Vacuum decay and quadratic gravity

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Abstract

Metastable states are classically stable at zero temperature but can decay due to quantum tunneling. The rate of this process is exponentially small and it may be computed in Euclidean space in the Coleman-de Luccia formalism. The exponential suppression is determined by the Euclidean action computed on a trajectory with definite boundary conditions, known as Coleman-de Luccia instanton, or bounce. In some theories, the bounce may not exist or its on-shell action may be ill-defined or infinite, thus hindering the vacuum decay process.
The issue of vacuum stability is, in fact, not just speculation: the Standard Model vacuum state is itself metastable. The Higgs field may tunnel outside its potential well, with catastrophic consequences for all observers. Luckily, the typical lifetime of such a state is predicted to be very long. Still, unknown high energy physics can change it by several orders of magnitude, and particle physics theories as well as cosmological models that predict large decay rates are ruled out thanks to the anthropic principle. Moreover, gravitational effects play an important role in this process, especially in the early Universe. It is thus important to examine in detail vacuum decay phenomena in gravitational settings and to keep the underlying field theory as general as possible.

This thesis aims at exploring existence conditions for the Coleman-de Luccia instanton in gravitational settings. The first two chapters are dedicated to outlining the basic formalism and describing preexisting results about vacuum decay in cosmology. The Euclidean path integral approach for decay rate calculations, which was first discussed by Callan and Coleman, is introduced in Chapter 1. A quantum mechanical description of the problem is formulated and then extended to field theory. A detailed analysis of bounce calculations and their physical interpretation as bubbles of true vacuum follows. The Higgs field stability within the Standard Model is also addressed. Gravitational effects on the vacuum decay process are considered in Chapter 2, by focusing on the decay from Minkowski and de Sitter space, as they have important cosmological consequences respectively in the current Universe (due to the smallness of the cosmological constant) and at early times. The implications on Higgs decay are discussed in both settings. The last two chapters are dedicated to new results. Vacuum decay in field theories with a scalar field and quadratic gravity is investigated. An Einstein-Hilbert term, a non-minimal coupling, and a quadratic Ricci scalar are considered while keeping the scalar field potential general. The focus is on decay from Minkowski and de Sitter space, due to their importance in cosmology. Scalar fields with Einstein-Hilbert gravity are discussed in Chapter 3, by showing that the bounce at large Euclidean radii has an analytical form that is almost entirely independent of the potential, which is called the "asymptotic bounce". Bounds on the Hubble parameter in the de Sitter case are also explored, by giving an analytical explanation to numerical evidence present in the literature. These properties are used, in Chapter 4, to test for stabilization of the false vacuum state in quadratic gravity. Conclusions follow.

Contents

Introduction

The advent of quantum mechanics allowed scientists to open the door to the world of the very small. Phenomena that were left unexplained by classical physics - unless assuming superluminal motion - found their interpretation in this new theory, which replaces continuum with discrete, and determinism with probability. Who encounters quantum physics for the first time recognizes that most of their everyday experience has nothing to do with the quantum regime. Anyway, the macroscopic world arises from quantum mechanics in the large scale limit, similarly as thermodynamics arise from statistical physics [1, 2]. Nonetheless, it predicts also new phenomena, such as quantum tunneling. Even the smallest quantum correction to a classical theory could entail tunneling, which, thus, is a non-perturbative phenomenon. The underlying reason is that wave functions can propagate through a potential barrier and thus a particle may be measured on either side of it with some finite probability, which depends on the barrier width and height and on the energy of the system. Even if the particle is in a vacuum state, such amplitude is non-vanishing. If the region beyond the potential barrier has lower energy than the vacuum (which is, thus, a *false* vacuum), the wave function centered in such state penetrates through the barrier and its tails grow in time, thus predicting an increasingly larger probability to detect the particle outside the well (see Fig.1.2 or [3] and references therein). The quantum instability of such classically stable configurations is called metastability: the typical time scale on which it occurs must be compared with the other ones of the system, in order to determine whether quantum tunneling might be observed or not. Much of our intuition of this phenomenon comes from one-dimensional quantum mechanics and quantum tunneling calculations and experiments trace back to the early '900 [4–6]. In this way, George Gamow was able to give a mathematical explanation to the decay of alpha nuclei, which was observed some years earlier [4, 5]. Solving the Schroedinger equation in the WKB approximation, it may be found that, in the long time limit, such decay is exponentially suppressed [7]. Its multidimensional generalization was formulated by Banks, Bender, and Wu [8, 9], according to whom the total tunneling amplitude depends on all possible escape paths, and the dominant contribution comes from the most probable one. A further improvement was provided by Coleman and Callan in the '70s [10, 11] and relies on the Euclidean path integral formulation, which ensures a straightforward generalization to field theory. Their work allows determining the decay rate of a false vacuum state as a combination of the Euclidean action computed on a specific trajectory, the Coleman-de Luccia instanton (also known as bounce), and its fluctuations around it. The former, in particular, determines the tunneling exponent, i.e. the argument of the exponential in the decay rate formula, thus controlling strong enhancements or suppressions of the false vacuum lifetime. As the equations of motion are highly non-linear, the Coleman-de Luccia instanton is usually found with numerical methods [12–21], despite some analytical solutions exist [22–28]. If the energy difference among the false vacuum state and the true vacuum is sufficiently small (the so-called thin-wall approximation [10, 22, 24, 28–32]) one can estimate the solution as a function of a parameter, which is usually referred to as the bounce radius. Minimizing the action with respect to such radius allows finding the tunneling exponent. The fluctuation determinant instead is connected to the interpretation of the decay rate as an imaginary part of the energy and is usually computed with the steepest descent method. The physical interpretation of the tunneling process in field theory corresponds to the nucleation of a bubble of the true vacuum phase in some region of spacetime. Bubbles are nucleated with a frequency given by the decay rate and then they fastly expand, possibly colliding with each other (there is a rich literature about bubble dynamics, as their collisions have important physical implications [33–39]). The Callan and Coleman formulation of the problem allows also easily including gravitational effects, which were seen to produce quenching of the false vacuum state under some circumstances. This result is particularly important for cosmological calculations, in which one computes the decay rate of a scalar field theory interacting with gravity.

False vacuum decay is not just speculation and, in fact, it is observed in many different areas in physics, from condensed matter to nuclear physics to cosmology. Last but not least, our currently accepted particle theory, the Standard Model, contains such a feature. In fact, the Higgs quartic coupling runs to negative values at high energies, and so our vacuum state is actually metastable [40–44]. The decay of the Higgs field would completely change the interactions that we experience, having catastrophic consequences for us. Luckily, its lifetime is much longer than the age of the Universe. Nonetheless, high energy physics in the form of new fields or Planck suppressed interactions may change it by several orders of magnitude [3, 45–48]. In particular, gravity should be important at high energies, and thus it needs to be accounted for in the calculation of the decay rate [46, 48–50], most importantly in strong gravity regimes, such as during inflation [51–53]. It has also been shown that inhomogeneities in the form of black holes may substantially increase it [54–58]. False vacuum stability can thus constrain viable particle physics theories and cosmological models, as they may predict a large decay rate, which is inconsistent with the anthropic principle.

This thesis aims at exploring existence conditions for the Coleman-de Luccia instanton in gravitational settings, focussing in particular on real scalar fields interacting with modified gravity. As a starting point, single scalar field theories on a fixed, flat background are considered. It can be found that the bounce has an analytical solution at large Euclidean radii, i.e. when it approaches the false vacuum, that was named "the asymptotic bounce". The method to find the asymptotic bounce relies on its very definition as a limiting undershoot trajectory, as described in [10]. The result is largely independent of the potential, apart from the mass term $m^2\phi^2$ and the cubic self-coupling $g\phi^3$. Massless scalar fields with small cubic interactions have a potential independent power-like behavior, while massive scalar fields are exponentially damped, as previously predicted by [59]. Such result is extended to include also Einstein-Hilbert gravity and modified gravity, in the form of a non-minimal coupling and a quadratic Ricci scalar, while keeping the scalar field potential general. This opens up a new way to test the viability of vacuum decay through the bounce. In fact, in this way, one can verify whether

1. the equations of motion have a solution such that all fields approach the false vacuum at spatial and temporal infinity;

2. this solution has well-defined and finite on-shell action.

This method, though, is not effective for scalar fields on de Sitter backgrounds, as spacetime in that case is compact. For this reason, another existence condition, specific of decay from de Sitter space, is also considered. This relates the shape of the scalar field potential to the Hubble parameter, resulting in a bound on the cosmological constant such that quantum tunneling phenomena are ruled out [60–68]. Literature on this topic is reviewed and improved, by giving analytical explanation to numerical evidence on such bound. This result is also extended to modified gravity. Compactness of de Sitter space provides also another constraint on decay that is specific of quadratic theories of gravity. This is related to having a constant Ricci scalar when the scalar field probes the false vacuum, while it is also a propagating degree of freedom.

This thesis is organized as follows. In Chapter 1, the basic formalism for decay rate calculations is introduced by first discussing the one-dimensional quantum mechanics case, in order to provide the physical interpretation of the phenomenon, as well as the limitations of this approach. Then the Coleman-de Luccia formalism is discussed, focussing in particular on the bounce, which is the main ingredient of calculations in the following chapters. The role of imaginary energy contributions is considered, along with calculations needed to isolate it. The physical interpretation of quantum tunneling as bubble nucleation is given, as well as a discussion on Higgs decay within the Standard Model. In Chapter 2, gravitational effects on vacuum decay are considered, according to their early description [69], and also more recent improvements. Their relation to Higgs field metastability in the current Universe and at early times is also addressed. The following two chapters are dedicated to my original contribution to vacuum decay in modified theories of gravity. In Chapter 3, the asymptotic bounce is introduced, by computing it in scalar field theories with Einstein-Hilbert gravity, focusing in particular on decay from flat space false vacua. Bounds on the Hubble constant are also discussed when the false vacuum is de Sitter instead. In Chapter 4, constraints on vacuum decay in modified gravity theories are addressed. While some of them are specific of modified gravity theories, others derive from a generalization of the discussion in Chapter 3. The Conclusion follows.

Notation and conventions

The metric signature in Minkowski space is $(-,+,+,+)$. The Wick rotation to Euclidean space is $t \to -it$ with action redefined as $S \to iS$, S and t being the Minkowski action and time coordinate respectively. The reduced Planck mass $(8\pi G)^{-1/2}$ is indicated as M_P. The velocity of light is $c = 1$, while the reduced Planck constant $\hbar$ is set to 1 only from Sect.1.3 onwards. The use of a mass unit is avoided, except for numerical calculations. In that case, it is $G = 1$. Total derivatives are denoted as d while partial ones with ∂. spacial derivatives are indicated as ∇. Derivatives with respect to ρ and with respect to t are also indicated as $\dot{}$. An apex (n) indicates the $n-$th time or ρ derivative. Derivatives with respect to ϕ are also denoted as $'$. The false vacuum state ϕ_{fv} satisfies $\phi_{\rm fv} < \phi_{\rm tv}$ and it is set to zero unless differently stated. The cosmological constant is indicated as a scalar field vacuum energy density $V(\phi_{\rm fv})$ and thus satisfies $V(\phi_{\rm fv}) = \Lambda M_P^2$ where Λ is the conventional cosmological constant of mass dimension two in Einstein Hlbert gravity.

Chapter 1

Decay rate calculations in quantum mechanics and field theory

Vacuum decay is a phenomenon displayed by quantum mechanical systems having a potential well separated by a finite barrier from a region of lower energy (see Fig.1.1) and, as discussed in the Introduction, it is intimately related to quantum tunneling. The probabilistic description featuring the quantum world characterizes also the tunneling process, in which a particle initially placed in FV is measured in R, after some time. The probability to measure the particle in some position x, at a time t, is determined by the wavefunction of the system $\psi(x,t)$. Thus, by determining $\psi(x,t)$, one can compute when the probability to measure the particle in R is large. Consider a particle initially placed in the false vacuum state: it is described by a wavefunction $\psi(x,t)$ having large support in FV and a small oscillating tail on the other side of the barrier (first frame in Fig.1.2). One can show that, as time goes by, the wavefunction moves back and forth in the well and some portion of it escapes through the barrier when hit. The decay rate, which is customarily indicated by Γ, accounts for the relative change in the probability to measure the particle in the false vacuum state and (under suitable approximations[1].) it is exponentially small

$$\Gamma = Ae^{-B}. \tag{1.1}$$

A and B depends on the potential and may be computed by solving the Schroedinger equation. From a practical perspective, this is extremely inconvenient: exact solutions are seldom available and one needs to resorts to some approximations. Moreover, as the degrees of freedom in the system increase, calculations get more and more cumbersome. Hence, the decay rate cannot be computed in this way in field theory, being the infinite-dimensional generalization of quantum mechanics. It seems that this way to compute the decay rate is not part of a systematically improvable framework. An alternative was provided by Callan and Coleman in the '70s [10,11]. They showed that B may be equivalently computed as the Euclidean action on a particular trajectory called bounce [10]. This makes its calculation easier in the multidimensional case and, thus, also in field theory. Soon after that, they described

[1] This approximation is valid for $\omega^{-1} \ll t \ll t_{nl}$, where ω indicates the oscillation frequency of the wavefunction in the potential well and t_{nl} the time at which non-linear effects set in. For further details see [3]

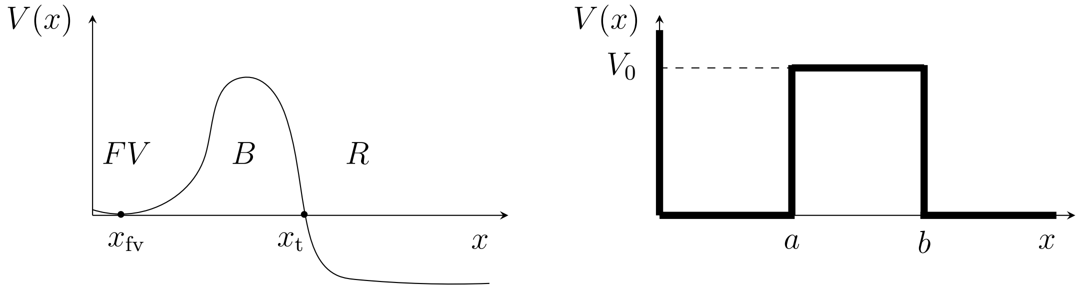

Figure 1.1: Left: potential with a metastable state in $x_{\rm fv}$. The false vacuum region is labelled as FV. The barrier B extends to $x_{\rm t}$ and separates the false vacuum from a region R of lower energy. Right: square potential barrier of height V_0, extending from $x = a$ to $x = b$.

how to compute also the prefactor A [11]. Both calculations are, in fact, part of the same theoretical framework, i.e., the Euclidean path integral approach. B corresponds to a stationary point of the action (the bounce) and A to its fluctuations around it, computed in the semiclassical approximation. Despite the computation seems straightforward, the connection between the the Schroedinger equation calculation and the path integral approach is quite delicate. This is mostly related to the interpretation of Γ as an imaginary part of the particle energy, while the Euclidean partition function is known to be a real quantity [3].
Despite the versality of the path integral approach, the physical interpretation of the tunneling process of a quantum mechanical particle is different than the one of a field. In the latter case, it consists in the nucleation of a new phase of lower energy, localized in space in a bubbly shape. The energy difference resides in the bubble wall, which expands at the speed of light once that the nucleation took place.

