Nonlinear Analysis 216 (2022) 112689

Contents lists available at ScienceDirect

Nonlinear Analysis

www.elsevier.com/locate/na

Lipschitz minimizers for a class of integral functionals under the bounded slope condition ☆

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ARTICLE INFO

Article history:
Received 29 April 2021
Accepted 4 November 2021
Communicated by Enrico Valdinoci

MSC:
49J45
26A45
49Q05
53C17

Keywords:
Minimization problem
Relaxation
Bounded slope condition
Functions of bounded variation
Heisenberg groups

ABSTRACT

We consider the functional $\int_\Omega g(\nabla u + \mathbf{X}^*)\, \mathrm{d}\mathcal{L}^{2n}$ where g is convex and $\mathbf{X}^*(x, y) = 2(-y, x)$ and we study the minimizers in $\mathrm{BV}(\Omega)$ of the associated Dirichlet problem. We prove that, under the bounded slope condition on the boundary datum, and suitable conditions on g, there exists a unique minimizer which is also Lipschitz continuous. The assumptions on g allow to consider both the case with superlinear growth and the one with linear growth. Moreover neither uniform ellipticity nor smoothness of g are assumed.

1. Introduction

In the present paper we are interested in the study of the Lipschitz regularity of minimizers of a class of functionals starting from the regularity of the boundary datum without assuming neither ellipticity nor the growth conditions on the Lagrangian: the literature on this subject is extremely rich, we address the interested reader to [1–9] and references therein for an overview. Our analysis moves from a recent paper by Pinamonti et al. [10] where the area functional for the t-graph of a function $u \in W^{1,1}(\Omega)$ in the sub-Riemannian Heisenberg group $\mathbb{H}^n = \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_t$

☆ S.D. was partially supported by the Academy of Finland (grant 288501 *'Geometry of subRiemannian groups'* and by grant 322898 *'Sub-Riemannian Geometry via Metric-geometry and Lie-group Theory'*) and by the European Research Council (ERC Starting Grant 713998 GeoMeG *'Geometry of Metric Groups'*). S.D. was also partially supported by the Swiss National Foundation grant 200020_191978. A.P. is partially supported by the INdAM-GNAMPA 2020 project, Italy Convergenze variazionali per funzionali e operatori dipendenti da campi vettoriali.

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https://doi.org/10.1016/j.na.2021.112689
0362-546X/

is investigated (see also further references in [10] on the Heisenberg's literature). Precisely, if $\Omega \subset \mathbb{R}^{2n}$ is open with Lipschitz boundary and $\mathbf{X}^*(x, y) := 2(-y, x) \in \mathbb{R}^{2n}$ they consider the functional $\mathscr{A} : W^{1,1}(\Omega) \to \mathbb{R}$ defined by

$$\mathscr{A}(u) = \int_\Omega |\nabla u + \mathbf{X}^*| \, \mathrm{d}\mathcal{L}^{2n}.$$

It was shown in [11] that because of the linear growth in the gradient variable, the natural variational setting for the functional $\mathscr{A}$ is $\mathrm{BV}(\Omega)$, the space of functions of bounded variation in Ω. More precisely, it has been proved that the L^1–relaxation of $\mathscr{A}$ is

$$\mathscr{A}(u) = \int_\Omega |\nabla u + \mathbf{X}^*| \, \mathrm{d}\mathcal{L}^{2n} + |D^s u|(\Omega), \quad u \in \mathrm{BV}(\Omega)$$

where $|D^s u|$ denotes the total variation of the singular part of the distributional derivative of u. In [10], the authors investigate a suitable Dirichlet problem for $\mathscr{A}$. Precisely, they show that the problem

$$\min \{\mathscr{A}(u) : u \in \mathrm{BV}(\Omega), \, u_{|\partial\Omega} = \varphi\}$$

has a unique solution which is also Lipschitz continuous if $\varphi \in L^1(\partial\Omega)$ satisfies the so-called *bounded slope condition* (see Section 4 for the definition).

In the present paper we are interested in the more general case of functionals of type

$$\mathscr{G}(u) = \int_\Omega g(\nabla u + \mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} \tag{1.1}$$

where $g : \mathbb{R}^{2n} \to \mathbb{R}$ is convex but not necessarily strictly convex. In particular, we want to study the Dirichlet problem associated with $\mathscr{G}$, i.e.

$$\min \left\{\mathscr{G}(u) : u \in W^{1,1}(\Omega), \, u_{|\partial\Omega} = \varphi\right\} \tag{1.2}$$

where $\varphi \in W^{1,1}(\partial\Omega)$. It is worth to remark that, while in the superlinear case the existence of a solution of (1.2) is guaranteed by the Direct Method of the Calculus of Variations, when we consider g with linear growth it may happen that the minimum is not achieved and we follow a widely used approach considering the relaxed functional in $\mathrm{BV}(\Omega)$.

In the first part of Section 3, we start by proving a representation formula for the relaxed functional of $\mathscr{G}$ in the L^1-topology and then we use the fact that the functional

$$\mathcal{G}_{\varphi,\Omega}(u) = \int_\Omega g(\nabla u + \mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) d|D^s u| + \int_{\partial\Omega} g^\infty((\varphi - u_{|\partial\Omega})\nu_\Omega) \, d\mathcal{H}^{2n-1} \tag{1.3}$$

admits a minimum in $\mathrm{BV}(\Omega)$. Here $g^\infty : \mathbb{R}^{2n} \to [0, +\infty)$ denotes the recession function of g (see (3.2)) and ν_Ω is the unit outer normal to $\partial\Omega$.

The most part of Section 3 is devoted to proving the Lipschitz regularity of a special minimizer of (1.3) using the assumption that the boundary datum φ satisfies the Bounded Slope Condition. Our approach is inspired by some classical and well known results in the Calculus of Variations (see [12,13] and also [14–17]). In all the cited results the focus is on the existence of minimizers where the space of competitor functions coincides with the set of Lipschitz functions and the main idea (see [18, Chapter 1]) is that the Bounded Slope Condition assumed on the boundary data allows the use of a compactness argument even with no growth assumptions on the Lagrangian. In recent years the use of the *bounded slope condition* has been renewed and applied to obtain various regularity results of minimizers that a priori exist in Sobolev spaces [1,3,4,7–9,19,20]. We point out that crucial points in this approach are: the validity of comparison principles between minimizers; the invariance of minimizers under translations of the domain; the fact that, if the boundary datum φ is affine, φ itself is a minimizer. In the recent results cited above comparison principles are extended to Sobolev functions and to problems where minimizers are not unique. Moreover, barriers that are different from affine functions are used. In [2,5,21], [22,23] Lagrangians of the form $f(\nabla u) + g(x, u)$ have been considered and the fact that the invariance of minimizers with respect to translations of the domain is peculiar of functionals depending only on the gradient has been overcome in various ways, thanks to additional special structure assumed on the function f and/or g.

In the present paper, as in [10], a different kind of functional is considered. One of the main difficulties of Section 3 is due to the fact that we deal with BV-functions. For some aspects, this obstacle has been overcome in [10] but here we treat also more general situations. In Section 3.2 we state a Comparison Principle for BV minimizers, Theorem 3.13, that relies on the validity of a general inequality proved in Theorem 3.8. We underline that Theorem 3.8 as here stated can have its own independent interests since it applies to more general functionals and its proof only relies on fine properties of BV functions. In Proposition 3.18 we also overcome the non-smoothness of the Lagrangian by proving the uniqueness of the affine function as solution of a Dirichlet problem with the affine map itself as boundary datum. In this section we also introduce two assumptions on g that we denote by (A) and (B). Roughly speaking, a function g satisfying these properties is not too far from being strictly convex. However, the epigraphs of g and g^∞ may only have $n-1$-dimensional flat faces with radial directions. Radial functions g defined by $g(z) = f(|z|)$ with a convex f with linear growth satisfy both (A) and (B), see Example 3.1 and notice that this class of functions includes the Lagrangian of the t-graphs of minimal area; a non radial function satisfying both (A) and (B) is given by $g(x, y) = \sqrt{x^2/a^2 + y^2/b^2}$, see Example 3.2.

Section 4 is completely devoted to the proof of uniqueness and Lipschitz regularity of minimizers in the BV class. Our main result is the following (see Theorem 4.4).

Theorem. *If $\varphi \in L^1(\partial\Omega)$ satisfies the bounded slope condition with constant $Q > 0$ and g satisfies properties (A) and (B), then $\mathcal{G}_{\varphi,\Omega}$ has a unique minimizer $u \in BV(\Omega)$ with $u_{|\partial\Omega} = \varphi$ and it satisfies $Lip(u) \le \overline{Q}$ where $\overline{Q}$ is a constant depending on Q and Ω.*

We notice that in our setting uniqueness is far from being obvious since our assumptions include non strictly convex Lagrangians whose epigraph may have unbounded exposed faces and, at least in the linear case, we deal with BV-functions. The problem of uniqueness has been previously addressed in [6,24] for the Euclidean setting, in [10] for the Heisenberg case, and in [25] for relaxed functionals.

In Section 5 we describe how to modify the previous proofs in order to deal with the case for which g has superlinear growth. In particular, the main result is as follows, see Theorem 5.1.

Theorem. *Assume g has superlinear growth at infinity and satisfies condition (A), and assume $\varphi \in L^1(\Omega)$ satisfies the bounded slope condition at $\partial\Omega$. Then $\mathscr{G}(u)$ has a unique minimizer in $\varphi + W_0^{1,1}(\Omega)$ which is Lipschitz.*

We conclude this introduction underlying some significant aspects of the results that we present in this paper. First of all we recall that regularity results are usually obtained under ellipticity and growth conditions on the Lagrangian. In the present paper, the *bounded slope condition* allows us to drop these assumptions and to prove Lipschitz regularity up to the boundary. As we mentioned above, the use of the *bounded slope condition* is strictly related to the invariance of the minimizers w.r.t. translation. This property is quite strong and it is satisfied for example by functionals depending only on the gradient or, as mentioned above, by functionals of sum type under very special assumptions on the structure of the Lagrangian. In particular, due to the x-dependence of the Lagrangian, the functional considered here does not satisfy it. Anyhow it is interesting that, as it will be pointed out in the proof Theorem 4.4, it turns out that a slight modification of the translated minimizer is still a minimizer and this property is crucial to complete the proof.

It is worth recalling that, in the framework of classical problems of the Calculus of Variations, the Lipschitz regularity of minimizers is the first ingredient to prove higher regularity. The assumptions of our main Theorem are wide enough to take into account Lagrangians that are not smooth so we cannot expect more regularity in such a general case.

As last remark we notice that our regularity result in particular implies the non occurrence of the Lavrentiev phenomenon. This result is classically obtained under suitable assumptions that control both from below and from above the growth of the functional and it has been proved also for some special classes of problems. To be more precise it has been proved that autonomous multidimensional scalar functionals do not exhibit the Lavrentiev phenomenon (see [26] for some special cases and [27,28] for more general results). A recent result on a class of functional that includes those considered in this paper is contained in [29].

2. Preliminaries

2.1. Functions of bounded variation and traces

The aim of this section is to recall some basic properties of the space of functions of bounded variation; we refer to the monographs [30,31] for a more extensive account on the subject as well as for proofs of the results we are going to recall.

Let Ω be an open set in $\mathbb{R}^n$. We say that $u \in L^1(\Omega)$ has *bounded variation* in Ω if

$$\sup\left\{\int_\Omega u \operatorname{div}\varphi\, dx \mid \varphi \in C^1_c(\Omega,\mathbb{R}^n), \|\varphi\|_\infty \le 1\right\} < +\infty; \tag{2.1}$$

equivalently, u has bounded variation if there exist a $\mathbb{R}^n$-valued Radon measure $Du := (Du_1, \dots, Du_n)$ in Ω which represents the distributional derivatives of u, i.e.,

$$\int_\Omega u \frac{\partial\varphi}{\partial x_i}\, \mathrm{d}\mathcal{L}^n = -\int_\Omega \varphi\ \mathrm{d}D_i u \quad \forall \varphi \in C^1_c(\Omega),\ \forall i = 1, \dots, n.$$

The space of functions with bounded variation in Ω is denoted by $\mathrm{BV}(\Omega)$. By definition, $W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$ and $Du = \nabla u\, \mathcal{L}^n$ for any $u \in W^{1,1}(\Omega)$.

We denote by $|Du|$ the total variation of the measure Du; $|Du|$ defines a finite measure on Ω and the supremum in (2.1) coincides with $|Du|(\Omega)$.

It is well-known that $\mathrm{BV}(\Omega)$ is a Banach space when endowed with the norm

$$\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega). \tag{2.2}$$

We say that $u \in L^1_{loc}(\Omega)$ has an *approximate limit* $z \in \mathbb{R}$ at $x \in \Omega$ if

$$\lim_{\rho\to 0^+} \fint_{B(x,\rho)} |u - z|\, \mathrm{d}\mathcal{L}^n = 0. \tag{2.3}$$

The set S_u of points where u has no approximate limit is called *approximate discontinuity set* of u; for any $x \in \Omega\backslash S_u$, we denote by $\tilde{u}(x)$ the unique z for which (2.3) holds. By the Lebesgue Theorem we have $\mathcal{L}^n(S_u) = 0$.
Moreover, we say that u has an *approximate jump point* at $x \in \Omega$ if there exist $\nu \in \mathbb{S}^{n-1}$ and $a, b \in \mathbb{R},\ a \ne b$ such that

$$\lim_{\rho\to 0^+} \fint_{B(x,\rho;\nu)^+} |u - a|\, \mathrm{d}\mathcal{L}^n = 0, \quad \lim_{\rho\to 0^+} \fint_{B(x,\rho;\nu)^-} |u - b|\, \mathrm{d}\mathcal{L}^n = 0$$

where

$$B(x,\rho;\nu)^+ := \{y \in B(x,\rho) \mid \langle y - x, \nu\rangle > 0\}$$
$$B(x,\rho;\nu)^- := \{y \in B(x,\rho) \mid \langle y - x, \nu\rangle < 0\}.$$

We observe that the triple (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν; we denote it by $(u^+(x), u^-(x), \nu_u(x))$. The set of approximate jump points of u is denoted by J_u; clearly, $J_u \subset S_u$.

Remark 2.1. Depending on the context, we will sometimes use the symbols u^+, u^- also to denote the positive part $u^+ := \max\{0, u\}$ and the negative part $u^- := \max\{0, -u\}$ of a real function u. This will not generate confusion.

