University of Trento
Department of Mathematics
Doctoral School - XXXIII cycle

Ph.D. Thesis

Portfolio optimization in presence of a self-exciting jump process:
from theory to practice

Advisor:
Prof. Luca Di Persio

Ph.D. Candidate:
Andrea Veronese

Academic Year 2019/2020

Portfolio optimization
in presence of a self-exciting jump process: from theory to practice

Andrea Veronese

Doctoral Thesis
Doctoral programme in Mathematics, XXXIII cycle
Department of Mathematics, University of Trento

Trento, Italy
April 27, 2022

Advisor:
Prof. Luca Di Persio

Ph.D. Candidate:
Andrea Veronese

A tratti percepisco tra indistinto brusio
Particolari in chiaro,
Di chiara luce splendidi,
Dettagli minimali in primo piano,
Più forti del dovuto e adesso so
Come fare non fare, quando dove perché
E ricordando che tutto va come va
Come fare non fare, quando dove perché
E ricordando che tutto va come va...

A tratti
C.S.I.

Contents

Overview

In writing this Thesis we aim at blending together recent advances in the analysis of stochastic self–exciting processes with recent mathematical challenges characterizing the well–known class of portfolio optimization problems "à la Merton". Such problems are particularly related to the optimal portfolio allocation over a fixed time horizon (not necessarily finite), with the portfolio's dynamic being possibly influenced by random components. More specifically, we focus our attention on portfolio's evolution steered by a specific class of self-exciting processes, which has been introduced in [HM16]. This has led us first to recall fundamental stochastic analysis results which have been applied then to our framework as to determine the right mathematical setting to exploit self–exciting model. In particular, we describe the case where the investor's goal is to maximize his utility, with the corresponding portfolio dynamic calibrated on market data.

Portfolio optimization problems have gained an increasing interest starting from the second half of the 20th century when H. Markowitz, see e.g. [Mar52], proposed a solution identifying the most efficient portfolio composition in terms of mean and variance for a set of securities, hence promoting a series of subsequent analyses based on considering portfolio tasks within the optimal control theory framework, see among others [Sha64; CR76; BL90].
Later on, R. Merton, see e.g. [Mer69], paved the way for applications of stochastic calculus to finance with the goal of providing the optimal allocation for a given portfolio with continuous dynamic over a finite time horizon. Further, in [Mer89], by exploiting results contained in [CH89], Merton solved the optimal consumption and portfolio policy in closed form in infinite horizon. Such seminal contributions have then known several variants along the years. The above mentioned results have guided us to add extra factors trying to enrich the financial scenario already treated to allow investor's preferences to be defined via a second control, namely cumulative consumption, analogously to [HH93]. This choice embodies the peculiarity of durability in the setting, see Section 1.3.
Moreover, our approach overcomes the limit imposed by considering *only* Lévy processes, allowing to model time-dependent increments and including a self-exciting process to describe the instantaneous frequency of jumps. For the sake of completeness, we underline that our proposal generalizes the ones in [BKR02] and [BKR01a] where an infinite-time horizon has been considered.

It is worth stressing that our risky model has relevant financial applications, as first noticed in [HO74], and then extensively used in economic scenarios, see e.g. [ASJ+09]. Previous choice has been made to accurately describe jump clustering characterizing observations in assets' level, also overcoming the difficulty of justifying extreme increments gathered in a small amount of

time, see e.g. [Haw18].
Our contribution is, to the best of our knowledge, the first successfully completed attempt along the previously described direction.

The Thesis is structured as follows: in the first chapter we provide the general notation and necessary definitions to have a self-contained work by also resuming and exposing the self-exciting model introduced in [HM16] and [HM19].
Then, we exploit the model presented in the first part as starting point for the innovative part of this work: the self-exciting model will describe the dynamic of the risky part of the portfolio. In this first part, we also report several propositions and results that will be later exploited along the whole Thesis. We conclude by presenting the stochastic control problem whose solution is defined by the investor's optimal risky allocation and consumption which maximize the investor's satisfaction at any given time considered.

In the second chapter, we tackle down the stochastic control problem previously formalized. The main result characterizing this part is the derivation of the Hamilton-Jacobi-Bellman (HJB) equation corresponding to the aforementioned optimization problem as a Partial-Integro Differential Equation (PIDE) with a gradient constraint. In doing so, we start from proving that the Dynamic Programming Principle (DPP) holds in our framework by extending and adapting the approach reported in [GS12, Section 3, Chapter 4]. Since DPP holds, we can then derive the HJB equation as a PIDE, by exploiting the Itô-Doeblin's lemma for discontinuous processes. The PIDE obtained is subject to a gradient constraint which will derived in Section 2.2.1.

In Chapter 3, we formulate the notions of viscosity solution for our specific problem and then, we will use them to cover the last part of the work. The goal of the chapter is prove that the value function V is the unique constrained viscosity solution for the optimization problem. In doing so, we report several equivalent formulations for viscosity solutions, each of which particularly addresses specific results later stated. We conclude by showing, based on a comparison principle technique, the uniqueness result for V.

Then, in Chapter 4, we derive the penalty approximation needed to solve the HJB equation, and we also discuss the numerical scheme used to solve the corresponding penalization problem.

Within the final part of the Thesis, we provide a complete treatment of a real-world case embedding previously stated tasks and solutions also developing an effective calibration procedure. In particular, we have chosen to apply the model to a period when financial market turmoil has intensified and broadened. Within this scenario, it has been observed overall increments of assets' volatility values, with related extreme returns clustering occurred. This has led to market instability, accompanied by fluctuations whose dynamics were hard to predict via *standard models*, even on small time windows and with respect to both their magnitude and frequency. As to give an example, approaches based on canonical Gaussian assumptions for log-returns failed by a large amount. Consequently, in Chapter 5, we extensively describe the approach in [CDPVnda] which will be later used to capture the *cluster* phenomenon we are interested in. To the best of our knowledge, we remark that such approach has not been used in presence of real data before. In particular, we provide an algorithm which collects a fast, but robust, calibration of a self-exciting jump model for the ENI asset listed in the *Italian Stock Exchange*.

For the sake of completeness, we remark that our proposal calibration is a two-step algorithm: first we operate a preliminary tuning of the model, which will turn out to be the a good starting point to retune the high-dimensional system through sequential Monte Carlo (SMC^2) algorithm and, finally, we compare the results obtained.

The second part of the approach mentioned is based on SMC^2 algorithm, which develops the Monte Carlo Particle Filtering algorithm, see e.g. [ABL02].
Let us underline that SMC^2 is more accurate compared to other currently widely used methodologies as, e.g., standard Monte Carlo based solutions and log-likelihood routines. Finally, the approach proposed gives more insights from the pure *practitioners'* point of view, when compared to more theoretically oriented papers as, e.g., [Cor+20].

In the last chapter of the Thesis, we exploit the obtained calibration results and, by using the numerical scheme defined in Chapter 4, we investigate and comment the numerical results obtained.

Part I

Portfolio optimization in the presence of a self-exciting jump diffusion process

Chapter 1

General framework

In what follows we provide an introduction on the theoretical model and on the framework exploited, with all the necessary machineries and results needed. The goal is to rigorously define both the dynamic of the risky quantities involved and the financial setting proposed in [HM16] and [HM19].

1.1 Notation and preliminary concepts

Consider a finite time horizon $T > 0$ and a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in(0,T]}, \mathbb{P}\big)$, supporting a Brownian motion $\boldsymbol{W} := (W_t)_{t\in[0,T]}$ and a Poisson-type stochastic process $\mathcal{N} := (\mathcal{N}_t)_{t\in(0,T]}$ both defined under real-world probability measure $\mathbb{P}$. The most common interpretation of the probability space is an experiment, thus we can represent $\omega \in \Omega$ as the experiment result and, therefore, Ω denotes all the possible realizations of the random experiment with a certain probability given by $\mathbb{P}$. Finally $\mathcal{F}$ is the σ-field representing the set of events $B \subset \Omega$ which we will work with. Therefore, we can define the filtration $(\mathcal{F}_t)_{t\in(0,T]}$ as complete and right continuous: the former condition is satisfied by the fact that $\mathcal{F}_0$ contains the null $\mathbb{P}$-set and the latter by the fact that

$$\mathcal{F}_t = \bigcap_{t<u} \mathcal{F}_u, \forall t \in [0, T]$$

In particular, we denote the *cumulative* filtration $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^N \vee \mathcal{F}_t^J$, $\forall\, t \in (0, T]$, where $(\mathcal{F}_t^W)_{t\in[0,T]}$ is the filtration generated by the Brownian Motion, while $(\mathcal{F}_t^{\mathcal{N}})_{t\in(0,T]}$ is the filtration generated by the Poisson process and, finally, $(\mathcal{F}_t^J)_{t\in(0,T]}$ is the filtration generated by the collection of jump-sizes occurred. Such jumps are described by a continuous random variable. The stochastic process $\mathcal{N}$ is characterized by the mean of the Poisson random measure $N(\cdot) : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{N}$, defined in the following, similarly to what has been done in [IW14].

Definition 1.1.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with finite measure ν on $(E, \mathcal{E})$. The Poisson random measure with intensity λ is a family of integer-valued random measure such that $N : \Omega \times E \to \mathbb{N}_0$. The following conditions are satisfied:*

1. *for each $B \in \mathcal{E}, N(B)$ is Poisson distributed, i.e.*

$$\mathbb{P}(N(B) = n) = e^{-\nu(B)} \frac{\nu(B)^n}{n!},$$

for all $n \in \mathbb{N}$;

2. *if $B_1, B_2, \ldots, B_k \subset E$ are disjoint sets, then $N(B_1), N(B_2), \ldots, N(B_k)$ are mutually independent.*

Therefore, we can derive that for almost all $\omega \in \Omega, N(\omega, \cdot)$ is an integer-valued random measure in E. For sake of simplicity, in the Thesis we will make use of the compact notation $\mathcal{N}_t := \int_0^t \int_{\mathbb{R}\setminus\{0\}} N(ds, dq)$.
Moreover, we will also deal with the compensated version of the previous measure, that is to say $\widetilde{N}(dt, dq) := N(dt, dq) - \lambda([0, t))\zeta(q)dtdq$. In particular, we define $\lambda([0, t))$ as the instantaneous frequency for $\mathcal{N}_t$ which, for the sake of simplicity, will be denoted by $\lambda_t := \lambda([0, t))$ in the remaining part of the Thesis. Moreover, we identify with $\boldsymbol{J} := (J_1, \ldots, J_{\mathcal{N}_t})$ the sequence of independent, identically distributed (i.i.d.) random variables defining the jumps occurred up to time t.
Namely, the random variable J_i describes the amplitude of the i-th jump occurred, which is distributed according to a double exponential density, defined by its continuous probability distribution as follows:

$$\zeta(q) := p\rho^+ e^{-\rho^+ q} \mathbb{1}_{\{q \geq 0\}} + (1 - p)|\rho^-|e^{-\rho^- q} \mathbb{1}_{\{q<0\}}. \tag{1.1}$$

We identify with $p \in (0, 1)$ the probability of obtaining a positive jump, while $1/\rho^+ \in \mathbb{R}_+$ (*resp.* $1/\rho^- \in \mathbb{R}_-$) is the expected value of the positive (*resp.* negative) jumps.
Previous choice of jumps distribution has been made to properly model jumps in finance (see e.g. [Den07]) also in case of extremely ample ones. Moreover, we remark that $\zeta(\cdot)$ in Equation (1.1) satisfies the following integrability conditions:

$$\begin{aligned} N(dt, \{0\}) = 0, \quad \int_{\mathbb{R}\setminus\{0\}} (|q|^2 \wedge 1)\zeta(q)dq < \infty, \quad \int_{\mathbb{R}\setminus\{0\}} |q(e^q - 1)|\zeta(q)dq \\ < \infty \quad \text{and} \quad \int_{|q|\geq 1} |e^q - 1|\zeta(q)dq < \infty. \end{aligned} \tag{1.2}$$

1.1.1 Instantaneous frequency

This section will be devoted to describe the instantaneous frequency we have introduced for the Poisson random measure in Definition 1.1.1.
Note that the instantaneous frequency will be extensively studied in 1.2. Let $\mathbf{N}$ be a Poisson random measure over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then we can introduce over the same space the stochastic process $\boldsymbol{\lambda} := (\lambda_t)_{t\in(0,T]}$, denoting the instantaneous frequency of jumps for each $t \in [0, T]$:

$$\lambda_t := \lim_{\Delta t \searrow 0} \frac{\mathbb{E}[\mathcal{N}_{t^-+\Delta t} - \mathcal{N}_{t^-}|\mathcal{F}_{t^-}]}{\Delta t}.$$

In what follows, we will exploit λ_t as solution of a mean-reverting Stochastic Differential Equation (SDE) driven by the Poisson random measure.

1.2 The model

In writing this Thesis we have been inspired by the model exposed in [HM16] and [HM19]. We use such model as a starting point to study a portfolio optimization problem characterized by a self-exciting dynamic, which represents the innovative part of this Thesis since, to the extend of our knowledge, a self-exciting dynamic has never been exploited in a portfolio optimization "à la Merton".

Before diving into the innovative part, we recall the model used to describe the risky asset and all the quantities needed in defining the tasks we will consider later on.

Let us introduce the instantaneous frequency of jumps. Let $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F})_{t\in(0,T]}\right)$ be the filtered probability space defined in Section 1.1.1, and let us introduce the stochastic process $\boldsymbol{\lambda} := (\lambda_t)_{t\in(0,T]}$ solution of the following SDE:

$$d\lambda_t = \alpha(\xi - \lambda_t)dt + \eta \int_{\mathbb{R}\setminus\{0\}} |q| N(dt, dq), \tag{1.3}$$

where q represents the jump size distributed according to $\zeta(\cdot)$ in Equation (1.1).
Therefore, $\boldsymbol{\lambda}$ turns to be a stochastic process depending on its current value: whether the current value of the process is less than the long-term mean (see ξ, then the drift will be positive, with reversion speed $\alpha \in \mathbb{R}_+$; if the current value of the process is greater than the long-term mean, the drift will be negative. Hence, ξ acts as an equilibrium level for the process itself. Finally, $\eta \in \mathbb{R}_+$ is the parameter governing the influence of the past jumps on the current frequency.
Moreover, we introduce the cumulative frequency up to time t:

$$\Lambda_t := \int_0^t \lambda_s ds. \tag{1.4}$$

Although the definition has not a particular economic meaning, it will be useful in terms of compact notation later on.

1.2.1 The dynamic for the risky asset

On the same probability space defined for Equation (1.3), let us introduce the stochastic process $\boldsymbol{S} := (S_t)_{t\in(0,T]}$, solution of the following SDE:

$$\begin{aligned}
\frac{dS_t}{S_{t^-}} &= \mu dt + \sigma dW_t + \int_{\mathbb{R}\setminus\{0\}} (e^q - 1)N(dt, dq) - \lambda_t \mathbb{E}[e^J - 1]dt \\
&= \mu dt + \sigma dW_t + \left(\int_{\mathbb{R}\setminus\{0\}} (e^q - 1)N(dt, dq) - \lambda_t \int_{\mathbb{R}\setminus\{0\}} (e^q - 1)\zeta(q)dqdt \right) \\
&= \mu dt + \sigma dW_t + \left(\int_{\mathbb{R}\setminus\{0\}} (e^q - 1)\widetilde{N}(dt, dq) \right),
\end{aligned} \tag{1.5}$$

where $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+$ are, respectively, the deterministic drift and the parameter governing the Brownian increments. All the remaining elements are defined as in Section 1.1.

1.2.2 Features of the model

In what follows, we exhibit some useful propositions to better comprehend the model, and the corresponding proofs are collected in order to have a self-contained work. Among the various propositions, we remark that the conditions of stability for the process $\boldsymbol{\lambda}$ will be used as constraints in the calibration procedure in Chapter 5.

Proposition 1.2.1. *The Equation* (1.5) *has the following closed-form exponential solution:*

$$S_t = S \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t - \Lambda_t \mathbb{E}\left[e^J - 1\right] + \sigma W_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} qN(dt, dq)\right), \tag{1.6}$$

with $S_0 = S$.

Furthermore, the corresponding log-return $R_t := \log(S_t/S_0)$ is defined by the equation

$$R_t = \left(\left(\mu - \frac{\sigma^2}{2}\right) t - \Lambda_t \mathbb{E}[e^J - 1] + \int_0^t \int_{\mathbb{R}\setminus\{0\}} qN(dt, dq)\right). \tag{1.7}$$

Proof. Let S_t satisfy Equation (1.5) and let us define $Z_t = f(t, S_t) := \log(S_t)$. Then, since $f(\cdot)$ satisfies the required regularity conditions, we can apply the Itô-Doeblin's formula for jump processes, see e.g. [Shr04, Section 11.5].

Thus, we obtain that Z_t is the solution of the following SDE:

$$dZ_t = \left(\mu - \frac{\sigma^2}{2} - \lambda_t \mathbb{E}[e^J - 1]\right) dt + \sigma dW_t + \int_{\mathbb{R}\setminus\{0\}} qN(dt, dq),$$

which can be rewritten, including the initial condition $Z_0 := \log(S)$, as the integral:

$$\frac{\log(S_t)}{\log(S)} = \left(\mu - \frac{\sigma^2}{2}\right) t - \mathbb{E}[e^J - 1] \int_0^t \lambda_s ds + \sigma W_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} qN(dt, dq).$$

By applying the inverse mapping of $f(\cdot)$, we can immediately obtain the solution reported in Equation (1.6). Finally, Equation (1.6) comes directly from the definition of R_t. □

Proposition 1.2.2. *The Equation* (1.3) *has solution:*

$$\lambda_t = \xi + (\lambda - \xi)e^{-\alpha t} + \eta \int_0^t \int_{\mathbb{R}\setminus\{0\}} e^{-\alpha(t-s)} |q| N(ds, dq). \tag{1.8}$$

with $\lambda_0 = \lambda \in \mathbb{R}_+$ denoting the initial value for the process $\boldsymbol{\lambda}$.

Proof. As for Equation (1.6), we start the proof by considering λ_t as solution of (1.3).
Then, we introduce a suitable change of variable $L_t := \lambda_t - \xi$, whose derivative is defined as:

$$dL_t = d\lambda_t = -\alpha L_t dt + \eta \int_{\mathbb{R}\setminus\{0\}} |q| N(dt, dq).$$

Successively, we apply another change of variable $Z_t := L_t e^{\alpha t}$ and, exploiting the Itô-Döeblin Lemma, we obtain:

$$\begin{aligned} dZ_t &= dL_t e^{\alpha t} + L_t e^{\alpha t}\alpha dt \\ &= \left(-\alpha L dt + \eta |q| N(dt, dq)\right) e^{\alpha t} + L_t e^{\alpha t}\alpha dt \\ &= e^{\alpha t}\eta \int_{\mathbb{R}\setminus\{0\}} |q| N(dt, dq)\ . \end{aligned}$$

Therefore, we assert

$$Z_t = Z_0 + \eta \int_0^t e^{\alpha s} \int_{\mathbb{R}\setminus\{0\}} |q| N(ds, dq).$$

By reverting the transformations applied, we obtain the solution reported in Equation (1.8), with initial value for the solution process being $\lambda_0 = \lambda$. $\square$

Proposition 1.2.3. *The first absolute moment of J is given by:*

$$\mathbb{E}\left[|J|\right] = \frac{p}{\rho^+} + \frac{(1-p)}{|\rho^-|} =: \mu_J. \tag{1.9}$$

While the second moment is:

$$\mathbb{E}\left[|J|^2\right] = \frac{2p}{(\rho^+)^2} + \frac{2(1-p)}{|\rho^-|^2}. \tag{1.10}$$

Proof. First of all, we compute $\mu_J := \mathbb{E}\left[|J|\right]$ as:

$$\begin{aligned} \mathbb{E}\left[|J|\right] = \int_{-\infty}^{\infty} |q|\zeta(q)dq &= p\int_0^{\infty} q\rho^+ e^{-\rho^+ q}dq - (1-p)\int_0^{-\infty} q\rho^- e^{-\rho^- q}dq \\ &= p\left[\frac{q^2}{2}\rho^+ e^{-\rho^+ q} - \frac{1}{\rho^+}e^{-\rho^+ q}\right]_{q=0}^{\infty} - (1-p)\left[\rho^- \frac{q^2}{2}e^{-\rho^- q} - \frac{1}{\rho^-}e^{-\rho^- q}\right]_{q=0}^{-\infty} \\ &= \frac{p}{\rho^+} - \frac{(1-p)}{\rho^-}. \end{aligned} \tag{1.11}$$

Then, the second moment is obtained as follows:

$$\begin{aligned} \mathbb{E}\left[|J|^2\right] = \int_{-\infty}^{\infty} |q|^2\zeta(q)dq &= p\int_0^{\infty} q^2\rho^+ e^{-\rho^+ q}dq + (1-p)\int_0^{-\infty} q^2\rho^- e^{-\rho^- q}dq \\ &= p\left[\frac{q^3}{3}\rho^+ e^{-\rho^+ q} + \frac{2}{\rho^+}\int \rho^+ e^{-\rho^+ q}q dq\right]_{q=0}^{\infty} \\ &\quad + (1-p)\left[\rho^- \frac{q^2}{2}e^{-\rho^- q} + \frac{2}{\rho^-}\int \rho^- e^{-\rho^- q}q dq\right]_{q=0}^{-\infty} \\ &= \frac{2p}{(\rho^+)^2} + \frac{2(1-p)}{(\rho^-)^2}, \end{aligned}$$

the latter deriving analogously as provided in treating Equation (1.11). In particular, $\mathbb{E}[J|J > 0] = p\int_0^{\infty} q\rho^+ e^{-\rho^+ q}dq$ and $\mathbb{E}[J|J \le 0] = -(1-p)\int_0^{-\infty} q\rho^- e^{-\rho^- q}dq$. $\square$

Proposition 1.2.4. *Let* $\mathcal{X}_0 := [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+$*,* S *and* λ *be defined according to the Equations* (1.18) *and* (1.8).
Therefore, we can define $\phi \in C_0^{1,2,1}(\mathcal{X}_0)$ *the linear integro-differential operator for* $\phi : \mathcal{X}_0 \to \mathbb{R}$ *with compact support as:*

$$\begin{aligned}\mathcal{L}\phi(t,S,\lambda) = &\left(\alpha(\xi-\lambda)+\eta\lambda\mu_J\right)\frac{\partial}{\partial\lambda}\phi + \left(\mu - \lambda\mathbb{E}[e^J-1]\right)S\frac{\partial}{\partial S}\phi \\ &+ \frac{1}{2}(S\sigma)^2\frac{\partial^2}{\partial S^2}\phi + \lambda\int_{\mathbb{R}\setminus\{0\}} \phi(t,S+q,\lambda+\eta|q|) - \phi(t,S,\lambda) \\ &- S(e^q-1)\frac{\partial}{\partial S}\phi(t,S,\lambda) - \eta|q|\frac{\partial}{\partial\lambda}\phi(t,S,\lambda)\zeta(q)dq.\end{aligned}$$

Proof. The proof follows directly from [Pro, Theorem 35], since $(\boldsymbol{S},\boldsymbol{\lambda})$ is a semimartingale. Therefore, we assert:

$$\begin{aligned}\mathcal{L}\phi(t,S,\lambda) := &\lim_{h\searrow 0}\frac{\mathbb{E}[\phi(t+h,S_{t+h},\lambda_{t+h})-\phi(t,S,\lambda)|\mathcal{F}_t]}{h} = \\ &\lim_{h\searrow 0}\frac{1}{h}\mathbb{E}\left[\left(\int_t^{t+h}\alpha(\xi-\lambda_s)ds + \eta\int_t^{t+h}\int_{\mathbb{R}\setminus\{0\}}|q|N(ds,dq)\right)\frac{\partial}{\partial\lambda}\phi\right. \\ &+\int_t^{t+h}\left(\mu-\lambda_s\mathbb{E}[e^J-1]\right)S_s\frac{\partial}{\partial S}\phi ds \\ &+\int_t^{t+h}\frac{1}{2}(S_s\sigma)^2\frac{\partial^2}{\partial S^2}\phi ds + \int_t^{t+h}\int_{\mathbb{R}\setminus\{0\}}\phi(s,S_s+q,\lambda_s+\eta|q|)-\phi(t,S,\lambda) \\ &\left.-S_s(e^q-1)\frac{\partial}{\partial S}\phi(s,S_s,\lambda_s) - \eta|q|\frac{\partial}{\partial\lambda}\phi(s,S_s,\lambda_s)N(ds,dq)\right|\mathcal{F}_t\Bigg]\end{aligned}$$

Then, since $\phi \in C^{1,2,1}(\overline{\mathcal{X}}_0)$, the proof follows directly by Taylor expansion of ϕ around (t,S,λ). $\square$

With the next proposition, we report the first two moments of instantaneous frequency, since they will be useful in Section 5.2.4.

Proposition 1.2.5. *The first two moments of* λ_t *are defined by* $\mathbb{E}\left[\lambda_t|\mathcal{F}_0\right]$ *and* $\mathbb{V}\left[\lambda_t|\mathcal{F}_0\right]$*, and their analytical solutions respectively read as:*

$$\mathbb{E}[\lambda_t|\mathcal{F}_0] = \lambda e^{-(\alpha-\eta\mu_J)t} + \lambda_\infty\left(1-e^{-(\alpha-\eta\mu_J)t}\right), \tag{1.12}$$

with $\lambda_\infty := \lim_{t\to\infty}\mathbb{E}[\lambda_t|\mathcal{F}_0] = \frac{\alpha\xi}{(\alpha-\eta\mu_J)}$*; and*

$$\mathbb{V}[\lambda_t|\mathcal{F}_0] = \eta^2\mathbb{E}[|J|^2]\left[\frac{\lambda_\infty}{2\alpha}(1-e^{-2\alpha t}) + \frac{(\lambda-\lambda_\infty)}{\alpha+\eta\mu_J}\left(e^{-(\alpha-\eta\mu_J)t}-e^{-2\alpha t}\right)\right]. \tag{1.13}$$

Proof. The first expected value for λ_t is obtained as follows:

$$\begin{aligned}\mathbb{E}[\lambda_t|\mathcal{F}_0] &= \mathbb{E}\left[\xi + (\lambda-\xi)e^{-\alpha t} + \int_0^t \eta e^{-\alpha(t-s)}|q|N(ds,dq)\middle|\mathcal{F}_0\right]\\ &= \xi + (\lambda-\xi)e^{-\alpha t} + \eta\mu_J\int_0^t e^{-\alpha(t-s)}\mathbb{E}[\lambda_s|\mathcal{F}_0]ds.\end{aligned}$$

Then, the analytical solution for $\mathbb{E}[\lambda_t|\mathcal{F}_0]$ follows by solving the corresponding ODE:

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}[\lambda_t|\mathcal{F}_0] &= -\alpha(\lambda-\xi)e^{-\alpha t} + \eta\mu_J\mathbb{E}[\lambda_t|\mathcal{F}_0] - \alpha\eta\mu_J\int_0^t e^{-\alpha(t-s)}\mathbb{E}[\lambda_s|\mathcal{F}_0]ds\\ &= -\alpha\underbrace{\left[(\lambda-\xi)e^{-\alpha t} + \eta\mu_J\int_0^t e^{-\alpha(t-s)}\mathbb{E}[\lambda_s|\mathcal{F}_0]ds\right]}_{=\mathbb{E}[\lambda_t|\mathcal{F}_0]-\xi} + \eta\mu_J\mathbb{E}[\lambda_t|\mathcal{F}_0]\\ &= \xi\alpha + \mathbb{E}[\lambda_t|\mathcal{F}_0](\eta\mu_J-\alpha).\end{aligned}$$

leading to

$$\mathbb{E}[\lambda_t|\mathcal{F}_0] = \lambda e^{-(\alpha-\eta\mu_J)t} + \frac{\alpha\xi}{(\alpha-\eta\mu_J)}\left(1-e^{-(\alpha-\eta\mu_J)t}\right). \tag{1.14}$$

Before focusing on the variance, we introduce a new process $\widetilde{\boldsymbol{\lambda}} := (\widetilde{\lambda}_t)_{t\in(0,T]}$, where $\widetilde{\lambda}_t := \lambda_t - \mathbb{E}[\lambda_t|\mathcal{F}_0]$.
Since $\widetilde{\boldsymbol{\lambda}}$ is the compensated version of the process $\boldsymbol{\lambda}$, thus a local martingale, we can exploit semimartingale quadratic variation properties, see e.g. [Pro, Theorem 22].
In particular, the quadratic variation of a continuous part of a semimartingale is null, therefore we have that $\mathbb{V}[\lambda_t|\mathcal{F}_0] = \mathbb{E}\left[[\widetilde{\lambda},\widetilde{\lambda}]_t|\mathcal{F}_0\right] = \mathbb{E}\left[[\lambda,\lambda]_t|\mathcal{F}_0\right]$, thus

$$\begin{aligned}[\lambda,\lambda]_t &= \Big[\xi + (\lambda-\xi)e^{-\alpha t} + \eta\int_0^t e^{-\alpha(t-s)}|q|N(ds,dq),\\ &\qquad \xi + (\lambda-\xi)e^{-\alpha t} + \eta\int_0^t e^{-\alpha(t-s)}|q|N(ds,dq)\Big]\\ &= \int_0^t \eta^2 e^{-2\alpha(t-s)}|q|^2 N(ds,dq).\end{aligned} \tag{1.15}$$

And

$$\begin{aligned}
\mathbb{V}[\lambda_t|\mathcal{F}_0] =& \mathbb{E}\left[[\lambda,\lambda]_t|\mathcal{F}_0\right] = \mathbb{E}[|J|^2]\eta^2\int_0^t e^{-2\alpha(t-s)}\mathbb{E}[\lambda_s|\mathcal{F}_0]ds \\
=& \mathbb{E}[|J|^2]\eta^2 e^{-2\alpha t}\int_0^t e^{2\alpha s}\left(\lambda e^{-(\alpha-\eta\mu_J)s} + \frac{\alpha\xi}{(\alpha-\eta\mu_J)}\left(1-e^{-(\alpha-\eta\mu_J)s}\right)\right)ds \\
=& \mathbb{E}[|J|^2]\eta^2 e^{-2\alpha t}\left[\frac{\lambda}{(\alpha+\eta\mu_J)}(e^{(\alpha+\eta\mu_J)t}-1) + \frac{\alpha\xi}{2\alpha(\alpha-\eta\mu_J)}(e^{2\alpha t}-1)\right. \\
& \left. -\frac{\alpha\xi}{(\alpha-\eta\mu_J)(\alpha+\eta\mu_J)}(e^{(\alpha+\eta\mu_J)t}-1)\right] \\
=& \mathbb{E}[|J|^2]\eta^2\left[\frac{\lambda}{(\alpha+\eta\mu_J)}(e^{-(\alpha-\eta\mu_J)t}-e^{-2\alpha t}) + \frac{\alpha\xi}{2\alpha(\alpha-\eta\mu_J)}(1-e^{-2\alpha t})\right. \\
& \left. -\frac{\alpha\xi}{(\alpha-\eta\mu_J)(\alpha+\eta\mu_J)}(e^{-(\alpha-\eta\mu_J)t}-e^{-2\alpha t})\right] \\
=& \mathbb{E}[|J|^2]\eta^2\left[\frac{(\lambda-\lambda_\infty)}{(\alpha+\eta\mu_J)}(e^{-(\alpha-\eta\mu_J)t}-e^{-2\alpha t}) + \frac{\lambda_\infty}{2\alpha}(1-e^{-2\alpha t})\right], \qquad (1.16)
\end{aligned}$$

where $\mathbb{E}[|J|^2]$ has the closed-form solution reported in Equation (1.10). □

Proposition 1.2.6. *The process* $\boldsymbol{\lambda}$ *is stable for* $\alpha - \eta\mu_J > 0$.

Proof. The previous proposition is directly derived from the forms of $\mathbb{E}[\lambda_t|\mathcal{F}_0]$ and $\mathbb{V}[\lambda_t|\mathcal{F}_0]$, where the two moments for λ_t are to be kept positive. In particular, we study the asymptotic behaviour of the process, denoted as λ_∞, which is computed as

$$\lambda_\infty := \lim_{t\to\infty}\mathbb{E}[\lambda_t|\mathcal{F}_0] = \frac{\alpha\xi}{(\alpha-\eta\mu_J)}.$$

The above ratio should be positive in order to ensure that the expectation of λ_t with $t\to\infty$ is well-defined, and converges to a finite positive value if $\alpha,\xi > 0$ and $\alpha-\eta\mu_J > 0$.
Note that α,ξ should be positive by the meaning of the parameters themselves, while the condition $\alpha > \eta\mu_J$ will be a useful constraint in the calibration procedure in Section 5.1.

Moreover, we assert that the process has finite variance as t goes to ∞ if the previous conditions are satisfied:

$$\lim_{t\to\infty}\mathbb{V}[\lambda_t|\mathcal{F}_0] = \mathbb{E}[|J|^2]\eta^2\frac{\lambda_\infty}{2\alpha}.$$

Therefore, we conclude by saying that the asymptotic results reported guarantee the stability of the process $\boldsymbol{\lambda}$ for $t\to\infty$. □

Proposition 1.2.7. *The solution for* $\mathbb{E}\left[\Lambda_t|\,\mathcal{F}_0\right]$ *is defined as:*

$$\mathbb{E}[\Lambda_t|\mathcal{F}_0] = \frac{\lambda(1-e^{-(\alpha-\eta\mu_J)t})}{(\alpha-\eta\mu_J)} + \lambda_\infty\left(t - \frac{1-e^{-(\alpha-\eta\mu_J)t}}{(\alpha-\eta\mu_J)}\right), \qquad (1.17)$$

where λ_∞ *is defined in Proposition 1.2.5.*

Proof. By Fubini's theorem

$$\begin{aligned}\mathbb{E}[\Lambda_t|\mathcal{F}_0] &= \mathbb{E}\left[\int_0^t \lambda_s ds \middle| \mathcal{F}_0\right] = \int_0^t \mathbb{E}[\lambda_s|\mathcal{F}_0]ds \\ &= \int_0^t \left(\lambda e^{-(\alpha-\eta\mu_J)s} + \frac{\alpha\xi}{(\alpha-\eta\mu_J)}\left(1-e^{-(\alpha-\eta\mu_J)s}\right)\right)ds.\end{aligned}$$

HenceEquation (1.17) is solved by computing the last integral. □

1.3 Portfolio optimization problem

Given the general framework introduced in Section 1.1, we can now focus on the optimal control problem. Throughout the Thesis we will consider a financial market where bonds remunerate the investor at an instantaneous constant risk-free rate $r > 0$, and where there exists a risky asset following the dynamic presented in Equation (1.5). Moreover, we will assume that the risk-free rate r satisfies the condition defined in [BKR01b], which corresponds to $r < \mu - \sigma^2/2$.

