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Algebraic representations of the weighted mean

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Abstract

We introduce a general framework for the algebraic representation of the weighted mean in the case in which priorities and values refer to a common aggregation domain, for instance in hierarchical multicriteria decision models of the AHP type.

The general framework proposed is based on the semifield structure of open interval domains and provides a natural algebraic description of weighted mean aggregation, including the essential mechanism of normalization which transforms priorities into weights. Such description necessarily involves two operations, addition (abelian semigroup) and multiplication (abelian group), which generalize the role of addition and multiplication in $\mathbb{P} = (0, \infty)$. In this sense, the semifield framework extends recent work by Cavallo, D'Apuzzo, and Squillante on the multiplicative group structure on the basis of the representation of pairwise comparison matrices and their associated priority vectors.

We consider open interval domains $S \subseteq \mathbb{R}$ whose semifield structures are generated by bijections $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. Continuous (thus strictly monotonic) bijections play a central role. In such case, continuous strict triangular conorms/norms and uninorms emerge naturally in the representation of the semifield structure and, in their weighted version, they also provide the representation of the weighted mean, in both the arithmetic and geometric forms.

Keywords: Aggregation on semifield domains, additive and multiplicative structures; Weighted mean, priority normalization; Triangular conorms (norms) and uninorms; Pairwise comparison matrices; Consistency and anti-consistency

1. Introduction

The classical weighted mean is a fundamental aggregation paradigm which is central to a wide range of generalized weighted aggregation models, see for instance Dombi [22], Yager [50], Fodor and Roubens [23], Mesiar [40], Grabisch

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[27], Calvo and Mesiar [10], Marques Pereira and Ribeiro [39] Calvo et al. [11], Yager [51], Mesiar et al. [41], Calvo and Beliakov [12], and Grabisch et al. [30, 31]. The mathematical framework of the weighted mean combines the different roles of priorities and values by means of interrelated sums and products which express a composite algebraic structure, particularly in relation with priority normalization, an essential feature of the weighted mean.

In this paper we consider an interesting special case of the weighted mean framework, in which priorities and values refer to a common aggregation domain. In such case the algebraic description of the weighted mean requires a combined additive and multiplicative structure, which suffices to capture the different aspects of normalization and aggregation.

The algebraic structure of the aggregation domain emerges clearly in the canonical instance of this framework, the weighted mean over $\mathbb{P} = (0, \infty)$, with priorities $u_i \in \mathbb{P}$ and values $x_i \in \mathbb{P}$, with $i = 1, \ldots, n \geq 2$,

$$\frac{u_1 \cdot x_1 + \ldots + u_n \cdot x_n}{u_1 + \ldots + u_n} = w_1 \cdot x_1 + \ldots + w_n \cdot x_n \tag{1}$$

where the normalized priorities produce the weights $w_i = u_i/(u_1 + \ldots + u_n) \in (0, 1)$, $i = 1, \ldots, n$ with $w_1 + \ldots + w_n = 1$.

As the weighted mean formula suggests, the aggregation domain is required to be an additive semigroup and a multiplicative group, plus the usual distributivity. Assuming the abelian nature of the two operations, we obtain the algebraic structure of a semifield, whose canonical instance is the set of strictly positive reals $\mathbb{P} = (0, \infty)$ with the usual addition and multiplication.

In the context of multicriteria decision models, we can find the canonical semifield framework of the weighted mean in the Analytic Hierarchy Process (AHP), see for instance Saaty [43, 44, 45, 46], Saaty and Vargas [47, 48], and Brunelli [8]. In the AHP the weighted mean plays a central role and $\mathbb{P} = (0, \infty)$ is the common aggregation domain of priorities and values. In hierarchical models of this kind the aggregation acts on priority values at various levels, and the normalized priorities of a higher level weight priority values of a lower level.

The multicriteria priorities in the AHP are formulated in terms of pairwise comparison matrices, which are strictly positive and reciprocal. In the context of the AHP the natural abelian multiplicative group structure of pairwise comparison matrices and their associated priority vectors has been investigated by Crawford and Williams [21], Barzilai et al. [3], and Barzilai and Golany [4]. These authors consider two main algebraic representations of the abelian multiplicative group structure, the canonical representation on $\mathbb{P}$ and an equivalent representation on $\mathbb{R}$. The two representations, so-called *multiplicative* ($\mathbb{P}$) and *additive* ($\mathbb{R}$), are further discussed in Barzilai and Golany [5] and Barziali [1, 2].

More recently, Cavallo and D'Apuzzo [15, 16, 18, 19] and Cavallo, D'Apuzzo, and Squillante [17] have extended the previous work on the two classical representations by introducing a general framework for representing pairwise comparison matrices and their associated priority vectors. The general framework proposed is based on the abelian group structure of open real intervals. In addition to the representations on $\mathbb{P}$ and $\mathbb{R}$, these authors have discussed two further representations referring to the bounded domains $(-1, 1)$ and $(0, 1)$.

The weighted mean aggregation scheme, however, can not be described by the multiplicative group structure alone, for it requires an additional additive semigroup structure. In the canonical framework of the AHP, for instance, the weighted mean involves the full semifield structure of $\mathbb{P} = (0, \infty)$, an abelian semigroup under addition and an abelian group under multiplication.

In this paper we introduce a general framework for the representation of the weighted mean, as well as the representation of pairwise comparison matrices and their associated priority vectors. The general framework is based on the semifield structure of open real intervals and extends the work by Cavallo, D'Apuzzo, and Squillante in so far as it incorporates a natural description of weighted mean aggregation. This necessarily involves two operations, addition (abelian semigroup) and multiplication (abelian group), which generalize the role of addition and multiplication in $\mathbb{P} = (0, \infty)$.

We consider open interval domains $S \subseteq \mathbb{R}$ whose semifield structures are generated by bijections $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. The additive and multiplicative structures generated by continuous (and thus strictly monotonic) bijections are represented by continuous strict triangular conorms/norms and continuous strict uninorms, respectively. In turn, the weighted mean is represented by weighted continuous strict triangular conorms/norms (weighted arithmetic mean) or by weighted continuous strict uninorms (weighted geometric mean).

We discuss in detail the normalization and aggregation aspects of the weighted mean in four basic algebraic representations, the *multiplicative* and *additive* representations on $\mathbb{P} = (0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$, plus the *bounded additive* representation on $(-1, 1)$ and the *bounded multiplicative* representation on $(0, 1)$.

The use of two operations, in the context of an abelian semiring structure, has already been considered by Hou [34, 35] and Cavallo [14]. In the framework proposed by these authors, however, the additive abelian semigroup is by no means isomorphic to the addition of strictly positive reals and the multiplicative abelian semigroup is poorer than the multiplication of strictly positive reals.

In recent years there has been growing interest in aggregation over (complete, distributive) lattices, see for instance Couceiro and Marichal [20], Lizasoain and Moreno [38], and Castiñeira, Calvo, and Cubillo [13]. In Lizasoain and Moreno [38], particularly, the authors consider the formulation of the ordered weighted mean (OWA) within the lattice framework. With respect to our approach, besides the different nature of the algebraic structures involved, in [38] weight normalization is assumed a priori, and the two operations are triangular norms and conorms. These, in our semifield approach, are alternative representations of addition, whereas multiplication is represented by uninorms.

The paper is organized as follows. In Section 2 we consider the semifield structure of a general aggregation domain and we define the module, the symmetric ratio, and the weighted mean as natural algebraic constructions involving addition and multiplication. In Section 3 we describe the canonical semifield representation on $\mathbb{P} = (0, \infty)$, with the usual addition and multiplication, and we derive the general form of equivalent semifield representations on open interval

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domains $S \subseteq \mathbb{R}$. In Section 4 we examine the four basic semifield representations on $S = \mathbb{P}$, $\mathbb{R}$, $(-1, 1)$, and $(0, 1)$. In Section 5 we review the description of reciprocal pairwise comparison matrices and the associated priority vectors, both in general and in the four basic representations, and we discuss the notions of consistency and anti-consistency. Finally, Section 6 contains some conclusive remarks.

2. The algebraic structure

We begin by considering the general semifield framework and we define some useful functions in terms of the natural algebraic structure involved. In particular, we introduce the weighted mean and we discuss its transformation properties. Comprehensive reviews of semirings and semifields can be founded in Hebisch and Weinert [32, 33], and Golan [26].

Definition 1. A *semifield* $(S, \oplus, \odot)$ is a set S equipped with two binary operations $\oplus$ and $\odot$, called addition and multiplication, such that:

1. $(S, \oplus)$ is an abelian semigroup:
   - $a \oplus b = b \oplus a$ (commutativity)
   - $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity)
2. $(S, \odot)$ is an abelian group with identity element e:
   - $a \odot b = b \odot a$ (commutativity)
   - $(a \odot b) \odot c = a \odot (b \odot c)$ (associativity)
   - $a \odot e = e \odot a = a$ (identity element)
   - $a \odot a^{-1} = a^{-1} \odot a = e$ (inverse)
3. Multiplication left and right distributes over addition:
   - $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ (distributivity)
   - $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ (distributivity)

In general, we use the notation $\oplus^n a = a \oplus \ldots \oplus a$ and $\odot^n a = a \odot \ldots \odot a$. As usual, $e = e^{-1}$ and $a \odot b^{-1}$ is denoted $a \div b$. The canonical semifield instance is $S = (0, \infty)$ with the usual addition and multiplication.

Definition 2. Given a semifield $(S, \oplus, \odot)$, the *module* $s : S \to S$ of an element $x \in S$ is defined as

$$s(x) = (x \oplus x^{-1}) \div (e \oplus e) \tag{2}$$

where $s(x) = s(x^{-1}) \in S$. Notice that $s(e) = e$.

Definition 3. Given a semifield $(S, \oplus, \odot)$, the *symmetric ratio* $r : S \times S \to S$ between two elements $x, y \in S$ is defined as

$$r(x, y) = (x \div y \oplus y \div x) \div (e \oplus e) \tag{3}$$

where $r(x, y) = r(y, x) \in S$. Notice that $r(x, y) = s(x \div y)$ and therefore $r(x, e) = s(x)$ for all $x, y \in S$.

In relation with a semifield $(S, \oplus, \odot)$, we consider the semivector space S^n equipped with the two standard operations, addition and multiplication by scalar, which fulfill the following axioms: for all a, $b \in S$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in S^n$,

$$\boldsymbol{x} \oplus \boldsymbol{y} = \boldsymbol{y} \oplus \boldsymbol{x} \qquad (\boldsymbol{x} \oplus \boldsymbol{y}) \oplus \boldsymbol{z} = \boldsymbol{x} \oplus (\boldsymbol{y} \oplus \boldsymbol{z}) \tag{4}$$

$$e \odot \boldsymbol{x} = \boldsymbol{x} \qquad (a \odot b) \odot \boldsymbol{x} = a \odot (b \odot \boldsymbol{x}) \tag{5}$$

$$a \odot (\boldsymbol{x} \oplus \boldsymbol{y}) = a \odot \boldsymbol{x} \oplus a \odot \boldsymbol{y} \qquad (a \oplus b) \odot \boldsymbol{x} = a \odot \boldsymbol{x} \oplus b \odot \boldsymbol{x}\,. \tag{6}$$

For simplicity the semifield elements $a, b \in S$ and the semivector space elements $\boldsymbol{x}, \boldsymbol{y} \in S^n$ will be called scalars and vectors, respectively.

Definition 4. Given a semifield $(S, \oplus, \odot)$ and a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in S^n$, the *plain mean* $\bar{x} \in S$ is defined as

$$\bar{x} = (x_1 \oplus \ldots \oplus x_n) \div (\oplus^n e)\,. \tag{7}$$

The semifield structure leads naturally to the fundamental notion of weighted mean at the basis of most aggregation schemes, whose complexity often depends on the way the weighted mean units are arranged and concatenated.