When u has bounded variation in Ω, the set of approximate jump points J_u enjoys much finer regularity properties. First, there holds

$$|Du|(S_u \setminus J_u) = \mathcal{H}^{n-1}(S_u \setminus J_u) = 0\,, \tag{2.4}$$

where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional *Hausdorff measure* on $\mathbb{R}^n$ (see e.g. [30] or [31]). Moreover, by the Federer–Vol'pert Thoerem, see [30, Theorem 3.78], J_u (and, consequently, S_u) is $(n-1)$-rectifiable, i.e., $\mathcal{H}^{n-1}(J_u) < \infty$ and there exist $N \subset \mathbb{R}^n$ and a countable family of hypersurfaces $\{S_j : j \in \mathbb{N}\}$ of class C^1 such that

$$J_u \subset N \cup \bigcup_{j=0}^{\infty} S_j \quad \text{and} \quad \mathcal{H}^{n-1}(N) = 0.$$

It turns out that ν_u corresponds ($\mathcal{H}^{n-1}$-a.e. and up to a sign) to a unit normal to J_u, i.e., for $\mathcal{H}^{n-1}$-a.e. $x \in J_u$, there holds

$$\nu_u(x) = \pm \nu_{S_i}(x) \text{ if } x \in S_i \setminus \bigcup_{j=0}^{i-1} S_j, \quad \forall i \in \mathbb{N}.$$

By the Radon–Nikodym Theorem, if $u \in \mathrm{BV}(\Omega)$ one can write $Du = D^a u + D^s u$, where $D^a u$ is the absolutely continuous part of Du with respect to $\mathcal{L}^n$ and $D^s u$ is the singular part of Du with respect to $\mathcal{L}^n$. We denote by $\nabla u \in L^1(\Omega)$ the density of $D^a u$ with respect to $\mathcal{L}^n$, so that $D^a u = \nabla u\, \mathcal{L}^n$. We are now in a position to state the following result:

Theorem 2.2. *Let $u \in \mathrm{BV}(\Omega)$; then u is* approximately differentiable *at a.e. $x \in \Omega$ with approximate differential $\nabla u(x)$, i.e.,*

$$\lim_{\rho \to 0^+} \fint_{B(x,\rho)} \frac{|u(y) - \tilde{u}(x) - \langle \nabla u(x), y - x \rangle|}{\rho}\, \mathrm{d}\mathcal{L}^n = 0 \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in \Omega.$$

Moreover, the decomposition $D^s u = D^j u + D^c u$ holds, where

$$D^j u := D^s u \llcorner J_u = (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u, \qquad D^c u := D^s u \llcorner (\Omega \setminus S_u)$$

are called respectively the jump part *and the* Cantor part *of the derivative Du.*

Notice that $D^a u, D^c u, D^j u$ are mutually singular; in particular

$$|D^a u| = |\nabla u| \mathcal{L}^n, \quad |D^j u| = |u^+ - u^-| \mathcal{H}^{n-1} \llcorner J_u$$

and

$$|Du| = |D^a u| + |D^c u| + |D^j u|$$

because the total variation of a sum of mutually singular measures is the sum of their total variations.

In what follows we recall a few basic facts about boundary trace properties of BV functions; we refer again to [30,31] for more details.

Let $\Omega \subset \mathbb{R}^n$ be a fixed open set with bounded Lipschitz regular boundary; the spaces $L^p(\partial\Omega), p \in [1, +\infty]$, will be always understood with respect to the (finite) measure $\mathcal{H}^{n-1} \llcorner \partial\Omega$. It is well-known that for any $u \in \mathrm{BV}(\Omega)$ there exists a (unique) function $u_{|\partial\Omega} \in L^1(\partial\Omega)$ such that, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$\lim_{\rho \to 0^+} \rho^{-n} \int_{\Omega \cap B(x,\rho)} |u - u_{|\partial\Omega}(x)|\, \mathrm{d}\mathcal{L}^n = \lim_{\rho \to 0^+} \fint_{\Omega \cap B(x,\rho)} |u - u_{|\partial\Omega}(x)|\, \mathrm{d}\mathcal{L}^n = 0.$$

The function $u_{|\partial\Omega}$ is called *trace* of u on $\partial\Omega$. The trace operator $u \mapsto u_{|\partial\Omega}$ is linear and continuous between $(\mathrm{BV}(\Omega), \|\cdot\|_{\mathrm{BV}})$ and $L^1(\partial\Omega)$; actually, it is continuous also when $\mathrm{BV}(\Omega)$ is endowed with the (weaker) topology induced by the so-called *strict convergence*, see [30, Definition 3.14].

Remark 2.3. It is well-known that, if $u_1, u_2 \in \mathrm{BV}(\Omega)$, then $\overline{u} := \max\{u_1, u_2\}$ and $\underline{u} := \min\{u_1, u_2\}$ belong to $\mathrm{BV}(\Omega)$; moreover, one can show that

$$\overline{u}_{|\partial\Omega} = \max\{u_{1|\partial\Omega}, u_{2|\partial\Omega}\}, \qquad \underline{u}_{|\partial\Omega} = \min\{u_{1|\partial\Omega}, u_{2|\partial\Omega}\}.$$

The proof of this fact follows in a standard way from the very definition of traces.

Since $Du \ll |Du|$ we can write $Du = \sigma_u |Du|$ for a $|Du|$-measurable function

$$\sigma_u : \Omega \to \mathbb{S}^{n-1}.$$

With this notation one also has

$$\int_\Omega u \operatorname{div} \varphi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \langle \sigma_u, \varphi \rangle \, \mathrm{d}|Du| + \int_{\partial\Omega} u_{|\partial\Omega} \langle \varphi, \nu_\Omega \rangle \, \mathrm{d}\mathcal{H}^{n-1}, \qquad \forall \varphi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n) \tag{2.5}$$

where ν_Ω is the unit outer normal to $\partial\Omega$.

Finally, we recall the following fact, whose proof essentially follows from (2.5).

Proposition 2.4 (*[31, Remark 2.13]*). *Assume that Ω and Ω_0 are open subsets of $\mathbb{R}^n$ with bounded Lipschitz boundary and such that $\Omega \Subset \Omega_0$. If $u \in \mathrm{BV}(\Omega)$ and $v \in \mathrm{BV}(\Omega_0 \setminus \overline{\Omega})$, then the function*

$$f(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ v(x) & \text{if } x \in \Omega_0 \setminus \overline{\Omega} \end{cases}$$

belongs to $\mathrm{BV}(\Omega_0)$ *and*

$$|Df|(\partial\Omega) = |D^j f|(\partial\Omega) = \int_{\partial\Omega} |u_{|\partial\Omega} - v_{|\partial\Omega}| \, \mathrm{d}\mathcal{H}^{n-1},$$

where we have used the notation $v_{|\partial\Omega}$ to mean $(v_{|\partial(\Omega_0 \setminus \overline{\Omega})}) \llcorner \partial\Omega$.

For any $z = (x, y) \in \mathbb{R}^{2n}$, we define $z^* := (-y, x)$. Let $\mathbf{X}^* : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be given by $\mathbf{X}^*(z) := 2z^*$. We conclude this section with the next lemma which can be extracted from the proof of [10, Thm. 5.5].

Lemma 2.5. *Let $R > 0$ and $u \in \mathrm{BV}(B_R(0))$ with $u = 0$ on $\partial B_R(0)$. Assume that there exists a $|Du|$-measurable function $\lambda : B_R(0) \to \mathbb{R}$ such that*

$$\frac{dDu}{d|Du|} = \lambda \mathbf{X}^* \quad |Du|\text{-a.e. on } B_R(0).$$

Then $u = 0$.

3. The linear growth case

Throughout this section we assume that $g : \mathbb{R}^{2n} \to \mathbb{R}$ is a positive convex function with linear growth, namely

$$\frac{1}{C}|z| \le g(z) \le C(1 + |z|), \tag{3.1}$$

for a constant $C \geq 1$ and for any $z \in \mathbb{R}^{2n}$. Moreover, defining the *recession function* of g as the function $g^\infty : \mathbb{R}^{2n} \to [0, +\infty)$ given by

$$g^\infty(p) := \lim_{t \to +\infty} \frac{g(tp)}{t}. \tag{3.2}$$

Note that, since $g(0) < \infty$, our definition of g^∞ coincides with the one given in [30, Definition 2.32]. As proved in [30], the recession function is positively homogeneous of degree 1, convex and lower semicontinuous. In particular, g^∞ satisfies the following inequalities

$$g^\infty(p) \le g^\infty(q) + g^\infty(p - q), \qquad \forall p, q \in \mathbb{R}^{2n}, \tag{3.3}$$

$$\frac{1}{C}|p| \leq g^{\infty}(p) \leq C|p|, \qquad \forall p \in \mathbb{R}^{2n}. \tag{3.4}$$

Since by [32, Proposition 2.32], g is Lipschitz continuous then denoting by L_g its Lipschitz constant we get

$$|g(tp) - g(tp+z)| \leq L_g|z|$$

which implies that for any $z, p \in \mathbb{R}^{2n}$ we have

$$g^{\infty}(p) = \lim_{t\to+\infty} \frac{g(tp+z)}{t}. \tag{3.5}$$

We consider the following conditions:

(A) If $\xi_1, \xi_2 \in \mathbb{R}^{2n}$ are such that
$$g\left(\frac{\xi_1+\xi_2}{2}\right) = \frac{g(\xi_1)+g(\xi_2)}{2}, \tag{3.6}$$
then there exists $\lambda \in \mathbb{R}$ such that $\xi_1 = \lambda\xi_2$.

(B) If $\xi_1, \xi_2 \in \mathbb{R}^{2n}$ and $p \in \partial g(\xi_2)$ are such that
$$g^{\infty}(\xi_1) = \langle p, \xi_1\rangle \tag{3.7}$$
then there exists $\lambda \in \mathbb{R}$ such that $\xi_1 = \lambda\xi_2$. Here $\partial g(q)$ denotes the subdifferential of g at the point q.

Example 3.1. Let $f\colon [0,+\infty) \to \mathbb{R}$ be a convex and strictly increasing function such that there exists $C > 1$ satisfying
$$\frac{1}{C}s \leq f(s) \leq C(s+1)$$
for any $s \in [0,+\infty)$. Consider the function $g\colon \mathbb{R}^{2m} \to \mathbb{R}$ defined by $g(z) := f(|z|)$. We claim that g satisfies conditions (A) and (B). Indeed, for any $\xi_1, \xi_2 \in \mathbb{R}^{2m}$ satisfying (3.6) we get

$$f\left(\frac{|\xi_1|}{2} + \frac{|\xi_2|}{2}\right) \leq \frac{1}{2}\left(f(|\xi_1|) + f(|\xi_2|)\right) = f\left(\frac{|\xi_1+\xi_2|}{2}\right) \leq f\left(\frac{|\xi_1|}{2} + \frac{|\xi_2|}{2}\right) \tag{3.8}$$

from which we infer $|\xi_1 + \xi_2| = |\xi_1| + |\xi_2|$ and the thesis follows. To prove condition (B), we start observing that by [33, Example 16.73] we have
$$\partial g(\xi) = \begin{cases} \left\{\frac{\alpha}{|\xi|}\xi \mid \alpha \in \partial f(|\xi|)\right\}, & \text{if } \xi \neq 0 \\ B(0,\rho), & \text{if } \xi = 0 \end{cases} \tag{3.9}$$
where $\rho \in [0,+\infty)$ is such that $\partial f(0) = [-\rho, \rho]$. Moreover a direct computation gives

$$g^{\infty}(\xi) = f^{\infty}(|\xi|) = \beta|\xi| \tag{3.10}$$

where, denoting by $f'(t)$ an arbitrary selection of $\partial f(t)$, $\beta = \lim_{t\to+\infty} \frac{f(t)}{t} = \lim_{t\to+\infty} f'(t)$. Hence the convexity of f implies also that $\beta \geq \alpha$ for every $\alpha \in \bigcup_{t\in[0,+\infty)} \partial f(t)$. Let us now consider $\xi_1, \xi_2 \in \mathbb{R}^{2n}$ and $p \in \partial g(\xi_2)$ such that $g^{\infty}(\xi_1) = \langle p, \xi_1\rangle$. If $\xi_2 = 0$ there is nothing to prove. If $\xi_2 \neq 0$ then $p = \alpha\frac{\xi_2}{|\xi_2|}$ for some $\alpha \in \partial f(|\xi_2|)$ and $\alpha > 0$. By (3.10) and the fact that f^{∞} is 1−homogeneous we get

$$g^{\infty}(\xi_1) = \langle \alpha\frac{\xi_2}{|\xi_2|}, \xi_1\rangle \leq \alpha|\xi_1| \leq \beta|\xi_1| = g^{\infty}(\xi_1)$$

and this implies that all the inequalities are in fact equalities and in particular the vectors ξ_1 and ξ_2 have to be parallel, concluding the proof. We notice that the Lagrangian describing the minimal area of t-graphs is included in this class.

Example 3.2. Let $a, b \in (0, +\infty)$. We claim that the function $g : \mathbb{R}^2 \to [0, +\infty)$ defined by

$$g(z_1, z_2) = \sqrt{\frac{z_1^2}{a^2} + \frac{z_2^2}{b^2}} \tag{3.11}$$

satisfies (3.1), conditions (A) and (B). Indeed, for any $z \in \mathbb{R}^2$

$$\min\left\{\frac{1}{a}, \frac{1}{b}\right\} |z| \leq g(z) \leq \max\left\{\frac{1}{a}, \frac{1}{b}\right\} |z| \tag{3.12}$$

and g is convex and it satisfies (A) by a direct computation. In order to prove condition (B) we start observing that, being g 1-homogeneous and in $C^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$, we have $g^\infty(z) = g(z)$ for any $z \in \mathbb{R}^2$ and $\partial g(z) = \left\{\left(\frac{z_1}{a^2 g(z)}, \frac{z_2}{b^2 g(z)}\right)\right\}$ for any $z \in \mathbb{R}^2 \setminus \{(0,0)\}$. Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $(\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $(p_1, p_2) = \left(\frac{\eta_1}{a^2 g(\eta)}, \frac{\eta_2}{b^2 g(\eta)}\right)$ be such that $g^\infty(\xi) = \langle p, \xi \rangle$, namely

$$\sqrt{\frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2}} = \frac{\eta_1 \xi_1}{a^2 g(\eta)} + \frac{\eta_2 \xi_2}{b^2 g(\eta)} \tag{3.13}$$

which immediately implies that $\xi_1 \eta_2 = \eta_1 \xi_2$ and the thesis follows. On the other hand, let $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2)$ be such that

$$\eta \in \partial g((0,0)) \qquad \text{and} \qquad g(\xi) = \langle \eta, \xi \rangle. \tag{3.14}$$

Since the function $f(z) = g(z) - \langle p, z \rangle$ is convex, 1-homogeneous, nonnegative and $f(\xi) = f((0,0)) = 0$, then one has $\xi = (0,0)$.

Let $\Omega \subset \mathbb{R}^{2n}$ be bounded, open and with Lipschitz boundary. We consider the functional $\mathscr{G}_\Omega : W^{1,1}(\Omega) \to [0, +\infty]$ defined by

$$\mathscr{G}_\Omega(u) := \int_\Omega g(\nabla u + \mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} \tag{3.15}$$

where we recall that $\mathbf{X}^*(z) = 2(-y, x)$, with $z = (x, y)$, $x, y \in \mathbb{R}^n$. In the following proposition, we underline some basic properties of the operator z^*, see [10, Lemma 3.1] for a proof.

Proposition 3.3. *The following properties hold:*

(i) *if $z_1, z_2 \in \mathbb{R}^{2n}$ are linearly dependent, then $z_1 \cdot z_2^* = 0$;*
(ii) *$z_1 \cdot z_2 = z_1^* \cdot z_2^*$ for each $z_1, z_2 \in \mathbb{R}^{2n}$;*
(iii) *if $\Omega \subset \mathbb{R}^{2n}$ is open and $f \in C^\infty(\Omega)$, then $\operatorname{div}(\nabla f)^* = 0$ on Ω.*

The following result, which generalizes [10, Proposition 5.1], states that if $\mathscr{G}_\Omega$ has a minimizer with some additional integrability, then it is unique.

