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# Consensus dynamics, network interaction, and Shapley indices in the Choquet framework

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**Abstract**

We consider a set $N = \{1, ..., n\}$ of interacting agents whose individual opinions are denoted by $x_i$, $i \in N$ in some domain $\mathbb{D} \subseteq \mathbb{R}$. The interaction among the agents is expressed by a symmetric interaction matrix with null diagonal and off-diagonal coefficients in the open unit interval. The interacting network structure is thus that of a complete graph with edge values in $(0, 1)$.

In the Choquet framework, the interacting network structure is the basis for the construction of a consensus capacity $\mu$, where the capacity value $\mu(S)$ of a coalition of agents $S \subseteq N$ is defined to be proportional to the sum of the edge interaction values contained in the subgraph associated to $S$. The capacity $\mu$ is obtained in terms of its 2-additive Möbius transform $m_\mu$, and the corresponding Shapley power and interaction indices are identified.

We then discuss two types of consensus dynamics, both of which refer significantly to the notion of context opinion. The second type converges simply the plain mean, whereas the first type produces the Shapley mean as the asymptotic consensual opinion. In this way it provides a dynamical realization of Shapley aggregation.

*Keywords*: consensus reaching, linear dynamical models, network interaction, Choquet capacities, Möbius transforms, Shapley power and interaction indices.

## 1 Introduction

In the study of multiagent opinion aggregation in linear dynamic models, the central notions of interaction and consensus have been the subject of a great

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deal of investigation. Fundamental contributions in this research field have been made by several authors, among which: Shapley [86] on cooperative game theory; French [34] and Harary [63] on social power theory; DeGroot [22], Chatterjee [15], Chatterjee and Seneta [16], Berger [7], Kelly [65], and French [35] on DeGroot's consensus formation model; Sen [85] on models of choice and welfare; Lehrer [66], Wagner [91, 92], Lehrer and Wagner [67], and Nurmi [83] on the rational choice model; Anderson and Graesser [4], Anderson [2, 3], and Graesser [61] on the information integration model; Davis [19, 20] on the social decision scheme model; and Abelson [1], Taylor [88], Friedkin [36, 37, 38, 39, 40, 41], Friedkin and Johnsen [42, 43, 44, 45], and Marsden and Friedkin [71, 72] on social influence network theory.

In this paper we focus on the classical DeGroot approach to consensus dynamics and network interaction, and we describe how the Shapley aggregation of individual opinions emerges naturally in the non-additive framework of Choquet integration.

Consider a set $N = \{1, ..., n\}$ of interacting agents whose individual opinions are denoted by $\{x_i \in \mathbb{D} \subseteq \mathbb{R},\ i \in N\}$. The interaction structure on $N$ is expressed by the symmetric interaction matrix $\mathbf{V} = [v_{ij},\ i, j \in N]$, with interaction coefficients $v_{ij} = v_{ji} \in (0, 1)$ for $i \neq j$ and $v_{ii} = 0$, with $i, j \in N$. In other words, the interaction matrix $\mathbf{V}$ is the adjacency matrix of a complete graph in which: each node $\{i\}$ encodes an individual opinion $x_i$, and each edge $\{i, j\}$ encodes an interaction coefficient $v_{ij} = v_{ji}$, for all $i \neq j \in N$.

In the Choquet framework, the interaction structure of the set $N$ of agents is the basis for the construction of the following capacity $\mu : 2^N \longrightarrow [0, 1]$. Let $S \subseteq N$ be a coalition of agents. The capacity value $\mu(S)$ of the coalition $S \subseteq N$ is defined to be proportional to the sum of the edge interaction values contained in the subgraph associated to $S$, $\mu(S) = \sum_{\{i,j\}\subseteq S} v_{ij}/\mathcal{N}$, where the normalization factor is given by $\mathcal{N} = \sum_{\{i,j\}\subseteq N} v_{ij}$. We can also write $\mathcal{N} = v/2$, with $v = \sum_{i,j=1}^{n} v_{ij}$. The measure $\mu$ satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$, and is monotonic. Moreover, it is superadditive, with null singletons.

An important role is played in this paper by two equivalent representations of the capacity $\mu$, its Möbius transform $m_\mu$ and the Shapley interaction representation, usually denoted $I_\mu$. The Shapley power index $I_\mu(\{i\}) = \phi_\mu(i)$ coincides with $v_i/v$ with $v_i = \sum_{j=1}^{n} v_{ij}$, and it is proportional to the average degree of interaction between agent $i$ and the remaining agents $j \neq i \in N$. On the other hand, the Shapley interaction index $I_\mu(\{i, j\}) = \sigma_\mu(i, j)$ coincides with $2v_{ij}/v$, which is proportional to the interaction coefficient $v_{ij}$, with $i \neq j \in N$.

The Shapley power indices $\phi_\mu(i)$ with $i \in N$ are thus natural weights for the consensual aggregation of the $n$ opinion values. Indeed, in our interacting network model we obtain the Shapley mean $\mathcal{S}_\mu(\boldsymbol{x}) = \sum_{i\in N} \phi_\mu(i)\, x_i$ as the consensual opinion with an appropriate definition of the convex linear dynamics.

Another relevant instance of the Shapley aggregation in our interactive network model is as follows. Consider the context opinions $\bar{x}_i = \sum_{j\in N\setminus\{i\}} w_{ij}\, x_j$ where $w_{ij} = v_{ij}/v_i$ for $i \in N$, $j \in N \setminus \{i\}$ and $\sum_{j\in N\setminus\{i\}} w_{ij} = 1$ for $i \in N$. Each $\bar{x}_i$ represents the context opinion as seen by agent $i$, i.e. the weighted average opinion of the remaining agents. Notice that the context weights correspond to a local normalization of the interaction coefficients between agent $i$ and the remaining agents.

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We will show that the context opinions $\bar{x}_i$, $i \in N$ have an interesting property: their Shapley mean value $\mathcal{S}_\mu(\bar{x}_1, \ldots, \bar{x}_n) = \sum_{i \in N} \phi_\mu(i)\, \bar{x}_i$ coincides with the Shapley mean value of the original opinions $x_i$, that is $\mathcal{S}_\mu(x_1, \ldots, x_n) = \sum_{i \in N} \phi_\mu(i)\, x_i$.

