THE IDENTITY $G(D)f = F$ FOR A LINEAR PARTIAL DIFFERENTIAL OPERATOR $G(D)$. LUSIN TYPE AND STRUCTURE RESULTS IN THE NON-INTEGRABLE CASE

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Abstract. We prove a Lusin type theorem for a certain class of linear partial differential operators $G(D)$, reducing to [1, Theorem 1] when $G(D)$ is the gradient. Moreover we describe the structure of the set $\{G(D)f = F\}$, under assumptions of non-integrability on F, in terms of lower dimensional rectifiability and superdensity. Applications to Maxwell type system and to multivariable Cauchy-Riemann system are provided.

1. introduction

To introduce the subject of this work, let us first consider a particularly simple situation. Let $\Omega \subset \mathbb{R}^2$ be an open set and let $F = (F_1, F_2)^t \in C^1(\Omega, \mathbb{R}^2)$ be an irrotational field, i.e., such that

$$\frac{\partial F_1}{\partial x_2}(x) - \frac{\partial F_2}{\partial x_1}(x) \neq 0, \text{ for all } x = (x_1, x_2) \in \Omega. \tag{1.1}$$

Suppose then that we are interested in finding f that maximizes the size of the set

$$A_{f,F} := \{x \in \Omega \mid \nabla f(x) = F(x)\}.$$

Looking for f in $C^1(\Omega)$ is expected to be "more productive" than looking for it in $C^2(\Omega)$, which is a smaller and more rigid class of candidate functions. But in any case the non-integrability condition (1.1) has the effect of strongly shaping the structure of $A_{f,F}$. Indeed, if $\mathcal{L}^2$ denotes the Lebesgue measure in $\mathbb{R}^2$ and $B_r(x)$ is the open disc of radius r centered at $x \in \mathbb{R}^2$, then the following properties are verified:

- Whatever the choice of $f \in C^1(\Omega)$, the set $A_{f,F}$ has no 3-density points, that is

  $$\limsup_{r\to 0+} \frac{\mathcal{L}^2(B_r(x) \setminus A_{f,F})}{r^3} > 0, \text{ for all } x \in A_{f,F},$$

  cf. Theorem 2.1 in [5]. Recall that, despite this, one has

  $$\sup_{f\in C^1(\Omega)} \mathcal{L}^2(A_{f,F}) = \mathcal{L}^2(\Omega)$$

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by Theorem 1 in [1].
- For all $f \in C^2(\Omega)$, this structure theorem even holds: The set $A_{f,F}$ is covered by a finite family of C^1 curves regularly imbedded in $\mathbb{R}^2$, by Proposition 3.1 and Theorem 4.1 of [9].

The purpose of this paper is to provide a wide generalization of the properties above. To make the main statements below understandable, we need to introduce some notation on systems of linear partial differential operators. Let $G = [G_{jl}]$ and $S = [S_{jl}]$ be two matrices of polynomials in $\mathbb{C}[\xi_1, \dots, \xi_n]$ of dimension $N \times k$ and $h \times N$, respectively. If $(x_1, \dots, x_n)$ denotes the standard coordinates of $\mathbb{R}^n$, let $G_{jl}(D)$ be the linear partial differential operator with constant coefficients obtained by replacing each ξ_j in $G_{jl}(\xi_1, \dots, \xi_n)$ with $-i\partial/\partial x_j$ and define $G(D)$ as the system $[G_{jl}(D)]$. Analogously we can define $S(D)$. Let p (resp. q) be the greatest of the degrees of the polynomials G_{jl} (resp. S_{jl}) and assume $p, q \geq 1$. Now consider an open set $\Omega \subset \mathbb{R}^n$, $F \in C^q(\Omega, \mathbb{C}^N)$ and recall the following obvious property: If there exist $f \in C^{p+q}(\Omega, \mathbb{C}^k)$ such that the set

$$A_{f,F} := \{x \in \Omega \,|\, (G(D)f)(x) = F(x)\}$$

has an interior point x_0, then one has $(S(D)F)(x_0) = 0$ whenever S is chosen in such a way that $SG = 0$.

We are finally ready to summarize the main results of this work:

Section 3. Let us assume that $F \in C^q(\Omega, \mathbb{C}^N)$ satisfies the following condition of non-integrability:

$$\text{There exists } S \text{ such that } SG = 0 \text{ and } (S(D)F)(x) \neq 0 \text{ for all } x \in \Omega.$$

If $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ and $B_r(x)$ is the open ball of radius r centered at $x \in \mathbb{R}^n$, then the following facts concerning the structure of $A_{f,F}$ hold:

- Whatever the choice of $f \in C^p(\Omega, \mathbb{C}^k)$, the set $A_{f,F}$ has no $(n+q)$-density points, that is
$$\limsup_{r \to 0+} \frac{\mathcal{L}^n(B_r(x) \setminus A_{f,F})}{r^{n+q}} > 0, \text{ for all } x \in A_{f,F},$$
cf. Corollary 3.2 below.
- For all $f \in C^{p+q}(\Omega, \mathbb{C}^k)$, the set $A_{f,F}$ is covered by a finite family of $(n-1)$-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^n$, cf. Theorem 3.1 below.

It is worth adding that the proof of Theorem 3.1 is based on the implicit function theorem and that a similar argument has been used in [10] to simplify the proof of the main result of [3], about the Hausdorff dimension of the tangency set of a submanifold with respect to a non-involutive distribution. It should be stressed, however, that in this application to the context of Frobenius theorem, the differential system to be studied is not linear, but only semilinear (c.f. [10, Section 3]).

Section 4. The main goal of this section is to prove Theorem 4.1, which provides a Lusin type result for any linear partial differential operator $G(D) = (G_1(D), \ldots, G_N(D))^t$ subject to the following assumptions:

- There exist $\alpha^{(1)}, \ldots, \alpha^{(N)} \in \mathbb{N}^n$ such that
$$G_r(D)x^{\alpha^{(s)}} = \begin{cases} 0 & \text{if } s \neq r \\ c_r \in \mathbb{C} \setminus \{0\} & \text{if } s = r \end{cases}$$
and $\min_j |\alpha^{(j)}| \geq p := \max_j \deg G_j$. One can easily verify that under this assumption, if $r \neq s$ then $G_r \neq G_s$ and one has also that $\alpha(r) \leq \alpha(s)$ cannot occur (in particular $\alpha(r) \neq \alpha(s)$), cf. Remark 4.1.
- There exist a nonnegative integer $m \leq p$, a constant $c_* > 0$ and an bounded open set $\Omega \subset \mathbb{R}^n$ such that
$$\|G(D)\varphi\|_{\infty,\Omega} \geq c_* \max_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} \|\partial^\alpha \varphi\|_{\infty,\Omega}$$
for all $\varphi \in C_c^\infty(\Omega, \mathbb{C})$.

Then, this result states that, for every bounded function $F \in C(\Omega, \mathbb{C}^N)$ and for every $\varepsilon \in (0, 1)$, there exist an open set $\mathcal{O} \subset \Omega$, $f \in C_0^m(\Omega, \mathbb{C})$ and $\Phi \in C_0(\Omega, \mathbb{C}^N)$ satisfying the following properties:

(1) $\mathcal{L}^n(\mathcal{O}) \leq \varepsilon \mathcal{L}^n(\Omega)$;
(2) $\Phi = F$ in $\Omega \setminus \mathcal{O}$;
(3) The equality $G(D)f = \Phi$ holds in the sense of distributions;
(4) In the special case $m = p$ one has $G(D)f = \Phi$ in the usual sense, hence $G(D)f = F$ in $\Omega \setminus \mathcal{O}$.

Section 5. We give two examples of application, respectively to Maxwell type system (cf. Corollary 5.2) and to multivariable Cauchy-Riemann system (cf. Corollary 5.3).

2. Notation and preliminaries

2.1. **General notation.** The constants depending only on $p, q, \ldots$ are indicated by $C(p, q, \ldots)$, while $B_r(x)$ is the open ball in $\mathbb{R}^n$ with center x and radius r. The open cube of side $2r$ centered at x in $\mathbb{R}^n$, that is $(-r, r)^n + x$, is denoted by $Q_r(x)$. For $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, we set $|z| := (|z_1|^2 + \cdots + |z_N|^2)^{1/2}$. If $E \subset \mathbb{R}^n$ and $f : E \to \mathbb{C}^N$, then we define
$$\|f\|_{\infty,E} := \sup_{x \in E} |f(x)|.$$

The coordinates of $\mathbb{R}^n$ are denoted by $(x_1, \dots, x_n)$ and we set for simplicity $\partial_j := \partial/\partial x_j$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, define

$$|\alpha| := \alpha_1 + \ldots + \alpha_n, \quad \alpha! := \alpha_1! \cdots \alpha_n!, \quad \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Similarly, if $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ then we write

$$\xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}.$$

We can also define a partial order in $\mathbb{N}^n$ by saying that $(\alpha_1, \dots, \alpha_N) \leq (\beta_1, \dots, \beta_N)$ if $\alpha_j \leq \beta_j$ for all j. If $\alpha, \beta \in \mathbb{N}^n$ satisfy $\alpha \leq \beta$ and $\alpha \neq \beta$, then we simply write $\alpha < \beta$.