Proposition 3.4. *Let $p \in [1, 2]$, let $p' := \frac{p}{p-1}$, let $\varphi \in W^{1,p'}(\Omega)$ and assume g satisfies condition (A). Let $u \in W^{1,p'}(\Omega)$ and $v \in W^{1,p}(\Omega)$ be two minimizers of*

$$\min\left\{\mathscr{G}_\Omega(u) : u \in \varphi + W_0^{1,p}(\Omega)\right\},$$

then $u = v$ a.e. in Ω.

Proof. First of all we use a standard argument in order to prove that $\nabla u + \mathbf{X}^*$ and $\nabla v + \mathbf{X}^*$ are linearly dependent a.e. on Ω. Using the convexity of g, we have

$$g\left(\frac{\nabla u + \mathbf{X}^*}{2} + \frac{\nabla v + \mathbf{X}^*}{2}\right) \leq \frac{g(\nabla u + \mathbf{X}^*) + g(\nabla v + \mathbf{X}^*)}{2} \quad \text{a.e. on } \Omega.$$

Hence, from the minimality of u and v we get

$$\mathcal{G}(u) \leq \int_\Omega g\left(\frac{\nabla u + \mathbf{X}^*}{2} + \frac{\nabla v + \mathbf{X}^*}{2}\right) \mathrm{d}\mathcal{L}^{2n} \leq \frac{1}{2}\int_\Omega [g(\nabla u + \mathbf{X}^*) + g(\nabla v + \mathbf{X}^*)]\, \mathrm{d}\mathcal{L}^{2n} = \mathcal{G}(u).$$

Then

$$g\left(\frac{\nabla u + \mathbf{X}^*}{2} + \frac{\nabla v + \mathbf{X}^*}{2}\right) = \frac{g(\nabla u + \mathbf{X}^*) + g(\nabla v + \mathbf{X}^*)}{2}, \quad \text{a.e. on } \Omega.$$

Using (A) we deduce that $\nabla u + \mathbf{X}^*$ and $\nabla v + \mathbf{X}^*$ are linearly dependent a.e. on Ω. The conclusion now follows proceeding exactly as in the second part of [10, Proposition 5.1]). □

Remark 3.5. Notice that inequality (3.1) can be replaced by

$$\frac{1}{C}|z| - C \leq g(z) \leq C(1 + |z|), \tag{3.16}$$

in which the map g is not necessarily positive. This comes by the fact that, since we are studying minimizers, the function g can be replaced by $g + M$, for any $M \in \mathbb{R}$.

In order to prove the existence of a minimizer for $\mathcal{G}_\Omega$ we first compute its L^1 relaxed functional, namely

$$G_\Omega(u) := \overline{\mathcal{G}_\Omega}(u) = \inf\left\{\liminf_h \mathcal{G}_\Omega(u_h) : u_h \in W^{1,1}(\Omega),\ u_h \to u \text{ in } L^1(\Omega)\right\}. \tag{3.17}$$

The following proposition provides an integral representation of G_Ω.

Proposition 3.6. *Let g be a convex function satisfying (3.1) and let $\Omega \subseteq \mathbb{R}^{2n}$ be open with Lipschitz boundary. Then the following facts hold.*

(i) *for any $u \in \mathrm{BV}(\Omega)$ one has*

$$G_\Omega(u) = \int_\Omega g(\nabla u + \mathbf{X}^*)\, \mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u|. \tag{3.18}$$

(ii) *For any $\varphi \in L^1(\partial\Omega)$ the functional*

$$G_{\varphi,\Omega}(u) := \int_\Omega g(\nabla u + \mathbf{X}^*)\, \mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| + \int_{\partial\Omega} g^\infty((\varphi - u_{|\partial\Omega})\nu_\Omega)\, \mathrm{d}\mathcal{H}^{2n-1} \tag{3.19}$$

admits a minimizer in $\mathrm{BV}(\Omega)$.

Proof. (i) By [34, Remark 2.17], it is enough to check (H1)-(H5) of the reference and observing that thanks to (3.5), g^∞ does not depend on x. Consider $f : \Omega \times \mathbb{R}^{2n} \to \mathbb{R}$ defined by $f(x, z) = g(z + X^*(x))$. For the sake of clarity, we here list the precise properties used in [34] for our specific case.

(H1) f is continuous;
(H2) $f(x, \cdot)$ is quasiconvex;
(H3) there exists a bounded and continuous $h : \Omega \to [0, +\infty)$ and a constant $M > 0$ such that

$$\frac{1}{M}h(x)\|z\| \leq f(x, z) \leq Mh(x)(1 + \|z\|),$$

for all $x \in \Omega$, and $z \in \mathbb{R}^{2n}$;
(H4) for every compact set $K \subseteq \Omega$, there exists a continuous function $\omega : \mathbb{R} \to \mathbb{R}$ with $\omega(0) = 0$ and

$$|f(x, z) - f(x', z)| \leq \omega(|x - x'|)(1 + |z|),$$

for all $x, x' \in \Omega$ and $z \in \mathbb{R}^{2n}$. In addition, for every $x_0 \in \Omega$ and $\varepsilon > 0$, there exists δ such that

$$f(x, z) - f(x_0, z) \geq -\varepsilon h(x)(1 + |z|),$$

for every $x \in \Omega, z \in \mathbb{R}^{2n}$ with $|x - x_0| < \delta$;

(H5) There exist $C', L > 0$ and $0 \le m < 1$ such that

$$\left| f^{\infty}(x,z) - \frac{f(x,tz)}{t} \right| \le C' h(x) \frac{|z|^{1-m}}{t^m},$$

whenever $x \in \Omega$, and $z \in \mathbb{R}^{2n}$ and $t > 0$ are such that $t|z| > L$.

It is clear by construction that f is continuous and $f(x,\cdot)$ is convex. This yields (H1) and (H2).

We set $h \equiv 1$. Then property (H3) comes directly from (3.1). To prove (H4), we first recall that g is Lipschitz with Lipschitz constant equal to L_g and therefore

$$|f(x,z) - f(x',z)| \le L_g |x - x'| \le L_g |x - x'|(1 + |z|).$$

In particular, if $x_0 \in \Omega$ and $\delta > 0$, then, whenever $|x - x_0| \le \frac{\delta}{L_g}$ we get

$$f(x,z) - f(x_0,z) \ge -L_g |x - x_0|(1 + |z|) \ge -\delta(1 + |z|),$$

which completes the proof of (H4). Finally, (H5) comes from the fact that (3.1) implies

$$|f^{\infty}(x,z) - f(x,z)| \le C(|z| + 1).$$

(ii) Let $\Omega_0 \subset \mathbb{R}^{2n}$ be an open Lipschitz domain with $\Omega \Subset \Omega_0$. Let $\varphi \in L^1(\partial\Omega)$ and $\Phi \in W^{1,1}(\Omega_0 \setminus \Omega)$ such that $\Phi = \varphi$ on $\partial\Omega$ and $\Phi = 0$ on $\partial\Omega_0$. We set

$$\mathrm{BV}_{\Phi}(\Omega_0) := \{u \in \mathrm{BV}(\Omega_0) : u = \Phi \text{ on } \Omega_0 \setminus \overline{\Omega}\}.$$

By [35, Theorem 1.3] (see also [36, Theorem 1.1]) we know that $\mathcal{G}_{\Omega_0}$ has a minimum on $\mathrm{BV}_{\Phi}(\Omega_0)$. Now observe that for any $u \in \mathrm{BV}_{\Phi}(\Omega_0)$ we have

$$\begin{aligned}\mathcal{G}_{\Omega_0}(u) = \int_{\Omega} g(\nabla u + \mathbf{X}^*)\, \mathrm{d}\mathcal{L}^{2n} + \int_{\Omega} g^{\infty}\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| \\ + \int_{\partial\Omega} g^{\infty}((\varphi - u_{|\partial\Omega})\nu_{\Omega})\, \mathrm{d}\mathcal{H}^{2n-1} + \mathcal{G}_{\Omega_0 \setminus \Omega}(\Phi)\end{aligned} \tag{3.20}$$

where ν_{Ω} is the outer unit normal to Ω and $u_{|\partial\Omega}$ is the trace of u on $\partial\Omega$. Since the last term on the right-hand side of (3.20) is constant we can write for any $u \in \mathrm{BV}_{\Phi}(\Omega_0)$,

$$\mathcal{G}_{\Omega_0}(u) = \mathcal{G}_{\varphi,\Omega}(u_{|\Omega}) + \text{constant}. \tag{3.21}$$

Conversely, for any $u \in \mathrm{BV}(\Omega)$ the extended function

$$u_0 = \begin{cases} u & \text{on } \Omega, \\ \Phi & \text{on } \Omega_0 \setminus \overline{\Omega} \end{cases}$$

belongs to $\mathrm{BV}_{\Phi}(\Omega_0)$ and

$$\mathcal{G}_{\Omega_0}(u_0) = \mathcal{G}_{\varphi,\Omega}(u) + \text{constant}.$$

Since $\mathcal{G}_{\Omega_0}$ admits a minimizer in $\mathrm{BV}_{\Phi}(\Omega_0)$, we have then proved that, for any $\varphi \in L^1(\partial\Omega)$, the functional $\mathcal{G}_{\varphi,\Omega}$ admits a minimizers in $\mathrm{BV}(\Omega)$. □

The following result will be crucial later on, it relies on the approach developed in [31] for the area functional (see also [36]).

Proposition 3.7. *For any $\varphi \in L^1(\partial\Omega)$,*

$$\min_{u \in \mathrm{BV}(\Omega)} \mathcal{G}_{\varphi,\Omega}(u) = \inf\left\{\mathcal{G}_{\Omega}(u) \;:\; u \in W_0^{1,1}(\Omega) + \varphi\right\}. \tag{3.22}$$

Proof. First we observe that $\mathcal{G}_\Omega(u) = \mathcal{G}_{\varphi,\Omega}(u)$ for any $u \in W_0^{1,1}(\Omega) + \varphi$, therefore

$$\inf\left\{\mathcal{G}_\Omega(u) \ : \ u \in W_0^{1,1}(\Omega) + \varphi\right\} \geq \min_{u\in \mathrm{BV}(\Omega)} \mathcal{G}_{\varphi,\Omega}(u). \tag{3.23}$$

Let $u \in \mathrm{BV}(\Omega)$ and define $u_0 \in \mathrm{BV}_\Phi(\Omega_0)$ as above. Then by [36, Lemma 2.1] there exists a sequence (u_h) in $C_c^\infty(\Omega_0)$ such that $u_h = \Phi$ on $\Omega_0 \setminus \overline{\Omega}$, $u_h \to u_0$ in $L^1(\Omega_0)$ and $\int_{\Omega_0} \sqrt{1+|\nabla u_h|^2} \to \int_{\Omega_0} \sqrt{1+|\nabla u_0|^2}$ as $h \to \infty$. Then, by Reshetnyak's continuity theorem (see e.g. [37, Theorem 1.1]) we get

$$\mathcal{G}_{\Omega_0}(u_0) = \lim_h \mathcal{G}_{\Omega_0}(u_h)$$

in particular

$$\begin{aligned}\mathcal{G}_{\varphi,\Omega}((u_0)_{|\Omega}) = \lim_h \mathcal{G}_{\varphi,\Omega}((u_h)_{|\Omega}) &= \lim_h \mathcal{G}_{\Omega}((u_h)_{|\Omega}) \\ &\geq \inf\left\{\mathcal{G}_\Omega(u) \ : \ u \in W_0^{1,1}(\Omega) + \varphi\right\}\end{aligned}$$

and the conclusion follows. □

3.1. A fundamental inequality

This subsection is devoted to proving the fundamental inequality (3.24), which will be useful when dealing with comparison principles for minimizers of the functional $\mathcal{G}_\Omega$. This inequality is a generalization of the well known inequality for the perimeters that can be found, for the Euclidean case, in [30, Proposition 3.38 (d)] and has been extended for perimeters in the Heisenberg case in [10]. We underline also that, when dealing with Sobolev function with given boundary datum, this inequality turns out to be an equality whose proof is quite straightforward (see [23, Lemma 5.1]).

We state the inequality in a quite general setting that includes the case of functionals that are not necessarily obtained by means of a relaxing argument but also fits to the relaxed functional considered in this paper. To this aim, we consider an open bounded subset of $A \subset \mathbb{R}^n$ with Lipschitz boundary, and two functions f_i, $i = 1, 2$ such that

(i) $f_1 : A \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function,
(ii) $f_2 : \mathbb{R}^n \to [0, +\infty)$ is convex, positively homogeneous of degree 1, and $f_2(\xi) = 0$ if and only if $\xi = 0$.

Then we define the functional $\mathcal{F}_A : \mathrm{BV}(A) \to \mathbb{R} \cup \{+\infty\}$ by

$$\mathcal{F}_A(u) = \int_A f_1(x, u, \nabla u)\, \mathrm{d}\mathcal{L}^n + \int_A f_2\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u|.$$

Theorem 3.8. *Let $A \subseteq \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary and let $f_1 : A \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f_2 : A \to [0, +\infty)$ be two functions satisfying respectively assumptions* i), ii) *above. Then, for any $u_1, u_2 \in \mathrm{BV}(A)$, we have*

$$\mathcal{F}_A(u_1 \vee u_2) + \mathcal{F}_A(u_1 \wedge u_2) \leq \mathcal{F}_A(u_1) + \mathcal{F}_A(u_2)\,. \tag{3.24}$$

Proof. Let us define

$$\begin{aligned}
X &:= \int_A f_1\left(x, u_1 \vee u_2, \nabla(u_1 \vee u_2)\right) \mathrm{d}\mathcal{L}^n + \int_A f_1\left(x, u_1 \wedge u_2, \nabla(u_1 \wedge u_2)\right) \mathrm{d}\mathcal{L}^n, \\
Y &:= \int_A f_2\left(\frac{dD^s(u_1 \vee u_2)}{d|D^s(u_1 \vee u_2)|}\right) \mathrm{d}|D^c(u_1 \vee u_2)| + \int_A f_2\left(\frac{dD^s(u_1 \wedge u_2)}{d|D^s(u_1 \wedge u_2)|}\right) \mathrm{d}|D^c(u_1 \wedge u_2)|, \\
Z &:= \int_A f_2\left(\frac{dD^s(u_1 \vee u_2)}{d|D^s(u_1 \vee u_2)|}\right) \mathrm{d}|D^j(u_1 \vee u_2)| + \int_A f_2\left(\frac{dD^s(u_1 \wedge u_2)}{d|D^s(u_1 \wedge u_2)|}\right) \mathrm{d}|D^j(u_1 \wedge u_2)|\,.
\end{aligned}$$