The convex linear dynamics of our model follows the classical DeGroot's paradigm. Consider a general row stochastic matrix $\mathbf{C} = [c_{ij},\ i, j \in N]$ and the associated convex linear dynamical law $x_i \ \longmapsto\ x'_i = \sum_{j \in N} c_{ij} x_j$, where $c_{ij} \geq 0$ and $\sum_{j \in N} c_{ij} = 1$, with $i, j \in N$. In this linear dynamical law, the coefficient $c_{ij}$ represents the influential weight accorded by individual $i$ to individual $j$, with $i, j \in N$.

In each iteration the new opinion $x'_i$ of agent $i \in N$ is a convex combination of his/her present opinion $x_i$ and the present opinions $x_{j \neq i}$ of the remaining agents. The present opinions $x_{j \neq i}$ are weighted with the coefficients $c_{ij}$, $j \neq i$, which are the $n-1$ degrees of freedom of the convex combination associated with agent $i \in N$. As a result, the weight of the present opinion $x_i$, i.e. the coefficient $c_{ii}$, is constrained by $c_{ii} = 1 - \sum_{j \in N \setminus i} c_{ij}$, with $i, j \in N$.

In this paper the values of the interaction coefficients are assumed to be constant in time and are given exogenously, as in DeGroot's classical model of consensus dynamics. Alternatively, the interaction coefficients can be computed endogenously in terms of the individual opinions $x_i$ themselves. The endogenous definition of the interaction coefficients can be done in various ways.
For instance, in the soft consensus model - see Fedrizzi, Fedrizzi, and Marques Pereira [28, 29], and Fedrizzi, Fedrizzi, Marques Pereira, and Brunelli [30, 31] - the interaction coefficients $v_{ij}$, with $i \neq j$, are defined by filtering the square difference values $(x_i - x_j)^2$ with a decreasing sigmoid function $\sigma(t) = 1/(1+e^{\beta(t-\alpha)})$. As a result, agents with similar opinions $((x_i - x_j)^2 < \alpha)$ interact strongly, whereas agents with dissimilar opinions $((x_i - x_j)^2 > \alpha)$ interact weakly. A similar idea has inspired the more recent models of bounded confidence, see Deffuant, Neau, Amblard, and Weisbuch [21], Dittmer [24], and Hegselmann and Krause [62]. In the soft consensus model, therefore, the Shapley power index $\phi_\mu(i)$ reflects the local degree of consensus around agent $i \in N$ and is thus a natural endogenous weight for the consensual aggregation of the $n$ opinion values.

Recent reviews on network models of linear consensus dynamics can be found in Jia, Mir Tabatabaei, Friedkin, and Bullo [64], Dong, Zha, Zhang, Kou, Fujita, Chiclana, and Herrera-Viedma [25], Dong, Zhang, Kou, Ding, and Liang [26], and Ureña, Kou, Dong, Chiclana, and Herrera-Viedma [90].

The paper is organized as follows. After the introductory notes, with reference to the classical literature on linear models of opinion formation and consensus dynamics, we present in Section 2 a brief review of the basic facts on the Choquet framework, including the fundamental notions on Choquet capacities, Möbius transforms and the Shapley (power and interaction) indices. Then, in Section 3, we introduce the interactive network structure and the associated consensus capacity, defined in terms of its 2-additive Möbius transform. In Section 4 we discuss the general properties of the consensus dynamics and we examine two particular cases, one of which obtains the Shapley mean as the asymptotic consensual opinion. Finally, in Section 5, we describe an illustrative example, followed by some conclusive remarks.

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## 2 Choquet capacities and Shapley indices

In this section we present a brief review of the basic notions on Choquet integration, focusing on the additive and 2-additive cases as described by their Möbius representations. Comprehensive reviews of Choquet integration can be found in Grabisch and Labreuche [53, 54], Grabisch, Kojadinovich, and Meyer [50], plus also Wang and Klir [93], Grabisch, Nguyen and Walker [59], Grabisch, Murofushi and Sugeno [58], Murofushi and Sugeno [82], Grabisch and Labreuche [52]. For recent reviews of the 2-additive case see Miranda, Grabisch, and Gil [77], and Mayag, Grabisch, and Labreuche [73, 74].

The Choquet integral is defined with respect to a general non-additive capacity and corresponds to a large class of aggregation functions, including the classical weighted means - the additive capacity case - and the ordered weighted means - the symmetric capacity case, see Yager [94] and Fodor, Marichal, and Roubens [32].

General reviews of aggregation functions, including Choquet integration, can be found in Fodor and Roubens [33], Marichal [68], Calvo, Mayor, and Mesiar [13], Calvo, Kolesárova, Komorníková, Mesiar [12], Beliakov, Pradera, and Calvo [6], Torra and Narukawa [89], Mesiar, Kolesárová, Calvo, and Komorníková [75], Grabisch, Marichal, Mesiar, and Pap [55, 56], and Beliakov, Bustince, and Calvo [5].

Consider a domain $\mathbb{D} \subseteq \mathbb{R}$ and a finite set of interacting agents $N = \{1, 2, \ldots, n\}$ whose individual opinions are denoted by $\{x_i \in \mathbb{D} \subseteq \mathbb{R},\ i \in N\}$. Single agents are indexed by $i, j \in N$ and the subsets $S, T \subseteq N$ with cardinalities $s, t \leq n$ are usually called coalitions.

The concepts of capacity and Choquet integral in the definitions below are due to Choquet [17], Sugeno [87], Chateauneuf and Jaffray [14], Murofushi and Sugeno [80, 81], Denneberg [23], Grabisch [46, 47], and Marichal [68].

**Definition 1.** *A* capacity *on the set $N$ is a set function $\mu : 2^N \longrightarrow [0, 1]$ which satisfies the following boundary and monotonicity conditions*

*(i)* $\mu(\emptyset) = 0,\ \mu(N) = 1$

*(ii)* $S \subseteq T \subseteq N \ \Rightarrow\ \mu(S) \leq \mu(T)$.

Capacities are also known as *fuzzy measures* [87] or *non-additive measures* [23].