If Ω is an open subset of $\mathbb{R}^n$, $p \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, then we set

$$C^p(\Omega, \mathbb{C}) := \{u + iv \,|\, u, v \in C^p(\Omega)\}, \quad C^p_c(\Omega, \mathbb{C}) := \{u + iv \,|\, u, v \in C^p_c(\Omega)\}$$

and

$$C^p(\Omega, \mathbb{C}^k) := \{(f_1, \dots, f_k)^t \,|\, f_1, \dots, f_k \in C^p(\Omega, \mathbb{C})\},$$

$$C^p_c(\Omega, \mathbb{C}^k) := \{(f_1, \dots, f_k)^t \,|\, f_1, \dots, f_k \in C^p_c(\Omega, \mathbb{C})\}.$$

For $\alpha \in \mathbb{N}^n$ and $f = (f_1, \dots, f_k)^t \in C^{|\alpha|}(\Omega, \mathbb{C}^k)$, we set

$$\partial^\alpha f := (\partial^\alpha f_1, \dots, \partial^\alpha f_k)^t.$$

The norm in $C^p(\Omega, \mathbb{C}^k)$ is defined as

$$C^p(\Omega, \mathbb{C}^k) \ni f \mapsto \|f\|_{C^p(\Omega, \mathbb{C}^k)} := \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq p}} \|\partial^\alpha f\|_{\infty, \Omega}.$$

The closure of $C^\infty_c(\Omega, \mathbb{C}^k)$ in $(C^p(\Omega, \mathbb{C}^k); \|\cdot\|_{C^p(\Omega, \mathbb{C}^k)})$ will be denoted by $C^p_0(\Omega, \mathbb{C}^k)$. For simplicity, we will write $C(\Omega, \mathbb{C}^k)$, $C_c(\Omega, \mathbb{C}^k)$ and $C_0(\Omega, \mathbb{C}^k)$ in place of $C^0(\Omega, \mathbb{C}^k)$, $C^0_c(\Omega, \mathbb{C}^k)$ and $C^0_0(\Omega, \mathbb{C}^k)$, respectively. If $E \subset \mathbb{R}^n$ is a Lebesgue measurable set and $u, v : E \to \mathbb{R}$ are Lebesgue integrable (resp. summable, locally summable) on E, then we say that $u + iv$ is Lebesgue integrable (resp. summable, locally summable) on E and define (omitting for simplicity to specify explicitly the measure, which is obviously the Lebesgue measure $\mathcal{L}^n$)

$$\int_E (u + iv) := \int_E u \, + \, i \int_E v.$$

The space of these locally summable functions will be denoted by $L^1_{\mathrm{loc}}(E, \mathbb{C})$. If $f_1, \dots, f_k : E \to \mathbb{C}$ are Lebesgue integrable (resp. summable, locally summable) on E, then we say that $f = (f_1, \dots, f_k)^t$ is Lebesgue integrable (resp. summable, locally summable) on E and define

$$\int_E f := \left(\int_E f_1, \dots, \int_E f_k \right)^t.$$

We also set $L^1_{\mathrm{loc}}(E, \mathbb{C}^k) := \{(f_1, \dots, f_k)^t \,|\, f_j \in L^1_{\mathrm{loc}}(E, \mathbb{C}) \text{ for } 1 \leq j \leq k\}$.

2.2. **Linear partial differential operators.** Let

$$P(\xi_1,\dots,\xi_n)=\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|\le p}} c_\alpha\,\xi^\alpha\in\mathbb{C}[\xi_1,\dots,\xi_n].$$

If $c_\alpha\neq 0$ for some $\alpha\in\mathbb{N}^n$ with $|\alpha|=p$, then the number p is said to be the total degree of P and is denoted by $\deg P$. As usual, $P(D)$ is the differential operator obtained by replacing each variable ξ_j with $-i\partial_j$, namely

$$P(D):=\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|\le p}}(-i)^{|\alpha|}c_\alpha\,\partial^\alpha. \tag{2.1}$$

If $M=[M_{ij}]$ is a matrix of polynomials in $\mathbb{C}[\xi_1,\dots,\xi_n]$, then $M(D):=[M_{ij}(D)]$. Moreover, we set

$$\deg M:=\max_{(i,j)}\deg M_{ij}.$$

Also define

$$P^*(\xi):=P(-\xi)=\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|\le p}}(-1)^{|\alpha|}c_\alpha\,\xi^\alpha\in\mathbb{C}[\xi_1,\dots,\xi_n]$$

and observe that if $P,Q\in\mathbb{C}[\xi_1,\dots,\xi_n]$ then these identities follows by a standard argument:

$$(P+Q)^*=P^*+Q^*,\quad (PQ)^*=P^*Q^*. \tag{2.2}$$

If $\varphi\in C^p_c(\mathbb{R}^n,\mathbb{C})$, $\psi\in C^p(\mathbb{R}^n,\mathbb{C})$ and $P\in\mathbb{C}[\xi_1,\dots,\xi_n]$ then $(P(D)\psi)\varphi$ and $(P^*(D)\varphi)\psi$ are obviously Lebesgue summable on $\mathbb{R}^n$ and a standard computation shows that

$$\int(P(D)\psi)\varphi=\int(P^*(D)\varphi)\psi. \tag{2.3}$$

2.3. **Distributions.** Let Ω be an open subset of $\mathbb{R}^n$. We recall that a linear functional $T:C^\infty_c(\Omega,\mathbb{C})\to\mathbb{C}$ is said to be a distribution on Ω if one has $\lim_{j\to\infty}T(\varphi_j)=T(\varphi)$ for every sequence $\{\varphi_j\}_{j=1}^\infty\subset C^\infty_c(\Omega,\mathbb{C})$ and $\varphi\in C^\infty_c(\Omega,\mathbb{C})$ such that

(i) There exists a compact set $K\subset\Omega$ such that $\operatorname{supp}\varphi_j\subset K$, for all j;
(ii) One has $\lim_{j\to\infty}\|\partial^\alpha\varphi_j-\partial^\alpha\varphi\|_{\infty,\Omega}=0$, for all $\alpha\in\mathbb{N}^n$.

If conditions (i) and (ii) are satisfied we say that the sequence $\{\varphi_j\}_{j=1}^\infty$ converges to φ in $C^\infty_c(\Omega,\mathbb{C})$. The set of all distributions on Ω, denoted by $\mathcal{D}'(\Omega)$, is obviously a vector space with addition and scalar multiplication defined by

$$(T_1+T_2)(\varphi)=T_1(\varphi)+T_2(\varphi),\quad(\lambda T)(\varphi)=\lambda T(\varphi)$$

for all $T_1,T_2,T\in\mathcal{D}'(\Omega)$, $\lambda\in\mathbb{C}$ and $\varphi\in C^\infty_c(\Omega,\mathbb{C})$. For every $u\in L^1_{\mathrm{loc}}(\Omega,\mathbb{C})$ one can define $T_u\in\mathcal{D}'(\Omega)$ as

$$T_u(\varphi):=\int_\Omega u\varphi,\quad \varphi\in C^\infty_c(\Omega,\mathbb{C}).$$

We recall that, if $P \in \mathbb{C}[\xi_1, \dots, \xi_n]$, $T \in \mathcal{D}'(\Omega)$ and set

$$[P(D)T](\varphi) := T(P^*(D)\varphi), \quad \varphi \in C_c^\infty(\Omega, \mathbb{C})$$

then $P(D)T \in \mathcal{D}'(\Omega)$. In particular, if $u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ then one has

$$[P(D)T_u](\varphi) = \int_\Omega (P^*(D)\varphi)u, \quad \varphi \in C_c^\infty(\Omega, \mathbb{C}).$$

Hence, in the special case when $u \in C^p(\Omega, \mathbb{C})$ with $p = \deg P$, recalling (2.3), we find the following regularity identity

$$P(D)T_u = T_{P(D)u}. \tag{2.4}$$

We shall use the weak topology in $\mathcal{D}'(\Omega)$, so that $\lim_{j\to\infty} T_j = T$ means that

$$\lim_{j\to\infty} T_j(\varphi) = T(\varphi), \text{ for all } \varphi \in C_c^\infty(\Omega, \mathbb{C}).$$

The map

$$L^1_{\mathrm{loc}}(\Omega, \mathbb{C}) \ni u \mapsto T_u \in \mathcal{D}'(\Omega)$$

is continuous. More precisely, if $\{u_j\}_{j=1}^\infty$ converges to u in $L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$, namely $u_j, u \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C})$ and

$$\lim_{j\to\infty} \int_K |u_j - u| = 0$$

for all compact set $K \subset \Omega$, then one has

$$\lim_{j\to\infty} T_{u_j} = T_u. \tag{2.5}$$

If $G = [G_{lm}]$ is a matrix of polynomials in $\mathbb{C}[\xi_1, \dots, \xi_n]$ of dimension $N \times k$ and

$$f = (f_1, \dots, f_k)^t \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^k), \quad \Phi = (\Phi_1, \dots, \Phi_N)^t \in L^1_{\mathrm{loc}}(\Omega, \mathbb{C}^N)$$

then we say that the equality $G(D)f = \Phi$ holds in the sense of distributions if

$$\sum_{m=1}^k G_{lm}(D)T_{f_m} = T_{\Phi_l}$$

for all $l = 1, \dots, N$, that is

$$\sum_{m=1}^k \int_\Omega (G^*_{lm}(D)\varphi) f_m = \int_\Omega \varphi \Phi_l \tag{2.6}$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{C})$ and $l = 1, \dots, N$.

2.4. **Superdensity.** A point $x \in \mathbb{R}^n$ is said to be a m-density point of $E \subset \mathbb{R}^n$ (where $m \in [n, +\infty)$) if

$$\mathcal{L}^n(B_r(x) \setminus E) = o(r^m) \qquad (\text{as } r \to 0+).$$

The set of all m-density points of E is denoted by $E^{(m)}$.

Remark 2.1. The following properties hold:

- Every interior point of $E \subset \mathbb{R}^n$ is an m-density point of E, for all $m \in [n, +\infty)$. Thus, whenever E is open, one has $E \subset E^{(m)}$ for all $m \in [n, +\infty)$;

- If $E \subset \mathbb{R}^n$ and $n \le m_1 \le m_2 < +\infty$, then $E^{(m_2)} \subset E^{(m_1)}$. In particular, one has $E^{(m)} \subset E^{(n)}$ for all $m \in [n, +\infty)$;
- If $A, B \subset \mathbb{R}^n$ then $(A \cap B)^{(m)} = A^{(m)} \cap B^{(m)}$, for all $m \in [n, +\infty)$;
- For all $A \subset \mathbb{R}^n$ and $m \in [n, +\infty)$, the set $A^{(m)}$ is $\mathcal{L}^n$-measurable (cf. [8, Proposition 3.1]).

Remark 2.2. Let $E \subset \mathbb{R}^n$. Then one has the following inequality

$$\frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} \ge 1 - \frac{\mathcal{L}^n(B_r(x) \setminus E)}{\mathcal{L}^n(B_r(x))} \qquad (r > 0)$$

where equality holds if E is $\mathcal{L}^n$-measurable. Hence:

- If $x \in E^{(n)}$, then x is a Lebesgue density point of E;
- If E is $\mathcal{L}^n$-measurable and x is a Lebesgue density point of E, then $x \in E^{(n)}$.

In particular, if E is $\mathcal{L}^n$-measurable, then $x \in E^{(n)}$ if and only if x is a point of Lebesgue density of E. Moreover

$$\mathcal{L}^n(E \Delta E^{(n)}) = 0$$

e.g., cf. Corollary 1.5 in [19, Chapter 3]. Then one has $\mathcal{L}^n(B_r(x) \setminus E^{(n)}) = \mathcal{L}^n(B_r(x) \setminus E)$, for all $r > 0$, hence

$$(E^{(n)})^{(n)} = E^{(n)}.$$

A remarkable family of superdense sets is the class of finite perimeter sets. From Theorem 1 in [12, Section 6.1.1] it actually follows that almost every point in a set $E \subset \mathbb{R}^n$ (with $n \ge 2$) of finite perimeter is a m_0-density point with

$$m_0 := n + 1 + \frac{1}{n-1}$$

which is also the maximum order of density common to all sets of finite perimeter. More precisely one has this result, cf. [5, Lemma 4.1] and [8, Proposition 4.1].