In this chapter, the vacuum decay formalism and its physical interpretation are discussed. The one-dimensional quantum mechanical description is outlined first (Sect.1.1) and then the path integral approach is considered (Sect.1.2). This pedagogical introduction to vacuum decay is based on [3, 10, 11, 29, 70], to which the interested reader is referred to for a thorough discussion on this topic. In Sect.1.3-1.4 the calculation of the tunneling exponent B and the physical interpretation of this decay process as the formation of bubbles are reported. This chapter closes with a discussion on the Higgs metastability in the Standard Model (Sect.1.5).

1.1 Vacuum decay in quantum mechanics

Consider a one-dimensional quantum mechanical theory described by the Lagrangian

$$\mathcal{L} = \frac{p^2}{2m} - V(x) \qquad \text{with} \qquad p = m\,\frac{dx}{dt} \tag{1.2}$$

where m is the particle mass, p is the momentum, and $V(x)$ is as depicted in Fig.1.1, on the left. The state of the system is described by a solution $\psi(x,t)$ to the time-dependent Schroedinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{1.3}$$

where $\hat{H}$ is the Hamiltonian of the system in an operatorial form

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + V(x), \tag{1.4}$$

$\hbar$ is the reduced Planck constant and i is the imaginary unit. A particle initially placed in the false vacuum state is described by a wavefunction with large support in the false vacuum region, while small beyond it (first frame of Fig.1.2). It is, in general, a sum of all energy eigenstates $\phi(x)_E$, that may be determined by solving the (time-independent) Schroedinger equation

$$E\,\phi(x)_E = \hat{H}\phi(x)_E \tag{1.5}$$

where E is the energy related to $\phi(x)_E$. From the structure of Eq.(1.5), one can see that $\phi(x)_E$ should be oscillating in the regions FV and R and exponential-like in B. It is characterized by five (energy-dependent) integration constants: the wavefunction normalization and four real numbers, which may be determined by continuity and derivability of $\phi(x)_E$ across the three regions FV, B and R. Consider, for example, a square potential (see Fig.1.1, on the right). The energy eigenstates are (here $p = \sqrt{2mE}$ and $\kappa = \sqrt{2mV_0 - p^2}$)

$$\phi(x)_p = \begin{cases} \frac{1}{N}\sin(px) & x < a \\ \frac{1}{N}\left(A_p e^{\kappa(x-a)} + B_p e^{-i\kappa(x-a)}\right) & a < x < b \\ \frac{1}{N}\left(C_p e^{ip(x-b)} + D_p e^{-ip(x-b)}\right) & x > b. \end{cases} \tag{1.6}$$

Setting the wavefunction norm to unity amounts to having N^2 proportional to both C_p and D_p [3]. In order to have an initial wavefunction with large support in FV, N should be very small, implying (nearly) vanishing C_p or D_p . Calculations show that this happens when the energy is complex, and, in particular, a negative (positive) imaginary part is related to a vanishing D_p (C_p). According to Eq.(1.6), it corresponds to outgoing (ingoing) boundary conditions. Thus, the vacuum decay process is characterized by the contribution to $\psi(x,t)$ of energy eigenstates, with an additional negative imaginary part.
In order to calculate the decay rate, one needs to compute the probability to find the particle in FV, $P_{\rm fv}$, which is defined as

$$P_{\rm fv}(t) \equiv \int_{\rm fv} dx|\psi(x,t)|^2. \tag{1.7}$$

To proceed further, the wavefunction of the system $\psi(x,t)$ needs to be determined. To do that, consider the initial state $\psi(x,0)$. This is, in general, the superposition

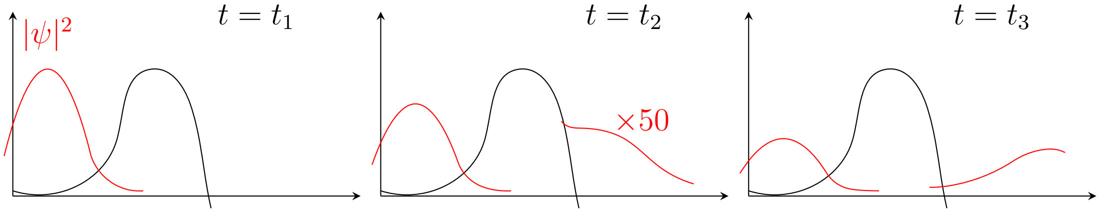

Figure 1.2: Leaking of the wavefunction ψ through the potential barrier at different times $t_1 < t_2 < t_3$. For a more detailed analysis of the process see [3].

of many $\phi(x)_E$, but, for simplicity, one can set it to be close to only one energy eigenstate with energy E_0. Hence, $\psi(x,t)$ is

$$\psi(x,t) = \phi(x)e^{-iEt} \qquad \text{with} \qquad E = E_0 - i\frac{\Gamma}{2}, \qquad \Gamma > 0 \tag{1.8}$$

where the subscript of $\phi(x)_E$ has been suppressed. Using Eq.(1.7) and Eq.(1.8) one finds that

$$\Gamma = -\frac{1}{P_{\text{fv}}}\frac{dP_{\text{fv}}}{dt} \tag{1.9}$$

that is, the imaginary part of the energy is related to the (relative) depletion of the false vacuum state in time, and thus it is a good definition of the decay rate.
One can simplify the above expression for Γ by integrating (in x) the quantum mechanical flux equation

$$\partial_t(\psi^*\psi) = \frac{i\hbar}{m}(\psi^*\partial_x\psi - \psi\partial_x\psi^*), \tag{1.10}$$

which holds for any solution to the time-depedent Schroedinger equation Eq.(1.3), getting

$$\Gamma = \frac{Np}{m}|\phi(x)|^2 \tag{1.11}$$

where N is a normalization factor and

$$p = -\frac{i}{2}\frac{\phi^*\partial_x\phi - \phi\partial_x\phi^*}{\phi^*\phi}. \tag{1.12}$$

Thus, the decay rate depends on how the wavefunction is spread in space, which, in turn, depends on the potential shape. In the WKB approximation [71],

$$\phi(x) = \frac{N}{\sqrt{|p(x)|}}\exp\left(\frac{i}{\hbar}\int_a^x p(x)dx\right) \qquad p(x) = \sqrt{2m(E_0 - V(x))} \tag{1.13}$$

where E_0 is defined in Eq.(1.8)[2]. To determine the wavefunction flow into the region R, set $x = x_{\text{t}}$ in Eq.(1.11). As a result, at leading order (and setting $E_0 = 0$), the decay rate is exponentially small

$$\Gamma \propto \exp\left(-\frac{2}{\hbar}\int_{x_{\text{fv}}}^{x_{\text{t}}}\sqrt{2V(x)}dx\right) \tag{1.14}$$

[2]As explained in detail in [3], using the real part of the energy in place to the complex one leads to a considerable loss in precision, but this still provides a good qualitative answer.

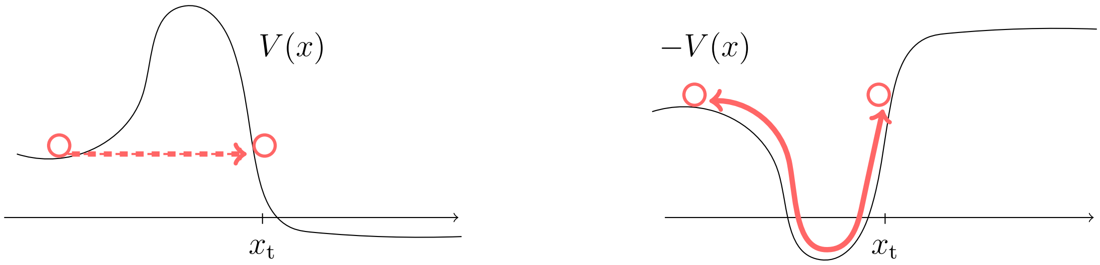

Figure 1.3: Left: potential as a function of x. The pink arrow represents the tunneling process after which the particle is measured at x_t. Right: the potential as seen by the particle in Euclidean space. The pink arrow represents the particle following the bounce trajectory as it evolves from the false vacuum to the tunneling point x_t and back.

where the integral is taken over the classically forbidden region. Comparing Eq.(1.14) with Eq.(1.1) one finds the tunneling exponent

$$B = \frac{2}{\hbar}\int_{x_{\rm fv}}^{x_{\rm t}} \sqrt{2V(x)}dx. \tag{1.15}$$

Notice that computing the transmission coefficient T, defined as

$$T = \frac{\phi(x_{\rm t})}{\phi(x_{\rm fv})}, \tag{1.16}$$

as Gamow did in the '900, gives the same result. The exponential suppression of the decay rate is confirmed also by numerical calculations, under appropriate approximations [3]. For a discussion on deviations from the exponential behaviour the interested reader is referred to [72–76] (and [77,78] for a discussion in a cosmological context).

As one increases the number of degrees of freedom in the system, finding a solution to the Schroedinger equation gets more and more complicated. This mirrors the possibility of having many escape paths through the potential barrier, that now is multidimensional. The generalization was provided by Banks, Bender and Wu [8,9], who predicted that the tunneling exponent B is determined by the most probable escape path (as the others should be exponentially suppressed)

$$B = \frac{2}{\hbar}\int_{x_{\rm fv}}^{x_{\rm t}} \sqrt{2V(x)}dx \tag{1.17}$$

where dx indicates integration on such path. Coleman and Callan noted that the same formula arises if one considers the Euclidean version of the theory, evaluated on a particular trajectory. Wick rotating Eq.(1.2) (i.e. taking $t \to -it$, which makes t the Euclidean time) and defining the Euclidean action S_E as

$$S_E \equiv -iS = -i\int dt L \tag{1.18}$$

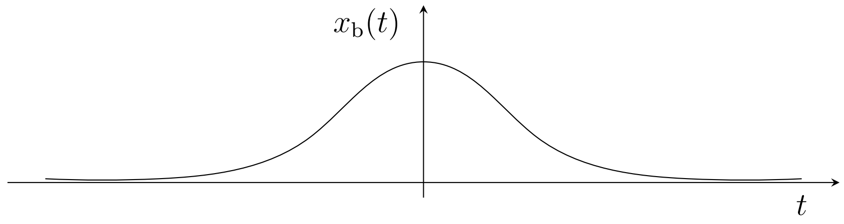

Figure 1.4: Bounce trajectory as a function of the Euclidean time.

one gets

$$S_E = \int \frac{m^2}{2}\dot{x}^2 + V(x)\,dt, \tag{1.19}$$

where the integration is over the real line. The equation of motion is the one of a particle moving in a reversed potential $-V(x)$ (Fig.1.3)

$$\frac{d^2x}{dt^2} = \frac{dV}{dx}. \tag{1.20}$$

Hence, the presence of a potential barrier becomes a "classical" obstacle that the scalar field has to overcome in order to reach the false vacuum. This theory has an instanton solution $x_{\rm b}(t)$ that starts at $x_{\rm fv}$ with vanishing velocity in the infinite past, reaches the (equipotential) tunneling point $x_{\rm t}$ at some time (say at $t = 0$), and bounces back (see Fig.1.3-1.4). Notice that, according to Eq.(1.20), vanishing initial velocity requires infinite time to interpolate between the fixed point $x_{\rm fv}$ and $x_{\rm t}$. The trajectory $x_{\rm b}(t)$ is called Coleman-de Luccia instanton, or bounce. The action evaluated on the bounce is

$$S_E(x_{\rm b}) = 2\int_{x_{\rm fv}}^{x_{\rm t}} \sqrt{2V(x)}dx \tag{1.21}$$

and thus it corresponds to $\hbar B$ in Eq.(1.14). The generalization to the case $V(x_{\rm fv}) \neq 0$ gives instead

$$\hbar B = S_E(x_{\rm b}) - S_E(x_{\rm fv}) \tag{1.22}$$

where $x_{\rm b}$ is again the bounce trajectory. In the multidimensional case now

$$\frac{d^2\vec{x}}{dt^2} = \frac{dV}{d\vec{x}}. \tag{1.23}$$

and B again corresponds to Eq.(1.17). This picture can be furtherly extended to compute also the prefactor A. The standard (but not the only one [3]) method to do that is the Euclidean path integral approach, which will be discussed in the next section.

1.2 The Euclidean path integral approach

The starting point is the Euclidean partition function $\mathcal{Z}$ of a theory with (Euclidean) action S_E, which measures the probability to detect a particle at some point $x_{\rm F}$ at some final time $t_{\rm F}$, given that it is at $x_{\rm I}$ at an initial time $t_{\rm I}$

$$\mathcal{Z} = \left\langle x_{\rm F}, t_{\rm F} \left| e^{-HT} \right| x_{\rm I}, t_{\rm I} \right\rangle = \mathcal{N} \int_{x_{\rm I}, t_{\rm I}}^{x_{\rm F}, t_{\rm F}} \mathcal{D}x \exp\left(-\frac{S_E(x)}{\hbar}\right) = \sum_E \phi_E(x_{\rm F}) \phi_E^*(x_{\rm I}) e^{-iEt} \tag{1.24}$$

where $\mathcal{N}$ is a normalization constant. We choose $x_{\rm F} = -x_{\rm I} = \frac{T}{2}$, so that $x_{\rm F} - x_{\rm I} = T$. In the case of interest, $x_{\rm I} = x_{\rm F} = x_{\rm fv}$ and $T \to +\infty$, as this measures the probability to find the particle in the false vacuum state after a long time, given that it was there initially. The $T \to +\infty$ limit singles out the lowest energy eigenstate E_0

$$\mathcal{Z} \approx_{T \to +\infty} |\langle E_0 | x_{\rm fv} \rangle|^2 e^{-E_0 T} \tag{1.25}$$

as the others are exponentially suppressed. The energy E_0 is expected to have an imaginary part, in analogy with the quantum mechanical case. The path integral cannot, in general, be solved exactly. To tackle the problem one can rely on the saddle-point approximation, in which the partition function is computed by expanding the action around its fixed points. Given a classical trajectory $x_j(t)$ and a fluctuation $\delta x_j(t)$, the action may be written as

$$S_E(x_j + \hbar \delta x_j) = S_E(x_j) + \frac{\hbar^2}{2} S_E''(x_j) \tag{1.26}$$

with

$$S_E''(x_j) = \int dt \delta x_j(t) \left(-\frac{d^2}{dt^2} + V''(x_j)\right) \delta x_j(t). \tag{1.27}$$

$\delta x_j(t)$ should vanish at the boundary

$$\delta x_j\left(\frac{T}{2}\right) = \delta x_j\left(-\frac{T}{2}\right) = 0. \tag{1.28}$$

Then the fixed point $x_j(t)$ contributes to the partition function as

$$\mathcal{Z}_j = \mathcal{N} \exp\left(-\frac{S_E(x_j)}{\hbar}\right) \int \mathcal{D}x \exp\left(-\frac{\hbar}{2} S_E''(x_j)\right) (1 + O(\hbar)) \tag{1.29}$$

If $x_{\rm fv}$ were a true vacuum state, the only contribution to the partition function would have been the trivial, classically allowed, solution

$$x(t) = x_{\rm fv} \tag{1.30}$$

which contributes with $S_E(x_{\rm fv}) = 0$ (setting $V(x_{\rm fv}) = 0$) at leading order in the path integral. In this case, instead, there is, at least, another contribution, i.e. the bounce. In general, there may be multiple trajectories, each of them contributing to some $\mathcal{Z}_j$, and the overall partition function $\mathcal{Z}$ is the sum of all $\mathcal{Z}_j$. Non-vanishing values of $S_E(x_j)$ give rise to non-perturbative and exponentially suppressed quantum corrections to the vacuum energy, as they are, to leading order, in the form $\exp\left(-S_E(x_j)\hbar^{-1}\right)$. Additional corrections to Eq.(1.29) are small in the semiclassical limit $\hbar \to 0$, and are neglected in the saddle-point approximation to this order [3]. Fluctuations around x may be parametrized in terms of eigenvectors of the operator S_E'', i.e.

$$\delta x_j = \sum_n c_n x_n \tag{1.31}$$

where

$$\left(-\frac{d^2}{dt^2} + V''(x_j)\right) x_n = \lambda_n x_n. \tag{1.32}$$

Moreover setting

$$\mathcal{D}x = \frac{dc_n}{\sqrt{2\pi\hbar}} \tag{1.33}$$

gives

$$\mathcal{Z}_j = \mathcal{N} e^{-\frac{S_E(x_j)}{\hbar}} \int \frac{dc_n}{\sqrt{2\pi\hbar}} \exp\left(-\frac{\hbar}{2}\sum_n \lambda_n c_n^2\right)(1 + O(\hbar)) = \mathcal{N} e^{-\frac{S_E(x_j)}{\hbar}} \prod \lambda_n^{-1/2} \tag{1.34}$$

where the last equation corresponds to integration along the real axis and it holds if all eigenvalues are positive. It can also be written as

$$\mathcal{Z}_j = \mathcal{N} e^{-S_E(x_j)} \det\left(-\frac{d^2}{dt^2} + V''(x)\right)^{-1/2} \tag{1.35}$$

under the same assumptions.