Definition 5. Given a semifield $(S, \oplus, \odot)$, the *weighted mean* $A_{\boldsymbol{u}} : S^n \to S$ with respect to a priority vector $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ is defined as

$$A_{\boldsymbol{u}}(\boldsymbol{x}) = (u_1 \odot x_1 \oplus \ldots \oplus u_n \odot x_n) \div (u_1 \oplus \ldots \oplus u_n) \tag{8}$$

in relation with the vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in S^n$. The weighted mean can be written as

$$A_{\boldsymbol{u}}(\boldsymbol{x}) = w_1 \odot x_1 \oplus \ldots \oplus w_n \odot x_n \tag{9}$$

where the normalized priorities produce the weights $w_i = u_i \div (u_1 \oplus \ldots \oplus u_n)$, $i = 1, \ldots, n$ with $w_1 \oplus \ldots \oplus w_n = e$.

In particular, with $u_i = e$ for $i = 1, \ldots, n$, we obtain the *plain mean*

$$A_{\boldsymbol{e}}(\boldsymbol{x}) = (x_1 \oplus \ldots \oplus x_n) \div (\oplus^n e) = \bar{x}\,. \tag{10}$$

Notice that $s(x)$ corresponds to the plain mean between x and x^{-1}. Analogously, $r(x, y)$ corresponds to the plain mean between $x \div y$ and $y \div x$.

In the definition of the weighted mean the priorities take unconstrained values in S and therefore the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ is not normalized. Normalization is obtained by dividing the vector $\boldsymbol{u}$ by the scalar $u_1 \oplus \ldots \oplus u_n \in S$.

A natural extension of the weighted mean in Definition 5 is obtained in the case in which the scalars $x_1, \ldots, x_n \in S$ are substituted by vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in S^m$, and the weighted mean aggregation acts componentwise.

The transformation properties of the weighted mean under translations and dilations in the semifield domain S are described in the following result.

Proposition 1. *Given a semifield $(S, \oplus, \odot)$ and the weighted mean $A_{\boldsymbol{u}} : S^n \to S$ with respect to a priority vector $\boldsymbol{u} \in S^n$, it holds that:*

- *additive transformations on $\boldsymbol{x}$: with $\boldsymbol{y} = (x_1 \oplus t, \ldots, x_n \oplus t) \in S^n$,*
$$A_{\boldsymbol{u}}(\boldsymbol{y}) = A_{\boldsymbol{u}}(\boldsymbol{x}) \oplus t \tag{11}$$
for all $\boldsymbol{x} \in S^n$ and $t \in S$.
- *additive transformations on $\boldsymbol{u}$: with $\boldsymbol{v} = (u_1 \oplus t, \ldots, u_n \oplus t) \in S^n$,*
$$A_{\boldsymbol{v}}(\boldsymbol{x}) = A_{\boldsymbol{w}}(A_{\boldsymbol{u}}(\boldsymbol{x}), \bar{x}) \tag{12}$$
where $\boldsymbol{w} = (\bar{u}, t) \in S^2$, for all $\boldsymbol{x} \in S^n$ and $t \in S$.
- *multiplicative transformations on $\boldsymbol{x}$: with $\boldsymbol{y} = (x_1 \odot t, \ldots, x_n \odot t) \in S^n$,*
$$A_{\boldsymbol{u}}(\boldsymbol{y}) = A_{\boldsymbol{u}}(\boldsymbol{x}) \odot t \tag{13}$$
for all $\boldsymbol{x} \in S^n$ and $t \in S$.
- *multiplicative transformations on $\boldsymbol{u}$: with $\boldsymbol{v} = (u_1 \odot t, \ldots, u_n \odot t) \in S^n$,*
$$A_{\boldsymbol{v}}(\boldsymbol{x}) = A_{\boldsymbol{u}}(\boldsymbol{x}) \tag{14}$$
for all $\boldsymbol{x} \in S^n$ and $t \in S$.

Proof: Concerning the translations on $\boldsymbol{u}$, we have

$$\begin{aligned}
A_{\boldsymbol{v}}(\boldsymbol{x}) &= \frac{(u_1 \oplus t) \odot x_1 \oplus \ldots \oplus (u_n \oplus t) \odot x_n}{(u_1 \oplus \ldots \oplus u_n) \oplus (\oplus^n t)} \\
&= \frac{(u_1 \odot x_1 \oplus \ldots \oplus u_n \odot x_n) \;\oplus\; (\oplus^n t) \odot \bar{x}}{(u_1 \oplus \ldots \oplus u_n) \;\oplus\; (\oplus^n t)} \\
&= \frac{(u_1 \oplus \ldots \oplus u_n) \odot A_{\boldsymbol{u}}(\boldsymbol{x}) \;\oplus\; (\oplus^n t) \odot \bar{x}}{(u_1 \oplus \ldots \oplus u_n) \;\oplus\; (\oplus^n t)} \\
&= \frac{(\oplus^n \bar{u}) \odot A_{\boldsymbol{u}}(\boldsymbol{x}) \;\oplus\; (\oplus^n t) \odot \bar{x}}{(\oplus^n \bar{u}) \;\oplus\; (\oplus^n t)} \\
&= \frac{\bar{u} \odot A_{\boldsymbol{u}}(\boldsymbol{x}) \;\oplus\; t \odot \bar{x}}{\bar{u} \oplus t} = A_{\boldsymbol{w}}(A_{\boldsymbol{u}}(\boldsymbol{x}), \bar{x})
\end{aligned} \tag{15}$$

where $\boldsymbol{w} = (\bar{u}, t) \in S^2$. In the proof we have used that $\oplus^n t = (\oplus^n e) \odot t$ and analogously for $\bar{u}$, that is, $\oplus^n \bar{u} = (\oplus^n e) \odot \bar{u} = u_1 \oplus \ldots \oplus u_n$. The remaining statements are straightforward: additive transformations on $\boldsymbol{x}$, multiplicative transformations on $\boldsymbol{x}$, and multiplicative transformations on $\boldsymbol{u}$.

□

In the following sections we discuss the representation of the additive and multiplicative semifield structures, plus that of the weighted mean itself, in the context of various open interval domains $S \subseteq \mathbb{R}$.

3. Algebraic representations

In this section we discuss the general semifield framework for the algebraic representations of the weighted mean on open interval domains $S \subseteq \mathbb{R}$. We begin with the canonical semifield representation on $S = (0, \infty) = \mathbb{P}$ and we derive equivalent semifield representations on other open interval domains. In each of these algebraic representations the semifield structure of $S \subseteq \mathbb{R}$ is generated by a bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$ from the canonical semifield structure of $\mathbb{P}$.

The material presented considers both the additive and multiplicative structures of open real interval semifields $S \subseteq \mathbb{R}$, thereby extending the purely multiplicative description of the various representations presented in Cavallo and D'Apuzzo [15, 16, 18] and Cavallo et al. [17].

In discussing the semifield structure of open real intervals $S \subseteq \mathbb{R}$, we also refer to the order and topology of $\mathbb{R}$, which allows the use of the notions of monotonicity and continuity in relation to bijections $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. In this respect, the terms positive (negative), increasing (decreasing), and monotonic are used in the weak sense. Otherwise these properties are said to be strict.

3.1. The canonical semifield representation

The canonical instance of the semifield structure is the so-called *canonical representation*, whose domain $S = (0, \infty) = \mathbb{P}$ is equipped with the usual addition and multiplication,

$$a \oplus b = a + b \qquad a \odot b = a \cdot b \qquad \text{for all } a, b \in \mathbb{P}\,. \tag{16}$$

The identity element is $e = 1$ and every element a has an inverse $a^{-1} = 1/a$. In this representation equations of the form $a \oplus b = c$ and $a \odot b = c$ take the form $a + b = c$ and $a\, b = c$, respectively.

In the open interval domain $S = (0, \infty) = \mathbb{P}$ the usual addition corresponds to a continuous strict t-conorm and the usual multiplication corresponds to a continuous strict uninorm, with neutral element $e = 1$.

The involution $\varsigma : \mathbb{P} \to \mathbb{P}$ with $\varsigma(x) = 1/x$ transforms the usual addition in the continuous strict t-norm $a \oplus b = \varsigma^{-1}(\varsigma(a) + \varsigma(b)) = ab/(a + b)$, whereas multiplication remains the continuous strict uninorm $a \odot b = \varsigma^{-1}(\varsigma(a)\varsigma(b)) = ab$.

The standard description of t-norms, t-conorms, and uninorms refers to closed interval domains, for instance $\bar{S} = [0, \infty]$ or $\bar{S} = [0, 1]$, which include the neutral and absorbing elements of t-norms and t-conorms. Here however, consistently with our semifield framework and particularly with its multiplicative group structure, we refer instead to open interval domains. In this way we also avoid the conjunctive/disjunctive boundary completions regarding uninorms.

In the canonical representation, the module $s : \mathbb{P} \to \mathbb{P}$ and the symmetric ratio $r : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ take the form

$$s(x) = \frac{1}{2}\Big(x + \frac{1}{x}\Big) \qquad r(x,y) = \frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x}\Big) \qquad \text{for all } x, y \in \mathbb{P}. \tag{17}$$

In the canonical representation the module has a number of important properties. In particular, $s(x) \geq 1$ for all $x \in \mathbb{P}$ and $s(x) = 1$ if and only if $x = 1$. Moreover, the module $s(x)$ increases as $x \in \mathbb{P}$ moves away from the identity 1, with left-right symmetry due to $s(x) = s(1/x)$.

Proposition 2. *The module $s : \mathbb{P} \to \mathbb{P}$ as in (17) has the following properties,*

- $s(x) = 1$ *if and only if* $x = 1$
- $s(x) \geq 1$ *for all* $x \in \mathbb{P}$
- $s(x) + s(y) \geq 2\, s\Big(\frac{x+y}{2}\Big)$ *for all* $x, y \in \mathbb{P}$
- $s(x) + s(y) \leq 1 + s(x)s(y)$ *for all* $x, y \in \mathbb{P}$
- $s(x\,y) \geq s(x)\,s(y)$ *if and only if* $x, y \leq 1$ *or* $x, y \geq 1$, *and*
  $s(x\,y) \leq s(x)\,s(y)$ *if and only if* $x \leq 1 \leq y$ *or* $y \leq 1 \leq x$, *with* $x, y \in \mathbb{P}$
- *with* $x \leq y$, $s(x) \geq s(y)$ *if and only if* $x\,y \leq 1$, *and*
  $s(x) \leq s(y)$ *if and only if* $x\,y \geq 1$, *with* $x, y \in \mathbb{P}$.

Proof: The first statement is immediate and the second statement follows from the inequalities

$$(x-1)^2 \geq 0 \quad \Rightarrow \quad x + x^{-1} \geq 2\,. \tag{18}$$

The last inequality can be written as $x - 1 \geq 1 - x^{-1}$ which means (say $0 < x^{-1} < 1 < x$) that the segment from x^{-1} to 1 is shorter than the segment from 1 to x. In other words the plain mean of x and x^{-1} is always to the right of 1.

Regarding the third statement, we begin by noting that

$$\Big(s(x) + s(y) - 2\,s\Big(\frac{x+y}{2}\Big)\Big)(x+y) = r(x,y) - 1 \tag{19}$$

which can be verified straightforwardly. Then, since $r(x,y) = s(x/y) \geq 1$, we obtain directly

$$s(x) + s(y) \geq 2\,s\Big(\frac{x+y}{2}\Big)\,. \tag{20}$$

The fourth statement is obtained from the inequalities

$$0 \leq u + v - uv \leq 1 \qquad u, v \in [0,1] \tag{21}$$

substituting u and v by $1/s(x)$ and $1/s(y)$, respectively.