Proposition 2.1. *The following facts hold ($n \ge 2$):*

(1) *If E is a set of locally finite perimeter in $\mathbb{R}^n$, then $\mathcal{L}^n$-almost every point in E belongs to $E^{(m_0)}$;*
(2) *For all $m > m_0$ there exists a compact set F_m of finite perimeter in $\mathbb{R}^n$ such that $\mathcal{L}^n(F_m) > 0$ and $F_m^{(m)} = \emptyset$.*

2.5. **A class of cut-off functions.** Consider $r > 0$, $\rho \in (0, 1)$ and a function $\psi \in C^\infty(\mathbb{R})$ such that

$$0 \le \psi \le 1, \quad \psi|_{(-\infty,0]} \equiv 1, \quad \psi|_{[1,+\infty)} \equiv 0.$$

If define $\varphi_{\rho,r} : \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi_{\rho,r}(x) := \prod_{j=1}^{n} \psi\left(\frac{|x_j| - \rho r}{(1-\rho)r}\right), \text{ for all } x = (x_1, \dots, x_n) \in \mathbb{R}^n,$$

then one obviously has

$$\varphi_{\rho,r} \in C^\infty(\mathbb{R}^n, [0,1]), \quad \varphi_{\rho,r}|_{Q_{\rho r}(0)} \equiv 1, \quad \varphi_{\rho,r}|_{\mathbb{R}^n \setminus Q_r(0)} \equiv 0. \tag{2.7}$$

Moreover, a standard computation yields

$$\partial^\alpha \varphi_{\rho,r}(x) = \frac{1}{(1-\rho)^{|\alpha|} r^{|\alpha|}} \prod_{j=1}^{n} \psi^{(\alpha_j)}\left(\frac{|x_j| - \rho r}{(1-\rho)r}\right),$$

for all $\alpha \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, hence

$$\|\partial^\alpha \varphi_{\rho,r}\|_{\infty,\mathbb{R}^n} \le \frac{C(\alpha)}{(1-\rho)^{|\alpha|} r^{|\alpha|}} \tag{2.8}$$

for all $\alpha \in \mathbb{N}^n$, where $C(\alpha)$ is a number depending only on α (and n).

3. Structure of the set of solutions of differential identities subject to a condition of non-integrability

Throughout this section, $G = [G_{jl}]$ and $S = [S_{jl}]$ are matrices of polynomials in $\mathbb{C}[\xi_1, \dots, \xi_n]$ of dimension $N \times k$ and $h \times N$, respectively. Let

$$p := \deg G \ge 1.$$

Moreover Ω is an open subset of $\mathbb{R}^n$. Finally, for any given $f \in C^p(\Omega, \mathbb{C}^k)$ and $F \in C(\Omega, \mathbb{C}^N)$, we define

$$A_{f,F} := \{x \in \Omega \mid (G(D)f)(x) = F(x)\}.$$

Remark 3.1 (*Integrability condition).* Let us consider $F \in C^q(\Omega, \mathbb{C}^N)$. Then a necessary condition for the existence of $f \in C^{p+q}(\Omega, \mathbb{C}^k)$ such that the set $A_{f,F}$ has an interior point x_0 is obviously that

$$(S(D)F)(x_0) = 0, \text{ for all } S \text{ such that } SG = 0 \text{ and } 1 \le \deg S \le q. \tag{3.1}$$

In Theorem 3.2 below, we shall prove a generalization of this property, by suitably adapting the proof of [5, Theorem 2.1].

Remark 3.2. In general, the problem of determining S such that $SG = 0$ is not easy and for an account about its resolution we refer the reader to algebraic analysis literature, e.g., [4] (and the references therein), where it is addressed also through the use of specific softwares. In this regard it must be remembered that a particularly significant case is when S is the matrix yielded by the first syzygies of G, also considered in Corollary 3.1, Corollary 3.3, Example 3.1, Section 5.2 (Maxwell type system) and Section 5.3 (multivariable Cauchy-Riemann system) below. In this case, the identity $(S(D)F)(x) = 0$ for each $x \in \Omega$, under the further assumption that Ω is convex, is a necessary and sufficient condition for the existence of a solution f to the partial differential equation $G(D)f = F$ in the framework corresponding to a large class of sheaves of functions, cf. [4, Theorem 2.1.1].

3.1. Structure of $A_{f,F}$ under assumptions of non-integrability of F. The case of $f \in C^{p+q}(\Omega, \mathbb{C}^k)$.

Theorem 3.1. *Let $F = (F_1, \dots, F_N)^t \in C^q(\Omega, \mathbb{C}^N)$, $f \in C^{p+q}(\Omega, \mathbb{C}^k)$ and define $A^*_{f,F}$ as the set of all $x \in A_{f,F}$ saisfying the following property: There exists S, possibly depending on x, such that*

$$1 \leq \deg S \leq q,\ SG = 0 \text{ and } (S(D)F)(x) \neq 0. \tag{3.2}$$

Then the set $A^*_{f,F}$ is covered by a finite family of $(n-1)$-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^n$. In particular, if $\mathcal{H}^{n-1}(A_{f,F} \setminus A^*_{f,F}) = 0$ then $A_{f,F}$ is $(n-1)$-rectifiable (cf. [13, 17]*).*

Proof. First of all, let us define

$$\Phi = (\Phi_1, \dots, \Phi_N)^t := F - G(D)f \in C^q(\Omega, \mathbb{C}^N).$$

Also consider $x \in A^*_{f,F}$ and let $S = [S_{jl}]$ (possibly depending on x) be such that (3.2) is satisfied. Then

$$S(D)F = S(D)(\Phi + G(D)f) = S(D)\Phi.$$

Since $A_{f,F} = \Phi^{-1}(0)$, it follows, in particular, that

$$S(D)F = S^\circ(D)\Phi, \text{ in } A_{f,F}$$

where $S^\circ = [S^\circ_{jl}]$ is the matrix of dimension $h \times N$ whose entries are defined as

$$S^\circ_{jl}(\xi_1, \dots, \xi_n) := S_{jl}(\xi_1, \dots, \xi_n) - S_{jl}(0, \dots, 0) \in \mathbb{C}[\xi_1, \dots, \xi_n]$$

for all $(j,l) \in \{1, \dots, h\} \times \{1, \dots, N\}$. Hence, by also recalling the non-integrability condition (3.2), we find

$$(S^\circ(D)\Phi)(x) \neq 0.$$

This implies the following

> Property: For all $x \in A^*_{f,F}$ there exist $\alpha \in \mathbb{N}^n$ with $1 \leq |\alpha| \leq q$ and $j \in \{1, \dots, N\}$ such that $\partial^\alpha \Phi_j(x) \neq 0$.

Now let $\partial^0\Phi := \Phi$, while, for $l = 1, \dots, q$, let $\partial^l\Phi$ denote the vector field obtained by ordering (in some arbitrarily chosen way) the set

$$\{\partial^\alpha \Phi_j \,|\, \alpha \in \mathbb{N}^n \text{ with } |\alpha| = l,\ j = 1, \dots, N\}.$$

Then, for $l = 0, \dots, q-1$, define

$$A^{(l)}_{f,F} := \left\{x \in A^*_{f,F} \,|\, \partial^{l+1}\Phi(x) \neq 0,\ \partial^l\Phi(x) = 0\right\} \tag{3.3}$$

and observe that

$$A^*_{f,F} = \bigcup_{l=0}^{q-1} A^{(l)}_{f,F}, \tag{3.4}$$

by Property above. By virtue of (3.4) we are reduced to prove that if $l \in \{0, \dots, q-1\}$ then $A^{(l)}_{f,F}$ is covered by a finite family of $(n-1)$-dimensional regularly imbedded C^1

submanifolds of $\mathbb{R}^n$. For this purpose, let K_l denote the number of components of $\partial^l\Phi$ and define

$$U^{(l)} = (u_1^{(l)}, \dots, u_{K_l}^{(l)})^t := \operatorname{Re}\partial^l\Phi \in C^{q-l}(\Omega, \mathbb{R}^{K_l}),$$

$$V^{(l)} = (v_1^{(l)}, \dots, v_{K_l}^{(l)})^t := \operatorname{Im}\partial^l\Phi \in C^{q-l}(\Omega, \mathbb{R}^{K_l}).$$

Moreover, set for simplicity

$$\left\{\operatorname{rk}(DU^{(l)}) \geq 1\right\} := \left\{x \in \Omega \,|\, DU^{(l)}(x) \text{ has rank} \geq 1\right\},$$

$$\left\{\operatorname{rk}(DV^{(l)}) \geq 1\right\} := \left\{x \in \Omega \,|\, DV^{(l)}(x) \text{ has rank} \geq 1\right\},$$

$$\left\{U^{(l)} = 0\right\} := \left\{x \in \Omega \,|\, U^{(l)}(x) = 0\right\}, \quad \left\{V^{(l)} = 0\right\} := \left\{x \in \Omega \,|\, V^{(l)}(x) = 0\right\}$$

and observe that, by definition (3.3), one has

$$A_{f,F}^{(l)} \subset \left\{\operatorname{rk}(DU^{(l)}) \geq 1\right\} \cup \left\{\operatorname{rk}(DV^{(l)}) \geq 1\right\}$$

and

$$A_{f,F}^{(l)} \subset \left\{U^{(l)} = 0\right\} \cap \left\{V^{(l)} = 0\right\}.$$

Hence

$$A_{f,F}^{(l)} \subset \left(\left\{U^{(l)} = 0\right\} \cap \left\{\operatorname{rk}(DU^{(l)}) \geq 1\right\}\right) \cup \left(\left\{V^{(l)} = 0\right\} \cap \left\{\operatorname{rk}(DV^{(l)}) \geq 1\right\}\right).$$

Moreover

$$\begin{aligned}\left\{U^{(l)} = 0\right\} \cap \left\{\operatorname{rk}(DU^{(l)}) \geq 1\right\} &= \left\{U^{(l)} = 0\right\} \cap \bigcup_{j=1}^{K_l} \left\{x \in \Omega \,|\, \nabla u_j^{(l)}(x) \neq 0\right\} \\ &\subset \bigcup_{j=1}^{K_l} \left\{x \in \Omega \,|\, u_j^{(l)}(x) = 0,\, \nabla u_j^{(l)}(x) \neq 0\right\}\end{aligned}$$

and analogously

$$\left\{V^{(l)} = 0\right\} \cap \left\{\operatorname{rk}(DV^{(l)}) \geq 1\right\} \subset \bigcup_{j=1}^{K_l} \left\{x \in \Omega \,|\, v_j^{(l)}(x) = 0,\, \nabla v_j^{(l)}(x) \neq 0\right\}.$$