Observe that (3.24) will follow if we show that

$$X + Y + Z \leq \mathcal{F}_A(u_1) + \mathcal{F}_A(u_2). \tag{3.25}$$

Without loss of generality, we may assume that $u_1 = \tilde{u}_1$ on $A \setminus S_{u_1}$ and $u_2 = \tilde{u}_2$ on $A \setminus S_{u_2}$. Setting

$$A_+ := (A \setminus (S_{u_1} \cup S_{u_2})) \cap \{u_1 \geq u_2\}, \quad A_- := (A \setminus (S_{u_1} \cup S_{u_2})) \cap \{u_1 < u_2\}$$

we have (see e.g. [30, Example 3.100])

$$\begin{aligned} \nabla(u_1 \vee u_2) &= \nabla u_1 \chi_{A_+} + \nabla u_2 \chi_{A_-} \quad \mathcal{L}^n\text{-a.e. in } A \\ \nabla(u_1 \wedge u_2) &= \nabla u_2 \chi_{A_+} + \nabla u_1 \chi_{A_-} \quad \mathcal{L}^n\text{-a.e. in } A, \end{aligned}$$

where χ_E denotes the characteristic function of a set E, and similarly

$$\begin{aligned} D^c(u_1 \vee u_2) &= D^c u_1 \llcorner A_+ + D^c u_2 \llcorner A_-; \\ D^c(u_1 \wedge u_2) &= D^c u_2 \llcorner A_+ + D^c u_1 \llcorner A_-. \end{aligned}$$

Therefore

$$\begin{aligned} X &= \int_{A_+} f_1(x, u_1, \nabla u_1)\, \mathrm{d}\mathcal{L}^n + \int_{A_-} f_1(x, u_2, \nabla u_2)\, \mathrm{d}\mathcal{L}^n \\ &\quad + \int_{A_+} f_1(x, u_2, \nabla u_2)\, \mathrm{d}\mathcal{L}^n + \int_{A_-} f_1(x, u_1, \nabla u_1)\, \mathrm{d}\mathcal{L}^n \\ &= \int_{A\setminus(S_{u_1}\cup S_{u_2})} f_1(x, u_1, \nabla u_1)\, \mathrm{d}\mathcal{L}^n + \int_{A\setminus(S_{u_1}\cup S_{u_2})} f_1(x, u_2, \nabla u_2)\, \mathrm{d}\mathcal{L}^n \\ &= \int_A f_1(x, u_1, \nabla u_1)\, \mathrm{d}\mathcal{L}^n + \int_A f_1(x, u_2, \nabla u_2)\, \mathrm{d}\mathcal{L}^n. \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} Y &= \int_{A_+} f_2\left(\frac{dD^c u_1}{d|D^c u_1|}\right) \mathrm{d}|D^c u_1| + \int_{A_-} f_2\left(\frac{dD^c u_2}{d|D^c u_2|}\right) \mathrm{d}|D^c u_2| \\ &\quad + \int_{A_+} f_2\left(\frac{dD^c u_2}{d|D^c u_2|}\right) \mathrm{d}|D^c u_2| + \int_{A_-} f_2\left(\frac{dD^c u_1}{d|D^c u_1|}\right) \mathrm{d}|D^c u_1| \\ &= \int_{A\setminus(S_{u_1}\cup S_{u_2})} f_2\left(\frac{dD^s u_1}{d|D^s u_1|}\right) \mathrm{d}|D^c u_1| + \int_{A\setminus(S_{u_1}\cup S_{u_2})} f_2\left(\frac{dD^s u_2}{d|D^s u_2|}\right) \mathrm{d}|D^c u_2| \\ &= \int_A f_2\left(\frac{dD^s u_1}{d|D^s u_1|}\right) \mathrm{d}|D^c u_1| + \int_A f_2\left(\frac{dD^s u_2}{d|D^s u_2|}\right) \mathrm{d}|D^c u_2|, \end{aligned} \tag{3.27}$$

where to obtain the last equality in (3.26) and in (3.27), we used the fact that $\mathcal{L}^n(S_{u_1} \cup S_{u_2}) = 0$ (see [30, Proposition 3.64]) and the fact that, since $u_1, u_2 \in \mathrm{BV}(A)$, then $|D^c u_1|(S_{u_2}) = |D^c u_2|(S_{u_1}) = 0$.[1]

Recall that, by [30, Eq. (3.90)], one has

$$\begin{aligned} D^j u_1 &= (u_1^+ - u_1^-)\nu_1 \mathcal{H}^{n-1} \llcorner J_{u_1} \\ D^j u_2 &= (u_2^+ - u_2^-)\nu_2 \mathcal{H}^{n-1} \llcorner J_{u_2}, \end{aligned}$$

where ν_1, ν_2 are the unit normals to the $(n-1)$-rectifiable sets J_{u_1}, J_{u_2}. Without loss of generality, we may assume that $u_1^+ \geq u_1^-$ and $\nu_1 = \nu_2$, $\mathcal{H}^{n-1}$-a.e. on $J_{u_1} \cap J_{u_2}$; in this way, the $(n-1)$-rectifiable set $T := J_{u_1} \cup J_{u_2}$ is associated with the unit normal ν_T defined by

$$\nu_T := \nu_1 \text{ on } J_{u_1}, \qquad \nu_T := \nu_2 \text{ on } T \setminus J_{u_1}.$$

[1] This last fact follows from Proposition 3.92 item c) and Remark 2.50 in [30]

We extend $u_1^\pm : J_{u_1} \to \mathbb{R}$ and $u_2^\pm : J_{u_2} \to \mathbb{R}$ to the whole T by setting

$$u_1^\pm := \begin{cases} u_1^\pm & \text{on } J_{u_1} \\ 0 & \text{on } T \setminus J_{u_1}, \end{cases} \qquad u_2^\pm := \begin{cases} u_2^\pm & \text{on } J_{u_2} \\ 0 & \text{on } T \setminus J_{u_2}. \end{cases}$$

In this way one has

$$D^j(u_1 + u_2) = (u_1^+ - u_1^- + u_2^+ - u_2^-)\,\nu_T\,\mathcal{H}^{n-1} \llcorner T.$$

By [30, Theorem 3.99], $|u_1 - u_2| \in BV(A)$ and

$$D^j(|u_1 - u_2|) = (|u_1^+ - u_2^+| - |u_1^- - u_2^-|)\,\nu_T\,\mathcal{H}^{n-1} \llcorner T. \tag{3.28}$$

We can then write

$$\begin{aligned} D^j(u_1 \vee u_2) &= D^j\left(\tfrac{u_1+u_2}{2} + \tfrac{|u_1-u_2|}{2}\right) = \tfrac{1}{2}D^j(u_1+u_2) + \tfrac{1}{2}D^j(|u_1-u_2|) \\ D^j(u_1 \wedge u_2) &= D^j\left(\tfrac{u_1+u_2}{2} - \tfrac{|u_1-u_2|}{2}\right) = \tfrac{1}{2}D^j(u_1+u_2) - \tfrac{1}{2}D^j(|u_1-u_2|). \end{aligned}$$

By using this decomposition and (3.28), we have

$$\begin{aligned} Z =& \int_T f_2\left(\frac{dD^j(u_1 \vee u_2)}{d|D^j(u_1 \vee u_2)|}\right) \mathrm{d}|D^j(u_1 \vee u_2)| + \int_T f_2\left(\frac{dD^j(u_1 \wedge u_2)}{d|D^j(u_1 \wedge u_2)|}\right) \mathrm{d}|D^j(u_1 \wedge u_2)| \\ =& \frac{1}{2}\int_T f_2\left((u_1^+ - u_1^- + u_2^+ - u_2^- + |u_1^+ - u_2^+| - |u_1^- - u_2^-|)\,\nu_T\right) \mathrm{d}\mathcal{H}^{n-1} \\ &+ \frac{1}{2}\int_T f_2\left((u_1^+ - u_1^- + u_2^+ - u_2^- - |u_1^+ - u_2^+| + |u_1^- - u_2^-|)\,\nu_T\right) \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{3.29}$$

Let for shortness $\alpha, \beta : T \to \mathbb{R}$ be the functions defined by

$$\begin{aligned} \alpha &:= u_1^+ - u_1^- + u_2^+ - u_2^- + |u_1^+ - u_2^+| - |u_1^- - u_2^-|, \\ \beta &:= u_1^+ - u_1^- + u_2^+ - u_2^- - |u_1^+ - u_2^+| + |u_1^- - u_2^-|. \end{aligned}$$

To estimate Z, we are going to split T into several regions. Set

$$T' := \{x \in T : u_2^+(x) \ge u_2^-(x)\}, \quad \text{and} \quad T'' := \{x \in T : u_2^+(x) < u_2^-(x)\}.$$

Then, taking into account that $u_1^- \le u_1^+$ on T, one can easily check that both α and β are positive on T'. Being f_2 positively homogeneous, then one has

$$\begin{aligned} &\frac{1}{2}\int_{T'} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{T'} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{2}\int_{T'} (\alpha+\beta) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{T'} (u_1^+ - u_1^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{T'} (u_2^+ - u_2^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{T'} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{T'} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{3.30}$$

We now subdivide T'' into the union of the following disjoint subsets:

$$\begin{aligned} T''_{++} &:= \{x \in T'' : u_1^+(x) \ge u_2^+(x), u_1^-(x) \ge u_2^-(x)\}, \quad T''_{--} := \{x \in T'' : u_1^+(x) < u_2^+(x), u_1^-(x) < u_2^-(x)\} \\ T''_{+-} &:= \{x \in T'' : u_1^+(x) \ge u_2^+(x), u_1^-(x) < u_2^-(x)\}, \quad T''_{-+} := \{x \in T'' : u_1^+(x) < u_2^+(x), u_1^-(x) \ge u_2^-(x)\}. \end{aligned}$$

Notice that, for every $x \in T''_{++}$, one has $\alpha(x) = 2(u_1^+(x) - u_1^-(x))$ and $\beta(x) = 2(u_2^+(x) - u_2^-(x))$, conversely, for every $x \in T''_{--}$, one has $\alpha(x) = 2(u_2^+(x) - u_2^-(x))$ and $\beta(x) = 2(u_1^+(x) - u_1^-(x))$. Using this information, we easily obtain

$$\begin{aligned} &\frac{1}{2}\int_{T''_{++} \cup T''_{--}} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{T''_{++} \cup T''_{--}} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{T''_{++} \cup T''_{--}} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{T''_{++} \cup T''_{--}} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{3.31}$$

We now consider T''_{+-}. The estimate on T''_{-+} can be done in a completely analogous way. We first write $T''_{+-} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where

$$\Gamma_1 \coloneqq \{x \in T''_{+-} : u_1^+(x) \geq u_2^-(x), u_2^+(x) \geq u_1^-(x)\}, \quad \Gamma_2 \coloneqq \{x \in T''_{+-} : u_1^+(x) \geq u_2^-(x), u_2^+(x) < u_1^-(x)\}$$
$$\Gamma_3 \coloneqq \{x \in T''_{+-} : u_1^+(x) < u_2^-(x), u_2^+(x) \geq u_1^-(x)\}, \quad \Gamma_4 \coloneqq \{x \in T''_{+-} : u_1^+(x) < u_2^-(x), u_2^+(x) < u_1^-(x)\}.$$

Notice that, for every $x \in T''_{+-}$, one has that $\alpha(x) = 2(u_1^+(x) - u_2^-(x))$ and $\beta(x) = 2(u_2^+(x) - u_1^-(x))$ and, by construction, α is positive on $\Gamma_1 \cup \Gamma_2$ and strictly negative on $\Gamma_3 \cup \Gamma_4$, while β is positive on $\Gamma_1 \cup \Gamma_3$ and strictly negative on $\Gamma_2 \cup \Gamma_4$. Using the positive homogeneity of f_2, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_1} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{\Gamma_1} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_1} (u_1^+ - u_2^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_1} (u_2^+ - u_1^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_1} (u_1^+ - u_1^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_1} (u_2^+ - u_2^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_1} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_1} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned} \tag{3.32}$$

Taking into account that α and β are strictly negative on Γ_4, we also have

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_4} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{\Gamma_4} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_4} (u_2^- - u_1^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_4} (u_1^- - u_2^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_4} (u_1^- - u_1^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_4} (u_2^- - u_2^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_4} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_4} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}.
\end{aligned} \tag{3.33}$$

Recall that, by (3.1), the map f_2 is positive, and therefore, for any $0 \leq \lambda_1 \leq \lambda_2$ and any $x \in \mathbb{R}^{2n}$, one has $f_2(\lambda_1 x) \leq f_2(\lambda_2 x)$. We can make the estimate on Γ_2, taking into account that α is positive and β is strictly negative:

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_2} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{\Gamma_2} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_2} (u_1^+ - u_2^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_2} (u_1^- - u_2^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_2} (u_1^+ - u_1^- + u_1^- - u_2^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_2} (u_1^- - u_2^- + u_2^- - u_2^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq \int_{\Gamma_2} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_2} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned} \tag{3.34}$$

where in the last inequality we used the fact that $(u_1^- - u_2^-)_{|T''_{+-}} < 0$ and $(u_2^- - u_2^+)_{|T''} > 0$. Analogously, for Γ_3, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Gamma_3} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{\Gamma_3} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_3} (u_2^- - u_1^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_3} (u_2^+ - u_1^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&= \int_{\Gamma_3} (u_2^- - u_2^+ + u_2^+ - u_1^+) f_2(-\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_3} (u_2^+ - u_1^+ + u_1^+ - u_1^-) f_2(\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} \\
&\leq \int_{\Gamma_3} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\Gamma_3} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1},
\end{aligned} \tag{3.35}$$

where in the last inequality we have used the fact that $(u_2^- - u_2^+)_{|T''} > 0$ and $(u_2^+ - u_1^+)_{|T''_{+-}} \le 0$. Combining (3.32), (3.34), (3.35) and (3.33) one obtains

$$\begin{aligned}&\frac{1}{2}\int_{T''_{+-}} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{T''_{+-}} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{T''_{+-}} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{T''_{+-}} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{3.36}$$

In a completely analogous fashion, we can also write

$$\begin{aligned}&\frac{1}{2}\int_{T''_{-+}} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{T''_{-+}} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{T''_{-+}} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{T''_{-+}} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{3.37}$$

As a direct consequence of (3.31), (3.36) and (3.37), we then have

$$\begin{aligned}&\frac{1}{2}\int_{T''} f_2(\alpha\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \frac{1}{2}\int_{T''} f_2(\beta\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}\\ &\le \int_{T''} f_2((u_2^+ - u_2^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1} + \int_{T''} f_2((u_1^+ - u_1^-)\nu_T)\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{3.38}$$

The thesis is then obtained by combining (3.26), (3.27), (3.29), (3.30) and (3.38). □

Remark 3.9. The functional $\mathcal{G}_\Omega$ considered in the present paper is a special case of the functional $\mathcal{F}_A$ where the set $\Omega \subset \mathbb{R}^{2n}$ plays the role of A, $f_1(x,u,\xi) = g(\xi + X^*(x))$ and $f_2(\xi) = g^\infty(\xi)$. Proposition 3.6 shows that $\mathcal{G}_\Omega$ is the relaxation of a functional defined in $W^{1,1}(\Omega)$. We notice that in this particular case the proof of Theorem 3.8 could be simplified by a relaxation argument.