A capacity $\mu$ is said to be additive over $N$ if $\mu(S \cup T) = \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$, with $S \cap T = \emptyset$. Alternatively, the capacity $\mu$ is subadditive over $N$ if $\mu(S \cup T) \leq \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$, with at least two such coalitions for which $\mu$ is subadditive in the strict sense. Analogously, the capacity $\mu$ is superadditive over $N$ if $\mu(S \cup T) \geq \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$, with at least two such coalitions for which $\mu$ is superadditive in the strict sense.

**Definition 2.** *Let $\mu$ be a capacity on $N$. The* Choquet integral $\mathcal{C}_\mu : \mathbb{D}^n \longrightarrow \mathbb{D}$ *with respect to $\mu$ is defined as*

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\mu(A_{(i)}) - \mu(A_{(i+1)})\right] x_{(i)} \qquad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{D}^n \tag{1}$$

*where $(\cdot)$ indicates a permutation on $N$ such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.*

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In the additive case, since

$$\mu(A_{(i)}) = \mu(\{(i)\}) + \mu(\{(i+1)\}) + \ldots + \mu(\{(n)\}) = \mu(\{(i)\}) + \mu(A_{(i+1)}) \quad (2)$$

the Choquet integral reduces to a weighted mean,

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} = \sum_{i=1}^{n} \mu(\{(i)\}) x_{(i)} = \sum_{i=1}^{n} \mu(\{i\}) x_i \quad (3)$$

where the weights are given by $w_i = \mu(\{i\})$, for $i \in N$, with $\sum_{i=1}^{n} w_i = 1$.

**Definition 3.** *Let $\mu$ be a capacity on $N$. The* Shapley power index *[86, 78, 79, 49, 68, 60, 51] of agent $i \in N$ with respect to the capacity $\mu$ is defined as*

$$\phi_\mu(i) = \sum_{T \subseteq N \setminus \{i\}} \frac{(n-1-t)!\, t!}{n!} [\mu(T \cup \{i\}) - \mu(T)] \qquad i \in N \quad (4)$$

*where $t$ denotes the cardinality of coalition $T \subseteq N$.*

The Shapley power index $\phi_\mu(i)$ amounts to a weighted average of the marginal contribution of agent $i$ with respect to all coalitions $T \subseteq N \setminus \{i\}$ and can be interpreted as an effective importance weight. Moreover, it can be shown [86, 60] that

$$\phi_\mu(i) \in [0,1]\,, \qquad \sum_i \phi_\mu(i) = 1 \qquad i \in N\,. \quad (5)$$

In the additive case, in particular, we have that $\phi_\mu(i) = \mu(\{i\})$, for $i \in N$.

**Definition 4.** *A capacity $\mu$ can be equivalently represented by its* Möbius transform *$m_\mu$ [84, 14, 49, 68, 76, 57], which is defined as*

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \qquad T \subseteq N \quad (6)$$

*where $s$ and $t$ denote the cardinality of the coalitions $S$ and $T$, respectively. Conversely, given the Möbius transform $m_\mu$, the associated capacity $\mu$ is obtained as*

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad T \subseteq N\,. \quad (7)$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad \sum_{T \subseteq N} m_\mu(T) = 1 \quad (8)$$

and the monotonicity conditions can be expressed as follows [14, 76, 57]. For each $i \in N$ and each coalition $T \subseteq N \setminus \{i\}$, the corresponding monotonicity condition is written as

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \qquad T \subseteq N \setminus \{i\} \qquad i \in N\,. \quad (9)$$

This form of the monotonicity conditions derives from the original monotonicity conditions in Definition 1, expressed as $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for all $i \in N$ and $T \subseteq N \setminus \{i\}$.

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According to the decomposition of the capacity $\mu$ in (7), the Shapley power indices as in (4) can also be expressed in terms of the Möbius transform [49, 68],

$$\phi_\mu(i) = \sum_{T \subseteq N \setminus \{i\}} \frac{m_\mu(T \cup \{i\})}{t+1} \qquad i \in N\,. \tag{10}$$

Analogously, the Choquet integral as in Definition 2 can be expressed in terms of the Möbius transform in the following way [49, 68],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{T \subseteq N} m_\mu(T) \min_{i \in T}(x_i)\,. \tag{11}$$

Defining a capacity $\mu$ on a set $N$ of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [48] introduced the concept of k-additive capacities, see also Grabisch [49], and Miranda and Grabisch [76, 57].
In particular, the 2-additive case - see Miranda, Grabisch, and Gil [77], and Mayag, Grabisch, and Labreuche [73, 74] - provides a good trade-off between the range of the model and its complexity: only $n(n+1)/2$ real coefficients are required to define a 2-additive capacity.

The Choquet integral with respect to a 2-additive capacity is a rich and effective modelling tool, see for instance Berrah and Clivillé [8], Clivillé, Berrah, and Maurice [18], Berrah, Maurice, and Montmain [9], Marques Pereira, Ribeiro, and Serra [70], and Bortot and Marques Pereira [10, 11].

**Definition 5.** *A capacity $\mu$ is said to be* k-additive *[48] if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.*

In particular, in the 1-additive (or simply additive) case, the decomposition formula (7) takes the simple form

$$\mu(T) = \sum_{i \in T} m_\mu(\{i\}) \qquad T \subseteq N\,, \tag{12}$$

and the boundary and monotonicity conditions (8), (9) reduce to

$$m_\mu(\emptyset) = 0 \qquad \sum_{i \in N} m_\mu(\{i\}) = 1 \tag{13}$$

$$m_\mu(\{i\}) \geq 0 \qquad i \in N\,. \tag{14}$$

Moreover, for additive capacities, the Shapley power indices in (10) are simply

$$\phi_\mu(i) = m_\mu(\{i\}) \qquad i \in N \tag{15}$$

and the Choquet integral in (11) reduces to

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i \in N} m_\mu(\{i\})\, x_i. \tag{16}$$

In the 2-additive case, the decomposition formula (7) takes the form

$$\mu(T) = \sum_{i \in T} m_\mu(\{i\}) \; + \sum_{\{i,j\} \subseteq T} m_\mu(\{i,j\}) \qquad T \subseteq N\,. \tag{17}$$