Suppose now there is an investor in the market who wants to maximize his satisfaction over some finite time horizon T. Let $\boldsymbol{\pi} := (\pi_t)_{t\in[0,T]}$ and $\pi_t \in [0,1]$ be the fraction of the investor's wealth concerning the risky asset at time t and let us assume there are no transaction costs.
Let $\boldsymbol{C} := (C_t)_{t\in[0,T]}$ be the cumulative consumption process. Namely C_t represents the total amount of consumption at time t, see e.g. the original formulation in [HH93], and the application over a Lévy process in [BKR01a; Hol10] for more details. Thus, we denote with dC_t the increment of consumption occurred at time t when the investor decides to consume part of his wealth.
According to [Rog13, Section 1.2], the investor's wealth dynamic is derived as

$$\begin{aligned}dX_t^{\pi,C} =& rX_t^{\pi,C}dt + \pi_t X_t^{\pi,C}(dS_t/S_t - rdt) - dC_t \\ =& \left(r(1-\pi_t)X_t^{\pi,C} + \mu\pi_t X_t^{\pi,C}\right)dt - dC_t + \pi_t X_t^{\pi,C}\sigma dW_t \qquad (1.18) \\ &+ \pi_{t^-}X_{t^-}^{\pi,C}\left(\int_{\mathbb{R}\setminus\{0\}}(e^q-1)N(dt,dq) - \lambda_t\int_{\mathbb{R}\setminus\{0\}}(e^q-1)\zeta(q)dqdt\right), \\ =& \left(r(1-\pi_t)X_t^{\pi,C} + \mu\pi_t X_t^{\pi,C}\right)dt - dC_t + \pi_t X_t^{\pi,C}\sigma dW_t \qquad (1.19) \\ &+ \pi_{t^-}X_{t^-}^{\pi,C}\int_{\mathbb{R}\setminus\{0\}}(e^q-1)\widetilde{N}(dt,dq),\end{aligned}$$

whose solution reads as follows

$$\begin{aligned}
X_t^{\pi,C} =& x - \int_s^t dC_u + \int_s^t (r + (\mu - r)\pi_u) X_u^{\pi,C} du + \int_s^t \sigma \pi_u X_u^{\pi,C} dW_u && (1.20)\\
&+ \int_s^t \pi_{u^-} X_{u^-}^{\pi,C} \left(\int_{\mathbb{R}\setminus\{0\}} (e^q - 1) N(du, dq) - \int_s^t \lambda_u \int_{\mathbb{R}\setminus\{0\}} (e^q - 1)\zeta(q) dq du \right), \\
=& x - \int_s^t dC_u + \int_s^t (r + (\mu - r)\pi_u) X_u^{\pi,C} du + \int_s^t \sigma \pi_u X_u^{\pi,C} dW_u \\
&+ \int_s^t \pi_{u^-} X_{u^-}^{\pi,C} \int_{\mathbb{R}\setminus\{0\}} (e^q - 1) \widetilde{N}(du, dq) && (1.21)
\end{aligned}$$

where x represents the wealth at time s, and $\boldsymbol{\pi}, \boldsymbol{C}$ represent the control processes. The average past consumption, $\boldsymbol{Y} := (Y_t^{\pi,C})_{t\in[0,T)}$, evolves according to the following Stochastic Differential Equation:

$$dY_t^{\pi,C} = -\beta Y_t^{\pi,C} dt + \beta dC_t, \tag{1.22}$$

whose solution reads as:

$$Y_t^{\pi,C} = ye^{-\beta(t-s)} + \beta e^{-\beta t} \int_s^t e^{\beta u} dC_u. \tag{1.23}$$

Here β denotes the *decaying speed* and $Y_s = y > 0$ is the past average consumption at time s. Equation (1.23) has been considered since it enriches the problem with the notion of durability. In particular, it causes the effect that a consumption at a certain date leads to decaying effects persisting in the future. Note that, since $\boldsymbol{Y}$ decays exponentially with velocity given by the parameter β, higher values for β imply higher emphasis on the *recent past* and less emphasis on consumption in the *distant past*. Thus, durability spurs us to consider $\boldsymbol{Y}$ instead of simply $\boldsymbol{C}$. Indeed, it is reasonable to assume that the utility provided by a consumption in perishable goods will decrease in time.

The investor aims at maximizing $S : \mathcal{X} \to \mathbb{R}_+$, being $\mathcal{X} := [0,T] \times \mathbb{R}_+^3$, where

$$S(t, x, y, \lambda) = \mathbb{E}\left[\int_t^T e^{-\delta u} f(Y_u^{\pi,C}) du + e^{-\delta T} h(X_T^{\pi,C}, Y_T^{\pi,C}, \lambda_T) \right] \to \max, \tag{1.24}$$

for a positive constant $\delta > 0$. The first component of Equation (1.24), $f : \mathbb{R} \to \mathbb{R}$, is the *running utility* function, whilst the second one is the *terminal utility* $h : \mathbb{R}^3 \to \mathbb{R}$.
In the following we will denote by $\overline{\mathcal{X}}$ the closure of the set $\mathcal{X}$.

Then, we denote by $\boldsymbol{\pi}^*$, *resp.* $\boldsymbol{C}^*$, the optimal allocation control process, *resp.* consumption process, that lets S attain the maximum. We thus define the so-called *value function* as

$$V(t, x, y, \lambda) = \sup_{(\boldsymbol{\pi},\boldsymbol{C})\in\mathcal{A}_{[t,T]}(x,y)} S(t, x, y, \lambda), \tag{1.25}$$

over the admissible set for (x, y), defined as:

$$\mathcal{A}_{[t,T]}(x, y) := \{(\boldsymbol{\pi}, \boldsymbol{C}) : (\pi_s, C_s) \in [0, 1] \times [0, X_s^{\pi,C}], \ \forall\, s \in [t, T]\}. \tag{1.26}$$

The admissible set is explicitly defined only for the variables x, y since they directly depend on the controls, however we should remark that λ should be a positive value as required by its dynamic. It is worth to note that x represents the maximum level of admissible consumption, meaning that the investor cannot spend more than the wealth he owns and cannot borrow money. Indeed, the condition $\sup_{s\in[t,T]} C_s < X_s^{\pi,C}$ ensures that the process $\boldsymbol{C}$ is admissible for the initial wealth x and consumption y, which leads to the fact that the processes $\boldsymbol{X}$ and $\boldsymbol{Y}$ remain non-negative for all the time considered, see [HH93]. We remark that we have extended the value function presented in [HH93]. In particular, the terminal utility depends on the instantaneous frequency: such a choice seems unusual, but it is due to the fact that the investor will decide if and how to reinvest the wealth at time T, and his decision will be certainly affected by the turmoil in the market, which is expressed in the model through the mean of λ_T.

1.3.1 Main definitions and assumptions

Before stating the main results of the Thesis, let us recall some definitions that will be used in the following.

Definition 1.3.1. *For $k \in \mathbb{N}$, denote by*

$$\mathcal{C}_k(\overline{\mathcal{X}}) = \left\{ \varphi \in \mathcal{C}(\overline{\mathcal{X}}) \quad : \quad \sup_{\overline{\mathcal{X}}} \frac{|\varphi(t,x,y,\lambda)|}{(1+x+y+\lambda)^k} < \infty \right\},$$

the set of continuous real-valued function with domain $\overline{\mathcal{X}}$ with at most polynomial growth of order k. Notice that $\mathcal{C}_{\widetilde{k}}(\overline{\mathcal{X}}) \subseteq \mathcal{C}_k(\overline{\mathcal{X}})$ if $\widetilde{k} < k$.

Definition 1.3.2. *For all $k \geq 0$, be*

$$\mathcal{C}'_k(\overline{\mathcal{X}}) = \{\varphi \in \overline{\mathcal{X}} : \varphi \in \mathcal{C}_{k'}(\overline{\mathcal{X}}) \textit{ for some k' < k}\}.$$

Note that $\mathcal{C}_{k'}(\overline{\mathcal{X}}) \subset \mathcal{C}'_k(\overline{\mathcal{X}})$ for all $k' < k$.

Throughout the Thesis, we will assume the following to hold.

Assumption A1. *1. $\boldsymbol{C}$ is a finite-variation, non-decreasing, càdlàg process, adapted to the filtration $(\mathcal{F}_t)_{t\in(0,T]}$, as defined in Section 1.1. We further require that $\boldsymbol{C}$ has finite first moment, i.e. $\mathbb{E}[C_t] < \infty$, $\forall\, t \in [0,T]$ and $C_0 = 0$.*

2. *$\boldsymbol{\pi}$ is a $(\mathcal{F}_t)_{t\in(0,T]}$-adapted càdlàg process assuming values in $[0,1]$.*

3. *The terminal utility function $h : \mathbb{R} \to \mathbb{R}$ is affine in the third component, i.e. there exist two functions $h_1 : \mathbb{R} \to \mathbb{R}$ and $h_2 : \mathbb{R} \to \mathbb{R}$ such that*

$$h(x,y,\lambda) = h_1(x,y) + h_2(\lambda).$$

Furthermore, h is a continuous, monotone non-decreasing w.r.t. the first and second components and non-increasing w.r.t. the third component, non-negative and concave function over $\mathbb{R}^3_+$.

4. *The running utility function $f : \mathbb{R} \to \mathbb{R}$ is a continuous, monotone non-decreasing, non-negative and concave function over $\mathbb{R}_+$.*

5. *For all $x, y, \lambda \in \mathbb{R}_+$, there exist positive constants $C_f, C_h \in \mathbb{R}_+$ and $\gamma \in (0,1)$ such that:*

$$f(y) \le C_f(1+y)^\gamma,$$

and

$$h(x,y,\lambda) \le C_h(1+x+y+\lambda)^\gamma.$$

Assumption A2. *There exists $\gamma \in (0,1)$ such that*

$$\begin{aligned} k(\gamma) := \alpha\gamma|\xi - 1| + \gamma\eta\mu_J + \max_{\pi\in[0,1]} \Bigg[&\gamma(r + (\mu - r)\pi) + \frac{1}{2}\gamma(\gamma-1)(\sigma\pi)^2 \\ &+ \sup_{t\in(0,T]} \lambda_t \int_{\mathbb{R}\setminus\{0\}} \left[(1+\pi(e^q-1)+\eta|q|)^\gamma - 1 - \gamma\pi(e^q-1) - \gamma\eta|q|\right]\zeta(q)dq \Bigg] < \delta, \end{aligned}$$

where

$$\mu_J := \mathbb{E}[|J|] = \frac{p}{\rho^+} - \frac{(1-p)}{\rho^-}\,.$$

By Taylor expansion, we remark that the integral term of $k(\gamma)$ is well-defined in a neighbourhood of 0, whereas, outside, it is finite. Therefore, we can conclude that $k(\gamma)$ is finite for $\gamma \in (0,1)$.
Furthermore, it is worth to note that in the case of null integral operator, $k(\gamma) : \mathbb{R}^+ \to \mathbb{R}_+$, with $k(0) = 0$ and it is increasing, which is not the case if the integral operator is non-null, for more details see e.g. [Alv94].

Therefore $k(\gamma) : (0,\infty) \to \mathbb{R}$ can be negative and non-monotone.

Assumption A3. *Let $\gamma \in (0,1)$, and f, h be the functions in Equation* (1.25)*, then the following homogeneity conditions hold*

$$f(\alpha y) = \alpha^\gamma f(y), \quad h(\alpha x, \alpha y, \lambda) = \alpha^\gamma h_1(x,y) + h_2(\lambda) \quad \forall\, \alpha \ge 0\,,\ (x,y,\lambda) \in \mathbb{R}^3_+.$$

The Assumption A1 ensures that the problem is well-posed and the safe investments are preferable to risky investments, while the Assumption A3 guarantees the possibility to reduce the dimension of the problem, which will be a key assumption in the numerical scheme presented in Chapter 6.

1.3.2 A priori results for the value function

In this part we report some features for the value function, which will be exploited in the Section 3.3.

Proposition 1.3.1. *The value function V defined in Equation* (1.25) *satisfies*

$$0 \le V(t,x,y,\lambda) \le C_2(1+x+y+\lambda)^\gamma\,, \quad \forall\,(t,x,y,\lambda) \in \mathcal{X}\,,$$

with $C_2 > 0$ a time-independent constant.

Proof. First of all, let us remark that, according to Assumption A1, f and h are non-negative, therefore we can immediately conclude that, using Equation (1.25), V is also non-negative.

Regarding the upper bound, suppose $(t,x,y,\lambda) \in \mathcal{X}$ and $(\boldsymbol{\pi}, \boldsymbol{C}) \in \mathcal{A}_{[t,T]}(x,y)$, and define the process

$$Z_t = X_t^{\pi,C} + \frac{Y_t^{\pi,C}}{\beta} + \lambda_t,$$

where $\beta > 0$.

Since $X_t^{\pi,c} \geq 0$ and $Y_t^{\pi,c} \geq ye^{-\beta(t-s)} > 0$, it follows that $Z_t > 0$.

Using the Equations (1.18) and (1.22), we get

$$\begin{aligned} dZ_t =& \Big((r+(\mu-r)\pi_t)X_t^{\pi,C} - Y_t^{\pi,C} + \alpha(\xi-\lambda_t)\Big)\,dt + \eta \int_{\mathbb{R}\{0\}} |q| N(dt,dq) \\ &+ \sigma\pi_t X_t^{\pi,C} dW_t + \pi_{t^-} X_{t^-}^{\pi,C} \left(\int_{\mathbb{R}\setminus\{0\}} (e^q-1)\widetilde{N}(dt,dq)\right). \end{aligned}$$

We apply the Itô-Doeblin's lemma and we consider that $\pi_t \frac{X_t^{\pi,C}}{Z_t} \in [0,1]$, then we exploit the martingale property of both the Brownian motion and the Poisson random compensated measure, to get

$$\begin{aligned} \mathbb{E}[Z_t^\gamma] =& z^\gamma + \mathbb{E}\bigg[\int_s^t \gamma Z_u^{\gamma-1}\big(((r+(\mu-r)\pi_u)X_u^{\pi,C} - Y_u^{\pi,C} + \alpha(\xi-\lambda_u)\big)\,du \\ &+ \frac{1}{2}\gamma(\gamma-1)\int_s^t (\sigma\pi_u X_u^{\pi,C})^2 Z_u^{\gamma-2} du + \eta Z_u^{\gamma-1}\int_s^t \int_{\mathbb{R}\setminus\{0\}} |q| N(du,dq) \\ &+ \int_s^t \lambda_u \int_{\mathbb{R}\setminus\{0\}} ((Z_u + \pi_u X_u(e^q-1) + |q|\eta)^\gamma - Z_u^\gamma \\ &- \gamma\pi_u Z_u^{\gamma-1} X_u(e^q-1) - \gamma\eta|q|Z_u^{\gamma-1})\,\zeta(q)dqdu\bigg] \\ =& z^\gamma + \mathbb{E}\left[\int_s^t Z_u^\gamma \left(\gamma(r+(\mu-r)\pi_u)\frac{X_u^{\pi,C}}{Z_u} - \gamma\frac{Y_u^{\pi,C}}{Z_u} + \gamma\frac{\alpha}{Z_u}(\xi-\lambda_u)\right.\right. \\ &+ \frac{1}{2}\gamma(\gamma-1)(\sigma\pi_u)^2\left(\frac{X_u^{\pi,C}}{Z_u}\right)^2 + \frac{\eta}{Z_u}\int_s^t \int_{\mathbb{R}\setminus\{0\}} |q| N(du,dq) \\ &\left.\left.+ \lambda_u \int_{\mathbb{R}\setminus\{0\}} \left(\left(1 + \pi_u\frac{X_u^{\pi,C}}{Z_u}(e^q-1) + \frac{\eta|q|}{Z_u}\right)^\gamma - 1 - \gamma\pi_u\frac{X_u^{\pi,C}}{Z_u}(e^q-1) - \frac{\gamma}{Z_u}\eta|q|\right)\zeta(q)dq\right) du\right] \\ \leq& z^\gamma + \mathbb{E}\left[\int_s^t Z_u^\gamma du\right] k(\gamma), \end{aligned}$$

where $k(\gamma)$ is reported in Assumption A2.

By exploiting the Grönwall's inequality, we obtain

$$\mathbb{E}[Z_t^\gamma] \leq z^\gamma e^{k(\gamma)(t-s)},$$

which, using the fact that $z := x + \frac{y}{\beta} + \lambda$, in turn implies that

$$\mathbb{E}[X_t^\gamma] \le C_X(x + \frac{y}{\beta} + \lambda)^\gamma e^{k(\gamma)(t-s)}, \tag{1.27}$$

and

$$\mathbb{E}[Y_t^\gamma] \le C_Y(x + \frac{y}{\beta} + \lambda)^\gamma e^{k(\gamma)(t-s)}, \tag{1.28}$$

with $C_X = \max\{1; \beta^{-\gamma}\}$ and $C_Y = \max\{1; \beta^{\gamma}\}$ Therefore, by Assumption A1, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T e^{-\delta s} f(Y_s^{\pi,C})ds + h(X_s^{\pi,C}, Y_T^{\pi,C}, \lambda_T)\right] \\
\le&\mathbb{E}\left[\int_t^T e^{-\delta s} C_f(1 + Y_s^{\pi,C})^\gamma ds + C_h(1 + X_T^{\pi,C} + Y_T^{\pi,C} + \lambda_T)^\gamma\right] \\
\le&\mathbb{E}\left[C_f \int_t^T e^{-\delta s}\left(1 + (Y_s^{\pi,C})^\gamma\right) ds + C_h\left(1 + (X_T^{\pi,C})^\gamma + (Y_T^{\pi,C})^\gamma + (\lambda_T)^\gamma\right)\right] \\
\le&C_f \int_t^T e^{-\delta s}\left(1 + C_Y(x+y+\lambda)^\gamma e^{k(\gamma)(t-s)}\right) ds \\
&+ C_h\left(1 + (C_X + C_Y)(x+y+\lambda)^\gamma e^{k(\gamma)(t-s)}\right) \\
\le&C_1 + C_1(x+y+\lambda)^\gamma \\
\le&C_2(1+x+y+\lambda)^\gamma,
\end{aligned}$$

where the constants C_1 and C_2 are time-dependent, but we can choose them independently since we are considering a bounded time interval.

Therefore, maximizing over the admissible set, we get

$$V(t,x,y,\lambda) \le C_2(1+x+y+\lambda)^\gamma.$$

□

The next proposition is adapted from [BKR01b, Lemma 4.2], and it will be useful in proving that the function V is a viscosity solution of the optimization problem considered.

Proposition 1.3.2. *Let $(t,x,y,\lambda), (t,\widetilde{x},\widetilde{y},\lambda) \in \overline{\mathfrak{X}}$, satisfying $\widetilde{x} = x - c$ and $\widetilde{y} = y + \beta c$ for some positive constant c, then $V(t,x,y,\lambda) \ge V(t,\widetilde{x},\widetilde{y},\lambda)$.*

Proof. Let (x,y) and $(\tilde{x},\tilde{y}) \in \mathbb{R}^2$ such that $\tilde{y} := y + \beta c$ and $\tilde{x} := x - c$, for some $c \in (0,x]$, then $V(s,x,y) \ge V(s,\tilde{x},\tilde{y})$.
Let us suppose $\boldsymbol{X}$ and $\boldsymbol{Y}$, take value x and y at time s, respectively. If at time s there was a consumption gulp c, we have that $\boldsymbol{X}$ and $\boldsymbol{Y}$ have initial values $\widetilde{x}$ and $\tilde{y}$, respectively, and we denote such processes as $\widetilde{\boldsymbol{X}}$ and $\widetilde{\boldsymbol{Y}}$, respectively. Let us introduce $\mathcal{A}_{[s,T]}(\widetilde{x},\widetilde{y}) \subseteq \mathcal{A}_{[s,T]}(x,y)$,

the set of controls which allows an initial consumption c. Therefore:

$$\begin{aligned} V(t,x,y,\lambda) &= \sup_{(\boldsymbol{\pi},\boldsymbol{C})\in\mathcal{A}_{[t,T]}(x,y)} \mathbb{E}\left[\int_t^T e^{-\delta u} f(Y_u^{\pi,C})du + h(X_T^{\pi,C}, Y_T^{\pi,C}, \lambda_T)\right] \\ &\geq \sup_{(\boldsymbol{\pi},\boldsymbol{C})\in\mathcal{A}_{[t,T]}(x,y)} \mathbb{E}\left[\int_t^T e^{-\delta u} f(\widetilde{Y}_u^{\pi,C})du + h(\widetilde{X}_T^{\pi,C}, \widetilde{Y}_T^{\pi,C}, \lambda_T)\right] \\ &= V(t,\widetilde{x},\widetilde{y},\lambda), \end{aligned}$$

where we denote $\widetilde{\boldsymbol{X}}, \widetilde{\boldsymbol{Y}}$ the processes with initial values $\widetilde{x}$ and $\widetilde{y}$. respectively. □

The following theorem is the main regularity result for the current section.

Theorem 1.3.1. *The value function V in Equation* (1.25) *is uniformly continuous on compact subsets of* $\overline{\mathfrak{X}}$.

In order to prove Theorem 1.3.1, we will prove separately continuity in time and space. Firstly we will prove continuity in space.

Proposition 1.3.3. *The value function V is uniformly continuous in $(x,y,\lambda)\in\mathbb{R}_+^3$.*

Proof. For the sake of readability, we will split the proof in two main steps.

Step 1 For a fixed $t\in[0,T]$, there exists a modulus of continuity $\omega_t:\mathbb{R}_+^3\to[0,\infty)$ such that

1. ω_t is continuous in $(0,0,0)$,
2. $\omega_t(0,0,0)=0$, and
3. ω_t ensures $\forall\,(x,y,\lambda),(\widetilde{x},\widetilde{y},\widetilde{\lambda})\in\mathbb{R}_+^3$,

$$|V(t,x,y,\lambda)-V(t,\widetilde{x},\widetilde{y},\widetilde{\lambda})| \leq \omega_t(|x-\widetilde{x}|,|y-\widetilde{y}|,|\lambda-\widetilde{\lambda}|)\,.$$

By using Proposition 1.3.1, it follows that, for all $(t,x,y,\lambda)\in\overline{\mathfrak{X}}$, the value function V assumes finite values.

Fix $t\in(0,T]$ and assume $(\boldsymbol{\pi},\boldsymbol{C})$ be admissible controls in the sense of Equation (1.26), and compare the paths of $\boldsymbol{X},\boldsymbol{Y}$ and $\widetilde{\boldsymbol{X}},\widetilde{\boldsymbol{Y}}$ starting, respectively, from the initial values x,y and $\widetilde{x},\widetilde{y}$.

Define the stopping time τ as

$$\tau := \begin{cases} \inf\{s\in[t,T]:\widetilde{X}_s^{\pi,C}<0\} & \text{if } \widetilde{X}_s^{\pi,C}<0 \text{ for some } s\in[t,T]\,, \\ \infty & \text{if } \widetilde{X}_s^{\pi,C}\geq 0 \text{ for all } s\in[t,T]\,, \end{cases}$$

and set

$$\widetilde{C}_s := C_s\mathbb{1}_{s<\tau} + (\Delta\widetilde{X}_\tau^{\pi,C} + \widetilde{X}_{\tau^-}^{\pi,C} + C_\tau)\mathbb{1}_{s\geq\tau}, \tag{1.29}$$

and

$$\Gamma_s := C_s - \widetilde{C}_s = (C_s - \Delta\widetilde{X}_s^{\pi,C} - \Delta\widetilde{X}_{\tau-}^{\pi,C} - C_\tau)\mathbb{1}_{s\geq\tau}.$$

Notice that, when $\widetilde{x} > x$, we have that $\tau = \infty$, $C_s = \widetilde{C_s}$, and $\Gamma_s = 0$. Further, we assert that since

$$\begin{aligned}
&\Delta \widetilde{X}^{\pi,C}_{\tau} + \widetilde{X}^{\pi,C}_{\tau^-} + C_\tau \\
&= \left(-\Delta C_\tau + \pi_{\tau^-} \widetilde{X}^{\pi,C}_{\tau^-}(e^q - 1)\right) + \widetilde{X}^{\pi,C}_{\tau^-} + (C_{\tau^-} + \Delta C_\tau) \\
&= C_{\tau^-} + \widetilde{X}^{\pi,C}_{\tau^-}(\pi_{\tau^-} e^q - \pi_{\tau^-} + 1) \\
&\geq C_{\tau^-}\,,
\end{aligned}$$

it follows that $\widetilde{C}_s$ and Γ_s are non-decreasing.

We note that $\widetilde{X}^{\pi,\widetilde{C}}_s = \widetilde{X}^{\pi,C}_s \mathbb{1}_{s<\tau}$ since for $s < \tau$ we have $\widetilde{C}_s = C_s$ and $\widetilde{X}^{\pi,\widetilde{C}}_s = \widetilde{X}^{\pi,C}_s$ and

$$\begin{aligned}
\widetilde{X}^{\pi,\widetilde{C}}_{\tau} &= \widetilde{X}^{\pi,\widetilde{C}}_{\tau^-} + \pi_\tau(e^q - 1)\widetilde{X}^{\pi,\widetilde{C}}_{\tau^-} - \Delta \widetilde{C}_\tau \\
&= \widetilde{X}^{\pi,\widetilde{C}}_{\tau^-} + \pi_\tau(e^q - 1)\widetilde{X}^{\pi,C}_{\tau^-} - (\Delta \widetilde{X}^{\pi,C}_{\tau^-} + \widetilde{X}^{\pi,\widetilde{C}}_{\tau^-} + C_\tau - C_{\tau^-}) \\
&= \pi_\tau(e^q - 1)\widetilde{X}^{\pi,C}_{\tau^-} - \Delta \widetilde{X}^{\pi,C}_{\tau} - \Delta C_\tau \\
&= 0.
\end{aligned} \tag{1.30}$$

The first equality in Equation (1.30) comes from the fact that each discontinuous variation in the process $\widetilde{\boldsymbol{X}}$ is due either to a jump and to the control process increment, whereas the second equation is due to the fact that $\Delta \widetilde{C}_\tau = \widetilde{C}_\tau - \widetilde{C}_{\tau^-}$ by using Definition 1.29. The third equation is a direct consequence of the fact that $\tau^- < \tau$ therefore, since $\widetilde{C}_{\tau^-} = C_\tau$, we have that $\widetilde{X}^{\pi,C}_{\tau^-} = \widetilde{X}^{\pi,\widetilde{C}}_{\tau^-}$. The last equality follows from the first one since we rewrite $\Delta \widetilde{X}^{\pi,C}_{\tau} = \widetilde{X}^{\pi,C}_{\tau} - \widetilde{X}^{\pi,C}_{\tau^-}$. Given that $\widetilde{X}^{\pi,C}_{\tau} = 0$ and $\widetilde{C}_s$ is constant for $s \geq \tau$, therefore the entire process is null for $s \geq \tau$, and $\widetilde{X}^{\pi,\widetilde{C}}_{\tau} = \widetilde{X}^{\pi,\widetilde{C}}_{\tau} \mathbb{1}_{s<\tau}$.

Thus $(\boldsymbol{\pi}, \widetilde{\boldsymbol{C}}) \in \mathcal{A}_{[t,T]}(\widetilde{x}, \widetilde{y})$ since for all $s \in [t,T]$ we have $\widetilde{X}^{\pi,\widetilde{C}}_{\tau} = \widetilde{X}^{\pi,\widetilde{C}}_{\tau} \mathbb{1}_{s<\tau} \geq 0$. We have $(\pi, \Gamma) \in \mathcal{A}_{[t,T]}(|x - \widetilde{x}|, |y - \widetilde{y}|)$, which holds for $\widetilde{x} > x$, this gives $\Gamma \equiv 0$. For $x \geq \widetilde{x}$ we also have

$$\begin{aligned}
(X - \widetilde{X})^{\pi,\Gamma}_s :&= X^{\pi,C}_s - \widetilde{X}^{\pi,\widetilde{C}}_s \\
&= X^{\pi,C}_s - \widetilde{X}^{\pi,C}_s \mathbb{1}_{s<\tau} \\
&\geq 0.
\end{aligned}$$

And we see that $|X^{\pi,C}_s - \widetilde{X}^{\pi,\widetilde{C}}_s| = |X - \widetilde{X}|^{\pi,\Gamma}_s$ holds for the next three cases

1. $\widetilde{x} \geq x$,
2. $\widetilde{x} \leq x$ and $\tau \geq s$,
3. $\widetilde{x} \leq x$ and $\tau < s$.

Similarly, by exploiting the triangle inequality and the explicit form for $\boldsymbol{Y}$ we assert that

$$|Y_s^{\pi,C} - \widetilde{Y}_s^{\pi,\widetilde{C}}| \leq |Y - \widetilde{Y}|_s^{\pi,\Gamma},$$

and we can extend it for λ also, the latter being not controlled.

By using the previous results, we finally conclude that

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T e^{-\delta s} f(Y_s^{\pi,C})ds + h(X_s^{\pi,C}, Y_T^{\pi,C}, \lambda_T)\right] \\
&\leq \mathbb{E}\left[\int_t^T e^{-\delta s} f(\widetilde{Y}_s^{\pi,\widetilde{C}})ds + h(\widetilde{X}_s^{\pi,C}, \widetilde{Y}_T^{\pi,C}, \lambda_T)\right] \\
&+ \mathbb{E}\left[\int_t^T e^{-\delta s}\omega_f(|Y - \widetilde{Y}|_s^{\pi,\Gamma} ds + \omega_h(|X - \widetilde{X}|_T^{\pi,\Gamma}, |Y - \widetilde{Y}|_T^{\pi,\Gamma}, |\lambda_T - \widetilde{\lambda}_T|)\right] \\
&\leq V(t, \widetilde{x}, \widetilde{y}, \widetilde{\lambda}) + \omega_t(|x - \widetilde{x}|, |y - \widetilde{y}|, |\lambda - \widetilde{\lambda}|),
\end{aligned} \tag{1.31}$$

where ω_f, ω_h are, respectively, the moduli of continuity of f and h.
We remark that f and h admit moduli of continuity being both continuous, concave and non-decreasing by Assumption 3, such moduli are assumed to be non-decreasing in x, y and λ.

Maximizing over the admissible set $A_{[t,T]}(x, y)$ and exploiting Inequality (1.31), we get

$$|V(t, x, y, \lambda) - V(t, \widetilde{x}, \widetilde{y}, \widetilde{\lambda})| \leq \omega_t(|x - \widetilde{x}|, |y - \widetilde{y}|, |\lambda - \widetilde{\lambda}|).$$

Moreover we observe that $\omega_t(0, 0, 0) = 0$, so the remaining part of the proof will be devote to show that ω_t is continuous in $(0, 0, 0)$. For any given $\varepsilon > 0$, there exist K_ε and C_ε such that:

1. $\omega_f(y) < \varepsilon + K_\varepsilon y^\gamma$,
2. $\omega_h(x, y, \lambda) < \varepsilon + C_\varepsilon(x^\gamma + y^\gamma + \lambda^\gamma)$.

Note that such constants exist thanks to Assumption 5. By using Gröwnwall's lemma and Inequality (1.27) we get

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T e^{-\delta s}\omega_f(Y_s^{\pi,C})ds + \omega_h(X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s)\right] \\
&\leq K\varepsilon + \mathbb{E}\left[K_\varepsilon \int_t^T e^{-\delta s}(Y_s^{\pi,C})^\gamma ds\right] \\
&+ K\varepsilon + \mathbb{E}\left[C_\varepsilon\left((X_T^{\pi,C})^\gamma + (Y_T^{\pi,C})^\gamma + (\lambda_T)^\gamma\right)\right] \\
&\leq K\varepsilon + KK_\varepsilon(x + y + \lambda)^\gamma + K\varepsilon + KC_\varepsilon(x + y + \lambda)^\gamma,
\end{aligned}$$

for some positive constant K independent of $t, \varepsilon, x, y, \lambda, \pi, C$. Taking the maximum over the controls π and C, we finally obtain

$$\omega_t(x, y, \lambda) \leq K(C_\varepsilon + K_\varepsilon)(x + y + \lambda)^\gamma + 2K\varepsilon,$$

which, for sufficiently small x, y and λ, can be reduced to

$$\omega_t(x, y, \lambda) \leq 3K\varepsilon. \tag{1.32}$$

Since K in Equation (1.32) is independent of t and ε is arbitrary, we see that ω_t is a modulus of continuity for V for any fixed $t \in [0, T]$.

Step 2 Define $\omega : \overline{\mathfrak{X}} \to [0, \infty)$ as

$$\omega(x, y, \lambda) := \sup_{t\in[0,T]} \omega_t(x, y, \lambda),$$

where $\omega_t(\cdot)$ is introduced in inequality (1.31) and satisfies the requirements in Proposition 1.3.3. In this way, we finally claim that the modulus continuity is t-independent.

□

Now, we are to prove the time-continuity of V. It is worth to stress that the next continuity results are among the main differences from the results obtained in [BKR01b]. In fact, in [BKR01b], the authors consider an infinite horizon optimal control and, in doing so, the resulting value function is time-independent. On the contrary, as typical when considering a finite horizon optimal control, in the case considered in this current research, the value function inherits the time-continuity of the driving process. Before stating the main proposition, let us assert an auxiliary result.

Proposition 1.3.4. *Consider $(t, x, y, \lambda) \in \overline{\mathfrak{X}}$ and $(\boldsymbol{\pi}, \boldsymbol{C})$ admissible controls.*

(i) *For $t < T$, consider a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t^+$. Then:*

(1) $\lim_{n\to\infty} \mathbb{E}\left[\int_t^{t_n} e^{-\delta s} f(Y_s^{\pi,C})ds\right] = 0$*;*

(2) if $(\boldsymbol{\pi}^*, \boldsymbol{C}^*)$ is an optimal control, then:

$$\lim_{n\to\infty} \mathbb{E}\left[V(t_n, X_{t_n}^{\pi^*,C^*}, Y_{t_n}^{\pi^*,C^*}, \lambda_{t_n})\right] = \lim_{n\to\infty} \mathbb{E}\left[V(t_n, x, y, \lambda)\right] . \tag{1.33}$$

(ii) *For $t < T$, consider $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to t^-$. Then:*

(3) $\lim_{n\to\infty} \mathbb{E}\left[\int_{t_n}^{t} e^{-\delta s} f(Y_s^{n,\pi_n,C_n})ds\right] = 0$*;*

(4) denoting by $\Delta_n := (C_n)_t - (C_n)_{t_n^-}$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[V(t, X_t^{n,\pi_n,C_n}, Y_t^{n,\pi_n,C_n}, \lambda_t^n) - V(t, x - \Delta C_n, y + \beta\Delta C_n, \lambda)\right] = 0\,. \tag{1.34}$$

Proof. We start by proving (1): we immediately see that the equation holds for a given $Y_t^{\pi,C} = y$. Moreover, for a sufficiently large $M \in \mathbb{N}$, we have:

$$\mathbb{E}\left[\int_t^{t_n} e^{-\delta s} f(Y_s^{\pi,C})ds\right] \leq \sum_{m=1}^{M} \mathbb{E}\left[\int_t^{t_n} f(Y_s^{\pi,C})ds | C_T < M\right] \mathbb{P}(C_T < M) + \epsilon/2,$$

since $\mathbb{P}(C_T < M) \leq 1$; and

$$\mathbb{E}\left[\int_{t_n}^{t} e^{-\delta s} f(Y_s^{n,\pi_n,C_n}) ds\right] < \int_{t_n}^{t} e^{-\delta s} f(y + \beta M) ds \to 0$$

for $n \to \infty$, therefore (1) holds. We assert (2) holds by proving that, given $\epsilon > 0$ and sufficiently large n,

$$\mathbb{E}[V(t_n, X_{t_n}^{\pi^*,C^*}, Y_{t_n}^{\pi^*,C^*}, \lambda_{t_n})] \leq \mathbb{E}[V(t_n, x, y, \lambda) + \epsilon; \tag{1.35}$$

$$\mathbb{E}[V(t_n, X_{t_n}^{\pi^*,C^*}, Y_{t_n}^{\pi^*,C^*}, \lambda_{t_n})] > \mathbb{E}[V(t_n, x, y, \lambda) - \epsilon. \tag{1.36}$$

Let C_s on $[t, T]$ be $C_s := \max\{C_t^*; C_s^* - \epsilon\}$. Since $\boldsymbol{C}^*$ is right-continuous a.s., $\boldsymbol{C}$ is constant at some interval starting at t a.s., therefore $C_s < C_s^*$ for all $s \in [t, T]$, and

$$\Omega_m := \left\{\omega \in \Omega : m - 1 \leq \max\left\{\sup_{s\in(t,T]} \frac{|X_s^{\pi^*,C} - X_t^{\pi^*,C}|}{s - t}; |C_s^* - C_t^*|; \sup_{s\in(t,T]} |\lambda_s - \lambda_t|\right\} < m\right\}$$

for all $m \in \mathbb{N}$.