Thus

$$A_{f,F}^{(l)} \subset \left(\bigcup_{j=1}^{K_l} \Gamma_j^{(l)}\right) \cup \left(\bigcup_{j=1}^{K_l} \Lambda_j^{(l)}\right)$$

where

$$\Gamma_j^{(l)} := \left\{x \in \Omega \,|\, u_j^{(l)}(x) = 0,\ \nabla u_j^{(l)}(x) \neq 0\right\},$$

$$\Lambda_j^{(l)} := \left\{x \in \Omega \,|\, v_j^{(l)}(x) = 0,\ \nabla v_j^{(l)}(x) \neq 0\right\}.$$

We conclude by observing that, for all $j = 1, \dots, K_l$, the sets $\Gamma_j^{(l)}$ and $\Lambda_j^{(l)}$ are $(n-1)$-dimensional regularly imbedded C^{q-l} submanifolds of $\mathbb{R}^n$ (by the implicit function theorem). $\square$

Corollary 3.1. *Let us consider the special case when $k = 1$, namely $G := (G_1, \dots, G_N)^t$ and $p := \deg G \geq 1$. Moreover, let $F = (F_1, \dots, F_N)^t \in C^p(\Omega, \mathbb{C}^N)$, $f \in C^{2p}(\Omega, \mathbb{C})$ and assume that for all $x \in A_{f,F}$ there exist $j, l \in \{1, \dots, N\}$ such that*

$$(G_j(D)F_l)(x) \neq (G_l(D)F_j)(x). \tag{3.5}$$

Then $A_{f,F}$ is covered by a finite family of $(n-1)$-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^n$. In particular, $A_{f,F}$ is $(n-1)$-rectifiable so that its Hausdorff dimension is less or equal to $n-1$.

Proof. Let $S = [S_{jl}]$ be the matrix yielded by the first syzygies of G (cf. Example 2.1.1 in [4]), which can be obtained as follows. First of all, set $h := N(N-1)/2$ and let $\{(r_j, s_j)\}_{j=1}^h$ be the set of all the couples

$$(r, s) \in \mathbb{N}^2, \text{ with } 1 \leq r < s \leq N$$

ordered in some way (e.g. lexicographically). Then $S = [S_{jl}]$ is the matrix of polynomials in $\mathbb{C}[\xi_1, \dots, \xi_n]$ of dimension $h \times N$ such that

$$S_{jl} := \begin{cases} G_{s_j} & \text{if } l = r_j \\ -G_{r_j} & \text{if } l = s_j \\ 0 & \text{otherwise.} \end{cases} \qquad (j = 1, \dots, h).$$

Observe that $\deg S = p$ and $(S(D)F)(x) \neq 0$ for all $x \in A_{f,F}$, by (3.5). Since $SG = 0$, the conclusion follows at once from Theorem 3.1. □

Remark 3.3. The result established in Theorem 3.1 is very general, but we expect that in many special cases its conclusions can be correspondingly improved. This expectation is confirmed, for example, by [9, Corollary 4.1] where a suitable adaptation of the argument used to prove Theorem 3.1 allowed to get the following result of interest for the context of the Heisenberg group $\mathbb{H}^m$ over $\mathbb{R}^{2m+1}$ (cf. also [2]): Consider $F : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ defined as

$$F(x_1, \dots, x_{2m}) := (2x_{m+1}, \dots, 2x_{2m}, -2x_1, \dots, -2x_m)^t$$

and let Ω be an open subset of $\mathbb{R}^{2m}$. Then, for all $f \in C^2(\Omega)$, the set $A_{f,F}$ is covered by a finite family of m-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2m}$. In particular, $A_{f,F}$ is m-rectifiable so that its Hausdorff dimension is less or equal to m.

3.2. **Structure of $A_{f,F}$ under assumptions of non-integrability of F. The case of $f \in C^p(\Omega, \mathbb{C}^k)$.** In Section 3.1 we have proved that (under assumptions of non-integrability of F) the set $A_{f,F}$ is $(n-1)$-rectifiable, whenever $f \in C^{p+q}(\Omega, \mathbb{C}^k)$. If we extend the class of functions f under consideration to $C^p(\Omega, \mathbb{C}^k)$, then, as we shall see below, it can happen to bump into f such that $\mathcal{L}^n(A_{f,F}) > 0$ (cf. Theorem 4.1 and Corollary 4.1). However, even in this case, the non-integrability condition strongly shapes the structure of $A_{f,F}$. Indeed the following result, which is the main goal of the present section, holds (cf. Corollary 3.2): A point x_0 at which the condition (3.1) is not verified cannot be a $(n+q)$-density point of $A_{f,F}$. From this property it follows in particular that no function $f \in C^p(\Omega, \mathbb{C}^k)$

can exist such that $G(D)f$ equals $F \in C^1(\Omega, \mathbb{C}^N)$ at almost every point of a subset of Ω with locally finite perimeter (cf. Corollary 3.4).

Theorem 3.2. *Let Ω be an open subset of $\mathbb{R}^n$ and consider*

$$f = (f_1, \dots , f_k)^t \in C(\Omega, \mathbb{C}^k), \quad \Phi = (\Phi_1, \dots , \Phi_N)^t \in C(\Omega, \mathbb{C}^N)$$

such that $G(D)f = \Phi$ holds in the sense of distributions. Moreover, let $F = (F_1, \dots , F_N)^t \in C^q(\Omega, \mathbb{C}^N)$ with $q \geq 1$ and define

$$B_{F,\Phi} := \{x \in \Omega \mid \Phi(x) = F(x)\}.$$

If $x_0 \in \Omega \cap B_{F,\Phi}^{(n+q)}$, then one has $(S(D)F)(x_0) = 0$ for all S such that $1 \leq \deg S \leq q$ and $SG = 0$.

Proof. Let us consider $x_0 \in \Omega \cap B_{F,\Phi}^{(n+q)}$, $\rho \in (1/2, 1)$ and $r \in (0, 1)$ small enough so that $\overline{Q_r(x_0)} \subset \Omega$. Recall from Section 2.5 that a function $\varphi_{\rho,r,x_0} \in C_c^\infty(\mathbb{R}^n, [0, 1])$ has to exist such that

$$\varphi_{\rho,r,x_0}|_{Q_{\rho r}(x_0)} \equiv 1, \quad \varphi_{\rho,r,x_0}|_{\mathbb{R}^n \setminus Q_r(x_0)} \equiv 0$$

and

$$\|\partial^\alpha \varphi_{\rho,r,x_0}\|_{\infty,\mathbb{R}^n} \leq \frac{C(\alpha)}{(1-\rho)^{|\alpha|} r^{|\alpha|}} \tag{3.6}$$

for all $\alpha \in \mathbb{N}^n$, where $C(\alpha)$ is a number depending only on α (and n). In the formulas below we set for simplicity

$$Q_r := Q_r(x_0), \quad Q_{\rho r} := Q_{\rho r}(x_0), \quad \varphi_{\rho,r} := \varphi_{\rho,r,x_0}.$$

Then, by (2.3), we obtain (for all $j \in \{1, \dots , h\}$)

$$\begin{aligned} \int_{Q_r} (S(D)F)_j \varphi_{\rho,r} &= \sum_{l=1}^N \int_{Q_r} (S_{jl}(D)F_l)\varphi_{\rho,r} = \sum_{l=1}^N \int_{Q_r} (S^*_{jl}(D)\varphi_{\rho,r})F_l \\ &= \sum_{l=1}^N \int_{Q_r \setminus B_{F,\Phi}} (S^*_{jl}(D)\varphi_{\rho,r})F_l + \sum_{l=1}^N \int_{Q_r \cap B_{F,\Phi}} (S^*_{jl}(D)\varphi_{\rho,r})\Phi_l \end{aligned}$$

that is

$$\int_{Q_r} (S(D)F)_j \varphi_{\rho,r} = I_{r,j} + \sum_{l=1}^N \int_{Q_r \setminus B_{F,\Phi}} (S^*_{jl}(D)\varphi_{\rho,r})(F_l - \Phi_l) \tag{3.7}$$

where

$$I_{r,j} := \sum_{l=1}^N \int_{Q_r} (S^*_{jl}(D)\varphi_{\rho,r})\Phi_l.$$

Observe that, by (2.2), (2.3), (2.6) and the assumption $SG = 0$, one has

$$\begin{aligned} I_{r,j} &= \sum_{m=1}^{k}\sum_{l=1}^{N}\int_{Q_r}(G^*_{lm}(D)S^*_{jl}(D)\varphi_{\rho,r})f_m \\ &= \sum_{m=1}^{k}\int_{Q_r}\left((\sum_{l=1}^{N}S_{jl}G_{lm})^*(D)\varphi_{\rho,r}\right)f_m \\ &= \sum_{m=1}^{k}\int_{Q_r}\left((SG)^*_{jm}(D)\varphi_{\rho,r}\right)f_m = 0. \end{aligned}$$

Hence it follows from (3.6) and (3.7) that

$$\begin{aligned} \left|\int_{Q_r}(S(D)F)_j\varphi_{\rho,r}\right| &= \left|\sum_{l=1}^{N}\int_{Q_r\setminus B_{F,\Phi}}(S^*_{jl}(D)\varphi_{\rho,r})(F_l-\Phi_l)\right| \\ &\le C_1\,\mathcal{L}^n(Q_r\setminus B_{F,\Phi})\sum_{\substack{\alpha\in\mathbb{N}^n\\|\alpha|\le q}}\frac{1}{(1-\rho)^{|\alpha|}r^{|\alpha|}} \\ &\le \frac{C_2\,\mathcal{L}^n(Q_r\setminus B_{F,\Phi})}{(1-\rho)^q r^q} \end{aligned}$$

where C_1 and C_2 are positive constants which do not depend on r and ρ. On the other hand

$$\begin{aligned} \left|\int_{Q_r}(S(D)F)_j\varphi_{\rho,r}\right| &\ge \left|\int_{Q_{\rho r}}(S(D)F)_j\varphi_{\rho,r}\right| - \left|\int_{Q_r\setminus Q_{\rho r}}(S(D)F)_j\varphi_{\rho,r}\right| \\ &= \left|\int_{Q_{\rho r}}(S(D)F)_j\right| - \left|\int_{Q_r\setminus Q_{\rho r}}(S(D)F)_j\varphi_{\rho,r}\right| \end{aligned}$$

and thus

$$\begin{aligned} \rho^n\left|\fint_{Q_{\rho r}}(S(D)F)_j\right| &\le \frac{1}{\mathcal{L}^n(Q_r)}\left(\left|\int_{Q_r}(S(D)F)_j\varphi_{\rho,r}\right| + \left|\int_{Q_r\setminus Q_{\rho r}}(S(D)F)_j\varphi_{\rho,r}\right|\right) \\ &\le \frac{C\,\mathcal{L}^n(Q_r\setminus B_{F,\Phi})}{(1-\rho)^q r^{q+n}} + \frac{C\,(r^n-\rho^n r^n)}{r^n} \end{aligned}$$

where C does not depend on r and ρ. Passing to the limit for $r \to 0+$ and recalling that $x_0 \in B^{(n+q)}_{F,\Phi}$, we obtain

$$\rho^n\,|(S(D)F)_j(x_0)| \le C\,(1-\rho^n).$$

We conclude by passing to the limit for $\rho \to 1-$. □

Corollary 3.2. *Let $f \in C^p(\Omega,\mathbb{C}^k)$ and $F \in C^q(\Omega,\mathbb{C}^N)$ with $q \ge 1$. If $x_0 \in \Omega \cap A^{(n+q)}_{f,F}$, then one has $(S(D)F)(x_0) = 0$ for all S such that $1 \le \deg S \le q$ and $SG = 0$. As a consequence, if $x_0 \in \Omega$ and there exists S such that $1 \le \deg S \le q$, $SG = 0$ and $(S(D)F)(x_0) \ne 0$, then $x_0 \notin A^{(n+q)}_{f,F}$.*