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In order to prove the first part of the fifth statement let us assume $x, y \leq 1$ or $x, y \geq 1$. It follows that

$$(x^2-1)(y^2-1) \geq 0 \quad \Rightarrow \quad x\,y + \frac{1}{x\,y} \geq \frac{x}{y} + \frac{y}{x} \quad \Rightarrow \tag{22}$$

$$\Rightarrow \quad \frac{1}{2}\Big(x\,y + \frac{1}{x\,y}\Big) \geq \frac{1}{2}\Big(x + \frac{1}{x}\Big)\frac{1}{2}\Big(y + \frac{1}{y}\Big)\,. \tag{23}$$

Since $x, y \in \mathbb{P}$ these steps can be taken backwards, which proves sufficiency. The second part of the fifth statement has an analogous proof, with inequalities reversed. Finally, $s(x\,y) = s(x)\,s(y)$ if and only if $x = 1$ or $y = 1$ (or both).

In order to prove the first part of the sixth statement take $x, y \in \mathbb{P}$ with $x \leq y$ and assume $x\,y \leq 1$. It follows that

$$y - x \geq (y-x)\,x\,y \quad \Rightarrow \quad \frac{1}{x} - \frac{1}{y} \geq y - x \quad \Rightarrow \quad s(x) \geq s(y)\,. \tag{24}$$

Since $x, y \in \mathbb{P}$ these steps can be taken backwards, which proves sufficiency. The second part of the sixth statement has an analogous proof, with inequalities reversed. □

The various results in Proposition 2 can be expressed in terms of the symmetric ratio, since $r(x,y) = s(x/y)$ with $x, y \in \mathbb{P}$. In particular, substituting x by x/y in the first and second statements, we obtain $r(x,y) \geq 1$ for all $x, y \in \mathbb{P}$, and $r(x,y) = 1$ if and only if $x = y$. Moreover, substituting x, y by x/z, y/z in the third and fourth statements, we obtain the inequalities

$$2r(\frac{x+y}{2}, z) \leq r(x,z) + r(y,z) \leq 1 + r(x,z)r(y,z) \qquad \text{for all } x, y, z \in \mathbb{P}\,. \tag{25}$$

In the canonical representation the weighted mean $A_{\boldsymbol{u}} : \mathbb{P}^n \to \mathbb{P}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{P}^n$ takes the form

$$A_{\boldsymbol{u}}(\boldsymbol{x}) = \frac{u_1 x_1 + \ldots + u_n x_n}{u_1 + \ldots + u_n} = w_1 x_1 + \ldots + w_n x_n \qquad \text{for all } \boldsymbol{x} \in \mathbb{P}^n \tag{26}$$

where $w_i \in (0,1)$ for $i = 1, \ldots, n$ and $w_1 + \ldots + w_n = 1$. In particular, the plain mean reduces to

$$\bar{x} = \frac{x_1 + \ldots + x_n}{n}\,. \tag{27}$$

Definition 6. Given a bijection $\tau : \mathbb{P} \to \mathbb{R}$, we define the *generalized weighted mean* $A^\tau_{\boldsymbol{u}} : \mathbb{P}^n \to \mathbb{P}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{P}^n$,

$$A^\tau_{\boldsymbol{u}}(\boldsymbol{x}) = \tau^{-1}\Big(\frac{u_1\tau(x_1) + \ldots + u_n\tau(x_n)}{u_1 + \ldots + u_n}\Big) \qquad \text{for all } \boldsymbol{x} \in \mathbb{P}^n\,. \tag{28}$$

The generalized weighted mean can be written as

$$A^\tau_{\boldsymbol{u}}(\boldsymbol{x}) = \tau^{-1}(w_1\tau(x_1) + \ldots + w_n\tau(x_n)) \tag{29}$$

where $w_i = u_i/(u_1 + \ldots + u_n) \in (0,1)$ for $i = 1, \ldots, n$, with $w_1 + \ldots + w_n = 1$.

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In particular, with the canonical bijection $\tau(x) = \ln x$ we obtain the following generalized weighted mean $B_{\boldsymbol{u}} : \mathbb{P}^n \to \mathbb{P}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{P}^n$,

$$B_{\boldsymbol{u}}(\boldsymbol{x}) = (x_1^{u_1} \dots x_n^{u_n})^{\frac{1}{u_1+\dots+u_n}} = x_1^{w_1} \dots x_n^{w_n} \qquad \text{for all } \boldsymbol{x} \in \mathbb{P}^n\,. \tag{30}$$

The weighted mean $A_{\boldsymbol{u}}$ as in (26) corresponds to the standard weighted arithmetic mean in $\mathbb{P}$ and the generalized weighted mean $B_{\boldsymbol{u}}$ as in (30) corresponds to the standard weighted geometric mean in $\mathbb{P}$. Notice that the weighted geometric mean requires the use of exponential forms x^y with $x, y \in \mathbb{P}$, besides the two semifield operations of addition and multiplication, the former being involved in weight normalization. In other words, the weighted geometric mean requires the full semifield structure of $\mathbb{P}$, not only its multiplicative group structure.

In what follows we derive the general form of other semifield representations on an open interval domains $S \subseteq \mathbb{R}$.

3.2. Equivalent representations

Consider now an open interval domain $S \subseteq \mathbb{R}$. In relation with the canonical representation on $\mathbb{P}$, an *equivalent representation* on $S \subseteq \mathbb{R}$ can be obtained by pull-back with respect to a bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$, thereby endowing S with a semifield structure which is isomorphic to that of $\mathbb{P}$.

Definition 7. Given an open interval $S \subseteq \mathbb{R}$ and a bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$, the *semifield representation* on S generated by ϕ is defined as

$$a \oplus b = \phi^{-1}(\phi(a) + \phi(b)) \qquad a \odot b = \phi^{-1}(\phi(a) \cdot \phi(b)) \qquad \text{for all } a, b \in S\,. \tag{31}$$

In relation with the multiplicative structure we can show that $\phi^{-1}(1) \in S$ and $\phi^{-1}(1/\phi(a)) \in S$, for every $a \in S$, play the role of identity and inverse, respectively. In fact, denoting

$$e = \phi^{-1}(1) \qquad a^{-1} = \phi^{-1}\Big(\frac{1}{\phi(a)}\Big) \tag{32}$$

we have $a \odot e = \phi^{-1}(\phi(a) \cdot \phi(e)) = \phi^{-1}(\phi(a) \cdot 1) = a$ and $a \odot a^{-1} = \phi^{-1}(\phi(a) \cdot \phi(a^{-1})) = \phi^{-1}(\phi(a)/\phi(a)) = \phi^{-1}(1) = e$ for every $a \in S$. It follows that $a \div b = a \odot b^{-1} = \phi^{-1}(\phi(a) \cdot \phi(b^{-1})) = \phi^{-1}(\phi(a)/\phi(b))$ for all $a, b \in S$.

An central role is played by continuous and thus strictly monotonic bijections ϕ. In such case, addition $\oplus$ is represented by a continuous strict t-conorm (resp. t-norm) over $S \subseteq \mathbb{R}$ when the bijection ϕ is strictly increasing (resp. decreasing), see Ling [37]. In the same way, multiplication $\odot$ is represented by a continuous strict uninorm with identity $e = \phi^{-1}(1)$ when the bijection ϕ is strictly monotonic, either increasing or decreasing, see Fodor et al. [24]. Notice that the bijections ϕ and $1/\phi$ generate the same uninorm.

Comprehensive reviews of aggregation functions can be found in Fodor and Roubens [23], Klement et al. [36], Calvo et al. [9], Fodor and De Baets [25], Beliakov et al. [6], Torra and Narukawa [49], Mesiar et al. [42], Grabisch et al. [28], Grabisch and Labreuche [29]. We consider the fundamental results on the representation of t-norms, t-conorms, and uninorms in the context of our framework based on open interval domains $S \subseteq \mathbb{R}$.

In the semifield representation on an open interval $S \subseteq \mathbb{R}$ generated by a bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$, the same pull-back scheme can be shown to apply to the module, the symmetric ratio, and the weighted mean.

Proposition 3. *Consider an open interval $S \subseteq \mathbb{R}$ and the semifield representation generated by the bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. In this representation the module $s : S \to S$ and the symmetric ratio $r : S \times S \to S$ are given by*

$$s(x) = \phi^{-1}\Big(\frac{1}{2}\Big(\phi(x) + \frac{1}{\phi(x)}\Big)\Big) \qquad \textit{for all } x \in S \tag{33}$$

$$r(x,y) = \phi^{-1}\Big(\frac{1}{2}\Big(\frac{\phi(x)}{\phi(y)} + \frac{\phi(y)}{\phi(x)}\Big)\Big) \qquad \textit{for all } x, y \in S. \tag{34}$$

Proof: According to the definition (17) of the module and the representation (31) of addition and multiplication, with $\phi(x \oplus y) = \phi(x) + \phi(y)$, we obtain

$$\begin{aligned} s(x) &= (x \oplus x^{-1}) \div (e \oplus e) = \phi^{-1}\Big(\frac{\phi\,(x \oplus x^{-1})}{\phi\,(e \oplus e)}\Big) \\ &= \phi^{-1}\Big(\frac{\phi(x) + \phi(x^{-1})}{\phi(e) + \phi(e)}\Big) = \phi^{-1}\Big(\frac{\phi(x) + \frac{1}{\phi(x)}}{1+1}\Big) \end{aligned} \tag{35}$$

which corresponds to the desired result. The proof regarding the symmetric ratio is analogous. □

Proposition 4. *Consider an open interval $S \subseteq \mathbb{R}$ and the semifield representation generated by the bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. The module $s : S \to S$ satisfies the following inequalities when ϕ is strictly increasing,*

- *$s(x) = e$ if and only if $x = e$*
- *$s(x) \geq e$ for all $x \in S$*
- *$s(x) \oplus s(y) \geq (e \oplus e) \odot s((x \oplus y) \div (e \oplus e))$ for all $x, y \in S$*
- *$s(x) \oplus s(y) \leq e \oplus s(x) \odot s(y)$ for all $x, y \in S$*
- *$s(x \odot y) \geq s(x) \odot s(y)$ if and only if $x, y \leq e$ or $x, y \geq e$, and $s(x \odot y) \leq s(x) \odot s(y)$ if and only if $x \leq e \leq y$ or $y \leq e \leq x$, with $x, y \in S$*
- *with $x \leq y$, $s(x) \geq s(y)$ if and only if $x \odot y \leq e$, and $s(x) \leq s(y)$ if and only if $x \odot y \geq e$, with $x, y \in S$.*

The inequalities are reversed when ϕ is strictly decreasing.