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The graph interpretation of this definition, with singletons $\{i\}$ corresponding to nodes and pairs $\{i,j\}$ corresponding to edges between nodes, is the following: the value of the 2-additive capacity $\mu$ on a coalition $T$ is given by the sum of the nodes and edges contained in the subgraph associated with the coalition $T$, For 2-additive capacities the boundary and monotonicity conditions (8), (9) reduce to

$$m_\mu(\emptyset) = 0 \qquad \sum_{i\in N} m_\mu(\{i\}) \,+ \sum_{\{i,j\}\subseteq N} m_\mu(\{i,j\}) = 1 \tag{18}$$

$$m_\mu(\{i\}) \geq 0 \qquad m_\mu(\{i\}) + \sum_{j\in T} m_\mu(\{i,j\}) \geq 0 \qquad i \in N \qquad T \subseteq N\setminus\{i\}\,. \tag{19}$$

Moreover, the Shapley power indices in (10) are given by

$$\phi_\mu(i) = m_\mu(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus\{i\}} m_\mu(\{i,j\}) \qquad i\in N \tag{20}$$

and the Choquet integral in (11) reduces to

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i\in N} m_\mu(\{i\})\, x_i \,+ \sum_{\{i,j\}\subseteq N} m_\mu(\{i,j\})\, \min(x_i,x_j)\,. \tag{21}$$

An equivalent representation of the capacity $\mu$ and its Möbius transform $m_\mu$ is the so-called Shapley interaction representation.

**Definition 6.** *Let $\mu$ be a capacity on $N$ and $m_\mu$ its Möbius transform. The* Shapley interaction representation *[79, 49, 68, 60, 76, 57] of the capacity $\mu$ is defined in terms of the Möbius transform $m_\mu$ in the following way,*

$$I_\mu(S) = \sum_{T\subseteq N\setminus S} \frac{m_\mu(T\cup S)}{t+1} \qquad S\subseteq N\,. \tag{22}$$

For coalitions of cardinality one and two, we have that (see also Eq. (10))

$$I_\mu(\{i\}) = \sum_{T\subseteq N\setminus\{i\}} \frac{m_\mu(T\cup\{i\})}{t+1} = \phi_\mu(i)$$
$$I_\mu(\{i,j\}) = \sum_{T\subseteq N\setminus\{i,j\}} \frac{m_\mu(T\cup\{i,j\})}{t+1} \qquad i,j\in N\,. \tag{23}$$

In the additive case, in particular, we have

$$I_\mu(\{i\}) = m_\mu(\{i\}) = \phi_\mu(i) \qquad I_\mu(\{i,j\}) = 0 \qquad i,j\in N \tag{24}$$

and, in the 2-additive case,

$$I_\mu(\{i\}) = m_\mu(\{i\}) + \frac{1}{2}\sum_{j\in N\setminus i} m_\mu(\{i,j\}) = \phi_\mu(i) \tag{25}$$

$$I_\mu(\{i,j\}) = m_\mu(\{i,j\}) = \sigma_\mu(i,j) \qquad i,j\in N\,. \tag{26}$$

Notice that, in the context of 2-additive capacities, the values $m_\mu(\{i,j\})$ correspond precisely to the Shapley interaction indices $\sigma_\mu(i,j)$ between the various agents.

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## 3 The consensus capacity

Consider a set $N = \{1, ..., n\}$ of interacting agents whose individual opinions are denoted by $\{x_i \in \mathbb{D} \subseteq \mathbb{R},\ i \in N\}$. The interaction among the agents is expressed by the symmetric interaction matrix $\mathbf{V} = [v_{ij},\ i, j \in N]$, with interaction coefficients $v_{ij} = v_{ji} \in (0, 1)$ for $i \neq j$, and $v_{ii} = 0$. For convenience, we introduce the following notation,

$$v_i = \sum_{j=1}^{n} v_{ij}\ ,\ i \in N \qquad \text{and} \qquad v = \sum_{i=1}^{n} v_i = \sum_{i,j=1}^{n} v_{ij}. \tag{27}$$

The interaction matrix $\mathbf{V}$ is the adjacency matrix of a complete graph in which each node $\{i\}$ represents an individual agent and encodes the corresponding opinion $x_i$. Moreover, each edge $\{i, j\}$ represents the interaction between two individual agents and encodes the corresponding interaction coefficient $v_{ij}$, as illustrated in Fig. 1.

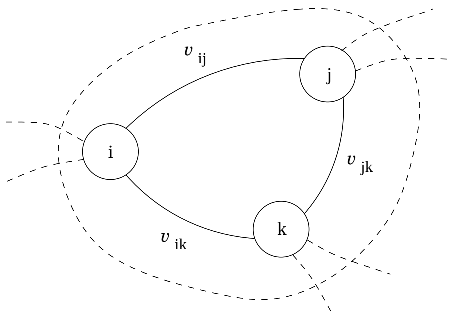


Figure 1: Graph representation of the interaction matrix $\mathbf{V}$.

In general, for any capacity $\mu$, we have $m_\mu(\{i\}) = \mu(\{i\})$, and therefore $0 \leq m_\mu(\{i\}) \leq 1$. Moreover, see [76, 57], we have $-1 \leq m_\mu(\{i, j\}) \leq 1$, subject to the monotonicity conditions. In the case of 2-additive capacities, the monotonicity conditions (19) are

$$m_\mu(\{i\}) \geq 0 \qquad m_\mu(\{i\}) + \sum_{j \in T} m_\mu(\{i, j\}) \geq 0 \qquad i \in N \qquad T \subseteq N \setminus (\{i\})$$

where the second group of inequalities means that the non negative value $m_\mu(\{i\})$ associated with node $i \in N$ must dominate the absolute value of the sum of all negative edges around the node.

Consider now a class of 2-additive capacities for which the Möbius value associated with each node is zero, $m_\mu(\{i\}) = 0$, and every edge of the graph is associated with a positive Möbius value, $m_\mu(\{i, j\}) > 0$. All capacities in this class are strictly superadditive, that is, $\mu(S \cup T) > \mu(S) + \mu(T)$ for all coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$.

On the basis of the graph representation of the set $N$ of interacting agents, we now define the consensus capacity in terms of its Möbius transform, associating the null value to each node and a positive value to each edge.

