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http://www.aimspress.com/journal/mine

Mathematics in Engineering, 4(2): 1–40.
DOI:10.3934/mine.2022013
Received: 09 January 2021
Accepted: 19 May 2021
Published: 02 July 2021

Research article

A geometric capacitary inequality for sub-static manifolds with harmonic potentials†

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† **This contribution is part of the Special Issue:** When analysis meets geometry – on the 50th birthday of Serrin's problem
Guest Editors: Giorgio Poggesi; Lorenzo Cavallina
Link: `www.aimspress.com/mine/article/5924/special-articles`

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Abstract: In this paper, we prove that associated with a sub-static asymptotically flat manifold endowed with a harmonic potential there is a one-parameter family $\{F_\beta\}$ of functions which are monotone along the level-set flow of the potential. Such monotonicity holds up to the optimal threshold $\beta = \frac{n-2}{n-1}$ and allows us to prove a geometric capacitary inequality where the capacity of the horizon plays the same role as the ADM mass in the celebrated Riemannian Penrose Inequality.

Keywords: sub–static metrics; splitting theorem; Schwarzschild solution; overdetermined boundary value problems

1. Introduction

In this paper, the object under investigation is a triple (M, g_0, u) satisfying the following two conditions:

(a) (M, g_0) is a smooth, connected, noncompact, complete, asymptotically flat, n-dimensional Riemannian manifold, with $n \geq 3$, with one end, and with nonempty smooth compact boundary ∂M, which is a priori allowed to have several connected components.

(b) $u \in C^\infty(M)$ satisfies the system

$$\begin{cases} u\mathrm{Ric}_{g_0} - \mathrm{D}^2_{g_0} u \geq 0 & \text{in } M, \\ \Delta_{g_0} u = 0 & \text{in } M, \\ u = 0 & \text{on } \partial M, \\ u \to 1 & \text{at } \infty, \end{cases} \tag{1.1}$$

where Ric_{g_0}, D_{g_0} and Δ_{g_0} are the Ricci tensor, the Levi–Civita connection, and the Laplace operator of the metric g_0, respectively.

If the equality holds in the first equation of (1.1), the triple (M, g_0, u) is said *static*. For clarity, we recall the definition to which we refer for asymptotically flat manifolds.

Definition 1.1. A smooth, connected, noncompact, n-dimensional Riemannian manifold (with or without compact boundary) (N, h), with $n \geq 3$, is said to be *asymptotically flat* if there exists a compact subset $K \subset N$ such that $N \setminus K$ is a finite disjoint union of *ends* N_k with the following properties. Every N_k is diffeomorphic to $\mathbb{R}^n$ minus a closed ball by a coordinate chart ψ_k and, if $\widetilde{h} := (\psi_k)_* h = \widetilde{h}_{ij} dx^i \otimes dx^j$, we have

$$\widetilde{h}_{ij} = \delta_{ij} + O(|x|^{-p}), \tag{1.2}$$

$$\partial_r \widetilde{h}_{ij} = O(|x|^{-(p+1)}), \tag{1.3}$$

$$\partial_r \partial_s \widetilde{h}_{ij} = O(|x|^{-(p+2)}), \tag{1.4}$$

$$\mathrm{R}_{\widetilde{h}} \in L^1(\mu_{\widetilde{h}}), \tag{1.5}$$

for some $p > (n-2)/2$. Here, δ is the Kronecker delta, and the coordinate charts ψ_k are called *charts at infinity*.

Throughout the paper, we will refer to a triple (M, g_0, u) that satisfies conditions (a) and (b) as to a *sub-static harmonic triple*. A fundamental sub-static harmonic triple is the so called *Schwarzschild solution*, which is given by

$$M = [(2m)^{\frac{1}{n-2}}, +\infty) \times \mathbb{S}^{n-1}, \qquad g_0 = \frac{dr \otimes dr}{1 - 2mr^{2-n}} + r^2 g_{\mathbb{S}^{n-1}}, \qquad u = \sqrt{1 - 2mr^{2-n}}. \tag{1.6}$$

It is well–known that both the metric g_0 and the potential u, which a priori are well defined only in $\mathring{M}$, extend smoothly up to the boundary and (M, g_0) is called *(spatial) Schwarzschild manifold*. The parameter $m > 0$ is the ADM mass m_{ADM} of the Schwarzschild manifold. We refer the reader to Section 5 for the definition of the m_{ADM} associated with a general asymptotically flat manifold. Here, we limit ourselves to recall that the decay conditions (1.2)–(1.5) guarantee that m_{ADM} is a geometric invariant [4, 8].

Associated with a sub-static harmonic triple, specifically with the potential u ranging in $[0, 1)$, let us consider the following family of functions depending on the parameter $\beta \geq 0$:

$$[0, 1) \ni t \longmapsto V_\beta(t) := (1 - t^2)^{-\beta(\frac{n-1}{n-2})} \int\limits_{\{u = t\}} |\mathrm{D}u|^{\beta+1} \, d\sigma.$$

In [2] it was proven that if (M, g_0, u) is a static triple, then, for every $\beta \geq 2$, the function V_β is strictly nonincreasig unless (M, g_0, u) is the Schwarzschild solution. The main purpose of this paper is to extend this result to the sub-static case and to the optimal threshold $\beta = \frac{n-2}{n-1}$. This is the content of Theorem 3.1, where the monotonicity of the above family - equipped with a corresponding rigidity statement - is expressed in terms of the functions $F_\beta(\tau)$, where $\tau = \frac{1+t^2}{1-t^2} \geq 1$, to be consistent with [3] and in light of the more advanced analysis contained therein. This generalisation suggests that our approach is robust enough and likely to be exported to other contexts. In a similar way, S. Brendle shows in [7] how some structure conditions for the metric are sufficient to prove an Alexandrov-type theorem and how such structure generalises to the sub-static case.

Let us now be slightly more detailed on how our Theorem 3.1 is proved. We adopt the main strategy proposed in [2], which essentially consists in obtaining the monotonicity as a consequence of a fundamental integral identity derived in a suitable conformally-related setting (see Proposition 4.3). A delicate point is justifying such identity in a region where critical points of the potential are present. One of the main differences with [2] is that, whereas in the static case the analyticity of the potential guarantees the local finiteness of the singular values, which made the argument simpler in many occurrences, in the present sub-static setting the metric and in turn the potential are not a priori analytic. Nevertheless, standard measure properties of the critical set of harmonic functions (summarised in Theorem 2.3) are enough to obtain the fundamental integral identity, which in turn implies the monotonicity of F_β and, coupled with Sard's Theorem, also its differentiability.

Observe that the difficulty in treating the critical points under the threshold $\beta = 1$ can be read off directly from formulæ (3.2) and (4.4), the first one displaying the derivative of F_β and the second one expressing the mean curvature on a equipotential set in terms of the Hessian of the potential itself. In fact, calling Φ_β the conformal version of F_β and looking at formula (4.25) containing the equivalent characterisation of Φ'_β derived from the integral identity (4.17), one realises that problems arise already when $\beta < 2$.

Let us stress that the monotonicity is obtained from the nonnegativity of the right-hand side of our fundamental integral identity. It is above the threshold $\beta = \frac{n-2}{n-1}$ that this is guaranteed, thanks to the Refined Kato Inequality for harmonic functions. The optimality of such inequality reflects a corresponding optimality of $\beta = \frac{n-2}{n-1}$ in our result. Moreover, let us remark that the (nonnegative) right-hand side of (4.17) is obtained as the divergence of a suitable modification of a specific vector filed with nonnegative divergence (see (4.18)), in the limit of a vanishing neighbourhood of the critical set. The crucial point in the construction is to maintain the divergence of the modified vector field nonnegative. It would be interesting to see whether a similar construction can be performed for other families of metrics, including special solutions as rigid case.

A straightforward application of the monotonicity of F_β is comparing $F_\beta(1)$ with $F_\beta(+\infty)$, in turn yielding a "capacitary version" of the Riemannian-Penrose inequality (Theorem 1.1 below). The capacity comes naturally into play when computing $F_\beta(1)$ and $F_\beta(+\infty)$, the latter value via the asymptotic expansions of the metric and of the potential. We recall that the capacity $\mathrm{Cap}(\partial M, g_0)$ of ∂M is defined as

$$\mathrm{Cap}(\partial M, g_0) := \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \inf\left\{ \int_M |\mathrm{D}_{g_0} v|^2_{g_0} \, d\mu_{g_0} \,:\, v \in \mathrm{Lip}_{loc}(M),\ v \geq 0,\ v = 0 \text{ on } \partial M,\ v \to 1 \text{ at} \infty \right\}.$$

Throughout the paper, we will use the short–hand notation C for the capacity. Comparing (1.6) with either (2.1) or (2.2), it is straightforward, in the case of the Schwarzschild solution, that $m_{\mathrm{ADM}} = C$. For a general sub-static harmonic triple, the following inequality holds.

Theorem 1.1 (Capacitary Riemannian Penrose Inequality)**.** *Let (M, g_0, u) be a sub-static harmonic triple with associated capacity C and suppose that ∂M is connected. Then*

$$C \geq \frac{1}{2}\left(\frac{|\partial M|}{|\mathbb{S}^{n-1}|}\right)^{\frac{n-2}{n-1}}. \tag{1.7}$$

Moreover, the equality in (1.7) *holds if and only if* (M, g_0) *is isometric to the Schwarzschild manifold with* $m_{\mathrm{ADM}} = C$.

Whereas the above inequality has been obtained as a consequence of the monotonicity of F_β, at every fixed $\beta \geq \frac{n-2}{n-1}$, we remark that one could possibly push the above described analysis one step forward, at the same time exploiting the full power of the optimality threshold. Indeed, we believe that considering p-harmonic functions defined at the exterior of a bounded domain Ω lying in M, it may be possible to derive, as done in [1] for the Euclidean case and in the simultaneous limit as $\beta \downarrow \frac{n-2}{n-1}$ and $p \downarrow 1$, a Minkowski-like inequality for $\partial\Omega$ (see [21] for a Minkowski-like inequality in the static, asymptotically flat case and [5] for the nonnegative Ricci case).

Concerning the treatment of general sub-static metrics and the derivation of related geometric inequalities, besides the already cited [7] we also would like to mention [19], where an integral formula is obtained and applied to prove Hentze-Karcher-type inequalities. For the case of asymptotically hyperbolic sub-static manifolds (specifically, for adS-Reissner-Nordström manifolds), we refer the interested reader to [13] and [27].

We remark that our results are not based on the Positive Mass Theorem. By contrast, we observe that using this celebrated result, more precisely a consequence of it contained in [17, Theorem 1.5], one can prove the following uniqueness statement. We refer the reader to Definition 1.1 for the notation and terminology.

Theorem 1.2 (Uniqueness Theorem for sub-static harmonic triples)**.** *Let (M, g_0, u) be a sub-static harmonic triple with associated capacity C. Suppose that there is a chart at infinity such that*

$$\mathrm{R}_{\widetilde{g}_0} = O(|x|^{-q}), \tag{1.8}$$

for some $q > n$. Then (M, g_0) is the Schwarzschild manifold with associated ADM mass given by C.

It remains an open question to see whether it is possible to remove the assumption on the decay of $\mathrm{R}_{\widetilde{g}_0}$ and get the same conclusion.

The paper is organised as follows. In Section 2, we recall and discuss some preparatory material, namely the asymptotic expansions of the metric and of the potential, and classical measure properties of the critical set of the potential, with a close look on related integral quantities. In Section 3, we prove the Monotonicity and Outer Rigidity Theorem 3.1, and the consequent Capacitary Riemannian Penrose Inequality contained in Theorem 1.1. To do this, we use from Section 4 some corresponding results obtained in a suitable conformally-related setting. The biggest technical effort is contained in such section. In the Appendix we also provide an alternative proof of the monotonicity of our monotone quantities. Finally, Section 5 is devoted to the proof of Theorem 1.2.

2. Preliminaries

Let (M, g_0, u) be a sub-static harmonic triple. We observe, as a first consequence of system (1.1), that the scalar curvature R_{g_0} is nonnegative. Since u satisfies the last three conditions of system (1.1), by the Maximum Principle we have

$$\mathring{M} = M \setminus \partial M = \{0 < u < 1\}.$$

Also, by the forth condition in (1.1), each level set of u is compact. Moreover, from the Hopf Lemma, it follows that $|\mathrm{D}_{g_0} u|_{g_0} > 0$ on ∂M. In particular, zero is a regular value of u. Furthermore, from the first two conditions in (1.1) restricted to ∂M it is easy to deduce that $\mathrm{D}^2_{g_0} u \equiv 0$ on ∂M. In turn, the function $|\mathrm{D}_{g_0} u|_{g_0}$ attains a positive constant value on each connected component of ∂M, and the boundary ∂M is a totally geodesic hypersurface in M.

We now deal with the asymptotic behaviour of the potential u at ∞. By Theorem 2.2 below, this is given by:

$$u = 1 - \frac{C}{|x|^{n-2}} + o_2(|x|^{2-n}) \quad \text{as} \quad |x| \to +\infty\,, \tag{2.1}$$

being

$$C = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int\limits_{\partial M} |\mathrm{D}_{g_0} u|_{g_0} d\sigma_{g_0}\,. \tag{2.2}$$

Here, σ_{g_0} is the canonical measure on the boundary ∂M seen as a Riemannian submanifold of (M, g_0), and we have used the standard notation o_2, which means that, in any chart at infinity ψ, denoted by $\widetilde{u}$ the function $u \circ \psi^{-1}$, the following conditions hold true.

$$\widetilde{u} = 1 - C|x|^{2-n} + o(|x|^{2-n})\,, \tag{2.3}$$
$$\partial_i \widetilde{u} = (n-2)\, C\, |x|^{-n}\, x^i + o(|x|^{1-n})\,, \tag{2.4}$$
$$\partial_i \partial_j \widetilde{u} = -(n-2)\, C\, |x|^{-n-2} (n\, x^i\, x^j - |x|^2 \delta_{ij}) + o(|x|^{-n})\,. \tag{2.5}$$

Let us remark that we can always suppose, without loss of generality, that the considered chart at infinity admits a diffeomorphic extension to the closure of the coordinate domain. We will make this implicit assumption throughout the paper, so that ∂K (see Definition 1.1) is a connected hypersurface of M and the quantities related to the metric can be pushed–forward in $\mathbb{R}^n$ outside an open ball and be smooth here. We also observe that formula (2.2) is nothing but an equivalent characterisation of the capacity of ∂M.

2.1. Asymptotic expansions

Let (N, h) be a smooth, connected, noncompact, complete, asymptotically flat, n-dimensional Riemannian manifold, with $n \geq 3$, with one end and with nonempty smooth compact boundary ∂N. We adopt the following notation.

- B and B_R a generic open ball and the open ball of radius $R > 0$ centred in the origin of $(\mathbb{R}^n, d_e)$, respectively;
- $|\cdot|$ the euclidean norm of $\mathbb{R}^n$;

- $|\mathbb{S}^{n-1}|$ the hypersurface area of the unit sphere inside $\mathbb{R}^n$ with the canonical metric;
- D_e and Δ_e the Levi–Civita connection and the Laplace operator of $(\mathbb{R}^n, g_{\mathbb{R}^n})$, respectively;
- D_h and Δ_h the Levi–Civita connection and the Laplace operator of (N, h), respectively;
- σ_e the canonical measure on a Riemannian submanifold of $(\mathbb{R}^n, g_{\mathbb{R}^n})$;
- σ_h the canonical measure on a Riemannian submanifold of (N, h);
- $|\cdot|_e$ the norm induced by $g_{\mathbb{R}^n}$ on the tangent spaces to the manifold $\mathbb{R}^n$;
- $|\cdot|_h$ the norm induced by h on the tangent spaces to the manifold N.
- If ψ is a chart at infinity of (N, h) according to Definition 1.1, we denote by $\widetilde{h}$ the push–forward metric $\psi_* h$ of h by ψ, having coordinate expression $\widetilde{h}_{ij}(x)\, dx^i \otimes dx^j$. In this context, $\mathrm{D}_{\widetilde{h}}$ and $\Delta_{\widetilde{h}}$ denote the Levi–Civita connection and the Laplace operator of $\widetilde{h}$, respectively, while $\sigma_{\widetilde{h}}$ is the canonical measure on a Riemannian submanifold of $(\mathbb{R}^n \setminus B, \widetilde{h})$ and $|\cdot|_{\widetilde{h}}$ is the norm induced by $\widetilde{h}$ on the tangent spaces. Moreover, $\mathrm{Ric}_{\widetilde{h}}$ and $\mathrm{R}_{\widetilde{h}}$ are the Ricci tensor and the scalar curvature of $\widetilde{h}$, respectively.

Proposition 2.1. *Let ψ be a chart at infinity of N (according to Definition 1.1). The decays*

$$\widetilde{h}^{ij} - \delta^{ij} = O_2(|x|^{-p}), \tag{2.6}$$
$$(\mathrm{Rm}_{\widetilde{h}})^l_{ijk} = O(|x|^{-(p+2)}), \tag{2.7}$$
$$(\mathrm{Ric}_{\widetilde{h}})_{ij} = O(|x|^{-(p+2)}), \tag{2.8}$$
$$\mathrm{R}_{\widetilde{h}} = O(|x|^{-(p+2)}), \tag{2.9}$$

hold true for some $p > \frac{n-2}{2}$. Moreover,

$$|\nu^i_{\widetilde{h}} - \nu^i_e| = O(|x|^{-p}) , \tag{2.10}$$
$$d\sigma_{\widetilde{h}} = (1 + O(|x|^{-p})\, d\sigma_e , \tag{2.11}$$

where ν_e is the ∞–pointing unit normal with respect to the Euclidean metric and σ_e the associated canonical measure on ∂B_R, while $\nu_{\widetilde{h}}$ is the ∞–pointing unit normal with respect to $\widetilde{h}$ and $\sigma_{\widetilde{h}}$ the associated canonical measure on ∂B_R.