Proof. We simply set $\Phi := G(D)f$ and apply Theorem 3.2. □

Corollary 3.3. *Let $k = 1$, namely $G := (G_1, \dots, G_N)^t$ and $p := \deg G \geq 1$. Moreover let $F = (F_1, \dots, F_N)^t \in C^p(\Omega, \mathbb{C}^N)$, $x_0 \in \Omega$ and assume that there exist $j, l \in \{1, \dots, N\}$ such that*

$$(G_j(D)F_l)(x_0) \neq (G_l(D)F_j)(x_0).$$

Then $x_0 \notin A^{(n+p)}_{f,F}$, whatever the choice of $f \in C^p(\Omega, \mathbb{C})$.

Proof. Let $S = [S_{jl}]$ be the matrix of polynomials in $\mathbb{C}[\xi_1, \dots, \xi_n]$ that we have considered in the proof of Corollary 3.1. Then $SG = 0$ and $(S(D)F)(x_0) \neq 0$, hence the conclusion follows at once from Corollary 3.2. □

From Corollary 3.2 (with $q = 1$) and (1) of Proposition 2.1, we obtain at once the following result.

Corollary 3.4. *Assume $\deg S = 1$ and $SG = 0$. Let Ω be an open subset of $\mathbb{R}^n$ and consider $F \in C^1(\Omega, \mathbb{C}^N)$ such that $S(D)F \neq 0$ at $\mathcal{L}^n$ a.e. point of a set E of locally finite perimeter in $\mathbb{R}^n$, with $E \subset \Omega$. Then there is no function $f \in C^p(\Omega, \mathbb{C}^k)$ such that $G(D)f = F$ at $\mathcal{L}^n$ a.e. point of E.*

Example 3.1. *From Corollary 3.3 with $N = n$ and $G_j(\xi_1, \dots, \xi_n) = i\xi_j$, we get immediately the following result already proved in* [5] *(cf. also* [7]*) which generalizes the classical Schwarz theorem about equality of mixed partial derivatives: Let $f \in C^1(\Omega)$ and $F \in C^1(\Omega, \mathbb{R}^n)$. Then, for all $x_0 \in \Omega \cap \{x \in \Omega : \nabla f(x) = F(x)\}^{(n+1)}$, one has $\partial_j F_l(x_0) = \partial_l F_j(x_0)$ for all $j, l = 1, \dots, n$. In particular, by recalling (1) of Proposition 2.1, one obtains this property: If the Jacobian matrix of $F \in C^1(\Omega, \mathbb{R}^n)$ is nonsymmetric at $\mathcal{L}^n$ a.e. point of a set E of locally finite perimeter in $\mathbb{R}^n$, with $E \subset \Omega$, then there is no function $f \in C^1(\Omega)$ such that $\nabla f = F$ at $\mathcal{L}^n$ a.e. point of E (cf.* [15, Corollary 2], [6, Theorem 1.3]*). These arguments can obviously be restated for $\mathbb{C}$ valued functions.*

4. A Lusin type result for a class of linear partial differential operators

The proofs of the following Lemma 4.1 and Theorem 4.1 are adaptations of those of [1, Lemma 7] and [1, Theorem 1].

Lemma 4.1. *Let $G_1, \dots, G_N \in \mathbb{C}[\xi_1, \dots, \xi_n]$ and define $G(D) := (G_1(D), \dots, G_N(D))^t$. Assume that there exist $\alpha^{(1)}, \dots, \alpha^{(N)} \in \mathbb{N}^n$ such that*

$$G_r(D)x^{\alpha^{(s)}} = \begin{cases} 0 & \text{if } s \neq r \\ c_r \in \mathbb{C} \setminus \{0\} & \text{if } s = r \end{cases}$$

and

$$\min_j |\alpha^{(j)}| \geq p := \max_j \deg G_j. \tag{4.1}$$

Moreover consider an open set $\Omega \subset \mathbb{R}^n$ such that $\mathcal{L}^n(\Omega) < +\infty$, a bounded function $f = (f_1, \ldots, f_N)^t \in C(\Omega, \mathbb{C}^N)$, $\varepsilon \in (0, 1/2)$ and $\eta > 0$. Then there exist a compact set $K \subset \Omega$, a function $v \in C_c^\infty(\Omega, \mathbb{C})$ and a constant c_0 which does not depend on f, ε, η such that

(1) $\mathcal{L}^n(\Omega \setminus K) \leq \varepsilon \mathcal{L}^n(\Omega)$;
(2) $\|G(D)v - f\|_{\infty,K} \leq \eta$;
(3) $\|G(D)v\|_{\infty,\Omega} \leq (1 + c_0\, \varepsilon^{-p})\|f\|_{\infty,\Omega}$.

Proof. According to the first steps in the proof of [1, Lemma 7], we can find $\delta \in (0, 1)$ and a compact set $K \subset \Omega$ with the following properties:

- The estimate (1) holds and
$$K = \bigcup_{j \in J} Q_j,$$
where $\{Q_j\}_{j \in J}$ is a finite family of closed cubes of side $(1 - \varepsilon/2n)\delta$, whose centers y_j belong to the lattice $(\delta\mathbb{Z})^n$;
- For $j \in J$, let T_j be the closed cube of side δ centered at y_j. Then, for all $j \in J$, one has $T_j \subset \Omega$ and

$$|f(x) - f(y)| \leq \eta, \text{ whenever } x, y \in T_j. \tag{4.2}$$

Now, for all $j \in J$ and $x \in \mathbb{R}^n$, set
$$\Phi_j(x) := \varphi_{\rho,\delta/2}(x - y_j), \text{ with } \rho := 1 - \varepsilon/2n,$$
and observe that

$$\Phi_j \in C^\infty(\mathbb{R}^n, [0, 1]), \quad \Phi_j|_{Q_j} \equiv 1, \quad \Phi_j|_{\mathbb{R}^n \setminus T_j} \equiv 0, \tag{4.3}$$

by (2.7). Moreover

$$\|\partial^\alpha \Phi_j\|_{\infty,\mathbb{R}^n} \leq C(\alpha)\varepsilon^{-|\alpha|}\delta^{-|\alpha|} \tag{4.4}$$

for all $\alpha \in \mathbb{N}^n$, by (2.8). Then define the function $v : \Omega \to \mathbb{R}$ as follows
$$v(x) := \sum_{j \in J} \Phi_j(x) \sum_{s=1}^{N} \frac{f_s(y_j)}{c_s}(x - y_j)^{\alpha^{(s)}}, \quad x \in \Omega.$$

One obviously has $v \in C_c^\infty(\Omega, \mathbb{C})$, by (4.3). To prove (2) and (3), we need the explicit expressions of the polynomials G_r, that is
$$G_r(\xi_1, \ldots, \xi_n) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq p}} c_\alpha^{(r)} \xi^\alpha \quad (c_\alpha^{(r)} \in \mathbb{C})$$

where the coefficients $c_\alpha^{(r)}$ are assumed to be zero when $|\alpha|$ exceeds the degree of G_r. Recalling (2.1), we find (for $x \in \Omega$)

$$[G_r(D)v](x) = \sum_{j\in J}\sum_{s=1}^N \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\le p}} \frac{(-i)^{|\alpha|}c_\alpha^{(r)} f_s(y_j)}{c_s}\partial^\alpha\left[\Phi_j(x)(x-y_j)^{\alpha^{(s)}}\right]$$

where, for suitable integer coefficients $k_\beta^{(\alpha)}$ (which coincide with 1 for $\beta = 0$ and $\beta = \alpha$), one has

$$\begin{aligned}\partial^\alpha\left[\Phi_j(x)(x-y_j)^{\alpha^{(s)}}\right] &= \sum_{\substack{\beta\in\mathbb{N}^n \\ \beta\le\alpha}} k_\beta^{(\alpha)}\partial^\beta\Phi_j(x)\,\partial^{\alpha-\beta}[(x-y_j)^{\alpha^{(s)}}] \\ &= \Phi_j(x)\,\partial^\alpha[(x-y_j)^{\alpha^{(s)}}] + \sum_{\substack{\beta\in\mathbb{N}^n \\ 0<\beta\le\alpha}} k_\beta^{(\alpha)}\partial^\beta\Phi_j(x)\,\partial^{\alpha-\beta}[(x-y_j)^{\alpha^{(s)}}].\end{aligned}$$

It follows that

(4.5)

$$\begin{aligned}[G_r(D)v](x) &= \sum_{j\in J}\Phi_j(x)\sum_{s=1}^N \frac{f_s(y_j)}{c_s}\,G_r(D)[(x-y_j)^{\alpha^{(s)}}] \\ &+ \sum_{j\in J}\sum_{s=1}^N \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\le|\alpha|\le p}} \frac{(-i)^{|\alpha|}c_\alpha^{(r)} f_s(y_j)}{c_s}\sum_{\substack{\beta\in\mathbb{N}^n \\ 0<\beta\le\alpha}} k_\beta^{(\alpha)}\partial^\beta\Phi_j(x)\,\partial^{\alpha-\beta}[(x-y_j)^{\alpha^{(s)}}] \\ &= \sum_{j\in J}\Phi_j(x) f_r(y_j) \\ &+ \sum_{j\in J}\sum_{s=1}^N \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\le|\alpha|\le p}}\sum_{\substack{\beta\in\mathbb{N}^n \\ 0<\beta\le\alpha}} \frac{(-i)^{|\alpha|}c_\alpha^{(r)}k_\beta^{(\alpha)}}{c_s}\,f_s(y_j)\,\partial^\beta\Phi_j(x)\,\partial^{\alpha-\beta}[(x-y_j)^{\alpha^{(s)}}].\end{aligned}$$

