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Vacuum decay in quadratic gravity

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Abstract Metastable states decay at zero temperature through quantum tunnelling at an exponentially small rate, which depends on the Coleman–de Luccia instanton, also known as bounce. In some theories, the bounce may not exist or its on-shell action may be ill-defined or infinite, thus hindering the vacuum decay process. In this paper, we test this possibility in modified theories of gravity interacting with a real scalar field. We consider an Einstein–Hilbert term with a non-minimally coupled scalar field and a quadratic Ricci scalar contribution. To tackle the problem, we use a new analytic method, with which we prove that the scalar field on the bounce has a universal behaviour at large Euclidean radii, almost independently of the potential. Our main result is that the quadratic Ricci scalar prevents the decay, regardless of the other terms in the action.

1 Introduction

Coleman and De Luccia discovered long ago how to describe the decay of a metastable state using the Euclidean path integral in the semi-classical approximation [1–3]. Such decay is driven by quantum fluctuations, which yield the spontaneous nucleation of a true vacuum phase in a bubbly shape. The nucleation rate is exponentially small, and the exponent is proportional to the Euclidean action of the theory evaluated on a particular $O(4)$-symmetric trajectory (called Coleman–de Luccia instanton or bounce) between the tunnelling point and the false vacuum. Since the equations of motion are nonlinear, the bounce is usually found numerically [4–13], although some analytical solutions exist [14–20]. Analytic approximate solutions can also be found when the energy difference between the false and the true vacuum is sufficiently small, in what is called the thin-wall approximation [3,14,16,18,21,22]). To demonstrate that a bounce does not exist is trickier. For instance, in the thin-wall approximation, this happens when the action has no minimum for finite non-vanishing values of the bounce radius. Most systems though cannot be studied in this approximation, and the question cannot be settled. Similar results may be derived in the case of a single scalar field non-minimally coupled to gravity but only if the gravitational backreaction is small [23,24].

Some cases of physical interest, such as modified gravity, have not been investigated much yet in the context of vacuum decay [25]. In this paper, we begin to explore systematically this class of theories and focus in particular on finding obstructions to the decay process when the false vacuum state has a flat geometry. One of our main results is that quadratic gravity terms forbid vacuum decay at zero temperature. Such term is usually required by renormalizability in quantized field theories on a gravitational (classical) background [26–28] and arises also as a low-energy limit of $f(R)$ gravity. Since the equations of motion, in this case, are quite involved, we introduce a new method that allows determining the bounce at large Euclidean radii, i.e. when it approaches the false vacuum, which will be called "asymptotic bounce" in the following. To illustrate it, we first focus on single scalar field theories on a fixed, flat background. We then extend it to Einstein–Hilbert gravity as well as modified gravity and determine it in the following cases

- a single scalar field theory with Einstein–Hilbert gravity, a non-minimal coupling $\xi\phi^2 R$ and a quadratic term R^2.
- a single scalar field theory with non-minimal coupling $\xi\phi^2 R$ and a quadratic term R^2.

Our findings allow us to verify whether

1. the equations of motion have a solution such that all fields approach the false vacuum at infinity;
2. this solution has well-defined and finite on-shell action;

and, thus, if the vacuum decay process is hindered in the aforementioned modified gravity theories. While these conditions constrain the Coleman–de Luccia bounce, they do not apply to static solutions such as the Hawking–Moss instanton, which contributes to vacuum decay when the false vacuum has a de Sitter geometry [29]. The violation of Condition 1. means that only bubbles of infinite

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Fig. 1 Scalar field potential with
two classical vacua (left panel)
and its Euclidean counterpart
(right panel). ϕ_{fv} is the value of
the scalar field at the false
vacuum, separated from the local
minimum of the potential by a
barrier, on top of which $\phi = \phi_{\mathrm{top}}$.
ϕ_0 marks the bounce initial
condition: the field undershoots
(overshoots) when released for
$\phi_{\mathrm{top}} \leq \phi \leq \phi_0$ ($\phi_0 \leq \phi \leq \phi_{\mathrm{tv}}$)
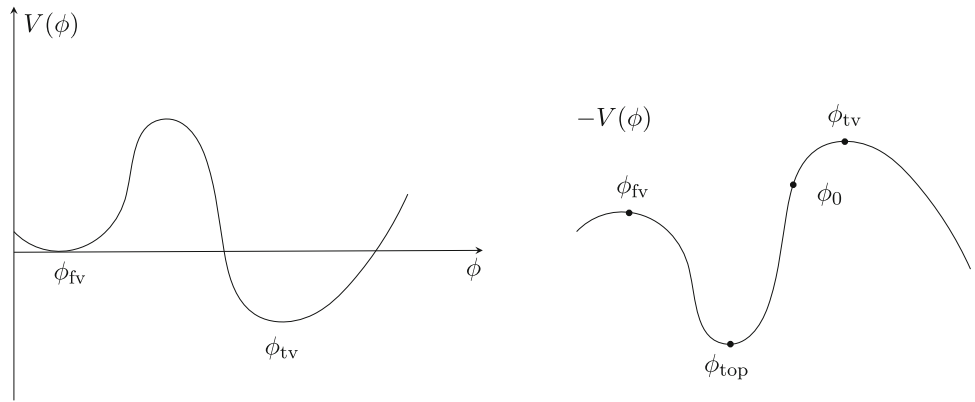
radius are critical, that is the only solution satisfying the boundary conditions is the false vacuum static solution, and thus, there is no phase transition to the true vacuum. If, instead, the on-shell action is infinite, the decay rate vanishes (the so-called vacuum quenching [3]). The bounce action may also be ill-defined near the false vacuum, i.e. at the upper bound of integration: this happens, for example, if our candidate metastable state is a minimum of the Euclidean potential, and thus a maximum in Minkowski space.[1] Such a state does not exhibit any metastability, but only an instability related to the local unboundedness of the potential around such a fixed point. As we will see, quadratic gravity falls precisely in this class of theories. The structure of the paper is the following: In Sec. 2, we introduce our method by first analysing a single scale field theory without gravity and then by including standard general relativity. In Sec. 3, we include quadratic corrections and a non-minimal coupling. In Sec. 4, we repeat our analysis without the Einstein–Hilbert term, i.e. with a scale-invariant gravitational sector. The asymptotic bounce has also numerical implications that are explored in Sec. 5. We finally conclude in Sec. 6 some remarks and future directions. To keep the discussion as simple as possible, we confine the most technical details in the appendices.

2 Scalar asymptotic bounce with Einstein–Hilbert gravity

2.1 Scalar asymptotic bounce

A metastable scalar field decays due to quantum tunnelling at an exponentially small rate Γ, given by [1,2,30,31]

$$\Gamma = A e^{-B} , \tag{1}$$

where B is the difference between the Euclidean action calculated on the bounce and the one computed in the false vacuum state. Consider a single scalar field theory with a metastable state (ϕ_{fv} in Fig. 1) and $O(4)-$symmetric Euclidean action

$$S = 2\pi^2 \int_0^{+\infty} \mathrm{d}t \, t^3 \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right) , \tag{2}$$

where the dot indicates a derivative with respect to the Euclidean radius t, and $\phi = \phi(t)$. For the time being, we assume also that $\phi_{\mathrm{fv}} = 0$ and $V(\phi_{\mathrm{fv}}) = 0$. The bounce is a solution to the equation of motion

$$\ddot{\phi} + \frac{3\dot{\phi}}{t} = \frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{3}$$

with boundary conditions

$$\lim_{t \to +\infty} \phi(t) = \phi_{\mathrm{fv}} \qquad \dot{\phi}(0) = 0. \tag{4}$$

According to Eq. (3), the scalar field is subjected to friction and evolves as a particle moving in the potential $-V(\phi)$. The bounce thus may be determined as the critical trajectory which separates undershoot trajectories (the scalar field does not quite reach the false vacuum, inverting its velocity somewhere between ϕ_{fv} and ϕ_{top}) from overshoot ones (the scalar field reaches the false vacuum with finite velocity).

We now derive the asymptotic behaviour of the bounce for the theory in Eq. (2). Let us consider a generic undershoot trajectory and look for an approximate solution for large $t \leq t^*$, where t^* marks the smaller radius at which the scalar field velocity changes

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[1] Despite this result seems trivial, as there is no potential barrier through which the scalar field can tunnel, early studies of the vacuum decay phenomenon actually focused on tunneling without barriers [16]. Scalar field decay in a quartic potential with negative coupling is an example of this behaviour.