Given any $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that

$$\begin{aligned}\mathbb{E}[V(t_n, X_{t_n}, Y_{t_n}, \lambda_{t_n})] &< \sum_{m=1}^{M} \mathbb{E}[V(t_n, X_{t_n}, Y_{t_n}, \lambda_{t_n})|\Omega_m]\mathbb{P}(\Omega_m) + \epsilon \\ &= \mathbb{E}[V(t_n, X_{t_n}, Y_{t_n}, \lambda_{t_n})| \cup_{m=1}^{M} \Omega_m]\mathbb{P}(\cup_{m=1}^{M}\Omega_m) + \epsilon.\end{aligned}$$

If $\omega \in \cup_{m=1}^{M}\Omega_m$, we know that $X_{t_n}(\omega) \leq X_t(\omega) + M(t_n - t)$, $Y_{t_n}(\omega) \leq Y_t(\omega) + \beta M(t_n - t)$ and $\lambda_{t_n}(\omega) \leq \lambda_t(\omega) + M(t_n - t)$. Assuming $\widetilde{\omega}$ the modulus of continuity of V in (x, y, λ), we have:

$$\begin{aligned}&\mathbb{E}[V(t_n, X_{t_n}, Y_{t_n}, \lambda_{t_n})| \cup_{m=1}^{M} \Omega_m]\mathbb{P}(\cup_{m=1}^{M}\Omega_m) \\ &\qquad \leq V(t_n, X_t + M(t_n - t), Y_t + \beta M(t_n - t), \lambda_t + M(t_n - t)) \\ &\qquad \leq V(t_n, X_t, Y_t, \lambda_t) + \widetilde{\omega}(M(t_n - t), \beta M(t_n - t), M(t_n - t))\end{aligned}$$

By Proposition 1.3.3, we see that Inequality (1.36) holds since $\widetilde{\omega}(0, 0, 0) = 0$ and $\widetilde{\omega}$ is continuous in $(0, 0, 0)$. We have

$$\begin{aligned}V(t, x, y, \lambda) &= \mathbb{E}\left[\int_{t}^{t_n} e^{-\delta s} f(Y_s^{\pi^*,C^*}) ds + V(t_n, X_{t_n}^{\pi^*,C^*}, Y_{t_n}^{\pi^*,C^*}, \lambda_{t_n})\right] \\ &\geq \mathbb{E}\left[\int_{t}^{t_n} e^{-\delta s} f(Y_s^{0,0}) ds + V(t_n, x_{t_n}^{0,0}, Y_{t_n}^{0,0}, \lambda_{t_n})\right]\end{aligned} \tag{1.37}$$

where $V(t_n, X_{t_n}^{0,0}, Y_{t_n}^{0,0}, \lambda_{t_n}) = V(t_n, xe^{r(t_n-t)}, ye^{-\beta(t_n-t)}, \lambda_{t_n})$. Then, we see that the integrals in Inequality (1.37) converge to 0 by (1.3.4) and, since V is continuous in (x, y, λ), we can assert that Inequality (1.36) holds. Now we have to prove that (1.3.4) holds. Let us define

$$\Omega_m := \begin{cases} \{\omega \in \Omega | \sup_{n\in\mathbb{N}}(C_n)_t \in [m - 1, m)\} \text{ if } m \in \mathbb{N}, \\ \{\omega \in \Omega | \sup_{n\in\mathbb{N}}(C_n)_t = \infty\} \text{ if } m = \infty \end{cases}$$

for all $m \in \mathbb{N} \cup \{\infty\}$. Then:

$$\mathbb{E}\left[\int_{t_n}^{t} e^{-\delta s} f(Y_s^{\pi,C})ds\right] \leq \mathbb{E}\left[\int_{t_n}^{t} f(Y_s^{\pi,C})ds| \cup_{m=1}^{M} \Omega_m\right] \mathbb{P}(\cup_{m=1}^{M}\Omega_m) + \epsilon.$$

Thus,

$$\mathbb{E}\left[\int_{t_n}^{t} e^{-\delta s} f(Y_s^{\pi,C})ds| \cup_{m=1}^{M} \Omega_m\right] < \int_{t_n}^{t} e^{-\delta s} f(y+\beta M)ds \to 0.$$

We see that (1.3.4) holds as $n \to \infty$. We conclude with the proof of (1.3.4): let $\widetilde{\omega}$ be the modulus of continuity for V such that:

$$\begin{aligned}
&\mathbb{E}\left[|V(t, X_t^{n,\pi_n,C_n}, Y_t^{n,\pi_n,C_n}, \lambda_t^n) - V(t, x-\Delta C_n, y+\beta\Delta C_n, \lambda)|\right] \\
&\quad \leq \mathbb{E}\left[\widetilde{\omega}(|X_t^{n,\pi_n,C_n} - x + \Delta C_n|, |Y_t^{n,\pi_n,C_n} - y - \beta\Delta C_n|, |\lambda_t^n - \lambda|)\right]
\end{aligned}$$

for all $n \in \mathbb{N}$.

Let us define

$$\begin{aligned}
\Omega_m = \{\omega \in \Omega \ :& \\
& m-1 \leq \max\left\{\sup_{n\in\mathbb{N}} \frac{|X_t^{n.\pi,C_n} + \Delta C_n - x|}{\sqrt{t-t_n}}; \sup_{n\in\mathbb{N}} \Delta C_n, \sup_{n\in\mathbb{N}} |\lambda_t^n - \lambda|\right\} < m\right\}
\end{aligned}$$

for all $m \in \mathbb{N}$.

Let now $\epsilon > 0$, then there exists $m \in \mathbb{N}$, such that

$$\begin{aligned}
&\mathbb{E}\left[\widetilde{\omega}(|X_t^{n,\pi_n,C_n} - x + \Delta C_n|, |Y_t^{n,\pi_n,C_n} - y - \beta\Delta C_n|, |\lambda_t^n - \lambda|)\right] \\
&< \mathbb{E}\left[\widetilde{\omega}(|X_t^{n,\pi_n,C_n} - x + \Delta C_n|, |Y_t^{n,\pi_n,C_n} - y - \beta\Delta C_n|, |\lambda_t^n - \lambda|)| \cup_{i=1}^{M} \Omega_m\right] \times \\
&\times \mathbb{P}(\cup_{i=1}^{M}\Omega_m) + \epsilon \\
&< \widetilde{\omega}\left(\sqrt{t-t_n}M, y|e^{-\beta(t-t_n)} - 1| + \beta e^{-\beta t}|e^{-\beta t_n} - e^{-\beta t}|M, \lambda|M-1|\right) + \epsilon.
\end{aligned}$$

The right-hand side of the above Inequality goes to 0 as $n \to \infty$, therefore Proposition 1.3.5 has been proved. □

Using Proposition 1.3.4 we can thus prove the time-continuity of the value function.

Proposition 1.3.5. *The value function V is continuous in time.*

Proof. Let $(t, x, y, \lambda) \in \overline{\mathcal{X}}$ and let $(t_n)_{n\in\mathbb{N}}$ be a right-continuous sequence in $(t, T]$ converging to t^+ as $n \to \infty$. By Proposition 2.1.1, we assert that

$$V(t, x, y, \lambda) = \mathbb{E}\left[\int_{t}^{t_n} e^{-\delta s} f(Y_s^{\pi^*,C^*})ds + V(t_n, X_{t_n}^{\pi^*,C^*}, Y_{t_n}^{\pi^*,C^*}, \lambda_{t_n})\right], \forall n \in \mathbb{N}. \quad (1.38)$$

By using Proposition 1.3.4, it follows that the running utility integral in Equation (1.38) converges to 0. Thus, $V(t_n, X_{t_n}^{\pi^*,C^*}, Y_{t_n}^{\pi^*,C^*}, \lambda_{t_n})$ converges to $\lim_{n\to\infty} V(t_n, x, y, \lambda)$, by Proposition 1.3.4. The right-continuity follows directly from the fact that the right-hand side of (1.38) converges to $\lim_{n\to\infty} V(t_n, x, y, \lambda)$.

Now we are left to prove that V is left-continuous at (t, x, y, λ). Let $(t_n)_{n\in\mathbb{N}}$ be a sequence on $[0, t)$ such that $t_n \to t^-$.

Denote by $\Delta t_n := t - t_n > 0$, so that we aim at proving that

1. $V(t_n, x, y, \lambda) \geq V(t, x, y, \lambda) - \epsilon$,

2. $V(t_n, x, y, \lambda) \geq V(t, x, y, \lambda) + \epsilon$,

for all $\epsilon > 0$ and a sufficiently large n.

For any admissible set of controls at t_n, let $\boldsymbol{X}^n, \boldsymbol{Y}^n$ and $\boldsymbol{\lambda}^n$ the processes at time t_n. Assume $(\boldsymbol{\pi}^*, \boldsymbol{C}^*)$ is an optimal control, and let

$$(\pi_n)_s = \begin{cases} \pi^*_{n,s+\Delta t_n} & \text{if } s \leq T - \Delta t_n, \\ 0 & \text{if } s > T - \Delta t_n, \end{cases}$$

and

$$(C_n)_s = \begin{cases} C^*_{s+\Delta t_n} & \text{if } s \leq T - \Delta t_n, \\ C^*_T & \text{if } s > T - \Delta t_n. \end{cases}$$

If we consider the initial time to be t_n, we see that $X_s^{n,\pi_n,C_n} = X_s^{n,\pi^*,C^*}, Y_s^{n,\pi_n,C_n} = Y_s^{n,\pi^*,C^*}$ and, obviously $\lambda_s^n = \lambda_s$ for all $n \in \mathbb{N}$ and $s \in [t, T - \Delta t_n]$. Moreover,

$$X_s^{n,\pi_n,C_n} = X_{T-\Delta t_n}^{n,\pi^*,C^*} + \int_{T-\Delta t_n}^{s} r X_u^{n,\pi^*,C^*} du = X_{T-\Delta t_n}^{n,\pi^*,C^*} e^{r(s-(T-\Delta t_n))},$$

$$Y_s^{n,\pi_n,C_n} = Y_{T-\Delta t_n}^{n,\pi_n,C_n} e^{-\beta(s-(T-\Delta t_n))},$$

for $s \in [T - \Delta t_n, T]$. By exploiting the previous equations, we have

$$
\begin{aligned}
V(t_n, x, y, \lambda) \geq & \mathbb{E}\left[\int_{t_n}^{T} e^{-\delta s} f(Y_s^{n,\pi_n,C_n}) ds + h(X_T^{n,\pi_n,C_n}, Y_T^{n,\pi_n,C_n}, \lambda_T^n)\right] \\
= & \mathbb{E}\left[\int_{t}^{T-\Delta t_n} e^{-\delta s} f(Y_s^{n,\pi_n,C_n}) ds + h(X_{T-\Delta t_n}^{n,\pi_n,C_n}, Y_{T-\Delta t_n}^{n,\pi_n,C_n}, \lambda_{T-\Delta t_n}^n)\right] \\
& + \mathbb{E}\left[\int_{T-\Delta t_n}^{T} e^{-\delta s} f(Y_s^{n,\pi_n,C_n}) ds + h(X_T^{n,\pi_n,C_n}, Y_T^{n,\pi_n,C_n}, \lambda_T^n)\right. \\
& \left. - h(X_{T-\Delta n}^{n,\pi_n,C_n}, Y_{T-\Delta n}^{n,\pi_n,C_n}, \lambda_{T-\Delta n}^n)\right] \\
= & V(t, x, y, \lambda) \\
& + \mathbb{E}\left[\int_{T-\Delta t_n}^{T} e^{-\delta s} f(Y_{T-\Delta t_n}^{n,\pi_n,C_n} e^{-\beta(s-T+\Delta t_n)}) ds\right] \\
& + \mathbb{E}\left[h(X_{T-\Delta t_n}^{n,\pi_n,C_n} e^{r\Delta t_n}, Y_{T-\Delta t_n}^{n,\pi_n,C_n} e^{-\beta \Delta t_n}, \lambda_T)\right. \\
& \left. - h\left(X_{T-\Delta t_n}^{n,\pi_n,C_n}, Y_{T-\Delta n}^{n,\pi_n,C_n}, \lambda_{T-\Delta t_n}^n\right)\right] \\
= & V(t, x, y, \lambda) \\
& + \mathbb{E}\left[\int_{T-\Delta t_n}^{T} e^{-\delta s} f\left(Y_{T-\Delta t_n}^{n,\pi_n,C_n} e^{-\beta(s-T+\Delta t_n)}\right) ds\right] \\
& + \mathbb{E}\left[h\left(X_T^{\pi^*,C^*} e^{r\Delta t_n}, Y_T^{\pi^*,C^*} e^{-\beta \Delta t_n}, \lambda_T - \lambda_{T-\Delta t_n}\right)\right] \\
& - \mathbb{E}\left[h(X_T^{\pi^*,C^*}, Y_T^{\pi^*,C^*}, \lambda_T)\right]
\end{aligned} \tag{1.39}
$$

We see that the second term on the right-hand side of Equation (1.39) is decreasing as $n \to \infty$, so it converges to 0 for all $\omega \in \Omega$, and consequently Inequality (1) holds.

Regarding Inequality (2), by Proposition 2.1.1, we have that

$$
V(t_n, x, y, \lambda) = \mathbb{E}\left[\int_{t_n}^{t} e^{-\delta s} f(Y_s^{\pi^*,C^*}) ds + V(t, X_t^{\pi^*,C^*}, Y_t^{\pi^*,C^*}, \lambda_t)\right], \forall n \in \mathbb{N}. \tag{1.40}
$$

By Proposition 1.3.4, the running utility integral converges to 0 so, for all $\epsilon > 0$, it can be proved that

$$
\mathbb{E}[V(t, X_t^{\pi_n^*,C_n^*}, Y_t^{\pi_n^*,C_n^*}, \lambda_t)] \leq V(t, x, y, \lambda) + \epsilon, \tag{1.41}
$$

for sufficiently large $n \in \mathbb{N}$. It thus follows that

$$
\begin{aligned}
& \mathbb{E}\left[V(t, X_t^{\pi_n^*,C_n^*}, Y_t^{\pi_n^*,C_n^*}, \lambda_t) - V(t, x, y, \lambda)\right] \\
= & \mathbb{E}\left[V(t, X_t^{\pi_n^*,C_n^*}, Y_t^{\pi_n^*,C_n^*}, \lambda_t) - V(t, X_t^{\pi_n^*,\Delta C_n}, Y_t^{\pi_n^*,\Delta C_n}, \lambda_t)\right] \\
& + \mathbb{E}\left[V(t, X_t^{\pi_n^*,\Delta C_n}, Y_t^{\pi_n^*,\Delta C_n}, \lambda_t) - V(t, x, y, \lambda)\right]
\end{aligned}
$$

with $\Delta C_n := (C_n)_t - (C_n)_{t_n^-}$. By using again Proposition 1.3.4, we infer that the first term converges to 0, and the second term converges to a negative value by Proposition 1.3.2. We can therefore conclude that Inequality (1.41) holds. □

Finally, we can prove the main continuity result, Theorem 1.3.1.

Proof of Theorem 1.3.1. Denote by $\overline{\mathcal{O}}$ a compact subset of $\overline{\mathcal{X}}$. Then, using the continuity of the value function in Propositions 1.3.3-1.3.5, the claim follows from the classical Heine-Cantor Theorem. □

Chapter 2

The Stochastic Control Problem

This chapter is based on the results reported in the working paper [CDPVndb].
Accordingly, we introduce all the notions and definitions needed to generalize the Merton's problem, see [Mer75], to then derive the equation providing the solution according to the framework proposed in Chapter 1.
In particular, in Section 2.1, we prove that Dynamic Programming Principle (DPP) holds in the case where a self-exciting Poisson point process is driving the discontinuous part of the portfolio, although it leads to a more complex and less tractable formulation of the problem.
Then, thanks to the previous result, we derive and study the so called Hamilton-Jacobi-Bellman (HJB) equation in 2.2. Finally, HJB equation is formally derived a Partial Integro Differential Equation (PIDE) subject to a gradient constraint which will be introduced in 2.2.1. uniqueness of the viscosity solution for the PIDE.

2.1 The Dynamic Programming Principle

The validity of the Dynamic Programming Principle is based on the fact that $(\boldsymbol{N}, \boldsymbol{\lambda})$ are jointly Markovian, and follows the arguments reported in [Zhu94; GS12; CB21] and [Ish04]. In order to have a self-contained work, we report the generalized version of the proof in [GS12]. Therefore, we decide to use a similar notation in the following section.

Firstly, in favour of a lighter notation, we will denote $z(t) := (X_t^{\pi,C}, Y_t^{\pi,C}, \lambda_t)$, where $X_t^{\pi,C}, Y_t^{\pi,C}$ and λ_t describe, respectively, the wealth of the investor reported in Equation (1.18), the average past consumption in Equation (1.22), and the jump frequency defined in Equation (1.3).
Then, we collect the control pair introducing $u(t) := (\pi_t, C_t)$ for all $t \in [0, T]$.

The value function defined in Equation (1.25), will be rewritten with a slightly different notation. In particular we will denote $V_0(z) := V(0, x, y, \lambda) = \sup_{u \in \mathcal{A}_{[t,T]}(z)} S_0(z(\cdot), u(\cdot))$, where $S_t(z(\cdot), u(\cdot)) := S(t, x, y, \lambda, \pi, C)$ defined in Equation (1.24), for each $s \in [0, T]$.
Moreover, we define $\widetilde{h}(z(\cdot), u(\cdot)) := h(x, y, \lambda)$ and $\widetilde{f}(z(\cdot), u(\cdot)) := f(y)$ to highlight the dependence of $\widetilde{h}$ and $\widetilde{f}$ to the control process $u(\cdot)$.

We proceed by defining $S_t(z(\cdot), u(\cdot)), t \in [0, T]$, which is a family of functionals of the form

$$S_t(z(\cdot), u(\cdot)) = \int_t^T e^{-\delta s} \widetilde{f}(z(s), u(s))ds + e^{-\delta(T-t)} \widetilde{h}(z(T), u(T)), \tag{2.1}$$

where $\widetilde{h}(\cdot), \widetilde{f}(\cdot)$ are given real functions bounded and continuous in their arguments as reported in Assumption 3. Then, functionals of the family expressed in Equation(2.1) can be rewritten as solutions of the following backward differential equation:

$$\begin{cases} -dS_t &= -\delta S_t(z(t), u(t))dt + \widetilde{f}(z(t), u(t))dt, \quad 0 \le t \le T, \\ S_T &= \widetilde{h}(z(T), u(T)). \end{cases} \tag{2.2}$$

Moreover, to stress the dependence of S_t on T, we denote it by $S_{[t,T]}(z(\cdot), u(\cdot))$.
In fact, $S_{[t,T]} \forall t \in [0, T]$ defines a class of functionals exploiting the realizations of the stochastic processes $(z(\cdot), u(\cdot))$, and satisfying the condition:

$$S_{[t_1,T]} = S^a_{[t_1,t_2]} + G[t_1, t_2] S_{[t_2,T]}$$

where

$$G[t_1, t_2] = G_{[t_1,t_2]} = \exp\left(-\delta(t_2 - t_1)\right)$$

and

$$S^a_{[t_1,t_2]} = \int_{t_1}^{t_2} G[t_1, s] \widetilde{f}(z(s), u(s))ds, \quad S^a_T = S^a_{[t,T]}.$$

We are now in position to apply the general method of constructing ε-optimal controls. Therefore, we can introduce the canonical subdivision $T^n := \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of the interval $[0, T]$ with $|T^n| \searrow 0$ and proceed to define the stepwise optimal controls corresponding to the ε-optimal solutions of Equation (1.18).
For a given T^n, we introduce the u-controlled sequences $\{Z_k\}_{k=0}^n$ following the same dynamic in Equation (1.18), by the iteration

$$\begin{aligned} Z_{k+1} = Z_k &+ \int_{t_k}^{t_{k+1}} a(s, Z_0, \ldots, Z_k, u_k)ds + \int_{t_k}^{t_{k+1}} b(s, Z_0, \ldots, Z_k)dW_s \\ &+ \int_{t_k}^{t_{k+1}} c(s, Z_0, \ldots, Z_k, u_k)dC + \int_{t_k}^{t_{k+1}} \int_{\mathbb{R}\setminus\{0\}} d(s, Z_0, \ldots, Z_k, q, u_k)N(ds, dq), \end{aligned} \tag{2.3}$$

where we ease the notation as

$$a(s, Z_0, \ldots, Z_k, u_k) = a(s, \check{Z}(\cdot)),$$

similarly we define the functionals $b(\cdot), c(\cdot), d(\cdot)$ similarly.
In particular, in our problem, we can use the formulation

$$a(s, \check{Z}(\cdot)) := \begin{pmatrix} r + (\mu - r - \lambda_k \mathbb{E}[e^J - 1])\pi_k X_k^{\pi,C} \\ -\beta Y_k^{\pi,C} \\ \alpha(\xi - \lambda_k) \end{pmatrix}, \quad b(s, \check{Z}(\cdot)) := \begin{pmatrix} \sigma \pi_k X_k^{\pi,C} \\ 0 \\ 0 \end{pmatrix},$$

$$c(s,\check{Z}(\cdot)) := \begin{pmatrix} -1 \\ \beta \\ 0 \end{pmatrix}, \quad d(s,\check{Z}(\cdot),q) := \begin{pmatrix} \pi_k X_k^{\pi,C}(e^q-1) \\ 0 \\ \eta|q| \end{pmatrix},$$

for $s \in [t_k, t_{k+1})$. Now, we can introduce the optimal control $\eta^* = \{\eta_i^*\}_{i=0}^n$, which is obtained by recursion: we start from η_n^*, namely the optimal control at time T, and we decrease the index down to $i=0$. Moreover, among all the generalized controls, there exists an optimal feedback control which can be defined as $\eta_k^* = g_k(Z_0, Z_1, \dots, Z_k)$ whereas $g_k(z_0, \cdots, z_k)$ refers to some Borel function of their arguments.
Finally, the optimal control can be achieved with the following routine.
Let $\overline{\eta} = (\eta_i)_{i=0}^n$ be an arbitrary generalized control for the collection $(Z_k)_{k=0}^n$, then we denote $\overline{S}(\overline{\eta}) = \mathbb{E}[S(\check{Z}(\cdot),\check{\eta}(\cdot))]$, $\check{Z}(t) = Z_k$, $\check{\eta}(t) = \eta_k$ for $t \in [t_k, t_{k+1})$. Consequently, we have the natural filtration defined as

$$\mathcal{F}_k := \sigma(\eta_0, \dots, \eta_{k-1}, W_s, N(s,B) : s \in [0,t_k), B \in \mathcal{B}(\mathbb{R})).$$

By setting $\widehat{S}_n(z) := \max_u \widetilde{h}(z,u)$, we can prove that the following inequality holds.

$$\begin{aligned}
\overline{S}(\overline{\eta}) =& \mathbb{E}\left[\mathbb{E}\left[S(\check{Z}(\cdot),\check{\eta}(\cdot))|\mathcal{F}_n\right]\right] \\
=& \mathbb{E}\left[S_0^a(\check{Z}(\cdot),\check{\eta}(\cdot)) + G[0,T]\mathbb{E}[\widetilde{h}(Z_n,\eta_n)|\mathcal{F}_n]\right] \\
\leq& \mathbb{E}\left[S_0^a(\check{Z}(\cdot),\check{\eta}(\cdot)) + G[0,T]\widehat{S}_n(Z_n)\right],
\end{aligned}$$

which is fundamental for the DPP to hold.

Let now $u = g_n(z)$ be a Borel function which allows to get the optimal u for the given z, attaining the maximum for $\widetilde{h}(z,u)$.
Thus:

$$\widetilde{h}(z, g_n(z)) = \max_u \widetilde{h}(z,u) = \widehat{S}_n(z) = V_n(z).$$

Now, we replace the control $\overline{\eta}$ with $\overline{\eta}^{(n)} := (\eta_0, \eta_1, \dots, \eta_{n-1}, \eta_n^*)$ where $\eta_n^* = g_n(Z_n)$, therefore the control $\overline{\eta}^{(n)}$ is at least as good as $\overline{\eta}$, that is to say $\overline{S}(\overline{\eta}) \leq \overline{S}(\overline{\eta}^{(n)})$.
Furthermore

$$\overline{S}(\check{\eta}^{(n)}) = \mathbb{E}\left[S^a_{[0,t_{n-1}]}(\check{Z}(\cdot),\check{\eta}(\cdot)) + G[0,t_{n-1}]\mathbb{E}[S_{t_{n-1}}(\check{Z}(\cdot),\check{\eta}(\cdot))|\mathcal{F}_{n-1}]\right],$$

$$\mathbb{E}\left[S_{t_{n-1}}(\check{Z}(\cdot),\check{\eta}(\cdot))|\mathcal{F}_{n-1}\right] = V_{n-1}(\check{Z}(\cdot),\check{\eta}_{n-1}),$$

where

$$V_{n-1}(\check{z}(\cdot),u) = S^a_{t_{n-1}}(z_{n-1},u) + G_{[t_{n-1},T]}(z_{n-1},u)\mathbb{E}[\widehat{S}_n(Z_n(z(\cdot),u))],$$

and $\check{z}(t) = z_k$, $\check{u}(t) = u_k$ for $t \in [t_k, t_{k+1})$. $Z_n(z(\cdot),u)$ are determined in accordance with the Equation (2.3):

$$\begin{aligned}
Z_n =& z_{n-1} + \int_{t_{n-1}}^{t_n} a(s, z_0, \dots, z_{n-1}, u)ds + \int_{t_{n-1}}^{t_n} b(s,\check{z}(\cdot),u)dW_s \\
&+ \int_{t_{n-1}}^{t_n} c(s,\check{z}(\cdot),u)dC + \int_{t_{n-1}}^{t_n}\int_{\mathbb{R}\setminus\{0\}} d(s,\check{z}(\cdot),q,u)N(ds,dq),
\end{aligned}$$

where we replaced $z(\cdot)$ and $u(\cdot)$ with the constant values z and u. Thus, we get:

$$\overline{S}(\overline{\eta}^{(n)}) = \mathbb{E}\left[S^a_{[0,t_{n-1}]}(\check{Z}(\cdot),\check{\eta}(\cdot)) + G[0,t_{n-1}]V_{n-1}(\check{Z}(\cdot),\eta_{n-1})\right],$$

which is nothing else that the initial situation in time span $[0, t_{n-1}]$ and $S^*_{n-1}(z,u)$. Similarly to what has been done before, we set $\eta^*_{n-1} = g_{n-1}(\check{Z}(\cdot))$; $\overline{\eta}^{(n-1)} = (\eta_0, \eta_1, \ldots, \eta^*_{n-1}, \eta^*_n)$, where $g_{n-1}(\check{z}(\cdot)) = g_{n-1}(z_0, z_1, \ldots, z_{n-1})$ is a Borel function of its arguments satisfying the equation

$$V_{n-1}(\check{z}(\cdot), g_{n-1}(\check{z}(\cdot))) = \max_u V_{n-1}(\check{z}(\cdot), u),$$

then, for any η_{n-1}, it holds that $\overline{S}(\overline{\eta}) \leq \overline{S}(\overline{\eta}^{(n-1)})$.
Set now $\widehat{S}_{n-1}(\check{z}(\cdot)) = \max_u V_{n-1}(\check{z}(\cdot), u)$, iterating, we get a sequence of functionals $\widehat{S}_k(\check{z}(\cdot)), V_k(\check{z}(\cdot))$ depending on the values of the process $z(\cdot)$ on the time arrival $[0, t_k]$ such that

$$\begin{cases} \widehat{S}_k(\check{z}(\cdot)) = \max_u V_k(\check{z}(\cdot), u), k = n, n-1, \ldots, 0, \\ V_k(\check{z}(\cdot), u) = \widetilde{h}(z, u), \\ V_k(z(\cdot), u) = S^a_{[t_k, t_{k+1}]}(z(\cdot), u(\cdot)) + G[t_k, t_{k+1}]\mathbb{E}[\widehat{S}_{k+1}(Z_{k+1}(\check{z}(\cdot)), u)], \end{cases} \tag{2.4}$$

with $V_k(\check{z}(\cdot), u)$ and $Z_k(z(\cdot))$ depending on the values of $z(\cdot)$ on $[0, t_k]$ if $g_k(z(\cdot))$ is a Borel function satisfying

$$V_k(z(\cdot), g_k(\check{z}(\cdot))) = \widehat{S}_k(\check{z}(\cdot)), \quad k = 0, 1, \ldots, n \tag{2.5}$$

which depends only on the values $z_0, z_1, \ldots, z_k$. Thus, for any control $\overline{\eta} = (\eta_i)_{i=1}^n$ and any $j \leq n$ the control $\overline{\eta}^{(j)} = (\eta_1, \ldots, \eta_{j-1}, \eta^*_j, \ldots, \eta^*_n)$ will be at least as good as the control $\overline{\eta}$: $\overline{S}(\overline{\eta}) \leq \overline{S}(\overline{\eta}^{(j)})$. In particular, the control $\overline{\eta}^{(0)} = (\eta^*_0, \ldots, \eta^*_n)$ is optimal for the sequence $(Z_i)_{i=0}^n$ and the optimal control for $Z_0 = z$ is equal to $\widehat{S}_0(z)$.

Furthermore, we can achieve a method for constructing ε-optimal feedback control for solution of the Equation (2.3) under the conditions of [GS12, Theorem 3.16].

For any $\varepsilon > 0$, there exists an ε-optimal feedback control of the form

$$\eta(t) = g_k(Z_0, Z_1, \ldots, Z_k),$$

for $t \in [t_k, t_{k+1})$.
The functions $S_k(z(\cdot))$ express the value of the optimal control utility for the sequence $(Z_k)_{k=0}^n$ such that the variables $(Z_j)_{j=k+1}^n$ are given by Equations (2.4) under the conditions $Z_0 = z_0, \ldots, Z_k = z_k$ up to time k. We now introduce the function representing the optimal cost of control for the solution of Equation (1.18) on the time interval $[t, T]$:

$$V(t, z(\cdot)) = \sup_{\eta(\cdot)\in\mathcal{A}_{[t,T]}(z)} \mathbb{E}[S_t(Z_t^{(\eta)}(\cdot), \eta(\cdot))]$$

assuming $Z_t^{(\eta)}(s) = z(s), s \in [0, t]$. We remark that $\mathcal{A}[t, T](z)$ refers to the uncountable set of measurable control functions on $[t, T]$ defined in (1.26),therefore, we introduce the least-upper-bound of this set of measurable functions as its essential l.u.b. without leading to a new notation. Thus

$$\sup_{\eta(\cdot)\in\mathcal{A}_{[t,T]}(z)} \mathbb{E}\left[S_t(Z_t^{(\eta)}(\cdot), \eta(\cdot))\right] = \operatorname*{ess\,sup}_{\eta(\cdot)\in\mathcal{A}_{[t,T]}(z)} \mathbb{E}\left[S_t(Z_t^{(n)}(\cdot), \eta(\cdot))\right].$$

Then, we can introduce the optimality principle also known as Bellman's principle, see e.g. [GS12, Theorem 3.19].

Theorem 2.1.1. *Let (a, b, c, d) be the functionals in Equation* (2.3).
Then, if (a, b, c, d) are linearly bounded and satisfy the uniform Lipschitz condition, we have, for all $s \in [t, T]$, that

$$V(t, z(\cdot)) = \sup_{\eta(\cdot)\in\mathcal{A}_{[t,s]}(z)} \mathbb{E}\left[S^a_{[t,s]}(Z^{(\eta)}_t(\cdot), \eta(\cdot)) + G[t, s]V(s, Z^{(\eta)}_t(\cdot))\right].$$

Proof. Let $t, z(\cdot)$ and $\varepsilon > 0$ be fixed. For any $\varepsilon > 0$, δ and $\eta(\cdot) \in \mathcal{A}_{[t,T]}(z)$, it can be proved that:

$$\begin{aligned}\widehat{S}(t, z) + \varepsilon &< \mathbb{E}[\widehat{S}_t(Z^{(n)}_t(\cdot), \eta(\cdot))]\\ &= \mathbb{E}\left[S^a_{[t,T]}(Z^{(\eta)}_t(\cdot), \eta(\cdot)) + G[t, s]S_s(Z^{(\eta)}_t(\cdot), \eta(\cdot))\right],\end{aligned}$$

whereas the expression on the right-hand side is at least

$$\mathbb{E}\left[S^a_{[t,s]}(Z^{(\eta)}_t(\cdot), \eta(\cdot)) + G[t, s]\widehat{S}(s, Z^{(\eta)}_t(\cdot))\right].$$

This implies that

$$\widehat{S}(t, z) \leq \sup_{\eta\in\mathcal{A}_{[t,s]}(z)} \mathbb{E}[S^a_{[t,T]}(Z^{(\eta)}_t(\cdot), \eta(\cdot)) + G[t, s]V(s, Z^{(\eta)}_t(\cdot))].$$

The reverse inequality is also valid: since $\mathbb{E}[S_{[s,T]}\left(Z^{(\eta^*)}_s(\cdot), \eta^*(\cdot)\right)]$ converges to $\widehat{S}(t, z(\cdot))$ as $|T^n| \to 0$ uniformly in $z(\cdot)$ on the ball $\|z(\cdot)\|_t \leq \rho$.
Note that we define η^* as an optimal control for the approximating sequence $(Z^{(\eta)}_k)^n_{k=0}$ corresponding to the given subdivision T^n. Moreover since $\sup_{0\leq s\leq T} |Z^{(\eta)}_t(s)|$ is a stochastically bounded random variable uniform in the above ball for all $N > 0$. Therefore, a subdivision T^n can be found such that

$$\mathbb{P}\left(\mathbb{E}[S_{[s,T]}(Z^{(\eta^*)}(\cdot), \eta^*(\cdot))|\mathcal{F}_s] > \widehat{S}(s, Z^{(\eta^*)}(s)) + \varepsilon\right) > 1 - \varepsilon.$$

We have that

$$\begin{aligned}\widehat{S}(t, z(\cdot)) &= \sup_{\eta\in\mathcal{A}_{[t,T]}(z)} \mathbb{E}[S_{[t,T]}(Z^{(\eta)}_t(\cdot), \eta(\cdot))]\\ &= \sup_{\eta\in\mathcal{A}_{[t,T]}(z)} \mathbb{E}[S^a_{[t,T]}(Z^{(\eta)}_t(\cdot), \eta(\cdot))] + G[t, s]\mathbb{E}[S_{[s,T]}(Z^{(\eta)}_t(\cdot), \eta(\cdot))|\mathcal{F}_s].\end{aligned}$$

Each one of the controls $\eta(\cdot) \in \mathcal{A}_{[t,T]}(z)$ can be decomposed in two components: $\eta(\cdot) = (\eta_1(\cdot), \eta_2(\cdot))$, where $\eta_1(\cdot) \in \mathcal{A}[t, s](z)$ and $\eta_2(\cdot) \in \mathcal{A}_{[s,T]}(z)$. Therefore

$$\begin{aligned}&\widehat{S}(t, z(\cdot))\\ &\geq \sup_{\eta\in\mathcal{A}_{[t,s]}(z)} \mathbb{E}[S^a_{[t,s]}(Z^{(\eta)}_t(\cdot), \eta(\cdot)) + G[t, s](\widehat{S}(s, Z^{(\eta^*)}(s)) + \varepsilon + C_1\varepsilon)]\\ &\geq C_2\varepsilon + \sup_{\eta\in\mathcal{A}_{[t,s]}(z)} \mathbb{E}[S^a_{[t,s]}(Z^{(\eta)}_t(\cdot), \eta(\cdot)) + G[t, s]\widehat{S}(s, Z^{(\eta^*)}_t(s))],\end{aligned}$$

where

$$C_1 = \sup_{z(\cdot),u(\cdot)} S_{[s,T]}(z(\cdot),u(\cdot)), \quad C_2 = C_1 + \sup_{z(\cdot),u(\cdot)} G[0,T] = C_1 + \exp -\delta T.$$

Since $\varepsilon > 0$ is arbitrarily small, the last inequality implies

$$V(t,z(\cdot)) \geq \sup_{\eta(\cdot)\in\mathcal{A}_{[t,T]}(z)} \mathbb{E}[S^a_{[t,s]}(Z_t^{(\eta)}(\cdot),\eta(\cdot)) + G[t,s]V(s,Z_t^{(\eta)}(s))],$$

which proves the theorem. □

Exploiting previous results, we have the validity of the DPP, the latter being the basis for Section 2.2.