Proof. From $\widetilde{h}^{ik}\widetilde{h}_{kj} = \delta^i_j$ it is easy to get

$$\partial_i \widetilde{h}^{kl} = -\widetilde{h}^{kr}\,\widetilde{h}^{ls}\,\partial_i \widetilde{h}_{rs}$$
$$\partial_i\partial_j \widetilde{h}^{kl} = \widetilde{h}^{ka}\,\widetilde{h}^{rb}\,\widetilde{h}^{ls}\,(\partial_i \widetilde{h}_{ab})\,(\partial_j \widetilde{h}_{rs}) + \widetilde{h}^{la}\,\widetilde{h}^{sb}\,\widetilde{h}^{kr}\,(\partial_i \widetilde{h}_{ab})\,(\partial_j \widetilde{h}_{rs}) - \widetilde{h}^{kr}\,\widetilde{h}^{ls}\,\partial_i\partial_j \widetilde{h}_{rs} .$$

These formulae coupled with (1.2), (1.3) and (1.4) give (2.6). Decay (2.7) is another direct consequence of Definition 1.1, keeping in mind that

$$(\mathrm{Rm}_{\widetilde{h}})^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js} ,$$
$$\Gamma^k_{ij} = \frac{\widetilde{h}^{kl}}{2}[\partial_i \widetilde{h}_{lj} + \partial_j \widetilde{h}_{li} - \partial_l \widetilde{h}_{ij}] . \tag{2.12}$$

Decays (2.8)–(2.9) are obtained by contractions of the Riemannian tensor. Now, observe that

$$\nu_{\widetilde{h}} = \frac{\widetilde{h}^{ij}\, x^i\, \frac{\partial}{\partial x^j}}{\sqrt{\widetilde{h}^{lk}\, x^l\, x^k}} ,$$

and that

$$|\nu_e^i - \nu_{\widetilde{h}}^i| = \left| \frac{x^i}{|x|} - \frac{\widetilde{h}^{ij}\, x^j}{\sqrt{\widetilde{h}^{lk} x^l x^k}} \right| = \left| (\delta^{ij} - \widetilde{h}^{ij})\, \frac{x^j}{\sqrt{\widetilde{h}^{lk} x^l x^k}} + x^i \left(\frac{1}{|x|} - \frac{1}{\sqrt{\widetilde{h}^{lk} x^l x^k}} \right) \right|$$

$$\leq C \sum_j |\delta^{ij} - \widetilde{h}^{ij}| + \frac{\left| (\widetilde{h}^{lk} - \delta^{lk})\, x^l x^k \right|}{\sqrt{\widetilde{h}^{lk} x^l x^k} \left(\sqrt{\widetilde{h}^{lk} x^l x^k} + |x| \right)} . \tag{2.13}$$

Observe also that

$$\widetilde{h}_{ij}(x) v^i v^j \geq C^{-1} v^i v^j \delta_{ij} , \tag{2.14}$$

for some $C > 0$, for any $x \in \mathbb{R}^n \setminus B$. Since trivially $|x^k x^l| \leq |x|^2$, from (2.13) and (2.14), coupled with (2.6), we get decay (2.10). Concerning decay (2.11), recall first that, using a coordinate chart $(y^1, \ldots, y^{n-1})$ on ∂B_R, we have that $d\sigma_{\widetilde{h}} = \sqrt{\det \widetilde{h}^{\partial B_R}}\, dy^1 \ldots dy^{n-1}$ with $\widetilde{h}^{\partial B_R} = \widetilde{h}^{\partial B_R}_{\alpha\beta}\, dy^\alpha \otimes dy^\beta$, where $\widetilde{h}^{\partial B_R}_{\alpha\beta} = \widetilde{h}(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta})$. Now, using the specific local parametrization $x = x(y^1, \ldots, y^{n-1})$ of ∂B_R, given by the inverse of stereographic projection from its north pole with the diffeomorphism $p \in \mathbb{S}^{n-1} \to Rp \in \partial B_R$, we have that

$$\begin{aligned}
\widetilde{h}(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial y^\beta})(x(y)) &= \widetilde{h}_{ij}(x(y))\, \frac{\partial x^i}{\partial y^\alpha}(y)\, \frac{\partial x^j}{\partial y^\beta}(y) \\
&= (\widetilde{h}_{ij}(x(y)) \pm \delta_{ij})\, \frac{\partial x^i}{\partial y^\alpha}(y)\, \frac{\partial x^j}{\partial y^\beta}(y) = \frac{4R^2}{(|y|^2+1)^2} \left(\delta_{\alpha\beta} + O(R^{-p}) \right),
\end{aligned}$$

because

$$\left| (\widetilde{h}_{ij}(x(y)) - \delta_{ij})\, \frac{\partial x^i}{\partial y^\alpha}(y)\, \frac{\partial x^j}{\partial y^\beta}(y) \right| \leq \sum_{i,j} \left| \widetilde{h}_{ij}(x(y)) - \delta_{ij} \right| \left| \frac{\partial}{\partial y^\alpha} \right|_e \left| \frac{\partial}{\partial y^\beta} \right|_e = \frac{4R^2}{(|y|^2+1)^2}\, O(R^{-p}) .$$

Hence, on ∂B_R,

$$d\sigma_{\widetilde{h}} = \left(\frac{2R}{|y|^2+1} \right)^{n-1} \sqrt{\det\left(\delta_{\alpha\beta} + O(R^{-p})\right)}\, dy^1 \ldots dy^{n-1} = (1 + O(R^{-p})) d\sigma_e ,$$

where in the last identity we have used the Leibniz formula for the determinant and Taylor–expanded the square root. □

The following result is well–known. For completeness, we provide the statement, along with its proof, which is an extension of [20, Lemma A.2.] to every $n \geq 3$.

Theorem 2.2. *Let (N, h) be a smooth, connected, noncompact, complete, asymptotically flat, n-dimensional Riemannian manifold, with $n \geq 3$, with one end, and with nonempty smooth compact boundary ∂N. If $v \in C^\infty(N)$ is the solution to*

$$\begin{cases} \Delta_h v = 0 & \text{in } N , \\ v = 1 & \text{on } \partial N , \\ v \to 0 & \text{at } \infty . \end{cases} \tag{2.15}$$

then

$$v = \frac{C}{|x|^{n-2}} + o_2(|x|^{2-n}) \ \ \textit{as} \ \ |x| \to \infty , \qquad \textit{with} \qquad C = \frac{1}{(n-2)|\mathbb{S}^{n-1}|} \int_{\partial N} |\, \mathrm{D}_h v\, |_h\, d\sigma_h . \tag{2.16}$$

We remark that the asymptotic behaviour of the potential u at ∞, given by formula (2.1), is a simply consequence of the above theorem observing that $u = 1 - v$ when $(N, h) = (M, g_0)$.

Proof. Step 1*: Construction of a barrier function.* Let ψ be a chart at infinity for N. From now on by C we will denote some positive constant, which may change from line to line. By Definition 1.1, there exist $p > (n-2)/2$ and $R_1 \geq 1$ such that

$$\mathbb{R}^n \setminus B_{R_1} \subseteq \mathbb{R}^n \setminus \overline{B}$$

$$|\widetilde{h}_{ij} - \delta_{ij}| \leq C|x|^{-p} \qquad |\partial_k \widetilde{h}_{ij}| \leq C|x|^{-(p+1)} \qquad |\partial_k\partial_l \widetilde{h}_{ij}| \leq C|x|^{-(p+2)} \tag{2.17}$$

for every $x \in \mathbb{R}^n \setminus B_{R_1}$. By (2.6), the same conditions as in (2.17) are satisfied by $\widetilde{h}^{ij}(x)$ for all $x \in \mathbb{R}^n \setminus B_{R_1}$. Then, for every $f \in C^\infty(\mathbb{R}^n \setminus B_{R_1})$, writing

$$\Delta_{\widetilde{h}} f = \delta^{ij}\partial_i\partial_j f + \sigma^{ij}\partial_i\partial_j f + b^j\partial_j f\ , \tag{2.18}$$

where

$$\sigma^{ij} := \widetilde{h}^{ij} - \delta^{ij}\,, \qquad b^j := -\widetilde{h}^{kl}\Gamma^j_{kl} = \frac{1}{2}\widetilde{h}^{kl}\,\widetilde{h}^{ij}\,\partial_i\widetilde{h}_{lk} - \widetilde{h}^{ki}\,\widetilde{h}^{lj}\,\partial_i\widetilde{h}_{kl}\,,$$

we have that

$$|\sigma^{ij}| \leq C|x|^{-p}\,, \qquad |b^j| \leq C|x|^{-(p+1)}\,, \qquad |\partial_l b^j| \leq C|x|^{-(p+2)} \tag{2.19}$$

in $\mathbb{R}^n \setminus B_{R_1}$. For a fixed $0 < \varepsilon < p$ and for $a > 0$ to be chosen later, consider the function

$$\phi_a = a\left(\frac{1}{|x|^{n-2}} - \frac{1}{|x|^{n-2+\varepsilon}}\right).$$

By direct computation one can check that

$$\partial_j\phi_a = -a\left(\frac{n-2}{|x|^n} - \frac{n-2+\varepsilon}{|x|^{n+\varepsilon}}\right)x^j$$

$$\partial_i\partial_j\phi_a = a\left[\frac{n\,(n-2)}{|x|^{n+2}} - \frac{(n+\varepsilon)(n-2+\varepsilon)}{|x|^{n+2+\varepsilon}}\right]x^i\,x^j - a\left[\frac{n-2}{|x|^n} - \frac{n-2+\varepsilon}{|x|^{n+\varepsilon}}\right]\delta_{ij}\,,$$

and in turn that

$$|\partial_i\phi_a| \leq a\,C|x|^{1-n} \qquad \text{and} \qquad |\partial_i\partial_j\phi_a| \leq a\,C|x|^{-n}\,.$$

Therefore, by (2.18) and (2.19), we obtain that

$$\Delta_{\widetilde{h}}\phi_a = a\Big[-(n-2+\varepsilon)\varepsilon|x|^{-(n+\varepsilon)} + O(|x|^{-(n+p)})\Big]\,,$$

and hence there exists $R_2 > R_1$ independent of a such that $\Delta_{\widetilde{h}}\phi_a < 0$ in $\mathbb{R}^n \setminus B_{R_2}$, for every $a > 0$. We now choose $a > 0$ so that $\phi_a = 1$ on ∂B_{R_2}, that is $a = \left[\frac{1}{R_2^{n-2}} - \frac{1}{R_2^{n-2+\varepsilon}}\right]^{-1}$. Since ϕ_a is $\widetilde{h}$–superharmonic in $\mathbb{R}^n \setminus B_{R_2}$ and since $\widetilde{v} := v \circ \psi^{-1} < 1$ on ∂B_{R_2}, by the Maximum Principle

$$\widetilde{v} \leq \phi_a \quad \text{in} \quad \mathbb{R}^n \setminus B_{R_2}\,. \tag{2.20}$$

Step 2*: Asymptotic expansion of* v*.* Note that from (2.20) one gets in particular that $\widetilde{v} \leq C|x|^{2-n}$. We now apply Shauder's Interior estimates ([12, Lemma 6.20]) to $\Delta_{\widetilde{h}}\widetilde{v} = 0$ in $\mathbb{R}^n \setminus \overline{B}_{R_2}$, where the operator

$\Delta_{\widetilde{h}}$ is defined as in (2.18) and its coefficients satisfy the estimates in (2.19). Recalling that the Hölder norms are weighted by the (Euclidean) distance $d_e(\,\cdot\,, \partial B_{R_2})$ from ∂B_{R_2} and since $d_e(x, \partial B_{R_2}) \simeq |x|$ when $|x| >> 1$, from such estimates we get

$$|\partial_i \widetilde{v}(x)| \leq C|x|^{1-n} \qquad |\partial_i \partial_j \widetilde{v}(x)| \leq C|x|^{-n} \tag{2.21}$$

in $\mathbb{R}^n \setminus B_{R_2}$ (up to a bigger R_2). Combining (2.19) and (2.21), the equation $\Delta_{\widetilde{h}} \widetilde{v} = 0$ can be equivalent written as $\Delta_e \widetilde{v} = f$ where

$$|f(x)| \leq C|x|^{-(n+p)}\,. \tag{2.22}$$

We consider a smooth extension of $\widetilde{v}$ on $\mathbb{R}^n$, still denoted by $\widetilde{v}$, which is zero in a ball centred in the origin, and the smooth extension of f given by $\Delta_e \widetilde{v}$, still denoted by f. By a classical representation formula and due to (2.22), the function

$$w(x) = -\frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}\, dy\,,$$

is well–defined and fulfils $\Delta_e w = f$ on $\mathbb{R}^n$. Now, one can rewrite w in $\mathbb{R}^n \setminus \{O\}$ as

$$\begin{aligned} w(x) = & -\frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} \int_{\mathbb{R}^n} f(y)\, dy + \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} \int_{\mathbb{R}^n \setminus B_{\frac{|x|}{2}}(O)} f(y)\, dy \\ & - \frac{1}{n(n-2)\omega_n} \int_{B_{\frac{|x|}{2}}(x)} \frac{f(y)}{|x-y|^{n-2}}\, dy - \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n \setminus \left(B_{\frac{|x|}{2}}(x) \cup B_{\frac{|x|}{2}}(O)\right)} \frac{f(y)}{|x-y|^{n-2}}\, dy \\ & - \frac{1}{n(n-2)\omega_n} \int_{B_{\frac{|x|}{2}}(O)} \left[\frac{1}{|x-y|^{n-2}} - \frac{1}{|x|^{n-2}} \right] f(y)\, dy\,, \end{aligned}$$

and show that each summand can be bounded by $C|x|^{-(n-2+\gamma)}$, where $\gamma = \min\{1, p\}$ if $p \neq 1$ and $\gamma \in (1/2, 1)$ if $p = 1$, except the first one. Therefore, we have that

$$w(x) = -\frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} \int_{\mathbb{R}^n} f(y)\, dy + z(x)\,, \qquad |z(x)| \leq C|x|^{-(n-2+\gamma)}\,,$$

in $\mathbb{R}^n \setminus \{O\}$. Since the function $\widetilde{v} - w$ is harmonic and bounded on $\mathbb{R}^n$, then it is constant and this constant is zero, using the fact that $\widetilde{v} - w \to 0$ for $|x| \to \infty$. Hence

$$\widetilde{v} = \frac{C}{|x|^{n-2}} + z(x) \tag{2.23}$$

in $\mathbb{R}^n \setminus B_{R_2}$. We observe that

$$\Delta_{\widetilde{h}} z = \Delta_{\widetilde{h}} \left(\widetilde{v} - \frac{C}{|x|^{n-2}} \right) = -C\Big(\sigma^{ij} \partial_i \partial_j \frac{1}{|x|^{n-2}} + b^k \partial_k \frac{1}{|x|^{n-2}} \Big) =: -C\hat{z}\,,$$

and that

$$|\hat{z}(x)| \leq C|x|^{-(n+\gamma)}, \qquad |\partial_k \hat{z}(x)| \leq C|x|^{-(n+\gamma+1)}.$$

Therefore, applying Shauder's Interior estimates to $\Delta_{\widetilde{h}} z = -C\hat{z}$ in $\mathbb{R}^n \setminus \overline{B}_{R_2}$, we get

$$|z(x)| \leq C|x|^{-(n-2+\gamma)} \qquad |\partial_i z(x)| \leq C|x|^{-(n-1+\gamma)} \qquad |\partial_i\partial_j z(x)| \leq C|x|^{-(n+\gamma)} \tag{2.24}$$

in $\mathbb{R}^n \setminus B_{R_2}$ (up to a bigger R_2). From (2.23) and (2.24) we obtain in particular (2.16).

Step 3: Characterization of C. First of all we remark that $0 < v < 1$ on $\mathring{N}$, $v : N \to (0, 1]$ is proper, and, from the Hopf Lemma, $|\mathrm{D}_h v|_h > 0$ on ∂N. In particular, 1 is a regular value of v. Let K be the compact set on the complement of which the chart ψ is defined. For every $R > R_2$, applying the Divergence Theorem to the function v on $K \cup \{|\psi| < R\}$ we obtain that

$$0 = \int_{K\cup\{|\psi|<R\}} \Delta_h v \, d\mu_h = \int_{\partial N} h(\mathrm{D}_h v, \nu_h) \, d\sigma_h + \int_{\{|\psi|=R\}} h(\mathrm{D}_h v, \nu_h) \, d\sigma_h ,$$

where ν_h is the outward unit normal vector field with respect to h along ∂N and $\{|\psi| = R\}$. Then, it follows that

$$\int_{\partial N} |\mathrm{D}_h v|_h \, d\sigma_h = \int_{\partial N} h(\mathrm{D}_h v, \nu_h) \, d\sigma_h = -\int_{\{|\psi|=R\}} h(\mathrm{D}_h v, \nu_h) \, d\sigma_h = -\int_{\partial B_R} \widetilde{h}\Big(\mathrm{D}_{\widetilde{h}} \widetilde{v}, \nu_{\widetilde{h}}\Big) \, d\sigma_{\widetilde{h}} ,$$

where $\widetilde{v} = v \circ \psi^{-1}$. Now, thanks to (1.2), (2.6), (2.10) and (2.11), which are true for γ too, and also by identity (2.23) and the second in (2.24), and keeping in mind that $|\partial_i \widetilde{v}| \leq C|x|^{1-n}$, we have that

$$\begin{aligned}
\int_{\partial B_R} \widetilde{h}\Big(\mathrm{D}_{\widetilde{h}} \widetilde{v}, \nu_{\widetilde{h}}\Big) \, d\sigma_{\widetilde{h}} &= \int_{\partial B_R} g_{\mathbb{R}^n}\Big(\mathrm{D}_e \widetilde{v}, \nu_e\Big) \, d\sigma_e + O(R^{-\gamma}) \\
&= \int_{\partial B_R} g_{\mathbb{R}^n}\left(\mathrm{D}_e\Big(C|x|^{2-n} + z\Big), \frac{x}{|x|}\right) d\sigma_e + O(R^{-\gamma}) \\
&= \int_{\partial B_R} g_{\mathbb{R}^n}\left(\mathrm{D}_e\Big(C|x|^{2-n}\Big), \frac{x}{|x|}\right) d\sigma_e + O(R^{-\gamma}) \\
&= -C(n-2)|\mathbb{S}^{n-1}| + O(R^{-\gamma}).
\end{aligned}$$

Hence

$$\int_{\partial N} |\mathrm{D}_h v|_h \, d\sigma_h = -\lim_{R\to\infty} \int_{\partial B_R} \widetilde{h}\Big(\mathrm{D}_{\widetilde{h}} \widetilde{v}, \nu_{\widetilde{h}}\Big) \, d\sigma_{\widetilde{h}} = C(n-2)|\mathbb{S}^{n-1}| .$$

□

2.2. *Measure of and integration on the level sets of the potential*

Let (N, h) and $\iota : S \hookrightarrow N$ be respectively a m–dimensional Riemannian manifold and a s–dimensional Riemannian submanifold of N. Let k be a positive real number. We set

$\mathcal{B}(S)$ the smallest σ–algebra containing all open sets of S;
$(S, \Lambda(S), \mu_{\iota^* h})$ the canonical space of measure on the Riemannian manifold $(S, \iota^* h)$ (see [14, Section 3.4]);
$\mathcal{H}^k_S$ the k-dimensional Hausdorff measure on (S, d_S), being d_S the distance function of S;
$\mathcal{H}^k_{S;N}$ the k-dimensional Hausdorff measure on $(S, d_{S;N})$ where $d_{S;N}$ is the distance function of N restricted to $S \times S$;
$\mathcal{H}^k_N \llcorner S$ the k-dimensional Hausdorff measure of N restricted to S.