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Proof: Proposition 4 follows immediately from Proposition 2. The bijection ϕ is an isomorphism between the semifields $S \subseteq \mathbb{R}$ and $\mathbb{P}$, with

$$\phi(a \oplus b) = \phi(a) + \phi(b) \qquad \phi(a \odot b) = \phi(a) \cdot \phi(b) \tag{36}$$

$$\phi(s(c)) = \frac{1}{2}\Big(\phi(c) + \frac{1}{\phi(c)}\Big) \qquad \text{for all } a, b, c \in S. \tag{37}$$

A strictly increasing ϕ is an isomorphism of the ordered semifields and therefore the two ordered semifield structures $S \subseteq \mathbb{R}$ and $\mathbb{P}$ satisfy the same inequalities. Which in the case of a strictly decreasing ϕ are simply reversed. □

Proposition 5. *Consider an open interval $S \subseteq \mathbb{R}$ and the semifield representation generated by the bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. The weighted mean $A_{\boldsymbol{u}} : S^n \to S$ associated with the priority vector $\boldsymbol{u} \in S^n$ is given by*

$$A_{\boldsymbol{u}}(\boldsymbol{x}) = \phi^{-1}\Big(\frac{\phi(u_1) \cdot \phi(x_1) + \ldots + \phi(u_n) \cdot \phi(x_n)}{\phi(u_1) + \ldots + \phi(u_n)}\Big) \qquad \textit{for all } \boldsymbol{x} \in S^n\,. \tag{38}$$

Proof: According to the general definition (8) of the weighted mean and the representation (31) of addition and multiplication,, with $\phi(x \oplus y) = \phi(x) + \phi(y)$ and $\phi(x \odot y) = \phi(x) \cdot \phi(y)$, we obtain

$$\begin{aligned} A_{\boldsymbol{u}}(\boldsymbol{x}) &= (u_1 \odot x_1 \oplus \ldots \oplus u_n \odot x_n) \div (u_1 \oplus \ldots \oplus u_n) \qquad (39)\\ &= \phi^{-1}\Big(\frac{\phi\,(u_1 \odot x_1 \oplus \ldots \oplus u_n \odot x_n)}{\phi\,(u_1 \oplus \ldots \oplus u_n)}\Big) \\ &= \phi^{-1}\Big(\frac{\phi(u_1 \odot x_1) + \ldots + \phi(u_n \odot x_n)}{\phi(u_1) + \ldots + \phi(u_n)}\Big) \\ &= \phi^{-1}\Big(\frac{\phi(u_1) \cdot \phi(x_1) + \ldots + \phi(u_n) \cdot \phi(x_n)}{\phi(u_1) + \ldots + \phi(u_n)}\Big)\,. \end{aligned}$$

□

Notice that the weighted mean can be written as

$$A_{\boldsymbol{u}}(\boldsymbol{x}) = \phi^{-1}(w_1 \cdot \phi(x_1) + \ldots + w_n \cdot \phi(x_n)) \qquad \text{for all } \boldsymbol{x} \in S^n \tag{40}$$

where $w_i = \phi(u_i)/(\phi(u_1) + \ldots + \phi(u_n)) \in (0, 1)$ for $i = 1, \ldots, n$ and $w_1 + \ldots + w_n = 1$. In particular, when $\boldsymbol{u} = \boldsymbol{e}$ the weighted mean (38) reduces to the plain mean

$$\bar{x} = \phi^{-1}\Big(\frac{\phi(x_1) + \ldots + \phi(x_n)}{n}\Big)\,. \tag{41}$$

In the case of continuous and thus strictly monotonic bijections ϕ, the weighted mean $A_{\boldsymbol{u}}$ is represented by a continuous strict weighted t-conorm (resp. t-norm) over $S \subseteq \mathbb{R}$ when the bijection ϕ is strictly increasing (resp. decreasing).

Definition 8. Consider an open interval $S \subseteq \mathbb{R}$ and the semifield representation generated by the bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. Given a bijection $\tau : \mathbb{P} \to \mathbb{R}$, we define the *generalized weighted mean* $A_{\boldsymbol{u}}^{\tau} : S^n \to S$ associated with the priority vector $\boldsymbol{u} \in S^n$ as

$$A_{\boldsymbol{u}}^{\tau}(\boldsymbol{x}) = \phi^{-1}\Big(\tau^{-1}\Big(\frac{\phi(u_1)\cdot\tau(\phi(x_1)) + \ldots + \phi(u_n)\cdot\tau(\phi(x_n))}{\phi(u_1) + \ldots + \phi(u_n)}\Big)\Big) \quad \text{for all } \boldsymbol{x} \in S^n . \tag{42}$$

The generalized weighted mean can be written as

$$A_{\boldsymbol{u}}^{\tau}(\boldsymbol{x}) = \phi^{-1}\Big(\tau^{-1}(w_1\tau(\phi(x_1)) + \ldots + w_n\tau(\phi(x_n)))\Big) \qquad \text{for all } \boldsymbol{x} \in S^n \tag{43}$$

where $w_i = \phi(u_i)/(\phi(u_1) + \ldots + \phi(u_n)) \in (0, 1)$ for $i = 1, \ldots, n$, with $w_1 + \ldots + w_n = 1$.

In particular, with the canonical bijection $\tau(x) = \ln x$ we obtain the following generalized weighted mean $B_{\boldsymbol{u}} : S^n \to S$ associated with the priority vector $\boldsymbol{u} \in S^n$,

$$\begin{aligned} B_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi^{-1}\Big(\big(\phi(x_1)^{\phi(u_1)}\ldots\phi(x_n)^{\phi(u_n)}\big)^{\frac{1}{\phi(u_1)+\ldots+\phi(u_n)}}\Big) \\ &= \phi^{-1}(\phi(x_1)^{w_1}\ldots\phi(x_n)^{w_n}) \qquad \text{for all } \boldsymbol{x} \in S^n \end{aligned} \tag{44}$$

where $w_i = \phi(u_i)/(\phi(u_1) + \ldots + \phi(u_n)) \in (0, 1)$ for $i = 1, \ldots, n$ and $w_1 + \ldots + w_n = 1$.

In the case of continuous and thus strictly monotonic bijections ϕ, the generalized weighted mean $B_{\boldsymbol{u}}$ is represented by a continuous strict weighted uninorm over $S \subseteq \mathbb{R}$ when the bijection ϕ is strictly monotonic, either increasing or decreasing. Notice that the bijections ϕ and $1/\phi$ generate the same uninorm.

3.3. Equivalent representations on $\mathbb{P}$

Consider the bijection

$$\phi : \mathbb{P} \to \mathbb{P} \qquad \phi(c) = \log_2(c+1) \quad c \in \mathbb{P} \qquad \phi^{-1}(C) = 2^C - 1 \quad C \in \mathbb{P} \tag{45}$$

and therefore the two semifield operations take the form

$$\begin{aligned} a \oplus b &= \phi^{-1}(\phi(a) + \phi(b)) \\ &= \phi^{-1}(\log_2(a+1) + \log_2(b+1)) \\ &= a + b + ab \end{aligned} \tag{46}$$

$$\begin{aligned} a \odot b &= \phi^{-1}(\phi(a)\cdot\phi(b)) \\ &= \phi^{-1}(\log_2(a+1)\cdot\log_2(b+1)) \\ &= (a+1)^{\log_2(b+1)} - 1 \end{aligned} \tag{47}$$

for all $a, b \in \mathbb{P}$. The identity element is

$$e = \phi^{-1}(1) = 2^1 - 1 = 1 \tag{48}$$

and the inverse of an element $a \in S$ is

$$a^{-1} = \phi^{-1}\Big(\frac{1}{\phi(a)}\Big) = \phi^{-1}\Big(\frac{1}{\log_2(a+1)}\Big) = 2^{\frac{1}{\log_2(a+1)}} - 1\,. \tag{49}$$

Consider now the bijection

$$\phi : \mathbb{P} \to \mathbb{P} \qquad \phi(c) = 2^c - 1 \quad c \in \mathbb{P} \qquad \phi^{-1}(C) = \log_2(C+1) \quad C \in \mathbb{P}\,. \tag{50}$$

Now the two semifield operations take the form

$$\begin{aligned} a \oplus b &= \phi^{-1}(\phi(a) + \phi(b)) \\ &= \phi^{-1}(2^a - 1 + 2^b - 1) \\ &= \log_2(2^a + 2^b - 1) \end{aligned} \tag{51}$$

$$\begin{aligned} a \odot b &= \phi^{-1}(\phi(a) \cdot \phi(b)) \\ &= \phi^{-1}((2^a - 1) \cdot (2^b - 1)) \\ &= \log_2(2^{ab} - 2^a - 2^b + 2) \end{aligned} \tag{52}$$

for all $a, b \in \mathbb{P}$. The identity element is

$$e = \phi^{-1}(1) = \log_2(1+1) = 1 \tag{53}$$

and the inverse of an element $a \in S$ is

$$a^{-1} = \phi^{-1}\Big(\frac{1}{\phi(a)}\Big) = \phi^{-1}\Big(\frac{1}{2^a - 1}\Big) = \log_2 \frac{2^a}{2^a - 1}\,. \tag{54}$$

In the next section we present four equivalent semifield representations of the weighted mean on the real open interval domains $S = (0, \infty)$, $S = (-\infty, \infty)$, $S = (-1, 1)$, and $S = (0, 1)$.

4. The basic algebraic representations

In this section we discuss the semifield representations of the weighted mean on the open interval domains $S \subseteq \mathbb{R}$ most commonly referred in the literature. We begin with the multiplicative representation on $S = (0, \infty) = \mathbb{P}$, which corresponds to the canonical representation described in the previous section, and we derive the equivalent representations over the domains $S = \mathbb{R}$ for the additive representation, $S = (-1, 1) = \mathbb{B}_\diamond$ for the bounded additive representation, and $S = (0, 1) = \mathbb{B}_\star$ for the bounded multiplicative representation, as depicted in Fig. 1. The associated bijections ϕ_+, $\phi_\diamond$, $\phi_\star$ are defined and the full structure of the corresponding representations is derived.

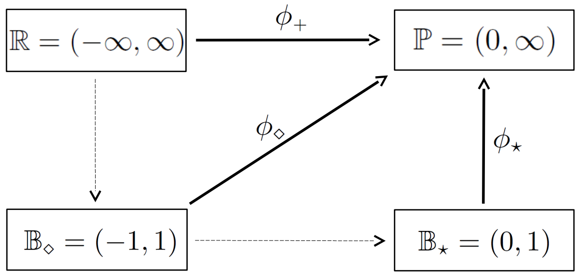

Figure 1: Diagram of the basis algebraic representations.

4.1. Multiplicative representation

The *multiplicative representation* corresponds to the canonical representation whose domain is $\mathbb{P} = (0, \infty)$. We briefly review the main aspects in order to offer the reader a direct comparison with the alternative representations.

The two semifield operations take the form

$$a \oplus b = a + b \qquad a \odot b = a \cdot b \tag{55}$$

for all $a, b \in \mathbb{R}$. In this representation equations of the form $a \oplus b = c$ and $a \odot b = c$ take the form $a + b = c$ and $a \cdot b = c$, respectively.

In the multiplicative representation the module $s : \mathbb{P} \to \mathbb{P}$ and the symmetric ratio $r : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ are as follows,

$$s(x) = \frac{1}{2}\Big(x + \frac{1}{x}\Big) \qquad r(x, y) = \frac{1}{2}\Big(\frac{x}{y} + \frac{y}{x}\Big) \tag{56}$$

and the weighted mean $A_{\boldsymbol{u}} : \mathbb{P}^n \to \mathbb{P}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{P}^n$ is expressed as

$$A_{\boldsymbol{u}}(\boldsymbol{x}) = \frac{u_1 x_1 + \ldots + u_n x_n}{u_1 + \ldots + u_n} = w_1 x_1 + \ldots + w_n x_n \qquad \text{for all } \boldsymbol{x} \in \mathbb{P}^n \tag{57}$$

where $w_i = u_i/(u_1 + \ldots + u_n) \in (0, 1)$ for $i = 1, \ldots, n$ and $w_1 + \ldots + w_n = 1$. In particular we obtain the plain mean

$$\bar{x} = \frac{x_1 + \ldots + \boldsymbol{x}_n}{n}\,. \tag{58}$$

In the multiplicative representation the generalized weighted mean $B_{\boldsymbol{u}} : \mathbb{P}^n \to \mathbb{P}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{P}^n$ is

$$B_{\boldsymbol{u}}(\boldsymbol{x}) = (x_1^{u_1} \ldots x_n^{u_n})^{\frac{1}{u_1 + \ldots + u_n}} = x_1^{w_1} \ldots x_n^{w_n} \qquad \text{for all } \boldsymbol{x} \in \mathbb{P}^n \tag{59}$$

where $w_i = u_i/(u_1 + \ldots + u_n) \in (0, 1)$ for $i = 1, \ldots, n$, with $w_1 + \ldots + w_n = 1$.

