 [[gr-qc/0111108](#)].
- [HW03] S. Hollands and R. M. Wald, “*On the renormalization group in curved space-time*,” Commun. Math. Phys. **237** (2003) 123 [[gr-qc/0209029](#)].
- [HW05] S. Hollands and R. M. Wald, “*Conservation of the stress tensor in interacting quantum field theory in curved spacetimes*,” Rev. Math. Phys. **17** (2005) 227 [[gr-qc/0404074](#)].
- [Hö03] L. Hörmander, *The analysis of linear partial differential operators I - distribution theory and fourier analysis*, (2003) Springer, 440p.
- [Hus94] D. Husemöller, *Fibre Bundles*, (1994) Springer, 356p.
- [Kel10] K. J. Keller, “*Dimensional Regularization in Position Space and a Forest Formula for Regularized Epstein-Glaser Renormalization*,” (2009), PhD thesis, U. Hamburg, [arXiv:1006.2148](#) [math-ph].
- [Kel09] K. J. Keller, “*Euclidean Epstein-Glaser Renormalization*,” J. Math. Phys. **50** (2009) 103503 [[arXiv:0902.4789](#) [math-ph]].
- [KMM17] I. Khavkine, A. Melati, V. Moretti, “*On Wick polynomials of boson fields in locally covariant algebraic QFT*”, Ann. Henri Poincaré (2019) 20: 929.
- [KM16] I. Khavkine and V. Moretti, “*Analytic Dependence is an Unnecessary Requirement in Renormalization of Locally Covariant QFT*,” Commun. Math. Phys. **344** (2016) no.2, 581 [[arXiv:1411.1302](#) [gr-qc]].
- [KMS93] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, (1993) Springer, 434p.
- [Lin13] F. Lindner, “*Perturbative Algebraic Quantum Field Theory at Finite Temperature*,” (2013), PhD thesis, U. Hamburg.
- [MS76] J. Milnor, J. D. Stasheff, *Characteristic classes*, (1974), Princeton University Press, 334p.
- [Mor99a] V. Moretti, “*Local zeta function techniques versus point splitting procedure: A Few rigorous results*,” Commun. Math. Phys. **201** (1999) 327 [[gr-qc/9805091](#)].
- [Mor99b] V. Moretti, “*One loop stress tensor renormalization in curved background: The Relation between zeta function and point splitting approaches, and an improved point splitting procedure*,” J. Math. Phys. **40** (1999) 3843 [[gr-qc/9809006](#)].
- [NS14] J. Navarro, J. B. Sancho, “*Peetre-Slovák theorem revisited*”, [arXiv:1411.7499](#) [math.DG].
- [OS75] K. Osterwalder and R. Schrader, “*Axioms for Euclidean Green’s Functions. 2.*,” Commun. Math. Phys. **42** (1975) 281.
- [OS73] K. Osterwalder and R. Schrader, “*Axioms For Euclidean Green’s Functions*,” Commun. Math. Phys. **31** (1973) 83.
- [Per02] G. Perelman, “*The Entropy formula for the Ricci flow and its geometric applications*,” [math/0211159](#) [math-dg].
- [Per03] G. Perelman, “*Ricci flow with surgery on three-manifolds*,” [math/0303109](#) [math-dg].
- [PPV11] E. Poisson, A. Pound and I. Vega, “*The Motion of point particles in curved spacetime*,” Living Rev. Rel. **14** (2011) 7 [[arXiv:1102.0529](#) [gr-qc]].
- [Rej16] K. Rejzner, “*Perturbative Algebraic Quantum Field Theory : An Introduction for Mathematicians*,” Math. Phys. Stud. (2016), Springer 180p.

- [Sch98] D. Schlingemann, “*From Euclidean field theory to quantum field theory*,” Rev. Math. Phys. **11** (1999) 1151 [[hep-th/9802035](#)].
- [Shu87] M. A. Shubin, *Pseudodifferential operators and spectral theory*, Springer-Verlag Berlin Heidelberg (1987), 302p.
- [Thu97] W. P. Thurston, “*Three-dimensional geometry and topology Vol. 1*,” Edited by S. Levy, (1997) Princeton Math. Series, 35, Princeton Univ. Press, 328p.
- [Wa79] R. M. Wald, “*On The Euclidean Approach To Quantum Field Theory In Curved Space-time*,” Comm. Math. Phys. **70** (1979) 221.
- [We108] R. O. Wells, *Differential Analysis on Complex Manifolds*, (2008) Springer, 299p.
- [Za15] J. Zahn, “*Locally covariant charged fields and background independence*,” Rev. Math. Phys. **27** (2015) no.07, 1550017 [[arXiv:1311.7661](#)] [math-ph].