sign ($\dot{\phi}(t^*) = 0$). In order to find the asymptotic bounce, we then take the limit $t^* \to \infty$. The right-hand side of Eq. (3), expanded around t^*, reads (the subscript $_*$ indicates quantities evaluated at t^*)

$$\frac{dV}{d\phi} = \left(\frac{dV}{d\phi}\right)_* + \sum_{n \geq 2} f_{n*} \frac{(t - t^*)^n}{n!} \tag{5}$$

where the general form of the coefficients f_{n*} is reported in Appendix A. We require that every f_n is such that the approximation

$$\frac{dV}{d\phi} \approx \left(\frac{dV}{d\phi}\right)_* \tag{6}$$

holds when t^* is large.[2] As explained in Appendix A, this means that we should require

$$\left(\frac{d^j V}{d\phi^j}\right)_* \ddot{\phi}_*^{j-2} t^{*2j-2} \ll 1 \tag{7}$$

for $j \geq 2$. Under these conditions, Eq. (3) becomes

$$\ddot{\phi} + \frac{3\dot{\phi}}{t} = \left(\frac{dV}{d\phi}\right)_* \tag{8}$$

and the solution reads

$$\phi(t) = \phi_* - \left(\frac{dV}{d\phi}\right)_* \frac{t^{*2}}{4} + \left(\frac{dV}{d\phi}\right)_* \frac{t^2}{8} + \left(\frac{dV}{d\phi}\right)_* \frac{t^{*4}}{8t^2}, \tag{9}$$

where we used $\phi(t_*) = \phi_*$ and $\dot{\phi}(t_*) = 0$ to fix the two integration constants. Taking the limit $\phi_* \to 0$ and $t^* \to +\infty$, we find the asymptotic behaviour of the bounce

$$\lim_{\substack{\phi_* \to 0 \\ t^* \to +\infty}} \dot{\phi}(t) = -\frac{C_0}{t^3}, \quad \lim_{\substack{\phi_* \to 0 \\ t^* \to +\infty}} \left(\frac{dV}{d\phi}\right)_* = \frac{4C_0}{t^{*4}},$$

$$\lim_{\substack{\phi_* \to 0 \\ t^* \to +\infty}} \phi(t) = \frac{C_0}{2t^2}, \tag{10}$$

from sufficiently large t up to $t \to +\infty$. The constant C_0 is determined by the second limit in Eqs. (10). As

$$\lim_{t^* \to +\infty} \ddot{\phi}_* t^{*4} = 4C_0 \tag{11}$$

we should also impose the conditions

$$\left(\frac{d^2 V}{d\phi^2}\right)_* t^{*2} \ll 1, \quad 4C_0 \left(\frac{d^3 V}{d\phi^3}\right)_* \ll 1, \tag{12}$$

to guarantee that the constraint Eq. (7) holds, implying that the scalar field should be massless with small cubic self-interactions. By using Eq. (10), it is trivial to verify that Condition 2. holds (while Condition 1. does not apply since we have only one equation of motion). The asymptotic t^{-2} behaviour of the scalar field near the bounce in Eq. (10) has been already observed by [32] in the case of Higgs decay with Einstein–Hilbert gravity.

A massive but light scalar field satisfies Eq. (10) for $t^* \ll m^{-1}$, while for $t^* \gg m^{-1}$ instead it should be proportional to e^{-mt} [33]. The proportionality constant may be determined with our method, but a finite mass makes calculations much more involved, as Eq. (12) suggests that the potential is important at each order in the Taylor expansion.

2.2 Asymptotic bounce with Einstein–Hilbert gravity

The above results can be readily generalized in the case of a single scalar field theory with Einstein–Hilbert gravity, when the false vacuum geometry is flat. The Euclidean action is

$$S = \int d^4 x \sqrt{g} \left[-\frac{M_{\rm P}^2 R}{2} + \frac{1}{2}(\partial\phi)^2 + V(\phi) \right]. \tag{13}$$

If the line element is $O(4)$-symmetric (in analogy with the flat space case Eq. (2))

$$ds^2 = dt^2 + \rho(t)^2 d\Omega_3^2, \tag{14}$$

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[2] We can formulate this condition also as $t - t^* \sim -At^*$ with A of order unity.

Equation (15) becomes

$$S = 2\pi^2 \int_0^{+\infty} dt \rho(t)^4 \left[-\frac{M_P^2 R}{2} + \frac{\dot{\phi}^2}{2} + V(\phi) \right].$$ (15)

The equations of motion are

$$\ddot{\phi} + 3\frac{\dot{\rho}\,\dot{\phi}}{\rho} = \frac{dV}{d\phi},$$ (16)

$$\dot{\rho}^2 = 1 + \frac{\rho^2}{3M_P^2}\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right).$$ (17)

As $V(\phi_{\text{fv}}) = 0$, we have that Eq. (17) yields

$$\rho(t) \approx t + \text{higher orders}$$ (18)

near the bounce at large times, thus readily giving Eq. (10) for a single scalar field theory with Einstein–Hilbert gravity. Note that Eq. (10) along with Eq. (18), together with Eq. (17), proves that Condition 1. holds. Moreover, the action (15), computed on the asymptotic bounce, is a convergent integral at $t \to +\infty$.

2.3 Examples

We now test our findings in two examples:[3]

- the Higgs potential

$$V(\phi) = \frac{\lambda(\phi)}{4}\phi^4$$ (19)

  where

$$\ell(\phi) = \lambda^* + \alpha' \ln(\phi)^2 + \beta \ln(\phi)^4$$ (20)

  and $\lambda^* = -0.0013$, $\alpha' = 1.4 \times 10^{-5}$, $\beta = 6.3 \times 10^{-8}$ (see [34]);
- a polynomial potential with vanishing quadratic term

$$V(\phi) = \alpha_1 \phi^5 + \alpha_2 \phi^4 + \alpha_3 \phi^3$$ (21)

  where we choose $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 10^{-6}$.

One can easily prove that Eq. (7) is satisfied for $\phi_* \to 0$ in the Higgs case for all α', β, λ^* and for $\alpha_3 C_0 \ll 1$ for the polynomial potential. In both cases, we compare t^* as a function of ϕ_* as given by our theoretical prediction Eq. (10) (red line in Fig. 2) with a numerical evaluation (black dots) and found good agreement among the two. The green line instead marks the position of the minimum for the numerical calculation, which corresponds to the maximum of $\frac{dV}{d\phi}$ in the Higgs case. We took C_0s to be the ones determined numerically with the method described in Sec. 5 and reported in Table 1.

## 3 The asymptotic bounce in modified gravity

As mentioned in Introduction, the decay of a metastable state in single scalar field theories with modified gravity may be hindered by the absence of a bounce with finite on-shell action. In this section, we consider Einstein–Hilbert gravity, a non-minimal coupling $\xi\phi^2 R$, and a quadratic Ricci scalar αR^2 and test for Conditions 1. and 2. in each case. These terms are usually required by renormalizability in quantized field theories on a gravitational (classical) background [26–28]. The Euclidean action is (the line element is again given by Eq. (14))

$$S = 2\pi^2 \int_0^{+\infty} dt \rho(t)^3 \left(\frac{\dot{\phi}^2}{2} + V(\phi) - \frac{M_P^2}{2}R - \frac{\xi}{2}\phi^2 R + \frac{\alpha}{36}R^2 \right)$$

where

$$R = \frac{-6(\dot{\rho}^2 - 1 + \rho\ddot{\rho})}{\rho^2}.$$ (22)

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[3] The mass unit is $G = (M_P^2 8\pi)^{-1} = 1$.
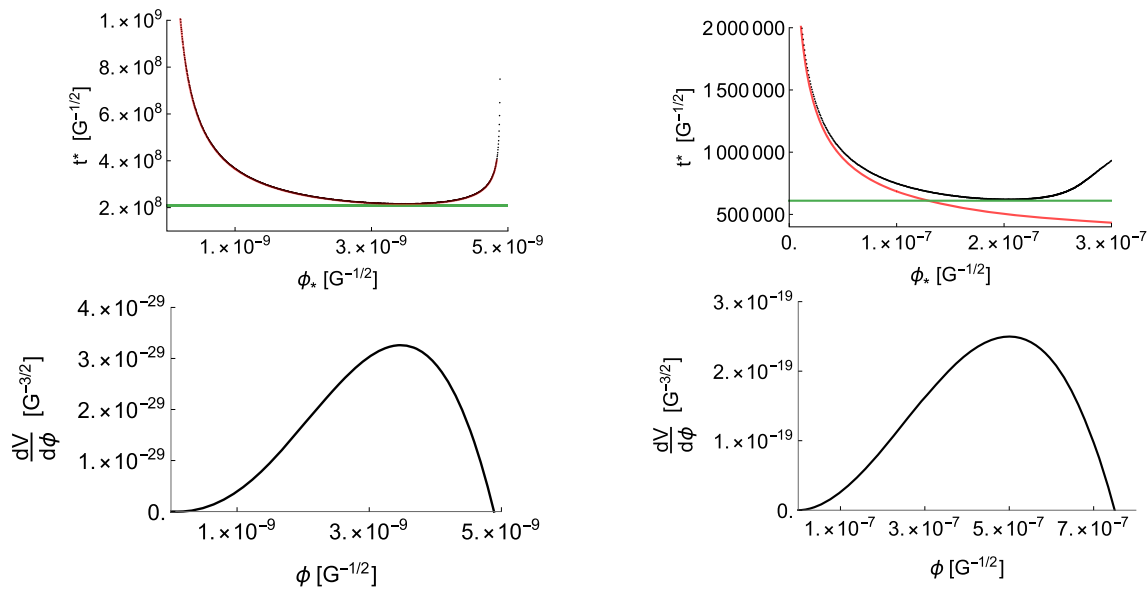
Fig. 2 Top: t^* as a function of ϕ_* for potentials Eq. (19) (on the left) and Eq. (21) (on the right), in the vicinity of the bounce ($\phi_* \to 0$). Bottom: $V'(\phi)$ as a function of ϕ

Table 1 On-shell action computed with the asymptotic bounce cut-off with the shooting method (*sm* in the Table) one

	ϕ_0	$\frac{|\Delta S|}{S_{sm}}$	S	C_0	$\bar{t}$	t^*	$\frac{(S_J - S_E)}{S_J}$
Higgs	0.071	10^{-4}	2063.3	17.6×10^3	10^3	10^{10}	$\times$
Polynomial $\alpha_3 = 10^{-6}$	24×10^{-4}	10^{-4}	6.5961	1617	10^4	10^6	$\times$
Polynomial $\alpha_3 = 10^{-5}$	76×10^{-4}	10^{-5}	6.6293	537	10^3	10^6	$\times$
Polynomial $\alpha_3 = 10^{-4}$	23×10^{-3}	10^{-3}	6.7442	172	10^3	10^5	$\times$
Polynomial $\alpha_3 = 10^{-3}$	68×10^{-3}	10^{-3}	7.1529	59.0	10^2	10^4	$\times$
Polynomial $\alpha_3 = 10^{-2}$	18×10^{-2}	10^{-3}	9.0185	23.3	10^2	10^4	$\times$
Higgs $+ 0.1\phi^2 R/2$	154×10^{-3}	10^{-5}	2049	8160	10^5	10^8	5×10^{-5}
Higgs $+ \phi^2 R/2$	17.7×10^{-3}	$\times 10^{-5}$	2094	7.04×10^5	10^5	10^{10}	3×10^{-3}
Higgs $+ 10\phi^2 R/2$	2.00×10^{-3}	10^{-3}	2140	6.42×10^5	10^6	10^{10}	3×10^{-2}