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4.2. Additive representation

The *additive representation*, whose domain is $S = (-\infty, \infty) = \mathbb{R}$, is generated by the bijection

$$\phi_+ : \mathbb{R} \to \mathbb{P} \qquad \phi_+(c) = e^c \quad c \in \mathbb{R} \qquad \phi_+^{-1}(C) = \ln C \quad C \in \mathbb{P} \tag{60}$$

and therefore the two semifield operations take the form

$$a \oplus b = \phi_+^{-1}(\phi_+(a) + \phi_+(b)) = \ln(e^a + e^b) \tag{61}$$

$$a \odot b = \phi_+^{-1}(\phi_+(a) \cdot \phi_+(b)) = \ln(e^a \cdot e^b) = a + b \tag{62}$$

for all $a, b \in \mathbb{R}$. The identity element is

$$e = \phi_+^{-1}(1) = \ln 1 = 0 \tag{63}$$

and the inverse is

$$a^{-1} = \phi_+^{-1}\Big(\frac{1}{\phi_+(a)}\Big) = \phi_+^{-1}\Big(e^{-a}\Big) = \ln e^{-a} = -a\,. \tag{64}$$

In relation to the identity element $e = 0$, we obtain $0 \oplus 0 = \ln 2$ and in general $\oplus^n 0 = \ln n$. Clearly $\odot^n 0 = 0$ by the identity element property.

In this representation equations of the form $a \oplus b = c$ and $a \odot b = c$ take the form $\ln(e^a + e^b) = c$ and $a + b = c$, respectively.

In the additive representation the module $s : \mathbb{R} \to \mathbb{R}$ and the symmetric ratio $r : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are expressed, according to (33) and (34), as follows,

$$s(x) = \ln\Big(\frac{e^x + e^{-x}}{2}\Big) \qquad\qquad r(x, y) = \ln\Big(\frac{e^{x-y} + e^{y-x}}{2}\Big) \tag{65}$$

and the weighted mean $A_{\boldsymbol{u}} : \mathbb{R}^n \to \mathbb{R}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{R}^n$, according to (38), takes the form

$$\begin{aligned} A_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi_+^{-1}\Big(\frac{\phi_+(u_1)\cdot\phi_+(x_1) + \ldots + \phi_+(u_n)\cdot\phi_+(x_n)}{\phi_+(u_1) + \ldots + \phi_+(u_n)}\Big) \\ &= \ln\Big(w_1 e^{x_1} + \ldots + w_n e^{x_n}\Big) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n \end{aligned} \tag{66}$$

where $w_i = e^{u_i}/(e^{u_1} + \ldots + e^{u_n}) \in (0, 1)$ for $i = 1, \ldots, n$ with $w_1 + \ldots + w_n = 1$. In particular we obtain the plain mean

$$\bar{x} = \ln\Big(\frac{e^{x_1} + \ldots + e^{x_n}}{n}\Big)\,. \tag{67}$$

In the additive representation the generalized weighted mean $B_{\boldsymbol{u}} : \mathbb{R}^n \to \mathbb{R}$ associated with the priority vector $\boldsymbol{u} \in \mathbb{R}^n$, according to (44), takes the form

$$\begin{aligned} B_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi_+^{-1}\Big(\Big(\phi_+(x_1)^{\phi_+(u_1)} \ldots \phi_+(x_n)^{\phi_+(u_n)}\Big)^{\frac{1}{\phi_+(u_1)+\ldots+\phi_+(u_n)}}\Big) \\ &= w_1 x_1 + \ldots + w_n x_n \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^n \end{aligned} \tag{68}$$

where $w_i = e^{u_i}/(e^{u_1} + \ldots + e^{u_n}) \in (0, 1)$ for $i = 1, \ldots, n$, with $w_1 + \ldots + w_n = 1$.

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4.3. Bounded additive representation

The *bounded additive representation*, whose domain is $S = (-1, 1) = \mathbb{B}_\diamond$, is generated by the bijection

$$\phi_\diamond : \mathbb{B}_\diamond \to \mathbb{P} \qquad \phi_\diamond(c) = \frac{1+c}{1-c} \quad c \in \mathbb{B}_\diamond \qquad \phi_\diamond^{-1}(C) = \frac{C-1}{C+1} \quad C \in \mathbb{P} \tag{69}$$

and therefore the two semifield operations take the form

$$a \oplus b = \phi_\diamond^{-1}(\phi_\diamond(a) + \phi_\diamond(b)) = \frac{(\frac{1+a}{1-a} + \frac{1+b}{1-b}) - 1}{(\frac{1+a}{1-a} + \frac{1+b}{1-b}) + 1} = \frac{1 + a + b - 3a\,b}{3 - a - b - a\,b} \tag{70}$$

$$a \odot b = \phi_\diamond^{-1}(\phi_\diamond(a) \cdot \phi_\diamond(b)) = \frac{(\frac{1+a}{1-a} \cdot \frac{1+b}{1-b}) - 1}{(\frac{1+a}{1-a} \cdot \frac{1+b}{1-b}) + 1} = \frac{a+b}{1+a\,b} \tag{71}$$

for all $a, b \in \mathbb{B}_\diamond$. The identity element is

$$e = \phi_\diamond^{-1}(1) = \frac{1-1}{1+1} = 0 \tag{72}$$

and the inverse is

$$a^{-1} = \phi_\diamond^{-1}\Big(\frac{1}{\phi_\diamond(a)}\Big) = \phi_\diamond^{-1}\Big(\frac{1-a}{1+a}\Big) = \frac{\frac{1-a}{1+a} - 1}{\frac{1-a}{1+a} + 1} = -a\,. \tag{73}$$

In relation to the identity element $e = 0$, we obtain $0 \oplus 0 = 1/3$ and in general $\oplus^n 0 = (n-1)/(n+1)$. Clearly $\odot^n 0 = 0$ by construction.

In this representation equations of the form $a \oplus b = c$ and $a \odot b = c$ take the form

$$\frac{1 + a + b - 3a\,b}{3 - a - b - a\,b} = c \qquad \frac{a+b}{1+a\,b} = c \tag{74}$$

respectively. Moreover, we obtain

$$(a \oplus b) \oplus c = \frac{1 - a\,b - a\,c - b\,c + 2\,a\,b\,c}{2 - a - b - c + a\,b\,c} = a \oplus (b \oplus c) \tag{75}$$

$$(a \odot b) \odot c = \frac{a + b + c + a\,b\,c}{1 + a\,b + a\,c + b\,c} = a \odot (b \odot c)\,. \tag{76}$$

In the bounded additive representation the module $s : \mathbb{B}_\diamond \to \mathbb{B}_\diamond$, according to (33), is expressed as follows

$$\begin{aligned} s(x) &= \phi_\diamond^{-1}\Big(\frac{1}{2}\Big(\phi_\diamond(x) + \frac{1}{\phi_\diamond(x)}\Big)\Big) \\ &= \frac{\frac{1+x^2}{1-x^2} - 1}{\frac{1+x^2}{1-x^2} + 1} = x^2 \qquad \text{for all } x \in \mathbb{B}_\diamond\,. \end{aligned} \tag{77}$$

Analogously, the symmetric ratio $r : \mathbb{B}_\diamond \times \mathbb{B}_\diamond \to \mathbb{B}_\diamond$, according to (34), takes the form

$$\begin{aligned} r(x,y) &= \phi_\diamond^{-1}\Big(\frac{1}{2}\Big(\frac{\phi_\diamond(x)}{\phi_\diamond(y)} + \frac{\phi_\diamond(y)}{\phi_\diamond(x)}\Big)\Big) \\ &= \frac{\frac{(1+x)^2(1-y)^2+(1-x)^2(1+y)^2}{2(1-x^2)(1-y^2)} - 1}{\frac{(1+x)^2(1-y)^2+(1-x)^2(1+y)^2}{2(1-x^2)(1-y^2)} + 1} \\ &= \Big(\frac{x-y}{1-x\,y}\Big)^2 \qquad \text{for all } x,y \in \mathbb{B}_\diamond\,. \end{aligned} \tag{78}$$

In the bounded additive representation the weighted mean $A_{\boldsymbol{u}} : \mathbb{B}_\diamond^n \to \mathbb{B}_\diamond$ associated with the priority vector $\boldsymbol{u} \in \mathbb{B}_\diamond^n$, according to (38), takes the form

$$\begin{aligned} A_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi_\diamond^{-1}\Big(\frac{\phi_\diamond(u_1)\cdot\phi_\diamond(x_1) + \ldots + \phi_\diamond(u_n)\cdot\phi_\diamond(x_n)}{\phi_\diamond(u_1) + \ldots + \phi_\diamond(u_n)}\Big) \\ &= \frac{\sum_{i=1}^n w_i(1+x_i)/(1-x_i) - 1}{\sum_{i=1}^n w_i(1+x_i)/(1-x_i) + 1} \qquad \text{for all } \boldsymbol{x} \in \mathbb{B}_\diamond^n \end{aligned} \tag{79}$$

where $w_i \in (0,1)$ for $i = 1,\ldots,n$ and $w_1 + \ldots + w_n = 1$, with

$$w_i = \frac{(1+u_i)/(1-u_i)}{\sum_{j=1}^n (1+u_j)/(1-u_j)} \in (0,1) \qquad i = 1,\ldots,n\,.$$

In particular, with $u_1 = \cdots = u_n = 0 \in \mathbb{B}_\diamond$, we obtain the plain mean

$$\bar{x} = \frac{\sum_{i=1}^n (1+x_i)/(1-x_i) - n}{\sum_{i=1}^n (1+x_i)/(1-x_i) + n}\,. \tag{80}$$

In the bounded additive representation the generalized weighted mean $B_{\boldsymbol{u}} : \mathbb{B}_\diamond^n \to \mathbb{B}_\diamond$ associated with the priority vector $\boldsymbol{u} \in \mathbb{B}_\diamond^n$, according to (44), takes the form

$$\begin{aligned} B_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi_\diamond^{-1}\Big(\big(\phi_\diamond(x_1)^{\phi_\diamond(u_1)}\ldots\phi_\diamond(x_n)^{\phi_\diamond(u_n)}\big)^{\frac{1}{\phi_\diamond(u_1)+\ldots+\phi_\diamond(u_n)}}\Big) \\ &= \frac{\prod_{i=1}^n(1+x_i)^{w_i} - \prod_{i=1}^n(1-x_i)^{w_i}}{\prod_{i=1}^n(1+x_i)^{w_i} + \prod_{i=1}^n(1-x_i)^{w_i}} \qquad \text{for all } \boldsymbol{x} \in \mathbb{B}_\diamond^n \end{aligned} \tag{81}$$

where $w_i \in (0,1)$ for $i = 1,\ldots,n$ and $w_1 + \ldots + w_n = 1$, with

$$w_i = \frac{(1+u_i)/(1-u_i)}{\sum_{j=1}^n (1+u_j)/(1-u_j)} \in (0,1) \qquad i = 1,\ldots,n\,.$$

4.4. Bounded multiplicative representation

The *bounded multiplicative representation* (also called fuzzy representation), whose domain is $S = (0,1) = \mathbb{B}_\star$, is generated by the bijection

$$\phi_\star : \mathbb{B}_\star \to \mathbb{P} \qquad \phi_\star(c) = \frac{c}{1-c} \quad c \in \mathbb{B}_\star \qquad \phi_\star^{-1}(C) = \frac{C}{1+C} \quad C \in \mathbb{P} \tag{82}$$

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and therefore the two semifield operations take the form

$$a \oplus b = \phi_\star^{-1}(\phi_\star(a) + \phi_\star(b)) = \frac{\frac{a}{1-a} + \frac{b}{1-b}}{1 + (\frac{a}{1-a} + \frac{b}{1-b})} = \frac{a + b - 2\,a\,b}{1 - a\,b} \tag{83}$$

$$a \odot b = \phi_\star^{-1}(\phi_\star(a) \cdot \phi_\star(b)) = \frac{\frac{a}{1-a} \cdot \frac{b}{1-b}}{1 + (\frac{a}{1-a} \cdot \frac{b}{1-b})} = \frac{a\,b}{a\,b + (1-a)(1-b)} \tag{84}$$

for all $a, b \in \mathbb{B}_\star$. The identity element is

$$e = \phi_\star^{-1}(1) = \frac{1}{1+1} = \frac{1}{2} \tag{85}$$

and the inverse is

$$a^{-1} = \phi_\star^{-1}\Big(\frac{1}{\phi_\star(a)}\Big) = \phi_\star^{-1}\Big(\frac{1-a}{a}\Big) = \frac{\frac{1-a}{a}}{1 + \frac{1-a}{a}} = 1 - a\,. \tag{86}$$

In relation to the identity element $e = 1/2$, we obtain $1/2 \oplus 1/2 = 2/3$ and in general $\oplus^n 1/2 = n/(n+1)$. Clearly $\odot^n 1/2 = 1/2$ by construction.