By definition of the Hausdorff measure and by [26, Proposition 12.7], $\mathcal{H}^k_{S;N}$, $\mathcal{H}^k_N \llcorner S$ and $\mathcal{H}^k_S$ coincide on $\mathcal{B}(S)$, and by [26, Proposition 12.6] and by [25, Proposition 2.17], $\mathcal{H}^s_S$ and $\mu_{\iota^* h}$ coincide on $\Lambda(S)$. The same results still hold when N is a manifold with boundary.

For the ease of the reader, we collect in the next theorem some results about the measure of the level sets of the potential u and

$$\mathrm{Crit}(u) := \{|\mathrm{D}_{g_0} u|_{g_0} = 0\},$$

which are well–known in the Euclidean setting (see, e.g., [15, 16]).

Theorem 2.3. *Let (M, g_0, u) be a sub-static harmonic triple. Then the following statements hold true.*

- (i) *For every $t \in [0, 1)$, the level set $\{u = t\}$ is compact and has finite $(n-1)$–Hausdorff measure in M;*
- (ii) $\mathrm{Crit}(u)$ *is a compact subset of M and its Hausdorff dimension in M is less than or equal to $(n-2)$;*
- (iii) *The set of the critical values of u has zero Lebesgue measure, and for every $t \in [0, 1)$ regular value of u there exists $\epsilon_t > 0$ such that $(t - \epsilon_t, t + \epsilon_t) \cap [0, 1)$ does not contain any critical value of u.*

Proof. Each level set of u is compact, due to the forth condition in (1.1), while the compactness of $\mathrm{Crit}(u)$ follows by (2.4). Now, consider the nontrivial case where $\mathrm{Crit}(u) \neq \emptyset$ and let p be a point of a critical level set $\{u = t\}$. Take a chart (U_p, ψ_p) centred at the point p with $\psi_p(U_p) = B_1$. Setting $\widetilde{g}_0 = (\psi_p)_* g_0 = \widetilde{g}_{0;ij} dx^i \otimes dx^j$, there exists $C > 0$ such that

$$C^{-1} v^i v^j \delta_{ij} \leq \widetilde{g}_{0;ij}(x) v^i v^j \leq C v^i v^j \delta_{ij}, \tag{2.25}$$

for each $(v^1, \ldots, v^n) \in \mathbb{R}^n$ and for each $x \in \overline{B_{\frac{1}{2}}}$. The same condition is satisfied by the coefficients $\widetilde{g}_0^{ij}$. In particular, setting $\widetilde{u} = u \circ \psi_p^{-1}$, we have that

$$\{|\mathrm{D}_{\widetilde{g}_0} \widetilde{u}|_{\widetilde{g}_0} = 0\} \cap B_{\frac{1}{2}} = \{|\mathrm{D}_e \widetilde{u}|_e = 0\} \cap B_{\frac{1}{2}}.$$

We observe that $\Delta_{\widetilde{g}_0} v = a_{ij} \partial_i \partial_j v + b_i \partial_i v = 0$ is an elliptic partial differential equation with coefficient C^∞ in $\overline{B_{\frac{1}{2}}}$. We recall that if v is a C^∞–solution of the above equation and if v vanishes to infinite order at a point $x_0 \in B_{\frac{1}{2}}$, i.e., for every $k > 0$

$$\lim_{r \to 0} \frac{1}{r^k} \int\limits_{B_r(x_0)} v^2 \, dx = 0,$$

then v is identically zero in $B_{\frac{1}{2}}$ (see [11, Theorem 1.2]). Applying this fact to $\widetilde{u}-t$, one can argue that $\widetilde{u}-t$ has finite order of vanishing at O. Then, by using [16, Theorem 1.7], there exists $0<\rho<\frac{1}{2}$ such that

$$\mathcal{H}^{n-1}(B_\rho \cap \{\widetilde{u}=t\}) < \infty\,.$$

Since $\widetilde{u}$ is nonconstant in $B_{\frac{1}{2}}$ and by the structure and regularity of $\Delta_{\widetilde{g}_0}$, [15, Theorem 1.1] yields

$$\mathcal{H}^{n-2}(B_\rho \cap \{|\mathrm{D}_e\widetilde{u}|_e = 0\}) < \infty\,.$$

Hence, since the restriction of ψ_p to $\psi_p^{-1}(B_{1/2})$ is bilipschitz due to (2.25) and since the measures $\mathcal{H}^k_{\psi_p^{-1}(B_{1/2})}$ and $\mathcal{H}^k_M \llcorner \psi_p^{-1}(B_{1/2})$ coincide on borel sets, statements (i) and (ii) are true locally. In turn, by compactness of $\{u=t\}$ and $\mathrm{Crit}(u)$, they are true globally. To prove (iii), observe that by Sard's Theorem, the set of the critical values of u has zero Lebesgue measure. Now, suppose by contradiction that there exists $\bar{t}\in[0,1)$ regular value such that, for all $m\geq\overline{m}$ with $\frac{1}{\overline{m}}<1-\bar{t}$, the interval $(\bar{t}-\frac{1}{m},\bar{t}+\frac{1}{m})\cap[0,1)$ contains critical values. Hence there is a sequence $\{t_m\}_{m\geq\overline{m}}$ of critical values such that $t_m\to\bar{t}$. In particular, there exists a sequence $\{p_m\}_{m\geq\overline{m}}$ of critical points contained in the set $\{0\leq u\leq\bar{t}+\frac{1}{m}\}$ and such that $u(p_m)=t_m$. Then, by compactness and up to a subsequence, $p_m\to p$. In turn, $0=|\mathrm{D}_{g_0}u|_{g_0}(p_m)\to|\mathrm{D}_{g_0}u|_{g_0}(p)$ and $t_m=u(p_m)\to u(p)$. Hence $|\mathrm{D}_{g_0}u|_{g_0}(p)=0$ and $u(p)=\bar{t}$, which is absurd. This concludes the proof of (iii). □

Remark 2.1. *It is useful to observe that:*

(i) for every $t\in(0,1)$*, the set* $\{u\geq t\}$ *is connected;*
(ii) for every $t\approx 1$*, the level set* $\{u=t\}$ *is regular and diffeomorphic to* $\mathbb{S}^{n-1}$*;*
(iii) For every $t\in(0,1)$*,* $\{u\geq t\}=\overline{\{u>t\}}$ *and* $\{0\leq u\leq t\}=\overline{\{0<u<t\}}$.

We check (ii) first. We start by observing that due to (2.1) $|\mathrm{D}_{g_0}u|_{g_0}\neq 0$ in $\{u\geq t_0\}$, for some $0<t_0<1$. This fact establishes a diffemorphism between $\{u\geq t_0\}$ and $\{u=t_0\}\times[t_0,1)$ and tells us at the same time that the level sets $\{u=t\}$ are pairwise diffeomorphic, for every $t\geq t_0$. It is thus sufficient to show that $\{u=t_0\}$ is connected. Suppose by contradiction that this is not the case. Without loss of generality we can assume that $\{u=t_0\}$ can be decomposed into the disjoint union of two connected sets C_1 and C_2, indeed the same argument works *a fortiori* if the connected components are more than two. Now, note that by definition of asymptotically flat manifold, there exists a compact set $K\subset M$ such that $M\setminus\mathring{K}$ is diffeomorphic to $\mathbb{R}^n\setminus\mathring{B}$ by a chart at infinity ψ, where B is a suitable ball, and we can suppose, up to a bigger t_0, that $\{u\geq t_0\}\subseteq M\setminus\mathring{K}$. Now, in view of the asymptotic expansion of u, there exist two positive constants $A<B$ such that

$$\frac{A}{|x|^{n-2}}\;\leq\;1-u\;\leq\;\frac{B}{|x|^{n-2}}.$$

In particular, setting $R_0=[B/(1-t_0)]^{1/(n-2)}$, we have that

$$\{|x|>R_0\}\;\subseteq\;\{u\geq t_0\}\;\simeq\;\{C_1\times[t_0,1)\}\sqcup\{C_2\times[t_0,1)\}.$$

At the same time, we have that $\{|x|>R_0\}$ is connected and each $C_i\times[t_0,1)$ is a closed set of M, so that indeed $\{|x|>R_0\}\subseteq C_i\times[t_0,1)$, for some $i\in\{1,2\}$.Therefore, we have that

$$\{C_1\times[t_0,1)\}\sqcup\{C_2\times[t_0,1)\}\;=\;\{u\geq t_0\}\;\subseteq\;M\setminus\mathring{K}$$

$$
\begin{aligned}
&= \big[(M \setminus \mathring{K}) \cap \{|x| \le R_0\}\big] \sqcup \big[(M \setminus \mathring{K}) \cap \{|x| > R_0\}\big] \\
&\subseteq \big[(M \setminus \mathring{K}) \cap \{|x| \le R_0\}\big] \sqcup \{C_i \times [t_0, 1)\},
\end{aligned}
$$

which gives the contradiction that the noncompact set $\{C_j \times [t_0, 1)\}$, where $j \in \{1, 2\} \setminus \{i\}$, is contained into the compact one $(M \setminus \mathring{K}) \cap \{|x| \le R_0\}$. Therefore, $\{u = t_0\}$ is connected. Now, setting $\widetilde{u} := u \circ \psi^{-1}$, we have that, up to a bigger t_0 and due to (2.5), the set $\{\widetilde{u} = t_0\}$ is a compact and connected hypersurface of $\mathbb{R}^n$ having strictly positive sectional curvature, as a Riemannian submanifold of $(\mathbb{R}^n, g_{\mathbb{R}^n})$. Hence, $\{\widetilde{u} = t_0\}$ is diffeomorphic to $\mathbb{S}^{n-1}$ by the Gauss map (see [10, Section 5.B] for more details). Statement (ii) thus follows, being $\{u = t_0\}$ and $\{\widetilde{u} = t_0\}$ diffeomorphic.

To see (i), observe first that if t is a regular value of u, then $E_t := \{u \ge t\}$ is a n–dimensional submanifold with boundary $\{u = t\}$. By Theorem 2.3 and by the Maximum Principle, every connected component C of E_t is unbounded. Since $u \to 1$ at ∞, we have that $u(C) = [t, 1)$, and hence $C \cap \{u = t_0\} \neq \emptyset$, for every $t_0 \in (t, 1)$. Then, E_t is connected by (ii). If t is a critical value of u, we let $\bar{t} > t$ be a regular value of u such that $\{u = \bar{t}\}$ is connected and let $\{t_m\}$ be a nondecreasing sequence of regular value of u such that $t_m < t$ and $t_m \to t$. Hence, $\{\,\{t_m \le u \le \bar{t}\}\,\}_{m\in\mathbb{N}}$ is a nonincreasing family of connected and compact sets in M, which is Hausdorff, and in turn the intersection $\{t \le u \le \bar{t}\}$ is still connected. In particular, we deduce that $E_t = \{t \le u \le \bar{t}\} \cup \{u \ge \bar{t}\}$ is connected.

To check (iii), note first that for every $t \in (0, 1)$ regular value of u, the equalities are always true. If $t \in (0, 1)$ is a critical value of u, by Theorem 2.3 and by the Maximum Principle the interior of $\{0 \le u \le t\}$ and the interior of $\{u \ge t\}$ are both disjoint from $\{u = t\}$, so that (iii) is still true.

Let (M, g_0, u) be a sub-static harmonic triple, and let $t \in [0, 1)$ be a real number. We consider the spaces of measure

$$
\Big(\{u = t\}, \mathcal{B}(\{u = t\}), \mathcal{H}^{n-1}_M \llcorner \{u = t\}\Big) \text{ and } \Big(S := \{u = t\} \setminus \mathrm{Crit}(u), \Lambda(S), \sigma_{g_0} := \mu_{\iota^* g_0}\Big).
$$

Let $f : S \to \mathbb{R}$ be a continuous and consider the zero–extension $\mathring{f}$ of f defined on $\{u = t\}$ as

$$
\mathring{f}(p) = \begin{cases} f(p) & \text{if } p \in S, \\ 0 & \text{if } p \in \{u = t\} \cap \mathrm{Crit}(u). \end{cases}
$$

By Theorem 2.3 and by definition of the Lebesgue integral, we have that $\mathring{f} \in L^1(\mathcal{H}^{n-1}_M \llcorner \{u = t\})$ iff $f \in L^1(\sigma_{g_0})$ and

$$
\int_{\{u=t\}} \mathring{f}\, d(\mathcal{H}^{n-1}_M \llcorner \{u = t\}) = \int_S f\, d\sigma_{g_0}. \tag{2.26}
$$

Similarly, if $f : \{u = t\} \to \mathbb{R}$ is a continuous function, then $f \in L^1(\mathcal{H}^{n-1}_M \llcorner \{u = t\})$ iff $f|_S \in L^1(\sigma_{g_0})$ and

$$
\int_{\{u=t\}} f\, d(\mathcal{H}^{n-1}_M \llcorner \{u = t\}) = \int_S f|_S\, d\sigma_{g_0}. \tag{2.27}
$$

In the rest of this paper, we will confuse the integrals of cases (2.26) *and* (2.27)*, denoting both by* $\int\limits_{\{u=t\}} f d\sigma_{g_0}$.

3. Monotonicity and Outer Rigidity Theorem

In this section, we state and prove our Monotonicity and Outer Rigidity Theorem, which is then used to prove the Capacitary Riemannian Penrose Inequality (1.7). *From now on and unless otherwise stated, (M, g_0, u) will always be a sub-static harmonic triple, and, when referring to such triple, the subscript g_0 will be dropped. The only exception is $|\mathbb{S}^{n-1}|$, which always stands for the Euclidean volume of $\mathbb{S}^{n-1}$.*

Theorem 3.1 (Monotonicity and Outer Rigidity Theorem). *Let (M, g_0, u) be a sub-static harmonic triple, and let $F_\beta : \tau \in [1, +\infty) \to [0, +\infty)$ be the function defined by*

$$F_\beta(\tau) := (1+\tau)^{\beta \frac{n-1}{n-2}} \int\limits_{\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\mathrm{D}u|^{\beta+1}\, d\sigma\,, \tag{3.1}$$

for every $\beta \geq 0$. Then, the following properties hold true.

(*i*) *Differentiability, Monotonicity and Outer Rigidity: for every $\beta > \frac{n-2}{n-1}$, the function F_β is continuously differentiable with nonpositive derivative in $(1, +\infty)$. Moreover, if there exists $\tau_0 \in (1, +\infty)$ such that $F'_\beta(\tau_0) = 0$ for some $\beta > \frac{n-2}{n-1}$, then, setting $t_0 = \sqrt{\frac{\tau_0-1}{\tau_0+1}}$, the Riemannian submanifold $\{u \geq t_0\}$ is isometric to*

$$\left([r_0, +\infty) \times \mathbb{S}^{n-1}\,, \frac{dr \otimes dr}{1 - 2Cr^{2-n}} + r^2 g_{\mathbb{S}^{n-1}}\right), \qquad r_0 = [C(1+\tau_0)]^{\frac{1}{n-2}}\,.$$

(*ii*) *Convexity: for every $\beta > \frac{n-2}{n-1}$, the function F_β is convex on $[1, \infty)$.*

We remark that the functions F_β *are well–defined*, in view of Theorem 2.3 and since the integrand function in (3.1) is bounded on every level set of u. Note that, from Theorem 3.1 and by a simple argument based on the Dominated Convergence Theorem, the monotonicity and the convexity of F_β extend to the case $\beta = \frac{n-2}{n-1}$. Moreover, on the values τ such that $\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}$ is regular and thus on a.e. $\tau > 1$ due to Theorem 2.3 (*iii*), the function F_β is twice differentiable for each $\beta > \frac{n-2}{n-1}$, with first and second derivative given by

$$F'_\beta(\tau) = -\beta\, \frac{(\tau+1)^{\beta\frac{n-1}{n-2} - \frac{3}{2}}}{\sqrt{\tau-1}} \int\limits_{\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\mathrm{D}u|^\beta \Big[\mathrm{H} - \frac{n-1}{n-2}\,\frac{2u}{1-u^2}\,|\mathrm{D}u|\Big] d\sigma\,, \tag{3.2}$$

$$\begin{aligned} F''_\beta(\tau) = \beta \frac{(\tau+1)^{\beta\frac{n-1}{n-2} - 3}}{\tau - 1} \Bigg\{ \Big(\beta - \frac{n-2}{n-1}\Big) & \int\limits_{\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\mathrm{D}u|^{\beta-1} \Big[\,\mathrm{H} - \frac{n-1}{n-2}\,\frac{2u}{1-u^2}\,|\mathrm{D}u|\,\Big]^2 d\sigma \\ + \beta \int\limits_{\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\mathrm{D}u|^{\beta-3}\, \Big|\mathrm{D}^{\mathrm{T}} |\mathrm{D}u|\Big|^2 d\sigma & + \int\limits_{\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\mathrm{D}u|^{\beta-1} \Big[\, |\mathrm{h}|^2 - \frac{1}{n-1}\, |\mathrm{H}|^2 \,\Big] d\sigma \\ + \int\limits_{\left\{u = \sqrt{\frac{\tau-1}{\tau+1}}\right\}} |\mathrm{D}u|^{\beta-1} \Bigg[\, \mathrm{Ric}(\nu,\nu) - \frac{\mathrm{D}^2 u\,(\nu,\nu)}{u} \,\Bigg] d\sigma \Bigg\}. & \end{aligned} \tag{3.3}$$

In the computation, we have used the first normal variation of the volume and the mean curvature of $\{u = t\}$, and the Divergence Theorem. The symbols H and h stand respectively for the mean curvature and the second fundamental form of the smooth $(n-1)$–dimensional submanifold $\{u = \sqrt{\frac{\tau-1}{\tau+1}}\}$, with respect to the ∞–pointing unit normal vector field $\nu = \frac{\mathrm{D}u}{|\mathrm{D}u|}$. Also, D^{T} denotes the tangential part of the gradient, that is

$$\mathrm{D}^{\mathrm{T}} f = \mathrm{D} f - g_0(\mathrm{D} f, \nu)\nu,$$

for every $f \in C^1(M)$.