The initial condition ϕ_0, the on-shell action computed with the asymptotic bounce cut-off S and its relative deviation with respect to the shooting method result are reported, along with C_0 and the order of magnitude of $\bar{t}$ and t^*

The equations of motion are

$$\dot{\rho}^2 = 1 + \rho^2 \frac{\frac{\dot{\phi}^2}{2} - V(\phi) + \frac{\alpha}{36}R^2 + \left(\frac{\alpha}{3}\dot{R} - 6\xi\,\phi\,\dot{\phi}\right)\frac{\dot{\rho}}{\rho}}{3\left(M_P^2 + \xi\phi^2 - \frac{\alpha}{9}R\right)}, \tag{23}$$

$$\ddot{\phi} + 3\frac{\dot{\rho}\,\dot{\phi}}{\rho} = \frac{dV}{d\phi} - \xi\phi R. \tag{24}$$

In addition, we consider the trace of the Einstein equation

$$0 = -\left[3M_P^2 + 3\xi(1 + 6\xi)\phi^2 + \alpha\square\right]R + 3\dot{\phi}^2(1 + 6\xi) + 12V(\phi) + 18\xi\phi\frac{dV}{d\phi}. \tag{25}$$

We first consider the non-minimal coupling case ($\xi \neq 0$, $M_P \neq 0$ and $\alpha = 0$), which has been already extensively analysed, in particular concerning Higgs decay [24,35–37]. No obstruction according to Conditions 1. and 2. has been found so far; thus, the same should hold for generic massless fields, since, as we have demonstrated, the asymptotic bounce is fairly independent of the potential. Still, we should verify whether Eq. (7) is sufficient for Eq. (10) to hold. Moreover, the asymptotic bounce itself may differ from the $\xi = 0$ case. We will turn then to quadratic gravity.

3.1 Non-minimal coupling only

As $\alpha = 0$, ϕ is the only propagating scalar degree of freedom, and Eq. (24) depends on gravity both through the friction term and the Ricci scalar. The former is approximately given by Eq. (18) near the bounce at large times. Using Eqs. (25) in (24) and expanding around the false vacuum ($\phi_{\mathrm{fv}} = 0$), we get

$$\ddot{\phi} + 3\frac{\dot{\phi}}{t} = \frac{\mathrm{d}V}{\mathrm{d}\phi} - \frac{\xi}{M_{\mathrm{P}}^2}\phi\dot{\phi}^2(1 + 6\xi). \tag{26}$$

Now we expand Eq. (26) around the turning point t^*, evaluate it for $t \leq t^*$, take the large t^* limit and see under which conditions the zeroth-order term dominates the Taylor expansion, and thus, when Eq. (10) holds. Barring numerical cancellations,[4] we can write

$$\sum_{n=0}^{+\infty} (\phi\dot{\phi}^2)_* \frac{(t - t^*)^n}{n!} \approx \sum_{n=2}^{+\infty} (\phi^3)_* \frac{(t - t^*)^{n-2}}{n!} \tag{27}$$

apart from numerical factors. This gives

$$\ddot{\phi} + 3\frac{\dot{\phi}}{t} \approx \sum_{n=0}^{+\infty} \left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)_* \frac{(t - t^*)^n}{n!} - \frac{\xi(1 + 6\xi)}{M_{\mathrm{P}}^2} \sum_{n=2}^{+\infty} \left(\frac{\phi^3}{t^{*2}}\right)_* \frac{(t - t^*)^n}{n!} \tag{28}$$

if we take $t - t^* \approx -At^*$, where A is a constant of order unity. As quartic interactions always satisfy Eq. (7), we conclude that such term does not give an appreciable contribution to the Taylor expansion at large times on the bounce. Thus, if Eq. (12) holds on $V(\phi)$, we can safely approximate the potential as Eq. (6), and thus, Eq. (10) holds. Notice that taking Eq. (10) and $\rho(t) \approx t$ in Eq. (23) gives consistently $\dot{\rho} \approx 1$, and the Lagrangian decays sufficiently fast so that its integral Eq. (22) converges for $t \to \infty$. Thus, as expected, no obstructions to decay are found in this case. The same calculations may be carried out in the $\phi_{\mathrm{fv}} \neq 0$ case and lead to analogous results, with a shifted integration constant

$$\lim_{\substack{\phi_* \to 0 \\ t^* \to +\infty}} \left(1 - \frac{6\xi^2\phi_{\mathrm{fv}}^2}{M_{\mathrm{P}}^2 + \xi(1 + 6\xi\phi_{\mathrm{fv}}^2)}\right)\left(\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)_* = \frac{4C_0}{t^{*4}}. \tag{29}$$

3.2 Quadratic gravity only

We now set $\xi = 0$, $\alpha \neq 0$. The scalar field equation of motion depends on gravity only through the friction term, while the trace equation Eq. (25) gives the dynamics of R when subjected to a scalar field source, given by the trace of the stress-energy tensor. In this case, we have two propagating degrees of freedom, the scalar field and the Ricci scalar. As we have coupled equations of motion, it seems that we cannot use the same argument as above, since there is no clear undershoot/overshoot distinction in this case. However, one finds that the scalar field equation of motion actually decouples near the bounce at all times. In fact, if Eq. (25) holds at some time t, we can determine R as given by Eq. (25) with

$$\Box \approx \frac{\mathrm{d}t^2}{\mathrm{d}t^2} + \frac{3}{t}\frac{\mathrm{d}}{\mathrm{d}t}. \tag{30}$$

The Ricci scalar depends only on deviations from the flat space solution (see Eq. (22)), and so, if small at some time, it should remain as such: in order to trigger large deviations from flat space in R, they should first appear in Eq. (30), but this means that higher-order deviations from flat space (in R) are determined by lower-order deviations (in the friction term), which is impossible. Thus, we can safely take Eq. (18) to hold in Eq. (23) at all times. To determine the Ricci scalar asymptotic bounce, we solve Eqs. (25) with (30). We find $R = R_{\mathrm{hom}} + R_{\mathrm{ps}}$ with

$$R_{\mathrm{hom}} = \epsilon^2 C_1 \frac{J_1\left(\epsilon t'\right)}{t},$$

$$R_{\mathrm{ps}} = -\epsilon^2 \frac{\pi}{2} \frac{J_1\left(\epsilon t'\right)}{t} \int^t F(y)Y_1\left(\epsilon^3 y'\right) y^2 \mathrm{d}y$$

$$+ \epsilon^2 \frac{\pi}{2} \frac{Y_1\left(\epsilon^3 t'\right)}{t} \int^t F(y)J_1\left(\epsilon y'\right) y^2 \mathrm{d}y. \tag{31}$$

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[4] Here we are using the fact that the second derivative of ϕ^3 contains $\phi\dot{\phi}^2$. We can trade one for the other because our calculations were carried out without accounting for numerical factors, and thus, they are independent of possible numerical cancellation among different terms generated by ϕ^3 in the Taylor expansion.

Here, J, Y are Bessel function of the first kind,

$$A = \sqrt{\frac{3M_{\mathrm{P}}^2}{|\alpha|}}, \quad t' = A(t+a), \quad \begin{cases} \epsilon = 1 & \alpha > 0 \\ \epsilon = i & \alpha < 0, \end{cases} \tag{32}$$

$C_{1,2}$ are constants, and the function $F(t)$ is given by

$$F(t) = \frac{3\dot{\phi}(t)^2}{\alpha} + \frac{12}{\alpha} V(\phi(t)). \tag{33}$$

Using Eq. (10), we find that the Ricci scalar can be approximated as

$$R = \epsilon^2 (C_1 + \tilde{C}_1) \frac{J_1(\epsilon At)}{t} + \tilde{C}_2 \frac{Y_1(\epsilon^3 At)}{t} + O(t^{-6}) \tag{34}$$

where

$$\tilde{C}_1 = -\frac{\pi}{2} \epsilon^2 \int_0^{+\infty} F(y) Y_1(\epsilon^3 y') y^2 \mathrm{d}y \tag{35}$$

$$\tilde{C}_2 = \frac{\pi}{2} \epsilon^2 \int_0^{+\infty} F(y) J_1(\epsilon y') y^2 \mathrm{d}y. \tag{36}$$

Higher-order terms $O(t^{-6})$ are computed by using the asymptotic forms of the Bessel functions J, Y for large arguments [38]

$$J_1^\infty(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{3\pi}{4}\right)$$

$$Y_1^\infty(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{3\pi}{4}\right). \tag{37}$$

The Ricci scalar is dominated by the first two terms in Eq. (34) unless both $C_1 + \tilde{C}_1$ and $\tilde{C}_2$ vanish. However, one can prove that $\tilde{C}_2$ is always non-vanishing. In fact, we have

$$\int_0^{+\infty} F(t) J_1(t') t^2 \mathrm{d}t < \int_0^{+\infty} t' t^2 F(t) \mathrm{d}t \tag{38}$$

and the right-hand side is negative definite if

$$\int_0^{+\infty} t^3 \left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) \mathrm{d}t < -\int_0^{+\infty} t^3 V(\phi) \mathrm{d}t. \tag{39}$$

To prove that, we consider an off-shell scalar field profile which corresponds to the bounce of the same theory without gravitational interaction. Then, the left-hand side of Eq. (39) is the bounce action of said theory, in the approximation of small gravitational backreaction. As computed in [23], this profile may be used to determine the on-shell action of the theory with also an Einstein–Hilbert term (i.e. the right-hand side).[5] One finds [23] that it is always larger than the flat space one, making $\tilde{C}_2 < 0$ for $\alpha > 0$ and $\tilde{C}_2 > 0$ for $\alpha < 0$.