In the formulae below, $C_1, C_2, \ldots$ will denote constants which do not depend on f, ε, δ, j. From the previous identity, it follows that (for all $j \in J$)

$$\begin{aligned}\|G(D)v\|_{\infty,T_j} &\le |f(y_j)| \\ &\quad + C_1\,|f(y_j)|\sum_{s=1}^N \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\le|\alpha|\le p}}\sum_{\substack{\beta\in\mathbb{N}^n \\ 0<\beta\le\alpha}} \|\partial^\beta\Phi_j\|_{\infty,T_j}\,\sup_{x\in T_j}|\partial^{\alpha-\beta}[(x-y_j)^{\alpha^{(s)}}]|\end{aligned}$$

where

$$\sup_{x\in T_j}|\partial^{\alpha-\beta}[(x-y_j)^{\alpha^{(s)}}]| \le C_2\,\delta^{|\alpha^{(s)}|-|\alpha|+|\beta|}.$$

Hence, by also recalling (4.4):

$$\|G(D)v\|_{\infty,T_j} \le |f(y_j)| + C_3\,|f(y_j)|\sum_{s=1}^N \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\le|\alpha|\le p}}\sum_{\substack{\beta\in\mathbb{N}^n \\ 0<\beta\le\alpha}} \varepsilon^{-|\beta|}\delta^{-|\beta|}\,\delta^{|\alpha^{(s)}|-|\alpha|+|\beta|}.$$

Since $\delta \in (0,1)$ and (4.1) holds, it follows that

$$\|G(D)v\|_{\infty,T_j} \leq |f(y_j)| \left(1 + C_3 \sum_{s=1}^{N} \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\leq|\alpha|\leq p}} \delta^{p-|\alpha|} \sum_{\substack{\beta\in\mathbb{N}^n \\ 0<\beta\leq\alpha}} \varepsilon^{-|\beta|}\right)$$

$$\leq |f(y_j)| \left(1 + C_4 \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\leq|\alpha|\leq p}} \delta^{p-|\alpha|} \sum_{h=1}^{|\alpha|} \varepsilon^{-h}\right)$$

where (since $0 < \varepsilon < 1/2$)

$$\sum_{h=1}^{|\alpha|} \varepsilon^{-h} = \frac{1-\varepsilon^{|\alpha|}}{1-\varepsilon}\, \varepsilon^{-|\alpha|} \leq 2\varepsilon^{-|\alpha|}.$$

Thus (since $0 < \varepsilon\delta < 1/2$)

$$\|G(D)v\|_{\infty,T_j} \leq |f(y_j)| \left(1 + C_5\delta^p \sum_{\substack{\alpha\in\mathbb{N}^n \\ 1\leq|\alpha|\leq p}} (\varepsilon\delta)^{-|\alpha|}\right) \leq |f(y_j)| \left(1 + C_6\delta^p \sum_{h=1}^{p} (\varepsilon\delta)^{-h}\right)$$

$$\leq |f(y_j)| \left(1 + C_7\delta^p(\varepsilon\delta)^{-p}\right)$$

that is

$$\|G(D)v\|_{\infty,T_j} \leq |f(y_j)| \left(1 + C_7\varepsilon^{-p}\right).$$

Hence (3) follows from the arbitrariness of $j \in J$ and recalling that v vanishes outside $\cup_{j\in J} T_j$, by (4.3).

To prove (2), observe that one has

$$[G(D)v](x) = f(y_j), \text{ for all } x \in Q_j \text{ and } j \in J$$

by (4.5) and (4.3). Recalling also (4.2), we get

$$\|G(D)v - f\|_{\infty,Q_j} \leq \eta, \text{ for all } j \in J$$

which yields the conclusion. □

Remark 4.1. Let $N \geq 2$ be an integer and consider N polynomials in $\mathbb{C}[\xi_1, \dots, \xi_n]$ given explicitly as

$$G_r(\xi_1, \dots, \xi_n) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|\leq p}} c_\alpha^{(r)} \xi^\alpha \qquad (r = 1, \dots, N).$$

Then the family $G_1, \dots, G_N$ satisfies the condition of Lemma 4.1 if and only if one has (for all $r = 1, \dots, N$)

$$\begin{cases} c_{\alpha^{(r)}}^{(r)} \neq 0 & \\ c_\alpha^{(r)} = 0 & \text{for all } \alpha < \alpha^{(r)} \\ c_\alpha^{(r)} = 0 & \text{for all } \alpha \leq \alpha^{(s)}, \text{ with } s \neq r \end{cases}$$

cf. (2.1). From this observation it follows that if $G_1, \dots, G_N$ satisfies the condition of Lemma 4.1 and $r \neq s$, then

$$G_r \neq G_s$$

and

$$\alpha(r) \leq \alpha(s) \text{ cannot occur (in particular } \alpha(r) \neq \alpha(s)).$$

Theorem 4.1. *Let $G_1, \dots, G_N \in \mathbb{C}[\xi_1, \dots, \xi_n]$ satisfy the same hypotheses as in Lemma 4.1. Moreover, consider a bounded open set $\Omega \subset \mathbb{R}^n$ and assume that there exist a non-negative integer $m \leq p = \deg G$ and a constant $c_* > 0$ such that*

$$\|G(D)\varphi\|_{\infty,\Omega} \geq c_* \max_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha|=m}} \|\partial^\alpha \varphi\|_{\infty,\Omega} \tag{4.6}$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{C})$. Then, for every bounded function $F \in C(\Omega, \mathbb{C}^N)$ and for every $\varepsilon \in (0,1)$, there exist an open set $\mathcal{O} \subset \Omega$, $f \in C_0^m(\Omega, \mathbb{C})$ and $\Phi \in C_0(\Omega, \mathbb{C}^N)$ with the following properties:

1. *$\mathcal{L}^n(\mathcal{O}) \leq \varepsilon \mathcal{L}^n(\Omega)$;*
2. *$\Phi = F$ in $\Omega \setminus \mathcal{O}$;*
3. *The equality $G(D)f = \Phi$ holds in the sense of distributions;*
4. *In the special case $m = p$ one has $G(D)f = \Phi$ in the usual sense, hence $G(D)f = F$ in $\Omega \setminus \mathcal{O}$.*

Proof. We can assume $F \neq 0$ (if $F = 0$ the result trivially holds with $\mathcal{O} = \emptyset$, $f \equiv 0$ and $\Phi = 0$), so that $\|F\|_{\infty,\Omega} > 0$. Then define $f_0 := F$ and let us prove, first of all, that there exist two sequences of functions

$$\{f_j\}_{j=1}^\infty \subset C(\Omega, \mathbb{C}^N) \cap L^\infty(\Omega, \mathbb{C}^N), \quad \{v_j\}_{j=1}^\infty \subset C_c^\infty(\Omega, \mathbb{C})$$

and a sequence $\{K_j\}_{j=1}^\infty$ of compact subsets of Ω satisfying the following properties, for all $j \geq 1$:

- (i) $\mathcal{L}^n(\Omega \setminus K_j) \leq 2^{-j} \varepsilon \, \mathcal{L}^n(\Omega)$;
- (ii) $\|G(D)v_j - f_{j-1}\|_{\infty,K_j} \leq 2^{-j(p+1)} \|F\|_{\infty,\Omega}$;
- (iii) $\|G(D)v_j\|_{\infty,\Omega} \leq (1 + c_0 \, 2^{jp} \varepsilon^{-p}) \|f_{j-1}\|_{\infty,\Omega}$, where c_0 is the constant in Lemma 4.1;
- (iv) $f_j(x) = f_{j-1}(x) - [G(D)v_j](x)$ for all $x \in K_j$ and $\|f_j\|_{\infty,\Omega} = \|f_{j-1} - G(D)v_j\|_{\infty,K_j}$.

Such a statement is proved by the following induction argument:

- First of all, use Lemma 4.1 to get a compact set $K_1 \subset \Omega$ and $v_1 \in C_c^\infty(\Omega, \mathbb{C})$ such that (i), (ii) and (iii) hold with $j = 1$. Then we get $f_1 \in C(\Omega, \mathbb{C}^N) \cap L^\infty(\Omega, \mathbb{C}^N)$ satisfying (iv) with $j = 1$, by extending the function

  $$f_0(x) - [G(D)v_1](x), \quad x \in K_1$$

  by means of Tietze's theorem [18, 20.4].

- Now suppose to have
$$\{f_j\}_{j=1}^H \subset C(\Omega, \mathbb{C}^N) \cap L^\infty(\Omega, \mathbb{C}^N), \quad \{v_j\}_{j=1}^H \subset C_c^\infty(\Omega, \mathbb{C})$$
and a family $\{K_j\}_{j=1}^H$ of compact subsets of Ω such that the properties (i-iv) above are satisfied for $j = 1, \dots, H$, where H is any positive integer. By using again Lemma 4.1 we can find a compact set $K_{H+1} \subset \Omega$ and $v_{H+1} \in C_c^\infty(\Omega, \mathbb{C})$ such that (i), (ii) and (iii) hold with $j = H+1$. Moreover, by Tietze's theorem [18, 20.4], we get $f_{H+1} \in C(\Omega, \mathbb{C}^N) \cap L^\infty(\Omega, \mathbb{C}^N)$ which satisfies (iv) with $j = H+1$.