To prove Theorem 3.1, we use the results of Section 4, which are obtained in the conformal setting defined by

$$g = (1-u^2)^{\frac{2}{n-2}} g_0 \,, \qquad \varphi = \log\Big(\frac{1+u}{1-u}\Big). \tag{3.4}$$

Denoting by ∇ and Δ_g the Levi–Civita connection and the Laplace–Beltrami operator of g, the triple (M, g, φ) satisfies the following system.

$$\begin{cases} \mathrm{Ric}_g - \coth(\varphi)\,\nabla^2\varphi + \dfrac{1}{n-2}\, d\varphi \otimes d\varphi - \dfrac{1}{n-2}\,|\nabla\varphi|_g^2\; g \geq 0 & \text{in } \mathring{M}, \\ \Delta_g \varphi = 0 & \text{in } M, \\ \varphi = 0 & \text{on } \partial M, \\ \varphi \to +\infty & \text{at } \infty. \end{cases} \tag{3.5}$$

Moreover, we have that

$$|\nabla\varphi|_g^2 = 4\,|\mathrm{D}u|^2(1-u^2)^{-2\frac{n-1}{n-2}} \longrightarrow (2\,C)^{-\frac{2}{n-2}}(n-2)^2 \text{ at } \infty\,, \tag{$\star$}$$

as we will see in the proof of Lemma 4.1.

Remark 3.1. Since $\mathrm{Crit}(\varphi) = \mathrm{Crit}(u) = \{|\nabla\varphi|_g = 0\}$ by the equality in $(\star)$ and since $\{\varphi = s\} = \{u = \tanh\left(\frac{s}{2}\right)\}$ by (3.4), using Theorem 2.3 and Remark 2.1 we deduce that: $\mathrm{Crit}(\varphi)$ *has zero μ_g–measure and zero $(n-1)$–Hausdorff measure in (M, g); the level sets of φ have finite $(n-1)$–Hausdorff measure in (M, g) and in particular the smooth $(n-1)$–dimensional submanifolds $\{\varphi = s\} \setminus \mathrm{Crit}(\varphi)$ have finite g–area, i.e., finite σ_g–measure. Moreover, $\{\varphi \geq s\}$ is connected for every $s \geq 0$ and there exists $s_0 \geq 0$ such that $\{\varphi = s\}$ is regular and diffeomorphic to $\mathbb{S}^{n-1}$, for every $s \geq s_0$. Similar comments as those at the end of Subsection 2.2 can be made, regarding the relation between integration and* $\mathrm{Crit}(\varphi)$.

Let $\Phi_\beta : [0, \infty) \to \mathbb{R}$ be the function defined by formula

$$\Phi_\beta(s) := \int\limits_{\{\varphi = s\}} |\nabla\varphi|_g^{\beta+1}\, d\sigma_g\,,$$

for every $\beta \geq 0$. For the convenience of the reader, we anticipate from Section 4 the properties of Φ_β that we are going to use.

($\circ$) For every $\beta \geq 0$, the function $\Phi_\beta(s)$ is continuous in $[0, +\infty)$.

($\diamond$) For every $\beta > \frac{n-2}{n-1}$, the function Φ_β is continuously differentiable in $(0, +\infty)$. The derivative Φ'_β is nonpositive, satisfies for every $S > s > 0$

$$\frac{\Phi'_\beta(S)}{\sinh(S)} - \frac{\Phi'_\beta(s)}{\sinh(s)} \geq 0\,, \tag{3.6}$$

and admits for every $s > 0$ the integral representation

$$\Phi'_\beta(s) = -\beta\, \sinh(s) \int\limits_{\{\varphi > s\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[\, (\beta - 2)\left|\nabla|\nabla\varphi|_g\right|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi, \nabla\varphi)\,\right]}{\sinh\varphi}\, d\mu_g \leq 0\,,$$

where Q is defined in (4.9).

($\diamond\,\diamond$) If there exists $s_0 > 0$ such that $\Phi'_\beta(s_0) = 0$ for some $\beta > \frac{n-2}{n-1}$, then $\{\varphi = s_0\}$ is connected and $(\{\varphi \geq s_0\}, g)$ is isometric to $([0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + g_{\{\varphi = s_0\}})$, where ρ is the g–distance function to $\{\varphi = s_0\}$ and φ is an affine function of ρ in $\{\varphi \geq s_0\}$. If Φ_β is constant for some $\beta > \frac{n-2}{n-1}$, then ∂M is connected and (M, g) is isometric to $([0, +\infty) \times \partial M, d\rho \otimes d\rho + g_{\partial M})$, where ρ is the g–distance function to ∂M and φ is an affine function of ρ.

In the above list we have gathered and summarised the results contained in Lemma 4.5, Proposition 4.6, and Corollary 4.7.

Proof of Theorem 3.1. Step 1*: Differentiability, Monotonicity and Convexity.* For every $\beta \geq 0$ and for all $\tau \in [1, +\infty)$, we note that

$$F_\beta(\tau) = 2^{\frac{\beta}{n-2}-1}\ \Phi_\beta\!\left(\log\left(\frac{\sqrt{\tau+1} + \sqrt{\tau-1}}{\sqrt{\tau+1} - \sqrt{\tau-1}}\right)\right). \tag{3.7}$$

Consequently, by ($\circ$), we deduce that *for every $\beta \geq 0$ the function F_β is continuous in* $[1, +\infty)$. By ($\diamond$) and by (3.7), we obtain immediately that for every $\beta > \frac{n-2}{n-1}$ the function F_β is continuously differentiable in $(1, +\infty)$, with

$$F'_\beta(\tau) = 2^{\frac{\beta}{n-2}-1}\ \frac{1}{\sqrt{\tau^2-1}}\ \Phi'_\beta\!\left(\log\left(\frac{\sqrt{\tau+1} + \sqrt{\tau-1}}{\sqrt{\tau+1} - \sqrt{\tau-1}}\right)\right). \tag{3.8}$$

In particular, from ($\diamond$) we get $F'_\beta \leq 0$. As for the convexity, noticing that

$$\sqrt{\tau^2 - 1} = \sinh\left(\log\left(\frac{\sqrt{\tau+1} + \sqrt{\tau-1}}{\sqrt{\tau+1} - \sqrt{\tau-1}}\right)\right)$$

and that the function $\log\big(\frac{\sqrt{\tau+1}+\sqrt{\tau-1}}{\sqrt{\tau+1}-\sqrt{\tau-1}}\big)$ is nondecreasing, from (3.6) we obtain that F'_β is nondecreasing in $(1, +\infty)$. Therefore also by the continuity of F_β at 1, F_β is convex in $[1, +\infty)$.

Step 2*: Outer Rigidity.* Let us assume that there exists $\tau_0 \in (1, \infty)$ such that $F'_\beta(\tau_0) = 0$ for some $\beta > \frac{n-2}{n-1}$. Then, by equality (3.8), $\Phi'_\beta(s_0) = 0$ for $s_0 = \log\big(\frac{\sqrt{\tau_0+1}+\sqrt{\tau_0-1}}{\sqrt{\tau_0+1}-\sqrt{\tau_0-1}}\big)$, and hence, by ($\diamond\,\diamond$), $(\{\varphi \geq s_0\}, g)$ is isometric to $([0, +\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + g_{\{\varphi = s_0\}})$, where ρ is the g–distance function to $\{\varphi = s_0\}$

and $\varphi = (n-2)\,(2C)^{-\frac{1}{n-2}}\rho + s_0$, because $|\nabla\rho|_g \equiv 1$ and in view of the limit in ($\star$). Setting $t_0 = \tanh\frac{s_0}{2}$, consider N the submanifold with boundary $\{\varphi \geq s_0\} = \{u \geq t_0\}$. Writing

$$\begin{array}{lllll} N & [0,+\infty)\times\partial N & & [s_0,+\infty)\times\partial N & & [t_0,1)\times\partial N \\ g & d\rho\otimes d\rho + g_{\partial N} & & \dfrac{d\varphi\otimes d\varphi}{(n-2)^2\,(2C)^{-\frac{2}{n-2}}} + g_{\partial N} & & \dfrac{2^{2\,\frac{n-1}{n-2}}\,C^{\frac{2}{n-2}}}{(n-2)^2\,(1-u^2)^2}\,du\otimes du + g_{\partial N} \\ p \;\mapsto & (\rho,q) & \mapsto & (\varphi = (n-2)\,(2C)^{-\frac{1}{n-2}}\,\rho, q) & \mapsto & \left(u = \tanh\dfrac{\varphi}{2}, q\right), \end{array}$$

the Riemannian manifolds in the first row, whose metrics are indicated in the second row, are pairwise isometric through the applications written in the third row. We recall that the application $p \to (\rho,q)$ in the third row is the inverse of the diffeomorphism given by the normal exponential map, i.e., the application which associates to every point p of N the couple having as first coordinate the g–distance of p from ∂N and as second coordinate the point q of ∂N that realizes such distance. Then, in view of (3.4) and with the same notation as above, the following Riemannian manifolds are isometric.

$$\begin{array}{llll} N & [t_0,1)\times\partial N & & [r_0,+\infty)\times\partial N \\ g_0 & \dfrac{2^{2\,\frac{n-1}{n-2}}\,C^{\frac{2}{n-2}}}{(n-2)^2\,(1-u^2)^{2\,\frac{n-1}{n-2}}}\,du\otimes du + (1-u^2)^{-\frac{2}{n-2}}g_{\partial N} & & \dfrac{dr\otimes dr}{1-2\,C\,r^{2-n}} + (2\,C)^{-\frac{2}{n-2}}\,r^2 g_{\partial N} \\ p \;\mapsto & (u,q) & \mapsto & \left(r = \left(\dfrac{2C}{1-u^2}\right)^{\frac{1}{n-2}}, q\right), \end{array} \tag{3.9}$$

where $r_0 = (\frac{2C}{1-t_0^2})^{\frac{1}{n-2}}$. Doing some computations, we obtain that

$$|\mathrm{Rm}|^2\,(p) = (2C)^{\frac{4}{n-2}}\,r^{-4}\,(p)\,\left|\mathrm{Rm}_{g_{\partial N}} + \frac{1-2\,C\,r^{2-n}}{2^{\frac{n}{n-2}}\,C^{\frac{2}{n-2}}}\,g_{\partial N} \owedge g_{\partial N}\right|^2_{g_{\partial N}}(q) + c\,r^{-2n}\,(p)\,, \tag{3.10}$$

where the convection followed for the Riemannian curvature tensor is that given in [23], c is a suitable positive constant and q is the point of ∂N that realizes the g–distance of p from ∂N. Denoting by Θ the diffeomorphism from N to $[r_0,+\infty)\times\partial N$ introduced in (3.9), for every $q_0 \in \partial N$ we consider the curve

$$\gamma : r \in [r_0,+\infty) \to \Theta^{-1}(r,q_0) \in M$$

and observe from (3.10) that

$$(2C)^{-\frac{4}{n-2}}\,r^4\,|\mathrm{Rm}|^2\,(\gamma(r)) \xrightarrow{r\to+\infty} \left|\mathrm{Rm}_{g_{\partial N}} + \frac{(2C\,)^{-\frac{2}{n-2}}}{2}g_{\partial N} \owedge g_{\partial N}\right|^2_{g_{\partial N}}(q_0)\,. \tag{3.11}$$

At the same time, we have that

$$r^4\,|\mathrm{Rm}|^2\,(\gamma(r)) \xrightarrow{r\to+\infty} 0\,. \tag{3.12}$$

This is because g_0 is asymptotically flat according to Definition 1.1 and by (2.3), which yields in particular

$$|\mathrm{Rm}| = O(|x|^{-(p+2)}) \qquad \text{and} \qquad \frac{r}{|x|} \xrightarrow{|x|\to+\infty} 1\,,$$

for some $p > \frac{n-2}{2}$. Combining (3.11) and (3.12), the arbitrariness of the point q_0 in ∂N gives that

$$\mathrm{Rm}_{g_{\partial N}} = -\frac{(2C)^{-\frac{2}{n-2}}}{2} g_{\partial N} \owedge g_{\partial N} \,.$$

Hence the sectional curvature of the Riemannian manifold $(\partial N, g_{\partial N})$ is constant and identically equal to $(2C)^{-\frac{2}{n-2}}$. Then, being all the level sets $\{u = t\}$ with $t \approx 1$ regular and diffeomorphic to $\mathbb{S}^{n-1}$ as observed in Remark 2.1, for [10, Section 3.F] $(\partial N, g_{\partial N})$ and $(\mathbb{S}^{n-1}, (2C)^{\frac{2}{n-2}} g_{\mathbb{S}^{n-1}})$ are isometric. Then, $(\{u \geq t_0\}, g_0)$ is isometric to the submanifold $\big([r_0, +\infty) \times \mathbb{S}^{n-1}, \frac{dr \otimes dr}{1-2Cr^{2-n}} + r^2 g_{\mathbb{S}^{n-1}}\big)$ of the Schwarzschild manifold with associated ADM mass given by C. □

<u>*Proof of Theorem 1.1. Spep* 1*: Inequality*.</u> By Theorem 3.1, we have that $F_\beta(\tau_0) \geq \lim_{\tau \to +\infty} F_\beta(\tau)$, for every $\tau_0 > 1$. In particular, since F_β is continuous in $[1, +\infty)$ due to the step 1 of Theorem 3.1, we have that

$$F_\beta(1) \geq \lim_{\tau \to +\infty} F_\beta(\tau)\,, \tag{3.13}$$

for every $\beta > \frac{n-2}{n-1}$. Since $\mathrm{D}^2 u \equiv 0$ on ∂M and since ∂M of M is connected, $|\mathrm{D}u|$ is constantly equal to $\frac{(n-2)\,C\,|\mathbb{S}^{n-1}|}{|\partial M|}$, by formula (2.2). In particular, we have that

$$F_\beta(1) = \frac{2^{\beta\,\frac{n-1}{n-2}}\,(n-2)^{\beta+1}\,C^{\beta+1}\,|\mathbb{S}^{n-1}|^{\beta+1}}{|\partial M|^\beta} \tag{3.14}$$

By $(\star)$, we know that

$$\frac{|\mathrm{D}u|}{(1-u^2)^{\frac{n-1}{n-2}}} \longrightarrow 2^{-\frac{n-1}{n-2}}\,(n-2)\,C^{-\frac{1}{n-2}} \qquad \text{at } \infty$$

Therefore, fixed $\varepsilon > 0$, there exists $1 < \tau_0 < +\infty$ such that

$$|\mathrm{D}u| \geq \,(1-u^2)^{\frac{n-1}{n-2}}\Big(2^{-\frac{n-1}{n-2}}\,(n-2)\,C^{-\frac{1}{n-2}} - \varepsilon\Big)$$

in $\{u \geq \sqrt{\frac{\tau_0 - 1}{\tau_0+1}}\,\}$ and the level sets $\{u = \sqrt{\frac{\tau-1}{\tau+1}}\,\}$ are regular for all $\tau \geq \tau_0$. Therefore, for every $\tau \geq \tau_0$ we have that

$$\begin{aligned}
F_\beta(\tau) &= (1+\tau)^{\beta\,\frac{n-1}{n-2}} \int\limits_{\{u = \sqrt{\frac{\tau-1}{\tau+1}}\}} |\mathrm{D}u|^{\beta+1}\,d\sigma \\
&\geq (1+\tau)^{\beta\,\frac{n-1}{n-2}} \int\limits_{\{u = \sqrt{\frac{\tau-1}{\tau+1}}\}} (1-u^2)^{\beta\,\frac{n-1}{n-2}}\Big(2^{-\frac{n-1}{n-2}}\,(n-2)\,C^{-\frac{1}{n-2}} - \varepsilon\Big)^\beta\,|\mathrm{D}u|\,d\sigma \\
&= 2^{\beta\,\frac{n-1}{n-2}}\Big(2^{-\frac{n-1}{n-2}}\,(n-2)\,C^{-\frac{1}{n-2}} - \varepsilon\Big)^\beta \int\limits_{\partial M} |\mathrm{D}u|\,d\sigma \\
&= 2^{\beta\,\frac{n-1}{n-2}}\,(n-2)\,C\Big(2^{-\frac{n-1}{n-2}}\,(n-2)\,C^{-\frac{1}{n-2}} - \varepsilon\Big)^\beta\,|\mathbb{S}^{n-1}|\,,
\end{aligned}$$

where in the second equality we have used the Divergence Theorem couple with the fact that u is harmonic, and in the third equality we have used formula (2.2). Since ε is arbitrary, we get

$$\lim_{\tau\to+\infty} F_\beta(\tau) \geq (n-2)^{\beta+1}\, C^{1-\frac{\beta}{n-2}}\, |\mathbb{S}^{n-1}|\,.$$

In a similar way we can obtain the reverse inequality, so that

$$\lim_{\tau\to+\infty} F_\beta(\tau) = (n-2)^{\beta+1}\, C^{1-\frac{\beta}{n-2}}\, |\mathbb{S}^{n-1}|\,. \tag{3.15}$$

Joining the formulas in (3.13), (3.14) and (3.15), we obtain the desired inequality (1.7).

Step 2: Rigidity. If (M, g_0) is isometric to the Schwarzschild manifold with ADM mass $m > 0$, then the right–hand side and the left–hand side of (1.7) are both equal to m, by direct computation.