Finally, using Eqs. (37) in (34) shows that R diverges for $t \to +\infty$ for $\alpha < 0$, thus implying that Condition 1. is violated. For $\alpha > 0$, instead, R undergoes damped oscillations around the fixed point $R = 0$. Such oscillations are, to the leading order, the same of a free massive scalar field around its minimum, and they make the action undefined in the upper limit of integration, thereby violating Condition 2. It seems thus that adding a gravitational degree of freedom does not lead to vacuum decay from the state $\phi = \phi_{\mathrm{fv}}$, $R = 0$, independently on the value of α. This is related to the boundary value nature of the problem: it makes the bounce solutions non-perturbative as new interactions or additional scalar fields are turned on.[6] In the present case, the stress-energy

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[5] Actually, the approach undertaken in [23] is slightly different, and it considers the flat-space bounce as a background for a perturbative expansion, in order to determine the action of a scalar field theory with an Einstein–Hilbert term as

$$S = S_0 + \frac{A}{M_P^2} + O(M_P^{-4})$$

where S_0 is the on-shell action of the flat space theory and A is some real constant. Some concerns have been raised in the literature [36,39] about this point, which is furtherly addressed in Appendix C.

[6] This is analogous to what happens to massive scalar fields with quartic self-interactions. If the mass vanishes, there is a bounce given by

$$\phi(t) = \sqrt{\frac{2}{\lambda}} \frac{2\mathcal{R}}{t^2 + \mathcal{R}^2}$$

where $\mathcal{R}$ is the bounce radius $\phi_0 = 2\mathcal{R}$, but it disappears as soon as the mass is turned on.

tensor naturally couples gravity and matter, making the new (gravitational) degree of freedom necessarily interacting with the scalar field. One may then ask whether the obstruction arises merely by the addition of the new degree of freedom or, instead, also by non-derivative terms in Eq. (25). If the former holds, one might guess that a bounce is forbidden also in theories with a scalar field, an Einstein–Hilbert term and a generic coupling αR^n, with integer n, as they also have two degrees of freedom. Equation (25) in that case is

$$\alpha n(n-1)R^{n-3}\left(R\ddot{R} + \frac{3\dot{\rho}}{\rho}R\dot{R} + (n-2)\dot{R}^2\right) = -6M_p^2 R + 6\dot{\phi}^2 + 24V(\phi). \tag{40}$$

Near the bounce at large times one has

$$\left(\ddot{R} + \frac{3\dot{\rho}}{\rho}\dot{R}\right)R^{n-2} \ll R \tag{41}$$

for $n > 2$ and $\dot{R}^2 R^{n-3} \ll R$ for $n > 3$. Then, for $n > 3$, R is approximately given by

$$R \approx \frac{\dot{\phi}^2 + 4V(\phi)}{M_P^2} \tag{42}$$

which does not allow to exclude in principle a bounce. Then, obstructions related to Conditions 1. and 2. that arise in the $n = 2$ case seem not to be merely related to the presence of derivatives of R in the Einstein trace equation. For $n = 3$ instead one has

$$\alpha \dot{R}^2 = -M_p^2 R + \dot{\phi}^2 + 4V(\phi) \tag{43}$$

near the bounce at large times. Now one would need to determine which terms dominate Eq. (43) and thus the asymptotic bounce of R. Our calculations in Appendix rely though on a specific form of the equation of motion. Moreover, we weren't able to find a full solution to Eq. (40). This makes a solution not feasible with our present means.

3.3 Non-minimal coupling and quadratic gravity

We have seen that including a squared Ricci scalar results in a bounce action that is ill-defined for $\alpha > 0$. Thus, we expect that setting $\xi \neq 0$, along with $\alpha \neq 0$ and $M_P^2 \neq 0$, does not change much the situation and we now show that this is the case. Equations (24) and (25) are coupled, so we need first to disentangle them to read the scalar field asymptotic bounce and use it to find the Ricci scalar. To do that, we use Eqs. (24) in (25) to replace non-derivative terms in R. We find

$$R = C_1 + \int^t t'^{-3} \int^{t'} F(t'')t''^3 dt' dt'' \tag{44}$$

with

$$\alpha F(t) = 3\dot{\phi}^2(1+6\xi) + 12V(\phi) + 18\xi\phi\frac{dV}{d\phi}$$
$$- 3\xi\left((1+6\xi)\phi + \frac{M_P^2}{\phi}\right)\left(\ddot{\phi} + 3\frac{\dot{\phi}}{t} - \frac{dV}{d\phi}\right). \tag{45}$$

Defining $f(t)$ as

$$\int^t t'^{-3}\int^{t'} F(t'')t''^3 dt' dt'' \equiv f(t)F(t) \tag{46}$$

and replacing R in Eqs. (24) with (44), we find

$$\ddot{\phi}F_1(\phi,t) + \frac{3\dot{\phi}}{t}F_2(t) = \frac{dV}{d\phi}F_3(t) + 12\xi\phi V(\phi)f(t) \tag{47}$$

with

$$F_1(\phi,t) = 1 + \frac{3\xi}{\alpha}\phi f(t)\left((1+6\xi)\phi + \frac{M_P^2}{\phi}\right) \approx \frac{3\xi}{\alpha}M_P^2 f(t) + 1 \tag{48}$$

$$F_2(\phi,t) = 1 - \frac{3\xi}{\alpha}(1+6\xi)\phi\dot{\phi}tf(t) + \frac{3\xi}{\alpha}\phi f(t)\left((1+6\xi)\phi + \frac{M_P^2}{\phi}\right) \approx \frac{3\xi}{\alpha}M_P^2 f(t) + 1 \tag{49}$$

$$F_3(\phi,t) = 1 + \frac{18\xi^2}{\alpha}\phi^2 f(t) + \frac{3\xi}{\alpha}\phi f(t)\left((1+6\xi)\phi + \frac{M_P^2}{\phi}\right) \approx \frac{3\xi}{\alpha}M_P^2 f(t) + 1 \tag{50}$$

near the bounce at large times. Therefore, the scalar field equation of motion may be approximated as:

$$\ddot{\phi} + \frac{3\dot{\phi}}{t} = \frac{dV}{d\phi} \tag{51}$$

for small ϕ, as if the non-minimal coupling was negligible. Thus, we find again that $\phi(t) \propto t^{-2}$, which leads to the same obstructions to vacuum decay as in the previous case.

4 Scale-invariant gravity

In this section, we consider a scale-invariant gravitational sector (that is we set $M_P = 0$), leaving $V(\phi)$ as the only possible source for mass scales in the theory. From a cosmological perspective, an extremely rich phenomenology arises if also the scalar field sector is scale-invariant [40–42]; therefore, it is important to study the stability of these configurations against vacuum decay. As we will see, our findings regarding the asymptotic bounce consistently differ from the ones found in the previous section.

4.1 Non-minimal coupling

We begin with the simplest case, when also $\alpha = 0$. By combining Eq. (25) with Eq. (24) to eliminate R, we find an analogous equation to Eq. (3), namely

$$\ddot{u} + 3\frac{\dot{\rho}\,\dot{u}}{\rho} = \frac{dW}{du} \equiv \frac{4}{1+6\xi}\left(u\frac{dV}{du} - 2V(u)\right), \tag{52}$$

where $u \equiv \phi^2$. As for Eq. (3), if u is massless with sufficiently small cubic interactions with respect to the potential $W(u)$, we have that the asymptotic behaviour is as in Eq. (10) and thus

$$\phi(t) = \frac{\sqrt{C_0}}{t}. \tag{53}$$

Plugging this solution in Eq. (23), we see that there is an inconsistency in boundary conditions for gravity, as using $\rho(t) = t$ and Eq. (10) we find at leading order in the large t limit

$$\dot{\rho}^2 = 3 + \frac{1}{3\xi} - \frac{V(\phi)t^4}{3\xi C_0} + O\left(t^{-1}\right). \tag{54}$$

Thus, there is no bounce if the false vacuum lives on flat space, unless also the scalar field sector is scale-invariant, with potential[7]

$$V \simeq \frac{1}{C_0}\left(6\xi + 1\right)\phi^4. \tag{55}$$

This situation changes if the scalar field has a non-vanishing false vacuum value. In fact, if the potential is such that $V(\phi_{fv}) = 0$, $\phi_{fv} \neq 0$ and $V(u)$ satisfies Eq. (12), we have that, on the bounce at large times,

$$\phi(t) \approx \sqrt{\phi_{fv}^2 + \frac{C_0}{2t^2}}. \tag{56}$$

From the discussion above, we expect that the asymptotic bounce is reached only in a narrow region around ϕ_{fv} (otherwise we would get again Eq. (54)), namely

$$\phi_{fv}^2 \gg \frac{C_0}{2t^2} \tag{57}$$

and thus we can replace Eq. (56) with

$$\phi(t) \approx \phi_{fv} + \frac{C_0}{4\phi_{fv}t^2}. \tag{58}$$

Using Eqs. (24), (58) and Eq. (18), we get

$$\dot{\rho}(t) = 1 + O\left(t^{-4}\right). \tag{59}$$

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[7] Our analysis actually excludes scale-invariant potentials as there is no undershoot/overshoot distinction in that case. The asymptotic bounce (and thus possible violations of conditions 1. and 2.) should thus be found with other methods.