Corollary 3.10. *Let $\Omega \subseteq \mathbb{R}^{2n}$ be an open and bounded set with Lipschitz boundary and let $g\colon \Omega \to [0,+\infty)$ be a convex function satisfying (3.1). Then, for every $\varphi_1,\varphi_2 \in L^1(\partial\Omega)$ and every $u_1,u_2 \in BV(\Omega)$ one has*

$$\mathcal{G}_{\Omega,\varphi_1\vee\varphi_2}(u_1 \vee u_2) + \mathcal{G}_{\Omega,\varphi_1\wedge\varphi_2}(u_1 \wedge u_2) \le \mathcal{G}_{\Omega,\varphi_1}(u_1) + \mathcal{G}_{\Omega,\varphi_2}(u_2). \tag{3.39}$$

Proof. Let $u_1,u_2 \in BV(\Omega)$ and $\varphi_1,\varphi_2 \in L^1(\partial\Omega)$.

First of all Theorem 3.8 and Proposition 3.6 imply

$$\mathcal{G}_\Omega(u_1 \vee u_2) + \mathcal{G}_\Omega(u_1 \wedge u_2) \le \mathcal{G}_\Omega(u_1) + \mathcal{G}_\Omega(u_2). \tag{3.40}$$

Fix any bounded open and Lipschitz set $\Omega_0 \Supset \Omega$. By [31, Theorem 2.16], we can find $w_1,w_2 \in W^{1,1}(\Omega_0 \setminus \overline{\Omega})$ with $w_{1|\partial\Omega} = \varphi_1$ and $w_{2|\partial\Omega} = \varphi_2$. Set now

$$v_1 := \begin{cases} w_1 & \text{on } \Omega_0 \setminus \overline{\Omega} \\ u_1 & \text{on } \Omega \end{cases} \quad \text{and} \quad v_2 := \begin{cases} w_2 & \text{on } \Omega_0 \setminus \overline{\Omega} \\ u_2 & \text{on } \Omega. \end{cases}$$

By [30, Theorem 3.84], $v_1,v_2 \in BV(\Omega_0)$ and, moreover, if ν_Ω denotes the exterior normal to Ω, one has

$$Dv_i = Du_i \llcorner \Omega + Dw_i \llcorner (\Omega_0 \setminus \overline{\Omega}) + (w_i - u_i)\nu_\Omega \mathcal{H}^{2n-1} \llcorner \partial\Omega, \qquad \text{for } i = 1,2,$$

from which we can compute, up to $|D^s v|$-negligible sets, the polar vector:

$$\frac{dD^s v_i}{d|D^s v_i|} = \begin{cases} \dfrac{dD^s u_i}{d|D^s u_i|} & \text{on } \Omega \\ 0 & \text{on } \Omega_0 \setminus \overline{\Omega} \\ \dfrac{(w_i - u_i)}{|w_i - u_i|}\nu_\Omega & \text{on } \partial\Omega. \end{cases}$$

Then, using the previous expression, the fact that g^∞ is homogeneous and the definition of w_i, we get

$$\begin{aligned}\mathcal{G}_{\Omega_0}(v_i) &= \int_{\Omega_0} g(\nabla v_i + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_{\Omega_0} g^\infty\left(\frac{dD^s v_i}{d|D^s v_i|}\right)\mathrm{d}|D^s v_i| \\ &= \mathcal{G}_\Omega(u_i) + \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_i) + \int_{\partial\Omega} g^\infty((w_i - u_i)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{2n-1} \\ &= \mathcal{G}_\Omega(u_i) + \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_i) + \int_{\partial\Omega} g^\infty((\varphi_i - u_i)\nu_\Omega)\,\mathrm{d}\mathcal{H}^{2n-1} \\ &= \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_i) + \mathcal{G}_{\Omega,\varphi_i}(u_i), \qquad \text{for } i = 1, 2.\end{aligned}$$

Similarly, using Remark 2.3 we also have

$$\begin{aligned}&\mathcal{G}_{\Omega_0}(v_1 \vee v_2) = \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_1 \vee w_2) + \mathcal{G}_{\Omega,\varphi_1\vee\varphi_2}(u_1 \vee u_2) \quad \textit{and} \\ &\mathcal{G}_{\Omega_0}(v_1 \wedge v_2) = \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_1 \wedge w_2) + \mathcal{G}_{\Omega,\varphi_1\wedge\varphi_2}(u_1 \wedge u_2).\end{aligned}$$

Taking into account that (3.39) is an equality when the maps are Sobolev (see [23, Lemma 5.1]), we can then conclude combining the previous identities with Theorem 3.8 to get

$$\begin{aligned}&\mathcal{G}_{\Omega,\varphi_1\vee\varphi_2}(u_1 \vee u_2) + \mathcal{G}_{\Omega,\varphi_1\wedge\varphi_2}(u_1 \wedge u_2) \\ &= \mathcal{G}_{\Omega_0}(v_1 \vee v_2) + \mathcal{G}_{\Omega_0}(v_1 \wedge v_2) - \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_1 \vee w_2) - \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_1 \wedge w_2) \\ &\le \mathcal{G}_{\Omega_0}(v_1) + \mathcal{G}_{\Omega_0}(v_2) - \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_1) - \mathcal{G}_{\Omega_0\setminus\overline{\Omega}}(w_2) = \mathcal{G}_{\Omega,\varphi_1}(u_1) + \mathcal{G}_{\Omega,\varphi_2}(u_2). \quad \square\end{aligned}$$

3.2. The set of minimizers and comparison principles

Given a bounded open set $\Omega \subset \mathbb{R}^{2n}$ with Lipschitz regular boundary and a function $\varphi \in L^1(\partial\Omega)$ we define

$$\mathscr{M}_\varphi := \operatorname*{argmin}_u \mathcal{G}_{\varphi,\Omega}(u).$$

We have already proved that $\mathscr{M}_\varphi \subset \mathrm{BV}(\Omega)$ is nonempty.

Using Theorem 3.8 and Corollary 3.10, the proof of Proposition 3.11 below is completely analogous to [10, Proposition 4.3] and we omit it.

Proposition 3.11. *Let $\varphi_1, \varphi_2 \in L^1(\partial\Omega)$ be such that $\varphi_1 \le \varphi_2$ $\mathcal{H}^{2n-1}$-a.e. on $\partial\Omega$ and assume that $u_1 \in \mathscr{M}_{\varphi_1}$ and $u_2 \in \mathscr{M}_{\varphi_2}$. Then $(u_1 \vee u_2) \in \mathscr{M}_{\varphi_2}$ and $(u_1 \wedge u_2) \in \mathscr{M}_{\varphi_1}$.*

In [9] (see also [10]), it has been proved that the set of minimizers of a superlinear convex functional has a maximum u (resp. a minimum u) defined as the pointwise supremum (infimum) of the minimizers. These special minimizers are then used to prove one-sided Comparison Principles.

Proposition 3.12. *Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz regular boundary and let $\varphi \in L^1(\partial\Omega)$. Then, there exists $\overline{u}, \underline{u} \in \mathscr{M}_\varphi$ such that the inequalities*

$$\underline{u} \le u \le \overline{u}, \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega \tag{3.41}$$

hold for any $u \in \mathscr{M}_\varphi$.

Proof. We start by proving that $\mathscr{M}_\varphi$ is bounded in $\mathrm{BV}(\Omega)$. Define $J := \min_{u\in BV(\Omega)} \mathcal{G}_{\varphi,\Omega}(u) < +\infty$. By (3.4) and denoting by $\tilde{C} = \sup_\Omega |\mathbf{X}^*|$ we get

$$\begin{aligned}
|Du|(\Omega) &= \int_\Omega |\nabla u| \,\mathrm{d}\mathcal{L}^{2n} + |D^s u|(\Omega) \\
&\le C \int_\Omega g(\nabla u + \mathbf{X}^*) \,\mathrm{d}\mathcal{L}^{2n} + \tilde{C}|\Omega| + \int_\Omega \mathrm{d}|D^s u| \\
&\le C \int_\Omega g(\nabla u + \mathbf{X}^*) \,\mathrm{d}\mathcal{L}^{2n} + \tilde{C}|\Omega| + C\int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| + C\int_{\partial\Omega} g^\infty((\varphi - u_{|\partial\Omega})\nu_\Omega) \,\mathrm{d}\mathcal{H}^{2n-1} \\
&= CJ + \tilde{C}|\Omega|, \qquad \forall u \in \mathscr{M}_\varphi,
\end{aligned} \tag{3.42}$$

where $|\Omega| := \mathcal{L}^{2n}(\Omega)$. Moreover, by [31, Theorem 1.28 and Remark 2.14] there exists $c = c(n) > 0$ such that

$$\begin{aligned}
\|u\|_{L^1(\Omega)} &\le |\Omega|^{1/2n} \|u\|_{L^{2n/(2n-1)}(\Omega)} \\
&\le c\,|\Omega|^{1/2n}\left(|Du|(\Omega) + \int_{\partial\Omega} |u| \,\mathrm{d}\mathcal{H}^{2n-1}\right) \\
&\le c|\Omega|^{1/2n}\left(|Du|(\Omega) + \int_{\partial\Omega} |\varphi - u_{|\partial\Omega}| \,\mathrm{d}\mathcal{H}^{2n-1} + \int_{\partial\Omega} |\varphi| \,\mathrm{d}\mathcal{H}^{2n-1}\right) \\
&= c|\Omega|^{1/2n}\left(|Du|(\Omega) + \int_{\partial\Omega} |(\varphi - u_{|\partial\Omega})\nu_\Omega| \,\mathrm{d}\mathcal{H}^{2n-1} + \int_{\partial\Omega} |\varphi| \,\mathrm{d}\mathcal{H}^{2n-1}\right) \\
&\le c|\Omega|^{1/2n}\left(C\int_\Omega g(\nabla u + X^*) \,\mathrm{d}\mathcal{L}^{2n} + \tilde{C}|\Omega| + C\int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right)|D^s u|(\Omega)\right. \\
&\qquad \left. + C\int_{\partial\Omega} g^\infty((\varphi - u_{|\partial\Omega})\nu_\Omega) \,\mathrm{d}\mathcal{H}^{2n-1} + \int_{\partial\Omega} |\varphi| \,\mathrm{d}\mathcal{H}^{2n-1}\right) \\
&= c|\Omega|^{1/2n}\left(CJ + \tilde{C}|\Omega| + \int_{\partial\Omega} |\varphi| \,\mathrm{d}\mathcal{H}^{2n-1}\right), \qquad \forall u \in \mathscr{M}_\varphi,
\end{aligned}$$

where in the second last inequality we argued as in (3.42). This, together with (3.42), implies that $\mathscr{M}_\varphi$ is bounded in $\mathrm{BV}(\Omega)$.

Therefore, by [30, Theorem 3.23], $\mathscr{M}_\varphi$ is pre-compact in $L^1(\Omega)$, i.e., for every sequence (u_h) in $\mathscr{M}_\varphi$ there exist $u \in \mathrm{BV}(\Omega)$ and a subsequence (u_{h_k}) such that $u_{h_k} \to u$ in $L^1(\Omega)$. By (3.21), $\mathcal{G}_{\varphi,\Omega}$ is lower semicontinuous with respect to the L^1-convergence, hence we have also

$$\mathcal{G}_{\varphi,\Omega}(u) \le \liminf_{k\to\infty} \mathcal{G}_{\varphi,\Omega}(u_{h_k}) = J,$$

so that $u \in \mathscr{M}_\varphi$. We have proved that $\mathscr{M}_\varphi$ is compact in $L^1(\Omega)$. Now, the functional

$$\mathrm{BV}(\Omega) \ni u \longmapsto I(u) := \int_\Omega u \;\mathrm{d}\mathcal{L}^{2n}$$

is continuous in $L^1(\Omega)$, hence it admits maximum $\overline{u}$ and minimum $\underline{u}$ in $\mathscr{M}_\varphi$: let us prove that $\overline{u}, \underline{u}$ satisfy (3.41) for any $u \in \mathscr{M}_\varphi$.

Assume by contradiction there exists $u \in \mathscr{M}_\varphi$ such that $\Omega' := \{z \in \Omega : u(z) > \overline{u}(z)\}$ has strictly positive measure. Then, by Proposition 3.11, $u \vee \overline{u}$ is in $\mathscr{M}_\varphi$. Moreover

$$\int_\Omega (u \vee \overline{u}) \,\mathrm{d}\mathcal{L}^{2n} = \int_{\Omega'} u \,\mathrm{d}\mathcal{L}^{2n} + \int_{\Omega\setminus\Omega'} \overline{u} \,\mathrm{d}\mathcal{L}^{2n} > \int_\Omega \overline{u} \,\mathrm{d}\mathcal{L}^{2n}$$

yielding a contradiction. The fact that $u \ge \underline{u}$ follows in a similar way. □

The following result is a Comparison Principle inspired by the results obtained in [9] for superlinear functionals in Sobolev spaces and it can be proved exactly as in [10, Theorem 4.5].

Theorem 3.13. *Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz regular boundary; let $\varphi, \psi \in L^1(\partial\Omega)$ be such that $\varphi \le \psi$ $\mathcal{H}^{2n-1}$-a.e. on $\partial\Omega$. Consider the functions $\overline{u}$, $\underline{u} \in \mathscr{M}_\varphi$ and $\overline{w}$, $\underline{w} \in \mathscr{M}_\psi$ such that*[2]

$$\begin{aligned} &\underline{u} \le u \le \overline{u} \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega,\ \forall u \in \mathscr{M}_\varphi \\ &\underline{w} \le w \le \overline{w} \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega,\ \forall w \in \mathscr{M}_\psi . \end{aligned} \tag{3.43}$$

Then

$$\overline{u} \le \overline{w} \quad \text{and} \quad \underline{u} \le \underline{w} \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega \tag{3.44}$$

and, in particular,

$$\begin{aligned} &u \le \overline{w} \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega,\ \forall u \in \mathscr{M}_\varphi \\ &\underline{u} \le w \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega,\ \forall w \in \mathscr{M}_\psi. \end{aligned}$$

Upon observing that $\mathcal{G}_{\varphi+\alpha,\Omega}(u+\alpha) = \mathcal{G}_{\varphi,\Omega}(u) \quad \forall\, u \in \mathrm{BV}(\Omega)$, the following result can be proved exactly as in [10, Corollary 4.6].

Corollary 3.14. *Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz regular boundary and $\varphi, \psi \in L^\infty(\partial\Omega)$; let $\overline{u}, \underline{u} \in \mathscr{M}_\varphi$ and $\overline{w}, \underline{w} \in \mathscr{M}_\psi$ be as in (3.43). Then, for every $\alpha \in \mathbb{R}$, one has*

$$\begin{aligned} &\overline{u} + \alpha, \underline{u} + \alpha \in \mathscr{M}_{\varphi+\alpha} \\ &\underline{u} + \alpha \le u \le \overline{u} + \alpha \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega,\ \forall u \in \mathscr{M}_{\varphi+\alpha} \end{aligned} \tag{3.45}$$

and

$$\begin{aligned} &\|\overline{u} - \overline{w}\|_{L^\infty(\Omega)} \le \|\varphi - \psi\|_{L^\infty(\partial\Omega)} \\ &\|\underline{u} - \underline{w}\|_{L^\infty(\Omega)} \le \|\varphi - \psi\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{3.46}$$

In particular, the implications

$$\begin{aligned} &\overline{u}_{|\partial\Omega} = \varphi,\ \overline{w}_{|\partial\Omega} = \psi \quad \Rightarrow \quad \|\overline{u} - \overline{w}\|_{L^\infty(\Omega)} = \|\varphi - \psi\|_{L^\infty(\partial\Omega)}, \\ &\underline{u}_{|\partial\Omega} = \varphi,\ \underline{w}_{|\partial\Omega} = \psi \quad \Rightarrow \quad \|\underline{u} - \underline{w}\|_{L^\infty(\Omega)} = \|\varphi - \psi\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{3.47}$$

hold.