To proceed further in the calculation of $\mathcal{Z}$, one needs to evaluate all fixed points x_{j} with the required boundary conditions. As mentioned above, there is the trivial solution

$$x(t) = x_{\mathrm{fv}} \tag{1.36}$$

which has vanishing on-shell action $S(x_{\mathrm{fv}}) = 0$. The path integral in the saddle-point approximation, to the order reported in Eq.(1.29), gives the partition function of an harmonic oscillator [4]

$$Z_{\mathrm{fv}} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \tag{1.37}$$

[3]Nonetheless, they might be important in compensating the imaginary part of the energy, in order to give an overall real $\mathcal{Z}$ [3].

[4]For more details see [79].

with $\omega^2 \equiv V''(x_{\rm fv})$. The bounce solution also contributes, and it has finite and positive action

$$Z_{\rm b} = e^{-B} K \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \tag{1.38}$$

where e^{-B} is the leading order contribution to the path integral

$$B = \frac{S_E(x_{\rm b})}{\hbar} \tag{1.39}$$

and K (formally[5]) corresponds to

$$K = \left(\frac{\det(-\frac{d^2}{dt^2} + V'')}{\det(-\frac{d^2}{dt^2} + \omega^2)}\right)^{-1/2}. \tag{1.40}$$

To evaluate K one needs to determine the eigenvalues λ_n in the fluctuation operator $S''_E(x_{\rm b})$ (see Eq.(1.2)). Note that, as the bounce velocity satisfies

$$\left(-\frac{d^2}{dt^2} + V''(x_{\rm b})\right)\dot{x}_{\rm b}(t) = 0, \tag{1.41}$$

S''_E has a zero mode λ_0. This is related to time translational invariance in the $T \to +\infty$ limit: a generic bounce solution $x_{\rm b}(t)$ may be centered at any time t_0, where the bounce center is defined as the value of t satisfying

$$\dot{x}_{\rm b}(t_0) = 0. \tag{1.42}$$

The generic bounce can thus be written as a function of $t - t_0$ only. This makes the integration on c_0 formally infinite. To evaluate it, consider the normalized zero mode

$$x_0(t) \equiv \frac{1}{\sqrt{S_E(x_b)}}\frac{dx_b}{dt_0} \tag{1.43}$$

The change in path obtained by changing t_0 contributes to the zero mode as

$$\delta x(t) = x_0(t) dc_0 \tag{1.44}$$

and may be written also as

$$\delta x(t) = \frac{dx_{\rm b}}{dt_0} dt_0. \tag{1.45}$$

Then the integration over c_0 can be traded with one over t_0, finding

$$\frac{dc_0}{\sqrt{2\pi\hbar}} = \frac{dt_0\sqrt{S_E(x_{\rm b})}}{\sqrt{2\pi\hbar}}. \tag{1.46}$$

[5]This formal expression derives from integration along the real axis, if all eigenvalues of S'' are positive. Actually, it turns out in the following that S'' has a negative eigevalue and thus the integration contour should be changed. Details on this topic are given below.

Integration over the zero mode thus amounts to

$$\int \frac{dt_0\sqrt{S_E(x_{\rm b})}}{\sqrt{2\pi\hbar}} = \frac{\sqrt{S_E(x_{\rm b})}}{\sqrt{2\pi\hbar}}T \tag{1.47}$$

and it diverges linearly in T. $\mathcal{Z}_{\rm b}$ becomes

$$Z_{\rm b} = e^{-B} K\, T \frac{\sqrt{S_E(x_{\rm b})}}{\sqrt{2\pi\hbar}} \left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \tag{1.48}$$

where now the determinant in K is taken on all modes except λ_0. Despite the leading order in the path integral makes the bounce action exponentially suppressed with respect to the false vacuum one, the degeneracy factor arising due to the zero mode makes it larger in the $T \to +\infty$ limit, with fixed $\hbar$.

Another effect related to the large T limit is the existence of multinstanton contributions $x_n(t)$, with $n \geq 2$ and integer. These are approximate fixed points of the classical action and correspond to gluing together n bounce solutions widely separated in time. The on-shell action in this case is nB, while the degeneracy factor amounts to time translation of all n objects, preserving their order. Integrating over all instanton centers (t_1, $t_2 > t_1$, $\ldots$, $t_n > t_{n-1} > \cdots > t_1$) one gets

$$\int_{-T/2}^{T_2} dt_1 \int_{t_1}^{T_2} dt_2 \int_{t_2}^{T_2} dt_3 \cdots = \frac{T^n}{n!} \tag{1.49}$$

which gives

$$Z_n = e^{-nB}\frac{T^n}{n!}\left(\frac{S_E(x_{\rm b})}{2\pi\hbar}\right)^{n/2} \det\left(-\frac{d^2}{dt^2} + V''(x_n)\right)^{-1/2}. \tag{1.50}$$

The third factor arises from trading the integration on c_0 to the one on the bounce center, and now the determinant is taken on all eigenvalues of the fluctuation operator apart from the zero mode. Moreover, it can be shown that[6]

$$Z_n = e^{-nB}\frac{T^n}{n!}\left(\frac{\omega}{\pi\hbar}\right)^{1/2} e^{-\omega/2\, T} K^n \tag{1.51}$$

i.e., K, as defined in Eq.(1.40), may be written as

$$K^n = \left(\frac{\det(-\frac{d^2}{dt^2} + V''(x_n))}{\det(-\frac{d^2}{dt^2} + \omega^2)}\right)^{-1/2}. \tag{1.52}$$

It should be noted that when the density of bounces gets large (i.e., n gets large), such configurations are no longer approximately critical points of the action and thus they do not give an appreciable contribution to the path integral. Anyway, it is convenient to keep these (negligible) terms in order to resum their contribution in the partition function [29]. Summing all contributions

$$\mathcal{Z} = \mathcal{Z}_{\rm fv} + \mathcal{Z}_{\rm b} + \sum_{n\geq 2} Z_n \tag{1.53}$$

[6] The reader is referred to [29, 79] for further details on the calculation.

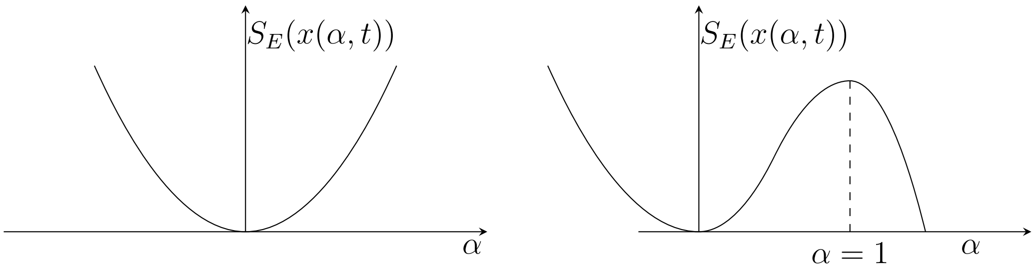

Figure 1.5: Left: the action as a function of α for a theory with a stabilized false vacuum state. Right: the action as a function of α with a metastable false vacuum state.

gives

$$\mathcal{Z} = \left(\frac{\omega}{\pi\hbar}\right)^{1/2} \exp\left(-\frac{\omega T}{2} + KTe^{-B}\frac{\sqrt{S_E(x_\mathrm{b})}}{\sqrt{2\pi\hbar}}\right). \tag{1.54}$$

The only source of imaginary contributions to the false vacuum energy (as defined in Eq.(1.25)) is given by K. Defining Γ as in the Schroedinger picture

$$\Gamma = -2\,\mathrm{Im}(E) \tag{1.55}$$

gives

$$\Gamma = 2\hbar e^{-B}\,\mathrm{Im}(K). \tag{1.56}$$

The real part of K, instead, contributes to an exponentially suppressed shift in the (real part of) E_0. It is related to quantum tunneling without vacuum decay. It appears, for example, in the calculation of the lowest energy levels in the double-well potential (for a thorough discussion see [29]). Note also that ordinary perturbative corrections are normally far larger than the ones described here, but they are not related to any tunneling phenomenon (which is, instead, non-perturbative).

Let's compute explicitly $\mathrm{Im}(K)$ and, thus, Γ. In order to do that, the path integration is to be carried out along an appropriate contour. As mentioned above, there is a vanishing eigenvalue, with eigenvector $\dot{x}_\mathrm{b}$. As $\dot{x}_\mathrm{b}$ vanishes at the bounce center, it has one node, and thus there should be a nodeless function with lower (i.e. negative) eigenvalue λ_{-1}. If one was to integrate along the real axis, the integral would be undefined, as Eq.(1.2) would diverge upon integration on dc_{-1}. Anyway, the saddle-point approximation is effective only by using the steepest descent contour: this is defined as the contour that increases the real part of the action as quickly as possible or, equivalently, that keeps the imaginary part of the action constant. In this way, approximating the integral as its value on the saddle-points is particularly effective. This definition implies that, as the contour hits a fixed point, it makes an abrupt 90 degrees turn [29]. In order to identify such contour let's first parametrize paths as a family of functions $x(\alpha,t)$ in which changes in α deform the path along the negative mode direction and $\alpha = 0$ corresponds to the false vacuum fixed point

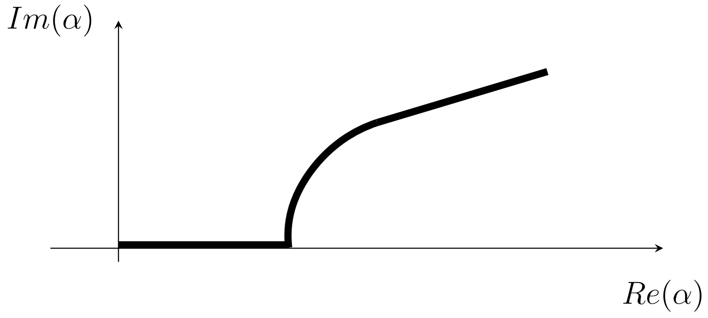

Figure 1.6: Steepest descent contour in the α-plane.

while $\alpha = 1$ the bounce fixed point [29] (Fig.1.5, right). Other deformations orthogonal to α are expected to contribute only to the real part of K, as systems that do not show vacuum decay have only positive modes and the steepest descent contour lies on the real axis. In this case, instead, the contour lies along the real axis near the false vacuum and juts out into the complex plane near the bounce fixed point (see Fig.1.6). In some sense, the steepest descent contour might be thought of as the one derived from stabilizing the false vacuum state: a theory with a stable false vacuum state may be continuously deformed into one showing metastability, and the steepest descent contour moves accordingly from one ending on the real axis to one ending in the complex plane (for an example, see [3, 29]), so to keep the path integral finite. As the saddle-point approximation probes only the vicinity of fixed points, the imaginary part of K should be related to integration in the imaginary direction near $\alpha = 1$. One gets

$$\mathrm{Im}(\mathcal{Z}) = \mathcal{N}\,\mathrm{Im}\left(\int \mathcal{D}z_\perp d\alpha\, \exp\left(-\frac{1}{\hbar}\left(S_E(x,1) + \frac{(\alpha-1)^2}{2}S_E''\right)\right)\right) \tag{1.57}$$

where $\mathcal{D}z_\perp$ indicates integration on directions orthogonal to α. Now S_E'' indicates the second derivative of the Euclidean action with respect to α. As [29]

$$\frac{dx(\alpha,t)}{d\alpha} = x_{-1}(t), \tag{1.58}$$

upon integration along half the gaussian peak, one gets

$$\mathrm{Im}(K) = \frac{1}{2\sqrt{|\lambda_{-1}|}}\left(\frac{\det(-\frac{d^2}{dt^2}+V'')}{\det(-\frac{d^2}{dt^2}+\omega^2)}\right)^{-1/2} \tag{1.59}$$

where now the determinant contains only positive modes. Γ is thus given by

$$\Gamma = e^{-B}\left(\frac{S_E(x_\mathrm{b})}{2\pi\hbar|\lambda_{-1}|}\right)^{1/2}\left(\frac{\det\left(-\frac{d^2}{dt^2}+V''\right)}{\det(-\frac{d^2}{dt^2}+\omega^2)}\right)^{-1/2}. \tag{1.60}$$

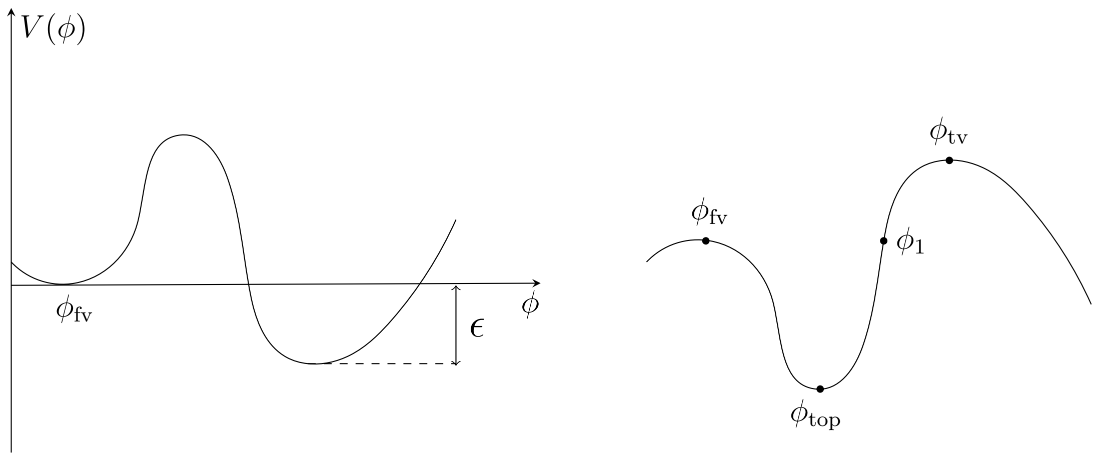

Figure 1.7: Left: potential with a false vacuum state at $\phi_{\rm fv}$, separated by an energy difference ϵ to the true one. Right: the same potential as seen by a particle in Euclidean space. Here $\phi_{\rm top}$ marks the scalar field value at the top of the potential barrier, ϕ_1 the equipotential point to $\phi_{\rm fv}$ and $\phi_{\rm tv}$ the true vacuum state.