Equation (57) implies that, if ϕ_{fv} is much smaller than the potential barrier width ℓ[8] the scalar field has not yet reached the asymptotic bounce regime when $V(\phi) > 0$. This possibly makes $\dot{\rho}$ vanish, if additionally $V(\phi) \gg \dot{\phi}^2$, making ϕ self-accelerated.[9] Imposing $\dot{\rho} = 0$ for $\phi \approx \phi_{\text{fv}} + \ell$ and using $V(\phi) \gg \dot{\phi}^2$ gives

$$\rho_\ell^2 = \frac{3\xi(\phi_{\text{fv}} + \ell)^2}{V(\phi_{\text{fv}} + \ell)}. \tag{60}$$

This behaviour is avoided if $\rho_\ell > \bar{t}$, where $\bar{t}$ indicates the time by which

$$\frac{\dot{\phi}^2}{2} = V(\phi) \tag{61}$$

on the asymptotic bounce, which in turn gives

$$C_0 = -2\phi_{\text{fv}}\bar{t}^3\sqrt{2V(B\phi_{\text{fv}})} \tag{62}$$

where B is some constant of order unity. Setting also

$$u_{\text{fv}} \approx \frac{C_0}{2\bar{t}^2} \tag{63}$$

we find the condition on ϕ_{fv}, ℓ and ξ in order for the bounce to exist

$$\frac{6\xi(\phi_{\text{fv}} + \ell)^2 V(B\phi_{\text{fv}})}{\phi_{\text{fv}}^2 V(\phi_{\text{fv}} + \ell)} + 1 < 0. \tag{64}$$

This condition is marginally satisfied for $\phi_{\text{fv}} \approx \ell$ and $\xi \approx \dfrac{1}{24}$. Decreasing ξ shrinks the range of ϕ_{fv} for which a bounce is forbidden. Moreover, Eq. (63) underestimates the actual $\bar{t}$, thus having the same effect. We tested this result taking as $V(\phi)$ the Higgs potential Eq. (19) with $\phi \to \phi - \phi_{\text{fv}}$. We computed numerically the values of ϕ_{fv} for which $F(\phi_{\text{fv}}, \xi) = 0$, with $F(\phi_{\text{fv}}, \xi)$ defined as

$$F(\phi_{\text{fv}}, \xi) \equiv \frac{6\xi(\phi_{\text{fv}} + \ell)^2 V(B\phi_{\text{fv}})}{\phi_{\text{fv}}^2 V(\phi_{\text{fv}} + \ell)} + 1 \tag{65}$$

and reported the result in Fig. 3 (on the right) as a function of ξ. We found that the zero of $F(\phi_{\text{fv}}, \xi)$ decreases for increasing ξ, and they lie approximately at $\phi_{\text{fv}} \approx \ell \approx 5 \times 10^{-9}G^{-1/2}$. We also found the bounce numerically, varying $\phi_{\text{fv}} \in [10^{-8}, 10^{-3}]$ and $\xi \in [0.01, 10]$. The on-shell action is reported in Fig. 3, on the left. We found that the action sharply increases for $\phi_{\text{fv}} \geq 10^{-8}$, and there is no bounce for lower ϕ_{fv}. The value of ϕ_{fv} for which the bounce disappears increases for increasing ξ. They are larger than the prediction reported in Fig. 3 as $\bar{t}$ is actually an underestimation of the matching time.

These considerations suggest that a non-minimally coupled Higgs field has no bounce, since it has a vacuum expectation value at $v = 246\,\text{GeV} \approx 10^{-17}G^{1/2} \ll \ell$. Actually, v is generated by the interplay of a mass term and the quartic interaction: the Higgs mass affects the asymptotic bounce and in principle changes these results. Nonetheless, much of our reasoning is focused on the behaviour near the potential barrier, which is unaffected by the mass term in a large range of field values. Still, the condition in Eq. (64) depends on the scalar field behaviour near the false vacuum, and thus, it should be reconsidered in the massive scalar field case. This will be addressed in future work.

4.2 Quadratic gravity

We consider here the effect of quadratic gravitational terms on the bounce, by setting $M_{\text{P}} = 0$, $\xi = 0$, $\alpha \neq 0$. As in Sect. 3, the scalar field on the bounce at large times is independent of R, and so, if Eq. (7) holds, then we have Eq. (10). The solution to the trace equation when the scalar field is given by Eqs. (10) and (18) holds, is

$$R(t) = C_1 + \frac{C_2}{2t^2} + \frac{3C_0^2}{8t^4\alpha} + 12\int^t \frac{dt'}{t'^3}\int^{t'} dt''t''^3 V\left(\frac{C_0}{2t^2}\right) \tag{66}$$

with C_1 and C_2 are real constant. Using Eqs. (66), (18) and Eq. (10) in Eq. (23), we find $\dot{\rho} \neq 1$ at large times on the bounce. This result is independent of the values of C_1 and C_2. Thus, there is no bounce for scale-invariant gravity with a quadratic Ricci scalar and flat Euclidean spacetime in the false vacuum.

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[8] The potential barrier width is defined here as the range of ϕ such that $V(\phi) > 0$.

[9] Having $\dot{\rho} = 0$ somewhere on the bounce forbids to satisfy the requirement $\dot{\rho} > 0$ at large t. In fact, this means that values for t such that $\dot{\rho} = 0$ should come in pairs but, after the first zero, one has

$$\frac{d}{dt}\left(\frac{\dot{\phi}^2}{2}\right) = \frac{dV(\phi)}{dt} - 3\frac{\dot{\rho}\dot{\phi}^2}{\rho} > \frac{dV(\phi)}{dt}$$

and thus one cannot have another one.
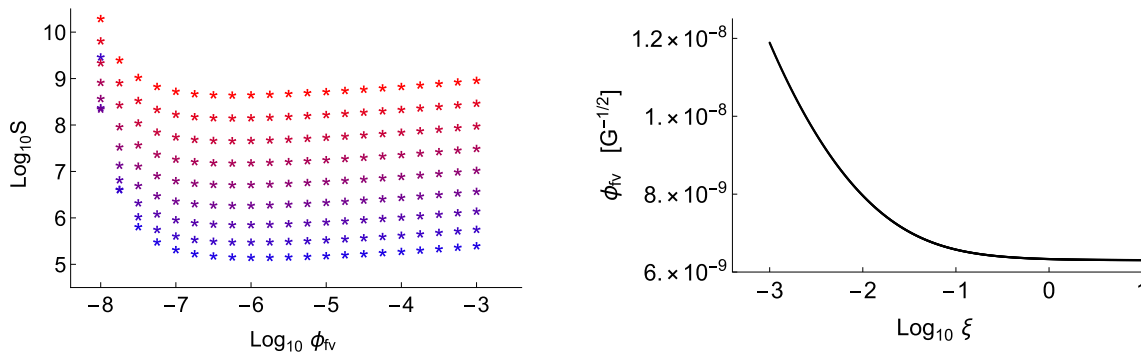
Fig. 3 Left: numerical bounce action as a function of ϕ_{fv} (in units $G = 1$). The non-minimal coupling is changed from $\xi = 0.01$ (red) to $\xi = 10$ (blue). The action sharply increases for $\phi_{\text{fv}} \approx 10^{-8}$. The bounce disappears for lower values of ϕ_{fv}. Right: zeros of $F(\phi_{\text{fv}}, \xi)$ as a function of ξ

4.3 Non-minimal coupling and quadratic gravity

We consider now a theory with a scalar field non-minimally coupled to gravity and a quadratic Ricci term, namely we set $\xi \neq 0$, $\alpha \neq 0$, $M_{\text{P}} = 0$, and repeat the calculations of the $M_{\text{P}} \neq 0$ case. In order to have a finite bounce action, $f(t)$ should satisfy

$$\lim_{t \to +\infty} \phi^2 f(t) = 0 \tag{67}$$

as $F(t)$ is monotonically decreasing at large times on the bounce

$$f(t) = \frac{\int_t^{+\infty} t'^{-3} \int_{t'}^{+\infty} t''^3 F(t'')}{F(t)}$$

$$\leq \int_t^{+\infty} t'^{-3} \int_{t'}^{+\infty} t''^3 = \frac{t^2}{8}. \tag{68}$$

We find

$$F_1(\phi, t) \approx F_2(\phi, t) \approx F_3(\phi, t) \approx 1 \tag{69}$$

so again

$$\ddot{\phi} + \frac{3\dot{\phi}}{t} = \frac{\mathrm{d}V}{\mathrm{d}\phi} \tag{70}$$

for small ϕ. Thus, we expect that ϕ satisfies Eq. (10) at large times on the bounce and (from Eq. (44))

$$R = \frac{C_2}{2t^2} + \frac{3(1 + 6\xi)C_0^2}{8\alpha t^4} \tag{71}$$

assuming that the potential contributes only to higher orders in inverse powers of t. Plugging these solutions in Eq. (23), we find that Condition 1. is violated. The situation might improve by considering a non-vanishing vacuum value for the scalar field $\phi_{\text{fv}} \neq 0$. This would amount to adding a linear non-minimal coupling to gravity, ϕR, and having an Einstein–Hilbert term on the bounce at large times, given by $\frac{\xi \phi_{\text{fv}}^2}{2} R$. The first one changes the right-hand side of Eq. (24) by a multiplication constant, while we already say that Condition 2. is violated for $M_{\text{P}} \neq 0$. Thus, we expect that there is no bounce also in this case.

5 Numerical implications of the asymptotic bounce

Having the asymptotic bounce at disposal allows improving existing numerical methods and introducing some new others. In this section, the possible implications of the asymptotic bounce for the shooting method are considered, and an alternative numerical method to find the bounce is discussed.