**Definition 7.** *The* consensus capacity $\mu : 2^N \longrightarrow [0,1]$ *expressed in terms of its Möbius transform is defined as follows. The Möbius transform is primarily defined sa*

$$m_\mu(\{i\}) = 0 \qquad m_\mu(\{i,j\}) = 2v_{ij}/v \qquad m_\mu(T) = 0 \qquad t \geq \tag{28}$$

*and the associated 2-additive consensus capacity $\mu$ in our model is thus given by*

$$\mu(T) = \sum_{i \in T} m_\mu(\{i\}) \;+ \sum_{\{i,j\} \subseteq T} m_\mu(\{i,j\}) = \sum_{\{i,j\} \subseteq T} 2v_{ij}/v \qquad T \subseteq N\,. \tag{29}$$

*The consensus capacity $\mu$ satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$. Moreover, it is strictly monotonic, with null singletons, and it is strictly superadditive.*

The individual share of the value $\mu(T)$ of a coalition $T \subseteq N$ is given by

$$\tilde{\mu}(T) = \mu(T)/t \tag{30}$$

where $t$ denotes the cardinality of the coalition $T \subseteq N$, and is related in an interesting way with the average edge value $\nu(T)$ within $T$, given by

$$\nu(T) = \mu(T)/\frac{1}{2}\,t(t-1)\,. \tag{31}$$

It follows that

$$\tilde{\mu}(T) = \frac{1}{2}(t-1)\nu(T)\,. \tag{32}$$

Notice that, given two coalitions $S \subseteq T \subseteq N$, it is always the case that the value $\mu(T)$ of the larger coalition is greater or equal than the value $\mu(S)$ of the smaller coalition, but it might not be the case that the individual share $\tilde{\mu}(T)$ in the larger coalition is also greater or equal than the individual share $\tilde{\mu}(S)$ in the smaller coalition. In other words, the question of whether or not it is worthwhile to form a larger coalition depends crucially on the way the average edge value changes, even though there is an overall bias to extend the coalition given by the dependency on the cardinality.

**Proposition 1.** *The Shapley power indices and the Shapley interaction indices associated with the consensus capacity $\mu$ are*

$$I_\mu(\{i\}) = v_i/v = \phi_\mu(i) \tag{33}$$

$$I_\mu(\{i,j\}) = 2\,v_{ij}/v = \sigma_\mu(i,j) \tag{34}$$

**Proof**: It follows immediately from equations (25) and (26)

$$\begin{aligned} I_\mu(\{i\}) &= m_\mu(i) + \frac{1}{2} \sum_{j \in N \setminus \{i\}} m_\mu(\{i,j\}) \\ &= \frac{1}{2} \sum_{j \in N \setminus \{i\}} 2v_{ij}/v = v_i/v = \phi_\mu(i) \qquad (35) \\ I_\mu(\{i,j\}) &= m_\mu(\{i,j\}) = 2\,v_{ij}/v = \sigma_\mu(i,j) \qquad (36) \end{aligned}$$

where $i \in N$ and $j \in N \setminus \{i\}$. □

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Apart from the normalization factor $2/v$, the Shapley interaction index $\sigma_\mu(i,j)$ coincides with the interaction coefficient $v_{ij}$, while the Shapley power index $\phi_\mu(i)$ is proportional to the average degree of interaction between agent $i$ and the remaining agents, which is given by

$$\sum_{j\in N\setminus\{i\}} v_{ij}/(n-1) = v_i/(n-1) \qquad i \in N\,. \tag{37}$$

Therefore, the Shapley power index $\phi_\mu(i)$ reflects the local degree of interaction around agent $i$ and is thus a natural weight for the consensual aggregation of the $n$ opinion values. In other words, the Shapley mean $\mathcal{S}_\mu(\boldsymbol{x}) = \sum_{i\in N} \phi_\mu(i)\, x_i$ is a natural linear aggregation function.

In the Choquet framework, there are also two natural non-linear aggregation functions, the Choquet integral $\mathcal{C}_\mu(\boldsymbol{x})$ and the associated ordered weighted averaging function $\mathcal{B}_\mu(\boldsymbol{x})$, which corresponds to the Choquet integral with respect to the symmetrized capacity, see Yager [94], and Fodor, Marichal, and Roubens [32]. We will briefly comment on the Ordered Weighted Averaging (OWA) functions in relation with the illustrative example discussed in the final part of the paper. Comprehensive reviews of OWA functions can be found in Yager and Kacprzyk [95], Yager, Kacprzyk and Beliakov [96], and Emrouznejad and Marra [27].

## 4 The consensus dynamics

We now examine the consensus dynamics in our network model and discuss the convergence to a consensual opinion in relation with Shapley mean $\mathcal{S}_\mu$.

**Definition 8.** *On the basis of the individual opinions $x_i \in \mathbb{D} \subseteq \mathbb{R}$, $i \in N$, the* context opinions *are defined as*

$$\bar{x}_i = \sum_{j\in N\setminus\{i\}} w_{ij}\, x_j\ ,\ i \in N \tag{38}$$

*where $w_{ij} = v_{ij}/v_i$, $j \in N \setminus \{i\}$ and $\sum_{j\in N\setminus\{i\}} w_{ij} = 1$ for $i \in N$.*

Each $\bar{x}_i$ represents the context opinion as seen by agent $i \in N$, i.e. the weighted average opinion of the remaining agents. Moreover, the context weights correspond to a local normalization of the interaction coefficients between agent $i \in N$ and the remaining agents.
The context opinions $\bar{x}_i$, $i \in N$ have the following interesting property.

**Proposition 2.** *The Shapley mean of the context opinions,*

$$\mathcal{S}_\mu(\bar{x}_1,\ldots,\bar{x}_n) = \sum_{i\in N} \phi_\mu(i)\, \bar{x}_i \tag{39}$$

*coincides with the Shapley mean of the individual opinions $x_i$, that is,*

$$\mathcal{S}_\mu(x_1,\ldots,x_n) = \sum_{i\in N} \phi_\mu(i)\, x_i\,. \tag{40}$$