1.3 Vacuum decay in field theory

The Euclidean path integral approach allows extending the computation of the vacuum decay rate in quantum mechanics to field theory settings. Let's consider a real scalar field $\phi(t,\vec{x})$ in Minkowski space, described by the action ($\hbar = 1$ is set from this section onwards)

$$S = \int d^4x \left[-\frac{(\partial_t\phi)^2}{2} + \frac{(\nabla\phi)^2}{2} + V(\phi) \right] \tag{1.61}$$

where the scalar field potential has a false vacuum state at $\phi_{\rm fv}$ (see Fig.1.7) and integration is taken over the whole spacetime. This state is classically stable at zero temperature, but there is a finite probability that the scalar field appears on the other side of the potential barrier due to quantum fluctuations. Wick rotating to Euclidean space and defining the Euclidean action as in Eq.(1.18) gives

$$S_E = \int d^4x \left[\frac{(\partial_t\phi)^2}{2} + \frac{(\nabla\phi)^2}{2} + V(\phi) \right]. \tag{1.62}$$

The bounce should approach the false vacuum state at spatial and temporal infinity[7]

$$\lim_{|\vec{x}|\to+\infty} \phi_{\rm b}(t,\vec{x}) = \phi_{\rm fv} \qquad \lim_{t\to+\infty} \phi_{\rm b}(t,\vec{x}) = \phi_{\rm fv} \tag{1.63}$$

and it bounces back when the tunneling point $\phi_{\rm t}$ is reached, say at $t = 0$[8],

$$\frac{\partial\phi_{\rm b}}{\partial t}(0,\vec{x}) = 0. \tag{1.64}$$

[7]While the latter condition stems from a direct generalization from the quantum mechanical case, the former derives from requiring finiteness of the on-shell action.

[8]Translational invariance allows to freely set the bounce center position also in this case.

The tunneling exponent in given by its on-shell action, in analogy with the quantum mechanical case. As a simplifying assumption, the scalar field is taken as to depend only on the four-dimensional Euclidean radius ρ

$$\rho = \sqrt{x^2 + t^2}. \tag{1.65}$$

In fact, it was rigorously shown in [80] that $O(4)$-symmetric bounces have smaller action than the others and thus they are exponentially larger. The action becomes

$$S_E = 2\pi^2 \int d\rho\, \rho^3 \left[\frac{\dot{\phi}^2}{2} + V(\phi)\right] \tag{1.66}$$

where a dot denotes derivatives with respect to ρ. The boundary conditions are

$$\lim_{\rho\to+\infty} \phi_{\rm b}(\rho) = \phi_{\rm fv} \qquad \dot{\phi}_{\rm b}(0) = 0 \tag{1.67}$$

where $\dot{}$ indicates derivatives with respect to the Euclidean radius ρ. The equation of motion is

$$\ddot{\phi} + \frac{3\dot{\phi}}{\rho} = V' \tag{1.68}$$

where $'$ denotes derivatives of the potential in the scalar field. Eq.(1.68) contains a friction-like term, and thus the tunneling point is displaced away from the equipotential one (ϕ_1 in Fig.1.7). To determine where $\phi_{\rm t}$ lies, one can use a simple qualitative argument [10]. If the scalar field is released at some $\phi_{\rm in}$ sufficiently near the true vacuum, it overshoots: it stays up to large radii in the region $\phi_1 < \phi_{\rm in} < \phi_{\rm tv}$, thus making the friction term negligible, and then it runs towards the false vacuum, climbs the hill, and reaches $\phi_{\rm fv}$ with finite velocity. Such velocity is smaller the smaller is $\phi_{\rm in}$. If it is released sufficiently to the left instead it undershoots: it does not have enough energy to climb the hill, so, at some ρ^*, such that $\phi_{\rm fv} < \phi(\rho^*) < \phi_{\rm top}$, it stops, inverts its velocity and start oscillating in the potential well. The turning point $\phi(\rho^*)$ is closer to $\phi_{\rm fv}$ (and lies at larger ρ^*), the larger is the initial condition. By continuity, the bounce corresponds to the trajectory bracketed between undershots and overshots: it may be thought of as an overshoot trajectory with vanishing velocity or a undershoot one with $\phi(\rho^*) \to \phi_{\rm fv}$, $\rho^* \to +\infty$.

This qualitative picture is extremely useful in a numerical perspective, and it underlies the so-called shooting method. In the shooting method, one solves the equations of motion with initial conditions

$$\dot{\phi}(0) = 0 \qquad \phi(0) = \phi_{\rm in} \tag{1.69}$$

up to some finite ρ^* at which either $\dot{\phi} = 0$ (the trajectory undershoots) or at which $\phi = \phi_{\rm fv}$ (the trajectory overshoots). This allows determining the bounce with arbitrary precision, computational limits aside. In general, to use this method the bounce should be found to a good approximation up to large radii, as $\phi(0)$ on the bounce can be large with respect to $\phi_{\rm fv}$ and/or the friction term can be very effective in slowing down the scalar field. Moreover, one should compute the bounce with sufficient precision to get a good estimate of the on-shell action S_E. The Lagrangian must be integrated up to a cut-off, which should be carefully chosen. This method is viable as long as there is a clear undershoot/overshoot distinction among trajectories and, thus, it may be applied in principle to more general theories than the ones in Eq.(1.66). There are also alternative numerical methods [12–18,21] but they are, usually, specific of single scalar field theories (an exception being [19,20]).

As the equations of motion are highly non-linear the bounce is usually found numerically, but an approximate solution may be determined if the energy difference ϵ among the false vacuum and the true one is small [10]

$$V(\phi) = U(\phi) + O(\epsilon) \tag{1.70}$$

where $U(\phi)$ is a potential with degenerate minima ϕ_{fv} and ϕ_{tv}. In this case, the scalar field spends a long time in the vicinity of them and the transition in between is relatively fast, leading to a thin wall separating both, at some radius $\rho = \mathcal{R}$, that, as it will be seen, is large with respect to the range of variation of ϕ. If $\mathcal{R}$ is indeed large, the friction term is small, and the system is approximately conservative, implying

$$\frac{1}{2}\dot{\phi}^2 = U(\phi). \tag{1.71}$$

The bounce then may be approximated as

$$\phi_{\text{b}}(\rho) = \begin{cases} \phi_{\text{tv}} & \rho \ll \mathcal{R} \\ \phi(\rho) & \rho \sim \mathcal{R} \\ \phi_{\text{fv}} & \rho \gg \mathcal{R}. \end{cases} \tag{1.72}$$

where $\phi(\rho)$ satisfies Eq.(1.71). Using Eq.(1.66), Eq.(1.71) and Eq.(1.72) one finds the tunneling exponent as a function of $\mathcal{R}$ [10]

$$B = B_{out} + B_{in} + B_{wall} = -\frac{\pi^2}{2}\mathcal{R}^4\epsilon + 2\pi^2\mathcal{R}^3 S_1 \tag{1.73}$$

where

$$S_1 = \int_{\phi_{\text{fv}}}^{\phi_{\text{tv}}} d\phi \sqrt{U(\phi) - U(\phi_{\text{fv}})} \tag{1.74}$$

computed on $\phi(\rho)$. R should be fixed by requiring that it minimizes the action

$$\frac{dB}{d\mathcal{R}} = 0 \tag{1.75}$$

thus giving

$$\mathcal{R} = \frac{3S_1}{\epsilon} \tag{1.76}$$

which is large as ϵ is small. Using Eq.(1.76) in Eq.(1.73) gives

$$B = \frac{27\pi^2 S_1^4}{2\epsilon^3}. \tag{1.77}$$

Notice that taking the limit $\epsilon \to 0$ corresponds to an infinite radius and a vanishing decay rate. In this case the two vacua are exactly degenerate and the field configuration is just the false vacuum one. As mentioned in Sect.1.2, quantum mechanical systems have also tunneling solutions among equivalent potential wells, corresponding to a shared ground state. In quantum field theory, instead, there is spontaneous symmetry breaking, and the two vacuum states correspond to inequivalent representations of the same theory, making B infinite and thus tunneling forbidden [29].

As discussed above, the calculation of B in the field theory framework is analogous to the quantum mechanical case and quite straightforward, as it just requires to solve the Euclidean equations of motion with some definite boundary conditions and computing the on-shell action. The calculation of the prefactor A, the fluctuation determinant, instead is not, and there are the same difficulties as in the quantum mechanical case that are related to the existence of zero modes and negative modes. The former comes from invariance under translations of the bounce in the four-dimensional space. The eigenvectors are

$$\phi_\mu = \frac{\partial \phi_{\rm b}}{\partial x^\mu}. \tag{1.78}$$

The resulting factor in the partition function amounts to

$$TV \frac{S_E(\phi_{\rm b})^2}{4\pi^2}. \tag{1.79}$$

Again, this implies the presence of (at least) a negative mode, as the scalar field velocity vanishes at some value of ρ. In fact, one can prove that there is only one of them in the thin-wall approximation [29]. Now integration over fluctuations with positive eigenvalues contains a factor

$$\kappa = \det(\partial_\mu \partial^\mu + V''(\phi_{\rm b}))^{1/2} \tag{1.80}$$

which has divergences that should be renormalized. The resulting decay rate may be written as a function of the renormalized action [29]. The instability itself might be generated by quantum fluctuations, and it gives the full theory (described by an effective action $S_{\rm eff}$) a bounce-like solution, while it lacks in the classical one. There is still no clear prescription as regards the computation of the decay rate in this case, despite similar arguments might apply [3].

1.4 The fate of the false vacuum

The Euclidean picture of vacuum decay suggests that a scalar field tunnels through a potential barrier thanks to quantum fluctuations, reappearing at the tunneling point $\phi_{\rm t}$ with vanishing velocity, after a time $\tau = \Gamma^{-1}$. When the tunneling event has occurred, it evolves classically towards the true vacuum state $\phi_{\rm tv}$, according to the equation of motion (here t_M indicates the time coordinate in Minkowski space)

$$(-\partial^2_{t_M} + \nabla^2)\phi = V' \tag{1.81}$$

The initial conditions to such evolution

$$\phi(0) = \phi_{\rm t} \qquad \dot{\phi}(0) = 0 \tag{1.82}$$

are the same required for the bounce at the tunneling point. Thus, the bounce profile for $t > 0$ corresponds to the physical (and classical) evolution of the scalar field in Minkowski space after a Wick rotation. As $\phi_{\rm b}$ depends on ρ only, it may be written as a function of Minkowski coordinates (r, t_M) as

$$\phi_{\rm b}(\rho) = \phi\left(\sqrt{r^2 - t_M^2}\right) \tag{1.83}$$

where r indicates the three-dimensional radius. Eq.(1.83) holds outside the lightcone $r = t_M$. The $O(4)$ symmetry of the bounce is turned into a $O(3,1)$ symmetry in Minkowski space, thus leading to a bubbly shape in three space dimensions, that evolves in time as dictated by Eq.(1.83). At fixed t_M, the bubble is formed by an inner region where the scalar field is near $\phi_{\rm t}$, a wall, and an outer region where $\phi = \phi_{\rm fv}$. One can prove [10] that the total energy to form the bubble is zero, as the negative energy in the inner region is compensated by the positive energy on the bubble wall: the cost to turn a homogeneous false vacuum configuration to an inhomogeneous one is expressed by the decay rate Γ. The larger is the bounce radius, the more inhomogeneous is the configuration, and thus the smaller is the decay rate, as predicted by Eq.(1.77) in the thin-wall approximation. As time goes by, the bubble wall (located at $r^2 - t_M^2 = \mathcal{R}^2$) expands with velocity

$$v = \frac{dr}{dt} = \sqrt{1 - \frac{\mathcal{R}^2}{r^2}} \tag{1.84}$$

where r is the three dimensional radius. v is very large for microscopic $\mathcal{R}$. In Coleman's words, once that an observer has been hit by the bubble wall [79]

> A time $\mathcal{R}$ later, that is to say, on the order of $10^{-10} - 10^{-30}$ sec later, he is inside the bubble and dead. (In the true vacuum, the constant of nature, the masses, and couplings of the elementary particles, are all different from what they were in the false vacuum, and thus the observer is no longer capable of functioning biologically, or even chemically).

but anyway

> Since even 10^{-10} sec is considerably less than the response time of a single neuron, there is literally nothing to worry about; if a bubble is coming toward us, we shall never know what hit us.

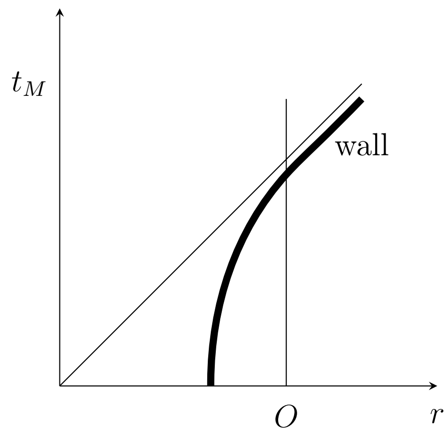

Figure 1.8: Evolution of the bubble wall in Minkowski coordinates (r, t_M) as seen by a static observer O. The diagonal line marks the lightcone position.

1.5 Higgs decay

The formalism to compute the decay rate in Euclidean space has an important application in particle physics, as it turns out that the Standard Model vacuum is, in fact, a false one. The Standard Model is the currently accepted theory describing the interactions observed in particle accelerators. There are three interactions (the electromagnetic, the weak, and the strong interactions) mediated by four gauge bosons: two of them are massive (W and Z bosons) and participate in the weak interaction and two (the photon and the gluon) are massless. Matter is constituted by twelve elementary particles, six quarks and six leptons, along with their antiparticles. The symmetry of the Standard Model is the gauge group $SU(2) \times U(1) \times SU(3)$. In order to have massive W and Z bosons, the $SU(2) \times U(1)$ symmetry is spontaneously broken to $U(1)_{EM}$ by the non-vanishing vacuum expectation value of a complex scalar doublet, the Higgs field H, that may be written as

$$H = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi + i\phi_3 \end{pmatrix}. \tag{1.85}$$

The Higgs potential is

$$V(H) = \mu^2 H H^\dagger + \lambda (H H^\dagger)^2 \tag{1.86}$$

where $\mu^2 < 0$ and $\lambda > 0$. Using the the unitary gauge [9], the only non-vanishing degree of freedom is ϕ. The potential becomes

$$V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \tag{1.87}$$

and it has vacuum expectation value given by

$$v = \sqrt{\frac{|\mu^2|}{\lambda}} \sim 246\,\mathrm{GeV}. \tag{1.88}$$

[9] In the unitary gauge $\phi_+ = 0$ and $\phi_0 = \phi_0^\dagger$

At large energies the (tree level) potential Eq.(1.87) is stable as $\lambda > 0$. Nonetheless, it has been observed that large logarithmic loop corrections may induce running of the quartic coupling to negative values at high energies if the Higgs and the Top mass lie in the appropriate range [40–44]. If this was the case, the Standard Model vacuum state would actually be metastable. The masses are [81]

$$M_h = 125.18 \pm 0.16\,\text{GeV} \qquad M_\text{t} = 173.1 \pm 0.9\,\text{GeV} \tag{1.89}$$

and place the Standard Model squarely in the metastable region. It may seem at odds that a perturbative expansion leads to a large change in the potential. In fact, large logarithms are usually related to a breakdown to said expansion. This problem is cured by making the running scale μ appearing at one-loop ϕ-dependent (the so-called renormalization group improvement), either to make the logarithms vanishing or to keep them sufficiently small (for a pedagogical introduction on this matter see [81]). Moreover, the potential barrier is very sensitive to approximations [81], and to predict precisely its position one needs to solve the renormalization group evolution of all Standard Model β-function, up to a suitable loop order. High precision calculation of the quartic coupling running are relatively recent (see for example [40,82,83]) and place the potential barrier to lie approximately at $\phi_\text{top} \sim 5 \times 10^{10}$ GeV, the barrier height being $V(\phi_\text{top}) \sim 3.5 \times 10^{38}\text{GeV}^4$. Moreover, the renormalization group improvement ensures that the perturbative expansion is reliable up to the Planck scale. A ready-to-use formula for the Higgs potential with two-loop corrections is [84]

$$V(\phi) = \frac{\lambda(\phi)}{4}\phi^4 \tag{1.90}$$

with

$$\lambda(\phi) = \lambda^* + \gamma \ln{(\phi)}^2 + \beta \ln(\phi)^4 \tag{1.91}$$

where ϕ is measured in Planck mass units $G = 1$. For $M_H = 125$ GeV and $M_\text{t} = 173$ GeV one finds

$$\lambda^* = -0.0013 \qquad \gamma = 1.4 \times 10^{-5} \qquad \beta = 6.3 \times 10^{-8}. \tag{1.92}$$