A cut-off for the shooting method

In the shooting method, one finds the bounce numerically as the trajectory bracketed between undershoots and overshoots. This allows determining it with arbitrary precision, computational limits aside. In general, using this method implies a large range of integration, as the bounce initial condition $\phi(0)$ can be large ($O(0.1M_P)$) and/or the friction term can be very effective in slowing

down the scalar field. Moreover, one should compute the bounce with sufficient precision to get a good estimate of the on-shell action S_E. The Lagrangian must be integrated up to a cut-off, which should be carefully chosen. Knowing the asymptotic bounce allows for a different method to find the (numerical) on-shell action instead of truncation by matching the numerical bounce with the asymptotic one at some $\bar{t}$. The action may be computed as

$$S_E = S_{C,1} + S_{C,2} \tag{72}$$

with

$$S_{C,1} \equiv 2\pi^2 \int_0^{\bar{t}} \sqrt{\bar{g}}\mathcal{L}(\bar{\phi}, \bar{R})\mathrm{d}t \tag{73}$$

$$S_{C,2} \equiv 2\pi^2 \int_{\bar{t}}^{\infty} t^3 \mathcal{L}(\phi_{+\infty}, R_{+\infty})\mathrm{d}t \tag{74}$$

where $\bar{R}$ and $\bar{\phi}$ are determined numerically and $\phi_{+\infty}$, $R_{+\infty}$ are, respectively, the scalar field and the Ricci scalar as given by the asymptotic bounce, which in our case depends on an integration constant C_0. C and $\bar{t}$ may be determined by continuity as

$$\bar{t} + 2\frac{\phi(\bar{t})}{\dot{\phi}(\bar{t})} = 0 \tag{75}$$

and $C = -\bar{t}^3 \dot{\phi}(\bar{t})$. Overshoot trajectories satisfy Eq. (75) at some finite $\bar{t}$ and then

$$\phi(t) < \phi_{+\infty}(t) \qquad \text{for } t > \bar{t}. \tag{76}$$

Instead undershoot trajectories near the bounce have

$$\phi(t) > \phi_{+\infty}(t) \tag{77}$$

at sufficiently large times, and thus, there may be no $\bar{t}$ for which Eq. (75) holds. Then, $\bar{t}$ may be determined as the point of closest approach

$$3 - 2\frac{\phi(\bar{t})\ddot{\phi}(\bar{t})}{\dot{\phi}(\bar{t})^2} = 0. \tag{78}$$

The matching time as defined here separates the bounce-like behaviour of undershots and overshots from the region in which they part, and thus, it should get infinitely large on the bounce. Nonetheless, $\bar{t}$ is always finite off the bounce, and in particular $\bar{t} \leq t^*$.

The on-shell action S_C with the asymptotic bounce cut-off and the one found with the shooting method S_{sm} in scalar field theories with Einstein–Hilbert gravity and a non-minimal coupling are compared in Table 1. We found that $\bar{t} \ll t^*$ and $\bar{t}$ have a maximum on the bounce as a function of ϕ_{in} (Fig. 4). There is a small relative deviation among S_C and S_{sm}, and calculations in the Jordan and in the Einstein frame are in good agreement. In the Higgs case, C_0 roughly corresponds to the one derived from minimization of the on-shell action, with a small backreaction [23]

$$C_0 = 2 \lim_{t\to +\infty} h(t)t^2 = \frac{4\sqrt{2}}{|\lambda|}\mathcal{R} = 17.4 \times 10^5 \quad \text{with} \quad \mathcal{R} = 350.$$

An alternative numerical method

The discussion above suggests that, for every trajectory with initial conditions

$$\phi(0) = \phi_{in} \qquad \dot{\phi}(0) = 0 \tag{79}$$

that, in general, has infinite on-shell action, there is another one that is on-shell only for $t < \bar{t}$ and that has finite S_C. S_C should have a stationary point on the bounce, as Eq. (10) holds on-shell for $\phi(0) = \phi_0$. One can show that S_C has a saddle point there (see Appendix B), and thus, it is not suitable for minimization to find the bounce. Instead, by slightly changing this functional, one can turn the saddle point into a minimum (or a maximum), at least in the case of single-scalar field theories with Einstein–Hilbert gravity and a non-minimal coupling. The full calculation is reported in Appendix B. The new functional is given by

$$S_{C,1} \equiv 2\pi^2 \int_0^{\bar{t}} \bar{\rho}(t)^3 \mathcal{L}(\bar{\phi}, \bar{R})\mathrm{d}t$$
$$S_{C,2} \equiv 2\pi^2 \int_{\bar{t}}^{\infty} t^3 \left(\frac{C^2}{2t^6} + V\left(\frac{C}{2t^2}\right)\right)\mathrm{d}t \tag{80}$$

and it has a minimum on the bounce for $\xi \geq -\frac{1}{6}$ and a maximum otherwise (see Figs. 5, 6). The computational time is of the same order of magnitude of the shooting method one (see Table 2), which gives no clear advantage in using this method over the standard shooting method.

Fig. 4 Matching time $\bar{t}$ as a function of the initial condition ϕ_{in} for a single-scalar field theory with Einstein–Hilbert gravity. The maximum marks the bounce initial condition ϕ_0. Plots from red to blue correspond to theories with potential Eq. (21) (from $\alpha_3 = 10^{-6}$ to $\alpha_3 = 10^{-2}$), the green one for the Higgs theory with Einstein–Hilbert gravity (the potential is Eq. (19))
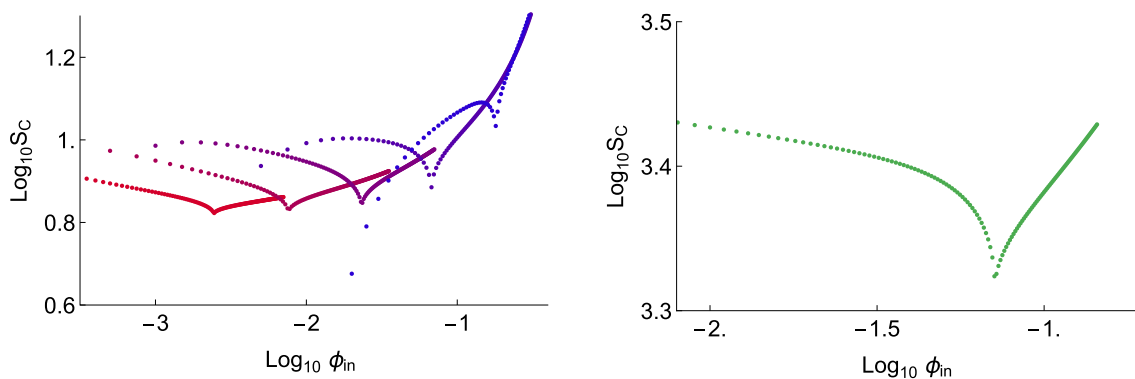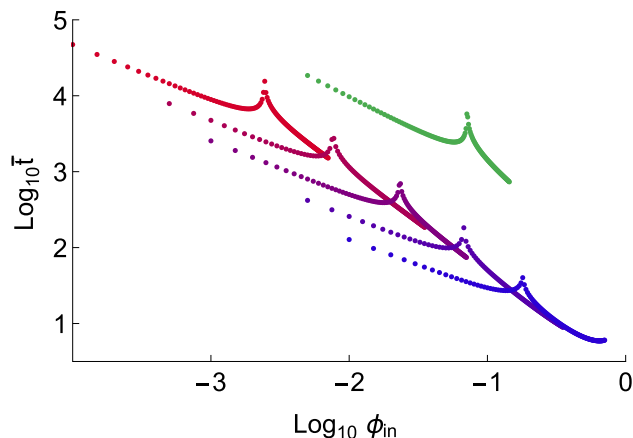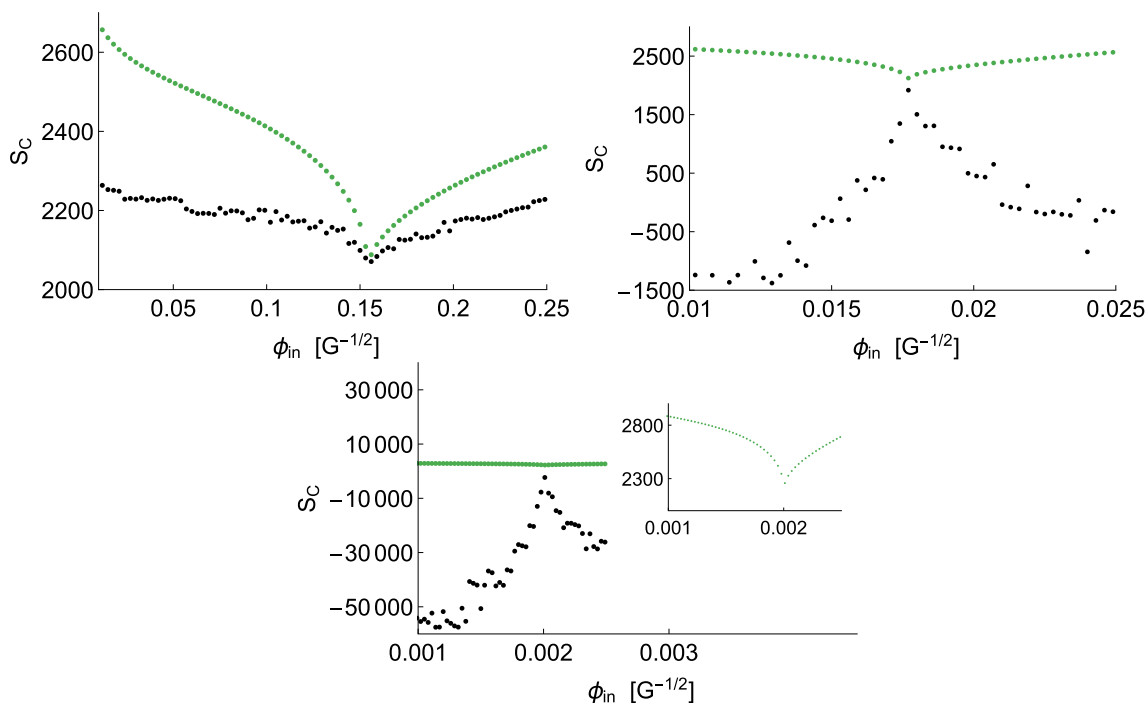
Fig. 5 S_C as a function of ϕ_{in} in units of G for a single scalar field with potential Eq. (19), Einstein–Hilbert gravity and a non-minimal coupling (top left: $\xi = 0.1$, top right: $\xi = 1$, bottom $\xi = 10$)

6 Discussion

Vacuum decay is a very important test to assess the quantum stability of a field configuration with gravity, but it is often very difficult to perform calculations. Most of these are numerical with poor analytical control. In this paper, we looked for an analytical method that potentially can overcome this issue and applied it to quadratic gravity with massless scalar fields. We used it to find the large-time behaviour of the bounce (what we called the "asymptotic bounce"), provided that some restrictions on the scalar field potential (in particular a masslessness condition) hold. We showed that this result can be extended to include modified gravity scenarios too. In this way, we can test for the finiteness of the action and the consistency of bounce boundary conditions for all fields, which we called Conditions 1. and 2. in the Introduction. In particular, we have studied well-founded modifications of gravity, which are required by renormalizability, namely the non-minimal coupling among the scalar field and the Ricci scalar R together with the R^2 term. We separately analysed the effect of such terms on the asymptotic bounce and tested for Conditions 1. and 2. for each theory. Our analysis shows that:

- In the presence of Einstein–Hilbert gravity, a bounce is allowed when a non-minimal coupling is included, while it is forbidden with a squared Ricci term;
- when the gravitational sector is scale-invariant (no Einstein–Hilbert term), a bounce is allowed when a non-minimal coupling is included if the field acquires a non-vanishing false vacuum value, which is larger than the width of the potential barrier. However, it is still forbidden when one adds a R^2 term.