Proposition 2.1.1 (Dynamic Programming Principle)**.** *For any* $(t,x,y,\lambda) \in \mathcal{X}$ *and* $\Delta t \in [0,T-t]$ *we have*

$$V(t,x,y,\lambda) = \sup_{(\boldsymbol{\pi},\boldsymbol{C})\in\mathcal{A}_{[t,t+\Delta t]}(x,y)} \mathbb{E}\left[\int_t^{t+\Delta t} e^{-\delta u} f(Y_u^{\pi,C})du + V(t+\Delta t, X^{\pi,C}_{t+\Delta t}, Y^{\pi,C}_{t+\Delta t}, \lambda_{t+\Delta t})\right]. \tag{2.6}$$

Proof. By Theorem 2.1.1, we see that the regularity conditions required hold in the portfolio model exploited, therefore the value function $V(t,x,y,\lambda)$ is the solution of the optimization problem. □

In the next section, we will prove that the value function can be connected to the so-called *Hamilton-Jacobi-Bellman* (HJB) equation via the Dynamic Programming Principle above stated.

2.2 The Hamilton-Jacobi-Bellman equation related to the problem

By exploiting the results in Section 2.1, we can use a technique similar to the one presented in [ØS05], where the Hamilton-Jacobi-Bellman equation is obtained for a portfolio presenting discontinuous paths.
In particular, Itô-Doeblin's lemma will be exploited for the value function represented in Equation (1.25). We remark that the value function satisfies the regularity requirements necessary to apply the lemma.

It holds that

$$
\begin{aligned}
V_{t+\Delta t} =& V_t + \int_t^{t+\Delta t} \left(\frac{\partial V_u}{\partial t} + \frac{\partial V_u}{\partial x} \left(r(1-\pi_u) X_u^{\pi,C} \right) - \frac{\partial V_u}{\partial y} \beta Y_u^{\pi,C} + \frac{\partial V_u}{\partial x} \mu \pi_u X_u^{\pi,C} \right. \\
&+ \left. \frac{\partial V_u}{\partial \lambda} \alpha(\xi - \lambda_u) - \frac{\partial V_u}{\partial x} \lambda_u \mathbb{E}[e^J - 1] \pi_u X_u^{\pi,C} + \frac{1}{2} \frac{\partial^2 V_u}{\partial x^2} (\sigma \pi_u X_u^{\pi,C})^2 \right) du \\
&+ \int_t^{t+\Delta t} - \frac{\partial V_{u-}}{\partial x} + \beta \frac{\partial V_{u-}}{\partial y} dC_u + \int_t^{t+\Delta t} \frac{\partial V_u}{\partial x} \sigma \pi_u X_u^{\pi,C} dW_u \\
&+ \int_t^{t+\Delta t} \int_{\mathbb{R}\setminus\{0\}} \frac{\partial V_{u-}}{\partial x} \pi_{u-} X_{u-}^{\pi} (e^q - 1) N(du, dq) + \eta \int_t^{t+\Delta t} \frac{\partial V_u}{\partial \lambda} \int_{\mathbb{R}\setminus\{0\}} |q| N(du, dq) \\
&+ \sum_{t \le u \le t+\Delta t} \left(\Delta V_u - \frac{\partial V_{u-}}{\partial x} \Delta X_u^{\pi,C} - \frac{\partial V_{u-}}{\partial y} \Delta Y_u^{\pi} - \frac{\partial V_{u-}}{\partial \lambda} \Delta \lambda_u \right),
\end{aligned} \tag{2.7}
$$

where $\frac{\partial^k V_t}{\partial a^k}$, $k \in \mathbb{N}$, refers to the k-order partial derivative of V_t with respect to a.

By exploiting the DPP and Equation (2.7), it holds that:

$$
\begin{aligned}
0 \le & \sup_{(\pi,C) \in \mathcal{A}_{[t,T]}(x,y)} \mathbb{E} \left[\int_t^{t+\Delta t} e^{-\delta u} f(Y_u^{\pi,C}) du + \int_t^{t+\Delta t} \left(\frac{\partial V_u}{\partial t} + \frac{\partial V_u}{\partial x} r(1-\pi_u) X_u^{\pi,C} \right. \right. \\
& + \left. \frac{\partial V_u}{\partial x} \mu \pi_u X_u^{\pi,C} - \frac{\partial V_u}{\partial y} \beta Y_u^{\pi,C} + \frac{1}{2} \frac{\partial^2 V_u}{\partial x^2} (\sigma \pi_u X_u^{\pi,C})^2 + \alpha(\xi - \lambda_u) \pi_u X_u^{\pi,C} \right) du \\
& + \int_t^{t+\Delta t} \left(- \frac{\partial V_{u-}}{\partial x} + \beta \frac{\partial V_{u-}}{\partial y} \right) dC + \eta \int_t^{t+\Delta t} \frac{\partial V_u}{\partial \lambda} \int_{\mathbb{R}\setminus\{0\}} |q| N(du, dq) \qquad (2.8) \\
& + \left. \sum_{t \le u \le t+\Delta t} \left(\Delta V_u - \frac{\partial V_{u^-}}{\partial x} \Delta X_u^{\pi,C} - \frac{\partial V_{u^-}}{\partial y} \Delta Y_u^{\pi,C} - \frac{\partial V_{u^-}}{\partial \lambda} \Delta \lambda_u \right) \right]. \qquad (2.9)
\end{aligned}
$$

Furthermore, if $\boldsymbol{C}$ is continuous in $[t, t + \Delta t]$, the last part of Equation (2.8) can be rewritten in the following form:

$$
\begin{aligned}
& \mathbb{E} \left[\sum_{t \le u \le t+\Delta t} \left(\Delta V_u - \frac{\partial V_{u^-}}{\partial x} \Delta X_u^{\pi,C} - \frac{\partial V_{u^-}}{\partial y} \Delta Y_u^{\pi,C} + \frac{\partial V_{u^-}}{\partial \lambda} \Delta \lambda_u \right) \right] \\
& = \int_t^{t+\Delta t} \mathfrak{J}^{\pi_u}(u, X_u^{\pi,C}, Y_u^{\pi,C}, \lambda_u, V) du,
\end{aligned} \tag{2.10}
$$

with

$$
\begin{aligned}
\mathfrak{J}^{\pi}(t, x, y, \lambda, V) :=& \lambda \int_{\mathbb{R}\setminus\{0\}} V(t, x + \pi x(e^q - 1), y, \lambda + \eta|q|) - V(t, x, y, \lambda) \\
& - \pi x(e^q - 1) V_x(t, x, y, \lambda) - \eta |q| V_\lambda(t, x, y, \lambda) \zeta(q) dq,
\end{aligned} \tag{2.11}
$$

where $\zeta(\cdot)$ is the probability distribution defined in Equation (1.1).
Equation (2.11) follows by

$$\Delta V_u = V(u, X^{\pi,C}_{u^-} + \Delta X^{\pi,C}_u, Y^{\pi,C}_{u^-}, \lambda_u + \Delta\lambda_u) - V(u, X^{\pi,C}_{u^-}, Y^{\pi,C}_{u^-}, \lambda_u),$$
$$\Delta X^{\pi,C}_u = \pi_{u^-} X^{\pi,C}_{u^-}(e^q - 1),$$
$$\Delta\lambda_u = \eta|q|,$$

and

$$\Delta Y^{\pi,C}_u = 0.$$

By setting $dC \equiv 0$, dividing Equation (2.8) by Δt, and letting $\Delta t \searrow 0$, we obtain the following result:

$$\begin{aligned} 0 \le e^{-\delta t} f(y) + \frac{\partial V_t}{\partial t} - \beta y \frac{\partial V_t}{\partial y} + \sup_{\pi\in[0,1]} \Bigg[& \frac{\partial V_t}{\partial x} r(1-\pi)x + \frac{\partial V_t}{\partial x}\pi\mu x \\ & + (\alpha(\xi - \lambda) + \eta\lambda\mu_J)\frac{\partial V_t}{\partial \lambda} + \frac{1}{2}\frac{\partial^2 V_t}{\partial x^2}(\sigma\pi x)^2 + \partial^\pi(t,x,y,\lambda,V) \Bigg]. \end{aligned} \tag{2.12}$$

2.2.1 Heuristic derivation of necessary conditions for C

Further considerations on the consumption process will lead to an additional constraint on the HJB equation in (2.8). We consider the consumption C_t to be either discontinuous at t, or differentiable at t a.s. and continuous on $[t, t+\Delta t]$. Then, there is a constant $c > 0$, such that $C'(t) = 0$ or $C'(t) > c$ for differentiable $\boldsymbol{C}$.

Suppose now $(\widehat{\boldsymbol{\pi}}, \widehat{\boldsymbol{C}}) \in \mathcal{A}_{[t,t+\Delta t]}(x, y)$ be an optimal control.
We take into account the three different cases:

1. a consumption gulp occurs at time t,
2. $\widehat{\boldsymbol{C}}$ is continuous on $[t, t+\Delta t]$ and $\widehat{C}'(t) > c$, and
3. $\widehat{\boldsymbol{C}}$ is continuous on $[t, t+\Delta t]$ and $\widehat{C}'(t) = 0$.

Consider Case 1, where a consumption gulp ΔC occurs at time t. By Proposition 1.3.2, it holds that

$$V(t, x, y, \lambda) = V(t, x - \Delta C, y + \beta\Delta C, \lambda). \tag{2.13}$$

Then, the gulp size should be chosen to maximize V, therefore the derivative of the right-hand side of Equation (2.13), with respect to ΔC must be 0, that is say:

$$-V_x(t, x - \Delta C, y + \beta\Delta C, \lambda) + \beta V_y(t, x - \Delta C, y + \beta\Delta C, \lambda) = 0. \tag{2.14}$$

We differentiate Equation (2.13) with respect to x and exploit Equation (2.14). Then, the following result holds

$$\begin{aligned} V_x(t, x, y, \lambda) =& \left(-V_x(t, x - \Delta C, y + \beta\Delta C, \lambda) + \beta V_y(t, x - \Delta C, y + \beta\Delta C, \lambda)\right)\frac{\partial \Delta C}{\partial x} \\ & + V_x(t, x - \Delta C, y + \beta\Delta C, \lambda) \\ =& V_x(t, x - \Delta C, y + \beta\Delta C, \lambda). \end{aligned} \tag{2.15}$$

A similar result can be obtained for $V_y(t,x,y,\lambda) = V_y(t,x-\Delta C, y+\beta\Delta C,\lambda)$.
From Equations (2.14) and (2.15) and the respective results for y we get

$$V_x(t,x,y,\lambda) = \beta V_y(t,x,y,\lambda).$$

Consider now Case 2, and let $c > 0$ such that $\widehat{C}'(t) > c$.

Since $(\widehat{\pi}, \widehat{C})$ is an optimal control, we know that the control processes maximize the right-hand side of the Equation (2.8).
Now, we apply a small perturbation from $(\widehat{\boldsymbol{\pi}}, \widehat{\boldsymbol{C}})$ to $(\boldsymbol{\pi}, \boldsymbol{C})$, where $C_s = \widehat{C}_s + C_s^{\Delta}$ and $\widehat{\pi}_s \equiv \pi_s$ for a function $C_s^{\Delta} : [t,T] \to \mathbb{R}$ defined by $C_s^{\Delta} := c'(t-s), c' \in \mathbb{R}$.
Let us focus on $(\boldsymbol{\pi}, \boldsymbol{C})$ defined $[t, t+\Delta t]$ by using the solution of (1.18) and (1.22), we see that $X_s^{\pi,C} = X_s^{\widehat{\pi},\widehat{C}} - \widehat{C}_s^{\Delta} + O(\Delta t^2)$ and $Y_s^{\pi,C} = Y_s^{\widehat{\pi},\widehat{C}} + \beta C_s^{\Delta}$.
Finally, we verify that $(\boldsymbol{\pi}, \boldsymbol{C}) \in \mathcal{A}_{[t,t+\Delta t]}(x,y)$. To do so, we plug $\boldsymbol{C}$ into Equation (2.8) to get an approximation of it, assuming g be a smooth function and Δt small enough.

$$\begin{aligned}
&\int_t^{t+\Delta t} g(s, X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s) ds - \int_t^{t+\Delta t} g(s, X_s^{\widehat{\pi},\widehat{C}}, Y_s^{\widehat{\pi},\widehat{C}}, \lambda_s) ds \\
&\quad \approx \int_t^{t+\Delta t} -C_s^{\Delta} g_x\left(s, X_s^{\widehat{\pi},\widehat{C}}, Y_s^{\widehat{\pi},\widehat{C}}, \lambda_s\right) + \beta C_s^{\Delta} g_y\left(s, X_s^{\widehat{\pi},\widehat{C}}, Y_s^{\widehat{\pi},\widehat{C}}, \lambda_s\right) ds \\
&\quad = O(\Delta t^2),
\end{aligned}$$

and

$$\begin{aligned}
&\int_t^{t+\Delta t} g(s, X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s) dC_s - \int_t^{t+\Delta t} g(s, X_s^{\widehat{\pi},\widehat{C}}, Y_s^{\widehat{\pi},\widehat{C}}, \lambda_s) d\widehat{C}_s \\
&\quad \approx c' \int_t^{t+\Delta t} g(s, X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s) ds - \int_t^{t+\Delta t} C_s^{\Delta} g_x\left(s, X_s^{\widehat{\pi},\widehat{C}}, Y_s^{\widehat{\pi},\widehat{C}}, \lambda_s\right) d\widehat{C}_s \\
&\quad + \int_t^{t+\Delta t} \beta C_s^{\Delta} g_y\left(s, X_s^{\widehat{\pi},\widehat{C}}, Y_s^{\widehat{\pi},\widehat{C}}, \lambda_s\right) d\widehat{C}_s \\
&\quad = c' \int_t^{t+\Delta t} g(s, X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s) ds + O(\Delta t^2).
\end{aligned}$$

By plugging the previous approximations in Equation (2.8) and using the fact that $\boldsymbol{C}^*$ maximizes the right-hand side of Equation (2.8), we ascertain that

$$\mathbb{E}\left[c' \int_t^{t+\Delta t} -V_x + \beta V_y ds\right] \leq 0. \tag{2.16}$$

For all $c' \in (-c, 0]$ and small enough Δt, we have $(\boldsymbol{\pi}, \boldsymbol{C}) \in \mathcal{A}_{[t,t+\Delta t]}(x,y)$ since $\boldsymbol{C}$ is increasing and $X_s^{\pi,C} \geq X_s^{\widehat{\pi},\widehat{C}} \geq 0$ for $c' \in (-c, 0]$.
Assume now $c' \in \mathbb{R}_+$, consequently $C_s > \widehat{C}_s$ for $s \in [t, t+\Delta t]$. Since C_s is strictly increasing, we have that $(\boldsymbol{\pi}, \boldsymbol{C}) \in \mathcal{A}_{[t,t+\Delta t]}(x,y)$ if and only if $X_s^{\pi,C} \geq 0$ on $[t, t+\Delta t]$. Since $x > 0$, we will have $(\boldsymbol{\pi}, \boldsymbol{C}) \in \mathcal{A}_{[t,t+\Delta t]}(x,y)$ for small Δt, and Inequality (2.16) holds for all $c' \in (-c, \infty)$ for $\Delta t \to 0$; finally by continuity we have $-V_x + \beta V_y = 0$.

For what concerns Case 3, we know that $(\widehat{\boldsymbol{\pi}}, \boldsymbol{C})$ maximizes the r.h.s. of Equation (2.8) and we have that $\beta V_y - V_x \leq 0$ (as in the previous case), but with the peculiarity that c' should be positive, in order to ensure that $\boldsymbol{C}$ is not decreasing. Equation (2.10) is still valid by the continuity of $\boldsymbol{C}$. Therefore, we insert Equation (2.10) and $(\boldsymbol{\pi}, \boldsymbol{C}) = (\widehat{\boldsymbol{\pi}}, \widehat{\boldsymbol{C}})$ into Equation (2.8) and we note that all the terms are of order $O(\Delta t)$, except for $\int_{t-}^{t+\Delta t} -V_x + \beta V_y dC$, which is of order $O(C_{t+\Delta t} - C_t)$. Moreover $C_{t+\Delta t} - C_t$ is of smaller order that Δt since $C'(t) = 0$. Letting $\Delta t \to 0$ in the Equation (2.8), we note that the inequality (2.12) holds as equality. By recollecting the previous results and lightening up the notation, we define $\boldsymbol{Z} := (\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{\lambda})$ then, for a generic function $v \in C^{1,2,1,1}(\mathcal{X})$ defined on the same domain as V, we have that v satisfies:

$$0 = \max\{G(D_Z v), v_t + F(t, z, D_Z v, D_Z^2 v, \mathfrak{J}^\pi(t, z, v))\}, \tag{2.17}$$

where

$$G(D_Z v) := \beta v_y(t, x, y, \lambda) - v_x(t, x, y, \lambda) \tag{2.18}$$

and

$$\begin{aligned} F(t, z, D_Z v, D_Z^2 v, \mathfrak{J}^\pi(t, z, v)) &:= e^{-\delta t} f(y) - \beta y v_y + (\alpha(\xi - \lambda) + \eta \lambda \mu_J) v_\lambda \\ &+ \max_{\pi \in [0,1]} \left[(r(1 - \pi) + \mu \pi) x v_x + \frac{1}{2} (\sigma \pi x)^2 v_{xx} + \mathfrak{J}^\pi(t, z, v) \right]. \end{aligned} \tag{2.19}$$

We remark that $D_Z v$, resp. $D_Z^2 v$, denote the first, resp. the second, order derivative of v with respect to z. Furthermore, $G(D_Z v) = 0$ in presence of consumption, whereas, in absence of consumption, Equation (2.19) turns into

$$v_t + F(t, z, D_Z v, D_Z^2 v, \mathfrak{J}^\pi(t, z, v)) = 0.$$

Chapter 3

Existence and uniqueness for the HJB solution

In this chapter we will prove that the value function V defined in Equation (1.25) solves the HJB equation in (2.8) in a *viscosity* sense.

To prove that, we start by showing the convergence of the integral part of the HJB equation. Then, we formulate several, although equivalent, definitions for viscosity solutions. Indeed, some theorems are easier to prove using a particular definition.

Therefore, we conclude with the main contribution of this chapter, we prove that the value function V is the unique constrained solution of the HJB equation by the mean of the comparison principle. The uniqueness result is based on [CDPVndb], where the approaches in [BKR01b; BKR02] and [Hol10] have been generalized. In fact, in the first two mentioned works, the authors consider a Lévy process over an infinite-horizon, while in the last one a Lévy process has been taken into account over a finite-horizon.

3.1 Definitions and convergence for the integral operator

Let us start by providing definitions that will be then used to prove uniqueness of the viscosity solution.

Definition 3.1.1. *For any $\kappa \in (0,1)$, $(t,z) \in \overline{\mathcal{X}}$, $\varphi \in C_1(\overline{\mathcal{X}}) \cap C^{1,2,1,1}(\overline{\mathcal{X}})$ and $P := (p_1, p_2, p_3) \in \mathbb{R}^3$, let us define*

$$\mathcal{J}^{\pi,\kappa}(t,z,\varphi,P) := \lambda \int_{|q|>\kappa} \left(\varphi(t, x+\pi x(e^q-1), y, \lambda+\eta|q|) - \varphi(t,z) - \pi x p_1(e^q-1) - p_3\eta|q|\right)\zeta(q)dq$$

and

$$\mathcal{J}^{\pi}_{\kappa}(t,z,\varphi) := \lambda \int_{|q|\le\kappa} \left(\varphi(t, x+\pi x(e^q-1), y, \lambda+\eta|q|) - \varphi(t,z) - \pi x(e^q-1)\varphi_x(t,z) - \eta|q|\varphi_\lambda(t,z)\right)\zeta(q)dq$$

Moreover, we must remark that $\mathfrak{J}^{\pi}(t,z,\varphi)$, defined in Equation (2.11), can be obtained as the sum of $\mathfrak{J}^{\pi}_{k}(t,z,\varphi)$ and $\mathfrak{J}^{\pi,\kappa}(t,z,\varphi,D_Z\varphi)$, therefore we obtain that:

$$\begin{aligned} F(t,&z,D_Z\varphi,D_Z^2\varphi,\mathfrak{J}^{\pi,\kappa}(t,z,\varphi,D_Z\varphi),\mathfrak{J}^{\pi}_{\kappa}(t,z,\varphi)) := \\ &F(t,z,D_Z\varphi,D_Z^2\varphi,\mathfrak{J}^{\pi,\kappa}(t,z,\varphi,D_Z\varphi)+\mathfrak{J}^{\pi}_{\kappa}(t,z,\varphi)). \end{aligned}$$

Although the previous statements seem not relevant at a first glance, they will be useful in the proof of Theorem 3.3.2. In the following the convergence for the integral operator corresponding to $\mathfrak{J}^{\pi}$ will be proved.

3.1.1 Convergence of integral part

The first part of this chapter is devoted to prove that the operator $\mathfrak{J}^{\pi}$ converges and therefore the integral is well-defined. In particular, we will prove that

$$\mathfrak{J}^{\pi}(t_n,z_n,\varphi)\to\mathfrak{J}^{\pi}(t,z,\varphi) \quad \textit{for} \quad (t_n,z_n)\to(t,z),$$

and

$$\mathfrak{J}^{\pi}(t,z,\varphi_n)\to\mathfrak{J}^{\pi}(t,z,\varphi)$$

for $\varphi_n\to\varphi$, $(\varphi_n)_x\to\varphi_x$, $(\varphi_n)_\lambda\to\varphi_\lambda$, and $(\varphi_n)_{xx}\to\varphi_{xx}$.

Lemma 3.1.1. *Let* $\varphi,\varphi_n\in C^{1,2,1,1}(\overline{\mathfrak{X}})\cap C_1(\overline{\mathfrak{X}})$ *and* $(t_n,z_n),(t,z)\in\overline{\mathfrak{X}} \quad \forall \quad n\in\mathbb{N}$. *If the assumptions*

1. $(t_n,\widetilde{z}_n)\to(t,\widetilde{z})$,
2. $\varphi_n(t_n,\widetilde{z}_n)\to\varphi(t,\widetilde{z})$,
3. $(\varphi_n)_x(t,\widetilde{z})\to\varphi_x(t,\widetilde{z})$,
4. $(\varphi_n)_\lambda(t,\widetilde{z})\to\varphi_\lambda(t,\widetilde{z})$,
5. $(\varphi_n)_{x\lambda}(t,\widetilde{z})\to\varphi_{x\lambda}(t,\widetilde{z})$,
6. $(\varphi_n)_{xx}(t,\widetilde{z})\to\varphi_{xx}(t,\widetilde{z})$,

hold for all the $(t,\widetilde{z})\in\overline{\mathfrak{X}}$, *then*

$$\mathfrak{J}^{\pi}(t_n,z_n,\varphi_n)\to\mathfrak{J}^{\pi}(t,z,\varphi)$$

as $n\to\infty$.

Proof. By the triangular inequality we have

$$\begin{aligned} &|\mathfrak{J}^{\pi}(t_n,z_n,\varphi)-\mathfrak{J}^{\pi}(t,z,\varphi)| \\ &\leq|\mathfrak{J}^{\pi}(t_n,z_n,\varphi_n)-\mathfrak{J}^{\pi}(t_n,z_n,\varphi)|+|\mathfrak{J}^{\pi}(t_n,z_n,\varphi)-\mathfrak{J}^{\pi}(t,z,\varphi)|. \end{aligned} \tag{3.1}$$

Given inequality (3.1), we must prove that the two terms on the right-hand side of the inequality go to 0 as $n \to \infty$, in order to to have the desired convergence.
Therefore, we start by focusing on the first term:

$$\begin{aligned}
&|\mathcal{J}^{\pi}(t_n, z_n, \varphi_n) - \mathcal{J}^{\pi}(t, z_n, \varphi)| \\
&= \lambda_n \bigg| \int_{\mathbb{R}\setminus\{0\}} (\varphi_n - \varphi)(t_n, x_n + x_n\pi_n(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n) \\
&- x_n\pi(e^q - 1)(\varphi_n - \varphi)_x(t_n, z_n) - \eta|q|(\varphi_n - \varphi)_\lambda(t_n, z_n)\zeta(q)dq \bigg| \\
\leq &\lambda_n \Bigg(\bigg| \int_{|q|<1} (\varphi_n - \varphi)(t_n, x_n + x_n\pi_n(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n) \\
&- x_n\pi(e^q - 1)(\varphi_n - \varphi)_x(t_n, z_n) - \eta|q|(\varphi_n - \varphi)_\lambda(t_n, z_n)\zeta(q)dq \bigg| \\
&+ \bigg| \int_{1<|q|\leq R} (\varphi_n - \varphi)(t_n, x_n + x_n\pi_n(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n) \\
&- x_n\pi(e^q - 1)(\varphi_n - \varphi)_x(t_n, z_n) - \eta|q|(\varphi_n - \varphi)_\lambda(t_n, z_n)\zeta(q)dq \bigg| \\
&+ \bigg| \int_{|q|>R} (\varphi_n - \varphi)(t_n, x_n + x_n\pi_n(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n) \\
&- x_n\pi(e^q - 1)(\varphi_n - \varphi)_x(t_n, z_n) - \eta|q|(\varphi_n - \varphi)_\lambda(t_n, z_n)\zeta(q)dq \bigg| \Bigg),
\end{aligned} \tag{3.2}$$

for all $R > 1$.

As $n \to +\infty$, we ascertain that the first term on the right-hand side of inequality (3.1) goes to 0. This can be proved by applying the Taylor expansion on $(\varphi_n - \varphi)$ around (t_n, z_n) and by using the following reasons, firstly we exploit the integrability conditions for $\zeta(\cdot)$ defined in (1.2) and then, the fact that $(e^q - 1)^2 < 3q^2$ for $|q| < 1$.

Finally, by applying the Heine-Cantor Theorem, we obtain the following result:

$$\begin{aligned}
&\bigg| \int_{|q|<1} (\varphi_n - \varphi)(t_n, x_n + x_n\pi(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n) \\
&\qquad - x_n\pi_n(e^q - 1)(\varphi_n - \varphi)_x(t_n, z_n) - \eta|q|(\varphi_n - \varphi)_\lambda(t_n, z_n)\zeta(q)dq \bigg| \\
&\leq 3(\pi x_n)^2 \sup_{|q|<1} \Big| (\varphi_n - \varphi)_{xx}(t_n, x_n + \pi x_n(e^q - 1), y_n, \lambda_n + \eta|q|) \Big| \int_{|q|<1} q^2\zeta(q)dq \\
&\qquad + \eta\pi x_n \sup_{|q|<1} \left(\Big| (\varphi_n - \varphi)_{\lambda x}(t_n, x_n + \pi x_n(e^q - 1), y_n, \lambda_n + \eta|q|) \Big| \int_{|q|<1} q(e^q - 1)\zeta(q)dq \right) \\
&\to 0.
\end{aligned}$$

From the last inequality, it is easy to see that the right-hand side goes to 0 as $n \to +\infty$.

The second term of inequality (3.2) converges to 0 thanks to (1.2), and because of the compactness of $\{q \in \mathbb{R} : 1 \leq |q| < R\}$:

$$\begin{aligned}
\Bigg| \int_{1<|q|\leq R} & (\varphi_n - \varphi)(t_n, x_n + x_n\pi(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n) \\
& - x_n\pi(e^q - 1)(\varphi_n - \varphi)_x(t_n, z_n) - \eta|q|(\varphi_n - \varphi)_\lambda(t_n, z_n)\zeta(q)dq \Bigg| \\
\leq & 2 \sup_{1<|q|\leq R} \Bigg(\Big|(\varphi_n - \varphi)(t_n, x_n + x_n\pi(e^q - 1), y_n, \lambda_n + \eta|q|) \\
& + x_n\pi \Big|(e^q - 1)(\varphi_n - \varphi)_x(t_n, x_n + x_n\pi(e^q - 1), y_n, \lambda_n + \eta|q|)\Big| \int_{1<|q|\leq R} \zeta(q)dq \\
& + \eta\pi \Big|(\varphi_n - \varphi)_{\lambda x}(t_n, x_n + x_n\pi(e^q - 1), y_n, \lambda_n + \eta|q|)\Big| \int_{1<|q|<R} q(e^q - 1)\zeta(q)dq \Bigg) \\
& + \eta R \sup_{1<|q|<R} \Big|(\varphi_n - \varphi)_\lambda(t_n, x_n + x_n\pi(e^q - 1), y_n, \lambda_n + \eta|q|)\Big| \int_{1<|q|\leq R} \zeta(q)dq \\
\to & 0.
\end{aligned}$$

Therefore, also in this case, the right-hand side goes to 0 for $n \to \infty$.

When $R \to \infty$, the third term of inequality (3.2) converges uniformly to 0 as well; in particular $\lim_{n\to\infty}(\varphi_n - \varphi)_x(t_n, z_n) = 0$ by Heine-Cantor Theorem.
To prove it, let us recall that $\varphi, \varphi_n \in \mathcal{C}_1(\overline{\mathcal{X}})$, and let us define $g := K(1 + x + y + \lambda), K \in \mathbb{R}$. Noticing that $|\varphi_n - \varphi| \leq g,\ \forall\, n \in \mathbb{N}$ for a sufficiently large value of K.
Therefore, there exists a constant $H \in \mathbb{R}$ independent of n such that

$$\begin{aligned}
& \Big|(\varphi_n - \varphi)(t_n, x_n + x_n\pi_n(e^q - 1), y_n, \lambda_n + \eta|q|) - (\varphi_n - \varphi)(t_n, z_n)\Big| \\
& \leq g(t_n, x_n + x_n\pi_n(e^q - 1), y_n, \lambda_n + \eta|q|) - g(t_n, z_n) + H \\
& = K(x_n\pi(e^q - 1) + \eta|q|) + H.
\end{aligned}$$

Since $K(x_n\pi(e^q - 1) + \eta|q|) + H$ is integrable with respect to $\zeta(\cdot)$, we prove that the first term of (3.1) converges to 0 as $R \to \infty$ and that the convergence is uniform in n.

Now, we focus on the second term of the right-hand side of inequality (3.1); in the following we will prove it converges to 0 as $n \to \infty$.
Applying the mean value theorem, we obtain

$$\begin{aligned}
& \mathcal{J}^\pi(t_n, z_n, \varphi_n) - \mathcal{J}^\pi(t, z, \varphi) \\
& = \lambda_n \int_{\mathbb{R}\setminus\{0\}} I_t(t_{q,n}, z_{q,n}, q)(t_{\epsilon_n} - t) + I_x(t_{q,n}, z_{q,n}, q)(x_{\epsilon_n} - x) \qquad (3.3) \\
& + I_y(t_{q,n}, z_{q,n}, q)(y_{\epsilon_n} - y) + I_\lambda(t_{q,n}, z_{q,n}, q)(\lambda_{\epsilon_n} - \lambda)\zeta(q)dq, \qquad (3.4)
\end{aligned}$$

where we define I_t, I_x, I_y, I_λ as the partial derivative of the integrand part $I : \overline{\mathcal{X}} \times \mathbb{R} \to \mathbb{R}$ the operator representing the integrand part of $\mathcal{J}^\pi$, and $(t_{q,n}, z_{q,n})$ is some point on the line

connecting (t_n, z_n) with (t, z). Immediately we ascertain that the integral is well-defined since it is the sum of a difference of two integrable functions.
The derivative of I with respect to x is given by

$$\begin{aligned}I_x(t_{q,n}, x_{q,n}, y_{q,n}, \lambda_{q,n}, q) :=& v_x(t_{q,n}, x_{q,n} + \pi x_{q,n}(e^q - 1), y_{q,n}, \lambda_{q,n} + \eta|q|)(1 + \pi(e^q - 1)) \\ & - v_x(t_{q,n}, z_{q,n}) - \pi(e^q - 1)v_{xx}(t_{q,n}, z_{q,n}).\end{aligned}$$

We note that I_x is of order $|e^q - 1|$ by Taylor expansion, for large $|q|$, thus I_x is bounded for some integrable function. For a small q, we see, by Taylor expansion, that I_x is of order q^2, so also in this case I_x is bounded by some integrable function. Similarly, it can be proved the same result for I_y, I_λ and I_t.

Assume I_x, I_y, I_λ, I_t are bounded by a certain $g : \overline{\mathcal{X}} \to \mathbb{R}$, integrable with respect to $\zeta(\cdot)$. Thus, by the dominated convergence theorem, we have that the right-hand side of Equation (3.3) converges to 0 as $n \to \infty$. □

3.2 Definitions for viscosity solutions

We can formally define the value function V as solution to the HJB equation (2.17) based on test functions in $\mathcal{C}_1(\overline{\mathcal{X}})$.

Definition 3.2.1 (Viscosity solutions). *Given $\mathcal{D} \subseteq \overline{\mathcal{X}}$, a continuous function $v : \overline{\mathcal{X}} \to \mathbb{R}$ is a viscosity supersolution (resp. subsolution) of* (2.17) *at $(t, z) \in \mathcal{D}$, if any continuous differentiable function $\varphi \in \mathcal{C}^{1,2,1,1}(\overline{\mathcal{X}}) \cap \mathcal{C}_1(\overline{\mathcal{X}})$, with $\varphi : \overline{\mathcal{X}} \to \mathbb{R}$ such that $\varphi(t, z) = v(t, z)$ and $v - \varphi$ reaches the minimum (resp. maximum) at (t, z), satisfies the following conditions:*

1. *for all $(t, z) \in \mathcal{D}$, it holds that*

$$\max\{G(D_Z\varphi), \varphi_t + F(t, z, D_Z\varphi, D_Z^2\varphi, \mathcal{J}^\pi(t, z, \varphi))\} \leq 0 \quad (\geq 0)\,;$$

2. *$\forall\, z \in \mathbb{R}^3_+$ it holds that*

$$V(T, z) = \max_{c\in[0,x]} h(x - c, y + \beta c, \lambda)\,. \tag{3.5}$$

Proposition 3.2.1. *A continuous function $v : \overline{\mathcal{X}} \to \mathbb{R}$ is a constrained viscosity solution of Equation* (2.17) *if and only if v is a viscosity supersolution of* (2.17) *in $\mathcal{X}$ and v is a viscosity subsolution of* (2.17) *in $\overline{\mathcal{X}}$;*

Next result is the main theorem of the current section, providing that the value function V is the viscosity solution of Equation (2.17). Before doing so, we introduce an alternative definition of viscosity solution that is fundamental in the existence proof.

Proposition 3.2.2. *Let v be a continuous function over $\overline{\mathcal{X}}$, with $v : \overline{\mathcal{X}} \to \mathbb{R}_+$. Then, v is a viscosity supersolution of Equation* (2.17) *if and only if:*

$$\max\left\{G(D_Z\varphi); \varphi_t + F(t, z, D_Z\varphi, D_Z^2\varphi, \mathcal{J}^\pi(t, z, \varphi)\right\} \leq 0, \tag{3.6}$$

for all the $\varphi \in C^{1,2,1,1}(\overline{\mathcal{X}}) \cap C_1(\overline{\mathcal{X}})$ and $(t, z) \in \overline{\mathcal{X}}$ that satisfy the following conditions:

(1) (t,z) *is a global minimum of* $v-\varphi$ *over* $\mathcal{X}$*, and there exists a positive* $\widetilde{\epsilon}$ *such that* $(v-\varphi)(\widetilde{t},\widetilde{z}) > (v-\varphi)(t,z)+\widetilde{\epsilon}$ *for all other minima* $(\widetilde{t},\widetilde{z})$.

(2) $(v-\varphi)(t,z) = -a(t,z)$ *for some given function* $a:\overline{\mathcal{X}}\to\mathbb{R}_+$,

(3) φ *has compact support.*

Proof. Let $v:\overline{\mathcal{X}}\to\mathbb{R}_+$ be a continuous function. If v is a viscosity supersolution for all φ satisfying (1)-(3), then Equation (3.6) holds by Definition 3.2.1. Therefore, we focus on the opposite implication. Assume that Equation (3.6) holds for all φ which satisfies conditions (1)-(3). We prove that Equation (3.6) must hold for all the φ for which

(0) (t,z) is a global minimum of $v-\varphi$ over $\mathcal{X}$.