With this choice, the potential barrier reaches the top at $\phi_\text{top} \sim 4.1 \times 10^{10}$ GeV, with height $3.6 \times 10^{38}\text{GeV}^4$ and width $5.2 \times 10^{10}\text{GeV}$ (see Fig.1.9, on the left). As loop corrections do not predict any true vacuum state, the potential barrier is thick and thus the thin-wall approximation cannot be used. Anyway, the bounce solution to a $\lambda\phi^4$ theory has been explicitly computed [24] and it is

$$\phi_\text{b}(t) = \sqrt{\frac{2}{\lambda}}\frac{2\mathcal{R}}{t^2 + \mathcal{R}^2} \tag{1.93}$$

where $\mathcal{R}$ is the bounce radius, defined as[10]

$$\phi_\text{b}(\mathcal{R}) = \frac{\phi_\text{b}(0)}{2}. \tag{1.94}$$

[10]In Sect.1.3 $\mathcal{R}$ was used to indicate the bounce radius in the thin-wall approximation. The definition given here extends also to thick-wall bounces and it is consistent with the thin-wall ones.

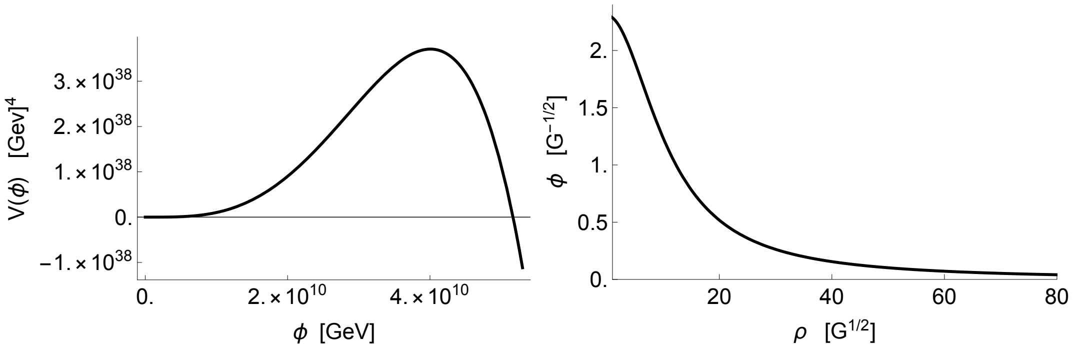

Figure 1.9: Left: Higgs potential at high energies, according to Eq.(1.90)-(1.91). Right: Higgs bounce profile.

As loop corrections softly break scale invariance, the bounce for a scalar field theory with potential Eq.(1.90) is Eq.(1.93) with $\mathcal{R}$ determined by minimization of the on-shell action

$$S_E = \frac{8\pi^2}{3|\lambda|} + S_q(\mathcal{R}). \tag{1.95}$$

The first term on the right-hand side of Eq.(1.95) is the bounce action for a single scalar field theory with quartic potential $\lambda\phi^4$. The second piece is small and accounts for the presence of a potential barrier arising from quantum corrections. One finds $\mathcal{R} \sim 11$ (in units $G = 1$), $B \sim 2025$, giving

$$\frac{\Gamma}{V} \sim A\, 10^{-879}. \tag{1.96}$$

Here V is the spacetime volume, which may be evaluated as $V = T_U^4$, T_U indicating the currently predicted lifetime of the Universe (thirteen billion years). The prefactor A is estimated as [85]

$$A \sim \mathcal{R}^{-4}. \tag{1.97}$$

The lifetime of the false vacuum state is thus

$$\Gamma^{-1} \sim 10^{640} T_U \tag{1.98}$$

which is consistent with the non-observation of vacuum decay in our Universe. As reported in Fig.1.9, the scalar field at the tunneling point $\phi_{\rm b}(0)$ probes Planck mass values, at which new physics might be important, as well as gravity. It has been found [48] that Planck-suppressed terms may considerably decrease the lifetime of the Higgs. Taking for example (again we set $G = 1$)

$$V(\phi) = \frac{\lambda(\phi)}{4}\phi^4 + \gamma_2\phi^6 + \beta_2\phi^8 \tag{1.99}$$

with $\gamma_2 = -0.25$ and $\beta_2 = 0.125$ gives

$$\Gamma^{-1} \sim 10^{-194} T_U. \tag{1.100}$$

Hence, the bounce is very sensitive to boundary conditions at $\phi_{\rm t}$. Instead, the infrared scalar field profile ($\rho \to +\infty$, corresponding to $\phi_{\rm b} \sim \phi_{\rm fv}$) is insensitive to this kind of high energy corrections. This point will be furtherly discussed in Chapter 3.

1.6 Summary

In this chapter, two different approaches to the calculation of the vacuum decay rate Γ in quantum mechanics were described. They both show that, in an appropriate time regime, Γ is exponentially small. The Euclidean path integral approach allows readily extending this result to field theory, and it has a physical interpretation in terms of true vacuum bubble nucleation. The tunneling exponent B may be computed as the on-shell action of an instanton in Euclidean space, called Coleman-de Luccia instanton or bounce. Fluctuations around the bounce are characterized by one negative eigevalue (for an extension to a wider class of systems see [86]). If the potential barrier separating the false vacuum and the true one is sufficiently small, B may be computed as a function of the bounce radius $\mathcal{R}$, and then minimized with respect to that. Recent experiments and calculations show that the Standard Model has an instability in the Higgs field potential. The decay rate is computed numerically, as the Higgs potential barrier is thick, and it is found to be very small. Planck-suppressed corrections though may change it by several orders of magnitude.

Chapter 2

Gravitational effects

As outlined in the previous Chapter, fields in a metastable state decay at rate Γ via bubble nucleation. Despite only scalar field theories with non-derivative interactions were considered, the path integral approach to the vacuum decay process is actually much more general. The tunneling exponent B is particularly simple to compute: given a theory described by an action S, it suffices to find trajectories with bounce-like boundary conditions in Euclidean space and to single out the one with the largest on-shell action. In this way, gravitational corrections may be easily included by choosing a line element and adding an Einstein-Hilbert term in S_E. They are expected to be important when high mass scales (not far from M_P) or large length scales are involved. The latter physically correspond to having a true vacuum bubble so large that it is affected by the spacetime curvature. To qualitatively understand the importance of gravitational corrections on vacuum decay, one may consider a spherical bubble of radius Λ containing a true vacuum phase of energy density ϵ. The resulting vacuum energy is [69]

$$E = \frac{4\pi}{3}\,\epsilon\,\Lambda^3 \tag{2.1}$$

which corresponds to a Schwartzschild radius $2\,GE$. This equals the bubble radius when

$$\Lambda = \left(\frac{8\pi G\epsilon}{3}\right)^{-1/2}. \tag{2.2}$$

For $\epsilon = 1$ GeV, one gets $\Lambda \sim 0.8$ km. This shows that gravitational effects may be relevant already on planetary and terrestrial scales.
The importance of gravitational corrections might be readily spotted also in the Coleman-de Luccia instanton, which probes the tunneling point beyond the potential barrier: if such point lies at high mass scales (such as in the Higgs decay example described in Chapter 1), then gravitational terms should be included in the action. Depending on the value of the scalar field potential $V(\phi)$ at $\phi_{\rm fv}$, the false vacuum state may live on a spacetime with various geometries. From a cosmological perspective, flat space and de Sitter space are particularly important. The former is usually regarded in the literature as an approximation in decays with a (very small) cosmological constant, hence describing the current Universe. The latter, instead, is

considered during inflation. Particle physics theories as well as cosmological models that predict large decay rates in these stages are ruled out thanks to the anthropic principle.

In this chapter, a pedagogical introduction to gravitational effects on the vacuum decay process is given. The standard setting of scalar fields interacting with Einstein-Hilbert gravity in (and beyond) the thin-wall approximation is explored in Sect.2.1-2.3. The focus is on decay from de Sitter and Minkowski space, due to their importance in cosmology. Then, modified gravity is considered in the form of a non-minimal coupling between the scalar field and the Ricci scalar (Sect.2.4). The Higgs decay from flat space (Sect.2.5) and during inflation (Sect.2.6) is discussed. This chapter closes with a Summary and discussion of results. A brief list of critical approaches to the Coleman-de Luccia formalism is included, as well as some literature to which the interested reader is referred.

2.1 Einstein-Hilbert gravity

Consider a scalar field $\phi(t, \vec{x})$ in Euclidean space and an $O(4)$-symmetric line element with scale factor $\rho(t)$

$$ds^2 = dt^2 + \rho(t)^2 d\Omega_3^2. \tag{2.3}$$

This symmetry is considered in the literature in analogy to the flat space case[1] [69]. The Euclidean action contains a real scalar field and Einstein-Hilbert gravity

$$S_E = 2\pi^2 \int dt\, \rho(t)^3 \left[-\frac{M_P^2 R}{2} + \frac{1}{2} g_{\mu\nu} \nabla^\mu \phi \nabla^\nu \phi + V(\phi) \right]. \tag{2.4}$$

Here M_P indicates the reduced Planck mass and $g_{\mu\nu}$ is the metric with line element Eq.(2.3). The Ricci scalar R is given by

$$R = -6\, \frac{\dot{\rho}^2 - 1 + \ddot{\rho}\rho}{\rho^2} \tag{2.5}$$

where now $\dot{}$ indicates derivatives with respect to t. The Hubble parameter is defined as[2]

$$H^2 \equiv -\frac{\dot{\rho}^2 - 1}{\rho^2}. \tag{2.6}$$

The scalar field is chosen to be homogeneous in space $\phi(t, \vec{x}) = \phi(t)$. The scalar field equation of motion is

$$\ddot{\phi} + \frac{3\dot{\rho}\,\dot{\phi}}{\rho} = V' \tag{2.7}$$

[1]However, there is so far no rigorous proof that $O(4)$-symmetric bounces dominate the decay process when gravity is included.

[2]This definition implies that H^2 is constant in de Sitter and AntideSitter spacetimes.

while the $tt-$component of the Einstein equations is

$$\dot{\rho}^2 = 1 + \frac{\rho^2}{3M_P^2}\left(\frac{\dot{\phi}^2}{2} - V(\phi)\right). \tag{2.8}$$

Using Eq.(2.7)-(2.8), Eq.(2.4) may be rewritten as [3]

$$S_E = -2\pi^2 \int dt\, \rho(t)^3 V(\phi). \tag{2.9}$$

In order to find the bounce, the flat space boundary conditions Eq.(1.67) should be supplemented with the ones for the scale factor as well. It may be proven that $\rho(t)$ should have at least one zero and at most two [87]. In order to keep the friction term in Eq.(2.7) finite, they should correspond to zeros of $\dot{\phi}$, also. One of them may be set at $t = 0$ exploiting time translational invariance of the bounce (matching the condition $\dot{\phi}(0) = 0$ in Eq.(1.67)), while the other gives rise to two possible scenarios, depending on whether the metric is compact or not. If there is only the $t = 0$ zero, spacetime is non-compact and t extends to $\pm\infty$. According to Eq.(2.7), $\dot{\phi}$ vanishes at a fixed point of the scalar field equation of motion for $t \to \pm\infty$, i.e. at the false vacuum state. This happens either when $V(\phi_{\rm fv}) = 0$ or $V(\phi_{\rm fv}) < 0$. In fact, plugging the false vacuum fixed point in Eq.(2.8), one finds[4]

$$\begin{cases} \rho(t) = t & V(\phi_{\rm fv}) = 0 \\ \rho(t) = \dfrac{\sinh(Ht)}{H} & V(\phi_{\rm fv}) < 0 \end{cases} \tag{2.10}$$

where

$$H^2 = \frac{V(\phi_{\rm fv})}{3M_P^2} \tag{2.11}$$

is the Hubble rate. The former is Euclidean flat space (corresponding to Minkowski space after a Wick rotation) and the latter Anti-de Sitter space. If instead $V(\phi_{\rm fv}) > 0$ the false vacuum state is de Sitter, which has compact spacetime, as

$$\rho(t) = \frac{\sin(Ht)}{H} \qquad \text{for} \qquad \phi = \phi_{\rm fv} \tag{2.12}$$

Again, H is given by Eq.(2.11). As a consequence, $\rho(t)$ on the bounce has another zero for some $t = t_{\rm max}$. According to Eq.(2.7), this implies that $\bar{\phi} \equiv \phi(t_{\rm max})$ should be such that

$$V'(\bar{\phi}) \neq 0. \tag{2.13}$$

[3]In this case one cannot set the vacuum energy $V(\phi_{\rm fv})$ to zero, as it produces gravitational effects. Thus, the general relation among B and S_E should be adopted

$$B = S_E(\phi_b) - S_E(\phi_{\rm fv})$$

where ϕ_b is the bounce.

[4]This solution has actually an additional arbitrary constant that amounts to a shift in the time coordinate and which may be freely set to zero. As, on the bounce, the time origin is already fixed by the condition $\dot{\phi}(0) = 0$, in that case the constant should be retained (see for example the bounce scale factor in Fig.2.4, which is $\rho(t) - t \approx \delta$ at large times).

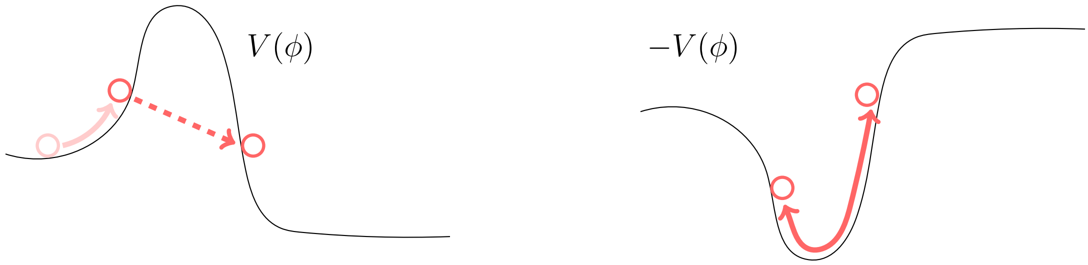

Figure 2.1: The vacuum decay process when the false vacuum has a de Sitter geometry. Left: potential as a function of ϕ. The pink dashed arrow represents the tunneling process. Before tunneling, the particle is in an excited state due to the Gibbons-Hawking temperature. Right: the potential as seen by the particle in Euclidean space. The pink arrow describes the scalar field on the bounce trajectory as it evolves from the false vacuum to the tunneling point and back.