Suppose now that the equality holds in (1.7). Then, by Step 1 and for every $\beta > \frac{n-2}{n-1}$, the function F_β is constant. In turn, Φ_β is constant, being

$$\Phi_\beta(s) = 2^{1-\frac{\beta}{n-2}}\, F_\beta\!\left(\frac{1+\tanh^2\left(\frac{s}{2}\right)}{1-\tanh^2\left(\frac{s}{2}\right)}\right).$$

Finally, $(\diamond\,\diamond)$ and the very same argument of the proof of the Outer Rigidity in Theorem 3.1 imply first that (M, g) is isometric to

$$([0,+\infty)\times\partial M, d\rho\otimes d\rho + g_{\partial M}),$$

where ρ is the g–distance to ∂M and φ is an affine function of ρ, and secondly that (M, g_0) is isometric to the Schwarzschild manifold with ADM mass C. □

4. Conformal setting

Let us consider the conformal change g of the metric g_0 introduced in (3.4) which is well–defined being $0 \leq u < 1$ in M. The metric g is complete, since any g–geodesic γ parametrized by g–arc length defined on a bounded interval $[0, a)$ can be extended to a continuous path on $[0, a]$. Indeed, if γ has infinity length with respect to g_0, there exists a sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $\gamma(t_m) \to \infty$ (being γ not contained in any compact set) and using, in the computation of g–length of γ, the passage from g to g_0, the asymptotic flatness of (M, g_0) and the asymptotic expansion of u in (2.3) we obtain that γ has infinity length with respect to g. Hence γ has finite length with respect to g_0 and, being g_0 complete, it follows that g is complete (see [24, Section 1.1] and [9]). We also recall that the metric g is *asymptotically cylindrical* (see [2, Section 3.1]). The other main element of the conformal setting is the C^∞–function φ, defined in (3.4). Now, the reverse changes are

$$g_0 = \Big(\cosh\frac{\varphi}{2}\Big)^{\frac{4}{n-2}} g\,, \qquad u = \tanh\frac{\varphi}{2}\,.$$

Recalling that we denote by the symbols ∇ and Δ_g the Levi–Civita connection and the Laplace–Beltrami operator of g, by the formulas in [6, Theorem 1.159], we obtain

$$\mathrm{D}u = \frac{1}{2}\Big(\cosh\frac{\varphi}{2}\Big)^{-\frac{2n}{n-2}}\, \nabla\varphi\,, \tag{4.1}$$

$$\mathrm{D}^2 u = \frac{1}{2}\,\frac{1}{\cosh^2\frac{\varphi}{2}}\,\nabla^2\varphi - \frac{n}{2(n-2)}\,\frac{\sinh\frac{\varphi}{2}}{\cosh^3\frac{\varphi}{2}}\,d\varphi\otimes d\varphi + \frac{1}{2(n-2)}\,\frac{\sinh\frac{\varphi}{2}}{\cosh^3\frac{\varphi}{2}}\,|\nabla\varphi|_g^2\,g\,, \tag{4.2}$$

$$\Delta u = \frac{1}{2}\Big(\cosh\frac{\varphi}{2}\Big)^{-\frac{2n}{n-2}}\Delta_g\varphi\,, \tag{4.3}$$

$$\begin{aligned}\mathrm{Ric} = \mathrm{Ric}_g - \tanh\Big(\frac{\varphi}{2}\Big)\nabla^2\varphi + \Big[\frac{1}{n-2}\tanh^2\Big(\frac{\varphi}{2}\Big) - \frac{1}{2}\frac{1}{\cosh^2\frac{\varphi}{2}}\Big]d\varphi\otimes d\varphi \\ - \frac{1}{(n-2)}\Big[\frac{1}{2}\,\frac{1}{\cosh^2\frac{\varphi}{2}} + \tanh^2\Big(\frac{\varphi}{2}\Big)\Big]|\nabla\varphi|_g^2\,g\,.\end{aligned}$$

Translating system (1.1) in terms of g and φ, we get system (3.5). Moreover, on $\{\varphi = s\}\setminus\mathrm{Crit}(\varphi)$ we consider the ∞–pointing normal unit vector fields

$$\nu = \frac{\mathrm{D}u}{|\mathrm{D}u|}\,, \qquad \nu_g = \frac{\nabla\varphi}{|\nabla\varphi|_g}\,,$$

the mean curvatures

$$\mathrm{H} = -\frac{\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u)}{|\mathrm{D}u|^3}\,, \qquad \mathrm{H}_g = -\frac{\nabla^2\varphi(\nabla\varphi,\nabla\varphi)}{|\nabla\varphi|_g^3}\,, \tag{4.4}$$

and the second fundamental forms

$$\mathrm{h}(X,Y) = \frac{\mathrm{D}^2u\,(X,Y)}{|\mathrm{D}u|}\,, \qquad \mathrm{h}_g(X,Y) = \frac{\nabla^2\varphi\,(X,Y)}{|\nabla\varphi|_g}\,,$$

for any X, Y tangent vector fields to the considered submanifold.
Reversing formulas (4.1), (4.2) and (4.3), we get

$$\begin{aligned}\nabla\varphi &= \frac{2}{(1-u^2)^{\frac{n}{n-2}}}\,\mathrm{D}u\,,\\ \nabla^2\varphi &= \frac{2}{1-u^2}\Big[\,\mathrm{D}^2u + \frac{n}{n-2}\,\frac{2u}{1-u^2}\,du\otimes du - \frac{1}{n-2}\,\frac{2u}{1-u^2}\,|\mathrm{D}u|^2\,g_0\,\Big],\\ |\nabla^2\varphi|_g^2 &= \frac{4}{(1-u^2)^{\frac{2n}{n-2}}}\,|\mathrm{D}^2u|^2 + \frac{16n}{n-2}\,\frac{u}{(1-u^2)^{\frac{3n-2}{n-2}}}\,\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u) + \frac{16n(n-1)}{(n-2)^2}\,\frac{u^2}{(1-u^2)^{4\left(\frac{n-1}{n-2}\right)}}\,|\mathrm{D}u|^4\,.\end{aligned}$$

These equalities, jointly with the asymptotic flatness of (M, g_0) and the asymptotic expansion of u given in Section 2, allow us to obtain an upper bound for the functions $|\nabla\varphi|_g$ and $|\nabla^2\varphi|_g$, and for the g–areas of the level sets of φ sufficiently "close" to infinity. This is the content of the following lemma.

Lemma 4.1. *There exists* $0 \le s_0 < +\infty$ *such that*

$$\sup_M|\nabla\varphi|_g + \sup_M|\nabla^2\varphi|_g + \sup_{s\ge s_0}\int_{\{\varphi=s\}} d\sigma_g < +\infty\,. \tag{4.5}$$

Proof. Let ψ be a chart at infinity. Considering $\widetilde{g}_0 = \psi_* g_0 = \widetilde{g}_{0;ij}dx^i\otimes dx^j$, by formulas (2.6) and (2.4), the coordinate expression of

$$|\nabla\varphi|_g^2 = \frac{4\,|\mathrm{D}u|^2}{(1-u^2)^{2\frac{n-1}{n-2}}}$$

is

$$\begin{aligned}\psi_*|\nabla\varphi|^2_g &= \frac{4\,\psi_*|\mathrm{D}u|^2}{(1-\widetilde{u}^2)^{2\frac{n-1}{n-2}}} = \frac{4\,\widetilde{g}_0^{ij}\,\partial_i\widetilde{u}\,\partial_j\widetilde{u}}{\{1-[1-C|x|^{2-n}+o(|x|^{2-n})]^2\}^{2\,(\frac{n-1}{n-2})}} \\ &= \frac{4\,[\delta^{ij}+O(|x|^{-p})]\,[(n-2)^2\,C^2\,|x|^{-2n}\,x^i\,x^j+o(|x|^{2-2n})]}{\big[2\,C\,|x|^{2-n}+o(|x|^{2-n})\big]^{2\,(\frac{n-1}{n-2})}} \\ &= \frac{4\,(n-2)^2\,C^2\,|x|^{2-2n}+o(|x|^{2-2n})}{(2\,C)^{2\,(\frac{n-1}{n-2})}\,|x|^{2-2n}\,(1+o(1))} = \frac{4\,(n-2)^2\,C^2+o(1)}{(2\,C)^{2\,(\frac{n-1}{n-2})}\,(1+o(1))}\,.\end{aligned}$$

Hence

$$|\nabla\varphi|^2_g \longrightarrow \frac{4\,(n-2)^2\,C^2}{(2\,C)^{2\,(\frac{n-1}{n-2})}} = (2\,C)^{-\frac{2}{n-2}}(n-2)^2 \ \text{ at } \infty\,. \tag{4.6}$$

Moreover, by limit (4.6) there exist a constant $L>0$ and a value $s_0>0$ of φ such that every $s\geq s_0$ is a regular value of φ and $(1-u^2)^{\frac{n-1}{n-2}}\leq L\,|\mathrm{D}u|$ on $\{\varphi\geq s_0\}$. Then

$$\int\limits_{\{\varphi=s\}} d\sigma_g = \int\limits_{\{u=\tanh\frac{s}{2}\}} (1-u^2)^{\frac{n-1}{n-2}}\,d\sigma \leq L \int\limits_{\{u=\tanh\frac{s}{2}\}} |\mathrm{D}u|\,d\sigma = L\int\limits_{\partial M}|\mathrm{D}u|\,d\sigma\,,$$

where in the last equality we have applied the Divergence Theorem. Consequently, we have that $\sup\limits_{s\geq s_0}\int\limits_{\{\varphi=s\}} d\sigma_g < +\infty$. Similarly, we have that

$$\begin{aligned}\psi_*|\mathrm{D}^2u|^2 &= \widetilde{g}_0^{i_1i_2}\widetilde{g}_0^{j_1j_2}(\mathrm{D}^2_{\widetilde{g}_0}\widetilde{u})_{i_1j_1}(\mathrm{D}^2_{\widetilde{g}_0}\widetilde{u})_{i_2j_2} = \widetilde{g}_0^{i_1i_2}\widetilde{g}_0^{j_1j_2}[\partial_{i_1}\partial_{j_1}\widetilde{u}-\Gamma^{k_1}_{\widetilde{g}_0;i_1j_1}\partial_{k_1}\widetilde{u}][\partial_{i_2}\partial_{j_2}\widetilde{u}-\Gamma^{k_2}_{\widetilde{g}_0;i_2j_2}\partial_{k_2}\widetilde{u}] \\ &= [\delta^{i_1i_2}\delta^{j_1j_2}+O(|x|^{-p})]\,[\partial_{i_1}\partial_{j_1}\widetilde{u}-O(|x|^{-(p+n)})]\,[\partial_{i_2}\partial_{j_2}\widetilde{u}-O(|x|^{-(p+n)})] \\ &= \big[(n-1)(n-2)C\big]^2|x|^{-2n}+o(|x|^{-2n})\end{aligned}$$

due to formulas (2.6), (2.5) and (5.3). Moreover,

$$\begin{aligned}\psi_*\mathrm{D}^2u(\mathrm{D}u,\mathrm{D}u) &= \mathrm{D}^2_{\widetilde{g}_0}\widetilde{u}(\mathrm{D}_{\widetilde{g}_0}\widetilde{u},\mathrm{D}_{\widetilde{g}_0}\widetilde{u}) = \widetilde{g}_0^{ir}\,\widetilde{g}_0^{js}\,\partial_r\widetilde{u}\,\partial_s\widetilde{u}\,(\partial_i\partial_j\widetilde{u}-\Gamma^k_{\widetilde{g}_0;ij}\partial_k\widetilde{u}) \\ &= -(n-1)\,(n-2)^3\,C^3|x|^{2-3n}+o(|x|^{2-3n})\,.\end{aligned}$$

All in all,

$$\begin{aligned}\psi_*|\nabla^2\varphi|^2_g &= \frac{4}{(1-\widetilde{u}^2)^{\frac{2n}{n-2}}}\,\psi_*|\mathrm{D}^2\widetilde{u}|^2 + \frac{16n}{n-2}\,\frac{\widetilde{u}}{(1-\widetilde{u}^2)^{\frac{3n-2}{n-2}}}\,\psi_*\mathrm{D}^2\widetilde{u}(\mathrm{D}\widetilde{u},\mathrm{D}\widetilde{u}) \\ &\quad + \frac{16n(n-1)}{(n-2)^2}\,\frac{\widetilde{u}^2}{(1-\widetilde{u}^2)^{4\left(\frac{n-1}{n-2}\right)}}\,\psi_*|\mathrm{D}\widetilde{u}|^4 \\ &= (n-1)\,(n-2)^2\,(2\,C)^{-\frac{4}{n-2}}\left\{\frac{n-1+o(1)}{1+o(1)} - \frac{2\,n\,\widetilde{u}+o(1)}{1+o(1)} + \frac{16\,n\,\widetilde{u}^2+o(1)}{1+o(1)}\right\},\end{aligned}$$

which gives

$$|\nabla^2\varphi|^2_g \longrightarrow (n-1)\,(15n-1)\,(n-2)^2\,(2\,C)^{-\frac{4}{n-2}} \qquad \text{at } \infty\,.$$

In particular, since φ is smooth, we have that

$$\sup_M |\nabla\varphi|_g \,+\, \sup_M |\nabla^2\varphi|_g \,<\, +\infty\,.$$

$\square$

Remark 4.1. Note that $\sup_{s\geq 0} \int_{\{\varphi=s\}} d\sigma_g \in (0,+\infty]$, since we cannot a priori exclude that there exist a critical value $\overline{s} > 0$ and a sequence $\{s_m\} \subset (0,+\infty)$ such that $s_m \to \overline{s}$ and

$$\int_{\{\varphi=s_m\}} d\sigma_g \to +\infty\,.$$

As it will be clear in the proof of the integral identity (4.17), which is at the core of the conformal–monotonicity result (Proposition 4.6), it is useful to introduce a suitable vector field with nonnegative divergence. To do this, let us focus on the set $\mathring{M} \setminus \mathrm{Crit}(\varphi)$ and notice first that the classical Bochner formula, applied to the g–harmonic function φ, becomes

$$\frac{1}{2}\,\Delta_g|\nabla\varphi|_g^2 \,=\, |\nabla^2\varphi|_g^2 \,+\, \mathrm{Ric}_g(\nabla\varphi,\nabla\varphi) + \, g(\nabla\Delta_g\varphi,\nabla\varphi) \,=\, |\nabla^2\varphi|_g^2 \,+\, \mathrm{Ric}_g(\nabla\varphi,\nabla\varphi)\,. \tag{4.7}$$

Then, we obtain

$$\begin{aligned}\Delta_g|\nabla\varphi|_g^\beta &= \mathrm{div}_g\Big(\nabla|\nabla\varphi|_g^\beta\Big) = \mathrm{div}_g\Big(\frac{\beta}{2}|\nabla\varphi|_g^{\beta-2}\nabla|\nabla\varphi|_g^2\Big)\\ &= \frac{\beta}{2}\Big[g(\nabla|\nabla\varphi|_g^{\beta-2},\nabla|\nabla\varphi|_g^2) + |\nabla\varphi|_g^{\beta-2}\Delta_g|\nabla\varphi|_g^2\Big]\\ &= \beta\,|\nabla\varphi|_g^{\beta-2}\Big[\,(\beta-2)\,\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + \mathrm{Ric}_g(\nabla\varphi,\nabla\varphi)\Big]\,,\end{aligned} \tag{4.8}$$

where in the third equality we have used (4.7). Now, observe from the nonnegativity of the tensor

$$Q \,:=\, \mathrm{Ric}_g - \coth(\varphi)\,\nabla^2\varphi + \frac{1}{n-2}\,d\varphi\otimes d\varphi - \frac{1}{n-2}\,|\nabla\varphi|_g^2\,\,g \tag{4.9}$$

(see (3.5)) that

$$Q(\nabla\varphi,\nabla\varphi) = \mathrm{Ric}_g(\nabla\varphi,\nabla\varphi) - \coth(\varphi)\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi) \geq 0\,. \tag{4.10}$$

Therefore, by adding and subtracting the term $\beta\,|\nabla\varphi|_g^{\beta-2}\,\coth(\varphi)\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)$ on the right–hand side of (4.8), we get

$$\begin{aligned}&\Delta_g|\nabla\varphi|_g^\beta - \beta\,|\nabla\varphi|_g^{\beta-2}\,\coth(\varphi)\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\\ &\qquad\qquad = \beta|\nabla\varphi|_g^{\beta-2}\Big[\,(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\Big]\,.\end{aligned} \tag{4.11}$$

Since

$$\beta\,|\nabla\varphi|_g^{\beta-2}\,\coth(\varphi)\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi) = \coth(\varphi)\,g(\nabla|\nabla\varphi|_g^\beta,\nabla\varphi)$$

and since, setting

$$Y_\beta := \frac{\nabla\, |\nabla\varphi|_g^\beta}{\sinh\varphi}\,, \tag{4.12}$$

there holds

$$\mathrm{div}_g\, Y_\beta = \frac{\Delta_g |\nabla\varphi|_g^\beta}{\sinh\varphi} - \frac{\cosh\varphi}{\sinh^2\varphi}\; g(\nabla|\nabla\varphi|_g^\beta, \nabla\varphi)\,,$$

from (4.11) we get

$$\sinh(\varphi)\,\mathrm{div}_g\, Y_\beta = \beta|\nabla\varphi|_g^{\beta-2}\Big[\,(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\Big]\,.$$

Note that, by the refined Kato inequality for harmonic function

$$\frac{n}{n-1}\Big|\nabla|\nabla\varphi|_g\Big|_g^2 \le |\nabla^2\varphi|_g^2\,, \tag{4.13}$$

we have that

$$\begin{aligned}(\beta-2)&\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\\ &= \Big(\beta - \frac{n-2}{n-1}\Big)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + \Big[|\nabla^2\varphi|_g^2 - \frac{n}{n-1}\Big|\nabla|\nabla\varphi|_g\Big|_g^2\Big] + Q(\nabla\varphi,\nabla\varphi) \ge 0\,,\end{aligned} \tag{4.14}$$

whenever $\beta \ge \frac{n-2}{n-1}$. Hence, $\mathrm{div}_g\, Y_\beta \ge 0$ for every $\beta \ge \frac{n-2}{n-1}$. This fact will be heavily used in the proof of the forthcoming results. It will also be useful to have a precise estimate of $\int_{\{|\nabla\varphi|_g=\delta\}} \Big|\nabla|\nabla\varphi|_g\Big|_g\, d\sigma_g$ in terms of a suitable power of δ, close to $\mathrm{Crit}(\varphi)$, that is when $\delta \to 0^+$. This is the content of the following lemma.

Lemma 4.2. *There exists $\delta_0 > 0$ such that*

$$\sup\left\{\delta^{-\frac{1}{n-1}} \int\limits_{\{|\nabla\varphi|_g=\delta\}} \Big|\nabla|\nabla\varphi|_g\Big|_g\, d\sigma_g \;:\; 0<\delta<\delta_0 \textit{ regular value of } |\nabla\varphi|_g\right\} < +\infty\,. \tag{4.15}$$

We recall that the set of the critical values of $|\nabla\varphi|_g^2$ has zero Lebesgue measure by Sard's Theorem, whereas we have no information regarding the local $\mathcal{H}$–dimension of $\mathrm{Crit}(|\nabla\varphi|_g^2)$.