In this representation equations of the form $a \oplus b = c$ and $a \odot b = c$ take the form

$$\frac{a + b - 2\,a\,b}{1 - a\,b} = c \qquad \frac{a\,b}{a\,b + (1-a)(1-b)} = c \tag{87}$$

respectively. Moreover, we obtain

$$(a \oplus b) \oplus c = \frac{a + b + c - 2\,a\,b - 2\,a\,c - 2\,b\,c + 3\,a\,b\,c}{1 - a\,b - a\,c - b\,c + 2\,a\,b\,c} = a \oplus (b \oplus c) \tag{88}$$

$$(a \odot b) \odot c = \frac{a\,b\,c}{a\,b\,c + (1-a)(1-b)(1-c)} = a \odot (b \odot c)\,. \tag{89}$$

In the bounded multiplicative representation the module $s : \mathbb{B}_\star \to \mathbb{B}_\star$, according to (33), is expressed as follows

$$\begin{aligned} s(x) &= \phi_\star^{-1}\Big(\frac{1}{2}\Big(\phi_\star(x) + \frac{1}{\phi_\star(x)}\Big)\Big) \\ &= \frac{\frac{2x^2-2x+1}{2x(1-x)}}{1 + \frac{2x^2-2x+1}{2x(1-x)}} = 1 - 2\,x(1-x) \qquad \text{for all } x \in \mathbb{B}_\star\,. \end{aligned} \tag{90}$$

Analogously, the symmetric ratio $r : \mathbb{B}_\star \times \mathbb{B}_\star \to \mathbb{B}_\star$, according to (34), takes the form

$$\begin{aligned} r(x,y) &= \phi_\star^{-1}\Big(\frac{1}{2}\Big(\frac{\phi_\star(x)}{\phi_\star(y)} + \frac{\phi_\star(y)}{\phi_\star(x)}\Big)\Big) \\ &= \frac{\frac{x^2(1-y)^2+(1-x)^2y^2}{2xy(1-x)(1-y)}}{1 + \frac{x^2(1-y)^2+(1-x)^2y^2}{2xy(1-x)(1-y)}} = \frac{x^2\,(1-y)^2 + (1-x)^2\,y^2}{(x\,(1-y) + (1-x)\,y)^2}\,. \end{aligned} \tag{91}$$

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In the bounded multiplicative representation the weighted mean $A_{\boldsymbol{u}} : \mathbb{B}_\star^n \to \mathbb{B}_\star$ associated with the priority vector $\boldsymbol{u} \in \mathbb{B}_\star^n$, according to (38), takes the form

$$\begin{aligned} A_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi_\star^{-1}\Big(\frac{\phi_\star(u_1)\cdot\phi_\star(x_1)+\ldots+\phi_\star(u_n)\cdot\phi_\star(x_n)}{\phi_\star(u_1)+\ldots+\phi_\star(u_n)}\Big) \\ &= \frac{\sum_{i=1}^n w_i\, x_i/(1-x_i)}{\sum_{i=1}^n w_i\, x_i/(1-x_i)+1} \qquad \text{for all } \boldsymbol{x} \in \mathbb{B}_\star^n \end{aligned} \tag{92}$$

where $w_i \in (0,1)$ for $i = 1,\ldots,n$ and $w_1 + \ldots + w_n = 1$, with

$$w_i = \frac{u_i/(1-u_i)}{\sum_{j=1}^n u_j/(1-u_j)} \in (0,1) \qquad i = 1,\ldots,n\,.$$

In particular, with $u_1 = \cdots = u_n = 1/2 \in \mathbb{B}_\star$, we obtain the plain mean

$$\bar{x} = \frac{\sum_{i=1}^n x_i/(1-x_i)}{\sum_{i=1}^n x_i/(1-x_i)+n}\,. \tag{93}$$

In the bounded multiplicative representation the generalized weighted mean $B_{\boldsymbol{u}} : \mathbb{B}_\star^n \to \mathbb{B}_\star$ associated with the priority vector $\boldsymbol{u} \in \mathbb{B}_\star^n$, according to (44), takes the form

$$\begin{aligned} B_{\boldsymbol{u}}(\boldsymbol{x}) &= \phi_\star^{-1}\Big(\big(\phi_\star(x_1)^{\phi_\star(u_1)}\ldots\phi_\star(x_n)^{\phi_\star(u_n)}\big)^{\frac{1}{\phi_\star(u_1)+\ldots+\phi_\star(u_n)}}\Big) \\ &= \frac{\prod_{i=1}^n x_i^{w_i}}{\prod_{i=1}^n x_i^{w_i} + \prod_{i=1}^n (1-x_i)^{w_i}} \qquad \text{for all } \boldsymbol{x} \in \mathbb{B}_\star^n \end{aligned} \tag{94}$$

where $w_i \in (0,1)$ for $i = 1,\ldots,n$ and $w_1 + \ldots + w_n = 1$, with

$$w_i = \frac{u_i/(1-u_i)}{\sum_{j=1}^n u_j/(1-u_j)} \in (0,1) \qquad i = 1,\ldots,n\,.$$

4.5. Changing the bijection

Naturally the algebraic form of addition and multiplication in a given representation on an open interval $S \subseteq \mathbb{R}$ depends significantly on the bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. For instance, taking $S = (0,1)$ as in the bounded multiplicative representation but considering a different bijection,

$$\phi : (0,1) \to \mathbb{P} \quad \phi(c) = \sqrt{\frac{c}{1-c}} \quad c \in (0,1) \quad \phi^{-1}(C) = \frac{C^2}{1+C^2} \quad C \in \mathbb{P} \tag{95}$$

the algebraic form of multiplication remains unchanged, but that of addition changes,

$$a \oplus b = \phi^{-1}(\phi(a)+\phi(b)) = \frac{a+b-2\,a\,b+2\sqrt{a\,b\,(1-a)(1-b)}}{1-a\,b+2\sqrt{a\,b\,(1-a)(1-b)}} \tag{96}$$

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$$a \odot b = \phi^{-1}(\phi(a) \cdot \phi(b)) = \frac{a\,b}{a\,b + (1-a)(1-b)} \tag{97}$$

where $a, b \in (0,1)$. Both the identity and the inverse remain unchanged,

$$e = \phi^{-1}(1) = \frac{1}{2} \qquad a^{-1} = \phi^{-1}\Big(\frac{1}{\phi(a)}\Big) = 1 - a\,. \tag{98}$$

Another example, with the same domain $S = (0,1)$, considers the bijection

$$\phi : (0,1) \to \mathbb{P} \quad \phi(c) = \frac{t\,c}{1-c} \quad c \in (0,1) \quad \phi^{-1}(C) = \frac{C}{t+C} \quad C \in \mathbb{P} \tag{99}$$

where $t \in \mathbb{P}$. In this case the algebraic form of addition remains unchanged, but that of multiplication changes,

$$a \oplus b = \phi^{-1}(\phi(a) + \phi(b)) = \frac{a + b - 2\,a\,b}{1 - a\,b} \tag{100}$$

$$a \odot b = \phi^{-1}(\phi(a) \cdot \phi(b)) = \frac{a\,b\,t}{a\,b\,t + (1-a)(1-b)} \tag{101}$$

where $a, b \in (0,1)$. Both the identity and the inverse change,

$$e = \phi^{-1}(1) = \frac{1}{1+t} \qquad a^{-1} = \phi^{-1}\Big(\frac{1}{\phi(a)}\Big) = \frac{1-a}{1 - a + a\,t^2}\,. \tag{102}$$

Analogous considerations can be made in relation to the semifiels representations on the various open intervals discussed in this section. In each case the representation is generated by a bijection traditionally referred in the literature, but different choices are possible.

5. Pairwise comparison matrices

In a large class of multicriteria aggregation models defined on open interval domains $S \subseteq \mathbb{R}$ equipped with semifield structures generated by bijections $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$, the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ associated with n criteria is derived from a pairwise comparison matrix $\mathbf{A} = [a_{ij} \in S]$ between criteria, with $i, j = 1, \ldots, n$.

Definition 9. Given a semifield $(S, \oplus, \odot)$, a matrix $\mathbf{A} = [a_{ij} \in S]$ is said to be *reciprocal* if

$$a_{ij} = a_{ji}^{-1} \qquad i, j = 1, \ldots, n\,. \tag{103}$$

Notice that diagonal elements of a reciprocal matrix are equal to the identity e.

- In the *multiplicative representation* on $\mathbb{P}$ the reciprocity of the pairwise comparison matrix $\mathbf{A} = [a_{ij}]$ is expressed as $a_{ij} = 1/a_{ji}$ with $i, j = 1, \ldots, n$. In both the *additive representation* on $\mathbb{R}$ and the *bounded additive representation* on $\mathbb{B}_\diamond$ reciprocity is expressed as $a_{ij} = -a_{ji}$ with $i, j = 1, \ldots, n$. In the *bounded multiplicative representation* on $\mathbb{B}_\star$ reciprocity is expressed as $a_{ij} = 1 - a_{ji}$ with $i, j = 1, \ldots, n$.

Definition 10. Given a semifield $(S, \oplus, \odot)$, a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is said to be *consistent* if

$$a_{ik} \odot a_{kj} = a_{ij} \qquad i, j, k = 1, \ldots, n\,. \tag{104}$$

Notice that consistency implies reciprocity, with $i = j = 1, \ldots, n$.

- In the *multiplicative representation* on $\mathbb{P}$ we have $a_{ik}\, a_{kj} = a_{ij}$ for all $i, j, k = 1, \ldots, n$. In an equivalent representation generated by the bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$ we obtain $a_{ij} = \phi^{-1}(\phi(a_{ik})\, \phi(a_{kj}))$ for all $i, j, k = 1, \ldots, n$. Accordingly, in the *additive representation* on $\mathbb{R}$ we have $a_{ik} + a_{kj} = a_{ij}$, in the *bounded additive representation* on $\mathbb{B}_\diamond$ we have

$$\frac{a_{ik} + a_{kj}}{1 + a_{ik}\, a_{kj}} = a_{ij} \qquad i, j, k = 1, \ldots, n \tag{105}$$

  and in the *bounded multiplicative representation* on $\mathbb{B}_\star$ we have

$$\frac{a_{ik}\, a_{kj}}{a_{ik}\, a_{kj} + (1 - a_{ik})(1 - a_{kj})} = a_{ij} \qquad i, j, k = 1, \ldots, n\,. \tag{106}$$

Definition 11. Given a semifield $(S, \oplus, \odot)$, a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is said to be *anti-consistent* if $a_{i1} \odot a_{i2} \odot \ldots \odot a_{in} = e$, for $i = 1, \ldots, n$.