We divide the proof in three parts: in the first one, we show that Equation (3.6) holds for all the functions satisfying conditions (1)-(2); the second part will be devoted to demonstrate that Equation (3.6) holds for all the functions which satisfy (1). We conclude with the third part, where we prove that Equation (3.6) holds for all functions satisfying only (0).

Part 1: Let us assume φ satisfies (1)-(2), and let

$$\varphi_\epsilon(\widetilde{t},\widetilde{z}) := \varphi(\widetilde{t},\widetilde{z})\eta_\epsilon$$

for all $\epsilon>0$, where $\eta_\epsilon:\overline{\mathcal{X}}\to[0,1]$ satisfying

(i) $\eta_\epsilon\in\mathcal{C}^\infty(\overline{\mathcal{X}})\cap\mathcal{C}_1(\overline{\mathcal{X}})$,

(ii) $\eta_\epsilon=1$ on $\overline{\mathcal{D}}_\epsilon$, where

$$\mathcal{D}_\epsilon := \left\{(\widetilde{t},\widetilde{z})\in\mathcal{X} : t\in[0,T), \widetilde{z}<z+\frac{1}{\epsilon}\right\},$$

(iii) $\eta(\widetilde{t},\widetilde{z})=0$ for $\widetilde{z}>z+\frac{2}{\epsilon}$.

We immediately note that $\varphi_\epsilon\in\mathcal{C}^\infty(\overline{\mathcal{X}})\cap\mathcal{C}_1(\overline{\mathcal{X}})$ must satisfy (2)-(3). Finally, we assert that φ_ϵ satisfies (1) by contradiction. Let

$$(v-\varphi_\epsilon)(\widetilde{t},\widetilde{z}) \le (v-\varphi_\epsilon)(t,z)+\widetilde{\epsilon}$$

for some $(\widetilde{t},\widetilde{z})\in\overline{\mathcal{X}}$. Therefore, $(\widetilde{t},\widetilde{z})\in\overline{\mathcal{X}}\backslash\overline{\mathcal{D}}_\epsilon$, because $v-\varphi_\epsilon=v-\varphi$ on $\overline{\mathcal{D}}_\epsilon$. Also,

$$(v-\varphi_\epsilon)(\widetilde{t},\widetilde{z}) \le (v-\varphi_\epsilon)(t,z)+\widetilde{\epsilon}$$

holds since φ satisfies (1).
Moreover,

$$(v-\varphi_\epsilon)(\widetilde{t},\widetilde{z}) < (v-\varphi_\epsilon)(t,z) = (v-\varphi)(t,z) = -a(t,z) < 0,$$

and since $v(\widetilde{t},\widetilde{z}) \geq 0$, we conclude $\varphi(\widetilde{t},\widetilde{z}) > 0$. Therefore, we have

$$\begin{aligned}(v-\varphi_\epsilon)(\widetilde{t},\widetilde{z}) &= v(\widetilde{t},\widetilde{z}) - \varphi(\widetilde{t},\widetilde{z})\eta_\epsilon(\widetilde{t},\widetilde{z}) \\ &\geq v(\widetilde{t},\widetilde{z}) - \varphi(\widetilde{t},\widetilde{z}) \\ &> (v-\varphi)(t,z) + \widetilde{\epsilon} \\ &= (v-\varphi_\epsilon)(t,z) + \widetilde{\epsilon},\end{aligned}$$

contradicting what obtained before. Thus, φ_ϵ satisfies (1). We proved that φ_ϵ satisfies (1)-(3), therefore φ_ϵ satisfies Equation (3.6). We assert that

$$\lim_{\epsilon\to 0} \mathcal{J}^\pi(t,z,\varphi_\epsilon) = \mathcal{J}^\pi(t,z,\varphi)$$

for all the admissible π. Therefore,

$$(\varphi_\epsilon)_t + F(t,z,D_Z\varphi_\epsilon, D_Z^2\varphi_\epsilon, \mathcal{J}^\pi(t,z,\varphi_\epsilon) \to \varphi_t + F(t,z,D_Z\varphi, D_Z^2\varphi\mathcal{J}^\pi(t,z,\varphi)$$

and $G(D_Z\varphi_\epsilon) \to G(D_Z\varphi)$ as $\epsilon \to 0$. We can conclude by saying that φ satisfies Equation (3.6).

Part 2: Assume φ satisfies (1), and define $\widetilde{\varphi} \in \mathcal{C}^{1,2,1,1}(\overline{\mathcal{X}}) \cap \mathcal{C}_1(\overline{\mathcal{X}})$ as

$$\widetilde{\varphi}(\widetilde{t},\widetilde{z}) - a(t,z) + v(t,z) - \varphi(t,z).$$

Since $\widetilde{\varphi}$ differs from φ only for a constant, we note that $\widetilde{\varphi}$ satisfies (1), condition (2) holds as well by insertion. inequality (3.6) holds for $\widetilde{\varphi}$ given the results in Part 1. Then

$$\widetilde{\varphi}_t + F(t,z,D_Z\widetilde{\varphi}, D_Z^2\widetilde{\varphi}, \mathcal{J}^\pi(t,z,\widetilde{\varphi}) = \varphi_t + F(t,z,D_Z\varphi, D_Z^2\varphi, \mathcal{J}^\pi(t,z,\varphi)$$

and $G(D_Z\varphi_\epsilon) \to G(D_Z\varphi)$. Therefore, we conclude that φ satisfies Equation (3.6).

Part 3: Finally, let φ satisfies (0), and let

$$\varphi_\epsilon(\widetilde{t},\widetilde{z}) := \varphi(\widetilde{t},\widetilde{z}) + \epsilon\eta_\epsilon(\widetilde{t}-t, \widetilde{z}-z)$$

for all $\epsilon > 0$, where

$$\eta_\epsilon(\widetilde{t},\widetilde{z}) := \eta\left(\frac{\widetilde{t}}{\epsilon}, \frac{\widetilde{z}}{\epsilon}\right)$$

and

$$\overline{\eta}(\widetilde{t},\widetilde{z}) := \begin{cases} \exp\left(-(\widetilde{x}^2 + \widetilde{y}^2 + \widetilde{\lambda}^2 + \widetilde{t}^2 - 1)\right) & \text{if } |(\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{\lambda})| < 1, \\ 0 & \text{if } |(\widetilde{t},\widetilde{x},\widetilde{y},\widetilde{\lambda})| \geq 1 \end{cases}$$

We see that φ_ϵ satisfies (1), since $\overline{\eta}$ has a strict maximum at (0,0,0,0). Given the result in Part 2, we assert that φ_ϵ satisfies Equation (3.6). Then, by directly computing $\epsilon\eta_\epsilon$, $\epsilon D\eta_\epsilon$ and $\epsilon D^2\eta_\epsilon$ we see they converge to 0, so we assert, by Lebesgue's dominated convergence, that

$$\widetilde{\varphi}_t + F(t,z,D_Z\widetilde{\varphi}, D_Z^2\widetilde{\varphi}, \mathcal{J}^\pi(t,z,\widetilde{\varphi})) \to \varphi_t + F(t,z,D_Z\varphi, D_Z^2\varphi\mathcal{J}^\pi(t,z,\varphi))$$

and $G(D_Z\widetilde{\varphi}) \to G(D_Z\varphi)$. Although it has not been explicitly stated, we briefly mention that the integral part is convergent by the conditions reported in (1.2), for a more detailed proof we refer to [BKR01b]. Therefore, φ must satisfy Equation (3.6) and the proposition is proved.

□

Theorem 3.2.1. *The value function V is a viscosity solution of Equation* (2.17).

Proof. For the sake of readability, we divide the proof into two main steps: in the first part, we show that V is a supersolution over $\mathcal{X}$, while, in the second part, we focus on V as subsolution over $\mathcal{X}$.

Step 1 - supersolution. Define a continuous function $\varphi \in \mathcal{C}^{1,2,1,1}(\mathcal{X}) \cap \mathcal{C}_1(\overline{\mathcal{X}})$, that $(t,z) \in \mathcal{X}$ is a global minimizer of $V - \varphi$ and $V(t,z) = \varphi(t,z)$ and that V satisfies inequality (3.19).

Since V is defined as a solution of Equation (2.19), we have

$$\varphi(t,x,y,\lambda) = V(t,x,y,\lambda) \geq V(t,x-c,y+\beta c,\lambda) \geq \varphi(t,x-c,y+\beta c,\lambda)$$

for all $c \in (0,x]$.

Dividing by c, for $c \to 0$ we obtain the inequality

$$\beta\varphi_y(t,x,y,\lambda) - \varphi_x(t,x,y,\lambda) = G(D_Z\varphi) \leq 0. \tag{3.7}$$

By using the DPP together with the definition of V with $\boldsymbol{C} \equiv 0$ and an admissible $\boldsymbol{\pi} \equiv \widetilde{\boldsymbol{\pi}} \in [0,1]$, we obtain that

$$V(t,x,y,\lambda) \geq \mathbb{E}\left[\int_t^{t+\Delta t} e^{-\delta s} f(Y^{\widetilde{\pi},0})ds + V(t+\Delta t, X^{\widetilde{\pi},0}_{t+\Delta t}, Y^{\widetilde{\pi},0}_{t+\Delta t}, \lambda_{t+\Delta t})\right].$$

for all $\Delta t \in [0, T-t]$.

Since $V - \varphi$ has a global minimum at (t,x,y,λ), we assert that

$$\varphi(t,x,y,\lambda) \geq \mathbb{E}\left[\int_t^{t+\Delta t} e^{-\delta s} f(Y^{\widetilde{\pi},0})ds + \varphi(t+\Delta t, X^{\widetilde{\pi},0}_{t+\Delta t}, Y^{\widetilde{\pi},0}_{t+\Delta t}, \lambda_{t+\Delta t})\right].$$

Finally, we apply Itô-Döeblin's lemma on $\varphi(t+\Delta t, X^{\widetilde{\pi},0}_{t+\Delta t}, Y^{\widetilde{\pi},0}_{t+\Delta t}, \lambda_{t+\Delta t})$, and we get

$$\begin{aligned}
0 \geq &\mathbb{E}\left[\int_t^{t+\Delta t} \varphi_t + e^{-\delta s} f(Y^{\widetilde{\pi},0}_s) + \varphi_x(r(1-\pi_s)+\pi_s\mu)X^{\widetilde{\pi},0}_s - \beta\varphi_y Y^{\widetilde{\pi},0}_s\right.\\
&\left.+\alpha(\xi-\lambda_s)\varphi_\lambda + \frac{1}{2}(\sigma\pi_s X^{\widetilde{\pi},0}_s)^2\varphi_{xx} + \mathcal{J}^{\widetilde{\pi}}(s, X^{\widetilde{\pi},0}_s, Y^{\widetilde{\pi},0}_s, \lambda_s, \varphi)ds\right]\\
\geq &\Delta t \inf_{s\in[t,t+\Delta t]}\left[\varphi_t + e^{-\delta s} f(Y^{\widetilde{\pi},0}_s) + \varphi_x(r(1-\pi_s)+\pi_s\mu)X^{\widetilde{\pi},0}_s - \beta\varphi_y Y^{\widetilde{\pi},0}_s\right.\\
&\left.+\alpha(\xi-\lambda_s)\varphi_\lambda + \frac{1}{2}(\sigma\pi_s X^{\widetilde{\pi},0}_s)^2\varphi_{xx} + \mathcal{J}^{\widetilde{\pi}}(s, X^{\widetilde{\pi},0}_s, Y^{\widetilde{\pi},0}_s, \lambda_s, \varphi)\right].
\end{aligned}$$

Since $X_s^{\pi,C}, Y_s^{\pi,C}$ and λ_s are càdlàg, and $\varphi, \mathcal{J}^\pi$ are smooth by construction, we can divide the last inequality by Δt. By letting $\Delta t \to 0$, we obtain

$$0 \geq \varphi_t + e^{-\delta t} f(y) + \varphi_x (r(1-\widetilde{\pi}) + \widetilde{\pi}\mu)x - \beta\varphi_y y + (\alpha(\xi - \lambda) + \eta\mu_J\lambda)\varphi_\lambda \tag{3.8}$$
$$+ \frac{1}{2}(\sigma\widetilde{\pi}x)^2\varphi_{xx} + \mathcal{J}^{\widetilde{\pi}}(t,x,y,\lambda,\varphi). \tag{3.9}$$

Inequality (3.8) holds for every $\widetilde{\pi} \in [0,1]$, thus

$$\varphi_t + F(t,x,y,\lambda, D_Z\varphi, D_Z^2\varphi, \mathcal{J}^{\widetilde{\pi}}(t,x,y,\lambda,\varphi)) \leq 0.$$

From Equations (3.7) and (3.8), we see that V is a viscosity supersolution.

Step 2 - subsolution We are now to show that V is a viscosity subsolution. Suppose that φ is a smooth function, and that $(t,x,y,\lambda) \in \mathcal{X}$ is a global maximizer of $V - \varphi$. Without loss of generality, we assume that $V(t,x,y,\lambda) = \varphi(t,x,y,\lambda)$, and that there is an $\widetilde{\epsilon} > 0$ such that $(V-\varphi)(\widetilde{t},\widetilde{z}) \leq (V-\varphi)(t,z) - \widetilde{\epsilon}$ for all the other possible maxima $(\tilde{t},\tilde{z}) \in \overline{\mathcal{X}}$. We operate by contradiction, by supposing that inequality (1) does not hold.

Recall that the value function V is continuous by Proposition 1.3.3 and 1.3.4. Since φ is continuous as well, there exists an $\epsilon > 0$ and a non-empty open ball $\mathcal{N}$ centered at (t,x,y,λ) such that

$$\beta\varphi_y - \varphi_x \leq 0 \tag{3.10}$$

then

$$\begin{aligned} -\epsilon > &\varphi_t - \beta\varphi_y\widetilde{y} + e^{-\delta\widetilde{t}} f(y) + (\alpha(\xi - \widetilde{\lambda}) + \widetilde{\lambda}\eta\mu_J)\varphi_\lambda \\ &+ \max_{\pi\in[0,1]} \left[(r(1-\pi) + \pi\mu)\widetilde{x}\varphi_x + \frac{1}{2}(\sigma\pi\widetilde{x})^2\varphi_{xx} + \mathcal{J}^\pi(\widetilde{t},\widetilde{z},\varphi) \right], \end{aligned} \tag{3.11}$$

for all $(\widetilde{t},\widetilde{z}) \in \mathcal{N} \cap \overline{\mathcal{X}}$.
We assume $(T,\widetilde{x},\widetilde{y},\widetilde{\lambda}) \notin \mathcal{N}$ for any $(\widetilde{x},\widetilde{y},\widetilde{\lambda}) \in \mathbb{R}_+^3$ and that $V \leq \varphi - \epsilon$ on $\partial\mathcal{N} \cap \overline{\mathcal{X}}$.

By the DPP, there exists a strategy $(\boldsymbol{\pi}, \boldsymbol{C}) \in \mathcal{A}_{[t,t+\Delta t]}(x,y)$ such that

$$V(t,x,y,\lambda) = \mathbb{E}\left[\int_t^{t+\Delta t} e^{-\delta s} f\left(Y_s^{\pi,C}\right) + \varphi\left(t+\Delta t, X_{t+\Delta t}^{\pi,C}, Y_{t+\Delta t}^{\pi,C}, \lambda_{t+\Delta t}\right)\right],$$

for all $\Delta t \in [0, T-t]$.
We define

$$\Delta t = \begin{cases} \frac{1}{2}\inf & \{s \in (0,T-t] : \varphi(t+s, X_{t+s}^{\pi,C}, Y_{t+s}^{\pi,C}, \lambda_{t+s}) \notin \mathcal{N}\} \\ & \text{if } \varphi(t+\Delta t, X_{t+\Delta t}^{\pi,C}, Y_{t+\Delta t}^{\pi,C}, \lambda_{t+s}) \notin \mathcal{N} \text{ for some } \Delta t \in [0,T-t], \\ \frac{1}{2}T & \text{if } \varphi(t+\Delta t, X_{t+\Delta t}^{\pi,C}, Y_{t+\Delta t}^{\pi,C}, \lambda_{t+s}) \in \mathcal{N} \text{ for all } \Delta t \in [0,T-t]. \end{cases}$$

First of all, we consider the case where $\Delta t = 0$: the self-exciting jump process does not jump at time t a.s., so exiting from the ball $\mathcal{N}$ is due to a consumption increment.

Suppose now that we have a jump of size $\Delta C > 0$. Recalling the derivative of the value function w.r.t. C is decreasing, we can choose arbitrary small V.
Then, we denote by l the segment joining (t, x, y, λ) and $(t, x-\Delta C, y+\beta C, \lambda)$, and let $(t, \widetilde{x}, \widetilde{y}, \widetilde{\lambda})$ be the intersection between the ball $\mathcal{N}$ and the line l. Since φ is decreasing along l in $\mathcal{N} \cap \overline{\mathcal{X}}$ we get

$$V(t, x, y, \lambda) = V(t, \widetilde{x}, \widetilde{y}, \widetilde{\lambda}) \leq \varphi(\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{\lambda}) - \epsilon \leq \varphi(t, x, y, \lambda) - \epsilon = V(t, x, y, \lambda) - \epsilon$$

which is a contradiction.
Now, let us consider $\Delta t > 0$, we see that $(V - \varphi)(t, x, y, \lambda) = 0$ by construction, and that $(V - \varphi) \leq 0$ elsewhere, thus we get

$$\varphi(t, x, y, \lambda) \leq \mathbb{E}\left[\int_t^{t+\Delta t} e^{-\delta s} f(Y_s^{\pi,C}) ds + \varphi(t + \Delta t, X_{t+\Delta t}^{\pi,C}, Y_{t+\Delta t}^{\pi,C}, \lambda_{t+\Delta t})\right].$$

By exploiting Itô-Döeblin's lemma on $\varphi(t + \Delta t, X_{t+\Delta t}^{\pi,C}, Y_{t+\Delta t}^{\pi,C}, \lambda_{t+\Delta t})$ and using inequalities (3.10) and (3.11), we get

$$\begin{aligned}
0 \leq &\mathbb{E}\Bigg[\int_t^{t+\Delta t} \varphi_t + e^{-\delta s} f(Y_s^{\pi,C}) + \varphi_x(r(1-\pi) + \pi\mu) X_s^{\pi,C} \\
&-\beta\varphi_y Y_s^{\pi,C} + (\alpha(\xi - \lambda) + \lambda\mu_J\eta)\varphi_\lambda \\
&+\frac{1}{2}\varphi_{xx}(\sigma\pi_s X_s^{\pi,C})^2 + \mathcal{J}^\pi(s, X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s, \varphi) ds + \int_t^{t+\Delta t} -\varphi_x + \beta\varphi_y d\widetilde{C}_s \\
&+ \sum_{\Delta C_s \neq 0} (\varphi(s, X_s^{\pi,C}, Y_s^{\pi,C}, \lambda_s) - \varphi(s, X_s^{\pi,C} - \Delta C, Y_s^{\pi,C} + \beta\Delta C, \lambda_s)\Bigg] \\
&\leq -\epsilon\Delta t,
\end{aligned}$$

with $\widetilde{\boldsymbol{C}}$ denoting the continuous part of the $\boldsymbol{C}$ process. This inequality is a contradiction as well, and V is also a subsolution for the Equation (2.17) on $\overline{\mathcal{X}}$. Therefore, we see that the continuous function V satisfies the viscosity solution definition over the domain $\mathcal{X}$.

□

3.3 The comparison principle

To prove uniqueness of the viscosity solution, we rely on the comparison principle. To this end, let us introduce the following (alternative) viscosity solution definitions, extending Definition 3.2.1 as follows.

Definition 3.3.1 (Strict viscosity solutions)**.** *Let $\mathcal{D} \in \overline{\mathcal{X}}$. Any function $v \in \mathcal{C}(\mathcal{X})$ is a strict supersolution (resp. subsolution) of Equation* (2.17) *in $\mathcal{D}$ if and only if, for any $\varphi \in \mathcal{C}^{1,2,1,1}(\overline{\mathcal{X}}) \cap \mathcal{C}_1(\overline{\mathcal{X}})$ and $(t, z) \in \mathcal{D}$ global minimum of $v - \varphi$ on $\mathcal{D}$, the following inequality holds:*

$$\max\{G(D_Z\varphi); \varphi_t + F(t, z, D_Z\varphi, D_Z^2\varphi, J^\pi(t, z, \varphi)\} \leq -\gamma \, (\geq \gamma),$$

for some γ real positive constant.

We are now in position to show the existence of a strict supersolutions arbitrary close to subsolutions in the classical sense.

Proposition 3.3.1. *Assume* $\widetilde{\gamma} > 0$ *such that* $\delta > k(\widetilde{\gamma})$ *(see A2) and let* $\overline{v} \in C_{\widetilde{\gamma}}(\overline{\mathcal{X}})$ *be a strict supersolution of Equation* (2.17) *in* $\mathcal{X}$*. Then, for* $\overline{\gamma} > max\{\gamma, \widetilde{\gamma}\}$ *such that* $\delta > \overline{\gamma}$*, we have*

$$w = (K + \chi^{\overline{\gamma}}(z))e^{-\delta t}, \quad \chi(z) = \left(1 + x + \frac{y}{2\beta} + \lambda\right).$$

Then, for a large enough K*,* $w \in \mathcal{C}^{\infty}(\overline{\mathcal{X}}) \cap \mathcal{C}_{\overline{\gamma}}(\mathcal{X})$ *is a strict subsolution of Equation* (2.17) *in any* $\mathcal{D} \subseteq \mathcal{X}$*. Furthermore, for* $\theta \in (0, 1]$*, the function*

$$v^{\theta} = (1 - \theta)\overline{v} + \theta w \in \mathcal{C}_{\overline{\gamma}}(\overline{\mathcal{X}}),$$

is a strict supersolution of Equation (2.17) *in any bounded set* $\mathcal{D} \subseteq \mathcal{X}$*.*

Proof. Firstly, we are intended to prove that

$$\max\{G(D_Z w); w_t + F(t, z, D_Z w, D_Z^2 w, \mathcal{J}^{\pi}(t, z, w))\} \leq -g, \tag{3.12}$$

for some $g \in \mathcal{C}(\overline{\mathcal{X}})$ strictly positive.
To prove it, let us observe that:

$$G(D_Z w) = -e^{-\delta t}\frac{\overline{\gamma}}{2}\chi^{\overline{\gamma}-1}.$$

Since $\pi\frac{x}{\chi} \in [0,1]$, we can rewrite the second term of (3.12) as follows:

$$
\begin{aligned}
&w_t + F(t,z,D_Z w, D_Z^2 w, \mathcal{J}^\pi(t,z,w)) \\
=&e^{-\delta t}\Bigg[f(y) - \delta(K+\chi^{\overline{\gamma}}) - \frac{1}{2}y\overline{\gamma}\chi^{\overline{\gamma}-1} + \overline{\gamma}(\alpha(\xi-\lambda)+\lambda\eta\mu_J)\chi^{\overline{\gamma}-1} \\
&+ \max_{\pi\in[0,1]}\Bigg[\overline{\gamma}(r+(\mu-r)\pi)x\chi^{\overline{\gamma}-1} + \frac{1}{2}\overline{\gamma}(\overline{\gamma}-1)(\sigma\pi x)^2\chi^{\overline{\gamma}-2} \\
&+\lambda\int_{\mathbb{R}\setminus\{0\}}(\chi+\pi x(e^q-1))^{\overline{\gamma}} - \chi^{\overline{\gamma}} - \overline{\gamma}\pi x\chi^{\overline{\gamma}-1}(e^q-1) - \overline{\gamma}\eta|q|\chi^{\overline{\gamma}-1}\zeta(q)dq\Bigg]\Bigg] \\
=&e^{-\delta t}\Bigg[f(y) - \delta K - \frac{1}{2}y\overline{\gamma}\chi^{\overline{\gamma}-1} + \overline{\gamma}(\alpha(\xi-\lambda)+\lambda\eta\mu_J)\chi^{\overline{\gamma}-1} \\
&+\Bigg(-\delta + \max_{\pi\in[0,1]}\Bigg[\overline{\gamma}(r+(\mu-r)\pi)\frac{x}{\chi} + \frac{1}{2}\overline{\gamma}(\overline{\gamma}-1)\left(\sigma\pi\frac{x}{\chi}\right)^2 \\
&+\ \lambda\int_{\mathbb{R}\setminus\{0\}}(1+\pi\frac{x}{\chi}(e^q-1))^{\overline{\gamma}} - 1 - \overline{\gamma}\pi\frac{x}{\chi}(e^q-1)\zeta(q)dq\Bigg]\Bigg)\chi^{\overline{\gamma}}\Bigg] \\
\leq&e^{-\delta t}\Bigg[f(y) - \delta K + \Bigg(-\delta + \max_{\pi\in[0,1]}\Bigg[\overline{\gamma}(r+(\mu-r)\pi) + \frac{1}{2}\overline{\gamma}(\overline{\gamma}-1)(\sigma\pi)^2 \\
&+\lambda\int_{\mathbb{R}\setminus\{0\}}(1+\pi x(e^q-1))^{\overline{\gamma}} - 1 - \overline{\gamma}\pi(e^q-1) - \frac{\overline{\gamma}}{\chi}\eta|q|\zeta(q)dq\Bigg]\Bigg)\chi^{\overline{\gamma}}\Bigg] \\
\leq&e^{-\delta t}\left(f(y) - \delta K - (\delta - k(\overline{\gamma}))\chi^{\overline{\gamma}}\right) \\
\leq&-1.
\end{aligned}
$$

where the last inequality comes naturally, choosing for example

$$K = \frac{1}{\delta}\left(e^{\delta T} + \sup_{(t,z)\in\overline{\mathcal{X}}}\left[f(y) - (\delta - k(\overline{\gamma}))\chi^{\overline{\gamma}}\right]\right).$$

Since we choose $\delta \geq k(\overline{\gamma})$ and $\overline{\gamma}$, we have a finite K. We note that Equation (3.12) holds for a proper function $g(\cdot,\cdot)$:

$$g(t,z) = \min\{1; \frac{\overline{\gamma}}{2}\chi^{\overline{\gamma}-1}(z)\}$$

. In the second part, we prove that v^θ is a strict supersolution of Equation (2.17) for all $\theta \in (0,1]$. Note that, for any $\varphi \in \mathcal{C}^{1,2,1,1}(\mathcal{X})$, (t^*, z^*) is a global minimum of $v - \varphi$ iff (t^*, z^*) is global minimum of $v^\theta - \varphi^\theta$, with $\varphi^\theta = (1-\theta)\varphi + w$. Since $\overline{v}$ is a supersolution of Equation (2.17) in $\mathcal{X}$ and H is linear, we have

$$G(D_Z\varphi^\theta) \leq -\theta\frac{\overline{\gamma}}{2}\chi^{\overline{\gamma}-1}.$$

Let π^* denote the maximizing value of π when evaluating F with φ^θ. Then

$$\begin{aligned}
&\varphi_t^\theta + F(t,z,D_Z\varphi^\theta, D_Z^2\varphi^\theta, \mathcal{J}^\pi(t,z,\varphi^\theta))\\
&=(1-\theta)f(y)e^{-\delta t} + (1-\theta)\varphi_t + (1-\varphi)\beta y\varphi_y + (1-\theta)(\alpha(\xi-\lambda)+\eta\lambda\mu_J)\varphi_\lambda\\
&\quad + (1-\theta)(r+(r+\mu)\pi^*)x\varphi_x + (1-\varphi)\frac{1}{2}(\sigma\pi^* x)^2\varphi_{xx}\\
&\quad + (1-\theta)\mathcal{J}^{\pi^*}(t,x,y,\lambda,\varphi) + \theta f(y)e^{-\delta t} + w_t + \theta\beta y w_y + \theta(\alpha(\xi-\lambda)+\eta\lambda\mu_J)w_\lambda\\
&\quad + \theta(r+(\mu+r)\pi^*)xw_x + \theta\frac{1}{2}(\sigma\pi^* x)^2 w_{xx} + \theta\mathcal{J}^{\pi^*}(t,x,y,\lambda,w)\\
&\le(1-\theta)F(t,z,\varphi_x,\varphi_y,\varphi_{xx},\mathcal{J}^\pi(t,z,\varphi)) + \theta F(t,z,D_Zw,D_Z^2w,\mathcal{J}^\pi(t,z,w))\\
&\le-\theta g
\end{aligned}$$

By combining the previous results we found:

$$\max\{G(D_Z\varphi); \varphi_t^\theta + F(t,z,D_Z\varphi^\theta, D_Z^2\varphi^\theta, \mathcal{J}^\pi(t,z,\varphi^\theta))\} \le -\theta g.$$

□

As typically done in order to prove the comparison principle, a characterization of viscosity solution in terms of subjets and superjets is employed. We thus provide next definition, as done in [CIL92].

Definition 3.3.2. *Denote by $\mathcal{S}^N$ the set of $N\times N$ symmetric matrices, $\mathcal{D}\subseteq\overline{\mathcal{X}}, v\in\mathcal{C}(\mathcal{D})$ and $(t,z)\in\mathcal{D}$. The second order superjet (resp. subjet), $J_{\mathcal{D}}^{2,+(-)}v(t,z)$ is the set of $(P,A)\in\mathbb{R}^3\times\mathcal{S}^3$ such that:*

$$v(t,z)\le(\ge 0)\, v(t,z)+\langle P,(s,u)-(t,z)\rangle+\frac{1}{2}\langle A((s,u)-(t,z)),(s,u)-(t,z)\rangle+o(|(s,z)-(t,u)|^2),$$

as $\mathcal{D}\ni(s,u)\to(t,z)$.

The closure $\overline{J}_{\mathcal{D}}^{2,+(-)}v(t,z)$ is the set of (P,A) for which there exists a sequence $(P_n,A_n)\in\overline{J}_{\mathcal{D}}^{2,+(-)}v(t_n,z_n)$ such that $(t_n,z_n,v(t_n,z_n),P_n,A_n)\to(t,z,v(t,z),P,A)$ as $n\to\infty$.

Now, we are about to prove the comparison principle for Equation (2.17), where we will use the maximum principle for semicontinuous functions inspired by [CIL92].

Proposition 3.3.2 (Comparison principle)**.** *Assume $\underline{v}\in\mathcal{C}_{\gamma^*}(\mathcal{X})$ is a subsolution of Equation* (2.17) *in $[0,T)\times\mathbb{R}_+^3$, that $\overline{v}\in\mathcal{C}_{\gamma^*}(\mathcal{X})$ is a supersolution of the Equation* (2.17) *in $\mathcal{X}$, and that $\underline{v}\le\overline{v}$ for $t=T$. Then, $\underline{v}\le\overline{v}$ everywhere in $\mathcal{X}$.*

Proof. Since $\underline{v},\overline{v}\in\mathcal{C}_{\gamma^*}(\mathcal{X})$, there exists a $\gamma'>0$ such that $\delta>k(\gamma')$ and $\underline{v},\overline{v}\in\mathcal{C}_{\gamma'}(\overline{\mathcal{X}})$. Let now w be defined as in 3.3.1, and choose $\widetilde{K}$ large enough such that

$$\overline{v}^\theta=(1-\theta)\overline{v}+\theta w$$

is a strict supersolution of Equation (2.17) on any bounded subset of $\mathcal{X}$, with $w>\underline{v}$ for $t=T$. By exploiting Proposition 3.3.1 and the fact that $\overline{\gamma}>\overline{\gamma}$, such a value for $\widetilde{K}$ exists. The proof is based on the comparison between $\overline{v}^\theta$ and $\underline{v}$, instead of the classical $\overline{v}$ and $\underline{v}$. Then, at last we let $\theta \to 0$, obtaining the classical comparison principle for $\overline{v}$ and $\underline{v}$.

We thus have that, as $x, y, \lambda \to \infty$,

$$\underline{v}(t,z) - \overline{v}^\theta(t,z) \to -\infty,$$

as $z \to -\infty$, and we have that $\underline{v}(T,z) - \overline{v}^\theta(T,z) < 0$. We will look for the maximum of $\underline{v}(t,z) - \overline{v}^\theta(t,z)$ on the domain $\mathcal{D}_T$, where we denote $\mathcal{D}_T := [0,T) \times \mathcal{D}$ and $\mathcal{D} := \{(x,y,\lambda) \in \mathbb{R}^3_+ : 0 < (x,y,\lambda) < R^3\}$ for some $R > 0$. Assume that $\underline{v} \leq \overline{v}^\theta$ is violated somewhere in $\mathcal{D}_T$, therefore there exists a M defined as

$$M := \max_{\overline{\mathcal{D}}}\{(\underline{v} - \overline{v}^\theta)(t,z) + \vartheta(t-T)\},$$

where $\vartheta > 0$ is chosen in such a way that $M > 0$. Assume $z^* := (x^*, y^*, \lambda^*)$, therefore $(t^*, z^*) \in \mathcal{D}_T$ satisfies

$$M = (\underline{v} - \overline{v}^\theta)(t^*, z^*) + \vartheta(t^* - T).$$

By choosing R large enough, we may assume that we are in presence of one of the following conditions:

1. $(t^*, z^*) \in \Gamma$, or
2. $(t^*, z^*) \in \mathcal{D}_T$,

where

$$\Gamma := \Big\{(t,z) \in \overline{\mathcal{D}}_T : \{x, \lambda = 0, y \in [0,R)\} \cup \{x \in [0,R), \lambda, y = 0\} \\ \cup \{x, y = 0, \lambda \in [0,R)\}\Big\}$$

defines the lower boundaries of $\mathcal{D}_T$. Notice that $t^* \neq T$, since we initially assume $\underline{v} \leq \overline{v}^\theta$ for $t = T$.

We can now have two different situations.