So, the false vacuum state won't be reached on the bounce. As $\dot{\phi} < 0$ on the bounce for $t > 0$, $\bar{\phi}$ should lie on the region where

$$V'(\bar{\phi}) > 0, \tag{2.14}$$

so that $\ddot{\phi}(t_{\max}) > 0$. At $\bar{\phi}$ the scale factor on the bounce is given by

$$\rho(t) = \frac{\sin(\bar{H}t)}{\bar{H}} \tag{2.15}$$

where $\bar{H}$ is

$$\bar{H}^2 = \frac{V(\bar{\phi})}{3M_P^2}. \tag{2.16}$$

The cosmological radius (and thus the outer boundary of spacetime) is given by $t_{\max} = \pi\bar{H}^{-1}$. Physically, this corresponds to a thermally assisted transition to the true vacuum phase (see Fig.2.1), with Gibbons-Hawking temperature $T = 2\pi\bar{H}$. Notice also that the "friction term" in Eq.(2.7) actually entails self-acceleration for $\bar{H}t \in [\frac{\pi}{2}, \pi]$. As a consequence, a richer zoology of instantons exists with respect to the non-compact case [60–63, 65–67]. These are usually found with the standard shooting method as trajectories separating undershots and overshots. The main difference to the flat space case lies in the fact that there might be a pattern of alternating undershots and overshots, as the initial condition is varied, and, thus, many bounce-like trajectories for a given value of $\bar{H}$. A list of instanton solutions when the false vacuum state has a de Sitter geometry follows:

- *Uptunneling*: these instantons have exchanged initial and final point, i.e. $\phi_b(0) = \bar{\phi}$ and $\phi(t_{\max}) = \phi_t$, corresponding to a transition from the true vacuum state to the false vacuum. One may guess the existence of such instantons exist by considering a single scalar field theory on a *fixed* de Sitter background with Hubble constant H, which has equation of motion[5]

$$\ddot{\phi} + \frac{3H\cos(Ht)}{\sin(Ht)}\dot{\phi} = V'. \tag{2.17}$$

[5]The action is $S_E = 2\pi^2 \int \frac{\sin(Ht)^3}{H^3}\left(\frac{\dot{\phi}^2}{2} + V(\phi)\right) dt$.

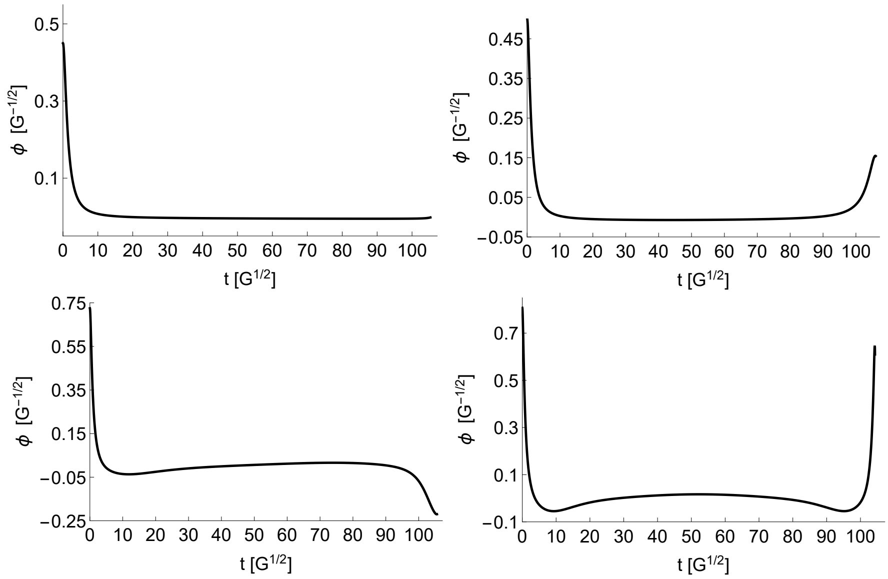

Figure 2.2: Oscillating instantons for $n = 0, 1, 2, 3$. The action is Eq.(2.4) and the potential is Eq.(4.68), with $V_0 = 10^{-4}$.

The symmetry $Ht \to \pi - Ht$ guarantees that for every downtunneling solution (i.e., the standard bounce) there is an uptunneling one.

- *Oscillating instantons*: They have been studied, for example, in [60–63,65–67] (see also Fig.2.2). Such solutions arise because, given an undershoot trajectory with some oscillations around $\phi_{\rm top}$, one can always smoothly change the initial condition such that the scalar field inverts its velocity at $t_{\rm max}$. This guarantees that, if an oscillating instanton with n oscillations is found, then all finite action solutions with less oscillations[6] are present, for a given $\bar{H}$. It is not clear if such solutions are related to only one negative eigenvalue in the fluctuation determinant or not. Numerical investigations point to the fact that bounces with n oscillations have n negative modes [88, 89], and, thus, their actual contribution to vacuum decay is ambiguous.

- *Coleman-de Luccia solution*: It corresponds to (a small modification of) the flat space solution $\bar{H} = 0$ for small $\bar{H}$ [61];

- *New instanton solutions*: New time dependent solutions might appear, as observed in [61] in the case of quartic potential barriers [7] (see Fig.2.3);

- *Hawking-Moss instanton*: this is the only time-independent solution contributing to vacuum decay. It corresponds to [68]

$$\phi(t) = \phi_{\rm top}. \tag{2.18}$$

[6]The existence of an instanton with zero oscillations instead is not guaranteed, as it requires also the presence of overshots.

[7]That is, potentials such that their quartic derivative, computed at $\phi_{\rm top}$, is the lowest non-vanishing derivative of $V(\phi)$.

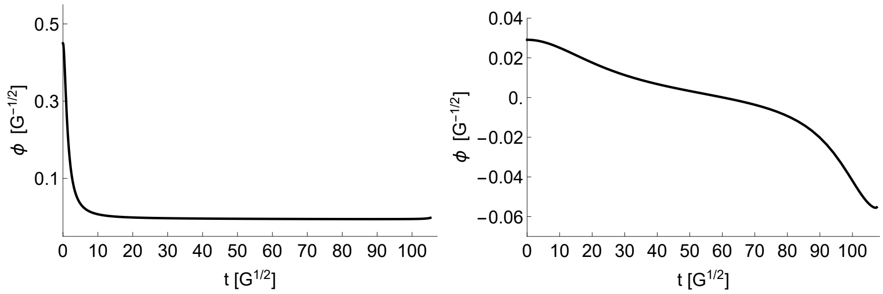

Figure 2.3: Lowest order ($n = 0$) oscillating instanton, and another instanton. The scalar field potential is $V(\phi) = 10^{-4} + 7.03\phi^2(-0.5\phi^2 - \phi^3 + 0.25\phi^4)$.

> The physical meaning of the Hawking-Moss instanton has not been completely understood yet. This was originally described as a thermal transition from the false vacuum [68]. Nonetheless, some concern arose on this interpretation as the final state has a larger effective cosmological constant (since $V(\phi_{\text{top}}) > V(\bar{\phi})$), and, thus, larger energy density. Moreover, it seems also to be independent of the distance in field space, as transitions to states with the same energy density should be equivalent. Recent discussions on this matter may be found in [64, 90].

The time-dependent solutions described above do not exist for any value of $\bar{H}$. In particular, it may be found that, for high enough Hubble rates, they completely disappear, and only the Hawking-Moss instanton survives [60]. In [60] a sufficient and a necessary condition were found. The first one reads

$$\frac{V''(\phi)}{4\bar{H}^2} + 1 < 0 \qquad \text{at } \phi = \phi_{\text{top}}. \tag{2.19}$$

In [60] it is described as the conditions such that undershots exist. While the authors infer that this is enough to guarantee the presence of a bounce, it was found numerically in [67] that overshots are not necessarily present. Thus, Eq.(2.19) guarantees the presence of instantons as long as they have at least one oscillation. The necessary condition instead is

$$\frac{V''(\phi)}{4H^2} + 1 < 0 \qquad \text{somewhere inside the barrier} \tag{2.20}$$

where

$$H^2 = \frac{V(\phi)}{3M_P^2}. \tag{2.21}$$

H is the true Hubble constant (see Eq.(2.6)) only when $\dot{\phi} = 0$, i.e. at the initial and final points of the bounce evolution. The potential barrier is defined here as the *whole* region probed by the scalar field on the bounce trajectory. Thus, in practice, one computes the left-hand side of Eq.(2.20) and determines whether it is nowhere negative for any ϕ such that $\phi_{\text{fv}} < \phi < \phi_{\text{tv}}$. If so, a Coleman-de Luccia bounce does not exist. This implies that a bounce cannot be excluded if the true vacuum state has $V(\phi_{\rm tv}) < 0$, as, for some ϕ in the potential barrier, one has $V(\phi) = 0$, which drives

$$\frac{V''(\phi)}{4\bar{H}^2} + 1 \tag{2.22}$$

to $-\infty$ if V'' has definite sign there. Eq.(2.20) holds also in scalar field theories on a fixed de Sitter background, despite H, in this case, is a free coupling constant of the theory. In that case, a bounce cannot be excluded in theories with no true vacuum state and

$$\lim_{\phi \to +\infty} V''(\phi) = -\infty \tag{2.23}$$

also, as it makes Eq.(2.20) unbounded from below. The Higgs field potential displays both features.

It has also been found numerically [62] that, if

$$|V''(\phi_{\rm top})| \neq 0, \tag{2.24}$$

and $V''(\phi)$ is monotonically decreasing for $\phi_{\rm fv} < \phi < \phi_{\rm top}$, the Coleman-de Luccia bounce disappears for values of H such that

$$\frac{V''(\phi_{\rm top})}{4\bar{H}^2} + 1 \tag{2.25}$$

vanishes. This, along with Eq.(2.20), lead the authors to think that Eq.(2.25) should be negative and thus the bound on $\bar{H}$ is an upper one. Nonetheless, it was numerically shown that, if the potential has a negative quartic derivative in $\phi_{\rm top}$, the Coleman-de Luccia bounce exists for positive values of Eq.(2.25) [67] and not negative ones. It was found that there are indeed undershoot trajectories for negative values of Eq.(2.25), but overshoot ones only appear for positive values. These findings thus suggest that, if Eq.(2.24) holds, and $V''(\phi)$ is monotonically decreasing for $\phi_{\rm fv} < \phi < \phi_{\rm top}$, the Coleman-de Luccia instanton exists for

$$\begin{cases} V''(\phi_{\rm top}) + 4\bar{H}^2 < 0 & V''''(\phi_{\rm top}) > 0 \\ V''(\phi_{\rm top}) + 4\bar{H}^2 > 0 & V''''(\phi_{\rm top}) < 0. \end{cases} \tag{2.26}$$

2.2 The thin-wall approximation

Gravitational corrections to vacuum decay may be computed analytically in the thin-wall approximation, as discussed in the seminal paper by Coleman and de Luccia [69]. Their results are explained and discussed in this section. Consider a single scalar field theory with Einstein-Hilbert gravity, and set (see Fig.1.7, left)

$$V(\phi) = U(\phi) + O(\epsilon) \tag{2.27}$$

where $U(\phi)$ is a potential with degenerate minima and ϵ is small. To find a solution to Eq.(2.7) one requires the friction term to be negligible, so that there are homogeneous scalar field configurations inside and outside the bubble, just as in the flat space case discussed in Sect.1.3

$$\phi = \phi_{\text{tv}} \qquad \text{inside the bubble} \tag{2.28}$$
$$\phi = \phi_{\text{fv}} \qquad \text{outside the bubble.} \tag{2.29}$$

In the previous chapter it corresponded to having constant and large radius $\mathcal{R}$ in the transition between the true vacuum and the false vacuum state. In this case, the friction term is given by

$$\frac{\dot{\rho}^2}{\rho^2} = \frac{1}{\rho^2} - \frac{V(\phi_{\text{fv/tv}})}{3M_P^2} \tag{2.30}$$

In order to keep it small, not only ρ at the transition needs to be large, but the vacuum energy in the true and false vacua states should be sufficiently small. The time at which the transition occurs will be indicated as $\mathcal{R}$, while the scale factor value at the transition is indicated by $\bar{\rho}$ ($\bar{\rho} \equiv \rho(\mathcal{R})$). In the following, decay from de Sitter space

$$V(\phi_{\text{tv}}) = 0 \qquad V(\phi_{\text{fv}}) = \epsilon \tag{2.31}$$

and Minkowski space

$$V(\phi_{\text{tv}}) = -\epsilon \qquad V(\phi_{\text{fv}}) = 0 \tag{2.32}$$

are considered. The scalar field is given by

$$\phi_b(t) = \begin{cases} \phi_{\text{tv}} & t \ll \mathcal{R} \\ \phi(t) & t \sim \mathcal{R} \\ \phi_{\text{fv}} & t \gg \mathcal{R} \end{cases} \tag{2.33}$$

and the scale factor may be found by solving the second equation in Eq.(2.8) with the scalar field as in Eq.(2.33). For $t \gg \mathcal{R}$ and $t \ll \mathcal{R}$ it just corresponds to the fixed point solutions outlined in Sect.2.1, which depend only on the value of $V(\phi_{\text{fv/tv}})$. In this way one finds B as the sum of a contribution outside the bubble B_{out}, one in the wall B_w, and one inside the bubble B_{in} [69]:

$$B = B_{in} + B_{out} + B_w \tag{2.34}$$

with

$$B_{out} = 0 \tag{2.35}$$

$$B_{in} = 12\pi^2 M_P^4 \left(\frac{1}{V(\phi_{\text{tv}})} \left(1 - \frac{\bar{\rho}^2 V(\phi_{\text{tv}})}{3M_P^2} \right)^{3/2} - 1 \right) - (\phi_{\text{tv}} \to \phi_{\text{fv}}) \tag{2.36}$$

$$B_w = 4\pi^2 \bar{\rho}^3 \int dt (U(\phi) - U(\phi_{\text{fv}})) = 2\pi^2 \bar{\rho}^3 S_1. \tag{2.37}$$

This determines B as a function of $\bar{\rho}$. As in Chapter 1, the tunneling exponent should be minimized with respect to $\bar{\rho}$

$$\frac{dB}{d\bar{\rho}} = 0 \tag{2.38}$$

giving

$$\bar{\rho} = \begin{cases} \dfrac{\rho_0}{1 + \rho_0^2/4/\Lambda^2} & V(\phi_{\text{fv}}) = \epsilon \\ \\ \dfrac{\rho_0}{1 - \rho_0^2/4/\Lambda^2} & V(\phi_{\text{fv}}) = 0 \end{cases} \tag{2.39}$$

where ρ_0 indicates the flat space radius and $\Lambda = \sqrt{3M_P^2/\epsilon}$. B then is

$$B = \begin{cases} \dfrac{B_0}{(1 + \rho_0^2/4/\Lambda^2)^2} & V(\phi_{\text{fv}}) = \epsilon \\ \\ \dfrac{B_0}{(1 - \rho_0^2/4/\Lambda^2)^2} & V(\phi_{\text{fv}}) = 0 \end{cases} \tag{2.40}$$

where B_0 indicates the flat space tunneling exponent. For $V(\phi_{\text{fv}}) = \epsilon$ case, corresponding to decay from de Sitter to Minkowski space, radii are smaller than in the fixed flat space case, and correspondingly B is smaller. Physically, this means that it is easier to build a bubble of smaller radius. For $V(\phi_{\text{fv}}) = 0$ radii are larger, and so is B, up to the scale at which B diverges thus having vacuum quenching, i.e., the false vacuum is stabilized. As thouroughly explained in [69], the interpretation of the process as bubble nucleation carries over from the flat spacetime case analized in Chapter 1, Sect.1.4. Moreover, one can show that energy is conserved, and it amounts to three pieces, of which two are purely gravitational: the flat space energy, the gravitational energy of the true vacuum state, and a volume term due to changes in the bubble shape induced by gravity. Thus there is no energy cost in bubble nucleation and the difficulty to form an inhomogeneous field configuration is contained in the decay rate.