In order to discuss the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ we must require that the group $(S, \odot)$ be divisible with unique roots. This means that the n^{th} root $y = \odot^{1/n} x \in S$ exists unique for all elements $x \in S$. In other words, for every $x \in S$ there exists a unique $y \in S$ such that $x = \odot^n y$.

In the context of divisibility with unique roots, we have $\odot^{1/n} e = e$ and the equation $\odot^n a = e$ has the unique solution $a = e$. Moreover it holds that $\odot^{1/n}(xy) = (\odot^{1/n} x) \odot (\odot^{1/n} y)$, since the n^{th} power of the right hand side is equal to xy, plus the fact that the n^{th} root is unique.

- In the *multiplicative representation* on $\mathbb{P}$ unique divisibility holds and $\odot^{1/n} x = x^{1/n}$. In the equivalent representation generated by a bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$ we obtain $\odot^{1/n} x = \phi^{-1}(\phi(x)^{1/n})$. Accordingly, in the *additive representation* on $\mathbb{R}$ we have $\odot^{1/n} x = x/n$, in the *bounded additive representation* on $\mathbb{B}_\diamond$ we have

$$\odot^{1/n} x = \frac{x^{1/n} - (1 - x)^{1/n}}{x^{1/n} + (1 - x)^{1/n}} \tag{107}$$

  and in the *bounded multiplicative representation* on $\mathbb{B}_\star$ we have

$$\odot^{1/n} x = \frac{x^{1/n}}{x^{1/n} + (1 - x)^{1/n}}\,. \tag{108}$$

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The priority vector $\boldsymbol{u}$ associated with a reciprocal matrix $\mathbf{A}$ is defined in terms of the multiplicative group structure of S, as in Crawford and Williams [21], Barzilai et al. [3], and Barzilai and Golany [4].

Definition 12. Given a semifield $(S, \oplus, \odot)$ and a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$, the *priority vector* $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is defined as

$$u_i = \odot^{1/n}(a_{i1} \odot a_{i2} \odot \ldots \odot a_{in}) \qquad i = 1, \ldots, n \tag{109}$$

with $u_1 \odot \ldots \odot u_n = e$ due to the reciprocity of the matrix $\mathbf{A}$.

- In the *multiplicative representation* on $\mathbb{P}$ the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij}]$ is

$$u_i = (a_{i1}a_{i2} \ldots a_{in})^{1/n} \qquad i = 1, \ldots, n\,. \tag{110}$$

In the equivalent representation generated by a bijection $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$, the priority vector $\mathbf{u} = (u_1, \ldots, u_n) \in S^n$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is given by

$$u_i = \phi^{-1}\Big((\phi(a_{i1})\phi(a_{i2}) \ldots \phi(a_{in}))^{1/n}\Big) \qquad i = 1, \ldots, n\,. \tag{111}$$

In this way we obtain the form of the priority vector in the various equivalent representations on $S = \mathbb{R}$, $\mathbb{B}_\diamond$, $\mathbb{B}_\star$. In the *additive representation* on $\mathbb{R}$ the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij}]$ is

$$u_i = \ln\Big((e^{a_{i1}}e^{a_{i2}} \ldots e^{a_{in}})^{1/n}\Big) = (a_{i1} + a_{i2} + \ldots + a_{in})/n \qquad i = 1, \ldots, n\,. \tag{112}$$

In the *bounded additive representation* on $\mathbb{B}_\diamond$ the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij}]$ is

$$u_i = \frac{\left(\frac{1+a_{i1}}{1-a_{i1}}\frac{1+a_{i2}}{1-a_{i2}}\cdots\frac{1+a_{in}}{1-a_{in}}\right)^{1/n} - 1}{\left(\frac{1+a_{i1}}{1-a_{i1}}\frac{1+a_{i2}}{1-a_{i2}}\cdots\frac{1+a_{in}}{1-a_{in}}\right)^{1/n} + 1} = \frac{\alpha_i^{1/n} - \beta_i^{1/n}}{\alpha_i^{1/n} + \beta_i^{1/n}} \qquad i = 1, \ldots, n \tag{113}$$

where $\alpha_i = (1+a_{i1})(1+a_{i2}) \ldots (1+a_{in})$ and $\beta_i = (1-a_{i1})(1-a_{i2}) \ldots (1-a_{in})$ with $i = 1, \ldots, n$. Finally, in the *bounded multiplicative representation* on $\mathbb{B}_\star$ the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n)$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij}]$ is

$$u_i = \frac{\left(\frac{a_{i1}}{(1-a_{i1})}\frac{a_{i2}}{(1-a_{i2})}\cdots\frac{a_{in}}{(1-a_{in})}\right)^{1/n}}{1 + \left(\frac{a_{i1}}{(1-a_{i1})}\frac{a_{i2}}{(1-a_{i2})}\cdots\frac{a_{in}}{(1-a_{in})}\right)^{1/n}} = \frac{\alpha_i^{1/n}}{\alpha_i^{1/n} + \beta_i^{1/n}} \qquad i = 1, \ldots, n \tag{114}$$

where $\alpha_i = (a_{i1}a_{i2} \ldots a_{in})$ and $\beta_i = (1 - a_{i1})(1 - a_{i2}) \ldots (1 - a_{in})$ with $i = 1, \ldots, n$.

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Definition 13. Given a semifield $(S, \oplus, \odot)$, the *consistent matrix* $\widehat{\mathbf{A}} = [\widehat{a}_{ij} \in S]$ associated with a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is defined as

$$\widehat{a}_{ij} = u_i \div u_j \qquad i, j = 1, \ldots, n \tag{115}$$

where $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ is the priority vector associated with $\mathbf{A}$, as in (109). On the other hand, the *anti-consistent matrix* $\widetilde{\mathbf{A}} = [\widetilde{a}_{ij} \in S]$ associated with $\mathbf{A}$ is defined as

$$\widetilde{a}_{ij} = a_{ij} \div (u_i \div u_j) \qquad i, j = 1, \ldots, n\,. \tag{116}$$

We obtain the *consistency decomposition formula*

$$a_{ij} = \widehat{a}_{ij} \odot \widetilde{a}_{ij} \qquad i, j = 1, \ldots, n\,. \tag{117}$$

It is straightforward to show that both $\widehat{\mathbf{A}}$ and $\widetilde{\mathbf{A}}$ are reciprocal,

$$\widehat{a}_{ji} = u_j \odot u_i^{-1} = (u_i \odot u_j^{-1})^{-1} = (\widehat{a}_{ij})^{-1} \tag{118}$$

$$\widetilde{a}_{ji} = a_{ji} \odot (\widehat{a}_{ji})^{-1} = (a_{ij})^{-1} \odot \widehat{a}_{ij} = (a_{ij} \odot (\widehat{a}_{ij})^{-1})^{-1} = (\widetilde{a}_{ij})^{-1}\,. \tag{119}$$

Moreover, $\widehat{\mathbf{A}}$ is consistent,

$$\widehat{a}_{ij} = u_i \div u_j = (u_i \div u_k) \odot (u_k \div u_j) = \widehat{a}_{ik} \odot \widehat{a}_{kj} \tag{120}$$

and $\widetilde{\mathbf{A}}$ is anti-consistent,

$$\begin{aligned} \widetilde{a}_{i1} \odot \ldots \odot \widetilde{a}_{in} &= (a_{i1} \odot \ldots \odot a_{in}) \div ((u_i \div u_1) \odot \ldots \odot (u_i \div u_n)) \\ &= u_i^n \div (u_i^n \div (u_1 \odot \ldots \odot u_n)) = u_1 \odot \ldots \odot u_n = e \end{aligned} \tag{121}$$

for $i = 1, \ldots, n$. The anti-consistent matrix associated with a reciprocal matrix has been introduced by Barzilai [2] under the name *totally inconsistent matrix*.

- In the *multiplicative representation* on $\mathbb{P}$ the decomposition formula(117) reduces to $a_{ij} = \widehat{a}_{ij} \cdot \widetilde{a}_{ij}$. Analogously, in the *additive representation* on $\mathbb{R}$ we have $a_{ij} = \widehat{a}_{ij} + \widetilde{a}_{ij}$ In the *bounded additive representation* on $\mathbb{B}_\diamond$ the decomposition formula is written as

$$a_{ij} = \frac{\widehat{a}_{ij} + \widetilde{a}_{ij}}{1 + \widehat{a}_{ij}\,\widetilde{a}_{ij}} \qquad i, j = 1, \ldots, n \tag{122}$$

  and in the *bounded multiplicative representation* on $\mathbb{B}_\star$ it is written as

$$a_{ij} = \frac{\widehat{a}_{ij}\,\widetilde{a}_{ij}}{\widehat{a}_{ij}\,\widetilde{a}_{ij} + (1 - \widehat{a}_{ij})(1 - \widetilde{a}_{ij})} \qquad i, j = 1, \ldots, n\,. \tag{123}$$

Proposition 6. *Consider a semifield $(S, \oplus, \odot)$. A reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is consistent if and only if it coincides with its associated consistent matrix $\widehat{\mathbf{A}} = [\widehat{a}_{ij} \in S]$, and $\mathbf{A}$ is anti-consistent if and only if it coincides with its associated anti-consistent matrix $\widetilde{\mathbf{A}} = [\widetilde{a}_{ij} \in S]$.*

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Proof: The first part of the proposition is a classical result which we indicate here only for the sake of completeness. Consistency means that $a_{ij} = a_{ik} \div a_{jk}$ for $i, j, k = 1, \ldots, n$. Therefore, on the basis of the general definition (115),

$$\begin{aligned} \widehat{a}_{ij} &= u_i \div u_j && (124) \\ &= \odot^{1/n}(a_{i1} \odot a_{i2} \odot \ldots \odot a_{in}) \div \odot^{1/n}(a_{j1} \odot a_{j2} \odot \ldots \odot a_{jn}) \\ &= \odot^{1/n}((a_{i1} \div a_{j1}) \odot \ldots \odot (a_{in} \div a_{jn})) \\ &= \odot^{1/n}(a_{ij} \odot \ldots \odot a_{ij}) = a_{ij} \qquad i, j = 1, \ldots, n\,. \end{aligned}$$

The second part of the proposition follows from the fact that anti-consistency means $u_i = e$ for $i = 1, \ldots, n$ and therefore $\widehat{a}_{ij} = e$ for $i, j = 1, \ldots, n$. For this reason, $\widetilde{a}_{ij} = a_{ij} \div \widehat{a}_{ij} = a_{ij}$ for $i, j = 1, \ldots, n$. □

Corollary 1. *Consider a semifield $(S, \oplus, \odot)$. A reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$ is both consistent and anti-consistent if and only if $a_{ij} = e$, for $i, j = 1, \ldots, n$.*

Proof: Consider the general decomposition formula $a_{ij} = \widehat{a}_{ij} \odot \widetilde{a}_{ij}$ for $i, j = 1, \ldots, n$. If the matrix $\mathbf{A}$ is consistent, then $a_{ij} = \widehat{a}_{ij}$ and thus $\widetilde{a}_{ij} = e$. On the other hand, if the matrix $\mathbf{A}$ is anti-consistent, then $a_{ij} = \widetilde{a}_{ij}$ and thus $\widehat{a}_{ij} = e$. Therefore if $\mathbf{A}$ is both consistent and anti-consistent, then $\widehat{a}_{ij} = \widetilde{a}_{ij} = e$ and thus $a_{ij} = e$ for $i, j = 1, \ldots, n$. Conversely, if $a_{ij} = e$ for $i, j = 1, \ldots, n$ it is straightforward to show that also $\widehat{a}_{ij} = \widetilde{a}_{ij} = e$ and thus $\mathbf{A}$ is both consistent and anti-consistent. □

Given a reciprocal matrix $\mathbf{A} = [a_{ij} \in S]$, the module of each single element of the associated anti-consistent matrix, $s(\widetilde{a}_{ij}) = r(a_{ij}, \widehat{a}_{ij})$ with $i, j = 1, \ldots, n$, can be regarded as a local measure of inconsistency - see Bortot and Marques Pereira [7] - in the sense that it reduces to the identity, $s(\widetilde{a}_{ij}) = e$, wherever the matrix $\mathbf{A}$ is locally consistent, $a_{ij} = \widehat{a}_{ij}$ for some $i, j = 1, \ldots, n$.