We recall below some notations introduced in [10], that will be useful also in the proof of the main theorem of the present paper. Given a subset $\Omega \subset \mathbb{R}^{2n}$, a function $u\colon \Omega \to \mathbb{R}$, a vector $\tau \in \mathbb{R}^{2n}$ and $\xi \in \mathbb{R}$ we set

$$\begin{aligned} &\Omega_\tau := \{z \in \mathbb{R}^{2n} : z + \tau \in \Omega\} \\ &u_\tau(z) := u(z+\tau), \quad z \in \Omega_\tau \\ &u^*_{\tau,\xi}(z) := u_\tau(z) + 2\langle \tau^*, z\rangle + \xi, \quad z \in \Omega_\tau . \end{aligned}$$

It is easily seen that, given Ω open and $u \in \mathrm{BV}(\Omega)$, then both u_τ and $u^*_{\tau,\xi}$ belong to $\mathrm{BV}(\Omega_\tau)$. Moreover, if Ω is bounded with Lipschitz regular boundary one has also

$$(u^*_{\tau,\xi})_{|\partial(\Omega_\tau)} = (u_{|\partial\Omega})_\tau + 2\langle \tau^*, \cdot\rangle + \xi = (u_{|\partial\Omega})^*_{\tau,\xi} . \tag{3.48}$$

Remark 3.15. The family of functions $u^*_{\tau,\xi}$ has a precise meaning from the viewpoint of Heisenberg groups geometry. Indeed, it is a matter of computations to observe that the t-subgraph $E^t_{u^*_{\tau,\xi}}$ of $u^*_{\tau,\xi}$ coincides with the left translation $(-\tau,\xi)\cdot E^t_u$ (according to the group law) of the t-subgraph E^t_u of u by the element $(-\tau,\xi) \in \mathbb{H}^n$. We address the interested reader to [10,11] for further informations.

[2] The existence of $\overline{u}$, $\underline{u}$, $\overline{w}$, $\underline{w}$ is guaranteed by Proposition 3.12.

Lemma 3.16. *Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz regular boundary, $\varphi \in L^1(\partial\Omega)$, $\tau \in \mathbb{R}^{2n}$ and $\xi \in \mathbb{R}$. Then*

$$\mathcal{G}_{\varphi^*_{\tau,\xi},\Omega_\tau}(u^*_{\tau,\xi}) = \mathcal{G}_{\varphi,\Omega}(u), \quad \forall u \in \mathrm{BV}(\Omega).$$

Proof. Using e.g. [30, Remark 3.18], we get $Du_\tau = \ell_{\tau\#}(Du)$, where ℓ_τ is the translation $z \mapsto z - \tau$ and $\ell_{\tau\#}$ denotes the push-forward of measures via ℓ_τ. In particular

$$\nabla u_\tau = (\nabla u)_\tau = \nabla u \circ \ell_\tau^{-1}, \quad D^s u_\tau = \ell_{\tau\#}(D^s u) \quad \text{and} \quad \frac{dD^s u_\tau}{d|D^s u_\tau|} = \frac{dD^s u}{d|D^s u|} \circ \ell_\tau^{-1}$$

hence

$$Du^*_{\tau,\xi} = (\nabla u \circ \ell_\tau^{-1} + 2\tau^*)\mathcal{L}^{2n} + \ell_{\tau\#}(D^s u).$$

Therefore

$$\begin{aligned}
&\mathcal{G}_{\varphi^*_{\tau,\xi},\Omega_\tau}(u^*_{\tau,\xi}) \\
&= \int_{\Omega_\tau} g((\nabla u \circ \ell_\tau^{-1}) + 2\tau^* + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_{\Omega_\tau} g^\infty\left(\frac{dD^s u}{d|D^s u|} \circ \ell_\tau^{-1}\right) \mathrm{d}|\ell_{\tau\#}(D^s u)| \\
&\quad + \int_{\partial\Omega_\tau} g^\infty((\varphi^*_{\tau,\xi} - (u^*_{\tau,\xi})_{|\partial\Omega_\tau})\nu_{\Omega_\tau})\,\mathrm{d}\mathcal{H}^{2n-1}.
\end{aligned}$$

We now use (3.48) and the equality

$$2\tau^* + \mathbf{X}^*(z) = 2(\tau + z)^* = (\mathbf{X}^* \circ \ell_\tau^{-1})(z), \quad \forall z \in \mathbb{R}^{2n}$$

to get, with a change of variable,

$$\begin{aligned}
&\mathcal{G}_{\varphi^*_{\tau,\xi},\Omega_\tau}(u^*_{\tau,\xi}) \\
&= \int_{\Omega_\tau} |\nabla u + \mathbf{X}^*| \circ \ell_\tau^{-1}\,\mathrm{d}\mathcal{L}^{2n} + \int_{\Omega_\tau} g^\infty\left(\frac{dD^s u}{d|D^s u|} \circ \ell_\tau^{-1}\right) \mathrm{d}|\ell_{\tau\#}(D^s u)| + \int_{\partial\Omega_\tau} g^\infty\left((\varphi - u_{|\partial\Omega})_\tau \nu_{\Omega_\tau}\right) \mathrm{d}\mathcal{H}^{2n-1} \\
&= \int_\Omega |\nabla u + \mathbf{X}^*|\,\mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| + \int_{\partial\Omega} g^\infty((\varphi - u_{|\partial\Omega})\nu_\Omega)\,\mathrm{d}\mathcal{H}^{2n-1} \\
&= \mathcal{G}_{\varphi,\Omega}(u)\,. \quad \square
\end{aligned}$$

Corollary 3.17. *If the same assumptions of Lemma 3.16 hold and if $\overline{u}$ and $\underline{u}$ are as in Proposition 3.12, then $(\overline{u})^*_{\tau,\xi}, (\underline{u})^*_{\tau,\xi} \in \mathscr{M}_{\varphi^*_{\tau,\xi}}$ and*

$$(\overline{u})^*_{\tau,\xi} \le u \le (\underline{u})^*_{\tau,\xi} \quad \mathcal{L}^{2n}\text{-a.e. in } \Omega_\tau, \forall u \in \mathscr{M}_{\varphi^*_{\tau,\xi}}.$$

The next proposition states that, whenever we fix an affine boundary datum L, the functional $\mathcal{G}_{L,\Omega}$ admits as unique minimizer the function L itself.

Proposition 3.18. *Let $L: \mathbb{R}^{2n} \to \mathbb{R}$ be given by $L(z) := \langle a, z\rangle + b$ with $a \in \mathbb{R}^{2n}$ and $b \in \mathbb{R}$ and assume g satisfies assumptions (A) and (B). Then L is the unique solution of the problem*

$$\min\{\mathcal{G}_{L,\Omega}(u) : u \in BV(\Omega)\}. \tag{3.49}$$

Proof. We divide the proof in several steps.

Step 1. We claim there exists $p: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $p(z) \in \partial g(z)$ for any $z \in \mathbb{R}^{2n}$ and with the property that

$$\int_\Omega \langle p(\mathbf{X}^*), \sigma_u\rangle\,\mathrm{d}|Du| = \int_{\partial\Omega} u_{|\partial\Omega} \langle p(\mathbf{X}^*), \nu_\Omega\rangle\,\mathrm{d}\mathcal{H}^{2n-1}, \tag{3.50}$$

for any $u \in BV(\Omega)$. If $g \in C^2(\mathbb{R}^{2n})$ formula (3.50) with $p = \nabla g$ follows using the Gauss–Green formula and the fact that, since $\operatorname{div} \mathbf{X}^* = 0$, also $\operatorname{div}\nabla g(\mathbf{X}^*) = 0$. We claim that (3.50) holds true again with $p = \nabla g$ if $g \in C^1(\mathbb{R}^{2n})$. Consider the convolutions $g_h := \rho_h * g$ where ρ_h is a convolution kernel, i.e. $\rho_h \in C_c^\infty(B(0, 1/h))$, $\rho_h \geq 0$ and $\int_{\mathbb{R}^{2n}} \rho_h = 1$. Then $g_h \in C^\infty(\mathbb{R}^{2n})$ and $\nabla g_h \to \nabla g$ uniformly on compact sets. It is now sufficient to pass to the limit in

$$\int_\Omega \langle \nabla g_h(\mathbf{X}^*), \sigma_u \rangle \, \mathrm{d}|Du| = \int_{\partial\Omega} u_{|\partial\Omega} \langle \nabla g_h(\mathbf{X}^*), \nu_\Omega \rangle \, \mathrm{d}\mathcal{H}^{2n-1}$$

using the Dominated Convergence Theorem. Finally we prove that (3.50) holds true for any convex function $g : \mathbb{R}^{2n} \to \mathbb{R}$ and for a suitable choice of p. We are going to use the Yosida approximation; see [38, Sec. IV.1] (see also [39, Theorem 2.1]) for details. Precisely, for any $\lambda > 0$ and for any $z \in \mathbb{R}^{2n}$ let

$$J_\lambda(z) = \min_{y \in \mathbb{R}^{2n}} \left\{ \frac{1}{2\lambda} \|y - z\|^2 + g(y) \right\},$$

and

$$g_\lambda(z) = g(J_\lambda(z)) + \frac{1}{2\lambda} \|z - J_\lambda(z)\|^2$$

Then $g_\lambda \in C^{1,1}(\mathbb{R}^{2n})$ and for any $z \in \mathbb{R}^{2n}$ there holds $\nabla f_\lambda(z) = A_\lambda(z)$ where A_λ is the Yosida approximation of the maximal monotone operator $A = \partial g$, $A_\lambda(z) := \lambda^{-1}(z - J_\lambda(z))$. Moreover, as λ decreases to zero, g_λ increases to g, and for any $z \in \mathbb{R}^{2n}$, $\|A_\lambda(z)\| \to \|\partial^0 g(z)\|$ and $A_\lambda(z) \to \partial^0 g(z)$, where $\partial^0 g(z)$ denotes the element of minimal norm of the closed convex set $\partial g(z)$. Finally, since g has linear growth we have $\|\partial^0 g(z)\| \leq c$ for some $c > 0$ and for every $z \in \mathbb{R}^{2n}$. The thesis now follows by taking $p : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by $p(z) := \partial g^0(z)$ and using the Dominated Convergence Theorem to pass to the limit in

$$\int_\Omega \langle A_\lambda(\mathbf{X}^*), \sigma_u \rangle \, \mathrm{d}|Du| = \int_{\partial\Omega} u_{|\partial\Omega} \langle A_\lambda(\mathbf{X}^*), \nu_\Omega \rangle \, \mathrm{d}\mathcal{H}^{2n-1}$$

as $\lambda \to 0$, obtaining (3.50).

Step 2. We claim that for any $w, z \in \mathbb{R}^{2n}$ we have

$$g^\infty(w) \geq \langle p(\mathbf{X}^*(z)), w \rangle. \tag{3.51}$$

Indeed, by convexity, for any $t > 0$

$$\frac{g(tw + \mathbf{X}^*(z))}{t} \geq \frac{g(\mathbf{X}^*(z))}{t} + \langle p(\mathbf{X}^*(z)), w \rangle,$$

and the conclusion follows letting $t \to \infty$ and using Eq. (3.5).

Step 3. We claim that $u = 0$ is a solution of the problem

$$\min\{\mathcal{G}_{0,\Omega}(u) : u \in BV(\Omega)\}.$$

Let $u \in BV(\Omega)$. Combining the convexity of g with (3.50) and (3.51) we obtain

$$\begin{aligned}
\mathcal{G}_{0,\Omega}(u) &\geq \int_\Omega g(\mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} + \int_\Omega \langle p(\mathbf{X}^*), \nabla u \rangle \, \mathrm{d}\mathcal{L}^{2n} + \int_\Omega \langle p(\mathbf{X}^*), \frac{dD^s u}{d|D^s u|} \rangle \, \mathrm{d}|D^s u| \\
&\quad + \int_{\partial\Omega} g^\infty(-u_{|\partial\Omega} \nu_\Omega) \, \mathrm{d}\mathcal{H}^{2n-1} \\
&= \int_\Omega g(\mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} + \int_\Omega \langle p(\mathbf{X}^*), \sigma_u \rangle \, \mathrm{d}|Du| + \int_{\partial\Omega} g^\infty(-u_{|\partial\Omega} \nu_\Omega) \, \mathrm{d}\mathcal{H}^{2n-1} \\
&\geq \int_\Omega g(\mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} + \int_{\partial\Omega} u_{|\partial\Omega} \langle p(\mathbf{X}^*), \nu_\Omega \rangle \, \mathrm{d}\mathcal{H}^{2n-1} - \int_{\partial\Omega} u_{|\partial\Omega} \langle p(\mathbf{X}^*), \nu_\Omega \rangle \, \mathrm{d}\mathcal{H}^{2n-1} \\
&= \int_\Omega g(\mathbf{X}^*) \, \mathrm{d}\mathcal{L}^{2n} \\
&= \mathcal{G}_{0,\Omega}(0)
\end{aligned} \tag{3.52}$$

which ends the proof of the minimality of $u = 0$.