Proof. Applying Sard's Theorem to the smooth function $|\nabla\varphi|_g^2$ there exists $\varepsilon_0 > 0$ such that ε_0 is a regular value of $|\nabla\varphi|_g^2$ and

$$\varepsilon_0 < \min\Big\{\min_{\{\varphi=0\}} |\nabla\varphi|_g^2,\ \text{the limit of } |\nabla\varphi|_g^2 \text{ at } \infty\Big\}\,,$$

where the limit in the previous expression is the (finite and positive) value computed in (4.6). In particular, $\{|\nabla\varphi|_g^2 \le \varepsilon_0\}$ is compactly contained in $\mathring{M}$, and for every $0<\delta<\delta_0$ regular value of $|\nabla\varphi|_g$ we have that

$$\delta^{-\frac{1}{n-1}} \int\limits_{\{|\nabla\varphi|_g=\delta\}} \Big|\nabla|\nabla\varphi|_g\Big|_g\, d\sigma_g = \int\limits_{\{|\nabla\varphi|_g=\delta\}} \sinh(\varphi)\,\frac{|\nabla\varphi|_g^{-\frac{1}{n-1}}\Big|\nabla|\nabla\varphi|_g\Big|_g}{\sinh(\varphi)}\, d\sigma_g$$

$$\leq c \int_{\{|\nabla\varphi|_g=\delta\}} \frac{|\nabla\varphi|_g^{-\frac{1}{n-1}} \big|\nabla|\nabla\varphi|_g\big|_g}{\sinh(\varphi)}\, d\sigma_g\,.$$

Now, consider the smooth vector field

$$Z := 2\,\frac{n-1}{n-2}\,Y_{\frac{n-2}{n-1}} = \frac{1}{\sinh\varphi}\,\frac{\nabla|\nabla\varphi|_g^2}{|\nabla\varphi|_g^{\frac{n}{n-1}}}\,,$$

with $\mathrm{div}_g Z \geq 0$. Set

$$U_\mu := \{|\nabla\varphi|_g^2 < \mu\} \quad \text{for every } \mu > 0\,. \tag{4.16}$$

Then, for every $0 < \varepsilon < \varepsilon_0$ regular value of the function $|\nabla\varphi|_g^2$, we apply the Divergence Theorem to the smooth vector field Z on $U_{\varepsilon_0} \setminus \overline{U_\varepsilon}$, and we get

$$\int_{\{|\nabla\varphi|_g^2=\varepsilon_0\}} \frac{1}{\sinh\varphi}\,\frac{\big|\nabla|\nabla\varphi|_g^2\big|_g}{|\nabla\varphi|_g^{\frac{n}{n-1}}}\,d\sigma_g - \int_{\{|\nabla\varphi|_g^2=\varepsilon\}} \frac{1}{\sinh\varphi}\,\frac{\big|\nabla|\nabla\varphi|_g^2\big|_g}{|\nabla\varphi|_g^{\frac{n}{n-1}}}\,d\sigma_g$$
$$= \int_{U_{\varepsilon_0}\setminus\overline{U_\varepsilon}} \mathrm{div}_g Z\, d\mu_g \geq 0\,.$$

Then, it follows

$$\int_{\{|\nabla\varphi|_g^2=\varepsilon_0\}} \frac{1}{\sinh\varphi}\,\frac{\big|\nabla|\nabla\varphi|_g^2\big|_g}{|\nabla\varphi|_g^{\frac{n}{n-1}}}\,d\sigma_g \geq \int_{\{|\nabla\varphi|_g^2=\varepsilon\}} \frac{1}{\sinh\varphi}\,\frac{\big|\nabla|\nabla\varphi|_g^2\big|_g}{|\nabla\varphi|_g^{\frac{n}{n-1}}}\,d\sigma_g\,.$$

Therefore, setting

$$c_1 := \int_{\{|\nabla\varphi|_g^2=\varepsilon_0\}} \frac{1}{\sinh\varphi}\,\frac{\big|\nabla|\nabla\varphi|_g^2\big|_g}{|\nabla\varphi|_g^{\frac{n}{n-1}}}\,d\sigma_g > 0\,,$$

we obtain

$$\frac{1}{\varepsilon^{\frac{1}{2}\frac{n}{n-1}}} \int_{\{|\nabla\varphi|_g^2=\varepsilon\}} \frac{\big|\nabla|\nabla\varphi|_g^2\big|_g}{\sinh\varphi}\,d\sigma_g \leq c_1\,.$$

Consequently, the desired statement follows keeping in mind that: if δ is a regular value of $|\nabla\varphi|_g$, then δ^2 is a regular value of $|\nabla\varphi|_g^2$; in $M \setminus \mathrm{Crit}(\varphi)$ we have $\nabla|\nabla\varphi|_g^2 = 2|\nabla\varphi|_g\nabla|\nabla\varphi|_g$. □

We underline that from now on we will use Remark 3.1 widely.

The following proposition contains the integral identity which is the main tool of our analysis.

Proposition 4.3. *Let (M, g_0, u) be a sub-static harmonic triple, and let g and φ be the metric and the function defined in* (3.4)*. Then, for every $\beta > \frac{n-2}{n-1}$ and for every $S > s > 0$ regular values of φ, it holds*

$$\int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g - \int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g$$

$$= \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{\beta-2}\Big[\,(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\Big]}{\sinh\varphi}\, d\mu_g\,, \quad (4.17)$$

where the tensor Q is defined as in (4.9)*.*

Proof. The case $\beta \geq 2$ is an easy adaptation of the argument used in [2] but it is anyway a consequence of the following argument. We focus on the unknown case $\frac{n-2}{n-1} < \beta < 2$. In $\mathring{M} \setminus \mathrm{Crit}(\varphi)$ we consider the smooth vector field Y_β, defined in (4.12) and satisfying

$$0 \leq \mathrm{div}_g\, Y_\beta = \frac{\beta\,|\nabla\varphi|_g^{\beta-2}\Big[\,(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\Big]}{\sinh\varphi}\,, \quad (4.18)$$

as already explained. Set

$$E_s^S := \{s < \varphi < S\}, \quad \text{for every } S > s > 0\,.$$

When $E_s^S \cap \mathrm{Crit}(\varphi) = \emptyset$, then the statement is a straightforward application of the Divergence Theorem. Now, suppose that $E_s^S \cap \mathrm{Crit}(\varphi) \neq \emptyset$. In this case we consider, for every $\varepsilon > 0$ sufficiently small, a smooth nondecreasing cut–off function $\chi_\varepsilon : [0, +\infty) \to [0, 1]$ satisfying the following conditions

$$\chi_\varepsilon(\tau) = 0 \text{ in } \Big[\,0, \frac{1}{2}\varepsilon\,\Big], \quad 0 \leq \chi_\varepsilon'(\tau) \leq c\varepsilon^{-1} \text{ in } \Big[\,\frac{1}{2}\varepsilon, \frac{3}{2}\varepsilon\,\Big], \quad \chi_\varepsilon(\tau) = 1 \text{ in } \Big[\,\frac{3}{2}\varepsilon, +\infty\,\Big),$$

where c is a positive real constant independent of ε. We then define the smooth function $\Xi_\varepsilon : M \to [0, 1]$ as

$$\Xi_\varepsilon = \chi_\varepsilon \circ |\nabla\varphi|_g^2\,,$$

and apply the Divergence Theorem to the smooth vector field $\Xi_\varepsilon\, Y_\beta$ in E_s^S. In this way, we get

$$\int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g - \int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g = \beta^{-1}\Bigg[\int\limits_{E_s^S} \Xi_\varepsilon\, \mathrm{div}_g\, Y_\beta\, d\mu_g + \int\limits_{E_s^S} g(\nabla\Xi_\varepsilon, Y_\beta)\, d\mu_g\Bigg]$$

$$= \int\limits_{E_s^S} \frac{\Xi_\varepsilon\, |\nabla\varphi|_g^{\beta-2}\Big[\,(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\Big]}{\sinh\varphi}\, d\mu_g$$

$$+ \int\limits_{\big(U_{\frac{3}{2}\varepsilon} \setminus \overline{U_{\frac{1}{2}\varepsilon}}\big) \cap E_s^S} \frac{\chi_\varepsilon'(\,|\nabla\varphi|_g^2\,)\, |\nabla\varphi|_g^{\beta-2}\,\Big|\nabla|\nabla\varphi|_g^2\Big|_g^2}{2\sinh\varphi}\, d\mu_g\,,$$

where U_μ is defined in (4.16). Note that $\{\chi_\varepsilon\}$ can always be chosen to be nondecreasing in ε so that, in turn, $\{\Xi_\varepsilon\}$ is nondecreasing. Therefore, applying the Monotone Convergence Theorem, when $\varepsilon \to 0^+$, the first term on the right of the second equality tends to

$$\int\limits_{E_s^S} \frac{|\nabla\varphi|_g^{\beta-2}\left[\,(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\right]}{\sinh\varphi}\, d\mu_g\,.$$

For obtaining the desired statement, we show

$$\lim_{\varepsilon\to 0^+} \int\limits_{\left(U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}\right)\cap E_s^S} \frac{\chi_\varepsilon'(\,|\nabla\varphi|_g^2\,)\,|\nabla\varphi|_g^{\beta-2}\left|\nabla|\nabla\varphi|_g^2\right|_g^2}{2\sinh\varphi}\, d\mu_g = 0\,. \tag{4.19}$$

First we observe that

$$\begin{aligned}\int\limits_{\left(U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}\right)\cap E_s^S} \frac{\chi_\varepsilon'(\,|\nabla\varphi|_g^2\,)\,|\nabla\varphi|_g^{\beta-2}\left|\nabla|\nabla\varphi|_g^2\right|_g^2}{2\sinh\varphi}\, d\mu_g &\leq \int\limits_{U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}} \frac{\chi_\varepsilon'(\,|\nabla\varphi|_g^2\,)\,|\nabla\varphi|_g^{\beta-2}\left|\nabla|\nabla\varphi|_g^2\right|_g^2}{2\sinh\varphi}\, d\mu_g \\ &\leq \frac{c}{2\varepsilon}\int\limits_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{\beta-2}{2}}\, ds \int\limits_{\{|\nabla\varphi|_g^2=s\}} \frac{\left|\nabla|\nabla\varphi|_g^2\,\right|_g}{\sinh\varphi}\, d\sigma_g\end{aligned}$$

where, keeping in mind the properties satisfied by χ_ε, in the first inequality we have used the nonnegativity of the integrand function and in the last one the Coarea Formula. Note that there exist ε_0, $c_1 > 0$ such that the inequality

$$\frac{1}{s^{\frac{1}{2}\frac{n}{n-1}}} \int\limits_{\{|\nabla\varphi|_g^2=s\}} \frac{\left|\nabla|\nabla\varphi|_g^2\,\right|_g}{\sinh\varphi}\, d\sigma_g \leq c_1$$

is true a.e. $s \in [\frac{1}{2}\varepsilon, \frac{3}{2}\varepsilon]$ for every $0 < \varepsilon < \frac{2}{3}\varepsilon_0$, by both Sard's Theorem applied to the smooth function $|\nabla\varphi|_g^2$ and by Lemma 4.2. Then, we get

$$\begin{aligned}\int\limits_{\left(U_{\frac{3}{2}\varepsilon}\setminus\overline{U_{\frac{1}{2}\varepsilon}}\right)\cap E_s^S} \frac{\chi_\varepsilon'(\,|\nabla\varphi|_g^2\,)\,|\nabla\varphi|_g^{\beta-2}\left|\nabla|\nabla\varphi|_g^2\right|_g^2}{2\sinh\varphi}\, d\mu_g &\leq \frac{c}{2\varepsilon}\int\limits_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{\beta-2}{2}}\, ds \int\limits_{\{|\nabla\varphi|_g^2=s\}} \frac{\left|\nabla|\nabla\varphi|_g^2\,\right|_g}{\sinh\varphi}\, d\sigma_g \\ &\leq \frac{c\,c_1}{2\varepsilon}\int\limits_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{\beta-2}{2}+\frac{1}{2}\frac{n}{n-1}}\, ds\end{aligned}$$

$$= \frac{c\,c_1}{2\varepsilon} \int\limits_{\frac{1}{2}\varepsilon}^{\frac{3}{2}\varepsilon} s^{\frac{1}{2}(\beta - \frac{n-2}{n-1})}\, ds$$
$$\leq c_2\, \varepsilon^{\frac{1}{2}(\beta - \frac{n-2}{n-1})}\,,$$

where $c_2 > 0$ is sufficiently big constant. This implies the limit in (4.19), because $\beta > \frac{n-2}{n-1}$. □

Corollary 4.4. *For every $S > s > 0$ regular values of φ there exists $r_{s,S} \geq 0$ such that*

$$\int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^{\frac{n-2}{n-1}}\ \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g - \int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^{\frac{n-2}{n-1}}\ \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g$$
$$= \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{-\frac{n}{n-1}} \Big[\, |\nabla^2\varphi|_g^2 - \frac{n}{n-1} \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + Q(\nabla\varphi, \nabla\varphi) \Big]}{\sinh\varphi}\, d\mu_g + r_{s,S}\ .$$

Proof. Let $\{\beta_m\}_{m\in\mathbb{N}}$ be a sequence such that $\beta_m > \frac{n-2}{n-1}$ and $\beta_m \to \frac{n-2}{n-1}$. Due to Proposition 4.3, we have

$$\int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^{\frac{n-2}{n-1}}\ \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g - \int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^{\frac{n-2}{n-1}}\ \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g$$
$$= \lim_{m\to+\infty} \Bigg[\int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^{\beta_m}\ \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g - \int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^{\beta_m}\ \mathrm{H}_g}{\sinh\varphi}\, d\sigma_g \Bigg]$$
$$= \lim_{m\to+\infty} \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{\beta_m-2} \Big[\, (\beta_m - 2) \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi, \nabla\varphi) \Big]}{\sinh\varphi}\, d\mu_g$$
$$\geq \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{-\frac{n}{n-1}} \Big[\, |\nabla^2\varphi|_g^2 - \frac{n}{n-1} \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + Q(\nabla\varphi, \nabla\varphi) \Big]}{\sinh\varphi}\, d\mu_g\,,$$

where the first equality is consequence of the Dominate Converge Theorem keeping in mind that s and S are regular values of φ while the inequality follows from Fatou's Lemma. Since $\{\beta_m\}_{m\in N}$ is arbitrary, the quantity

$$r_{s,S} := \lim_{\beta\to\frac{n-2}{n-1}^+} \Bigg\{ \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{\beta-2} \Big[\, (\beta - 2) \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi, \nabla\varphi) \Big]}{\sinh\varphi}\, d\mu_g \Bigg\}$$
$$- \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{-\frac{n}{n-1}} \Big[\, |\nabla^2\varphi|_g^2 - \frac{n}{n-1} \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + Q(\nabla\varphi, \nabla\varphi) \Big]}{\sinh\varphi}\, d\mu_g$$

is well–defined. Moreover, it is nonnegative as above and therefore we get the statement. □

Remark 4.2. For every $\beta > \frac{n-2}{n-1}$ and for every $s > 0$ regular value of the function φ:

$$\int_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi} \, d\sigma_g = \int_{\{\varphi>s\}} \frac{|\nabla\varphi|_g^{\beta-2} \Big[(\beta-2) \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi) \Big]}{\sinh\varphi} \, d\mu_g \, . \tag{4.20}$$

For every S big enough, which is a regular value of φ, by Lemma 4.5 with (4.6) we have

$$\Bigg| \int_{\{\varphi=S\}} |\nabla\varphi|_g^\beta \, \mathrm{H}_g \, d\sigma_g \Bigg| \leq \int_{\{\varphi=S\}} |\nabla\varphi|_g^{\beta-1} \, |\nabla^2\varphi|_g \, d\sigma_g \, \leq \widetilde{c} \, .$$

In particular,

$$\lim_{S\to+\infty} \frac{1}{\sinh(S)} \int_{\{\varphi=S\}} |\nabla\varphi|_g^\beta \, \mathrm{H}_g \, d\sigma_g = 0 \, .$$

Therefore, the desired identity can be obtained by the Monotone Convergence Theorem, by passing to the limit as $S \to +\infty$ in (4.17).

Remark 4.3. For every $\beta > \frac{n-2}{n-1}$, as consequence of integral identity (4.17), we have

$$|\nabla\varphi|_g^{\beta-2} \Big[(\beta-2) \Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi) \Big] \in L^1_{\mathrm{loc}}(\mathring{M}, \mu_g) \, . \tag{4.21}$$

Since

$$\int_K |\nabla\varphi|_g^{\beta-3} \, |\nabla^2\varphi(\nabla\varphi,\nabla\varphi)| \, d\mu_g \leq \int_K |\nabla\varphi|_g^{\beta-1} \Big|\nabla|\nabla\varphi|_g\Big|_g \, d\mu_g = \int_K |\nabla\varphi|_g^{\frac{\beta}{2}} \, |\nabla\varphi|_g^{\frac{\beta-2}{2}} \Big|\nabla|\nabla\varphi|_g\Big|_g \, d\mu_g$$

for every $K \subset \mathring{M}$ compact, by Hölder's Inequality from (4.21) with (4.14) we get that

$$|\nabla\varphi|_g^{\beta-3} \; \nabla^2\varphi(\nabla\varphi,\nabla\varphi) \in L^1_{\mathrm{loc}}(\mathring{M}, \mu_g) \, . \tag{4.22}$$

We need a final lemma before stating the (last and) most important result of this section.

Lemma 4.5. *Let (M, g_0, u) be a sub-static harmonic triple, and let g and φ be the metric and the function defined in* (3.4)*. Then, the following statements hold true.*

(i) *For every $\beta \geq 0$ and for every $S > s > 0$:*

$$\int_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi} \, d\sigma_g - \int_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi} \, d\sigma_g = \tag{4.23}$$

$$= \int_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{\beta-2} \Big[\beta \, \nabla^2\varphi(\nabla\varphi,\nabla\varphi) \, - \coth(\varphi) |\nabla\varphi|_g^4 \Big]}{\sinh\varphi} \, d\mu_g \, .$$

(*ii*) *For every* $\beta \geq 0$ *and for every* $s > 0$*:*

$$\int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\, d\sigma_g = \int\limits_{\{\varphi>s\}} \frac{|\nabla\varphi|_g^{\beta-2}\Big[\coth(\varphi)|\nabla\varphi|_g^4 - \beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\Big]}{\sinh\varphi}\, d\mu_g\,.$$

(*iii*) *The function* $\Phi_\beta : [0,\infty) \to \mathbb{R}$*, defined by formula*

$$\Phi_\beta(s) := \int\limits_{\{\varphi=s\}} |\nabla\varphi|_g^{\beta+1}\, d\sigma_g \tag{4.24}$$

for every $\beta \geq 0$*, is continuous and admits for every* $s > 0$ *the integral representation*

$$\Phi_\beta(s) := \sinh(s) \int\limits_{\{\varphi>s\}} \frac{|\nabla\varphi|_g^{\beta-2}\Big[\coth(\varphi)|\nabla\varphi|_g^4 - \beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\Big]}{\sinh\varphi}\, d\mu_g\,.$$

This lemma can be proved as [2, Proposition 4.1]. In the Appendix we provide an alternative proof which is self contained and does not make use of any fine property of the measure of $\mathrm{Crit}(\varphi)$: we just need to know very classical properties of it (see Remark 3.1).

Proposition 4.6. *Let* (M, g_0, u) *be a sub-static harmonic triple, let* g *and* φ *be the metric and the function defined in* (3.4)*, and let* $\Phi_\beta : [0,\infty) \to \mathbb{R}$ *be the function defined by formula* (4.24) *for every* $\beta \geq 0$*. Then for every* $\beta > \frac{n-2}{n-1}$*, the function* Φ_β *is continuously differentiable. The derivative* Φ'_β *is nonpositive and admits for every* $s > 0$ *the integral representation*

$$\Phi'_\beta(s) = -\beta\, \sinh(s) \int\limits_{\{\varphi>s\}} \frac{|\nabla\varphi|_g^{\beta-2}\Big[(\beta-2)\Big|\nabla|\nabla\varphi|_g\Big|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\Big]}{\sinh\varphi}\, d\mu_g \leq 0\,. \tag{4.25}$$

Moreover, if there exists $s_0 > 0$ *such that* $\Phi'_\beta(s_0) = 0$ *for some* $\beta > \frac{n-2}{n-1}$*, then* $(\{\varphi \geq s_0\}, g)$ *is isometric to* $([0,+\infty) \times \{\varphi = s_0\}, d\rho \otimes d\rho + g_{\{\varphi=s_0\}})$*, where* ρ *is the* g*–distance function to* $\{\varphi = s_0\}$ *and* φ *is an affine function of* ρ *in* $\{\varphi \geq s_0\}$*.*

The following proof is essentially the same as in [2]. For completeness, we include it here, in a slightly refined version.