Now let
$$\mathcal{O} := \Omega \setminus \bigcap_{j=1}^{\infty} K_j \tag{4.7}$$
and note that (1) follows at once from (i) above. Moreover, from (ii), (iii), (iv) and recalling that $\varepsilon \in (0,1)$, we get this estimate
$$\begin{aligned}
\sum_{j=1}^{\infty} \|G(D)v_j\|_{\infty,\Omega} &= \left(1 + \frac{c_0 2^p}{\varepsilon^p}\right) \|F\|_{\infty,\Omega} + \sum_{j=2}^{\infty} \left(1 + \frac{c_0 2^{jp}}{\varepsilon^p}\right) 2^{-(j-1)(p+1)} \|F\|_{\infty,\Omega} \\
&= \frac{\|F\|_{\infty,\Omega}}{\varepsilon^p} \sum_{j=1}^{\infty} \left(1 + c_0 2^{jp}\right) 2^{-(j-1)(p+1)} \\
&< \frac{\|F\|_{\infty,\Omega}}{\varepsilon^p} \sum_{j=0}^{\infty} \left(1 + c_0 2^p\right) 2^{-j} \\
&= \frac{2(1 + c_0 2^p)\|F\|_{\infty,\Omega}}{\varepsilon^p}
\end{aligned}$$
so that the series $\sum_{j=1}^{\infty} G(D)v_j$ converges totally in $L^\infty(\Omega, \mathbb{C}^N)$. Hence, if define $u_H := \sum_{j=1}^{H} v_l \in C_c^\infty(\Omega, \mathbb{C})$ (for $H = 1, 2, \dots$):

- There exists $\Phi = (\Phi_1, \dots, \Phi_N)^t \in C_0(\Omega, \mathbb{C}^N)$ such that
$$\lim_{H \to \infty} \|G(D)u_H - \Phi\|_{\infty,\Omega} = 0; \tag{4.8}$$
- By recalling assumption (4.6) and Poincaré's inequality (cf. Theorem 3 of [11, Sect. 5.6]), we find that $f \in C_0^m(\Omega, \mathbb{C})$ has to exist such that
$$\lim_{H \to \infty} \|u_H - f\|_{C^m(\Omega,\mathbb{C})} = 0. \tag{4.9}$$

Now, recalling (iv) above, one can easily prove by induction on k that if $x \in \cap_{j=1}^{\infty} K_j$ and $H \geq 1$ then the following identity
$$F(x) - [G(D)u_H](x) = f_k(x) - \sum_{l=k+1}^{H} [G(D)v_l](x)$$

holds for all $k = 0, \dots, H-1$. Hence, recalling also (4.7) and (ii) above, we obtain

$$\begin{aligned} \|F - G(D)u_H\|_{\infty,\Omega\setminus\mathcal{O}} &= \|F - G(D)u_H\|_{\infty,\cap_j K_j} \\ &= \|f_{H-1} - G(D)v_H\|_{\infty,\cap_j K_j} \\ &\le 2^{-H(p+1)}\|F\|_{\infty,\Omega}. \end{aligned} \tag{4.10}$$

From (4.8), (4.10) and the inequality

$$\|F - \Phi\|_{\infty,\Omega\setminus\mathcal{O}} \le \|F - G(D)u_H\|_{\infty,\Omega\setminus\mathcal{O}} + \|G(D)u_H - \Phi\|_{\infty,\Omega\setminus\mathcal{O}}$$

we get assertion (2).

By (4.8), (4.9), recalling the regularity identity (2.4) and the continuity property (2.5) for distributions, we obtain

$$G_j(D)T_f = \lim_{H\to\infty} G_j(D)T_{u_H} = \lim_{H\to\infty} T_{G_j(D)u_H} = T_{\Phi_j} \quad (j = 1, \dots, N)$$

which proves (3). Finally (4) follows immediately from (3) and (2). □

Remark 4.2. The conclusions of Theorem 4.1 do not extend to families of polynomials $G_1, \dots, G_N$ in which there are repeated elements (compare Remark 4.1). To prove it, let's assume that there is a repetition, namely $G_r = G_s$ with $r \neq s$, and consider any $F = (F_1, \dots, F_N)^t$ such that $F_r \equiv 0$ and $F_s \equiv 1$. Then at least one of statements (1),(2),(3) of Theorem 4.1 must fail to be true. Indeed (3) yields $T_{\Phi_r} = T_{\Phi_s}$, hence $\Phi_r = \Phi_s$. Then $1 = 0$ in $\Omega \setminus \mathcal{O}$, by (2). But this implies $\mathcal{O} = \Omega$, which contradicts (1).

From Theorem 4.1 we get immediately the following property.

Corollary 4.1. *Let $G_1, \dots, G_N \in \mathbb{C}[\xi_1, \dots, \xi_n]$ satisfy the same hypotheses as in Lemma 4.1. Moreover, consider a bounded open set $\Omega \subset \mathbb{R}^n$ and assume that there exists a constant $c_* > 0$ such that*

$$\|G(D)\varphi\|_{\infty,\Omega} \ge c_* \max_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=p}} \|\partial^\alpha\varphi\|_{\infty,\Omega}$$

for all $\varphi \in C_c^\infty(\Omega, \mathbb{C})$. Then, for every bounded function $F \in C(\Omega, \mathbb{C}^N)$, one has

$$\sup_{f\in C_0^p(\Omega,\mathbb{C})} \mathcal{L}^n(A_{f,F}) = \mathcal{L}^n(\Omega).$$

5. Examples of application

5.1. **Alberti's Theorem.** Given a positive integer k, let $\mathcal{T}_k$ denote the set of n-tuples $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k$ and set $N_k := \#\mathcal{T}_k$. Moreover let $j \mapsto \alpha^{(j)}$ be an arbitrarily chosen bijection from $\{1, \dots, N_k\}$ to Λ_k. Then we obtain the following well known result (cf. [1, 14]).

Corollary 5.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ and k be a positive integer. Then, for every bounded function $F = (F_1, \dots, F_{N_k})^T \in C(\Omega, \mathbb{R}^{N_k})$ and for every $\varepsilon \in (0, 1)$, there exist an open set $\mathcal{O} \subset \Omega$ and $f \in C_0^k(\Omega)$ with the following properties:*

(1) $\mathcal{L}^n(\mathcal{O}) \leq \varepsilon \mathcal{L}^n(\Omega)$;
(2) $\partial^{\alpha^{(j)}} f = F_j$ *in* $\Omega \setminus \mathcal{O}$, *for all* $j = 1, \ldots, N_k$.

Proof. The family of polynomials

$$G_j(\xi_1, \ldots, \xi_n) := i^{|\alpha^{(j)}|} \xi^{\alpha^{(j)}} \in \mathbb{C}[\xi_1, \ldots, \xi_n] \qquad (j = 1, \ldots, N_k)$$

verifies the assumptions of Lemma 4.1, in that

$$G_r(D) x^{\alpha^{(s)}} = \partial^{\alpha^{(r)}} x^{\alpha^{(s)}} = \begin{cases} 0 & \text{if } s \neq r \\ \alpha^{(r)}! & \text{if } s = r \end{cases}$$

and

$$\deg G_j = |\alpha^{(j)}| = k \qquad (j = 1, \ldots, N_k)$$

so that $p = \max_j \deg G_j = k$. Moreover condition (4.6) is trivially satified with $m = k = p$. The conclusion follows from (1) and (4) of Theorem 4.1. □

5.2. **Maxwell type system.** Let us recall that the electromagnetic field is characterized by the system

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j} \end{cases}$$

where $\mathbf{E}, \mathbf{B}, \rho$ and $\mathbf{j}$ are the electric field, the magnetic field, the electric charge density and the electric current density, respectively. The symbol of this system is the following matrix of polynomials in $\mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4]$

$$G(\xi_1, \xi_2, \xi_3, \xi_4) = [G_{jl}(\xi_1, \xi_2, \xi_3, \xi_4)] = \begin{bmatrix} i\xi_1 & i\xi_2 & i\xi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & i\xi_1 & i\xi_2 & i\xi_3 \\ 0 & -i\xi_3 & i\xi_2 & i\xi_4 & 0 & 0 \\ i\xi_3 & 0 & -i\xi_1 & 0 & i\xi_4 & 0 \\ -i\xi_2 & i\xi_1 & 0 & 0 & 0 & i\xi_4 \\ -i\xi_4 & 0 & 0 & 0 & -i\xi_3 & i\xi_2 \\ 0 & -i\xi_4 & 0 & i\xi_3 & 0 & -i\xi_1 \\ 0 & 0 & -i\xi_4 & -i\xi_2 & i\xi_1 & 0 \end{bmatrix},$$

where ξ_1, ξ_2, ξ_3 are the symbols of the spatial differential operators $-i\partial_{x_1}, -i\partial_{x_2}, -i\partial_{x_3}$, while ξ_4 is the symbol of the time differential operator $-i\partial_{x_4}$ (for consistency with the notation introduced in the previous sections, we denote the time variable with x_4). In this case, a remarkable example of S such that $SG = 0$ is the matrix associated to the first syzygies (cf. [4, Section 5.1])

$$\underline{S}(\xi_1, \xi_2, \xi_3, \xi_4) = [\underline{S}_{jl}(\xi_1, \xi_2, \xi_3, \xi_4)] = \begin{bmatrix} 0 & i\xi_4 & i\xi_1 & i\xi_2 & i\xi_3 & 0 & 0 & 0 \\ i\xi_4 & 0 & 0 & 0 & 0 & i\xi_1 & i\xi_2 & i\xi_3 \end{bmatrix}.$$

Corollary 5.2. *Let Ω be a bounded open subset of $\mathbb{R}^4$. Then, for every bounded function $F = (F_1, \dots, F_8)^t \in C(\Omega, \mathbb{C}^8)$ and for every $\varepsilon \in (0,1)$, there exist an open set $\mathcal{O} \subset \Omega$ and $f \in C_0^1(\Omega, \mathbb{C}^6)$ with the following properties:*

(1) $\mathcal{L}^4(\mathcal{O}) \leq \varepsilon \mathcal{L}^4(\Omega)$;
(2) $G(D)f = F$ *in* $\Omega \setminus \mathcal{O}$.

Moreover:

(3) *Let* $F \in C^1(\Omega, \mathbb{C}^8)$. *Then for all* $(x_0, t_0) \in \Omega \cap A^{(5)}_{f,F}$ *one has* $(S(D)F)(x_0, t_0) = 0$ *whenever* $\deg S = 1$ *and* $SG = 0$. *In particular,* $(\underline{S}(D)F)(x_0, t_0) = 0$, *i.e.,*
$$\begin{cases} \partial_t F_2(x_0,t_0) + \nabla_x \cdot (F_3, F_4, F_5)(x_0, t_0) = 0 \\ \partial_t F_1(x_0,t_0) + \nabla_x \cdot (F_6, F_7, F_8)(x_0, t_0) = 0 \end{cases}$$
for all $(x_0, t_0) \in \Omega \cap A^{(5)}_{f,F}$.
(4) *Let* $F \in C^1(\Omega, \mathbb{C}^8)$, $g \in C^2(\Omega, \mathbb{C}^6)$ *and assume that* $(\underline{S}(D)F)(x) \neq 0$ *for all* $x \in A_{g,F}$. *Then the set* $A_{g,F}$ *is covered by a finite family of* 3*-dimensional regularly imbedded* C^1 *submanifolds of* $\mathbb{R}^4$. *In particular,* $A_{g,F}$ *is* 3*-rectifiable so that its Hausdorff dimension is less or equal to* 3.