Situation 1. In the first case, $(t^*, z^*) \in \Gamma$. Since the boundaries are piecewise linear, there exist $h_0, k_0 \geq 0$ and a uniformly continuous function $\eta : \overline{\mathcal{D}} \to \mathbb{R}^4$ which satisfies

$$\mathcal{N}((t,z) + h\eta(t,z), hk_0) \subset \mathcal{D}_T, \tag{3.13}$$

for all $(t,z) \in \overline{\mathcal{D}}_T$ and $h \in (0, h_0]$, where $\mathcal{N}(c, \rho)$ denotes the ball in $\mathbb{R}^4$ with center $c \in \mathbb{R}^4$ and radius ρ. By using an approach similar to [Son86], let us define the auxiliary functions φ and Φ on $\overline{\mathcal{D}}_T \times \overline{\mathcal{D}}_T$. For any $\gamma > 1$ and $\varepsilon \in (0,1)$, let:

$$\begin{aligned}
\varphi((t_1,z_1),(t_2,z_2)) &:= |\gamma((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})) + \epsilon\eta(t^*, z^*)|^2 \\
&\quad + \epsilon|(t_1,z_1) - (t^*,z^*)|^2 - \vartheta(t_2 - T), \\
\Phi((t_1,z_1),(t_2,z_2)) &:= \underline{v}(t_1,z_1) - \overline{v}^\theta(t_2,z_2) - \varphi((t_1,z_1),(t_2,z_2)).
\end{aligned}$$

Then,

$$M_\gamma := \max_{\overline{\mathcal{D}}_T \times \overline{\mathcal{D}}_T} \Phi\left((t_1, z_1), (t_2, z_2)\right).$$

$M_\gamma > 0$ for any $\gamma > 1$ and $\epsilon \le \epsilon_0$, with $\epsilon_0 > 0$ is some fixed small number.
Let now $((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma})) \in \overline{\mathcal{D}}_T \times \overline{\mathcal{D}}_T$ be a maximizer of Φ, that is to say that $M_\gamma = \Phi((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma}))$. By using the Equation (3.13), we may assume that γ is large enough such that $(t^*, z^*) + \frac{\varepsilon}{\gamma}\eta(t^*, z^*) \in \overline{\mathcal{D}}_T$. By exploiting

$$\Phi((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma})) \ge \varphi((t^*, z^*), (t^*, z^*) + \frac{\epsilon}{\gamma}\eta(t^*, z^*)),$$

we obtain

$$\begin{aligned} &|\gamma((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma}) + \epsilon\eta(t^*, z^*)|^2 + \epsilon|(t_{1,\gamma}, z_{1,\gamma}) - (t^*, z^*)|^2 \\ &\le \underline{v}(t_{1,\gamma}, z_{1,\gamma}) - \overline{v}^\theta(t_{2,\gamma}, z_{2,\gamma}) - \underline{v}(t^*, z^*) \\ &+ \overline{v}^\theta((t^*, z^*) + \frac{\epsilon}{\gamma}\eta(t^*, z^*)) + \vartheta((t_{2,\gamma} - t^* - \frac{\epsilon}{\gamma}\eta(t^*, z^*)). \end{aligned} \tag{3.14}$$

The right-hand side of this inequality is bounded as $\gamma \to \infty$, thus $\gamma|(t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})|$ is bounded uniformly in γ. Therefore

$$\lim_{\gamma\to\infty} ((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})) = 0$$

and

$$\lim_{\gamma\to\infty} \left(\underline{v}(t_{1,\gamma}, z_{1,\gamma}) - \overline{v}^\theta(t_{2,\gamma}, z_{2,\gamma})\right) \le M.$$

Letting $\gamma \to \infty$ in inequality (3.14) and using the uniform continuity for $\underline{v}$ and $\overline{v}^\theta$ in $\overline{\mathcal{D}}_T$, we see that $\gamma\left((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})\right) + \epsilon\eta(t^*, z^*) \to 0$, $(t_{1,\gamma}, z_{1,\gamma})$, $(t_{2,\gamma}, z_{2,\gamma})) \to (t^*, z^*)$ and $M_\gamma \to M$, for $\gamma \to \infty$.
Then, by using the uniform continuity of η, we have

$$\begin{aligned} (t_{2,\gamma}, z_{2,\gamma}) &= (t_{1,\gamma}, z_{1,\gamma}) + \frac{\epsilon}{\gamma}\eta(t^*, z^*) + O\left(\frac{1}{\gamma}\right) \\ &= (t_{1,\gamma}, z_{1,\gamma}) + \frac{\epsilon}{\gamma}\eta(t_{1,\gamma}, z_{1,\gamma}) + O\left(\frac{1}{\gamma}\right), \end{aligned}$$

where we use (3.13) to get $(t_{2,\gamma}, z_{2,\gamma}) \in \overline{\mathcal{D}}$ for γ large enough. Furthermore,

$$D^2\varphi = 2\gamma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\epsilon \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

Moreover, we note that the conditions exposed in [CL83] hold, so for any $\varsigma \in (0, 1)$, there exist $A_1, A_2 \in \mathcal{S}^4$ such that

$$\begin{pmatrix} A_1 & 0 \\ 0 & -A_2 \end{pmatrix} \le 2\frac{\gamma}{\varsigma} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 2\epsilon \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.15}$$

$$0 \leq \max\{G(D_{Z_1}\varphi); +\varphi_{t_1} \\ + F(t_{1,\gamma}, z_{1,\gamma}, D_{Z_1}\varphi, \hat{A}_1, \mathfrak{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, D_{Z_1}\varphi), \mathfrak{J}^{\pi}_{\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi))\}, \tag{3.16}$$

and

$$-\theta g \geq \max\{G(-D_{Z_2}\varphi); -\varphi_{t_2} \\ + F(t_{2,\gamma}, z_{2,\gamma}, -D_{Z_2}\varphi, \hat{A}_2, \mathfrak{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, \overline{v}^{\theta}, -D_{Z_2}\varphi), \mathfrak{J}^{\pi}_{\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\varphi))\}, \tag{3.17}$$

where φ_t and $D_Z\varphi$ are computed at $((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma}))$, $\hat{A}_1$ and $\hat{A}_2$ are the parts of A_1 and A_2 corresponding to z_1 and z_2, respectively. Since $\hat{A}_i$ is a 4-dimensional matrix, we denote the elements by $a_{i,xx}$, $a_{i,xy}$ and $a_{i,x\lambda}$ and so on.
By multiplying the left-hand side by $(z_{1,\gamma}e_2', z_{2,\gamma}e_2')$ and the right-hand side by $(x_{1,\gamma}e_2', x_{2,\gamma}e_2')'$ (with $e_2 = (0,1,0,0)' \in \mathbb{R}^4$ and e' denotes the transpose of e), we get

$$x_{1,\gamma}^2 a_{1,xx} - x_{2,\gamma}^2 a_{2,xx} \leq \frac{\gamma}{\varsigma}(x_{1,\gamma} - x_{2,\gamma})^2 + \epsilon x_{1,\gamma}^2,$$

and therefore

$$\lim_{\epsilon\to 0}\lim_{\gamma\to\infty}(x_{1,\gamma}^2 a_{1,xx} - x_{2,\gamma}^2 a_{2,xx}) \leq 0.$$

Moreover Equation (3.17) implies that $G(-D_{Z_2}\varphi) \leq -\theta g$ and

$$-\theta g \geq -\varphi_{t_2} + F(t_{2,\gamma}, z_{2,\gamma}, -D_{Z_2}\varphi, \hat{A}_2, J^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, \overline{v}^{\theta}, -D_{Z_2}\varphi), \mathfrak{J}^{\pi}_{\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\varphi)) \tag{3.18}$$

In the following we show that $G(D_{Z_1}\varphi) < 0$ holds for a sufficiently large γ. We operate by contradiction: by saying that $G(D_{Z_1}\varphi) \geq 0$. Then it follows that

$$\begin{aligned} -\theta g &\geq G(-D_{Z_2}\varphi) - G(D_{Z_1}\varphi)) \\ &= -\beta(\varphi_{y_2} + \varphi_{y_1}) + (\varphi_{x_2} + \varphi_{x_1}) \\ &= -2\beta\epsilon(y_{1,\gamma} - y^*) + 2\epsilon(x_{1,\gamma} - x^*), \end{aligned}$$

which is converging to 0 as $\gamma \to \infty$.
This is a contradiction since g is strictly positive on $\mathcal{D}_T$, so the assertion holds. Equation (3.16) and $G(D_{Z_1}\varphi) < 0$ have as direct consequence that

$$\varphi_{t_1} + F(t_{1,\gamma}, z_{1,\gamma}, D_{Z_1}\varphi, \hat{A}_1, \mathfrak{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, D_{Z_1}\varphi), \mathfrak{J}^{\pi}_{\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi)) \geq 0. \tag{3.19}$$

By using inequalities (3.17) and (3.19), we have

$$
\begin{aligned}
\theta g \leq & \varphi_{t_1} + F(t_{1,\gamma}, z_{1,\gamma}, D_{Z_1}\varphi, \hat{A}_1, \mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, D_{Z_1}\varphi), \underline{v}, \mathcal{J}^{\pi}_{\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi)) \\
& + \varphi_{t_2} - F(t_{2,\gamma}, z_{2,\gamma}, -D_{Z_2}\varphi, \hat{A}_2, \mathcal{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, \overline{v}^{\theta}, -D_{Z_2}\varphi), \mathcal{J}^{\pi}_{\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\varphi)) \\
\leq & \left(e^{-\delta t_{1,\gamma}} f(y_{1,\gamma}) - e^{-\delta t_{2,\gamma}} f(y_{2,\gamma})\right) + (\varphi_{t_1} + \varphi_{t_2}) - \beta(y_{1,\gamma}\varphi_{y_1} + y_{2,\gamma}\varphi_{y_2}) \\
& + \alpha(\xi - (\lambda_{1,\gamma}\varphi_{\lambda_1} + \lambda_{2,\gamma}\varphi_{\lambda_2}) + \eta\mu_J(\lambda_{1,\gamma}\varphi_{\lambda_1} + \lambda_{2,\gamma}\varphi_{\lambda_2})) \qquad (3.20) \\
& + \max_{\pi\in[0,1]} [(r + (\mu - r)\pi)(x_{1,\gamma}\varphi_{x_1} + x_{2,\gamma}\varphi_{x_2}) \\
& + \frac{1}{2}(\sigma\pi)^2(x^2_{1,\gamma}a_{1,xx} - x^2_{2,\gamma}a_{2,xx}) \\
& + \mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, \overline{v}^{\theta}, D_{Z_1}\varphi) - \mathcal{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, \underline{v}, -D_{Z_2}\varphi)) \\
& + (\mathcal{J}^{\pi}_{\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi) - \mathcal{J}^{\pi}_{\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\varphi))] \\
\leq & \left(e^{-\delta t_{1,\gamma}} f(y_{1,\gamma}) - e^{-\delta t_{2,\gamma}} f(y_{2,\gamma})\right) + 2\epsilon t_{1,\gamma} - \vartheta - \beta\epsilon y_{1,\gamma}(y_{1,\gamma} - y^*) \\
& + \max_{\pi\in[0,1]} [2(r + (\mu - r)\pi)\epsilon x_{1,\gamma}(x_{1,\gamma} - x^*) \\
& + 2\alpha\epsilon(\xi - \lambda_{1,\gamma}(\lambda_{1,\gamma} - \lambda^*) + \eta\mu_J\lambda_{1,\gamma}(\lambda_{1,\gamma} - \lambda^*)) + \frac{1}{2}(\sigma\pi)^2(x^2_{1,\gamma}a_{1,xx} - x^2_{2,\gamma}a_{2,xx}) \\
& + \mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, \overline{v}^{\theta}, D_{Z_1}\varphi) - J^{\pi,\kappa}(t_{2,\gamma}, Z_{2,\gamma}, \underline{v}, -D_{Z_2}\varphi)) \\
& + (\mathcal{J}^{\pi}_{\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi) - \mathcal{J}^{\pi}_{\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\varphi))] \qquad (3.21)
\end{aligned}
$$

We conclude by remarking that [BKR01b] demonstrated that

$$\lim_{\epsilon\to 0}\lim_{\gamma\to\infty} \mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, \overline{v}^{\theta}, D_{Z_1}\varphi) - \mathcal{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, \underline{v}, -D_{Z_2}\varphi) \leq 0,$$

the convergence is uniform for positive κ. We know that

$$\mathcal{J}^{\pi}_{\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi) - \mathcal{J}^{\pi}_{\kappa}(t_{2,\gamma}, z_{2,\gamma}, \varphi) \to 0,$$

as $\kappa \to 0$. We conclude the first part of the proof by saying that, since $\gamma \to \infty, \epsilon \to 0, \kappa \to 0$, and we see that the right-hand side of inequality (3.20) convergences to something negative which is a contradiction to (3.20) itself.

Situation 2. The second case regards the internal part of $\mathcal{D}_T$. For any $\gamma > 1$ and $\epsilon \in (0, 1)$, we define, similarly to what we have done in the previous part, the functions $\varphi, \Phi : [0, T] \times \overline{\mathcal{D}}_T \times \overline{\mathcal{D}}_T \to \mathbb{R}$ as

$$\varphi((t_1, z_1), (t_2, z_2)) := \frac{\gamma}{2}|(t_1, z_1) - (t_2, z_2)|^2 - \vartheta(t_2 - T),$$

and

$$\Phi((t_1, z_1), (t_2, z_2) := \underline{v}(t_1, z_1) - \overline{v}^{\theta}(t_2, z_2) - \varphi((t_1, z_1), (t_2, z_2)).$$

Let

$$M_\gamma := \max_{\overline{\mathcal{D}}_T\times\overline{\mathcal{D}}_T} \Phi\left((t_1, z_1), (t_2, z_2)\right).$$

For all $\gamma > 1$ we have that $M_\gamma \geq M > 0$. Let us recall that $((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma}))$ maximizes φ over $\overline{\mathcal{D}}_T \times \overline{\mathcal{D}}_T$.
From

$$\varphi((t_{1,\gamma}, z_{1,\gamma}), (t_{1,\gamma}, z_{1,\gamma}) + \Phi((t_{2,\gamma}, z_{2,\gamma}), (t_{2,\gamma}, z_{2,\gamma}) \leq 2\Phi((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma}),$$

we get

$$\begin{aligned} &\gamma|((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})|^2 \\ &\leq \underline{v}(t_{1,\gamma}, Z_{1,\gamma}) - \overline{v}^\theta(t_{2,\gamma}, z_{2,\gamma}) - \underline{v}(t_{2,\gamma}, z_{2,\gamma}) + \overline{v}^\theta((t_{2,\gamma}, z_{2,\gamma})) + \vartheta((t_{2,\gamma} - t_{1,\gamma}). \end{aligned} \tag{3.22}$$

The right-hand side of this equation is bounded as $\gamma \to \infty$, since all the functions reported are uniformly continuous in $\overline{\mathcal{D}}_T \times \overline{\mathcal{D}}_T$.
Therefore, for $\gamma \to \infty$, we assert that $|((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})| \to 0$ as $\gamma \to \infty$. Finally, using $\underline{v}, \overline{v}^\theta$ in $\mathcal{D}_T$ again, we see that $\gamma|((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})| \to 0$ as $\gamma \to \infty$.

Now, by making use of $M \leq M_\gamma$ and the definition of M we get

$$\begin{aligned} 0 &= \lim_{\gamma\to\infty} \gamma|((t_{1,\gamma}, z_{1,\gamma}) - (t_{2,\gamma}, z_{2,\gamma})|^2 \\ &\leq \lim_{\gamma\to\infty} \underline{v}(t_{1,\gamma}, z_{1,\gamma}) - \overline{v}^\theta(t_{2,\gamma}, z_{2,\gamma}) + \vartheta(t_{2,\gamma} - T) - M \\ &\leq 0. \end{aligned}$$

Then, we conclude that $M_\gamma \to M$ as $\gamma \to \infty$. Since $M > 0$ and $\underline{v} \leq \overline{v}^\theta$ on Γ, we see that any limit point of $((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma}))$ belongs to $\mathcal{D}_T \times \mathcal{D}_T$. For large enough γ, we have that $((t_{1,\gamma}, z_{1,\gamma}), (t_{2,\gamma}, z_{2,\gamma})) \in \mathcal{D}_T$. Given

$$D^2\varphi = \gamma \begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

define $g_0 \equiv \gamma, g_1 \equiv 0$ and $g_2 \equiv 0$, and note that the conditions for Theorem 3.3.2 are satisfied. Thus, it follows that, for any $\zeta \in (0, 1)$, there exist matrices $A_1, A_2 \in \mathbb{S}^4$ such that

$$\begin{pmatrix} A_1 & 0 \\ 0 & -A_2 \end{pmatrix} \leq \frac{\gamma}{\varsigma} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

So

$$0 \leq \max\{G(D_{Z_1}\varphi); \varphi_{t_1} + F(t_1^*, z_1^*, D_{Z_1}\varphi, \hat{A}_1, \mathcal{J}^{\pi,\kappa}(t_1^*, z_1^*, \underline{v}, D_{Z_1}\varphi), \mathcal{J}^\pi_\kappa(t_1^*, z_1^*, \varphi))\}, \tag{3.23}$$

and

$$\begin{aligned} -\theta g \geq \max\{&G(-D_{Z_2}\varphi); \\ &-\varphi_{t_2} + F(t_2^*, z_2^*, -D_{Z_2}\varphi, \hat{A}_2, \mathcal{J}^{\pi,\kappa}(t_2^*, z_2^*, \overline{v}^\theta, -D_{Z_2}\varphi), \mathcal{J}^\pi_\kappa(t_2^*, z_2^*, -\varphi))\}, \end{aligned} \tag{3.24}$$

As in the first case, we obtain that

$$\lim_{\gamma\to\infty} (x_{1,\gamma}^2 a_{1,xx} - x_{2,\gamma}^2 a_{2,xx}) \leq 0,$$

$$G(-D_{Z_2}\varphi) \leq -\theta g$$

, and

$$-\varphi_{t_2} + F(t_2^*, z_2^*, -D_{Z_2}\varphi, \hat{A}_2, \mathcal{J}^{\pi,\kappa}(t_2^*, z_2^*, -D_{Z_2}\varphi), \mathcal{J}_\kappa^\pi(t_2^*, z_2^*, \varphi)) \leq -\theta g \tag{3.25}$$

We have $G(D_{Z_1}\varphi) < 0$, because $G(D_{Z_1}\varphi) \equiv G(-D_{Z_2}\varphi) < 0$. By exploiting (3.23), we get

$$0 \leq \varphi_{t_1} + F(t_1^*, z_1^*, D_{Z_1}\varphi, \hat{A}_1, \mathcal{J}^{\pi,\kappa}(t_1^*, z_1^*, \underline{v}, D_{Z_1}\varphi), \underline{v}, \mathcal{J}_\kappa^\pi(t_1^*, z_1^*, \varphi)) \geq 0 \tag{3.26}$$

$$\begin{aligned}
\theta g \leq & \varphi_{t_1} + F(t_1^*, z_1^*, D_{Z_1}\varphi, \hat{A}_1, \mathcal{J}^{\pi,\kappa}(t_1^*, z_1^*, \underline{v}, D_{Z_1}\varphi), \mathcal{J}_\kappa^\pi(t_1^*, z_1^*, \varphi)) \\
& + \varphi_{t_2} - F(t_2^*, z_2^*, -D_{Z_2}\varphi, \hat{A}_2, \mathcal{J}^{\pi,\kappa}(t_2^*, z_2^*, \overline{v}^\theta, -D_{Z_2}\varphi), \mathcal{J}_\kappa^\pi(t_2^*, z_2^*, -\varphi)) \\
\leq & (e^{-\delta t_{1,\gamma}} f(y_{1,\gamma}) - e^{-\delta t_{2,\gamma}} f(y_{2,\gamma})) - \vartheta + (\varphi_{t_1} + \varphi_{t_2}) && (3.27) \\
& - \beta(y_{1,\gamma}\varphi_{y_1} + y_{2,\gamma}\varphi_{y_2}) && (3.28) \\
& + \alpha(\xi - (\lambda_{1,\gamma}\varphi_{\lambda_1} + \lambda_{2,\gamma}\varphi_{\lambda_2}) + \eta\mu_J(\lambda_{1,\gamma}\varphi_{\lambda_1} + \lambda_{2,\gamma}\varphi_{\lambda_2})) \\
& + \max_{\pi\in[0,1]} [(r + (\mu - r)\pi)(x_{1,\gamma}\varphi_{x_1} + x_{2,\gamma}\varphi_{x_2}) \\
& + \frac{1}{2}(\sigma\pi)^2(x_{1,\gamma}^2 a_{1,xx} - x_{2,\gamma}^2 a_{2,xx}) \\
& + (\mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, \overline{v}^\theta, D_{Z_1}\varphi) - \mathcal{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, \underline{v}, -D_{Z_2}\varphi)) \\
& + (\mathcal{J}_\kappa^\pi(t_{1,\gamma}, z_{1,\gamma}, \lambda_{1,\gamma}, \varphi) - \mathcal{J}_\kappa^\pi(t_{2,\gamma}, z_{2,\gamma}, \lambda_{2,\gamma}, -\varphi))]
\end{aligned} \tag{3.29}$$

As in the first case, we have that

$$\lim_{\epsilon\to 0}\lim_{\gamma\to\infty} \mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, \overline{v}^\theta, D_{Z_1}\varphi) - \mathcal{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\underline{v}, D_{Z_2}\varphi) \leq 0,$$

and

$$\lim_{\kappa\to 0} \mathcal{J}^{\pi,\kappa}(t_{1,\gamma}, z_{1,\gamma}, \varphi) - \mathcal{J}^{\pi,\kappa}(t_{2,\gamma}, z_{2,\gamma}, -\varphi) = 0,$$

so letting $\gamma \to \infty, \kappa \to \infty$ in (3.27) we obtain the contradiction, which concludes the proof.

□

Finally, we conclude by proving that V is the unique viscosity solution to Equation (2.17).

Theorem 3.3.1. *Viscosity solutions for the terminal value problem, Equation* (3.5)*, and Equation* (2.17) *in* $\mathcal{C}'_{\gamma^*}(\mathcal{X})$ *are unique.*

Proof. Let $v_1, v_2 \in \mathcal{C}'_{\gamma^*}(\mathcal{X})$ be two viscosity solutions of Equations (3.5) and (2.17) on $\mathcal{X}$, i.e. $v_1, v_2 \in \mathcal{C}_{\gamma'}(\mathcal{X})$ for $\gamma' < \gamma^*$ with $\delta > k(\gamma')$.
We observe that v_1 is a viscosity subsolution of Equation (2.17) on $\mathcal{X}$, that v_2 is a supersolution of Equation (2.17) on $\mathcal{X}$, and that $v_1 = v_2$ at $t = T$. By Proposition 3.3.2 we get $v_1 \leq v_2 \in \mathcal{X}$, and we also have $v_1 \geq v_2$ by a similar argument, so $v_1 \equiv v_2$, and uniqueness follows. □

Chapter 4

Numerical scheme to solve HJB equation

In this part, we describe a finite difference scheme suitable to *penalty approximating* Equation (2.17). Firstly, we provide a description of the approach which will be used, then we will explicit the discretization adopted for solving the HJB equation characterizing the penalty approximation problem. In particular, we use a scheme similar to the one presented in [BJK10], whereas a difference-quadrature scheme is exploited for non-linear degenerate parabolic PIDE. We remark that the scheme presented in [BJK10] attains a solution over the domain $[0, T] \times \mathbb{R}^n$, therefore we will introduce some tailored boundary conditions for $(t, z) \in \partial\mathcal{X}$.

4.1 Penalty Approximation

The numerical scheme adopted is called *penalty approximation*. This particular technique allows to solve the non-linear parabolic integro-PDE arising from Equation (2.17) approximating the original constrained problem defined in Equation (1.25), denoted by OP. The goal is to approximate OP with a sequence of sub-problems indexed by ϵ, i.e. $(OP_\epsilon)_{\epsilon>0}$, which solutions converge to the original one. The main concern in applying such a method is to prove that the family of sub-problems converges to the solution of the original problem. However, since this is not the aim of the Thesis, see e.g. [NS79] and [DZ08], where this converging issue is extensively discussed. We remark that the gradient constraint allows the cumulative consumption process $\boldsymbol{C}$ to be discontinuous, therefore we derive a continuous version of the problem. In particular, for any $\epsilon > 0$, suppose $C_t = \int_s^t c(u)du$ for all $t \in [s, T]$, where $|c(u)| \leq 1/\epsilon$ a.e.. Let us define

$$V_\epsilon(t, x, y, \lambda) = \sup_{\pi \in \mathcal{B}^\epsilon_{t,x,y}} \mathbb{E}\left[\int_t^T e^{-\delta s} f(Y_s^{\pi,C})ds + h(X_T^{\pi,C}, Y_T^{\pi,C}, \lambda_T)\right],$$

where

$\mathcal{B}^\epsilon_{t,x,y} = \{(\pi, C) \in \mathcal{A}_{[t,T]}(x, y) :$ *$\boldsymbol{C}$ is absolutely continuous with derivative bounded by* $\leq 1/\epsilon\}$;

and, following the same steps in Section 2.2, we get

$$0 = e^{-\delta t} f(y) + (V_\epsilon)_t + (\alpha(\xi - \lambda) + \eta\lambda\mu_J)(V_\epsilon)_\lambda \tag{4.1}$$

$$+ \sup_{\pi\in[0,1]} \Big\{ (r(1-\pi) + \mu\pi)x(V_\epsilon)_x + \frac{1}{2}(\sigma\pi x)^2 (V_\epsilon)_{xx} \tag{4.2}$$

$$+ \mathcal{J}^\pi(t,x,y,\lambda,V_\epsilon) \Big\} + \sup_{c\in[0,1/\epsilon]} \{ c((-V_\epsilon)_x + \beta(V_\epsilon)_y) \}.$$

We can easily note that the last part can be solved analytically for c: if $-(V_\epsilon)_x + \beta(V_\epsilon)_y) < 0$ we have $c = 0$, otherwise c is the maximum value inside the admissible region, that is to say $c = 1/\epsilon$. Therefore, by using Equation (2.18), we can rewrite Equation (4.1) as

$$\begin{aligned} 0 =& e^{-\delta t} f(y) + (V_\epsilon)_t + \sup_{\pi\in[0,1]} \Big\{ (r(1-\pi) + \mu\pi)x(V_\epsilon)_x + \frac{1}{2}(\sigma\pi x)^2 (V_\epsilon)_{xx} \\ &+ \mathcal{J}^\pi(t,x,y,\lambda,V_\epsilon) \Big\} + \frac{1}{\epsilon}\max\{ G((V_\epsilon)_x, (V_\epsilon)_y); 0 \}, \end{aligned} \tag{4.3}$$

where the terminal condition has changed accordingly. We remark that existence and uniqueness results hold also for the penalty approximation problem, as well as many results in Chapter 3, so we will not report them.

4.2 Finite difference scheme for penalty approximation

In this section, we will focus on the numerical scheme providing the numerical solution for the HJB equation defined in Equation (4.3). First of all, we write down the HJB equation in a more convenient discrete form. Moreover, for a sake of lighter notation we denote a regular smooth function by v instead if v_ϵ; therefore Equation (4.3) can be defined as

$$\sup_{a\in\mathcal{A}} \mathcal{S}^a(t,x,y,v) = 0,$$

where

$$\begin{aligned} \mathcal{S}^a(t,z,v) =& v_t(t,z) + f(t,z) + \mathcal{L}^a[v](t,z) + \mathcal{I}^a[v](t,z), \\ \mathcal{L}^a[v](t,z) =& \frac{1}{2}(\sigma\pi x)^2 v_{xx} + (r + (\mu - r)\pi)xv_x - \beta y v_y + c(\beta v_y - v_x) + (\alpha(\lambda - \xi) + \eta\lambda\mu_J)v_\lambda, \\ \mathcal{I}^a[v](t,z) =& \mathcal{J}^a(t,z) \\ =& \lambda \int_{\mathbb{R}\setminus\{0\}} v(t, x + x\pi(e^q - 1), y, \lambda + \eta|q|) - v(t,z) \\ & - x\pi(e^q - 1)v_x(t,z) - \eta|q|v_\lambda(t,z)\zeta(q)dq, \\ f(t,z) =& e^{-\delta t} f(y), \\ a =& (\pi, c); \end{aligned}$$

with $\mathcal{A} = [0,1] \times [0, 1/\epsilon]$.

4.2.1 Discretizating the HJB equation

We redefine the discrete domain as $\overline{\mathcal{O}} = [0,T] \times [0,x_{max}] \times [0,y_{max}] \times [0,\lambda_{max}]$, with $x_{max}, y_{max}, \lambda_{max} > 0$. Let $N^t, N^x, N^y, N^\lambda \in \mathbb{N}$, and let us define

$$\Delta t = \frac{T}{N^t}, \Delta x = \frac{x_{max}}{N^x}, \Delta y = \frac{y_{max}}{N^y}, \Delta\lambda = \frac{\lambda_{max}}{N^\lambda},$$

and

$$t_m = m\Delta t, x_i = i\Delta x, y_j = j\Delta y, \lambda_l = l\Delta\lambda$$

for $m = 0,\ldots,N^t; i = 0,,\ldots,N^x; j = 0,\ldots,N^y; l = 0,\ldots,N^\lambda$, where the set of grid points is

$$\overline{\mathcal{G}} = \{(m,i,j,l) \in \mathbb{N}_0^4 : 0 \le m \le N^t, 0 \le i \le N^x, 0 \le j \le N^y, 0 \le l \le N^\lambda\},$$

and

$$\overline{\mathcal{H}} = \{(t,x,y,\lambda) \in \overline{\mathcal{O}} : t = m\Delta t, x = i\Delta x, y = j\Delta y, \lambda = l\Delta\lambda, (m,i,j,l) \in \overline{\mathcal{G}}\}.$$

Let us now introduce the discrete approximation of the value function $\hat{v} : \overline{\mathcal{H}} \to \mathbb{R}$ defined by:

$$\hat{v}^m_{i,j,l} = \hat{v}(t_m, x_i, y_j, \lambda_l) \approx v(t_m, x_i, y_j, \lambda_l) \quad \forall \quad (m,i,j,l) \in \overline{\mathcal{G}}.$$

Now, Equation (4.3) can be rewritten as

$$\sup_{a\in\mathcal{A}} \mathcal{S}^a_{\mathcal{H}}(t_m, x_i, y_j, \lambda_l, \hat{v}) = 0,$$

where

$$\begin{aligned}
\mathcal{S}^a_{\mathcal{H}}(t_m, x_i, y_j, \lambda_l, \hat{v}) &= \frac{1}{\Delta t}\left(\hat{v}^m_{i,j,l} - \hat{v}^{m-1}_{i,j,l}\right) + f^{m-1}_{i,j,l} + \mathcal{L}^a_{\mathcal{H}}[\hat{v}^m_{i,j,l}] + \mathcal{I}^a_{\mathcal{H}}[\hat{v}^m_{i,j,l}],\\
\mathcal{L}^a_{\mathcal{H}}[\hat{v}] &= \frac{1}{2}(\sigma\pi x)^2\Delta_{xx,\Delta x}\hat{v}\\
&+ (r + (\mu - r)\pi)x\delta^+_{x,\Delta x}\hat{v} - \beta y\delta^-_{y,\Delta y}\hat{v}\\
&+ (\alpha(\xi - \lambda) + \eta\mu_J\lambda)\delta^-_{\lambda,\Delta\lambda}\hat{v}\\
&+ \frac{1}{\epsilon}\max\left\{\beta\delta^+_{y,\Delta y}\hat{v} - \delta^-_{x,\Delta x}\hat{v}; 0\right\}.
\end{aligned}$$

where the finite differences used in the above scheme are defined by

$$\begin{aligned}
\delta^\pm_{r,h}\varphi(r,t,z) &:= \pm\frac{1}{h}\left(\varphi(r \pm h, t, z) - \varphi(r,t,z)\right),\\
\Delta_{rr,h}\varphi(r,t,z) &:= \frac{1}{h^2}\left(\varphi(r+h,t,z) - 2\varphi(r,t,z) + \varphi(r-h,t,z)\right),
\end{aligned}$$

for functions $\varphi : \mathbb{R} \times \overline{\mathcal{X}}$. Finally, a special mention should be set aside for the integral operator $\mathcal{I}^a$.

4.2.2 Quadrature rule for the integral operator

In our case, the finite integral cannot be solved analytically, so we make use of the difference-quadrature scheme defined in [BJK08; BJK10], alternatives being, e.g., the trapezoidal rule, the mid-point rule, and the Simpson's rule; for more details see, e.g., [RS90; DCS98; Kun62]). Anyway, we choose the scheme defined in [BJK10] since it provides a direct discretization of the non-local part of the PIDE which leads to a monotone scheme capable of handling singular measures. Therefore, we have:

$$\mathcal{I}^a[\varphi](t,z) = \mathcal{I}^{a,+}[\varphi](t,z) + \mathcal{I}^{a,-}[\varphi](t,z) - \widetilde{b}^a(t,z)\varphi_x(t,z) - \widetilde{c}^a(t,z)\varphi_\lambda(t,z)$$

where

$$\begin{aligned}
\widetilde{b}^a(t,z) &:= \int_{-\infty}^{+\infty} \partial_z^2 x\pi(e^q-1)\widetilde{k}(q)dq,\\
\widetilde{c}^a(t,z) &:= \int_{-\infty}^{+\infty} \partial_z^2 \eta|q|\widetilde{k}(q)dq,\\
\mathcal{I}^a[\varphi] &:= \pm\int_0^{\pm\infty} \partial_z^2[\varphi(t, x+\pi x(r^q-1), y, \lambda+\eta|q|)]\widetilde{k}(q)dq,
\end{aligned}$$

and the integrand measure is defined by:

$$\widetilde{k}(q) := \begin{cases} \int_{-\infty}^{q}\int_{-\infty}^{w} k(r)drdw \text{ if } q<0\\ \int_{q}^{\infty}\int_{w}^{\infty} k(r)drdw \text{ if } q>0. \end{cases}$$

Discretization of $\mathcal{I}^{a,\pm}$ follows directly from [BJK10], therefore

$$\mathcal{I}^{a,\pm}[\varphi](t,z) = \sum_{n=0}^{\infty} \Delta_{qq,\Delta q}[l_{\mathcal{H}}[\varphi](t, x+\pi x(e^q-1), y, \lambda+\eta|q|)]\widetilde{k}^{\pm}_{\mathcal{H},n},$$

where $q_n = n\Delta x$, and $\Delta z = \sqrt{\Delta x}$, $\mathcal{I}_{\mathcal{H}}$ is the second order interpolation operator, and

$$\widetilde{k}^{\pm}_{\mathcal{H},n} = \pm\int_{q_{\pm n}}^{q_{\pm(n+1)}} \widetilde{k}(q)dq,$$

where

$$I_{\mathcal{H}}\varphi(t,z) := \sum_{i=0}^{N^x} w_i(x)\varphi(t, x_i, y, \lambda),$$

and

$$|\varphi(t,x,y,\lambda) - I_{\mathcal{H}}\varphi(t,x,y,\lambda)| \leq K_I \Delta x^2 |D_X^2\varphi|_{L^\infty},$$

for all $x \in [0, x_{max}]$ and $\varphi \in \mathcal{C}^2(\overline{\mathcal{X}})$, where K_I is some constant and $w_i : \mathbb{R} \to [0,1], i = 0,\dots,N^x$, are some weighting functions satisfying the following conditions:

1. $w_i(x_i) = 1$,
2. $w_i(x_j) = 0$ for $i \neq j$,
3. $\sum_{i=0}^{N^x} w_i = 1$.

In particular we define w_i such that:

$$w_i(x) := \begin{cases} 0 & = \text{ if } |x - x_i| \geq \Delta x, \\ (x - x_{i-1})/\Delta x & = \text{ if } x_{i-1} < x < x_i, \\ (x_{i+1} - x_i)/\Delta x & = \text{ if } x_i < x < x_{i+1}. \end{cases}$$

Finally, $\widetilde{b}^a(t,z)\varphi_x$ is discretized by an upwind difference $\widetilde{b}^a(t,z)\delta^-_{x,\Delta x}\widehat{v}$, similarly for $\widetilde{c}^a(t,z)\varphi_\lambda$, we have $\widetilde{c}^a(t,z)\delta^-_{\lambda,\Delta\lambda}\widehat{v}$.

4.2.3 Boundary conditions

Since the problem is defined on $\overline{\mathfrak{X}}$, we need some boundary conditions in order to complete the numerical scheme presented in Section 4.2.2.
In particular, we will explicit the value function for the time variable $t = T$, and for the space variables $x = 0, y = 0, \lambda = 0, x = x_{max}, y = y_{max}$, and $\lambda = \lambda_{max}$.

Condition for terminal utility

At the final time T, we have the following condition:

$$\hat{v}^{N^t}_{i,j,l} = h(x_i, y_j, \lambda_l),$$

for all $(i,j,l) \in \{\mathbb{N}^3_0 : 0 \leq i \leq N^x, 0 \leq j \leq N^y, 0 \leq l \leq N^\lambda\}$.

Conditions for the wealth

We discuss in this part the boundary conditions referring to x. In the null wealth case, i.e. $x = 0$, we have that $\hat{v}^m_{0,j,l}$ collapses into:

$$\begin{aligned} \hat{v}^m_{0,j,l} = &\sup_{a\in\mathcal{A}} \bigg\{ \frac{1}{\Delta t}\left(\hat{v}^m_{0,j,l} - \hat{v}^{m-1}_{0,j,l}\right) + f^{m-1}_{0,j,l} - \beta y \delta^-_{y,\Delta y}\hat{v} \\ &+ (\alpha(\xi - \lambda) + \eta\mu_J\lambda)\delta^-_{\lambda,\Delta\lambda}\hat{v}\bigg\}. \end{aligned}$$

In the case of $x = x_{max}$, we can reasonably assume that $V_x \to 0$ as $x \to \infty$. Therefore, we obtain the following Neumann boundary condition:

$$\hat{v}^m_{N^x,j,l} = \hat{v}^m_{N_x-1,j,l}$$

for all $(m,j,l) \in \{\mathbb{N}^3_0 : 0 \leq m \leq N^t, 0 \leq j \leq N^y, 0 \leq l \leq N^\lambda\}$

Conditions for average past consumption

At the boundary case where we have null past average consumption, i.e. $y = 0$, we can not recover an explicit solution formula as we have done in Section 4.2.3. Anyway, we can investigate in depth such case.
We note that $\max\{G(D_Z v); 0\}$ is always positive, and the numerical scheme is obtained as

$$\sup_{\pi \in [0,1]} \mathcal{S}_{\mathcal{H}}^{(\pi,0)}(t_m, (x_i, 0, \lambda_l), \hat{v}) = 0.$$

Similarly to what has been done for the boundary conditions for x, we have, for $y = y_{max}$, that

$$\hat{v}_{i,N^y,l}^{m} = \hat{v}_{i,N_y-1,l}^{m},$$

for all $(m, i, l) \in \{\mathbb{N}_0^3 : 0 \le m \le N^t, 0 \le i \le N^x, 0 \le l \le N^\lambda\}$.