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**Proof**: From (33) and Definition 8 we have

$$\begin{aligned} \mathcal{S}_\mu(\bar{x}_1,\ldots,\bar{x}_n) &= \sum_{i=1}^n \phi_\mu(i)\,\bar{x}_i \\ &= \sum_{i=1}^n \Big[\frac{v_i}{v}\Big(\sum_{j=1}^n \frac{v_{ij}}{v_i}\,x_j\Big)\Big] \\ &= \sum_{i=1}^n\sum_{j=1}^n \frac{v_{ij}}{v}\,x_j = \sum_{j=1}^n \frac{v_j}{v}\,x_j \\ &= \sum_{j=1}^n \phi_\mu(j)\,x_j = \mathcal{S}_\mu(x_1,\ldots,x_n) \end{aligned} \tag{41}$$

which proves the result. □

Consider now a general row stochastic matrix $\mathbf{C} = [c_{ij},\ i,j \in N]$, with $c_{ij} \geq 0$ and $\sum_{j\in N} c_{ij} = 1$, for $i \in N$. Moreover, consider the associated convex linear dynamical law

$$\boldsymbol{x} \longmapsto \boldsymbol{x}' = \mathbf{C}\boldsymbol{x} \qquad\qquad x_i \longmapsto x'_i = \sum_{j\in N} c_{ij}\,x_j \tag{42}$$

where $\mathbf{C} \geq 0$, $\mathbf{C1} = \mathbf{1}$ and $\mathbf{1} = (1\ ...\ 1)$. In this linear dynamical law, the coefficient $c_{ij}$ represents the influential weight accorded by agent $i$ to agent $j$, with $i,j \in N$.
In each iteration, the new opinion $x'_i$ of agent $i \in N$ is a convex combination of his/her present opinion $x_i$ and the present opinions $x_{j\neq i}$ of the remaining agents. The present opinions $x_{j\neq i}$ are weighted with the coefficients $c_{ij}$ with $j \neq i$, which are the $n-1$ degrees of freedom of the convex combination associated with agent $i \in N$. As a result, the weight of the present opinion $x_i$, i.e. the coefficient $c_{ii}$, is constrained to be one minus the sum of the remaining coefficients,

$$x'_i = c_{ii}\,x_i + \sum_{j\in N\setminus\{i\}} c_{ij}\,x_j\ ,\ i \in N \tag{43}$$

$$c_{ii} = 1 - \sum_{j\in N\setminus\{i\}} c_{ij}\ ,\ i \in N\,. \tag{44}$$

In our interactive network model, we assume that

$$c_{ij} = \varepsilon\frac{v_{ij}}{\rho_i}\ ,\ i \in N\,,\ \ j \in N\setminus\{i\} \qquad\qquad c_{ii} = 1 - \varepsilon\frac{v_i}{\rho_i}\ ,\ i \in N \tag{45}$$

where $\varepsilon \in [0,1]$ and $\rho_i \geq v_i$ for $i \in N$. Note that the matrix $\mathbf{C}$ is not symmetric. The influential weight that agent $i$ accords to agent $j$ is $c_{ij} = \varepsilon\,v_{ij}/\rho_i$, whereas conversely we have $c_{ji} = \varepsilon\,v_{ji}/\rho_j$. Due to interaction symmetry, we obtain $c_{ij}/c_{ji} = \rho_j/\rho_i$, which means that the values $\rho_i$ for $i \in N$ control the ratio of the reciprocal influential weights between agents $i$ and $j$, with $i,j \in N$. Moreover, the self-influential weight $c_{ii}$ assigned by agent $i \in N$ to him/herself is related to the value $\rho_i$ by $(1-c_{ii})\,\rho_i = \varepsilon v_i$ with $i \in N$.

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The associated linear dynamical law can then be written as

$$\begin{aligned} x_i' &= \left(1 - \varepsilon \frac{v_i}{\rho_i}\right) x_i + \varepsilon \sum_{j \in N \setminus \{i\}} \frac{v_{ij}}{\rho_i} x_j \\ &= \left(1 - \varepsilon \frac{v_i}{\rho_i}\right) x_i + \varepsilon \frac{v_i}{\rho_i} \bar{x}_i \ , \ i \in N \end{aligned} \tag{46}$$

where $\bar{x}_i$ denotes the context opinion around agent $i \in N$.

**Proposition 3.** *For $\varepsilon \in (0, 1)$, the row stochastic transition matrix $\mathbf{C}$ is positive. It follows (Perron's theorem) that it has a simple maximal unit eigenvalue and a unique normalized positive eigenvector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$, $\boldsymbol{\psi}^T = \boldsymbol{\psi}^T \mathbf{C}$, with*

$$\psi_i = \frac{\rho_i}{\sum_{j \in N} \rho_j} \ , \ i \in N \qquad\qquad \sum_{i \in N} \psi_i = 1 \, . \tag{47}$$

**Proof**: It suffices to prove that

$$\sum_{i \in N} \rho_i \, c_{ij} = \rho_j \ , \ j \in N \tag{48}$$

which is obtained as follows,

$$\begin{aligned} \sum_{i \in N} \rho_i \, c_{ij} &= \rho_j \, c_{jj} + \sum_{i \in N \setminus \{j\}} \rho_i \, c_{ij} \\ &= \rho_j \left(1 - \varepsilon \frac{v_j}{\rho_j}\right) + \varepsilon \sum_{i \in N \setminus \{j\}} \rho_i \frac{v_{ij}}{\rho_i} \\ &= \rho_j - \varepsilon v_j + \varepsilon v_j \\ &= \rho_j \ , \ j \in N \end{aligned} \tag{49}$$

where we have used the fact that $v_{ij} = v_{ji}$ for $i, j \in N$. □

**Proposition 4.** *For $\varepsilon \in (0, 1)$, the convex linear dynamical law $\boldsymbol{x} \longmapsto \boldsymbol{x}' = \mathbf{C}\boldsymbol{x}$ leaves $\boldsymbol{\psi}^T \boldsymbol{x}$ invariant and converges to the consensual solution $\bar{x}\, \mathbf{1} = (\boldsymbol{\psi}^T \boldsymbol{x})\, \mathbf{1}$. In other words, the opinions $x_i$ converge asymptotically to the consensual opinion $\bar{x} = \sum_{i \in N} \psi_i x_i = \sum_{i \in N} \rho_i \, x_i / \sum_{j \in N} \rho_j$.*

**Proof**: The invariance of $\boldsymbol{\psi}^T \boldsymbol{x}$ is immediate,

$$\boldsymbol{\psi}^T \boldsymbol{x}' = \boldsymbol{\psi}^T (\mathbf{C}\boldsymbol{x}) = (\boldsymbol{\psi}^T \mathbf{C})\, \boldsymbol{x} = \boldsymbol{\psi}^T \boldsymbol{x} \tag{50}$$

and the convergence follows from the positivity of the transition matrix $\mathbf{C}$, see DeGroot [22] and references therein. □

We are interested in the following two different instances of the dynamical law above. In the first case (homogeneous dynamics), the free coefficients $c_{ij}$, with $j \neq i$, are assumed to be proportional to a local normalization of the corresponding interaction coefficients $v_{ij}$, with $j \neq i$. More specifically, they are assumed to be proportional to $v_{ij}/v_i$, with $j \neq i$. In the second case (inhomogeneous dynamics), on the other hand, the free coefficients $c_{ij}$, with $j \neq i$, are assumed to be simply proportional to the corresponding interaction coefficients $v_{ij}$, with $j \neq i$.