Proof. Let
$$H := (H_1, H_2, H_3, H_4)^t := (G_{11}, G_{41}, G_{51}, G_{61})^t$$
and
$$K := (K_1, K_2, K_3, K_4)^t := (G_{24}, G_{34}, G_{74}, G_{84})^t.$$
Then:

- The polynomials H_1, H_2, H_3, H_4 verify the assumptions of Lemma 4.1 with
$$\alpha^{(1)} = (1,0,0,0),\ \alpha^{(2)} = (0,0,1,0),\ \alpha^{(3)} = (0,1,0,0),\ \alpha^{(4)} = (0,0,0,1).$$
Moreover H satisfies condition (4.6) with $m = \deg H = 1$.
- Analogously, K_1, K_2, K_3, K_4 verify the the assumptions of Lemma 4.1 with
$$\alpha^{(1)} = (1,0,0,0),\ \alpha^{(2)} = (0,0,0,1),\ \alpha^{(3)} = (0,0,1,0),\ \alpha^{(4)} = (0,1,0,0),$$
and K satisfies condition (4.6) with $m = \deg K = 1$.

Hence, by Theorem 4.1, there exist two open sets $\mathcal{O}_1, \mathcal{O}_2 \subset \Omega$ and $f_1, f_4 \in C_0^1(\Omega, \mathbb{C})$ such that
$$\mathcal{L}^4(\mathcal{O}_1) \leq \frac{\varepsilon}{2} \mathcal{L}^4(\Omega), \quad H(D)f_1 = (F_1, F_4, F_5, F_6)^t \text{ in } \Omega \setminus \mathcal{O}_1$$
and
$$\mathcal{L}^4(\mathcal{O}_2) \leq \frac{\varepsilon}{2} \mathcal{L}^4(\Omega), \quad K(D)f_4 = (F_2, F_3, F_7, F_8)^t \text{ in } \Omega \setminus \mathcal{O}_2.$$

Statements (1) and (2) follow by setting $\mathcal{O} := \mathcal{O}_1 \cup \mathcal{O}_2$ and $f := (f_1, 0, 0, f_4, 0, 0)^t$. As for (3), it follows immediately from Corollary 3.2. Finally, we obtain (4) from Theorem 3.1. □

5.3. **Multivariable Cauchy-Riemann system.** Let $G_1, \dots, G_N \in \mathbb{C}[\xi_1, \dots, \xi_{2N}]$ be defined as

$$G_j(\xi_1, \dots, \xi_{2N}) := \frac{i}{2}\,\xi_{2j-1} - \frac{1}{2}\,\xi_{2j} \qquad (j = 1, \dots, N).$$

Then $G = (G_1, \dots, G_N)^t$ is the symbol of the Cauchy-Riemann system in N complex variables $z_j = x_{2j-1} + ix_{2j}$ $(j = 1, \dots, N)$, namely

$$G(D) = \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1} \\ \vdots \\ \frac{\partial}{\partial \bar{z}_N} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\,\partial_1 + \frac{i}{2}\,\partial_2 \\ \vdots \\ \frac{1}{2}\,\partial_{2N-1} + \frac{i}{2}\,\partial_{2N} \end{pmatrix}.$$

Observe that $p = \deg G = 1$. Analogously as we have done for the Maxwell type system, we can consider the matrix associated to the first syzygies, namely the one of dimension $\frac{N(N-1)}{2} \times N$ used in the proof of Corollary 3.1. Also in this case we denote such a matrix by $\underline{S}$.

Corollary 5.3. *Let Ω be a bounded open subset of $\mathbb{R}^{2N}$. Then, for every bounded function $F \in C(\Omega, \mathbb{C}^N)$ and for every $\varepsilon \in (0,1)$, there exist an open set $\mathcal{O} \subset \Omega$, $f \in C_0(\Omega, \mathbb{C})$ and $\Phi \in C_0(\Omega, \mathbb{C}^N)$ such that:*

(1) *$\mathcal{L}^{2N}(\mathcal{O}) \le \varepsilon \mathcal{L}^{2N}(\Omega)$;*
(2) *$\Phi = F$ in $\Omega \setminus \mathcal{O}$;*
(3) *The equality $G(D)f = \Phi$ holds in the sense of distributions.*

Moreover:

(4) *Let $F \in C^1(\Omega, \mathbb{C}^N)$. Then for all $x_0 \in \Omega \cap B^{(2N+1)}_{F,\Phi}$ (in particular, for all $x_0 \in \Omega \cap (\mathcal{O}^c)^{(2N+1)}$) one has $(S(D)F)(x_0) = 0$ whenever $\deg S = 1$ and $SG = 0$. In particular, $(\underline{S}(D)F)(x_0) = 0$, i.e.,*

$$\frac{\partial F_l}{\partial \bar{z}_j}(x_0) = \frac{\partial F_j}{\partial \bar{z}_l}(x_0)$$

for all $j, l \in \{1, \dots, N\}$ and for all $x_0 \in \Omega \cap B^{(2N+1)}_{F,\Phi}$ (in particular, for all $x_0 \in \Omega \cap (\mathcal{O}^c)^{(2N+1)}$).
(5) *Let $F \in C^1(\Omega, \mathbb{C}^N)$, $g \in C^2(\Omega, \mathbb{C})$ and assume that for all $x \in A_{g,F}$ one has $(\underline{S}(D)F)(x) \neq 0$, that is the matrix $[\frac{\partial F_l}{\partial \bar{z}_j}(x)]$ is not symmetric. Then the set $A_{g,F}$ is covered by a finite family of $(2N-1)$-dimensional regularly imbedded C^1 submanifolds of $\mathbb{R}^{2N}$. In particular, $A_{g,F}$ is $(2N-1)$-rectifiable so that its Hausdorff dimension is less or equal to $2N-1$.*

Proof. Observe that $G_1, \dots, G_N$ verify the hypothesis of Lemma 4.1 with

$$\alpha^{(j)} = (\alpha_1^{(j)}, \dots, \alpha_{2N}^{(j)}) \in \mathbb{N}^{2N}$$

defined by

$$\alpha_h^{(j)} := \begin{cases} 0 & \text{if } h \neq 2j \\ 1 & \text{if } h = 2j. \end{cases}$$

Moreover we can easily prove that (4.6) holds with $m = 0$, as follows. First of all, let us consider $R > 0$ such that

$$\Omega \subset B_R(0) = \left\{(x_1, \dots, x_{2N}) \in \mathbb{R}^{2N} \mid x_1^2 + \dots + x_{2N}^2 < R^2\right\}.$$

Moreover, let $\varphi \in C_c^\infty(\Omega, \mathbb{C})$ and define $\widetilde{\varphi} \in C_c^\infty(B_R(0), \mathbb{C})$ by

$$\widetilde{\varphi}(x) := \begin{cases} \varphi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in B_R(0) \setminus \Omega. \end{cases}$$

In the special case when $N = 1$, from a well known classical representation formula (cf. [16, Corollary 1.1.5]) we obtain

$$\widetilde{\varphi}(z) = -\frac{1}{2\pi i} \int_{B_R(0)} \frac{\partial \widetilde{\varphi}(\zeta)}{\partial \bar{\zeta}} (\zeta - z)^{-1} \, d\bar{\zeta} \wedge d\zeta$$

for all $z \in B_R(0)$, hence (4.6) with $m = 0$ follows at once. Then assume $N \geq 2$. In such a case let us consider the projection

$$\Pi : \mathbb{R}^{2N} \to \mathbb{R}^{2N-2}, \quad \Pi(x_1, \dots, x_{2N}) := (x_3, \dots, x_{2N}).$$

Moreover, for $y \in \mathbb{R}^{2N-2}$ such that $|y| < R$, let D_y denote the open disc in $\mathbb{R}^2$ of radius $(R^2 - |y|^2)^{1/2}$ centered at 0 and define the map

$$\sigma_y : D_y \to B_R(0), \quad \sigma_y(t_1, t_2) := (t_1, t_2, y).$$

Applying the representation formula mentioned above to the function $\widetilde{\varphi} \circ \sigma_y : D_y \to \mathbb{C}$ and observing that $(\widetilde{\varphi} \circ \sigma_y)|_{\partial D_y} = 0$, we get

$$\widetilde{\varphi} \circ \sigma_y(z) = -\frac{1}{2\pi i} \int_{D_y} \frac{\partial (\widetilde{\varphi} \circ \sigma_y)(\zeta)}{\partial \bar{\zeta}} (\zeta - z)^{-1} \, d\bar{\zeta} \wedge d\zeta \tag{5.1}$$

for all $z \in D_y$. Now, for all $x = (x_1, \dots, x_{2N}) \in B_R(0)$, one has

$$(x_1, x_2) \in D_{\Pi(x)}, \quad \widetilde{\varphi}(x) = \widetilde{\varphi}(x_1, x_2, \Pi(x)) = \widetilde{\varphi} \circ \sigma_{\Pi(x)}(x_1, x_2)$$

and hence also

$$\frac{\partial \widetilde{\varphi}(x)}{\partial \bar{z}_1} = \frac{\partial (\widetilde{\varphi} \circ \sigma_{\Pi(x)})(x_1, x_2)}{\partial \bar{\zeta}}.$$

Thus, if $x \in B_R(0)$ and set $y := \Pi(x)$ in (5.1), we find

$$\widetilde{\varphi}(x) = -\frac{1}{2\pi i} \int_{D_{\Pi(x)}} \frac{\partial \widetilde{\varphi}(\zeta, \Pi(x))}{\partial \bar{z}_1} (\zeta - (x_1 + i x_2))^{-1} \, d\bar{\zeta} \wedge d\zeta.$$

Since $D_{\Pi(x)}$ is contained in the disc $\widetilde{D}(x_1, x_2)$ of radius $2R$ centered at (x_1, x_2), it follows that

$$\begin{aligned}\|\widetilde{\varphi}\|_{\infty,B_R(0)} &\leq \frac{1}{\pi}\left\|\frac{\partial\widetilde{\varphi}}{\partial\bar{z}_1}\right\|_{\infty,B_R(0)}\int_{\widetilde{D}(x_1,x_2)}[(\zeta_1-x_1)^2+(\zeta_2-x_2)^2]^{-1/2}\,d\mathcal{L}^2(\zeta_1,\zeta_2)\\ &= 4R\left\|\frac{\partial\widetilde{\varphi}}{\partial\bar{z}_1}\right\|_{\infty,B_R(0)}\end{aligned}$$

namely

$$\|\varphi\|_{\infty,\Omega} \leq 4R\,\|G_1(D)\varphi\|_{\infty,\Omega}.$$

This yields (4.6) with $m = 0$, so the hypotheses of Theorem 4.1 are all verified and thus (1), (2), (3) follow. To prove (4), we observe that

$$\Omega\setminus\mathcal{O} \subset B_{F,\Phi} = \{x\in\Omega \,|\, \Phi(x) = F(x)\}$$

by (2). Hence

$$\Omega\cap(\mathcal{O}^c)^{(2N+1)} \subset \Omega\cap B_{F,\Phi}^{(2N+1)}$$

and we get the conclusion from Theorem 3.2. Finally, we immediately get (5) from Theorem 3.1. □

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