2.3 Thick-wall bubbles

As described in the previous chapter, the thin-wall approximation overlooks the undershoot/overshoot behaviour of trajectories, by determining a scalar field configuration with finite action that is on-shell for some value of the bubble radius. The underlying approximation to that calculation is that the wall separating the true vacuum and false vacuum configuration is steep. If this is not the case, the bounce may be found numerically with the shooting method (see Sect.1.3). In theories with a scalar field and Einstein-Hilbert gravity this amounts to setting initial conditions

$$\phi(0) = \phi_{in} \qquad \dot{\phi}(0) = 0 \qquad \rho(0) = 0 \tag{2.41}$$

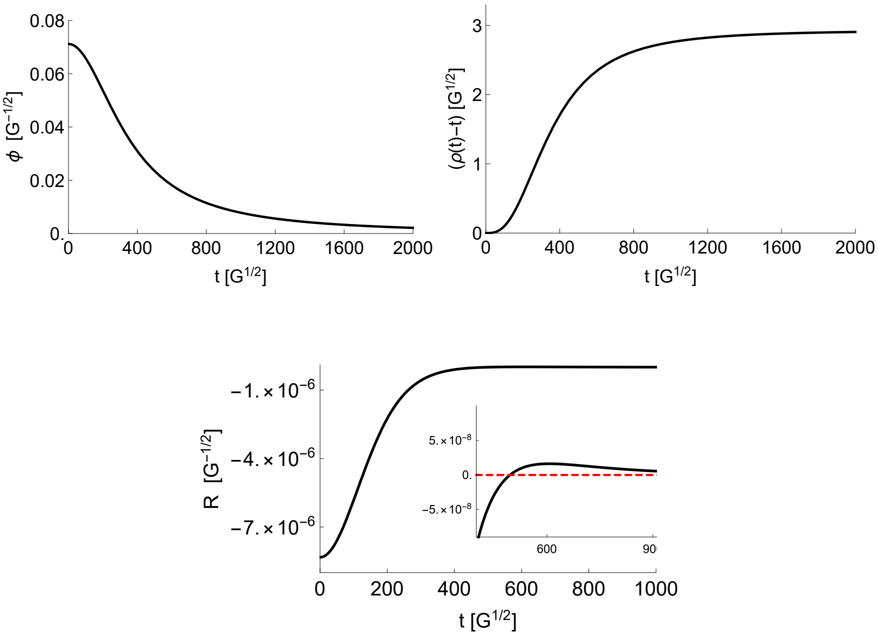

Figure 2.4: Higgs bounce with Einstein-Hilbert gravity. Top left: scalar field profile on the bounce. Top right: scale factor profile on the bounce, in the form $\rho(t) - t$. Bottom: Ricci scalar profile, on the bounce. The inset shows a magnification highlighting the bump that lowers B and the bounce radius in thick-wall bubbles with respect to the thin-wall ones.

and verifying whether

$$\phi(+\infty) = \phi_{\text{fv}} \qquad \dot{\phi}(+\infty) = 0 \tag{2.42}$$

if $V(\phi_{\text{fv}}) \leq 0$, or

$$\phi(t_{\text{max}}) = \bar{\phi} \qquad \dot{\phi}(t_{\text{max}}) = 0 \tag{2.43}$$

if $V(\phi_{\text{fv}}) > 0$. Once that the bounce is found, its on-shell action and thus the tunneling exponent may be computed via numerical integration. The behaviour of B and ρ as a function of B_0 and ρ_0 might change with respect to the thin-wall case. This possibility has been investigated in [70, 91] as regards decay from Minkowski space. It has been observed numerically that thick-wall bubbles have smaller radii compared to the thin-wall regime, as the bounce is further away from $V(\phi_{\text{tv}})$ and smeared in a wider region, resulting in a higher (negative) value of the potential at the transition point and thus a smaller Anti-de Sitter radius there. The decay exponent decrease correspondingly. The reason is that the Ricci scalar has a bump at positive values in the thick-wall regime, thus (partially) compensating the negative energy effect inside the bubble (see Fig.2.4). These considerations are particularly important in a cosmological perspective, as Higgs decay is expected to be driven by thick-wall bubbles, since the energy difference among the false vacuum and the true one is large. The Higgs bounce with Einstein-Hilbert gravity is reported in Fig.2.4.

2.4 Non-minimal coupling

A non-minimal coupling $\xi\phi^2 R$ has been recently considered in the literature, as it is required by perturbative renormalizability [92–94] and it is important in Higgs inflation (see [95,96] and references therein). While ξ should be small at low energies, it may get large at high ones due to quantum running of the couplings, thus possibly changing the decay rate by several order of magnitudes. It is thus important to consider also this coupling in the decay process. The action in this case is

$$S_E = 2\pi^2 \int dt\rho(t)^3 \left[-\frac{M_P^2 R}{2} - \xi\phi^2 R + \frac{\dot{\phi}^2}{2} + V(\phi) \right] \tag{2.44}$$

and the equations of motion are

$$\dot{\rho}^2 = 1 + \rho^2 \, \frac{\frac{\dot{\phi}^2}{2} - V(\phi) - 6\xi\,\phi\,\dot{\phi}\frac{\dot{\rho}}{\rho}}{3\,(M_P^2 + \xi\phi^2)} \tag{2.45}$$

$$\ddot{\phi} + 3\frac{\dot{\rho}\,\dot{\phi}}{\rho} = V' - \xi\phi R. \tag{2.46}$$

The trace of the Einstein equations is

$$\left(3M_P^2 + 3\xi(1+6\xi)\phi^2\right) R = 3\dot{\phi}^2(1+6\xi) + 12V(\phi) + 18\xi\phi V'. \tag{2.47}$$

Notice that the non-minimal coupling in Eq.(2.46) may be interpreted as a mass term for the scalar field, thus affecting the validity of Eq.(2.19) and Eq.(2.20) when the false vacuum state is de Sitter. In particular, the calculations underlying Eq.(2.20) rely on energy considerations, so its implications in this case might not be quite straightforward. To address this issue, one can redefine field variables, in order to recast the action in a more familiar form. The conformal transformation to the Einstein frame [97, 98] allows writing the action (which is called Einstein frame action) in the form of Eq.(2.7). The scalar field potential in the Einstein frame is

$$V_{EF}(\phi) = \frac{V(\phi)}{(1 - \xi\phi^2 M_P^{-2})^2}. \tag{2.48}$$

Notice that V_{EF} has the same sign of V, thus preserving the metastability of the false vacuum state in the Einstein frame. Moreover, the singular point $\phi = M\xi^{-1/2}$ should be excluded in the bounce evolution from the tunneling point to the false vacuum. Calculating $V''_{EF}(\phi)$ suggests that the non-minimal coupling interpretation as a scalar field mass might not be exhaustive.
Most of the literature on the role of the non-minimal coupling on vacuum decay has been focused on the Higgs field metastability with a flat space false vacuum [48,50,99] (an exception being [49]). This topic will be fully addressed in the next section.

2.5 Higgs decay from flat space

As described in Chapter 1, the Higgs field tunneling point lies at very high energies, at which new physics and gravity might be important. Gravitational corrections to Higgs decay in the form of an Einstein-Hilbert term in the action have been computed numerically (see for example [46]) and semi-analitically [100, 101]. Using the shooting method to find the bounce, the tunneling exponent slightly increases with respect to the flat space case

$$B \sim 2063 \implies \frac{B - B_0}{B_0} \sim 2\% \tag{2.49}$$

and the lifetime is

$$\Gamma^{-1} \sim 10^{662}\, T_U. \tag{2.50}$$

The numerical bounce is reported in Fig.2.4. As an alternative to the shooting method, one can compute the bounce semi-analytically, in the approximation of small gravitational backreaction [100, 101] (some concerns have been raised in the literature [50, 91] as regards the use of a perturbative expansion to determine the bounce action; they are partially addressed in Appendix A). Under this approximation, the action may be written as a function of the bounce radius $\mathcal{R}$ as

$$S_E = \frac{8\pi^2}{3|\lambda|} + S_q(\mathcal{R}) + \frac{32\pi^2}{45(\lambda \mathcal{R} M_P)^2} \tag{2.51}$$

where the third contribution is due to the Einstein-Hilbert term. Notice that the factor $\lambda \mathcal{R} M_P$ could have been easily inferred by dimensional consistency and by considering a transformation that leaves the equations of motion Eq.(2.7) unchanged

$$V(\phi) \to \alpha V(\phi) \qquad t \to t\,\alpha^{-1/2} \qquad \rho \to \rho\,\alpha^{-1/2} \tag{2.52}$$

where α is a real parameter. The action changes as $\alpha S_E \to S_E$[8] Minimizing S_E with respect to $\mathcal{R}$ allows to determine the bounce action. Notice that, disregarding quantum corrections, the action has no minimum for finite values of $\mathcal{R}$ and, thus, no bounce.

Including a non-minimal coupling $\xi\phi^2 R$ also marginally affects the Higgs in numerical calculations (see for example Fig.2 in [48,50]), and it amounts to multiplying the third factor in Eq.(2.51) by $(1 + 6\xi)^2$ [101]. In both methods, the bounce action, as a function of ξ, displays a minimum for nearly conformal values of the non-minimal coupling $\xi = -\frac{1}{6}$, and the on-shell action is close to the flat space one.

As mentioned in the introduction to this chapter, vacuum decay calculations on flat space are usually considered as an approximation to the actual geometry of the

[8] With similar arguments one finds that a scalar field with quartic potential and a small mass (as measured by $m^2\mathcal{R}^2$) has no bounce, as S_E should be

$$S_E = \frac{8\pi^2}{3|\lambda|}(1 + A m^2 \mathcal{R}^2 \lambda). \tag{2.53}$$

at lowest order in m, where A is a real constant.

Universe now, which is slightly accelerating [102], corresponding thus to a de Sitter Universe. Nonetheless, this acceleration is very small ($\Lambda \approx 10^{-122}$ in reduced Planck mass units, where $\sqrt{\Lambda} = H$). Thus, the bounce trajectory is approximately given by the flat space one until very large times $t \sim \bar{H}^{-1}$. As B is dominated by the bounce at small t, where the scalar field is far from the false vacuum, it seems that one can consistently disregard Λ altogether and that the Higgs metastability may be investigated in the flat spacetime approximation. This is extremely convenient, as the origin of the cosmological constant is still unknown. Nonetheless, in some theories, the bounce may altogether disappear for $\Lambda \neq 0$, no matter how small it is, due to the boundary value nature of the problem. This is even more significant as, turning on a cosmological constant, means turning the spacetime from non-compact to compact, which may questionably regarded as a small transformation of spacetime.

2.6 Higgs decay during inflation

Despite Higgs decay is very unlikely in our Universe now, the situation might have been different during inflation or reheating, thus possibly constraining cosmological models for the early Universe (see for example [52, 103–107], or [108] for a review on early Universe Higgs cosmology). During inflation, spacetime has a (quasi) de Sitter geometry with very large Hubble parameter $H_{inf} \sim 10^{13}$ GeV [81], whose precise value depends on the underlying cosmological model. H_{inf} is then used to compute the decay rate of the Higgs field, under the assumption that it does not participate to the inflationary dynamics. Historically, the argument that the curvature at the top of the potential barrier is much less than the Hubble constant was used to justify suppression of quantum tunnelling with respect to the thermal transition related to the Hawking-Moss instanton [68, 106], by the fact that the bubble radius is larger than the cosmological horizon. Nonetheless, this does not emerge in the arguments reported in Sect.2.1 relating the potential curvature to the Hubble constant, as the Higgs potential has no true vacuum state, and so it is not clear how far the potential barrier in Eq.(2.20) extends. Moreover, it has negative quartic derivative, which would put a lower (instead of upper) bound on H according to numerical evidence. Supposing that quantum tunneling is suppressed with respect to the thermal transition, the decay rate is determined as [81]

$$\Gamma \propto H^4 e^{-B_{HM}} \tag{2.54}$$

where B_{HM} is

$$B_{HM} = S_E(\phi_{\text{top}}, H) - S_E(\phi_{\text{fv}}, H). \tag{2.55}$$

Using Γ, one computes the average number of bubble produced during inflation, which should be $\lesssim 1$ [81]. Analogously, reheating mechanisms increase the temperature of the Universe leading to possibly thermal transitions beyond the potential barrier.

The two main limitations of this approach are that, first of all, inflation consists in a *quasi* de Sitter phase. Moreover, including modified gravity terms (which might be important in such strong gravity regimes) spoils the validity of Eq.(2.20) for the reasons mentioned in Sect.2.4. A stochastic approach towards the Higgs stability problem has been extensively considered in the literature [103, 104, 109] and allowed addressing the problem of de Sitter departure at the end of inflation [52]. The standard approach in this case is to consider the Higgs field as a spectator field that receives stochastic kicks at a scale set by the inflation scale H_{inf}. It has also been shown that a direct coupling among the inflation and the Higgs [109, 110] or a direct coupling to gravity [81] may change the picture.

2.7 Summary and discussion

In this chapter, gravitational corrections to the vacuum decay process are considered. After giving a plausibility argument for their importance, at least in a cosmological context, the scale factor behaviour on the bounce and its relation to the false vacuum value of the potential $V(\phi_{\rm fv})$ are described. The rich zoology of gravitational instanton when the false vacuum state is de Sitter is discussed. Following Coleman's seminal work [69], the tunneling exponent was determined in the thin-wall approximation and then thick-wall bubbles and modified gravity terms, in the form of a non-minimal coupling, were briefly considered. The Higgs decay on flat space and during inflation is addressed, and it is found to be small when the false vacuum has a flat geometry.
As described in this Chapter, the extension from flat to curved spacetimes proceeds by analogies rather than first principles, and, in fact, there are a number of issues when using the Euclidean path integral approach in this context. First of all, the Wick rotation to Euclidean space is ambiguous in non-static spacetimes [111], as well as the nucleation event [112, 113]. Moreover, the inclusion of gravitational terms in the action leads to the negative mode problem, which has been known for some time but still lacks a solution: it seems that, depending on the underlying scalar field potential, there is an infinite tower of imaginary contributions in the fluctuation determinant and, thus, it is not clear whether these solutions contribute to the decay rate or not (see for example [99, 114–122]). However, it seems that this problem may not arise in other approaches to vacuum decay, such as the Hamiltonian one [113, 123–125].There are also alternative methods to the vacuum decay problem that use a real time coordinate thus avoiding the Wick rotation to Euclidean space [126–131]. Research in this direction could improve our understanding of the vacuum decay process when gravitational effects are included.