Finally, consider a hierarchical multicriteria decision model of the AHP type, with semifield domain $(S, \oplus, \odot)$. The priority values of the various criteria are expressed by the priority vector $\boldsymbol{u} = (u_1, \ldots, u_n) \in S^n$ associated with the reciprocal pairwise comparison matrix $\mathbf{A} = [a_{ij} \in S]$ between criteria. In the same way, the vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in S^n$ contains the priority values of a given alternative with respect to the various criteria. In our general framework, the aggregated priority value of the given alternative is obtained by the weighted mean $A_{\boldsymbol{u}}(\boldsymbol{x}) \in S$ or the generalized weighted mean $B_{\boldsymbol{u}}(\boldsymbol{x}) \in S$, whatever the particular representation considered.

6. Concluding remarks

We have introduced a general framework for the algebraic representation of the weighted mean over open interval domains $S \subseteq \mathbb{R}$ endowed with semifield structures isomorphic to that of $\mathbb{P} = (0, \infty)$. The algebraic framework considers the case in which priorities and values refer to a common aggregation domain, and provides a natural description of weighted mean aggregation, in both the arithmetic and the geometric forms.

In either case the weighted mean necessarily involves two operations, addition (abelian semigroup) and multiplication (abelian group), which generalize the role of addition and multiplication in $\mathbb{P} = (0, \infty)$.

We have considered open interval domains $S \subseteq \mathbb{R}$ whose semifield structures are generated by bijections $\phi : S \subseteq \mathbb{R} \to \mathbb{P}$. Continuous and thus strictly monotonic bijections play a central role. In such case the additive semifield structure is represented by continuous strict triangular conorms/norms and the multiplicative semifield structure is represented by continuous strict uninorms. In turn, the weighted mean is represented by continuous strict weighted triangular conorms/norms or by continuous strict weighted uninorms, depending on its arithmetic or geometric character.

We have described the general semifield representation of the weighted mean and we have examined it in detail in the case of four basic algebraic representations on $S = (0, \infty) = \mathbb{P}$, $S = (-\infty, \infty) = \mathbb{R}$, $S = (0, 1)$, $S = (-1, 1)$. Then, extending the work by Cavallo, D'Apuzzo, and Squillante on the basic representations of pairwise comparison matrices and their associated priority vectors, we have discussed the general representation of the notion of anti-consistency introduced by Barzilai, as opposed to the classical notion of consistency. The two complementary notions are combined in the consistency decomposition formula.

A final note regards the representation of the generalized weighted means $A_{\boldsymbol{u}}^{\tau}$ as in (42), which might provide interesting material for future investigation, beyond the arithmetic and geometric instances examined here.

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References

[1] J. Barzilai, Deriving weights from pairwise comparison matrices, The Journal of the Operational Research Society 48 (12) (1997) 1226-1232.

[2] J. Barzilai, Consistency measures for pairwise comparison matrices, Journal of Multi-Criteria Decision Analysis 7 (3) (1998) 123-132.

[3] J. Barzilai, W. D. Cook, B. Golany, Consistent weights for judgements matrices of the relative importance of alternatives, Operations Research Letters 6 (3) (1987) 131-134.

[4] J. Barzilai, B. Golany, Deriving weights from pairwise comparison matrices: the additive case, Operations Research Letters 9 (6) (1990) 407-410.

[5] J. Barzilai, B. Golany, AHP rank reversal, normalization and aggregation rules, INFOR 32 (2) (1994) 57-64.

[6] G. Beliakov, A. Pradera, T. Calvo, Aggregation Functions: A Guide for Practitioners, Studies in Fuzziness and Soft Computing, Vol. 221, Springer, Heidelberg, 2007.

[7] S. Bortot, R. A. Marques Pereira, Inconsistency and non-additive capacities: the Analytic Hierarchy Process in the framework of Choquet integration, Fuzzy Sets and Systems 213 (2013) 6-26.

[8] M. Brunelli, Introduction to the Analytic Hierarchy Process, SpringerBriefs in Operations Research, Springer, Heidelberg, 2015.

[9] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: T. Calvo, G. Mayor, R. Mesiar (Eds.), Aggregation Operators: New Trends and Applications, Physica, Heidelberg, 2002, pp. 3-104.

[10] T. Calvo, R. Mesiar, Weighted triangular norms-based aggregation operators, Fuzzy Sets and Systems 137 (1) (2003) 3-10.

[11] T. Calvo, R. Mesiar, R. R. Yager, Quantitative weights and aggregation, IEEE Transactions on Fuzzy Systems 12 (1) (2004) 62-69.

[12] T. Calvo, G. Beliakov, Aggregation functions based on penalties, Fuzzy Sets and Systems 161 (10) (2010) 1420-1436.

[13] E. E. Castiñeira, T. Calvo, S. Cubillo, Multi-argument fuzzy measures on lattices of fuzzy sets, Knowledge-Based Systems 53 (2013) 27-39.

[14] B. Cavallo, A further discussion of "A Semiring based study of judgment matrices: properties and models" [Information Sciences 181 (2011) 2166-2176], Information Sciences 287 (2014) 61-67.

[15] B. Cavallo, L. D'Apuzzo, A General unified framework for pairwise comparison matrices in multicriterial methods, International Journal of Intelligent Systems 24 (4) (2009) 377-398.

[16] B. Cavallo, L. D'Apuzzo, Characterizations of consistent pairwise comparison matrices over abelian linearly ordered groups, International Journal of Intelligent Systems 25 (10) (2010) 1035-1059.

[17] B. Cavallo, L. D'Apuzzo, M. Squillante, About a consistency index for pairwise comparison matrices over a divisible alo-group, International Journal of Intelligent Systems 27 (2012) 153-175.

[18] B. Cavallo, L. D'Apuzzo, Deriving weights from a pairwise comparison matrix over an alo-group, Soft Computing 16 (2012) 353-366.

[19] B. Cavallo, L. D'Apuzzo, Reciprocal transitive matrices over abelian linearly ordered groups: characterizations and application to multi-criteria decision problems, Fuzzy Sets and Systems 226 (2015) 33-46.

[20] M. Couceiro, J. L. Marichal, Characterizations of discrete Sugeno integrals as polynomial functions over distributive lattices, Fuzzy Sets and Systems 161 (2010) 694-707.

[21] G. Crawford, C. Williams, A note on the analysis of subjective judgment matrices, Journal of Mathematical Psychology 29 (4) (1985) 387-405.

[22] J. Dombi, Basic concepts for a theory of evaluation: the aggregative operator, European Journal of Operational Research 10 (3) (1982) 282-293.

[23] J. Fodor, M. Roubens, Fuzzy Preference Modelling and Multicriteria Decision Support, Kluwer Academic Publishers, Dordrecht, 1994.

[24] J. Fodor, R. R. Yager, A. Rybalov, Structure of uninorms, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 5 (4) (1997) 411-427.

[25] J. Fodor, B. De Baets, Uninorm basics, in: P. P. Wang, D. Ruan, E. E. Kerre (Eds.), Fuzzy Logic—A Spectrum of Theoretical & Practical Issues, Springer, Heidelberg, 2007, pp.51–66.

[26] J. S. Golan, Semirings and their Applications, Kluwer Academic Publishers, Norwell MA, 1999.

[27] M. Grabisch, On equivalence classes of fuzzy connectives-the case of fuzzy integrals, IEEE Transactions on Fuzzy Systems 3 (1) (1995) 96-109.

[28] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Encyclopedia of Mathematics and its Applications, Vol. 127, Cambridge University Press, 2009.

[29] M. Grabisch, C. Labreuche, A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid, Annals of Operations Research 175 (1) (2010) 247-286.

[30] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, Aggregation functions: means, Information Sciences 181 (2011) 1-22.

[31] M. Grabisch, J. L. Marichal, R. Mesiar, E. Pap, Aggregation functions: Construction methods, conjunctive, disjunctive and mixed classes, Information Sciences 181 (2011) 23-43.

[32] U. Hebisch, H. J. Weinert, Semirings and semifields, in: M. Hazewinkel (Ed.), Handbook of Algebra 1, North-Holland (Elsevier Science), Amsterdam, 1996, pp. 425-462.

[33] U. Hebisch, H. J. Weinert, Semirings: Algebraic Theory and Applications in Computer Science, World Scientific, Singapore, 1998.

[34] F. Hou, A semiring-based study of judgment matrices: properties and models, Information Sciences 181 (2011) 2166-2176.

[35] F. Hou, Rank preserved aggregation rules and application to reliability allocation, Communications in Statistics-Theory and Methods, 41 (2012) 3831-3845.

[36] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.

[37] C. H. Ling, Representation of associative functions, Pubblicationes Mathematicae Debrecen, 12 (1965) 189-212.

[38] I. Lizasoain, C. Moreno, OWA operators defined on complete lattices, Fuzzy Sets and Systems 224 (2013) 36-52.

[39] R. A. Marques Pereira, R. A. Ribeiro, Aggregation with generalized mixture operators using weighting functions, Fuzzy Sets and Systems 137 (1) (2003) 43-58.

[40] R. Mesiar, Choquet-like integrals, Jounal of Mathematical Analysis and Appllications 194 (1995) 477–488.

[41] R. Mesiar, J. Špirková, L. Vavríková, Weighted aggregation operators based on minimization, Information Sciences 178 (4) (2008) 1133-1140.

[42] R. Mesiar, A. Kolesárová, T. Calvo, M. Komorníková, A review of aggregation functions, in: H. Bustince, F. Herrera, J. Montero (Eds.), Fuzzy Sets and their Extensions: Representation, Aggregation and Models, Springer, Heidelberg, 2008, pp.121-144.

[43] T. L. Saaty, A scaling method for priorities in hierarchical structures, Journal of Mathematical Psychology 15 (3) (1977) 234-281.

[44] T. L. Saaty, The Analytic Hierarchy Process: Planning, Priority Setting, Resource Allocation, McGraw-Hill, New York NY, 1980.

[45] T. L. Saaty, Axiomatic foundation of the analytic hierarchy process, Management Science 32 (7) (1986) 841-855.

[46] T. L. Saaty, Decision making with the analytic hierarchy process, International Journal of Services Sciences 1 (1) (2008) 83-98.

[47] T. L. Saaty, L. G. Vargas, Prediction, Projection and Forecasting, Kluwer Academic Publishers, Norwell MA, 1991.

[48] T. L. Saaty, L. G. Vargas, Models, Methods, Concepts & Applications of the Analytic Hierarchy Process, 2nd edition, International Series in Operations Research & Management Science, Vol. 175, Springer, New York NY, 2012. Original version in International Series in Operations Research & Management Science, Vol. 34, Kluwer Academic Publishers, Norwell MA, 2001.

[49] V. Torra, V. Narukawa, Modeling Decisions, Information Fusion and Aggregation Operators, Springer, Heidelberg, 2007.

[50] R. R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. on Systems, Man and Cybernetics 18 (1) (1988) 183–190.

[51] R. R. Yager, Weighted triangular norms using generating functions, International Journal of Intelligent Systems 19 (3) (2004) 217-231.