Proof. *Step* 1*: Continuous Differentiability and Monotonicity.* Let $\beta > \frac{n-2}{n-1}$. Note that the boundary ∂M is a regular level set of φ and then, by Theorem 2.3 and the relationship between $\mathrm{Crit}(u)$ and $\mathrm{Crit}(\varphi)$, there exists ϵ_0 such that the interval $[0, \epsilon_0]$ doesn't contain critical values of the function φ. Therefore, for every $0 < \epsilon \leq \epsilon_0$, applying first the Divergence Theorem to the smooth vector field $|\nabla\varphi|_g^\beta\, \nabla\varphi$ in $\{0 < \varphi < \epsilon\}$ and later the Coarea Formula, we get

$$\Phi_\beta(\epsilon) = \Phi_\beta(0) - \beta \int\limits_0^\epsilon ds \int\limits_{\{\varphi=s\}} |\nabla\varphi|_g^\beta\, \mathrm{H}_g\, d\sigma_g\,.$$

Being

$$\int\limits_{\{\varphi=s_1\}} |\nabla\varphi|_g^\beta \, \mathrm{H}_g \, d\sigma_g - \int\limits_{\{\varphi=s_2\}} |\nabla\varphi|_g^\beta \, \mathrm{H}_g \, d\sigma_g = \beta^{-1} \int\limits_{\{s_1<\varphi<s_2\}} \mathrm{div}_g(\nabla|\nabla\varphi|_g^\beta) \, d\mu_g$$

for every $0 \leq s_1 < s_2 \leq \epsilon_0$, by the Dominated Convergence Theorem the function

$$s \in [0, \epsilon_0] \to \int\limits_{\{\varphi=s\}} |\nabla\varphi|_g^\beta \, \mathrm{H}_g \, d\sigma_g \in \mathbb{R}$$

is continuous and therefore, by the Fundamental Theorem of Calculus Φ_β is continuously differentiable on the closed interval $[0, \epsilon_0]$.

Let s_0 be a regular value of the function φ. By Remark 4.3, we can define the function $\Psi_\beta : (0, +\infty) \to \mathbb{R}$ by

$$\Psi_\beta(s) = \begin{cases} \displaystyle\int\limits_{\{\varphi=s_0\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi} \, d\sigma_g + \int\limits_{\{s<\varphi<s_0\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[(\beta-2)\left|\nabla|\nabla\varphi|_g\right|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\right]}{\sinh\varphi} \, d\mu_g & \text{if } s \leq s_0 \\ \displaystyle\int\limits_{\{\varphi=s_0\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi} \, d\sigma_g - \int\limits_{\{s_0<\varphi<s\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[(\beta-2)\left|\nabla|\nabla\varphi|_g\right|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\right]}{\sinh\varphi} \, d\mu_g & \text{if } s > s_0\,, \end{cases}$$

which satisfies the following properties

(i) for every $s > 0$ regular value of the function φ, we have $\Psi_\beta(s) = \int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^\beta \, \mathrm{H}_g}{\sinh\varphi} \, d\sigma_g$;

(ii) the function Ψ_β is continuous on its definition interval $(0, +\infty)$.

The first statement follows immediately from Proposition 4.3. As for the second statement, we first observe that

$$\Psi_\beta(s) - \Psi_\beta(\overline{s}) = \int\limits_{\{s<\varphi<\overline{s}\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[(\beta-2)\left|\nabla|\nabla\varphi|_g\right|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\right]}{\sinh\varphi} \, d\mu_g \tag{4.26}$$

for every couple $0 < s < \overline{s} < +\infty$. Always by Remark 4.3 and by the Dominated Convergence Theorem, we can deduce the right and the left continuity of Ψ_β on the interval $(0, +\infty)$.

We consider $\Upsilon_\beta : s \in (0, +\infty) \to \frac{\Phi_\beta(s)}{\sinh s} \in \mathbb{R}$. For every $(s, \overline{s})$ couple of real number such that $0 < s < \overline{s} < +\infty$, we have

$$\begin{aligned} \frac{\Upsilon_\beta(\overline{s}) - \Upsilon_\beta(s)}{\overline{s} - s} &= \frac{1}{\overline{s} - s} \int\limits_{\{s<\varphi<\overline{s}\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[\beta\, \nabla^2\varphi(\nabla\varphi,\nabla\varphi) - \coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi} \, d\mu_g \\ &= \frac{1}{\overline{s} - s} \int_s^{\overline{s}} d\tau \int\limits_{\{\varphi=\tau\}} \frac{|\nabla\varphi|_g^{\beta-3}\left[\beta\, \nabla^2\varphi(\nabla\varphi,\nabla\varphi) - \coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi} \, d\sigma_g \end{aligned}$$

$$\stackrel{(\star)}{=} -\frac{\beta}{\overline{s}-s}\int_s^{\overline{s}} \Psi_\beta(\tau)\,d\tau - \frac{1}{\overline{s}-s}\int_s^{\overline{s}} \coth(\tau)\Upsilon_\beta(\tau)\,d\tau\,,$$

where the first equality follows from Lemma 4.5 (i), the second equality from the Coarea Formula keeping in mind (4.22). Moreover, the last equality follows from (i) and from Sard's Theorem. Using the continuity of both the functions Υ_β and Ψ_β, passing to the limit in ($\star$) for either $s \to \overline{s}$ or $\overline{s} \to s$ yields that the function Υ_β is C^1, and

$$\Upsilon'_\beta(\,\cdot\,) = -\beta\,\Psi_\beta(\,\cdot\,) - \coth(\,\cdot\,)\,\Upsilon_\beta(\,\cdot\,)\,.$$

Since $\Phi_\beta(s) = \sinh(s)\Upsilon_\beta(s)$ for every $s > 0$, then $\Phi_\beta \in C^1(0,+\infty)$ and $\Phi'_\beta(s) = -\beta\sinh(s)\Psi_\beta(s)$. Moreover, by (4.26), we can see

$$\begin{aligned}\frac{\Phi'_\beta(S)}{\sinh(S)} - \frac{\Phi'_\beta(s)}{\sinh(s)} &= -\beta\,\Psi_\beta(S) + \beta\,\Psi_\beta(s)\\ &= \beta\int_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[\,(\beta-2)\left|\nabla|\nabla\varphi|_g\right|_g^2 + |\nabla^2\varphi|_g^2 + Q(\nabla\varphi,\nabla\varphi)\,\right]}{\sinh\varphi}\,d\mu_g\end{aligned}$$

for every $0 < s < S < +\infty$.

Finally the integral representation (4.25) follows in the limit as $S \to +\infty$ of the above identity, by using the Monotone Convergence Theorem, and by the fact that

$$\lim_{S\to+\infty}\frac{\Phi'_\beta(S)}{\sinh(S)} = -\beta\lim_{S\to+\infty}\Psi_\beta(S) = -\beta\lim_{S\to+\infty}\int_{\{\varphi=S\}}\frac{|\nabla\varphi|_g^\beta\,\mathrm{H}_g}{\sinh\varphi}\,d\sigma_g = 0\,.$$

Step 2: Outer Rigidity. Let $\beta > \frac{n-2}{n-1}$ and suppose $\Phi'_\beta(s_0) = 0$ for some $s_0 > 0$. By (4.25) with (4.14) we deduce that

$$\Big(\beta - \frac{n-2}{n-1}\Big)\left|\nabla|\nabla\varphi|_g\right|_g^2 \equiv 0 \quad\text{and}\quad |\nabla^2\varphi|_g^2 - \frac{n}{n-1}\left|\nabla|\nabla\varphi|_g\right|_g^2 \equiv 0 \quad \text{in}\,\{\varphi \geq s_0\}\setminus\mathrm{Crit}(\varphi)\,.$$

Consequently $\nabla^2\varphi \equiv 0$ in $\{\varphi \geq s_0\}$ being $\mu_g(\mathrm{Crit}(\varphi)) = 0$, and hence $|\nabla\varphi|_g^2 \equiv a^2$ with $a > 0$ since $\{\varphi \geq s_0\}$ is connected, due to Remark 3.1. Then, $\{\varphi \geq s_0\}$, with the induced Riemanninan metric, is a noncompact, connected and complete Riemannian manifold (being properly embedded in M), with smooth, compact and totally geodesic boundary, and with $\mathrm{Ric}_g \geq 0$ (from the inequality in (3.5)). Applying [18, Theorem C], we can thus deduce that the level set $\{\varphi = s_0\}$ is connected (this is true in general and not only in the rigid case, if $s_0 \gg 0$, as observed in Remark 3.1), and that $\{\varphi \geq s_0\}$ is isometric to the product $[0,+\infty)\times\{\varphi = s_0\}$. Moreover, the isometry from the product $[0,+\infty)\times\{\varphi = s_0\}$ to $\{\varphi \geq s_0\}$ is given by the normal exponential map.

Now we want to prove that φ is an affine function of ρ on $\{\varphi \geq s_0\}$. First, we remark that every integral curve γ_p of $\nabla\varphi$ outgoing from a point p of $\{\varphi = s_0\}$ is defined on the interval $[0,+\infty)$, and it is contained in $\{\varphi \geq s_0\}$, by the completeness and since $|\nabla\varphi|_g > 0$. Furthermore, $\varphi\circ\gamma_p(t) = a^2 t + s_0$

for every $t \in [0, +\infty)$, and all the curves γ_p realize the distance between the hypersurfaces $\{\varphi = s_0\}$ and $\{\varphi = s_1\}$ with $s_1 > s_0$. Indeed, for any curve $\sigma : [0, l] \to \{\varphi \geq s_0\}$ parametrized by arc–length joining a point of $\{\varphi = s_0\}$ to a point of $\{\varphi = s_1\}$ we have

$$\begin{aligned} L_g(\sigma) &= \int_0^l |\dot\sigma(\tau)|_g \, d\tau \geq \left| \int_0^l g\Big(\dot\sigma(\tau), \frac{1}{a}\nabla\varphi(\sigma(t))\Big) \, d\tau \right| = \frac{1}{a}|\varphi \circ \sigma(l) - \varphi \circ \sigma(0)| \\ &= at = L_g(\gamma_{\sigma(0)}|_{[0,t]}) = L_g(\gamma_\cdot|_{[0,t]}), \end{aligned}$$

where s_1, s_0 and t satisfy $s_1 = a^2 t + s_0$. Since $\xi = \frac{\nabla\varphi}{a}$ is the unit inner normal vector field of the boundary $\{\varphi = s_0\}$ and we just know that the normal exponential map is a diffeomorphism, $\exp^\perp(t\xi_p)$ is a point having distance from $\{\varphi = s_0\}$ equal to t, and therefore

$$\varphi(\exp^\perp(t\xi_p)) = \varphi \circ \gamma_p\Big(\frac{t}{a}\Big) = at + s_0 = a\,\rho(\exp^\perp(t\xi_p)) + s_0\,.$$

This tell us that φ is an affine function of ρ on $\{\varphi \geq s_0\}$. □

While the previous proposition contains an outer rigidity result, with the following corollary we provide a global rigidity result.

Corollary 4.7. *Let (M, g_0, u) be a sub-static harmonic triple, let g and φ be the metric and the function defined in* (3.4)*, and let $\Phi_\beta : [0, \infty) \to \mathbb{R}$ be the function defined by formula* (4.24) *for every $\beta \geq 0$. If Φ_β is constant for some $\beta > \frac{n-2}{n-1}$, then ∂M is connected and (M, g) is isometric to* $([0, +\infty) \times \partial M, d\rho \otimes d\rho + g_{\partial M})$*, where ρ is the g–distance function to ∂M and φ is an affine function of ρ.*

Proof. We obtain immediately that $\Phi'_\beta(s) = 0$ for every $s > 0$. Thus, by formula (4.25) with (4.14) we have that

$$\int_{\{\varphi > s\}} \left\{ \Big(\beta - \frac{n-2}{n-1}\Big) \big|\nabla|\nabla\varphi|_g\big|_g^2 + \left[|\nabla^2\varphi|_g^2 - \frac{n}{n-1}\big|\nabla|\nabla\varphi|_g\big|_g^2 \right] + Q(\nabla\varphi, \nabla\varphi) \right\} d\mu_g = 0$$

for every $s > 0$. In turn, by the Monotone Convergence Theorem, we get

$$\int_M \left\{ \Big(\beta - \frac{n-2}{n-1}\Big) \big|\nabla|\nabla\varphi|_g\big|_g^2 + \left[|\nabla^2\varphi|_g^2 - \frac{n}{n-1}\big|\nabla|\nabla\varphi|_g\big|_g^2 \right] + Q(\nabla\varphi, \nabla\varphi) \right\} d\mu_g = 0\,.$$

Then, we deduce that

$$\Big(\beta - \frac{n-2}{n-1}\Big) \big|\nabla|\nabla\varphi|_g\big|_g^2 \equiv 0 \quad \text{and} \quad |\nabla^2\varphi|_g^2 - \frac{n}{n-1}\big|\nabla|\nabla\varphi|_g\big|_g^2 \equiv 0 \quad \text{in } M \setminus \mathrm{Crit}(\varphi)\,,$$

due to Kato Inequality for harmonic functions (4.13) and by (4.10). Consequently $\nabla^2\varphi \equiv 0$ in M. The very same argument of the proof of Outer Rigidity in Proposition 4.6 implies that ∂M is connected and (M, g) is isometric to

$$([0, +\infty) \times \partial M, d\rho \otimes d\rho + g_{\partial M})$$

where ρ is the g–distance to ∂M and φ is an affine function of ρ. □

5. A Black–Hole uniqueness theorem for sub-static manifolds

This section is devoted to the proof of the Black–Hole uniqueness result for a sub–static harmonic triple, Theorem 1.2. We first recall the classical definition of ADM mass, together with an alternative characterization of it.

Let (N, h) and ψ be an asymptotically flat manifold with one end and a chart at infinity of N, respectively. We consider $\widetilde{h} := \psi_* h = \widetilde{h}_{ij} dx^i \otimes dx^j$ and we set

$$m(r) = \frac{1}{2(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\partial B_r} (\partial_j \widetilde{h}_{ij} - \partial_i \widetilde{h}_{jj}) \nu_e^i d\sigma_e \,,$$

$$m_I(r) = -\frac{1}{(n-2)(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\partial B_r} \Big(\mathrm{Ric}_{\widetilde{h}} - \frac{1}{2}\mathrm{R}_{\widetilde{h}}\widetilde{h}\Big)(X, \nu_{\widetilde{h}})\, d\sigma_{\widetilde{h}} \,,$$

where ν_e and σ_e are the ∞–pointing unit normal and the canonical measure on ∂B_r as Riemannian submanifold of $(\mathbb{R}^n \setminus \overline{B}, g_{\mathbb{R}^n})$, respectively, and $\nu_{\widetilde{h}}$ and $\sigma_{\widetilde{h}}$ are the ∞–pointing unit normal and the canonical measure on ∂B_r as Riemannian submanifold of $(\mathbb{R}^n \setminus \overline{B}, \widetilde{h})$, respectively. Also, $\mathrm{Ric}_{\widetilde{h}}$ and $\mathrm{R}_{\widetilde{h}}$ are the Ricci tensor and the scalar curvature of $\widetilde{h}$ respectively, and X is the Euclidean conformal Killing vector field $x^i \frac{\partial}{\partial x^i}$. The ADM mass is well defined as

$$m_{\mathrm{ADM}} := \lim_{r \to +\infty} m(r) \,,$$

and independent of the chosen chart at infinity. Moreover (see [22]), it can be equivalently expressed as

$$m_{\mathrm{ADM}} = \lim_{r \to +\infty} m_I(r) \,. \tag{5.1}$$

From the alternative definition of ADM mass, given by (5.1), and using the Positive Mass Theorem, more precisely a consequence of it contained in [17, Theorem 1.5], one can prove the following uniqueness statement. For the notation and terminology, we refer the reader to Definition 1.1 and Section 2.