Step 4. We claim now that if $\Omega = B_R(0)$ then $u = 0$ is the unique solution of the problem

$$\min\{\mathcal{G}_{0,\Omega}(u) : u \in BV(\Omega)\}.$$

Let $u \in BV(\Omega)$ be another minimizer, i.e. $\mathcal{G}_{0,\Omega}(u) = \mathcal{G}_{0,\Omega}(0) = m$. By convexity we have

$$\begin{aligned}
m &\le \mathcal{G}_{0,\Omega}\left(\frac{u}{2}\right)\\
&= \int_\Omega g\left(\frac{1}{2}\nabla u + \mathbf{X}^*\right) \mathrm{d}\mathcal{L}^{2n} + \frac{1}{2}\int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| + \frac{1}{2}\int_{\partial\Omega} g^\infty(-u_{|\partial\Omega}\nu_\Omega)\, \mathrm{d}\mathcal{H}^{2n-1}\\
&\le \frac{1}{2}\int_\Omega g(\nabla u + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \frac{1}{2}\int_\Omega g(\mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \frac{1}{2}\int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u|\\
&\quad + \frac{1}{2}\int_{\partial\Omega} g^\infty(-u_{|\partial\Omega}\nu_\Omega)\, \mathrm{d}\mathcal{H}^{2n-1}\\
&= \frac{1}{2}m + \frac{1}{2}m = m.
\end{aligned}$$

As a consequence we get

$$g\left(\frac{\nabla u + \mathbf{X}^*}{2} + \frac{\mathbf{X}^*}{2}\right) = \frac{g(\nabla u + \mathbf{X}^*) + g(\mathbf{X}^*)}{2}, \quad \mathcal{L}^{2n}\text{-a.e. on } \Omega.$$

Using assumption (A), we conclude that

$$\nabla u = \lambda^a \mathbf{X}^*, \quad \mathcal{L}^{2n}\text{-a.e. on } \Omega$$

for some measurable function $\lambda^a : \Omega \to \mathbb{R}$. Rewriting (3.52) and using (3.51) we then obtain

$$\begin{aligned}
m &= \int_\Omega g\left(\nabla u + \mathbf{X}^*\right) \mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| + \int_{\partial\Omega} g^\infty(-u_{|\partial\Omega}\nu_\Omega)\, \mathrm{d}\mathcal{H}^{2n-1}\\
&\ge \int_\Omega g(\nabla u + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_\Omega \left\langle p(\mathbf{X}^*), \frac{dD^s u}{d|D^s u|}\right\rangle \mathrm{d}|D^s u| + \int_{\partial\Omega} g^\infty(-u_{|\partial\Omega}\nu_\Omega)\, \mathrm{d}\mathcal{H}^{2n-1}\\
&\ge \int_\Omega g(\mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_{\partial\Omega} (u_{|\partial\Omega}\langle p(\mathbf{X}^*), \nu_\Omega\rangle + g^\infty(-u_{|\partial\Omega}\nu_\Omega))\, \mathrm{d}\mathcal{H}^{2n-1}\\
&\ge \int_\Omega g(\mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n}\\
&= m.
\end{aligned}$$

This means that

$$g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) = \left\langle p(\mathbf{X}^*), \frac{dD^s u}{d|D^s u|}\right\rangle, \quad |D^s u|\text{-a.e. on } \Omega \tag{3.53}$$

and

$$u_{|\partial\Omega}\langle p(\mathbf{X}^*), \nu_\Omega\rangle + g^\infty(-u_{|\partial\Omega}\nu_\Omega) = 0, \quad \mathcal{H}^{2n-1}\text{-a.e. on } \Omega. \tag{3.54}$$

Combining assumption (B) with (3.53), we immediately deduce that

$$\frac{dD^s u}{d|D^s u|} = \lambda^s \frac{\mathbf{X}^*}{|\mathbf{X}^*|}, \quad |D^s u|\text{-a.e. on } \Omega$$

for some measurable function $\lambda^s : \Omega \to \mathbb{R}$. From (3.54) we get $u_{|\partial\Omega} = 0$. Indeed, at any point of $\partial\Omega$ where $u_{\partial\Omega} > 0$, condition (3.54) implies

$$g^\infty(-\nu_\Omega) = \langle p(\mathbf{X}^*), -\nu_\Omega\rangle$$

which means, thanks to assumption (B), that ν_Ω is parallel to $\mathbf{X}^*$, and this is impossible since $\Omega = B_R(0)$, namely $\mathbf{X}^* \perp \nu_\Omega$ everywhere on $\partial\Omega$. By means of the same argument we can also exclude $u_{|\partial\Omega} < 0$. Therefore, we can say that

$$\sigma_u = \lambda \mathbf{X}^*, \quad |Du|\text{-a.e. on } \Omega$$

for some measurable function $\lambda : \Omega \to \mathbb{R}$. Lemma 2.5 gives the conclusion.

Step 5. Now we prove that $u = 0$ is the unique solution of the problem

$$\min\{\mathcal{G}_{0,\Omega}(u) : u \in BV(\Omega)\}$$

for a general Ω. Indeed, let $u \in BV(\Omega)$ be such that $\mathcal{G}_{0,\Omega}(u) = \mathcal{G}_{0,\Omega}(0)$. Let $R > 0$ be such that $\Omega \subset\subset B_R(0)$. Let $u_0 : B_R(0) \to \mathbb{R}$ be given by

$$u_0(z) := \begin{cases} u(z) & \text{if } z \in \Omega \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned}\mathcal{G}_{0,B_R(0)}(u_0) &= \int_\Omega g(\nabla u + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right)\,\mathrm{d}|D^s u| + \int_{\partial\Omega} g^\infty(-u_{|\partial\Omega}\nu_\Omega)\,\mathrm{d}\mathcal{H}^{2n-1} \\ &\quad + \int_{B_R(0)\setminus\overline{\Omega}} g(\mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} \\ &= \mathcal{G}_{0,\Omega}(u) + \mathcal{G}_{0,B_R(0)\setminus\overline{\Omega}}(0) \\ &= \mathcal{G}_{0,B_R(0)}(0),\end{aligned}$$

Where in the last equality we used $\mathcal{G}_{0,\Omega}(u) = \mathcal{G}_{0,\Omega}(0)$. Hence, by step 3 we get $u_0 = 0$ from which the conclusion.

Step 6. We conclude the proof proving that $u = L$ is the unique solution of the problem

$$\min\{\mathscr{G}_{L,\Omega}(u) : u \in BV(\Omega)\}.$$

Let $\Omega_a := \Omega - a^*/2$, $u \in BV(\Omega)$ and $u_a : \Omega \to \mathbb{R}$ be given by $u_a(z) := u(z + a^*/2) - L(z)$. Then $u_a \in BV(\Omega_a)$. Hence we get, using step 2,

$$\begin{aligned}\mathcal{G}_{L,\Omega}(u) &= \int_\Omega g(\nabla u + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_\Omega g^\infty\left(\frac{dD^s u}{d|D^s u|}\right)\,\mathrm{d}|D^s u| + \int_{\partial\Omega} g^\infty((L - u_{|\partial\Omega})\nu_\Omega)\,\mathrm{d}\mathcal{H}^{2n-1} \\ &= \int_{\Omega_a} g(\nabla u_a + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} + \int_{\Omega_a} g^\infty\left(\frac{dD^s u_a}{d|D^s u_a|}\right)\,\mathrm{d}|D^s u_a| + \int_{\partial\Omega_a} g^\infty(-(u_a)_{|\partial\Omega}\nu_\Omega)\,\mathrm{d}\mathcal{H}^{2n-1} \\ &= \mathcal{G}_{0,\Omega_a}(u_a) \geq \mathcal{G}_{0,\Omega_a}(0) \\ &= \int_{\Omega_a} g(\mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} = \int_\Omega g(a + \mathbf{X}^*)\,\mathrm{d}\mathcal{L}^{2n} \\ &= \mathscr{G}_{L,\Omega}(L)\end{aligned}$$

which says that $u = L$ is a minimizer. Uniqueness easily follows by the fact that the equality $\mathcal{G}_{L,\Omega}(u) = \mathcal{G}_{L,\Omega}(0)$ implies, using the previous estimate, $\mathcal{G}_{0,\Omega_a}(u_a) = \mathcal{G}_{0,\Omega_a}(0)$ which in turn yields $u_a = 0$ from step 4. In order to conclude the proof it is sufficient to observe that $u_a = 0$ means $u = L$. □

Corollary 3.19. *Let $\Omega \subset \mathbb{R}^{2n}$ be a bounded open set with Lipschitz boundary, $\varphi \in L^1(\partial\Omega)$ and $L : \mathbb{R}^{2n} \to \mathbb{R}$ be an affine function, i.e., $L(z) = \langle a, z\rangle + b$ for some $a \in \mathbb{R}^{2n}, b \in \mathbb{R}$.*

(1) *Assume that $\varphi \leq L$ $\mathcal{H}^{2n-1}$-a.e. on $\partial\Omega$. Then, for any $u \in \mathscr{M}_\varphi$, we have $u \leq L$ $\mathcal{L}^{2n}$-a.e. in Ω.*
(2) *Assume that $\varphi \geq L$ $\mathcal{H}^{2n-1}$-a.e. on $\partial\Omega$. Then, for any $u \in \mathscr{M}_\varphi$, we have $u \geq L$ $\mathcal{L}^{2n}$-a.e. in Ω.*

Proof. Both claims follow immediately from Theorem 3.13 when we observe that the set $\mathscr{M}_L$ consists of just one element that is L itself, so that, following the notations of Proposition 3.12, $L = \overline{L} = \underline{L}$. □

4. The bounded slope condition

We recall the well-known definition of a boundary datum satisfying the Bounded Slope Condition (see [16]). We also refer to [18] for some classical results.

Definition 4.1. We say that a function $\varphi: \partial\Omega \to \mathbb{R}$ satisfies the bounded slope condition with constant $Q > 0$ (Q-B.S.C. for short or simply B.S.C. when the constant Q does not play any role) if for every $z_0 \in \partial\Omega$, there exist two affine functions $w_{z_0}^+$ and $w_{z_0}^-$ such that

$$w_{z_0}^-(z) \le \varphi(z) \le w_{z_0}^+(z) \quad \forall z \in \partial\Omega, \tag{4.1}$$

$$w_{z_0}^-(z_0) = \varphi(z_0) = w_{z_0}^+(z_0) \tag{4.2}$$

$$\mathrm{Lip}(w_{z_0}^-) \le Q \quad \text{and} \quad \mathrm{Lip}(w_{z_0}^+) \le Q, \tag{4.3}$$

where $\mathrm{Lip}(w)$ denotes the Lipschitz constant of w.

Moreover, we denote by f_1 and f_2 the functions defined, respectively, by $f_1(z) := \sup_{z_0\in\partial\Omega} w_{z_0}^-(z)$ and $f_2(z) := \inf_{z_0\in\partial\Omega} w_{z_0}^+(z)$. We underline that f_1 is convex, f_2 is concave and they are both Lipschitz continuous with Lipschitz constant not greater than Q.

The following result can be proved exactly as in [10, Lemma 6.2].

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^{2n}$ be an open bounded set with Lipschitz regular boundary; assume that $\varphi \in L^1(\partial\Omega)$ satisfies the Q-B.S.C. Then, if $u \in BV(\Omega)$ is a minimizer of $\mathcal{G}_{\varphi,\Omega}$, the following facts hold.*

(1) $u_{|\partial\Omega} = \varphi$;
(2) $f_1 \le u \le f_2$ $\mathcal{L}^{2n}$-*a.e. in* Ω;
(3) u *is also a minimizer of* $\mathcal{G}_\Omega$ *in* $\mathrm{BV}(\Omega)$.

The following fact is inspired by [10, Remark 6.4].

Remark 4.3. If $\Omega' \subset \Omega$ are open bounded domains with Lipschitz regular boundary and $u \in \mathrm{BV}(\Omega)$.

Write $\Gamma := \partial\Omega' \cap \Omega$ and $\partial\Omega = \Delta_1 \cup \Delta_2$, where

$$\Delta_1 := \partial\Omega \cap \partial\Omega' \quad \text{and} \quad \Delta_2 := \partial\Omega \setminus \partial\Omega'.$$

Notice that $\partial\Omega' = \Gamma \cup \Delta_1$. We also denote by $u_i, u_o : \Gamma \to \mathbb{R}$ the "inner" and "outer" (with respect to Ω') traces of u on Γ, i.e.,

$$u_i := (u_{|\partial\Omega'}) \llcorner \Gamma \quad \text{and} \quad u_o := (u_{|\partial(\Omega\setminus\overline{\Omega'})}) \llcorner \Gamma.$$

We use the notation $\mathcal{G}_{u,\Omega'}$ to denote the functional $\mathcal{G}_{u_o,\Omega'}$. Let us prove that, if u is a minimizer of $\mathcal{G}_{\varphi,\Omega}$ with $\varphi = u_{|\partial\Omega}$, then u is also a minimizer of $\mathcal{G}_{u,\Omega'}$. Assume by contradiction that u is not a minimizer of $\mathcal{G}_{u,\Omega'}$; then, there exists $v \in \mathrm{BV}(\Omega')$ such that

$$\begin{aligned} 0 &< \mathcal{G}_{u,\Omega'}(u) - \mathcal{G}_{u,\Omega'}(v) \\ &= \mathcal{G}_{\Omega'}(u) - \mathcal{G}_{\Omega'}(v) + \int_\Gamma g^\infty((u_o - u_i)\nu_{\Omega'})\, \mathrm{d}\mathcal{H}^{2n-1} \\ &\qquad - \int_\Gamma g^\infty((u_o - v_{|\partial\Omega'})\nu_{\Omega'})\, \mathrm{d}\mathcal{H}^{2n-1} - \int_{\Delta_1} g^\infty((\varphi - v_{|\partial\Omega'})\nu_\Omega)\, \mathrm{d}\mathcal{H}^{2n-1} \end{aligned} \tag{4.4}$$

where we used inequality (3.3). We would reach a contradiction if we show that the function $w \in \mathrm{BV}(\Omega)$ defined by

$$w := v \text{ on } \Omega', \quad w := u \text{ on } \Omega \setminus \Omega'$$

satisfies $\mathcal{G}_{\varphi,\Omega}(u) - \mathcal{G}_{\varphi,\Omega}(w) > 0$.

Let us compute

$$\begin{aligned} \mathcal{G}_{\varphi,\Omega}(u) = \mathcal{G}_\Omega(u) &= \mathcal{G}_{\Omega'}(u) + \mathcal{G}_{\Omega\setminus\overline{\Omega'}}(u) + \int_\Gamma g^\infty\left(\frac{dD^s u}{d|D^s u|}\right) \mathrm{d}|D^s u| \\ &= \mathcal{G}_{\Omega'}(u) + \mathcal{G}_{\Omega\setminus\overline{\Omega'}}(u) + \int_\Gamma g^\infty((u_o - u_i)\nu_\Gamma)\, \mathrm{d}\mathcal{H}^{2n-1} \end{aligned}$$

and

$$
\begin{aligned}
\mathcal{G}_{\varphi,\Omega}(w) &= \mathcal{G}_{\Omega'}(v) + \mathcal{G}_{\Omega\setminus\overline{\Omega'}}(u) + \int_{\Gamma} g^{\infty}\left(\frac{dD^s w}{d|D^s w|}\right) \mathrm{d}|D^s w| + \int_{\partial\Omega} g^{\infty}((\varphi - w_{|\partial\Omega})\nu_{\Omega}) \,\mathrm{d}\mathcal{H}^{2n-1} \\
&= \mathcal{G}_{\Omega'}(v) + \mathcal{G}_{\Omega\setminus\overline{\Omega'}}(u) + \int_{\Gamma} g^{\infty}((u_o - v_{|\partial\Omega'})\nu_{\Omega'}) \,\mathrm{d}\mathcal{H}^{2n-1} + \int_{\Delta_1} g^{\infty}((\varphi - v_{|\partial\Omega})\nu_{\Omega}) \,\mathrm{d}\mathcal{H}^{2n-1}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{G}_{\varphi,\Omega}(u) - \mathcal{G}_{\varphi,\Omega}(w) \\
&\quad = \mathcal{G}_{\Omega'}(u) - \mathcal{G}_{\Omega'}(v) + \int_{\Gamma}\Big(g^{\infty}((u_o - u_i)\nu_{\Omega'}) - g^{\infty}((u_o - v_{|\partial\Omega'})\nu_{\Omega'})\Big) \,\mathrm{d}\mathcal{H}^{2n-1} \\
&\qquad - \int_{\Delta_1} g^{\infty}((\varphi - v_{|\partial\Omega'})\nu_{\Omega}) \,\mathrm{d}\mathcal{H}^{2n-1} > 0,
\end{aligned}
$$

where we used (4.4) and $u_{|\partial\Omega'} = u_i$.

We are now in position to prove our main result, whose proof is actually very similar to the one given in [10].