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**Definition 9.** *In relation with the general form of the linear dynamical law (43)-(45), the* homogeneous dynamics *corresponds to the choice* $\rho_i = v_i$ *for* $i \in N$*, and therefore the transition coefficients are*

$$c_{ij} = \varepsilon v_{ij}/v_i \qquad i \neq j\ , \qquad c_{ii} = 1 - \varepsilon \qquad \varepsilon \in [0, 1] \tag{51}$$

*with* $\sum_{j\in N\setminus\{i\}} c_{ij} = \varepsilon$ *and* $\sum_{j\in N} c_{ij} = 1$ *for* $i \in N$*.*

The dynamical law can be written as

$$x_i \ \longmapsto\ x'_i = (1-\varepsilon)\, x_i + \varepsilon\, \bar{x}_i\ ,\ i \in N \qquad \varepsilon \in [0,1]\,. \tag{52}$$

For $\varepsilon \in (0, 1)$, the row stochastic transition matrix $\mathbf{C}$ is positive, and in the case $\varepsilon = 1$ it is positive outside the main diagonal. In any case the matrix $\mathbf{C}$ is non negative and irreducible. It follows (Frobenius' theorem) that it has a simple maximal unit eigenvalue and a unique normalized positive eigenvector $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$, $\boldsymbol{\varphi}^T\mathbf{C} = \boldsymbol{\varphi}^T$ with

$$\varphi_i = \frac{1}{2} \sum_{j\in N\setminus\{i\}} 2\, v_{ij}/v = \phi_\mu(i)\ ,\ i \in N \qquad\qquad \sum_{i\in N} \varphi_i = 1\,. \tag{53}$$

Moreover, this dynamical law leaves $\boldsymbol{\varphi}^T\boldsymbol{x}$ invariant (the Shapley mean) and, given that $\mathbf{C}^2$ is positive (see note), it converges to the consensual solution $\bar{x}\,\mathbf{1} = (\boldsymbol{\varphi}^T\boldsymbol{x})\,\mathbf{1}$. In other words, the opinions $x_i$ converge asymptotically to their Shapley mean $\sum_{i\in N} \varphi_i x_i = \sum_{i\in N} x_i \phi_\mu(i)$.

**Note** The sole exception is the case $\varepsilon = 1$ for $n = 2$. In this case, the iterations of the dynamics simply exchange the opinions $x_1$ and $x_2$, and thus no consensus is ever reached.

**Definition 10.** *In relation with the general form of the linear dynamical law (43)-(45), the* inhomogeneous dynamics *corresponds to the choice* $\rho_i = n-1 = \rho$ *for* $i \in N$*, and therefore the transition coefficients are*

$$c_{ij} = \varepsilon v_{ij}/(n-1) \qquad i \neq j\ , \qquad c_{ii} = 1 - \varepsilon v_i/(n-1) \qquad \varepsilon \in [0, 1] \tag{54}$$

*with* $\sum_{j\in N\setminus\{i\}} c_{ij} = \varepsilon v_i/(n-1)$ *and* $\sum_{j\in N} c_{ij} = 1$ *for* $i \in N$*.*

The dynamical law can be written as

$$x_i \ \longmapsto\ x'_i = [1 - \varepsilon v_i/(n-1)]\, x_i + [\varepsilon v_i/(n-1)]\, \bar{x}_i\ ,\ i \in N \qquad \varepsilon \in [0,1]\,. \tag{55}$$

For $\varepsilon \in (0, 1)$ and $\varepsilon = 1$, the row stochastic transition matrix $\mathbf{C}$ is positive and thus irreducible. It follows (Perron's theorem) that it has a simple maximal unit eigenvalue and a unique normalized positive eigenvector $\boldsymbol{\psi} = (\psi_1, \ldots, \psi_n)$, $\boldsymbol{\psi}^T\mathbf{C} = \boldsymbol{\psi}^T$, with

$$\psi_i = 1/n\ ,\ i \in N \qquad\qquad \sum_{i\in N} \psi_i = 1\,. \tag{56}$$

Moreover, this dynamical law leaves $\boldsymbol{\psi}^T\boldsymbol{x}$ invariant and, given that $\mathbf{C}$ is positive, it converges to the consensual solution $\bar{x}\,\mathbf{1} = (\boldsymbol{\psi}^T\boldsymbol{x})\,\mathbf{1}$. In other words, the opinions $x_i$ converge asymptotically to their plain mean $\sum_{i\in N} \psi_i x_i = \sum_{i\in N} x_i/n$.

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# 5 An illustrative example

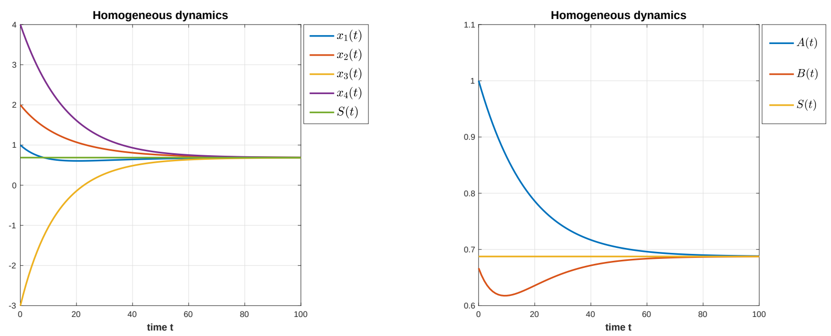
In this section we present an illustrative example of the consensual dynamics. We consider a network of four interacting agents whose initial individual opinions are assumed to be $\boldsymbol{x}(t=0) = \{1, 2, -3, 4\}$. For the homogeneous dynamics case we consider $\varepsilon = 0.06$, whereas for the inhomogeneous dynamics we consider $\varepsilon = 0.35$.