Proof of Theorem 1.2. By condition (1.8) and by the fact that $\mathrm{D}^2_{g_0} u \equiv 0$ on ∂M, which in turn implies $\mathrm{H}^{g_0}_{\partial M} \equiv 0$, we have that the hypothesis of [17, Theorem 1.5] are fulfilled, so that

$$m_{\mathrm{ADM}} \geq C.$$

Now, we want to show that the reverse inequality holds. Let ψ be a chart at infinity of M (according to Definition 1.1) and consider $\widetilde{g}_0 = \psi_* g_0$. Recalling that $\widetilde{u}$ stands for $u \circ \psi^{-1}$, we rewrite characterization (5.1) as

$$m_{\mathrm{ADM}} = \lim_{r \to +\infty} \Bigg\{ -\frac{1}{(n-2)(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\partial B_r} \Bigg(\mathrm{Ric}_{\widetilde{g}_0} - \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}}\Bigg)(X, \nu_e)\, d\sigma_{\widetilde{g}_0}$$

$$- \frac{1}{(n-2)(n-1)|\mathbb{S}^{n-1}|} \int\limits_{\partial B_r} \Bigg(\mathrm{Ric}_{\widetilde{g}_0} - \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}}\Bigg)(X, \nu_{\widetilde{g}_0} - \nu_e)\, d\sigma_{\widetilde{g}_0}$$

$$- \frac{1}{(n-2)(n-1)|\mathbb{S}^{n-1}|} \int_{\partial B_r} \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}}(X, \nu_{\widetilde{g}_0})\, d\sigma_{\widetilde{g}_0} + \frac{1}{2(n-2)(n-1)|\mathbb{S}^{n-1}|} \int_{\partial B_r} \mathrm{R}_{\widetilde{g}_0}\, \widetilde{g}_0(X, \nu_{\widetilde{g}_0})\, d\sigma_{\widetilde{g}_0} \Big\} .$$

We note first that since $\nu_e = \frac{x^i}{|x|} \frac{\partial}{\partial x^i} = \frac{1}{|x|} X$ and $\widetilde{u}\,\mathrm{Ric}_{\widetilde{g}_0} - \mathrm{D}^2_{\widetilde{g}_0} \widetilde{u} \geq 0$ from the first equation in (1.1), we have

$$\int_{\partial B_r} \left(\mathrm{Ric}_{\widetilde{g}_0} - \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}} \right)(X, \nu_e)\, d\sigma_{\widetilde{g}_0} = \frac{1}{r} \int_{\partial B_r} \left(\mathrm{Ric}_{\widetilde{g}_0} - \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}} \right)(X, X)\, d\sigma_{\widetilde{g}_0} \geq 0 . \tag{5.2}$$

Secondly, recalling that $(\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u})_{ij} = \partial_i \partial_j \widetilde{u} - \Gamma^k_{ij} \partial_k \widetilde{u}$, where Γ^k_{ij} are the Christoffel symbols related to $\widetilde{g}_0$, and using (1.3), (2.12), and the asymptotic expansions of $\widetilde{u}$, we get

$$|(\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u})_{ij} - (\mathrm{D}^2_e \widetilde{u})_{ij}| = |\Gamma^k_{ij} \partial_k \widetilde{u}| = O(|x|^{-(n+p)}) \tag{5.3}$$

$$(\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u})_{ij} = O(|x|^{-n}) . \tag{5.4}$$

Decay (5.4) coupled with (2.8) (2.10) and (2.11) yields

$$\Big| \int_{\partial B_r} \left(\mathrm{Ric}_{\widetilde{g}_0} - \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}} \right)(X, \nu_{\widetilde{g}_0} - \nu_e)\, d\sigma_{\widetilde{g}_0} \Big| \leq C \int_{\partial B_r} \frac{1}{|x|^{p+\min\{p+2,n\}-1}}\, d\sigma_e = \frac{C}{r^{p+\min\{p+2,n\}-n}} \to 0 , \tag{5.5}$$

being $p > \frac{n-2}{2}$. Thirdly, we observe that

$$\int_{\partial B_r} \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}}(X, \nu_{\widetilde{g}_0})\, d\sigma_{\widetilde{g}_0} \xrightarrow[r \to +\infty]{} -(n-1)(n-2)\, C |\mathbb{S}^{n-1}| . \tag{5.6}$$

Indeed

$$\begin{aligned} \int_{\partial B_r} \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}}(X, \nu_{\widetilde{g}_0})\, d\sigma_{\widetilde{g}_0} = & \int_{\partial B_r} \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u}}{\widetilde{u}}(X, \nu_{\widetilde{g}_0} - \nu_e)\, d\sigma_{\widetilde{g}_0} + \int_{\partial B_r} \frac{\mathrm{D}^2_{\widetilde{g}_0} \widetilde{u} - \mathrm{D}^2_e \widetilde{u}}{\widetilde{u}}(X, \nu_e)\, d\sigma_{\widetilde{g}_0} \\ & + \int_{\partial B_r} \frac{\mathrm{D}^2_e \widetilde{u}}{\widetilde{u}}(X, \nu_e)\, d\sigma_{\widetilde{g}_0} , \end{aligned}$$

and one can show, with similar estimates as before, that the first two terms of this sum tend to 0 for $r \to +\infty$. It is also easy to see, using (2.5) and (2.11), that

$$\int_{\partial B_r} \frac{\mathrm{D}^2_e \widetilde{u}}{\widetilde{u}}(X, \nu_e)\, d\sigma_{\widetilde{g}_0} \xrightarrow[r \to +\infty]{} -(n-1)(n-2)\, C |\mathbb{S}^{n-1}| .$$

Hence, (5.6) is proven. Gathering (5.2) , (5.5), and (5.6), we have finally obtained

$$m_{\mathrm{ADM}} \leq C + \limsup_{r \to +\infty} \frac{1}{2(n-2)(n-1)|\mathbb{S}^{n-1}|} \int_{\partial B_r} \mathrm{R}_{\widetilde{g}_0}\, g(X, \nu_{\widetilde{g}_0})\, d\sigma_{\widetilde{g}_0} . \tag{5.7}$$

We remark that the above inequality is true for any ψ chart at infinity of M. From now on we assume that ψ satisfies condition (1.8) regarding the decay rate of $\mathrm{R}_{\widetilde{g}_0}$ at ∞. Since

$$\begin{aligned}\widetilde{g}_0(X, \nu_{\widetilde{g}_0}) &= \widetilde{g}_{0;ij} X^i \nu^j_{\widetilde{g}_0} = (\,\delta_{ij} + O(|x|^{-p})\,) X^i (\nu^j_{\widetilde{g}_0} \pm \nu^j_e) \\ &= g_{\mathbb{R}^n}(X, \nu_e) + O(|x|^{-p+1}) = |x| + O(|x|^{-p+1})\,,\end{aligned}$$

also using (2.11) we obtain

$$\Big|\int_{\partial B_r} \mathrm{R}_{\widetilde{g}_0}\, \widetilde{g}_0(X, \nu_{\widetilde{g}_0})\, d\sigma_{\widetilde{g}_0}\Big| \leq C \int_{\partial B_r} r^{-q}(\,r + O(r^{-p+1})\,)\, d\sigma_e \leq C r^{-q+n} \xrightarrow[r\to+\infty]{} 0\,. \tag{5.8}$$

The fact that $m_{\mathrm{ADM}} \leq C$ thus follows from (5.7). All in all, the rigidity case $m_{\mathrm{ADM}} = C$ of [17, Theorem 1.5] holds, which implies that (M, g_0) is the Schwarzschild manifold. □

Acknowledgments

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA), which is part of the Istituto Nazionale di Alta Matematica (INdAM), and they are partially funded by the GNAMPA project "Aspetti geometrici in teoria del potenziale lineare e nonlineare". F. O. thanks L. Benatti and C. Mantegazza for useful discussions during the preparation of the manuscript.

Conflict of interest

The authors declare no conflict of interest.

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A. Appendix

In this appendix we provide a proof of Lemma 4.5 which is alternative and more self contained than the corresponding in [2]. We underline that we will use Remark 3.1 widely.

Proof of Lemma 4.5 (i). In $\mathring{M} \setminus \mathrm{Crit}(\varphi)$ and for every $\beta \geq 0$, we consider the smooth vector field

$$X_\beta := \frac{|\nabla\varphi|_g^\beta \, \nabla\varphi}{\sinh\varphi}\,,$$

which is such that

$$\mathrm{div}_g\, X_\beta = \frac{|\nabla\varphi|_g^{\beta-2}\left[\beta\, \nabla^2\varphi(\nabla\varphi,\nabla\varphi) \,-\coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi}$$

If $\{s \leq \varphi \leq S\} \cap \mathrm{Crit}(\varphi) = \emptyset$, then the statement is a straightforward application of the Divergence Theorem. Now, suppose that $\{s \leq \varphi \leq S\} \cap \mathrm{Crit}(\varphi) \neq \emptyset$. Since there always exists $\overline{s} \in (s, S)$ regular value of φ, up to splitting the right–hand side of (4.23) into two subintegrals, we can suppose without loss of generality that one among s and S is a regular value of φ. To fix the ideas, suppose that S is the regular value. We are going to change the function φ in a neighbourhood of the set $\mathrm{Crit}(\varphi)$. To do this, for every $\varepsilon > 0$ sufficiently small, applying Sard's Theorem to the smooth function φ, we can fix a positive real number $\delta(\varepsilon)$ such that $s + \delta(\varepsilon) < S$ is a regular value of φ and $\delta(\varepsilon) < d\,\varepsilon$, where $d > 0$ will be specified later. Then, considering a smooth nonincreasing cut–off function $\zeta_\varepsilon : [0,+\infty) \to [0,1]$ satisfying the conditions

$$\zeta_\varepsilon(\tau) = 1 \quad \text{in} \left[0, \frac{1}{2}\varepsilon\right], \quad |\zeta_\varepsilon'(\tau)| \leq \frac{c}{\varepsilon} \quad \text{in} \left[\frac{1}{2}\varepsilon, \frac{3}{2}\varepsilon\right], \quad \zeta_\varepsilon(\tau) = 0 \quad \text{in} \left[\frac{3}{2}\varepsilon, +\infty\right), \tag{A.1}$$

where c is a positive real constant independent of ε, we define

$$\varphi_\varepsilon := \varphi - \zeta_\varepsilon(|\nabla\varphi|_g^2)\,\delta(\varepsilon)\,.$$

Clearly,

$$\nabla\varphi_\varepsilon = \nabla\varphi - \delta(\varepsilon)\,\zeta_\varepsilon'(\,|\nabla\varphi|_g^2\,)\,\nabla|\nabla\varphi|_g^2, \tag{A.2}$$

and

$$\varphi = \varphi_\varepsilon \ \text{in} \left\{|\nabla\varphi|_g^2 \geq \frac{3}{2}\varepsilon\right\}. \tag{A.3}$$

Note that s is a regular value for the function φ_ε. To see this, let p be a point of $\{\varphi_\varepsilon = s\}$ and distinguish the two cases

$$|\nabla\varphi|_g^2\,(p) \leq \frac{1}{2}\varepsilon\,; \qquad\qquad |\nabla\varphi|_g^2\,(p) \overset{(\star)}{>} \frac{1}{2}\varepsilon.$$

In the first case, $\zeta_\varepsilon(|\nabla\varphi|_g^2) \equiv 1$ so that $s = \varphi_\varepsilon(p) = \varphi(p) - \delta(\varepsilon)$ and $\nabla\varphi_\varepsilon(p) = \nabla\varphi(p)$. Since $s + \delta(\varepsilon)$ is a regular value for φ, $\nabla\varphi_\varepsilon(p) \neq 0$. In the second case, observing that $s \leq \varphi(p) \leq s + \delta(\varepsilon)$ and therefore $p \in \{s \leq \varphi \leq S\}$, we have from (A.2) that in p

$$|\nabla\varphi_\varepsilon|_g \geq |\nabla\varphi|_g - \delta(\varepsilon)\,|\zeta_\varepsilon'|(\,|\nabla\varphi|_g^2\,)\big|\nabla|\nabla\varphi|_g^2\big|_g = |\nabla\varphi|_g\Big(1 - 2\delta(\varepsilon)\,|\zeta_\varepsilon'|(\,|\nabla\varphi|_g^2\,)\big|\nabla|\nabla\varphi|_g\big|_g\Big)$$

$$\geq |\nabla\varphi|_g\Big(1 - 2d\,\varepsilon\,\frac{c}{\varepsilon}\max_{\{s\leq\varphi\leq S\}}\Big|\nabla|\nabla\varphi|_g\Big|_g\Big),$$

where c is the constant appearing in (A.1). Now, observe that $\max_{\{s\leq\varphi\leq S\}}\Big|\nabla|\nabla\varphi|_g\Big|_g > 0$, since otherwise, due to the presence of critical points in $\{s \leq \varphi \leq S\}$, there should be a connected component of $\{s \leq \varphi \leq S\}$ where $\nabla\varphi \equiv 0$. But this is impossible because $\{s \leq \varphi \leq S\} = \overline{\{s < \varphi < S\}}$ (by Remark 2.1) and by the size of $\mathrm{Crit}(\varphi)$. Hence, choosing

$$d \leq \frac{1}{4\,c\max_{\{s\leq\varphi\leq S\}}\Big|\nabla|\nabla\varphi|_g\Big|_g},$$

from above we obtain $|\nabla\varphi_\varepsilon|_g(p) \geq \frac{|\nabla\varphi|_g}{2}(p)$. In particular, from $(\bigstar)$ we get that $|\nabla\varphi_\varepsilon|_g(p) > \frac{\varepsilon}{4}$.
Now, we apply the Divergence Theorem to the smooth vector field $\Xi_{4\varepsilon}X_\beta$ on $\{s < \varphi_\varepsilon < S\}$, where

$$\Xi_\varepsilon := 1 - \zeta_\varepsilon(|\nabla\varphi|_g^2).$$

Recalling that U_μ is defined as in (4.16), we obtain

$$\begin{aligned}
\int\limits_{\{\varphi_\varepsilon=S\}} g\Big(\Xi_{4\varepsilon}X_\beta, \frac{\nabla\varphi_\varepsilon}{|\nabla\varphi_\varepsilon|_g}\Big)d\sigma_g &- \int\limits_{\{\varphi_\varepsilon=s\}} g\Big(\Xi_{4\varepsilon}X_\beta, \frac{\nabla\varphi_\varepsilon}{|\nabla\varphi_\varepsilon|_g}\Big)\,d\sigma_g \\
&= \int\limits_{\{s<\varphi_\varepsilon<S\}} \Xi_{4\varepsilon}\frac{|\nabla\varphi|_g^{\beta-2}\left[\beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi) - \coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi}\,d\mu_g \\
&\quad - 2\int\limits_{\left(U_{6\varepsilon}\setminus\overline{U_{2\varepsilon}}\right)\cap\{s<\varphi_\varepsilon<S\}} \frac{\zeta'_{4\varepsilon}(\,|\nabla\varphi|_g^2\,)\,|\nabla\varphi|_g^\beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)}{\sinh\varphi}\,d\mu_g\,.
\end{aligned}$$

Note that $\{\varphi = S\}$ is compactly contained in $\{|\nabla\varphi|_g^2 > \frac{3}{2}\,\varepsilon\}$ for every ε sufficiently small, and $\Xi_{4\varepsilon} \equiv 0$ in $\{|\nabla\varphi|_g^2 \leq 2\,\varepsilon\} \supset \{|\nabla\varphi|_g^2 \leq \frac{3}{2}\,\varepsilon\}$. Then, by (A.3) we get

$$\begin{aligned}
\int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\,d\sigma_g &- \int\limits_{\left\{\varphi=s,|\nabla\varphi|_g^2\geq\frac{3}{2}\varepsilon\right\}} \Xi_{4\varepsilon}\,\frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\,d\sigma_g \\
&= \int\limits_{\{s<\varphi<S\}} \Xi_{4\varepsilon}\,\frac{|\nabla\varphi|_g^{\beta-2}\left[\beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi) - \coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi}\,d\mu_g \\
&\quad - 2\int\limits_{\left(U_{6\varepsilon}\setminus\overline{U_{2\varepsilon}}\right)\cap\{s<\varphi<S\}} \frac{\zeta'_{4\varepsilon}(\,|\nabla\varphi|_g^2\,)\,|\nabla\varphi|_g^\beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)}{\sinh\varphi}\,d\mu_g\,.
\end{aligned} \tag{A.4}$$

Looking at the left–hand side of (A.4), note that

$$\left|\int\limits_{\left(U_{6\varepsilon}\setminus\overline{U_{2\varepsilon}}\right)\cap\{s<\varphi<S\}} \zeta'_{4\varepsilon}(\,|\nabla\varphi|_g^2\,)|\nabla\varphi|_g^\beta\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\,d\mu_g\right| \leq \frac{c}{4\,\varepsilon}\int\limits_{U_{6\varepsilon}} |\nabla\varphi|_g^{\beta+2}|\nabla^2\varphi|_g\,d\mu_g$$

$$\leq C\frac{\varepsilon^{\frac{\beta}{2}+1}}{\varepsilon}\mu_g(U_{6\varepsilon}) \to 0\,,$$

where in the second inequality we have used Lemma 4.1 and the fact that U_ε is contained in a compact set for every $\varepsilon << 1$ (which is a consequence of (4.6)). Moreover, by the Dominated Convergence Theorem, we have that

$$\lim_{\varepsilon\to 0^+}\int\limits_{\{s<\varphi<S\}} \Xi_{4\varepsilon}\,\frac{|\nabla\varphi|_g^{\beta-2}\left[\beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\;-\coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi}\,d\mu_g$$
$$= \int\limits_{\{s<\varphi<S\}} \frac{|\nabla\varphi|_g^{\beta-2}\left[\beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\;-\coth(\varphi)|\nabla\varphi|_g^4\right]}{\sinh\varphi}\,d\mu_g\,.$$

Finally, note that $\{\zeta_\varepsilon\}$ can always be chosen to be nonincreasing in ε so that, in turn, $\{\Xi_\varepsilon\}$ in nondecreasing. Therefore, looking at the left–hand side of (A.4), we have that

$$\lim_{\varepsilon\to 0^+}\int\limits_{\left\{\varphi=s,|\nabla\varphi|_g^2\geq\frac{3}{2}\varepsilon\right\}} \Xi_{4\varepsilon}\,\frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\,d\sigma_g = \lim_{\varepsilon\to 0^+}\int\limits_{\{\varphi=s\}} \Xi_{4\varepsilon}\,\frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\,d\sigma_g$$
$$= \int\limits_{\{\varphi=s\}} \frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\,d\sigma_g\,,$$

by the Monotone Convergence Theorem. All in all, passing to the limit as $\varepsilon \to 0^+$ in (A.4), yields the desired identity. □

Proof of Lemma 4.5 (ii). Lemma 4.1 implies

$$\lim_{S\to+\infty}\int\limits_{\{\varphi=S\}} \frac{|\nabla\varphi|_g^{\beta+1}}{\sinh\varphi}\,d\sigma_g = 0\,.$$

Note that

$$\frac{|\nabla\varphi|_g^{\beta-2}\left[\;\coth(\varphi)|\nabla\varphi|_g^4 - \beta\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\,\right]}{\sinh\varphi} \in L^1\Big(\{\varphi\geq s\};\mu_g\Big)$$

because its absolute value belongs to $L^1_{\mathrm{loc}}(\{\varphi \geq s\},\mu_g)$ immediately and to $L^1(\{\varphi \geq S\},\mu_g)$ for S sufficiently big, applying the Coarea Formula coupled with (4.6) and Lemma 4.5. Therefore, passing to the limit as $S \to +\infty$ in (4.23) and using the Dominated Convergence Theorem gives the desired identity.

□

Proof of Lemma 4.5 (iii). Let $\beta \geq 0$. We are assuming that the boundary ∂M is a regular level set of φ so that there exists $\epsilon > 0$ such that $[0,\epsilon] \cap \mathrm{Crit}(\varphi) = \emptyset$. Therefore, applying the Divergence Theorem to the smooth vector field $|\nabla\varphi|_g^\beta\,\nabla\varphi$ in $\{0 < \varphi < \epsilon\}$ yields

$$\Phi_\beta(\epsilon) - \Phi_\beta(0) = \int\limits_{\{0<\varphi<\epsilon\}} \beta\,|\nabla\varphi|_g^{\beta-2}\,\nabla^2\varphi(\nabla\varphi,\nabla\varphi)\,d\mu_g\,.$$

In turn, the absolute continuity of the integral implies the continuity of Φ_β at 0. By point (i) and again by the absolute continuity of the integral, we obtain the right and the left continuity of the function

$$\Upsilon_\beta : s \in (0, +\infty) \to \frac{\Phi_\beta(s)}{\sinh s} \in \mathbb{R}\,.$$

Hence, Φ_β is continuous also in $(0, +\infty)$. The integral representation of Φ_β follows directly from point (ii). $\square$