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^{2n}$ be open, bounded and with Lipschitz regular boundary, let $\varphi : \partial\Omega \to \mathbb{R}$ satisfy the Q-B.S.C. for some $Q > 0$ and let $g : \mathbb{R}^{2n} \to \mathbb{R}$ be a convex function with linear growth satisfying conditions (A) and (B). Then, the minimization problem*

$$
\min\{\mathcal{G}_{\Omega} : u \in \mathrm{BV}(\Omega),\ u_{|\partial\Omega} = \varphi\} \tag{4.5}
$$

admits a unique solution $\hat{u}$. Moreover, $\hat{u}$ is Lipschitz continuous and $\mathrm{Lip}(\hat{u}) \le \overline{Q} = \overline{Q}(Q, \Omega)$.

Proof. We divide the proof into several steps.

Step 1. We denote by $\overline{u}$ the (pointwise a.e.) maximum of the minimizers of $\mathcal{G}_{\varphi,\Omega}$ in BV (see Proposition 3.12). Lemma 4.2 implies that $f_1 \le \overline{u} \le f_2$ $\mathcal{L}^{2n}$-a.e. in Ω and $\overline{u} = \varphi = f_1 = f_2$ on $\partial\Omega$, where f_1 and f_2 are defined as in Definition 4.1; in particular, $\overline{u}$ is also a minimizer for (4.5).

Let $\tau \in \mathbb{R}^{2n}$ be such that $\Omega \cap \Omega_\tau \ne \emptyset$; following the notations introduced before Lemma 3.16, we consider the function $\overline{u}^*_{\tau,0}$, which we denote by $\overline{u}^*_\tau$ to simplify the notation. Consider the set $\Omega \cap \Omega_\tau$. By Remark 4.3, $\overline{u}$ is a minimizer of $\mathcal{G}_{\overline{u},\Omega\cap\Omega_\tau}$ and, by Corollary 3.17 and Remark 4.3, $\overline{u}^*_\tau$ is a minimizer of $\mathcal{G}_{\overline{u}^*_\tau,\Omega\cap\Omega_\tau}$. Let $z \in \partial(\Omega \cap \Omega_\tau)$, then either $z \in \partial\Omega$ or $z \in \partial\Omega_\tau$.
If $z \in \partial\Omega$, then $z + \tau \in \overline{\Omega}$ and the inequality (36) in [10, Lemma 6.3] implies that

$$
\overline{u}(z) - Q|\tau| \le \overline{u}(z + \tau) \le \overline{u}(z) + Q|\tau|. \tag{4.6}
$$

Otherwise, $z \in \partial\Omega_\tau$ and $z = (z + \tau) - \tau \in \overline{\Omega}$, and Lemma 4.2 implies again (4.6).

So we have proved that (4.6) holds for any $z \in \partial(\Omega \cap \Omega_\tau)$, hence

$$
\overline{u}(z) - Q|\tau| + 2\langle\tau^*, z\rangle \le \overline{u}(z + \tau) + 2\langle\tau^*, z\rangle \le \overline{u}(z) + Q|\tau| + 2\langle\tau^*, z\rangle.
$$

Setting $M := Q + 2\sup_{z\in\Omega}|z|$, one has

$$
\overline{u}(z) - M|\tau| \le \overline{u}^*_\tau(z) \le \overline{u}(z) + M|\tau| \quad \text{for any } z \in \partial(\Omega \cap \Omega_\tau)
$$

and, by Corollary 3.14,

$$
\overline{u}(z) - M|\tau| \le \overline{u}^*_\tau(z) \le \overline{u}(z) + M|\tau| \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } z \in \Omega \cap \Omega_\tau.
$$

This is equivalent to

$$\overline{u}(z) - M|\tau| - 2\langle\tau^*, z\rangle \le \overline{u}(z+\tau) \le \overline{u}(z) + M|\tau| - 2\langle\tau^*, z\rangle \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } z \in \Omega \cap \Omega_\tau$$

and, setting $K := M + 2\sup_{z\in\Omega}|z|$,

$$\overline{u}(z) - K|\tau| \le \overline{u}(z+\tau) \le \overline{u}(z) + K|\tau| \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } z \in \Omega \cap \Omega_\tau.$$

Step 2. We claim that the inequality $|\overline{u}(z) - \overline{u}(\bar{z})| \le K|z - \bar{z}|$ holds for any Lebesgue points $z, \bar{z}$ of $\overline{u}$. We define $\tau := \bar{z} - z$; then $\Omega \cap \Omega_\tau \neq \emptyset$ and, arguing as in Step 1, we obtain

$$|\overline{u}(z'+\tau) - \overline{u}(z')| \le K|\tau| \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } z' \in \Omega \cap \Omega_\tau.$$

Let $\rho > 0$ be such that $B(z,\rho) \subset \Omega \cap \Omega_\tau$ and $B(\bar{z},\rho) \subset \Omega \cap \Omega_\tau$; then

$$\begin{aligned} |\overline{u}(z) - \overline{u}(\bar{z})| &= \left|\lim_{\rho\to 0}\left(⨍_{B(z,\rho)} \overline{u}(z')dz' - ⨍_{B(\bar{z},\rho)} \overline{u}(z')dz'\right)\right| \\ &\le \lim_{\rho\to 0} ⨍_{B(z,\rho)} \left|\overline{u}(z') - \overline{u}(z'+\tau)\right| dz' \le K|z-\bar{z}|. \end{aligned}$$

Step 3. We have proved that $\overline{u}$, the maximum of the minimizer of $\mathcal{G}_{\varphi,\Omega}$, has a representative that is Lipschitz continuous on Ω, with Lipschitz constant not greater than $K = Q + 4\sup_{z\in\Omega}|z|$. The same argument leads to prove that $\underline{u}$, the minimum of the minimizers of $\mathcal{G}_{\varphi,\Omega}$, has a representative that is Lipschitz continuous on Ω, with Lipschitz constant not greater than K. The uniqueness criterion in Proposition 3.4 (with $p = 1$) implies that $\overline{u} = \underline{u}$ $\mathcal{L}^{2n}$-a.e. on Ω. If u is another minimizer of $\mathcal{G}_{\varphi,\Omega}$, we have by Proposition 3.12 that $\underline{u} \le u \le \overline{u}$ $\mathcal{L}^{2n}$-a.e. on Ω. This concludes the proof. □

5. The superlinear growth case

In this section we consider the functional defined in (3.15) by

$$\mathscr{G}_\Omega(u) := \int_\Omega g(\nabla u + \mathbf{X}^*)\, d\mathcal{L}^{2n}, \qquad u \in \varphi + W_0^{1,1}(\Omega) \tag{5.1}$$

where φ satisfies, as in the previous sections, the Bounded Slope Condition of order Q and g has superlinear growth.

Our aim is to show that, for the functional $\mathscr{G}_\Omega$ defined in (5.1), we can get both regularity and uniqueness results using again the Bounded Slope Condition and arguing with the same approach that we used for the BV case.

Theorem 5.1. *Let $g: \mathbb{R}^{2n} \to \mathbb{R}$ be a convex function satisfying condition (A) and let $\varphi: \Omega \to \mathbb{R}$ satisfy the Bounded Slope Condition of order Q on the boundary of Ω. Assume also that g has superlinear growth, i.e., $g(\xi) \ge \psi(|\xi|)$ for a suitable $\psi: [0, +\infty) \to \mathbb{R}$ such that*

$$\lim_{t\to+\infty} \frac{\psi(t)}{t} = +\infty.$$

Then the functional

$$\mathscr{G}_\Omega(u) = \int_\Omega g(\nabla u + \mathbf{X}^*)\, d\mathcal{L}^{2n}, \qquad u \in \varphi + W_0^{1,1}(\Omega) \tag{5.2}$$

has a unique Lipschitz minimizer, i.e.: there exists $u \in \varphi + W_0^{1,\infty}(\Omega)$ such that $\mathscr{G}_\Omega(u) \le \mathscr{G}_\Omega(v)$ for every $v \in \varphi + W_0^{1,1}(\Omega)$.

Proof. The superlinearity of g and the lower semicontinuity of $\mathscr{G}_\Omega$ imply the existence of $u_0 \in \varphi + W_0^{1,1}(\Omega)$ such that $\mathscr{G}_\Omega(u_0) \le \mathscr{G}_\Omega(u)$ for every $u \in \varphi + W_0^{1,1}(\Omega)$.

In the same spirit of the previous sections, we denote by $\mathscr{M}_\varphi = \{v \in \varphi + W_0^{1,1}(\Omega) : \mathscr{G}_\Omega(v) \leq \mathscr{G}_\Omega(u), \forall u \in \varphi + W_0^{1,1}(\Omega)\}$. Thanks to the superlinearity of g we can argue as in the proof of Proposition 3.12 to state that there exist two functions $\overline{u}, \underline{u} \in \mathscr{M}_\varphi$ such that for every $u \in \mathscr{M}_\varphi$

$$\underline{u}(x) \leq u(x) \leq \overline{u}(x) \quad \text{for a.e. } x \in \Omega.$$

We remark that the results contained in Sections 3 and 4 can be restated replacing the space $BV(\Omega)$ with $\varphi + W_0^{1,1}(\Omega)$. All the proofs in fact can be repeated and simplified dropping both the terms where D^s appears and those that take into account the jumps at the boundary. Hence we can conclude that $\overline{u} \in \varphi + W_0^{1,\infty}(\Omega)$, where $K = Q + 2\max_\Omega |z|$. Proposition 3.4 then leads to uniqueness of minimizers. □

References

[1] P. Bousquet, Boundary continuity of solutions to a basic problem in the calculus of variations, Adv. Calc. Var. 3 (2010) 1–27.
[2] P. Bousquet, F. Clarke, Local Lipschitz continuity of solutions to a problem in the calculus of variations, J. Differential Equations 243 (2007) 489–503.
[3] A. Cellina, On the bounded slope condition and the validity of the Euler Lagrange equation, SIAM J. Control Optim. 40 (2001/02) 1270–1279, (electronic).
[4] F. Clarke, Continuity of solutions to a basic problem in the calculus of variations, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 (5) (2005) 511–530.
[5] A. Fiaschi, G. Treu, The bounded slope condition for functionals depending on x, u, and ∇u, SIAM J. Control Optim. 50 (2012) 991–1011.
[6] L. Lussardi, E. Mascolo, A uniqueness result for a class of non strictly convex variational problems, J. Math. Anal. Appl. 446 (2) (2017) 1687–1694.
[7] C. Mariconda, G. Treu, Existence and Lipschitz regularity for minima, Proc. Amer. Math. Soc. 130 (2002) 395–404.
[8] C. Mariconda, G. Treu, Lipschitz regularity for minima without strict convexity of the Lagrangian, J. Differential Equations 243 (2007) 388–413.
[9] C. Mariconda, G. Treu, Local Lipschitz regularity of minima for a scalar problem of the calculus of variations, Commun. Contemp. Math. 10 (2008) 1129–1149.
[10] A. Pinamonti, F. Serra Cassano, G. Treu, D. Vittone, *BV* Minimizers of the area functional in the Heisenberg group under the bounded slope condition, Ann. Sc. Norm. Super. Pisa Cl. Sci. 14 (3) (2015) 907–935.
[11] F. Serra Cassano, D. Vittone, Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in the Heisenberg group, Adv. Calc. Var. 7 (4) (2014) 409–492.
[12] D. Hilbert, Über das Dirichletsche prinzip, Math. Ann. 59 (1904) 161–186.
[13] A. Haar, Über das Plateausche Problem, Math. Ann. 97 (1927) 124–158.
[14] P. Hartman, On the bounded slope condition, Pacific J. Math. 18 (1966) 495–511.
[15] P. Hartman, Convex sets and the bounded slope condition, Pacific J. Math. 25 (1968) 511–522.
[16] G.J. Hartmann, G. Stampacchia, On some nonlinear elliptic differential equations, Acta Math. 115 (1966) 271–310.
[17] G. Stampacchia, On some regular multiple integral problems in the calculus of variations, Comm. Pure Appl. Math. 16 (1963) 383–421.
[18] E. Giusti, Direct Methods in the Calculus of Variations, World Scientific, Pub Co Inc, 2003.
[19] P. Bousquet, C. Mariconda, G. Treu, Hölder continuity of solutions to a basic problem in the calculus of variations, C. R. Math. Acad. Sci. Paris 346 (2008) 1301–1305.
[20] C. Mariconda, G. Treu, Hölder regularity for a classical problem of the calculus of variations, Adv. Calc. Var. 2 (2009) 311–320.
[21] P. Bousquet, L. Brasco, Global Lipschitz continuity for minima of degenerate problems, Math. Ann. 366 (1403) (2016) 1403–1450.
[22] F. Giannetti, G. Treu, On the Lipschitz regularity for minima of functional depending on $x, u, \nabla u$ under the bounded slope condition, 2021, preprint.
[23] C. Mariconda, G. Treu, A Haar-Rado type theorem for minimizers in Sobolev spaces, ESAIM Control Optim. Calc. Var. 17 (2011) 1133–1143.
[24] P. Marcellini, A relation between existence of minima for nonconvex integrals and uniqueness for nonstrictly convex integrals of the calculus of variations, in: Mathematical Theories of Optimization, Genova, in: Lecture Notes in Math., vol. 979, 1981, pp. 216–231, 1983.
[25] L. Beck, T. Schmidt, On the Dirichlet problem for variational integrals in BV, J. Reine Angew. Math. 674 (2013) 113–194.
[26] G. Buttazzo, M. Belloni, A survey on old and recent results about the gap phenomenon in the calculus of variations, recent developments in well-posed variational problems, Math. Appl. (331) (1995) 1–27.
[27] P. Bousquet, C. Mariconda, G. Treu, On the Lavrentiev phenomenon for multiple integral scalar variational problems, J. Funct. Anal. 266 (9) (2014) 5921–5954.
[28] P. Bousquet, C. Mariconda, G. Treu, A survey on the non occurence of the Lavrentiev gap for convex, autonomous multiple integral scalar variational problems., Set-Valued Var. Anal. 23 (1) (2015) 55–68.
[29] C. Mariconda, G. Treu, Non-occurrence of the Lavrentiev phenomenon for a class of convex nonautonomous Lagrangians, Open Math. 18 (1) (2020) 1–9.
[30] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Oxford Science Publications, 2000.

[31] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Birkhäuser, Boston, 1984.
[32] B. Dacorogna, Direct Methods in the Calculus of Variations, second ed., Appl. Math. Sci. 78 (2008).
[33] H.H. Bauschke, P.L. Combettes, Convex Analysis and Monotone Operator Theory in Hilbert Spaces, Springer, 2017.
[34] I. Fonseca, S. Müller, Relaxation of quasiconvex functionals in $\mathrm{BV}(\Omega, \mathbf{R}^p)$ for integrands $f(x, u, \nabla u)$, Arch. Ration. Mech. Anal. 123 (1) (1993) 1–49.
[35] M. Giaquinta, G. Modica, J. Soucek, Functionals with linear growth in the calculus of variations, Comment. Math. Univ. Carolinae 20 (1979) 143–156.
[36] M. Bildhauer, M. Fuchs, Relaxation of convex variational problems with linear growth defined on classes of vector-valued functions, Algebra I Analiz 14 (1) (2002) 26–45.
[37] D. Spector, Simple proofs of some results of Reshetnyak, Proc. Amer. Math. Soc. 139 (5) (2011) 1681–1690.
[38] R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, American Mathematical Society, 2014.
[39] H. Attouch, D. Aze, Approximation and regularization of arbitrary functions in Hilbert spaces by the Lasry-Lions method, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (3) (1993) 289–312.