We believe that our method can be extended to include more general scenarios that are certainly worth exploring. In fact, finding the asymptotic bounce for massive theories would allow us to extend the flat spacetime result of [33] to gravitational settings. This might have important implications on Higgs decay [23,24,30,32,35–37,43–58], in which the mass term is usually neglected. Moreover, vacuum decay from de Sitter spacetime is particularly important to test for Conditions 1. and 2 in the early and current Universe. Regarding the latter, we should mention that existing calculations regarding Higgs decay with a non-minimal coupling are carried out in the flat spacetime approximation in the false vacuum state (see, for example, [36,37]). Our method may provide new windows in which other decay channels (such as decay through the Hawking–Moss instantons) dominate.

Fig. 6 S_C as a function of ϕ_{in} in units of G for a single scalar field with potential Eq. (19), Einstein–Hilbert gravity and a non-minimal coupling (top left: $\xi = 0.1$, top right: $\xi = 1$, bottom $\xi = 10$)

Table 2 We report the computational time needed to find the bounce by minimization with the shooting method, in theories with a scalar field with potentials Eqs. (19) and (21) and Einstein–Hilbert gravity

	t_C	$t_{C,sm}$
Higgs	374s	134s
Polynomial $\alpha_3 = 10^{-6}$	350s	93s
Polynomial $\alpha_3 = 10^{-5}$	277s	102s
Polynomial $\alpha_3 = 10^{-4}$	276s	76s
Polynomial $\alpha_3 = 10^{-3}$	331s	113s
Polynomial $\alpha_3 = 10^{-2}$	346s	211s*

Appendix A: coefficients f_n

Here, coefficients f_n of Eq. (5) are computed. Radial derivatives of arbitrary order are denoted by an index (n), while radial derivatives of first and second orders are denoted by one dot or two dots, respectively. Derivatives of the potential with respect to the scalar field of order i are indicated as $\dfrac{d^i V}{d\phi^i}$. As f_ns are computed at ρ^*, all quantities are implicitly evaluated at the turning point (recall

that $\dot{\phi}_* = 0$). Using Eq. (3), one gets

$$\frac{\mathrm{d}^i V}{\mathrm{d}\phi^i}^{(n+1)} = \left(\frac{\mathrm{d}^{i+1} V}{\mathrm{d}\phi^{i+1}}\dot{\phi}\right)^{(n)}, \qquad \phi^{(n)} = \left(\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\dot{\phi}\right)^{(n-3)} + \sum_{i=2}^{n-1} B_i \frac{\phi^{(i)}}{\rho^{n-i}} \tag{A1}$$

where B_is are numerical factors, whose value is not relevant for the following discussion.

Using the first equation in Eq. (A1), the (n+1)-th derivative of V' may be written as

$$\frac{\mathrm{d}V}{\mathrm{d}\phi}^{(n+1)} = \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\phi^{(n+1)} + \cdots + \left(\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right)^{(n-1)}\ddot{\phi} = \ddot{\phi}\left(\left(\frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}\right)^{(n-3)}\ddot{\phi} + \cdots + \frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}\phi^{(n-1)}\right)$$
$$+ \phi^{(3)}\left(\left(\frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}\right)^{(n-4)}\ddot{\phi} + \cdots + \frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}\phi^{(n-2)}\right) + \cdots + \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\phi^{(n+1)} \tag{A2}$$

which can be further expanded using again Eq. (A1). Then, one gets

$$\frac{\mathrm{d}V}{\mathrm{d}\phi}^{(n+1)} = \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\phi^{(n+1)} + \frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}(\ddot{\phi}\phi^{(n-1)} + \phi^{(3)}\phi^{(n-2)} + \cdots + \phi^{(n+1)/2}\phi^{(n+1)/2})$$
$$+ \frac{\mathrm{d}^4 V}{\mathrm{d}\phi^4}(\ddot{\phi}^2\phi^{(n-3)} + \phi^{(3)}\ddot{\phi}\phi^{(n-4)} + \cdots + \phi^{(n+1)/3}\phi^{(n+1)/3}\phi^{(n+1)/3}) + \cdots. \tag{A3}$$

Each $\dfrac{\mathrm{d}^i V}{\mathrm{d}\phi^i}$ in Eq. (A3) is multiplied by $i - 1$ terms that are derivatives of $\ddot{\phi}$. They are of order $n + 5 - 2i$ or lower, and thus, these terms are non-vanishing only if $n + 5 - 2i > 1$. So, the highest-order derivative $\dfrac{\mathrm{d}^{\bar{i}} V}{\mathrm{d}\phi^{\bar{i}}}$ that appears in Eq. (A3) is the one satisfying $n + 5 - 2\bar{i} = 3$ for even n and $n + 5 - 2\bar{i} = 2$ for odd n. Expanding radial derivatives of $\ddot{\phi}$ in Eqs. (A3) using (A1), $V'^{(n+1)}$ may be expressed in terms of derivatives of the potential with respect to the scalar field, $\ddot{\phi} = V'_*$ and ρ^* only. From Eq. (A3), one finds that the highest-order derivative (the $\bar{i}$-th term) is multiplied only by radial derivatives of the scalar field of order 2 or 3 and thus it contributes as

$$\frac{\mathrm{d}^{\bar{i}} V}{\mathrm{d}\phi^{\bar{i}}}\frac{\ddot{\phi}^{\bar{i}-1}}{\rho} \quad \text{even } n, \qquad \frac{\mathrm{d}^{\bar{i}} V}{\mathrm{d}\phi^{\bar{i}}}\ddot{\phi}^{\bar{i}-1} \quad \text{odd } n \tag{A4}$$

to f_n.

The second-highest derivative $\bar{i} - 1$ is multiplied by radial derivatives of the scalar field of order 2, 3, 4, 5. Using Eq. (A1), derivatives of order 4 and 5 may be expressed in terms of lower derivatives. As can be seen from Eq. (A1), this results in an additional V'' contribution (numerical coefficients are omitted for simplicity)

$$\frac{\mathrm{d}^{\bar{i}-1} V}{\mathrm{d}\phi^{\bar{i}-1}}\frac{\ddot{\phi}^{\bar{i}-2}}{\rho^3}\left(1 + A_1 \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\rho^2\right) \quad \text{even } n,$$
$$\frac{\mathrm{d}^{\bar{i}-1} V}{\mathrm{d}\phi^{\bar{i}-1}}\frac{\ddot{\phi}^{\bar{i}-1}}{\rho^2}\left(1 + A_2 \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\rho^2\right) \quad \text{odd } n. \tag{A5}$$

The third-highest derivative $\bar{i} - 2$ is multiplied by radial derivatives of the scalar field of order 2, 3, 4, 5, 6, 7. Using Eq. (A1), to express derivatives of order 4, 5, 6, 7 in terms of lower order ones, one finds additional contributions with respect to the previous case

$$\frac{\mathrm{d}^{\bar{i}-2} V}{\mathrm{d}\phi^{\bar{i}-2}}\frac{\ddot{\phi}^{\bar{i}-3}}{\rho^5}\left(1 + A_3 \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\rho^2 + A_4 \left(\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\rho^2\right)^2 + A_5 \frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}\rho^4\ddot{\phi}\right) \quad \text{even } n,$$
$$\frac{\mathrm{d}^{\bar{i}-2} V}{\mathrm{d}\phi^{\bar{i}-2}}\frac{\ddot{\phi}^{\bar{i}-3}}{\rho^4}\left(1 + A_6 \frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\rho^2 + A_7 \left(\frac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\rho^2\right)^2 + A_8 \frac{\mathrm{d}^3 V}{\mathrm{d}\phi^3}\rho^4\ddot{\phi}\right) \quad \text{odd } n. \tag{A6}$$

In general, the $\bar{i} - i$th term has contributions from terms in Eq. (A3) that are multiplied with a radial derivative of the scalar field of order $n - 2i + 3$ or higher. In this way, the dependence of the $\bar{i} - i$th term on $\ddot{\phi}$ and $\dfrac{\mathrm{d}^j V}{\mathrm{d}\phi^j}$ can be fully determined. The dependence on ρ can be fixed by dimensional consistency. In particular, in each $\bar{i} - i$-th term, these contributions appear always in the combination $\dfrac{\mathrm{d}^j V}{\mathrm{d}\phi^j}\ddot{\phi}^{j-2}\rho^{2j-2}$ with $j \geq 2$.