The interaction coefficients $v_{ij},\ i,j \in N$ are expressed by the following symmetric interaction matrix,

$$\mathbf{V} = \begin{bmatrix} 0 & 1/16 & 6/16 & 4/16 \\ & 0 & 2/16 & 2/16 \\ & & 0 & 1/16 \\ & & & 0 \end{bmatrix}$$

We consider the functions $\mathcal{A}_\mu(\boldsymbol{x}(t))$, $\mathcal{B}_\mu(\boldsymbol{x}(t))$, $\mathcal{S}_\mu(\boldsymbol{x}(t))$ which aggregate the individual opinions at time $t = 0, 1, 2, \ldots$ Given the individual opinions $\boldsymbol{x}(t) = (x_1(t), \ldots, x_n(t))$, we define $A(t) = \mathcal{A}_\mu(\boldsymbol{x}(t))$ as the plain mean of the opinions $x_i(t)$ with weights $w_i = 1/n$.
Moreover, we define $B(t) = \mathcal{B}_\mu(\boldsymbol{x}(t))$ as the OWA mean of the opinions $x_i(t)$ with weights $w_i = 2(n-i)/n(n-1)$. These weights correspond to the OWA weights associated with a symmetric 2-additive capacity with null singletons, see for instance Bortot and Marques Pereira [11].
Finally, we define $S(t) = \mathcal{S}_\mu(\boldsymbol{x}(t))$ as the Shapley mean of the opinions $x_i(t)$ with weights $w_i = \phi_\mu(i)$.

In Fig. 2 we present the time evolution of the individual opinions $x_i(t)$, $i = 1, \ldots, 4$ and the time evolution of the aggregated opinions $A(t)$, $B(t)$, $S(t)$ in the homogeneous dynamics case.




(a) Individual opinions $x_i(t)$, $i = 1, \ldots, 4$ (b) Aggregated opinions $A(t), B(t)$, $S(t)$

Figure 2: Homogeneous dynamics

As we can see, the simulation results are consistent with Proposition 4. The homogeneous dynamical law leaves invariant the Shapley mean and the opinions $x_i$, $i \in N$ converge to the consensual Shapley mean. Regarding the time evolution of the aggregated opinions $A(t)$ and $B(t)$, we observe that they also converge to the consensual opinion, the Shapley mean.

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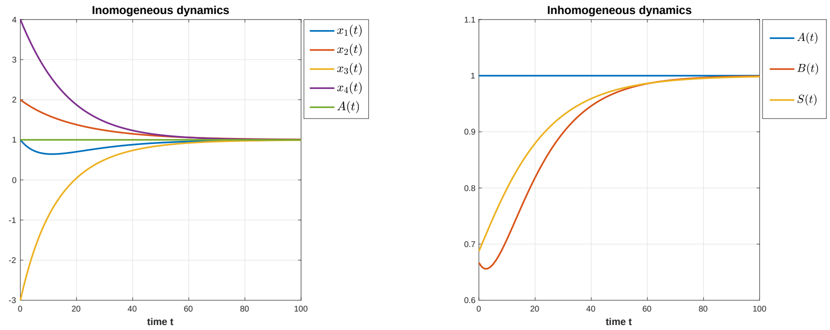
In Fig. 3 we present the time evolution of the individual opinions $x_i(t)$, $i = 1, \dots, 4$ and the time evolution of the aggregated opinions $A(t), B(t)$, $S(t)$ in the inhomogeneous dynamics case.




(a) Individual opinions $x_i(t)$, $i = 1, \dots, 4$ (b) Aggregated opinions $A(t), B(t)$, $S(t)$

Figure 3: Inhomogeneous dynamics

As expected, according to Proposition 4, the inhomogeneous dynamical law leaves invariant the plain mean and the individual opinions converge to the consensual plain mean. Regarding the time evolution of the aggregated opinions $B(t)$ and $S(t)$, we observe that they converge to the plain mean as well.

# 6 Concluding remarks

Summarizing, we have introduced an interactive network structure for a set of agents $N = \{1, ..., n\}$ whose opinions are denoted by $\{x_i \in \mathbb{D} \subseteq \mathbb{R},\ i \in N\}$.

The interaction among the agents is expressed by the symmetric interaction matrix $\mathbf{V} = [v_{ij},\ i, j \in N]$, with interaction coefficients $v_{ij} = v_{ji} \in (0, 1)$ for $i \neq j$, and $v_{ii} = 0$. In the Choquet framework, we have defined the associated consensus capacity in terms of its 2-additive Möbius transform, and we have identified the corresponding Shapley power and interaction indices.

We have considered a general row stochastic matrix $\mathbf{C} = [c_{ij},\ i, j \in N]$, with $c_{ij} \geq 0$ and $\sum_{j \in N} c_{ij} = 1$, for $i \in N$. The associated convex linear dynamical law is expressed by $\boldsymbol{x} \longmapsto \boldsymbol{x}' = \mathbf{C}\boldsymbol{x}$ where $\mathbf{C} \geq 0$, $\mathbf{C1} = \mathbf{1}$ and $\mathbf{1} = (1 \dots 1)$, as in the classical consensus reaching model introduced by DeGroot. In this linear dynamical law, the coefficient $c_{ij}$ represents the influential weight accorded by agent $i$ to agent $j$, with $i, j \in N$.

We have then discussed two instances of consensus dynamics, both of which refer significantly to the notion of context opinion.

The first case produces the Shapley mean as the asymptotic consensual opinion. In this way it provides a dynamical realization of the Shapley aggregation and connects naturally with the Choquet framework discussed earlier.

The second case produces the plain mean as the asymptotic consensual opinion and is thus less interesting. However, in an extended version with time dependent interaction coefficients, it corresponds to the purely consensual component of the non-linear dynamics in the soft consensus model (see e.g. [28]). In its complete form, the non-linear dynamics of the soft consensus model also takes into account the agents' initial opinions and in this sense it corresponds to an extended version of the linear social influence network dynamics introduced by Friedkin and Johnsen [44, 45].

## Compliance with Ethical Standards

**Conflict of interest** The authors declare that they have no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by the authors.

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