Conditions for instantaneous frequency

Finally, we are left with the boundary conditions for λ.

In the case of null instantaneous frequency of jumps, we note that the HJB equation can be simplified into the canonical case presented in [Mer75], whereas the risky asset dynamic is described by a continuous Lévy process. Therefore, the value function is obtained through

$$\sup_{a \in \mathcal{A}} \mathcal{S}_{\mathcal{H}}^{a}(t_m, (x_i, y_j, 0), \hat{v}) = 0.$$

Formally, we have:

$$\begin{aligned} \mathcal{S}_{\mathcal{H}}^{a}(t_m, x_i, y_j, 0, \hat{v}) &= \frac{1}{\Delta t}\left(\hat{v}_{i,j,0}^{m} - \hat{v}_{i,j,0}^{m-1}\right) + f_{i,j,0}^{m-1} \\ \mathcal{L}_{\mathcal{H}}^{a}[\hat{v}] &= \frac{1}{2}(\sigma\pi x)^2 \Delta_{xx,\Delta x}\hat{v} \\ &\quad + (r + (\mu - r)\pi)x\delta_{x,\Delta x}^{+}\hat{v} - \beta y \delta_{y,\Delta y}^{-}\hat{v} \\ &\quad + \frac{1}{\epsilon}\max\left\{\beta\delta_{y,\Delta y}^{+}\hat{v} - \delta_{x,\Delta x}^{-}\hat{v}; 0\right\}. \end{aligned}$$

Note that explicit solutions can be obtained for utility functions belonging to the *Hyperbolic absolute risk aversion* (HARA) family. Finally, we define the boundary case where the investor cannot bear more uncertainty, i.e. $\lambda = \lambda_{max}$. Also in this case, we make use of the Neumann condition:

$$\hat{v}_{i,j,N^\lambda}^{m} = \hat{v}_{i,j,N^{\lambda-1}}^{m},$$

for all $(m, i, j) \in \{\mathbb{N}_0^3 : 0 \le m \le N^t, 0 \le i \le N^x, 0 \le j \le N^y\}$.

Part II

Calibration and numerical results

Chapter 5

Calibration algorithms

In this part, we report the calibration algorithm for the self-exciting model on market data. In particular, we provide an effective calibration procedure for an Italian asset, namely ENI, by considering a time interval where a canonical Lévy process cannot explain clusters of extraordinary movements affecting the market. Indeed, we have chosen the COVID-19 breakout period, when we assume the self-exciting effects can catch better cluster phenomenon.

The calibration procedure will be divided in two parts due to the high-dimensionality of the problem. Firstly, we perform a preliminary calibration where we provide an estimate of the parameters by mean of a log-likelihood procedure. Then, we comment such results and use them as starting point to properly locate the prior distributions for a particular type of Sequential Monte Carlo algorithm, which will be explained in details. Among the various calibration approaches available, we calibrate the model over ENI asset, by using an asymmetric *Peaks over Threshold* (PoT) methodology, see e.g. [ELL11]. Then, by focusing exclusively on the jump part, we obtain the parameters defining either the double exponential distribution and the process $\boldsymbol{\lambda}$.

We collected the daily level for ENI asset from March 3, 2012 to March 3, 2022, obtaining 2535 values. The data are stored in a vector of $(n+1)$ observed levels for the risky asset $\boldsymbol{s}$. Then, we compute the vector $\boldsymbol{y} = (y_1, ..., y_n)$, collecting the log-returns, where $y_i := \log(\frac{s_i}{s_{i-1}})$. Note that each observation is indexed to time Δi, where we assume $\Delta := 1/252$, since we have collected daily data.

5.1 Log-likelihood algorithm

Due to the high-dimensionality of the problem, we firstly apply the PoT methodology to divide the total sample into a *pure* Gaussian and a mixture part. Therefore, we cut off the sample by obtaining a subset of returns delimited by the Normal quantiles of two optimal confidence levels $q(\alpha_1), q(\alpha_2)$, such that the sample without jumps replicates the third and forth central moments of a Gaussian random variable.
In such a way, we are excluding from the pure Gaussian sample those extreme returns which are diverging the sample's skewness and kurtosis from those desired. Table 5.1 reports the yearly mean $\hat{\mu}$, standard deviation $\hat{\sigma}$, skewness $\hat{S}$, and excess of kurtosis $\hat{K}$ of our sample.

parameter	$\hat{\mu}$	$\hat{\sigma}$	$\hat{S}$	$\hat{K}$
value	3.6929 %	27.5388 %	-1.3907	23.8088

Table 5.1: Descriptive statistics (yearly basis) for the sample of daily log-returns $\boldsymbol{y}$

There is no need of Jarque-Bera test to notice that the distribution is far for being Gaussian. Indeed, the negative value of $\hat{S}$ highlights slightly extended tails on the left side, the value $\hat{K}$ denotes *fat tails*, therefore the distribution is highly leptokurtic. Let us define $Y := \hat{\sigma} W_\Delta$, so $Y \sim \mathcal{G}(0, \hat{\sigma}^2 \Delta)$. Now let us define the distribution of the *centered* y_i in the following way:

$$\begin{cases} (y_i - \hat{\mu}) \sim Y & \text{if} \quad q(\alpha_1) \leq (y_i - \hat{\mu}) \leq q(\alpha_2), \\ (y_i - \hat{\mu}) \sim Y + J & \text{otherwise} , \end{cases}$$

where we call $q(\alpha)$ the quantile of order α of the continuous random variable Y.
This optimization problem has solution in $(\alpha_1 = 0.024, \alpha_2 = 0.975)$, such solution ensures a *purged* sample with skewness $\hat{S}' = 0.0045$ and excess of kurtosis $\hat{K}' = -0.0111$. We can now perform the Jarque-Bera test on the sample since the sample size is sufficient, see e.g. [GG08]. The results confirm our expectations: the statistic test is equal to 0.0208, with a corresponding p-value of 98.96%, thus the hypothesis of Normal distribution cannot be rejected for the sample considered, for almost every reasonable confidence level.

We report a graphical presentation in Figure 5.2, where log-returns of ENI have been plotted. Then, we mark the threshold levels provided by the solution of the optimization problem.

The log-returns detected as jumps are 111 and are collected in the mixture part. These observations allow us to calibrate the latent process $\boldsymbol{\lambda}$ for a triplet of parameters (α, ξ, η):

$$\lambda_i = \lambda_{i-1} + \alpha(\xi - \lambda_{i-1})\Delta + \eta |J_i| \mathbb{1}(\Delta \mathcal{N}_i = 1 | \lambda_{i-1}),$$

where we reasonably set $\lambda_0 = \xi$.

Given the frequency of the data collected, we assume that at most, one jump can occur between two observations, so we focus only on the possibility of having one jump or no jumps in Δ.
The jump distribution and the frequency parameters are then calibrated by jointly maximizing the following two log-likelihood functions:

$$\begin{cases} (p, \rho^+, \rho^-) & = \arg\max \sum_{i=1}^n \log \mathbb{P}(y_i | p, \rho^+, \rho^-) \mathbb{1}(\Delta \mathcal{N}_i = 1 | \lambda_{i-1}) \\ (\alpha, \xi, \eta) & = \arg\max \sum_{i=1}^n \log \mathbb{P}(\Delta \mathcal{N}_i | \lambda_{i-1}) \end{cases} .$$

Formally we have

$$(\Delta N_i | \lambda_{i-1} \Delta) = \begin{cases} 0, & \mathbb{P}(\Delta \mathcal{N}_i = 0 | \lambda_{i-1}) = 1 - \lambda_{i-1}\Delta + O((\lambda_{i-1}\Delta)^2) \\ 1, & \mathbb{P}(\Delta \mathcal{N}_i = 1 | \lambda_{i-1}) = \lambda_{i-1}\Delta - O((\lambda_{i-1}\Delta)^2). \end{cases}$$

In the implementation, we will reasonably assume $O((\lambda_{i-1}\Delta)^2)$ is negligible, therefore where we approximate the increment of the Poisson process with a Bernoulli process.

Finally, in order to confirm the entire procedure, we calibrate μ and σ appearing in Equation (1.5) by maximizing the log-likelihood of the distribution of z_i, where:

$$z_i := y_i - \left(\mu - \frac{\sigma^2}{2} - \lambda_{i-1}\mathbb{E}[e^J - 1]\right)\Delta - j_i \mathbb{1}(\Delta\mathcal{N}_i = 1|\lambda_{i-1}), \tag{5.1}$$

with j_i denoting the jump size if a jump occurred at i. Finally, we denote the time-dependent mean as

$$\mu_i^* = \left(\mu - \frac{\sigma^2}{2} - \lambda_{i-1}\mathbb{E}[e^J - 1]\right).$$

Thus, the solution of the problem is

$$(\mu, \sigma, p, \rho^+, \rho^-) = \arg\max \sum_{i=1}^{n} \log \mathbb{P}\left(z_i \middle| \theta\right),$$

where $\theta = (\mu, \sigma, p, \rho^+, \rho^-) \in \Theta$, and

$$\begin{aligned}\mathbb{P}(z_i|\theta) = \varphi(y_i - \mu_i^*|\Theta)\mathbb{P}(\Delta\mathcal{N}_i = 0|\lambda_{i-1}, \theta) \\ + \iota(y_i - \mu_i^*|\Theta)\mathbb{P}(\Delta\mathcal{N}_i = 1|\lambda_{i-1}, \theta),\end{aligned}$$

where $\varphi(\cdot)$ is the probability density function for a random variable distributed as $\mathcal{G}\left(0, \sigma^2\Delta\right)$ and $\iota(\cdot|\theta)$ is the probability density function for $Z := Y + J$. It is called *double exponential modified Gaussian* (DEMG) distribution, and it is an extension of the mixture of a Gaussian and an exponential distribution, known, in literature, as *exponential modified Gaussian* (EMG), see e.g. [AGG19; FD84].

Proposition 5.1.1. *Let Y, J be two continuous and independent random variables, such that $Y \sim \mathcal{G}(0, \sigma^2\Delta)$ and J is distributed according $\zeta(\cdot|p, \rho_+, \rho_+)$ as Equation* (1.1). *Given a set of parameters $\theta = (\sigma, p, \rho^+, \rho^-)$, the density function of $Z := Y + J$, has the following closed-form solution:*

$$\begin{aligned}\iota(z|\theta) = &- (1-p)\rho^- e^{\left(\frac{1}{2}\sigma^2\Delta(\rho^-)^2 - \rho^- z\right)}\Phi\left(\frac{-z + \sigma^2\Delta\rho^-}{\sigma\sqrt{\Delta}}\right) \\ &+ p\rho^+ e^{\frac{1}{2}\sigma^2\Delta(\rho^+)^2 - \rho^+ z}\left(1 - \Phi\left(\frac{-z + \sigma^2\Delta\rho^+}{\sigma\sqrt{\Delta}}\right)\right),\end{aligned}$$

where $\Phi(\cdot)$ denotes the distribution of a Gaussian random variable with null mean and variance unity.

Proof. The solution for the closed-form for $\iota(\cdot)$ can be obtained by the *convolutional method*, namely:

$$
\begin{aligned}
\iota(z|\theta) =&(\varphi * \zeta)(z|\theta) = \int_{-\infty}^{\infty} \varphi(z-t)\zeta(t)dt \\
=& -(1-p)\int_{-\infty}^{0} \rho^{-} e^{-\rho^{-} t} \frac{1}{\sqrt{2\pi\Delta}\sigma} e^{-\frac{(z-t)^2}{2\sigma^2\Delta}} dt \\
&+ p\int_{0}^{\infty} \rho^{+} e^{-\rho^{+} t} \frac{1}{\sqrt{2\pi\Delta}\sigma} e^{-\frac{(z-t)^2}{2\sigma^2\Delta}} dt \\
=& -(1-p)\rho^{-} e^{\frac{1}{2}\sigma^2\Delta(\rho^{-})^2 - \rho^{+} z} \times \\
&\times \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\Delta}\sigma} e^{-\frac{(t-(z-\sigma^2\Delta\rho^{-}))^2}{2\sigma^2\Delta}} dt \\
&+ p\rho^{+} e^{\frac{1}{2}\sigma^2\Delta(\rho^{+})^2 - \rho^{+} z} \times \\
&\times \int_{0}^{\infty} \frac{1}{\sqrt{2\pi\Delta}\sigma} e^{-\frac{(t-(z-\sigma^2\Delta\rho^{+}))^2}{2\sigma^2\Delta}} dt \\
=& -(1-p)\rho^{-} e^{(\frac{1}{2}\sigma^2\Delta(\rho^{-})^2 - \rho^{-} z)} \Phi\left(\frac{-z+\sigma^2\Delta\rho^{-}}{\sigma\sqrt{\Delta}}\right) \\
&+ p\rho^{+} e^{\frac{1}{2}\sigma^2\Delta(\rho^{+})^2 - \rho^{+} z} \left(1 - \Phi\left(\frac{-z+\sigma^2\Delta\rho^{+}}{\sigma\sqrt{\Delta}}\right)\right),
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative density function of a standard Gaussian random variable. $\square$

5.1.1 Log-likelihood calibration results

Now, we can report the results of the procedure described the previous section. The parameters values are collected in Table 5.2, where we denote the log-likelihood calibrated parameters by $\widetilde{\mu}, \widetilde{\sigma}, \widetilde{p}, \widetilde{\rho}^{+}, \widetilde{\rho}^{-}, \widetilde{\alpha}, \widetilde{\eta}, \widetilde{\xi}$, to discern from the calibrated parameters in Section 5.2.

parameter	$\widetilde{\mu}$	$\widetilde{\sigma}$	$\widetilde{p}$	$\widetilde{\rho}^{+}$	$\widetilde{\rho}^{-}$
value	4.42%	23.41%	0.5045	20.8960	-18.5073

parameter	$\widetilde{\alpha}$	$\widetilde{\xi}$	$\widetilde{\eta}$
value	16.7859	5.1858	420.0910

Table 5.2: Parameters of the model obtained with the calibration on ENI.

Results

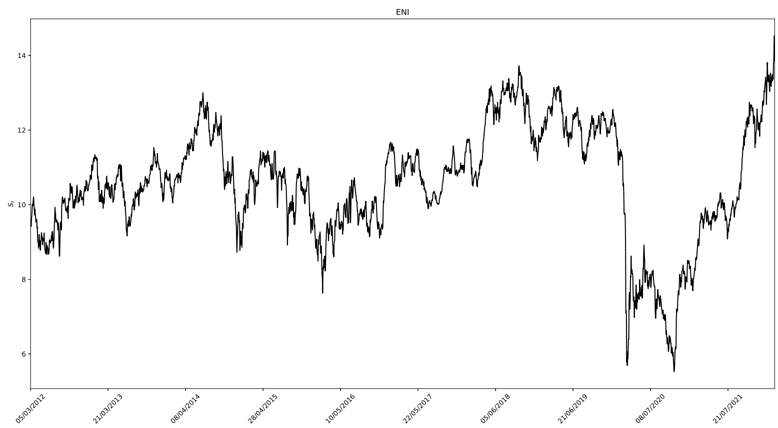


Figure 5.1: *Close* adjusted values for ENI from March 03, 2012 to March 03, 2022. Source: *Refinitiv*.

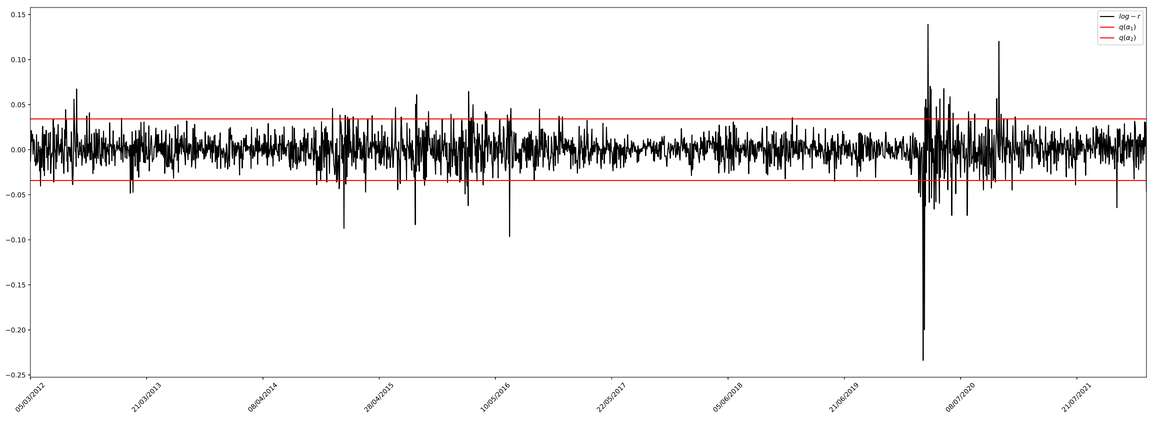


Figure 5.2: Log-returns for ENI asset. Red lines report the thresholds exploited to detect jumps.

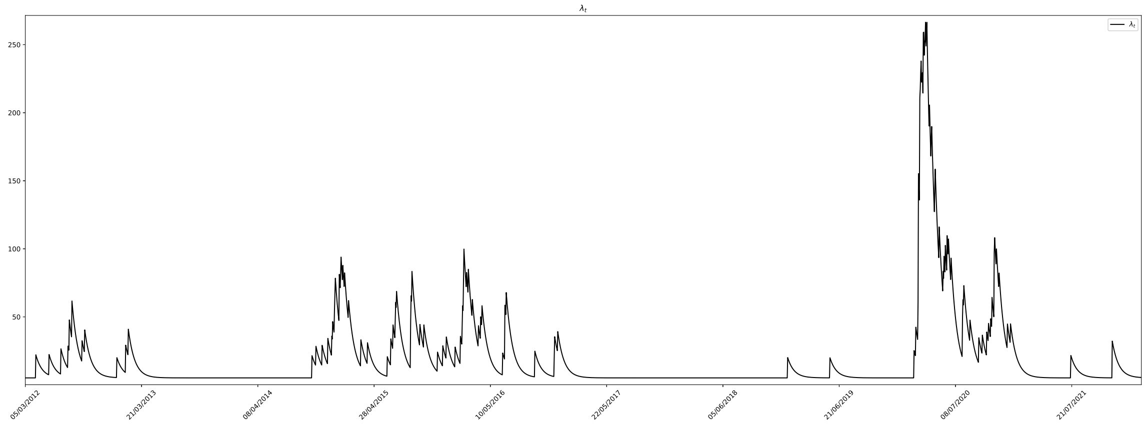


Figure 5.3: Latent process $\boldsymbol{\lambda}$ with $\alpha = 16.7859, \xi = 5.1858, \eta = 420.0910$.

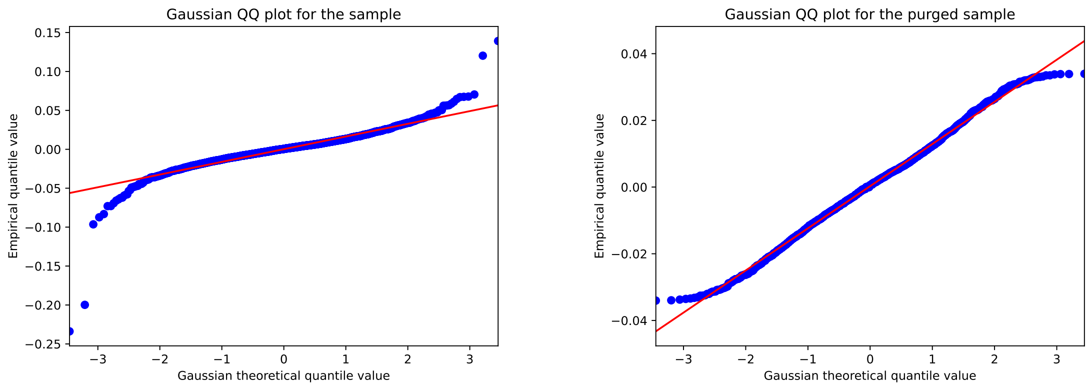


Figure 5.4: Quantile-Quantile plots for the log-returns distribution. On the left side the whole log-returns are plotted, on the right side the sample without jumps is reported.

In Figure 5.1 we report the ENI asset levels for the considered period. We can notice that there are some sudden increments and drops in prices in particular in the first months of 2020. Such movements are highlighted in Figure 5.2, where we mark in red the thresholds dividing the pure Gaussian from the mixture part.

Finally, in Figure 5.3, we report the latent process $\boldsymbol{\lambda}$ and, in particular, we notice that the model describes accurately the jump cluster at the beginning of 2020, where the instantaneous frequency reaches its peak in the period considered.

Note that the value reported for $\widetilde{\mu}$ and $\widetilde{\sigma}$ are on yearly basis. Moreover, we should remark that $\widetilde{\sigma}$ does not refer to the overall volatility of the asset in this case, differently to what happens in a canonical Geometric Brownian Motion. Although the algorithm proposed is immediate and with limited computational costs, we can observe that the Quantile-Quantile plot for the log-returns without jumps seems different from being a Gaussian. Therefore, we apply different tests to verify that the distribution is Normal, see for example Kolmogorov-Smirnov test, Lilliefors test, Shapiro-Wilk, Anderson-Darling test in [BZ14; AM07; RW+11]. In particular the Kolmogorov-Smirnov test reports poor results for the p-value. Moreover, the Jarque-Bera test requires, at least, 2000 elements, which is not always reasonable when dealing with daily data. Therefore, these reasons motivate us to look for an additional and robust calibration routine which will be presented in the next section.

5.2 The Sequential Monte Carlo algorithm

The previous preliminary results, reported in Section 5.1.1, will be the starting point and benchmark for a particular implementation of Sequential Monte Carlo algorithm, namely SMC^2 .
The choice of a new particle filtering calibration routine is due to the main drawbacks of the aforementioned log-likelihood method. In fact, it detects jumps only in the case we are exceeding the thresholds set, concretely meaning it cannot detect jumps of small sizes. This drawback spurs to look for further methods to retune the log-likelihood results presented in Section 5.1.
Aside the Particle Filtering method proposed above, let us underline that we have considered several alternative ways to calibrate the parameters, namely *neural networks* (NNs) approach. This family of methods has been already successfully exploited in recent literature, see e.g. [DPH16; DPH17; GR11; Hou+18; KA16], gaining an increasing attention by the *financial community*. However, we decide to use the SMC^2 algorithm since it is more flexible given the parameter uncertainty under analysis.

In this section, we report the procedure in [CDPVnda], where we focus on this particular family of particle filtering techniques (PF). Such PF relies on Bayesian statistical Inference and aims to compute the posterior distributions for some stochastic processes using a given set of particles, see e.g. [DM96; LC98], which paved the way in this field.
The filtering problem is high-dimensional and the diffusive part is driven by several sources of noise: this motivates a robust calibration procedure which acts simultaneously on $(\mathcal{N}, \boldsymbol{\lambda}, \boldsymbol{J})$. The proposed Sequential Monte Carlo approach is based on the joint use of the iterated bath importance sampling (IBIS) algorithm, see e.g. [Cho02], and the PFs technique, see e.g. [Fea+10]. The procedure samples the values for the marginal distribution of θ and then reweights iteratively these values by using the likelihood increments $p(y_t|y_{1:t-1}, \theta)$. We must remark that the reweighting algorithm plays a crucial role in the procedure since it avoids the degeneracy of the importance weights, see e.g. [DDFG01; Dju+03], but, on the other side, it yields a loss of heterogeneity in the population of particles including additional computational costs, for a detailed description see [Che+03; DJ+09]. Before applying the SMC^2 model, we will introduce the notation and the pseudocode in Sections 5.2.1 and 5.2.2, providing a description of the quantities involved. In Section 5.2.4, we implement a calibration routine based on the approach presented in [CJP13], and then, after we have correctly set the framework, we apply the calibration algorithm to the processes presented in Section 1.2.1 and comment the results in Section 5.2.4, remarking the differences with the the log-likelihood approach presented in Section 5.1.

5.2.1 Introduction to the notation for SMC²

We recall the colon notation for sets of random variables which will be used in this section, that is to say $x_t^{1:N_x} := \bigcup_{n=1}^{N_x} x_t^n$ is a set of N_x random variables $x_t^n, n = 1, \dots, N_x$ and, similarly, $x_{1:t}^{1:N_x} := \bigcup_{s=1}^{t} x_s^{1:N_x}$. Moreover, we denote by $p(\cdot)$ the probability density defined by the model, meanwhile π_t defines the posterior probability density at t, investigated by the algorithm. More precisely, for a generic state-space model, with parameters $\theta \in \Theta$, prior $p(\theta)$, latent Markov process $(x_t)_t$ and $p(x_1|\theta) = \mu_\theta(x_1)$, we have

$$p(x_{t+1}|x_{1:t}, \theta) = p(x_{t+1}|x_t, \theta) =: f_\theta(x_{t+1}|x_t), \quad t \geq 1, \tag{5.2}$$

with the observed process defined as

$$p(y_t|y_{1:t-1}, x_{1:t-1}, \theta) = p(y_t|x_t, \theta) =: g_\theta(y_t|x_t), \quad t \geq 1.$$

The goal of the algorithm is the iterated exploration of the posterior distributions

$$\pi_0(\theta) = p(\theta), \quad \pi_t(\theta, x_{1:t}) = p(\theta, x_{1:t}|y_{1:t}), \quad t \geq 1,$$

as well as computing the distribution for the observed process $p(y_{1:t})$. Therefore, in Section 5.2.4 we will perform some analysis on $p(y_{1:t})$, whereas we prove the goodness of results in the SMC² algorithm, in particular focusing on the out-of-sample prediction. SMC² is a generic black box tool which allows to perform particle Markov chain Monte Carlo (PMCMC), a class of MCMC algorithms that uses a particle filter of size N_x as a proposal method to overcome the intractability of the distribution $p(y_{1:t}|\theta)$, obtained as by-product of particle filtering outcome. Formally, we have:

$$p(y_{1:t}|\theta) = p(y_1|\theta) \prod_{s=2}^{t} p(y_s|y_{1:s-1}, \theta), \quad 1 \leq t \leq T,$$

with the variance increasing linearly in time, for more details see [CDMG11]. Each of the N_θ, θ-tuple particle θ^m is exploited by the particle filter to propagate N_x x-particles.
Differently from the canonical PF algorithms, SMC² produces unbiased estimates for the θ marginal likelihood instead of an exact filter, as the IBIS case. We report that, in [ADH10], it has been shown that, as N_x grows, the PMCMC algorithm behaves more similarly to a theoretical MCMC. This overcomes the intractability of $\pi_T(\theta)$ and, even more important, for a given value of N_x, the PMCMC algorithm returns stationary distribution for $\pi_T(\theta, x_{0:T})$.
Finally, we remark that such family of algorithms offers further advantages: it calibrates automatically the N_x tuning parameters and the proposal distributions for θ.

5.2.2 Pseudocode for SMC²

In the following section, we present a pseudocode for SMC² , the procedure is reported for a given k-tuple of parameters θ^m, therefore the operations involving the index m should be intended on $(\theta_1^m, \theta_2^m, \dots, \theta_k^m) =: \theta^m, \forall m \in \{1, \dots, N_\theta\}$.

The algorithm proposed acts on the tuple $(\theta, x_{1:t}^{1:N_x}, a_{1:t-1}^{1:N_x})$ in the following way: if the degeneracy criterion is met, the set of variables is replaced by new ones, otherwise the set of variables are kept with no changes. Before exploring the procedure, we remark that we denote by $K_t(\cdot)$ a PMCMC kernel which defines the rejuvenation step, while the proposal step is based on some kernel $T(\theta, d\widetilde{\theta})$ in the k-dimensional support.

Algorithm 1 SMC^2 : parameter and state estimation

procedure SMC^2

for $m \in \{1, \dots, N_\theta\}$ **do**

Sample θ^m from $p(\theta)$ and set $\omega^m \leftarrow 1$.

$t \leftarrow 1$.

Sample $x_i^{1:N_x,m} \sim \psi_{0,\theta^m}(\cdot)$ and compute

$$\widehat{p}(y_1|\theta^m) = \frac{1}{N_x}\sum_{n=1}^{N_x} \omega_{1,\theta}(x_1^{n,m}).$$

for $t \in \{2, \dots, T\}$ **do**

Sample $\left(x_t^{1:N_x,m}, a_{t-1}^{1:N_x,m}\right) \sim \psi_{t-1,\theta^m}$ and compute

$$\widehat{p}(y_t|y_{1:t-1}, \theta^m) = \frac{1}{N_x}\sum_{n=1}^{N_x} \omega_{t,\theta}(x_{t-1}^{a_{t-1}^{n,m}}, x_t^{n,m}).$$

Update the weights:

$$\omega^m \leftarrow \omega^m \widehat{p}(y_t|y_{1:t-1}, \theta^m).$$

if some degeneracy criterion is fulfilled (see Section 5.2.3), **then**

Sample $\left(\widetilde{\theta}^m, \widetilde{x}_{1:t}^{1:N_x,m}, \widetilde{a}_{1:t-1}^{1:N_x,m}\right)$ from

$$\frac{1}{\sum_{m=1}^{N_\theta}\omega^m}\sum_{m=1}^{N_\theta}\omega^m K_t\left\{\left(\theta^m, x_{1:t}^{1:N_x,m}, a_{1:t-1}^{1:N_x,m}\right), \cdot\right\}. \tag{5.3}$$

Replace the current state of the system with the set of *unweighted* particles:

$$\left(\omega^m, x_t^{1:N_x,m}, a_{t-1}^{1:N_x,m}, \theta^m\right) \leftarrow \left(\widetilde{\theta}^m, \widetilde{x}_t^{1:N_x,m}, \widetilde{a}_{t-1}^{1:N_x,m}, 1\right).$$

5.2.3 The degeneracy criterion

In the pseudocode, we remark that we may want to rejuvenate the sample $\left(\widetilde{\theta}^m, \widetilde{x}_{1:t}^{1:N_x,m}, \widetilde{a}_{1:t-1}^{1:N_x,m}\right)$ from Equation (5.3), whenever some criterion is met. We decide to use the most standard and studied approach, which regards the *Effective Sample Size* (hence ESS) metric, defined as:

$$ESS := \frac{(\sum_{m=1}^{N_\theta} \omega^m)^2}{\sum_{m=1}^{N_\theta} (\omega^m)^2}.$$

Therefore, the move step is achieved by using a Markovian kernel $K_t(\cdot)$ which ensures $p(\theta|y_{1:t})$ is invariant. In the calibration procedure, we set $K_t(\cdot)$ as a Metropolis-Hastings kernel, which proposes a new set of particles whenever $ESS < \gamma N_\theta$ for $\gamma \in (0,1)$. For a more detailed theoretical explanation, see Sections 3.7 and 4 in [CJP13], and [GK18].

5.2.4 SMC^2 calibration results

In the final part of this section, we take into account the calibration of the risky model defined in Section 1.2.1 by using the pseudocode in 5.2.2. We focus on the most recent thorny period where a deep instability was affecting the market. Therefore, we reduce the sample used in Section 5.1.1, by focusing on the interval from December 31, 2019 to March 3, 2022, just at the beginning of the global outbreak of the conflict in Ukraine. In order to have a test set, we reduce the sample by taking out the last 10 observations which will be used to briefly test the model.

Firstly, we define the marginal prior distributions for the model parameters, almost all of them are centered in the log-likelihood calibrated parameters defined in Section 5.1.1, while the probability of having a positive jump is bounded between 0 and 1. In details, we have:

$$\begin{aligned}
&\alpha \sim \mathcal{E}(1/\widetilde{\alpha}),\\
&\xi \sim \mathcal{E}(1/\widetilde{\xi});\\
&\eta \sim \mathcal{LN}(\log(\widetilde{\eta} - 50.0), 10.0),\\
&p \sim \mathcal{B}(2.0, 2.0),\\
&\rho^- \sim \mathcal{E}_-(1/\widetilde{\rho}_-),\\
&\rho^+ \sim \mathcal{E}(1/\widetilde{\rho}_+),\\
&\mu \sim \mathcal{G}(\widetilde{\mu}, 0.1),\\
&\sigma \sim \mathcal{LN}(\log(\widetilde{\sigma} - 0.02), 0.2),
\end{aligned}$$

where we denote, respectively, by $\mathcal{E}(a)$ an Exponential distribution of parameter $a \in \mathbb{R}_+$, by $\mathcal{E}_-(a)$ a (negative) Exponential distribution with parameter $a \in \mathbb{R}_-$, by $\mathcal{LN}(a,b)$ we denote a Log-Normal distribution with location $a \in \mathbb{R}$ and scale $b \in \mathbb{R}_+$. Finally, $\mathcal{B}(a,b)$ is a Beta distribution with concentration parameters $a, b \in \mathbb{R}_+$. The key feature of our approach is to consider the latent multidimensional process $x_{1:t}$ as collection of all the singular latent processes. Despite the great advantage of coping with a state-space with intractable densities, we have to deal with the fact that the joint density distribution cannot be attained in a closed-form. Therefore, we make use of the Dirac-delta distribution provided by the Github package *pyro-ppl*, see e.g. [PPJ19;

Bin+18].
In the case considered, at each $t \in \{1, \dots, T\}$, we have $x_t = (\lambda_t, \mathcal{N}_t, J_{\mathcal{N}_t})$, and we remark that this tuple evolves with $N_x = N_\theta = 1000$ particles. For what concerns the computational cost, we briefly point out that SMC^2 algorithm is memory-intensive, this forces us to use *Google Colaboratory*, as suggested also in [Car+18], where an analysis is performed through the use of *Colaboratory* for accelerating deep learning for computer vision and other GPU-centric applications.
In particular, in [CJP13], it is reported that the algorithm requires, up to iteration t, a computational cost of order $O(tN_\theta N_x)$, which eventually can be reduced to $O(N_\theta N_x)$ with little loss of generality.

Then, we extended the *Github* package *pyfilter* [Vic22] by developing those new classes required to run the algorithm in our case.

Given the previous considerations, we can now focus the analysis on the parameters calibrated and, to mark the stability of the algorithm, we plot the results obtained with two different calibration runs. The calibration is performed by setting the parameter γ defined in Section 5.2.3 equal to 0.2. In particular, we decide to focus on the Gaussian observations

$$\widetilde{y}_i = y_i + \lambda_{i-1}\mathbb{E}[e^J - 1]\Delta - j_i(\Delta\mathcal{N}_i = 1|\lambda_{i-1}) \tag{5.4}$$

with $\widetilde{y}_i \sim \mathcal{G}\left((\mu - 0.5\sigma^2)\Delta, \sigma^2\Delta)\right)$.