Consider

$$\lim_{t^* \to +\infty} \ddot{\phi}_* \rho^{4*} = 0. \tag{A7}$$

If all derivatives of the potential in the scalar field are finite for $\phi \to 0$, then

$$\frac{\mathrm{d}^j V}{\mathrm{d}\phi^j} \ddot{\phi}_*^{\;j-2} \rho^{*2j-2} \ll 1 \qquad \text{for } j > 2 \text{ and large } \rho^*. \tag{A8}$$

If also $\left(\dfrac{\mathrm{d}^2 V}{\mathrm{d}\phi^2}\right)_* \rho^{*2} \ll 1$, then

$$\left(\frac{\mathrm{d}V^{(n+1)}}{\mathrm{d}\phi}\right)_* \approx \sum_{i=0}^{i=\bar{\imath}-2} \tilde{A}_i \left(\frac{\mathrm{d}V^{\bar{\imath}-i}}{\mathrm{d}\phi^{\bar{\imath}-i}}\right)_* \frac{\ddot{\phi}_*^{\;\bar{\imath}-i-1}}{\rho^{*2i+1}} \quad \text{even } n,$$

$$\left(\frac{\mathrm{d}V^{(n+1)}}{\mathrm{d}\phi}\right)_* \approx \sum_{i=0}^{i=\bar{\imath}-2} \tilde{A}_i \left(\frac{\mathrm{d}V^{\bar{\imath}-i}}{\mathrm{d}\phi^{\bar{\imath}-i}}\right)_* \frac{\ddot{\phi}_*^{\;\bar{\imath}-i-1}}{\rho^{*2i}} \quad \text{odd } n, \tag{A9}$$

and the sum in Eq. (5) is negligible with respect to V'_* for large ρ^* if Eq. (A8) holds.

Appendix B: S_C has a saddle point on the bounce

S_C, as defined in Eq. (72), has a saddle point on the bounce, which can be turned into a minimum or maximum by slightly changing $S_{C,2}$. One has

$$\frac{\mathrm{d}S_C}{\mathrm{d}C} = 2\pi^2 \bar{t}^3 \left(V(\phi(\bar{t})) - V\left(\frac{C}{2\bar{t}^2}\right)\right) + \pi^2 \frac{C}{\bar{t}^2} + \pi^2 \int_{\bar{t}}^{+\infty} \frac{\mathrm{d}V}{\mathrm{d}\phi}\left(\frac{C}{2t^2}\right) t \,\mathrm{d}t + B_\phi + B_g. \tag{B1}$$

Here B_ϕ, B_g are boundary terms for the scalar field and for gravity, that appear by using the equations of motion in the variation of the first term of Eq. (72). There is one for the scalar field

$$B_\phi = \bar{t}^3 \dot{\phi}(\bar{t}) \frac{\delta\phi}{\delta C}(\bar{t}) = -\frac{\pi^2 C}{\bar{t}^2}, \tag{B2}$$

while the gravitational one can be computed from the Hawking–Gibbons–York boundary term [59,60] evaluated at $t = \bar{t}$

$$\delta S_{\mathrm{GHY}} = \oint_{\partial V} d^3 x \epsilon \sqrt{|h|} n_\alpha V^\alpha \tag{B3}$$

with

$$V^\alpha = g^{\mu\nu} \delta\Gamma^\alpha_{\mu\nu} + g^{\alpha\mu} \delta\Gamma^\nu_{\mu\nu}$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(\partial_\beta \delta g_{\gamma\mu} + \partial_\gamma \delta g_{\beta\mu} - \partial_\mu \delta g_{\gamma\beta}) + \frac{1}{2} \delta g^{\alpha\mu}(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\gamma\beta}) \tag{B4}$$

and $\delta g_{\alpha\beta}$ is the variation of the metric, that has inverse $\delta g^{\alpha\beta} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\mu\nu}$. Moreover, n_α is the unit normal to ∂V and h is the determinant of $h_{\alpha\beta}$, the induced metric on the boundary. ϵ is $+1$ if ∂V is timelike, -1 if it is spacelike. Choosing a timelike future-oriented one-form $n_\alpha = (1, 0, 0, 0)$, one gets

$$B_g = \frac{\pi}{16} \bar{t}^3 \left(2g^{\alpha\beta} \delta\dot{g}_{\alpha\beta} + \delta g^{\alpha\beta} \dot{g}_{\alpha\beta}\right) = 0 \tag{B5}$$

Thus, the first term in Eq. (B1) dominates, and it gives

$$\frac{\mathrm{d}S_C}{\mathrm{d}C} \approx 0 \qquad \frac{\mathrm{d}^2 S_C}{\mathrm{d}C^2} \approx 0. \tag{B6}$$

One can turn the saddle point into a minimum or a maximum in the case of a single-scalar field with Einstein–Hilbert gravity and a non-minimal coupling. To do that, S_C may be redefined as

$$S_C \equiv S_{C,1} + S_{C,2} \tag{B7}$$

where

$$S_{C,1} \equiv 2\pi^2 \int_0^{\bar{t}} \bar{\rho}^3 \left(\frac{\dot{\bar{\phi}}^2}{2} + V(\bar{\phi}) - \frac{M_{\mathrm{P}}^2}{2}\bar{R} - \frac{\xi}{2}\bar{\phi}^2\bar{R}\right) \mathrm{d}t$$

$$S_{C,2} \equiv 2\pi^2 \int_{\bar{t}}^{\infty} t^3 \left(\frac{C^2}{2t^6} + V\left(\frac{C}{2t^2}\right)\right) \mathrm{d}t. \tag{B8}$$

Now

$$\frac{dS_C}{dC} = -2\pi^2 \frac{M_P^2}{2} R \, \bar{t}^3 \frac{d\bar{t}}{dC} - 2\pi^2 \frac{\xi}{2} R \, \phi^2 \bar{t}^3 \frac{d\bar{t}}{dC} + \pi^2 \int_{\bar{t}}^{+\infty} \frac{dV}{d\phi} \left(\frac{C}{2t^2}\right) t \, dt. \tag{B9}$$

To determine $\bar{t}(C)$, the bounce velocity is matched with the numerical estimate

$$-\frac{C_0}{(\bar{t}+\delta)^3} = -\frac{C}{\bar{t}^3} \tag{B10}$$

where δ is a real number that satisfies $\rho(t) - t \approx \delta$ at large times on the bounce. It gives

$$C = C_0 \left(1 - \frac{3\delta}{\bar{t}}\right) + \text{h.o.} \tag{B11}$$

where higher orders are suppressed at large $\bar{t}$ and for $\delta \neq 0$. Thus,

$$\frac{d\bar{t}}{dC} = \frac{\bar{t}^2}{C_0 3\delta} \tag{B12}$$

to lowest order. Equation (B9) then gives

$$\frac{dS_C}{dC} \approx -\frac{\pi}{8} R(\bar{t}) \, \bar{t}^3 \frac{d\bar{t}}{dC} \approx -\frac{C_0 \pi^2}{3\delta \bar{t}} = -\frac{C_0(C_0 - C)\pi^2}{9\delta^2}(1 + 6\xi) \qquad \frac{d^2 S_C}{dC^2} \approx \frac{\pi^2}{9\delta^2}(1 + 6\xi) \tag{B13}$$

S_C has a minimum on the bounce for $\xi > -\frac{1}{6}$ and a maximum otherwise.

7 Appendix C: Bounce action from a perturbative expansion

Some concerns have been raised in the literature [36,39] as regards the use of a perturbative expansion to determine the bounce action of a scalar field theory with an Einstein–Hilbert term as done in [23]. To do that, one expands the scalar field and the scale factor around the bounce in flat space as

$$\phi(t) = \phi_0(t) + \kappa \phi_1(t) \tag{C1}$$

$$\rho(t) = t + \kappa \rho_1(t). \tag{C2}$$

Using the equations of motion, the action is expanded in the same way and one finds that the on-shell action is determined by

$$S = S_0 + \frac{A}{M_P^2 \mathcal{R}^2} \qquad \text{for} \qquad M_P^2 \mathcal{R}^2 \gg 1 \tag{C3}$$

with S_0 the flat space on-shell action, A a real constant and $\mathcal{R}$ the bounce radius, $\phi_0 = 2\mathcal{R}$. Then, it is minimized with respect to R. Actually, it has been shown [36] that ϕ_1 in a scalar field theory with negative quartic potential does not satisfy the proper boundary conditions and thus the perturbative expansion is unreliable. There might be, however, an alternative interpretation of the calculation that makes the final result justified. Consider two theories that have a bounce, and one of them is the vanishing coupling limit of the other (in this case it is the single scalar field theory arising from the $M_P \mathcal{R} \to +\infty$ limit of the same theory with Einstein–Hilbert gravity). By continuity, the action, as the coupling is turned on, changes by a small amount, for sufficiently small values of the coupling. Moreover, if the gravitational backreaction is small, one might choose an *off-shell* profile with the same shape as $\phi_0(t)$, as a function of an arbitrary parameter $\mathcal{R}$, and use it to determine a correction for $\rho_1(t)$ and thus S_E, which is to be minimized as a function of $\mathcal{R}$. If the action has a minimum, the off-shell profile is, to first order in $O(M_P^{-2})$, the bounce one of the full theory. In this way, one avoids to use the perturbation equation for the scalar field and uses the flat space solution only as a field profile to keep the action functional finite.

To keep the approximation under control is instead trickier. While the natural choice might be requiring $M_P \mathcal{R} \gg 1$, one can see that a transformation that leaves the equations of motion Eqs. 16)–(17) unchanged

$$V(\phi) \to \alpha V(\phi) \qquad t \to t \, \alpha^{-1/2} \qquad \rho \to \rho \, \alpha^{-1/2} \qquad \text{with real } \alpha \tag{C4}$$

changes the action as $\alpha S_E \to S_E$. Changing α, $\mathcal{R}$ can be arbitrarily small, increasing λ accordingly. In this case, the relevant parameter is thus $M_P \mathcal{R} \sqrt{\lambda}$. The detailed generalization to theories with multiple couplings λ_i seems to be more involved, despite one may guess that

$$M_P \mathcal{R} \sqrt{\lambda_i} \gg 1 \tag{C5}$$

keeps the gravitational backreaction small.

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