Firstly, we investigate the posterior distributions for 2 different runs to show the stability and then we compute the parameters distribution mean over several runs. In particular, we have:

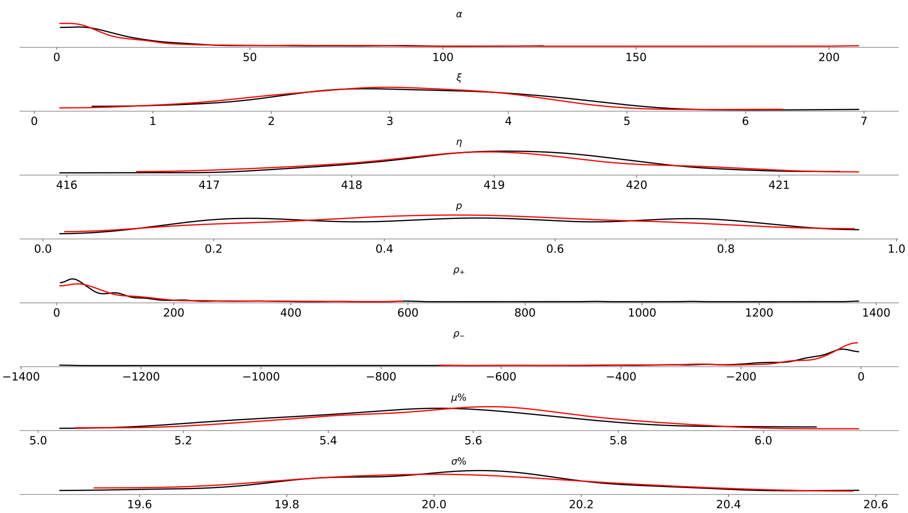


Figure 5.5: Posterior distributions for the model parameters for 2 different runs.

Therefore, we perform 100 runs and we store the distributions and their means. Then, for each parameter we compute the mean over all the 100 means obtained at the previous step. The results are reported in the following table.

parameter	μ	σ	p	ρ^+	ρ^-
value	5.77 %	20.13 %	0.4889	60.5654	-53.8959

parameter	α	ξ	η
value	9.3016	4.6756	419.2810

Table 5.3: Parameters mean obtained with the SMC^2 procedure on ENI asset.

In Table 5.3, the parameters ρ^- and ρ^+ assume values which are coherent with our expectations, in fact, the SMC^2 algorithm has detected more small jumps than the procedure presented in Section 5.1. Moreover, due to the increment of the number of jumps detected, also the impact of the Brownian Motion is reduced, as we observe a value of 20.13% for σ, whereas $\widetilde{\sigma}$ was 23.41% in Table 5.2. Further, we investigate the evolution of the most important latent process, namely $\boldsymbol{\lambda}$, and we observe that the particle filter detects similar behaviours over different runs:

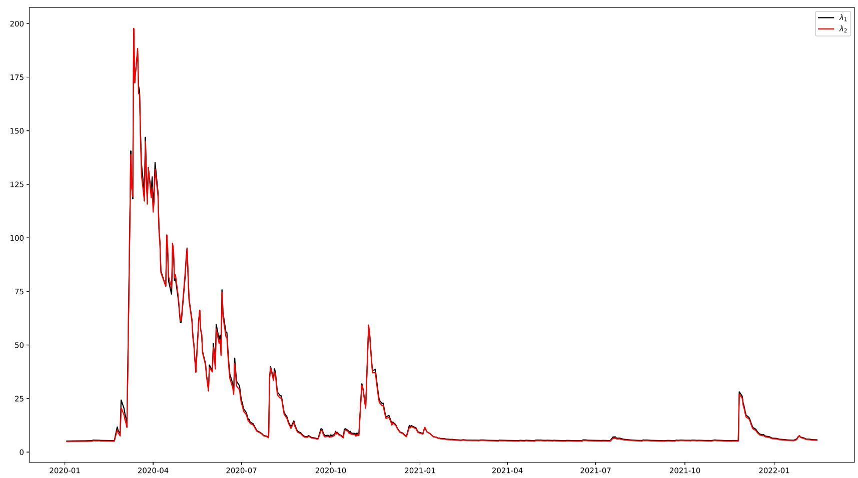


Figure 5.6: $\boldsymbol{\lambda}$ evolution for the observed period for different runs.

Also in this case, we note that the process $\boldsymbol{\lambda}$ describes the turmoil in the market during the COVID-19 breakout in 2020.

Finally, we briefly test our model to predict an out-of-sample return 10-days ahead and compare it with the forecast given by the historical rolling mean.

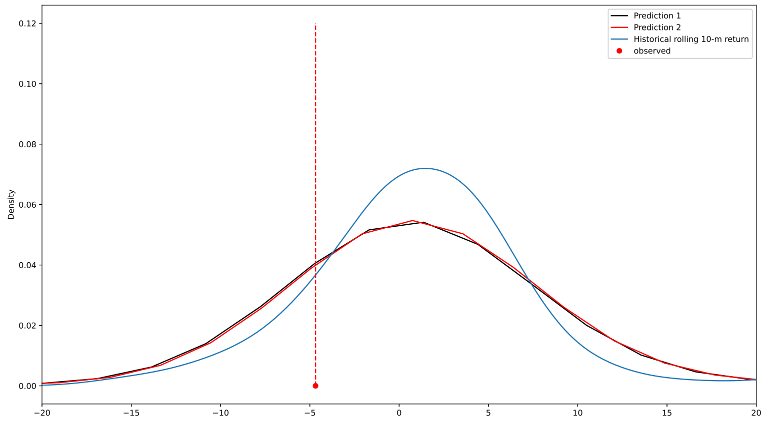


Figure 5.7: Log-return forecast 10-day ahead.

We clearly see that the model overperforms the forecast provided by the rolling mean on 10-day windows.

Now that the risky part of the portfolio is calibrated, we can focus on the simulation part. In the following chapter we represent the numerical solutions for the stochastic control problem described in Chapter 2.

Chapter 6

Simulation and results

In this last part we solve the *penalized* HJB equation (4.1) numerically. Therefore, firstly we explicit the utility functions used, and then we collect all the results from the simulations, in particular focusing on the optimal value function and optimal investment strategies.

6.1 Settings

Thanks to Assumption A3 defined in Section 1.3.1, we can reduce the dimension of the problem. Thus, we look for a solution on the space defined by $w = x/y$ and then, in order to graphically explore the results, we simulate separately $\boldsymbol{\lambda}$.
The procedure consists in iterating backwards over the time interval $[0, T]$ solving the value functions for optimal π using the finite difference scheme exposed in Chapter 4.

6.1.1 Utility functions

Although the numerical scheme is defined for value functions directly depending also on λ, we decided to omit this dependence and ease the graphical representation.
Therefore, we explicit the terminal utility function in Equation (6.1) as

$$h(x, y) := \frac{1}{\gamma}(x + \beta y)^{\gamma}, \tag{6.1}$$

where $\beta \in \mathbb{R}_+$, and $\gamma \in (0, 1)$.
Then, we introduce the form of the running utility which will be used in the simulation, as denoted by Equation (1.25).
Therefore, we define the functional f as

$$f(y) := \frac{y^{\gamma}}{\gamma}, \tag{6.2}$$

with $\gamma \in (0, 1)$.
Now, we can recollect the results obtained in Table 5.3 for Equations (1.18) and (1.3).

parameter	μ	σ	p	ρ^+	ρ^-
value	5.77 %	20.13 %	0.48	60.5654	-53.8960

parameter	α	ξ	η
value	9.3016	4.6756	419.2810

Table 6.1: Parameters of the model obtained with the calibration on ENI asset.

It is worth emphasizing that we decide to fix a maximum change in the portfolio allocation, i.e. $\Delta\pi = 0.1$ for each time step, in this way we bound the changes in the risky part of the portfolio. Indeed, it is not always possible to disinvest a remarkable amount of capital from one day to another.

Now we are left to set the risk-free short rate r. Although it is not completely realistic according to the current market conditions, we set a positive value for r. Such condition, $r > 0$, guarantees that the model does not collapse into a trivial solutions, that is to say that the entire wealth is invested in the risky part.
We set now the parameters characterizing the objective function, which are reported in Table 6.2.

parameter	r	γ	β	δ
value	3.00 %	3.00% %	2.00	0.10

Table 6.2: Objective function parameters.

And finally, we report the model settings in Table 6.3:

parameter	T	π_T	$\Delta\boldsymbol{\pi}$	w_{max}	N^w	N^t	λ
value	5.00	0.40	0.10	5.00	30.00	1260	5.00

Table 6.3: Model settings.

We decide to set the initial value for $\boldsymbol{\lambda}$ equal to 5.00, which is a reasonable starting point given the calibration outputs.

6.2 Results

The results are graphically represented by the mean of the control and the value functions. In particular, we use $\mathbb{E}[\lambda_t|\mathcal{F}_0]$ in the numerical scheme to solve the PIDE; therefore, we firstly represent the optimal control against the evolution for $\mathbb{E}[\lambda_t|\mathcal{F}_0]$, obtained as for Equation 1.17. Therefore, the control function is reported in Figure 6.1 and, in Figure 6.2, we represent the expectation for $\boldsymbol{\lambda}$ against the control value for some fixed values for w. Furthermore, we would like to remark that we choose a reasonable small amount for the risky allocation at time T, because we expect that investors get more risk adverse as time passes by, preferring safer investments as they get older.

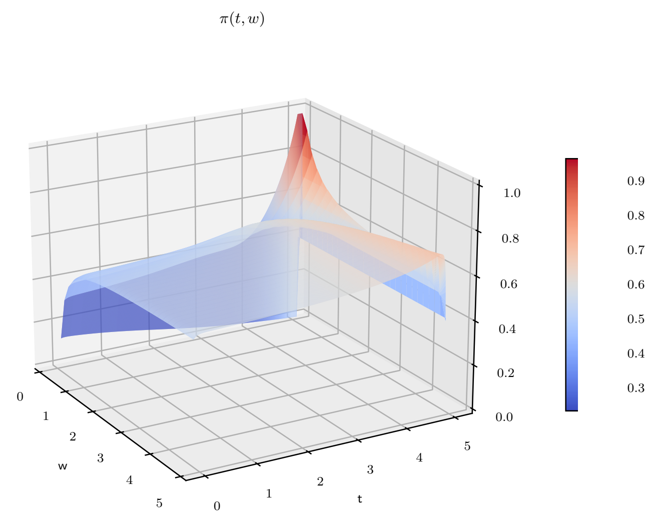


Figure 6.1: Control function for $\mathbb{E}[\boldsymbol{\lambda}]$.

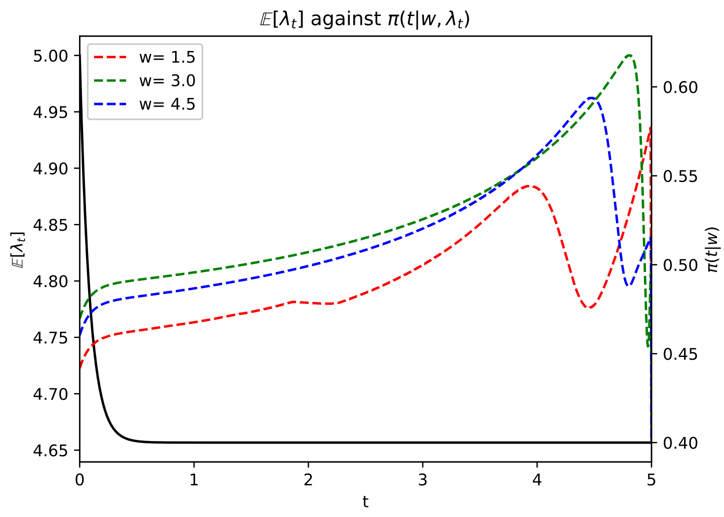


Figure 6.2: $\mathbb{E}[\boldsymbol{\lambda}]$ evolution over time against control function for $w = \{1.5, 3.0, 4.5\}$.

For what concerns the value function V, the result is coherent with the literature on the topic. In Figure 6.3 we report the value function results, where $\mathbb{E}[\lambda_t|\mathcal{F}_0]$ has been used in solving the PIDE. The surface declines slowly over time, although it is almost flat for different values of w.

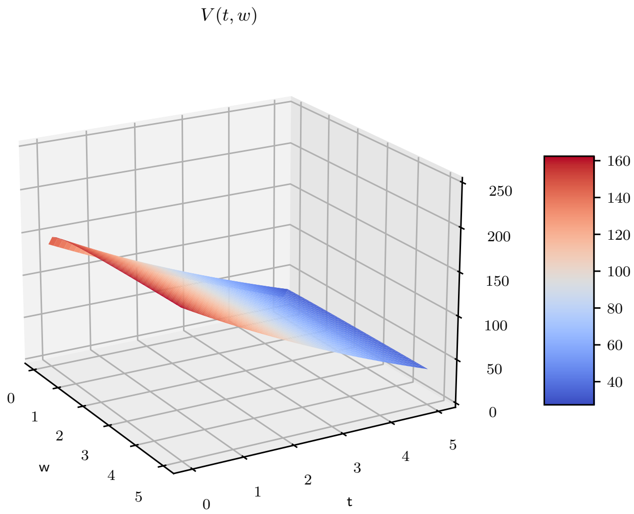
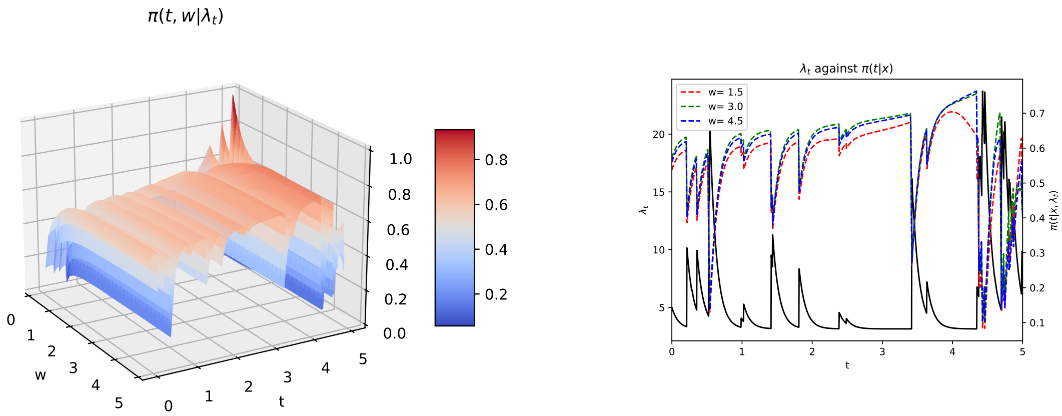
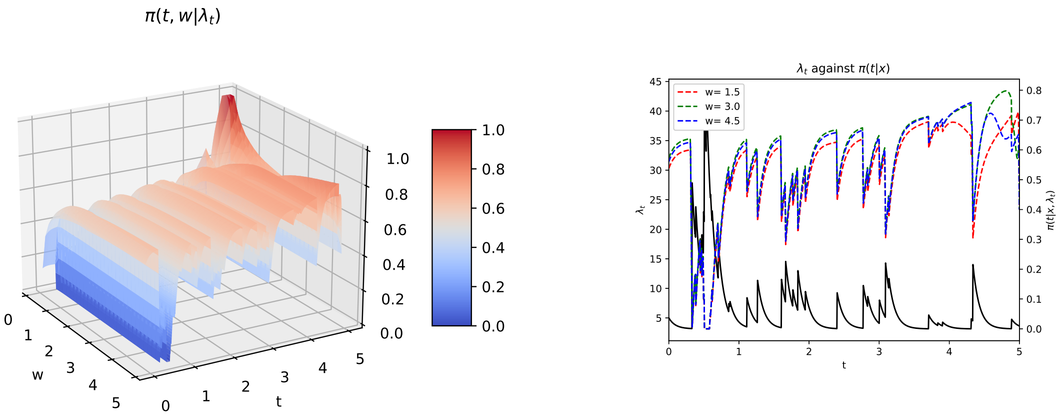


Figure 6.3: Value function representation for $\mathbb{E}[\boldsymbol{\lambda}]$.

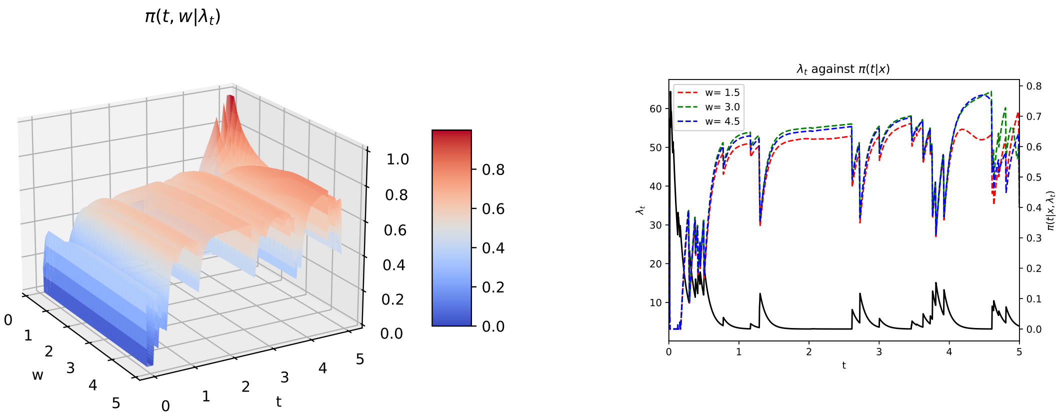
We consider also the evolution of $\boldsymbol{\pi}$ for different realizations of $\boldsymbol{\lambda}$, therefore we report some simulations for $\boldsymbol{\lambda}$ and the corresponding values for $\boldsymbol{\pi}$.



(a) First simulation



(b) Second simulation



(c) Third simulation

Figure 6.4: Different simulations for $\boldsymbol{\lambda}$ and corresponding values for $\boldsymbol{\pi}$.

In presence of jumps, and consequently for sudden increases for $\boldsymbol{\lambda}$, we note that the control function drops for all the value of w and then immediately raises, converging to the value of $\boldsymbol{\pi}$ reported in Figure 6.1. This effect is quite interesting and realistic, since it is due to some economic reasons. In fact, in presence of jumps the total uncertainty incorporated in the portfolio increases, forcing the investor to decrease the amount allocated in the risky asset since such a high risk cannot be born in a small time frame.

6.2.1 Sensitivity analysis for α

Finally, we conclude the chapter by performing some sensitivity analysis over α which governs the speed convergence to the long-term mean for the $\boldsymbol{\lambda}$ process, see Equation (1.3).
Thus, we plot some realizations for $\boldsymbol{\lambda}$ with different α, as reported in the legend.

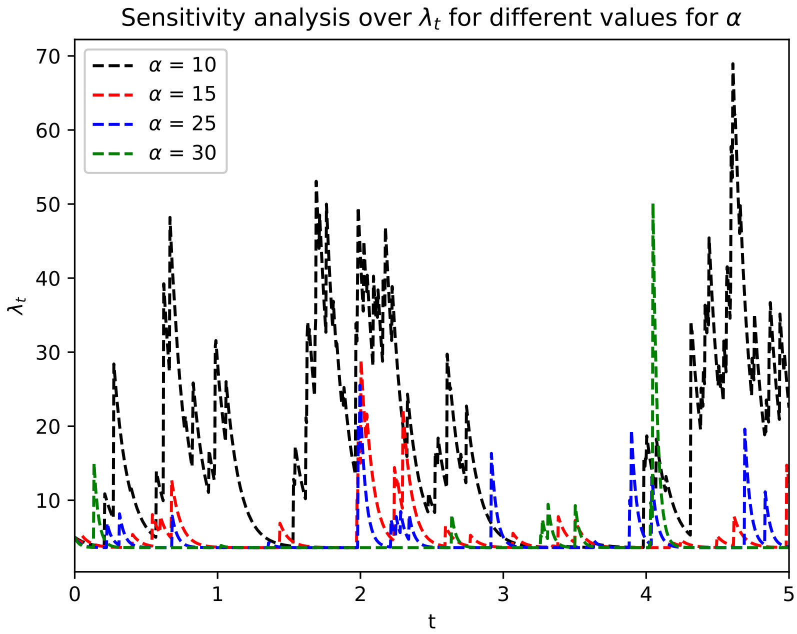


Figure 6.5: Sensitivity analysis for different values of α and
$(\lambda, \xi, \eta, p, \rho_+, \rho_-) = (10., 4.6756, 419.2810, 0.48, 60.5654, -53.8960)$.

We fix an initial frequency λ different from ξ, since we want to highlight convergence of the process and mark its self-exciting nature.
In particular, for the realizations reported we can appreciate that the self-exciting feature is more marked whenever we are in presence of small α since the jump effect is more persistent in time.

Conclusion

In the present work we investigated both the optimal risky allocation and consumption for a portfolio whose dynamics is steered by a self–exciting Poisson point process. The proposed model overcomes some drawbacks of the Lévy processes and generalizes the setting, indeed it can justify extreme returns clusters which can be observed in a small amount of time. In the first chapter we introduced the model, focusing in particular on the concept of instantaneous jump frequency. Then, we enunciated the portfolio optimization problem over a finite time horizon solved by the value function, proving its continuity over the whole domain and remarking all the necessary conditions ensuring its well posedness. The latter allows to prove that Dynamic Programming Principle holds within our framework, hence allowing us to derive the Hamilton-Jacobi-Bellman equation for the portfolio optimization problem. Then, we included an extra constraint on the gradient for the HJB equation because of the discontinuity of the second control, and we studied such equation exploiting results obtained in [CDPVndb]. In particular we proved that the V is the unique viscosity solution for our optimization problem, also providing uniqueness via comparison principle. Then, we concluded the first part of the Thesis by penalty approximating the HJB equation and presenting the corresponding backward numerical scheme, incorporating the gradient constraint as a penalty term.

Within the second part of the present work, we applied aforementioned theoretical results to the case of investors who might want to stay exposed in the market even during financial turmoil. Therefore, we calibrated the self-exciting process on an Italian asset, namely ENI, during the *COVID-19* breakout. We started by using a likelihood approach joint with a *Peaks over Threshold* (PoT) technique. Then, the previous results are used as a starting point to implement the model in [CDPVnda], where a *Sequential Monte Carlo* (SMC^2) algorithm has been used to solve the high-dimensional calibration problem in presence of a self-exciting process. Finally, we used the calibrated parameters to simulate the HJB equation, retrieving the optimal allocation and value function, and commenting the results.

Concluding, we remark that further developments and other interesting extensions could be investigated by including in the self-exciting framework a stochastic process modelling the risk-free rate or the volatility, this kind of choice increases the complexity of the problem and reduces the tractability of the HJB equation itself adding a new source of noise.

Bibliography

[ABL02] Torben G Andersen, Luca Benzoni, and Jesper Lund. “An empirical investigation of continuous-time equity return models”. In: *The Journal of Finance* 57.3 (2002), pp. 1239–1284.

[ADH10] Christophe Andrieu, Arnaud Doucet, and Roman Holenstein. “Particle markov chain monte carlo methods”. In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 72.3 (2010), pp. 269–342.

[AGG19] Sebastian Ament, John Gregoire, and Carla Gomes. “Exponentially-modified Gaussian mixture model: applications in spectroscopy”. In: *arXiv preprint arXiv:1902.05601* (2019).

[Alv94] O Alvarez. “A singular stochastic control problem in an unbounded domain”. In: *Communications in Partial Differential Equations* 19.11-12 (1994), pp. 2075–2089.

[AM07] Hervé Abdi and Paul Molin. “Lilliefors/Van Soest’s test of normality”. In: *Encyclopedia of measurement and statistics* (2007), pp. 540–544.

[ASJ+09] Yacine Aït-Sahalia, Jean Jacod, et al. “Testing for jumps in a discretely observed process”. In: *The Annals of Statistics* 37.1 (2009), pp. 184–222.

[Bin+18] Eli Bingham et al. “Pyro: Deep Universal Probabilistic Programming”. In: *Journal of Machine Learning Research* (2018).

[BJK08] Imran H Biswas, Espen R Jakobsen, and Kenneth H Karlsen. “Error estimates for a class of finite difference-quadrature schemes for fully nonlinear degenerate parabolic integro-PDEs”. In: *Journal of Hyperbolic Differential Equations* 5.01 (2008), pp. 187–219.

[BJK10] Imran H Biswas, Espen R Jakobsen, and Kenneth H Karlsen. “Difference-quadrature schemes for nonlinear degenerate parabolic integro-PDE”. In: *SIAM Journal on Numerical Analysis* 48.3 (2010), pp. 1110–1135.

[BKR01a] Fred E Benth, Kenneth H Karlsen, and Kristin Reikvam. “On the existence of optimal controls for a singular stochastic control problem in finance”. In: *Mathematical Finance*. Springer, 2001, pp. 79–88.

[BKR01b] Fred Espen Benth, Kenneth Hvistendahl Karlsen, and Kristin Reikvam. “Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach”. In: *Finance and Stochastics* 5.3 (2001), pp. 275–303.

[BKR02] Fred Espen Benth, Kenneth Hvistendahl Karlsen, and Kristin Reikvam. “Portfolio optimization in a Lévy market with intertemporal substitution and transaction costs”. In: *Stochastics: An International Journal of Probability and Stochastic Processes* 74.3-4 (2002), pp. 517–569.

[BL90] Fischer Black and Robert Litterman. “Asset allocation: combining investor views with market equilibrium”. In: *Goldman Sachs Fixed Income Research* 115 (1990).

[BZ14] Vance W Berger and YanYan Zhou. “Kolmogorov–smirnov test: Overview”. In: *Wiley statsref: Statistics reference online* (2014).

[Car+18] Tiago Carneiro et al. “Performance analysis of google colaboratory as a tool for accelerating deep learning applications”. In: *IEEE Access* 6 (2018), pp. 61677–61685.

[CB21] Yiling Chen and Baojun Bian. “Optimal dividend policy in an insurance company with contagious arrivals of claims”. In: *Mathematical Control & Related Fields* 11.1 (2021), p. 1.

[CDMG11] Frédéric Cérou, Pierre Del Moral, and Arnaud Guyader. “A nonasymptotic theorem for unnormalized Feynman-Kac particle models”. In: *Annales de l’IHP Probabilités et statistiques*. Vol. 47. 3. 2011, pp. 629–649.

[CDPVnda] Giuseppe Cordoni, Luca Di Persio, and Andrea Veronese. “A self-exciting jump calibration approach in Finance during COVID-19 outbreak”. N.D.

[CDPVndb] Giuseppe Cordoni, Luca Di Persio, and Andrea Veronese. “Portfolio optimization in presence of a self-exciting jump process”. N.D.

[CH89] John C Cox and Chi-fu Huang. “Optimal consumption and portfolio policies when asset prices follow a diffusion process”. In: *Journal of economic theory* 49.1 (1989), pp. 33–83.

[Che+03] Zhe Chen et al. “Bayesian filtering: From Kalman filters to particle filters, and beyond”. In: *Statistics* 182.1 (2003), pp. 1–69.

[Cho02] Nicolas Chopin. “A sequential particle filter method for static models”. In: *Biometrika* 89.3 (2002), pp. 539–552.

[CIL92] Michael G Crandall, Hitoshi Ishii, and Pierre-Louis Lions. “User’s guide to viscosity solutions of second order partial differential equations”. In: *Bulletin of the American mathematical society* 27.1 (1992), pp. 1–67.

[CJP13] Nicolas Chopin, Pierre E Jacob, and Omiros Papaspiliopoulos. “SMC2: an efficient algorithm for sequential analysis of state space models”. In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 75.3 (2013), pp. 397–426.

[CL83] Michael G Crandall and Pierre-Louis Lions. “Viscosity solutions of Hamilton-Jacobi equations”. In: *Transactions of the American mathematical society* 277.1 (1983), pp. 1–42.

[Cor+20] F. Cordoni et al. “A stochastic approach to path-dependent nonlinear Kolmogorov equations via BSDEs with time-delayed generators and applications to finance”. In: *Stochastic Processes and their Applications* 130.3 (2020), pp. 1669–1712. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-85068172305&doi=10.1016%2fj.spa.2019.05.013&partnerID=40&md5=15abb883ff31d99fbde497ea7435b0e1.

[CR76] John C Cox and Stephen A Ross. “The valuation of options for alternative stochastic processes”. In: *Journal of financial economics* 3.1-2 (1976), pp. 145–166.

[DCS98] Sever S Dragomir, Pietro Cerone, and Anthony Sofo. “Some remarks on the midpoint rule in numerical integration”. In: *RGMIA research report collection* 1.2 (1998).

[DDFG01] Arnaud Doucet, Nando De Freitas, and Neil Gordon. “An introduction to sequential Monte Carlo methods”. In: *Sequential Monte Carlo methods in practice*. Springer, 2001, pp. 3–14.

[Den07] Guohe Deng. “Pricing European option in a double exponential jump-diffusion model with two market structure risks and its comparisons”. In: *Applied Mathematics-A Journal of Chinese Universities* 22.2 (2007), pp. 127–137.

[DJ+09] Arnaud Doucet, Adam M Johansen, et al. “A tutorial on particle filtering and smoothing: Fifteen years later”. In: *Handbook of nonlinear filtering* 12.656-704 (2009), p. 3.

[Dju+03] Petar M Djuric et al. “Particle filtering”. In: *IEEE signal processing magazine* 20.5 (2003), pp. 19–38.

[DM96] Pierre Del Moral. “Nonlinear filtering using random particles”. In: *Theory of Probability & Its Applications* 40.4 (1996), pp. 690–701.

[DPH16] L. Di Persio and O. Honchar. “Artificial neural networks architectures for stock price prediction: Comparisons and applications”. In: *International Journal of Circuits, Systems and Signal Processing* 10 (2016), pp. 403–413. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-84998705847&partnerID=40&md5=f3fa06042299716f81a58ef587947ec2.

[DPH17] L. Di Persio and O. Honchar. “Analysis of recurrent neural networks for short-term energy load forecasting”. In: vol. 1906. 2017. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-85038849154&doi=10.1063%2f1.5012469&partnerID=40&md5=c9b0cb5acfc5507f154eb92ab7b7e915.

[DZ08] Min Dai and Yifei Zhong. “Penalty methods for continuous-time portfolio selection with proportional transaction costs”. In: *Available at SSRN 1210105* (2008).

[ELL11] Paul Embrechts, Thomas Liniger, and Lu Lin. "Multivariate Hawkes processes: an application to financial data". In: *Journal of Applied Probability* 48.A (2011), pp. 367–378.

[FD84] Joe P Foley and John G Dorsey. "A review of the exponentially modified gaussian (EMG) function: evaluation and subsequent calculation of universal data". In: *Journal of chromatographic science* 22.1 (1984), pp. 40–46.

[Fea+10] Paul Fearnhead et al. "Random-weight particle filtering of continuous time processes". In: *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 72.4 (2010), pp. 497–512.

[GG08] Yulia R Gel and Joseph L Gastwirth. "A robust modification of the Jarque–Bera test of normality". In: *Economics Letters* 99.1 (2008), pp. 30–32.

[GK18] Andrew Golightly and Theodore Kypraios. "Efficient SMC2 schemes for stochastic kinetic models". In: *Statistics and Computing* 28.6 (2018), pp. 1215–1230.

[GR11] S. Giebel and M. Rainer. "Stochastic processes adapted by neural networks with application to climate, energy, and finance". In: *Applied Mathematics and Computation* 218.3 (2011), pp. 1003–1007. URL: `https://www.scopus.com/inward/record.uri?eid=2-s2.0-80052264353&doi=10.1016%2fj.amc.2011.03.121&partnerID=40&md5=f5d3bce5a578d8d71581e7dce99cb6aa`.

[GS12] Iosif Il'ich Gihman and Anatolij Vladimirovič Skorohod. *Controlled stochastic processes*. Springer Science & Business Media, 2012.

[Haw18] Alan G Hawkes. "Hawkes processes and their applications to finance: a review". In: *Quantitative Finance* 18.2 (2018), pp. 193–198.

[HH93] Ayman Hindy and Chi-fu Huang. "Optimal consumption and portfolio rules with durability and local substitution". In: *Econometrica: Journal of the Econometric Society* (1993), pp. 85–121.

[HM16] Donatien Hainaut and Franck Moraux. "A multifactor self-exciting jump diffusion approach for modelling the clustering of jumps in equity returns". In: *Paris December 2016 Finance Meeting EUROFIDAI-AFFI*. 2016.

[HM19] Donatien Hainaut and Franck Moraux. "A switching self-exciting jump diffusion process for stock prices". In: *Annals of Finance* 15.2 (2019), pp. 267–306.

[HO74] Alan G Hawkes and David Oakes. "A cluster process representation of a self-exciting process". In: *Journal of Applied Probability* 11.3 (1974), pp. 493–503.

[Hol10] Nina Holden. "Portfolio optimization in a jump-diffusion market with durability and local substitution: A penalty approximation of a singular control problem". MA thesis. 2010.

[Hou+18] K. in 't Hout et al. "Special issue—Computational and algorithmic finance". In: *Journal of Computational Science* 24 (2018), pp. 180–181. URL: `https://www.scopus.com/inward/record.uri?eid=2-s2.0-85042005204&doi=10.1016%2fj.jocs.2017.12.009&partnerID=40&md5=79f0e1a23c9c659e4014405cd37a4df`

[Ish04] Yasushi Ishikawa. “Optimal control problem associated with jump processes”. In: *Applied Mathematics and Optimization* 50.1 (2004), pp. 21–65.

[IW14] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*. Vol. 24. Elsevier, 2014.

[KA16] K. Khashanah and T. Alsulaiman. “Network theory and behavioral finance in a heterogeneous market environment”. In: *Complexity* 21 (2016), pp. 530–554. URL: https://www.scopus.com/inward/record.uri?eid=2-s2.0-84995632406&doi=10.1002%2fcplx.21834&partnerID=40&md5=3eba71a2b9ece88ea710fa0ffdf72eb6.

[Kun62] Guy F Kuncir. “Algorithm 103: Simpson’s rule integrator”. In: *Communications of the ACM* 5.6 (1962), p. 347.

[LC98] Jun S Liu and Rong Chen. “Sequential Monte Carlo methods for dynamic systems”. In: *Journal of the American statistical association* 93.443 (1998), pp. 1032–1044.

[Mar52] Harry Markowitz. “Portfolio Selection, Journal of Finance”. In: *Markowitz HM—1952.—№* (1952), pp. 77–91.

[Mer69] Robert C Merton. “Lifetime portfolio selection under uncertainty: The continuous-time case”. In: *The review of Economics and Statistics* (1969), pp. 247–257.

[Mer75] Robert C Merton. “Optimum consumption and portfolio rules in a continuous-time model”. In: *Stochastic Optimization Models in Finance*. Elsevier, 1975, pp. 621–661.

[Mer89] Robert C Merton. “On the application of the continuous-time theory of finance to financial intermediation and insurance”. In: *The Geneva Papers on Risk and Insurance-Issues and Practice* 14.3 (1989), pp. 225–261.

[NS79] VH Nguyen and JJ Strodiot. “On the convergence rate for a penalty function method of exponential type”. In: *Journal of Optimization Theory and Applications* 27.4 (1979), pp. 495–508.

[ØS05] Bernt Karsten Øksendal and Agnes Sulem. *Applied stochastic control of jump diffusions*. Vol. 498. Springer, 2005.

[PPJ19] Du Phan, Neeraj Pradhan, and Martin Jankowiak. “Composable Effects for Flexible and Accelerated Probabilistic Programming in NumPyro”. In: *arXiv preprint arXiv:1912.11554* (2019).

[Pro] P Protter. *Stochastic integration and differential equations, 1990*. Springer, New York.

[Rog13] Leonard CG Rogers. *Optimal investment*. Springer, 2013.

[RS90] Qazi I Rahman and Gerhard Schmeisser. “Characterization of the speed of convergence of the trapezoidal rule”. In: *Numerische Mathematik* 57.1 (1990), pp. 123–138.

[RW+11] Nornadiah Mohd Razali, Yap Bee Wah, et al. "Power comparisons of shapiro-wilk, kolmogorov-smirnov, lilliefors and anderson-darling tests". In: *Journal of statistical modeling and analytics* 2.1 (2011), pp. 21–33.

[Sha64] William F Sharpe. "Capital asset prices: A theory of market equilibrium under conditions of risk". In: *The journal of finance* 19.3 (1964), pp. 425–442.

[Shr04] Steven E Shreve. *Stochastic calculus for finance II: Continuous-time models*. Vol. 11. Springer Science & Business Media, 2004.

[Son86] Halil Mete Soner. "Optimal control with state-space constraint. II". In: *SIAM journal on control and optimization* 24.6 (1986), pp. 1110–1122.

[Vic22] Gruselius Victor. *pyfilter*. `https://github.com/tingiskhan/pyfilter`. 2022.

[Zhu94] Hang Zhu. "Dynamic programming and variational inequalities in singular stochastic control." In: (1994).