Chapter 3

Vacuum decay of scalar fields with Einstein-Hilbert gravity

In the previous chapter, the path integral approach to vacuum decay was used to determine the decay rate in scalar field theories with gravitational interactions. The tunneling exponent, in particular, is computed by finding a trajectory, called Coleman-de Luccia instanton, with specific boundary conditions in Euclidean space. The resulting decay rate is to be compared with typical time scales of the system, in order to assess whether the tunneling phenomenon might be observed or not. In cosmological settings, such scale is the age of the Universe T_U. By comparing decay rates with T_U, one can rule out cosmological models and particle theories that are not consistent with the anthropic principle. Due to the boundary nature of the problem, the decay rate strongly depends on the underlying theory and, usually, it requires a full numerical calculation. In some theories, it might be even the case that the tunneling exponent is infinite, or the bounce trajectory does not exist, which, of course, cannot be proven with numerical methods. Analytic approximate solutions may be found in the thin-wall approximation (see Sect.2.2) or using a perturbative expansion when the gravitational backreaction is small [100, 101]: if the action has no minimum for finite non-vanishing values of the bounce radius $\mathcal{R}$, vacuum decay does not take place. Most systems, though, cannot be studied in this way and the question cannot be settled.

In this chapter, the problem of theory independent obstructions to the vacuum decay process is addressed. The focus is on single scalar field theories when the false vacuum has a flat or de Sitter geometry while the scalar field potential, $V(\phi)$, is arbitrary. Calculations and results reported in this Chapter concerning flat space false vacua are based on [132, 133], while the rest is original unpublished work. To do that, two methods are adopted: one consists in the analysis of the bounce trajectory near the spacetime boundary, which is particularly useful if the false vacuum is flat. The other aims to provide bounds on the Hubble constant, similar to the ones described in Chapter 2. The former will be discussed in Sect.3.1 in the case $V(\phi_{\rm fv}) = 0$: the asymptotic bounce trajectory (which will be, for short, called the "asymptotic bounce") may be computed in closed form for massless scalar fields with small cubic self-interactions (Sect.3.1.1) and massive scalar fields (Sect.3.1.2).

The extension to scalar field theories with Einstein-Hilbert gravity (Sect.3.2) is quite straightforward. In this way, one can verify whether

1. the equations of motion have a solution such that all fields approach the false vacuum at infinity;

2. this solution has well-defined and finite on-shell action.

The violation of Condition 1. means that only bubbles of infinite radius are on-shell, that is, the only solution satisfying the boundary conditions is the false vacuum static solution, and thus there is no phase transition to the true vacuum. If, instead, the on-shell action is infinite and positive, the decay rate vanishes (the so-called "vacuum quenching", see Chap.2.2). If infinite and negative, Γ is driven to $+\infty$, thus breaking the semi-classical regime. The bounce action may also be ill-defined near the false vacuum, i.e. at the upper bound of integration: this happens, for example, if our candidate metastable state is a minimum of the Euclidean potential, and thus a maximum in Minkowski space. Such a state does not exhibit any metastability, but only an instability related to the local unboundedness of the potential around such a fixed point[1]. Our results may be extended also to other theories of great physical interest, such as scalar field theories with derivative self-interactions (Sect.3.1.3) and in a number of space dimensions d other than three (Sec.3.1.4). The former have been found to be a candidate high-energy correction to solve the hierarchy problem [134–138]. The latter might be relevant in recent proposals for analogue experiments with $d < 3$ [127, 131, 139–143]. Also, it has been recently explored the possibility that our four-dimensional Universe lives on a five-dimensional bubble [144–147].
In all theories considered in this chapter, Conditions 1. and 2. turn out to be satisfied. Nonetheless, this method may be applied to other ones also, such as scalar fields with modified gravity, with more interesting results. This possibility will be explored in Chapter 4.

If $V(\phi_{\rm fv}) > 0$, instead, spacetime is compact, which implies that the boundary analysis does not constrain vacuum decay (Sect.3.3). As described in previous research (see Chapter 2), the existence of Coleman-de Luccia bounces depends on the Hubble constant value, which may be bounded from above or below, depending on the scalar field potential. In the following, such bounds will be discussed and improved. Potentials satisfying a monotonicity condition will be examined (Sect.3.3.1). This allows to recover previous results, such as Eq.(2.20), and also to provide an analytical explanation to numerical evidence present in the literature. Such bounds will be also extended to theories including an Einstein-Hilbert term (Sect.3.3.3) and modified gravity (Chapter 4).

[1] Despite this result seems trivial, as there is no potential barrier through which the scalar field can tunnel, early studies of the vacuum decay phenomenon actually focused on tunneling without barriers [24]. As an example, one can consider scalar field decay in a quartic potential with negative coupling, which has an analytic solution Eq.(1.93).

3.1 The asymptotic bounce

3.1.1 Massless scalar fields

The asymptotic behaviour of the bounce may be used to test for Conditions 1. and 2. and, thus, to determine if vacuum decay can happen. To compute it, consider first a scalar field theory with action Eq.(1.66). For simplicity, take $\phi_{\rm fv} = 0$. As a hand-waving argument, assume that the right-hand side of Eq.(1.68) is negligible with respect to the left-hand side, i.e.

$$\left|\frac{dV}{d\phi}\right| \ll \frac{3\dot{\phi}}{\rho} \approx \ddot{\phi}. \tag{3.1}$$

One gets

$$\dot{\phi}(\rho) = -\frac{C_0}{\rho^3} \tag{3.2}$$

where $C_0 > 0$ is an integration constant. The bounce then should satisfy

$$\phi(\rho) = \frac{C_0}{2\rho^2} \tag{3.3}$$

to lowest order for large ρ. In order for Eq.(3.1) and Eq.(3.3) to hold, a polynomial potential should depend on the scalar field as ϕ^n with $n > 3$ for large ρ on the bounce, i.e., near the false vacuum. Thus, Eq.(3.3) is consistent with the equation of motion if the potential contains only powers of the scalar field higher than three. Undershots and overshot arbitrarily close to the bounce satisfy Eq.(3.3) up to an arbitrarily large ρ^* that may be determined as the Euclidean radius such that the scalar field reaches the false vacuum ($\phi(\rho^*) = 0$) for overshots, or it turns around in the region $0 < \phi < \phi_{\rm top}$ ($\dot{\phi}(\rho^*) = 0$) for undershots.

There are two reasons why this result need a more sound derivation of Eq.(3.3), beyond the consistency requirements just described. Firstly, C_0 might vanish: in this case higher order terms in Eq.(3.3) are important. Secondly, the bounce cannot be determined numerically with infinite precision; thus, it is important to understand how nearby trajectories behave. Let's consider a generic undershot and look for an approximate solution for large $\rho \leq \rho^*$, where ρ^* marks the radius at which the scalar field turns around for the first time ($\dot{\phi}(\rho^*) = 0$). The right-hand side of Eq.(1.68), expanded around ρ^*, reads (the subscript $_*$ indicates quantities evaluated at ρ^*)

$$\frac{dV}{d\phi} = \left(\frac{dV}{d\phi}\right)_* + \sum_{n\geq 2} f_{n*}\frac{(\rho-\rho^*)^n}{n!} \tag{3.4}$$

where the general form of the coefficients f_{n*} is reported in Appendix B. By requiring that

$$\left(\frac{d^jV}{d\phi^j}\right)_* \ddot{\phi}_*^{j-2}\rho^{*2j-2} \ll 1 \tag{3.5}$$

for $j \geq 2$ one finds

$$\frac{dV}{d\phi} \approx \left(\frac{dV}{d\phi}\right)_* \tag{3.6}$$

in the large ρ^* limit[2]. Under these conditions, Eq.(1.68) becomes

$$\ddot{\phi} + \frac{3\,\dot{\phi}}{\rho} = \left(\frac{dV}{d\phi}\right)_* \tag{3.7}$$

and the solution reads

$$\phi(\rho) = \phi_* - \left(\frac{dV}{d\phi}\right)_* \frac{\rho^{*2}}{4} + \left(\frac{dV}{d\phi}\right)_* \frac{\rho^2}{8} + \left(\frac{dV}{d\phi}\right)_* \frac{\rho^{*4}}{8\rho^2}\,, \tag{3.8}$$

using $\phi(\rho_*) = \phi_*$ and $\dot{\phi}(\rho_*) = 0$ to fix the two integration constants. Taking the limit $\phi_* \to 0$ and $\rho^* \to +\infty$ one gets the asymptotic behaviour of the bounce

$$\begin{aligned} \lim_{\substack{\phi_* \to 0 \\ \rho^* \to +\infty}} \dot{\phi}(\rho) &= -\frac{C_0}{\rho^3}\,, \qquad \lim_{\substack{\phi_* \to 0 \\ \rho^* \to +\infty}} \rho^{*4} \left(\frac{dV}{d\phi}\right)_* = 4C_0\,, \\ \lim_{\substack{\phi_* \to 0 \\ \rho^* \to +\infty}} \phi(\rho) &= \frac{C_0}{2\rho^2}\,, \end{aligned} \tag{3.9}$$

from sufficiently large ρ up to $\rho \to +\infty$. The constant C_0 is determined by the second limit in Eqs.(3.9). Notice that this behaviour is consistent to the one found in the case of Higgs decay described in Chapter 1. As

$$\lim_{\rho^* \to +\infty} \ddot{\phi}_* \rho^{*4} = 4C_0, \tag{3.10}$$

Eq.(3.5) becomes

$$\left(\frac{d^2V}{d\phi^2}\right)_* \rho^{*2} \ll 1\,, \qquad 4C_0 \left(\frac{d^3V}{d\phi^3}\right)_* \ll 1\,, \tag{3.11}$$

implying that the scalar field should be massless with small cubic self-interactions. A massive but light scalar field satisfies Eq.(3.9) for $\rho^* \ll m^{-1}$, while for $\rho^* \gg m^{-1}$ instead it should be proportional to $e^{-m\rho}$ [59]. The proportionality constant may be determined with our method, but a finite mass makes calculations much more involved, as Eq. (3.11) suggests that the potential in that case is important at each order in the Taylor expansion. For this reason, the massive case will be separately analysed below. By using Eq.(3.9) it is trivial to verify that the action Eq.(2.4), computed on the asymptotic bounce, is a convergent integral at $t \to +\infty$, and thus Condition 2. holds.

3.1.2 Massive scalar fields

The asymptotic bounce associated to a scalar field with mass m and in the absence of gravitational interactions was found in [59] and reads

$$\phi(\rho) = C_0 e^{-m\rho}. \tag{3.12}$$

[2]This condition may also be formulated as $\rho - \rho^* \sim -A\rho^*$ with A of order $\lesssim O(1)$.

This result is based on the fact that the mass term dominates the potential at small values of ϕ, and thus the scalar field should be proportional to the Green function of a massive scalar field theory. Notice that the action Eq.(1.66) would diverge on the bounce if Eq.(3.3) held, giving

$$\int_0^{+\infty} d\rho\, \rho^3 \phi^2 \approx \ln(\rho)|_0^{+\infty}. \tag{3.13}$$

Eq.(3.12) may be obtained also with an asymptotic bounce analysis, which allows a straightforward generalization to theories interacting with gravity. In Sect.3.1.1 massive scalar fields were excluded, as they violate Eq.(3.11) at large ρ^*. This suggests that every order in the Taylor expansion is dominated by the mass term, and that each one is equally important. The magnitude of terms on left-hand side of Eq.(1.68) instead is unkown. To tackle the problem, the friction term may be absorbed in the potential to see how large it is compared to the others when $\rho^* \to +\infty$. In this way, the potential acquires an explicit dependence on ρ that should be accounted for in the Taylor expansion for finite and large ρ^* to find the asymptotic bounce. The results of the previous section should be thus extended to ρ-dependent potentials. This generalization is provided below.

Radius-dependent potential

The origin of a radial dependence in the scalar field potential may come about in various ways. Of course, a ρ-dependent coupling may be put by hand, if the underlying physical phenomenon that the field theory wants to describe contains such a feature. Another possibility is that the potential is independent of the radius, but such dependence is acquired when looking for the asymptotic bounce. Consider, for example, a theory with n scalar fields, in which all but one are decoupled from the others near the false vacuum i.e.

$$\frac{\partial V}{\partial \phi_i} \quad \text{is a function of } \phi_i \text{ only, for } i = 2, \dots, n \tag{3.14}$$

$$\frac{\partial V}{\partial \phi_1} \quad \text{is a function of all scalar fields.} \tag{3.15}$$

The equations of motion then are

$$\ddot{\phi}_i + \frac{3\dot{\phi}_i}{\rho} = \frac{\partial V}{\partial \phi_i}(\phi_i) \quad \text{for } i = 2, \dots, n \qquad \ddot{\phi}_1 + \frac{3\dot{\phi}_1}{\rho} = \frac{\partial V}{\partial \phi_1}(\phi_1, \phi_i). \tag{3.16}$$

To find the asymptotic bounce of ϕ_1, initial conditions $\phi_i(0)$ maybe chosen such that ϕ_i for $i = 2, \dots, n$ are on the bounce, and ϕ_1 lies on a undershoot trajectory. One has

$$\frac{\partial V}{\partial \phi_1}(\phi_1, \phi_i) = \frac{\partial V}{\partial \phi_1}(\phi_1, \phi_{i,\mathrm{b}}(\rho)) \equiv \frac{\partial V}{\partial \phi}(\phi, \rho) \tag{3.17}$$

where $\phi_{i,\mathrm{b}}(\rho)$ are the bounces for $\phi_2, \dots, \phi_n$ and ϕ_1 is labelled as ϕ. In this way, the potential depends on ρ both through the scalar field ϕ and explicitly.

Expand now the right-hand side of Eq.(1.68) in a Taylor series around the turning point ρ^*

$$\frac{\partial V}{\partial \phi} = \left(\frac{dV}{d\phi}\right)_* + \sum_{n\geq 1} f_{*n} \frac{(\rho-\rho^*)^n}{n!}, \tag{3.18}$$

$$f_{*n} \equiv \left(\frac{dV}{d\phi}\right)_*^{(n)} \tag{3.19}$$

with f_{*n} evaluated in the large ρ^* limit. The details of the calculation are reported in Appendix C and they are a generalization of the ones computed in the massless case. One finds that the zeroth-order term in the Taylor expansion dominates if, besides Eq. (3.5), one has finite

$$\left(\frac{\partial^{i+j} V}{\partial \phi^i \partial \rho^j}\right)_* \rho^{*j+2i-2} \ddot{\phi}_*^{\,i-2} \qquad \text{for } i \geq 2 \text{ and } j \geq 1 \tag{3.20}$$

with

$$\left(\frac{\partial^{j} V}{\partial \phi \partial \rho^j}\right)_* \rho^{*j} \ll \ddot{\phi}_* \qquad \text{for} \qquad j \geq 1, \tag{3.21}$$

and

$$\left(\frac{\partial^{i} V}{\partial \phi^i \partial \rho^j}\right)_* \rho^{*j} \ll \left(\frac{\partial^{i} V}{\partial \phi^i}\right)_* \qquad \text{for } i \geq 2 \text{ and } j \geq 1 \tag{3.22}$$

in the large ρ_* limit. Then, Eq.s (3.9) hold. If, instead, Eq.(3.5) is violated by some terms in the potential (such as a mass term), the right-hand side of Eq.(1.68) consists of the zeroth order of the expansion plus a resummed part that accounts for them. This will be important in the next section.

Massive scalar field

The asymptotic bounce of a massive scalar field theory may be determined with the help of the results just derived for a radius-dependent potential. $V(\phi)$ in this section is chosen such that it reduces to

$$V(\phi) \approx \frac{m^2}{2}\phi^2 \tag{3.23}$$

for $\phi \approx 0$ (again $\phi_{\rm fv} = 0$). As anticipated above, it is convenient to include the friction term and the mass term as contributions to the right-hand side of Eq.(1.68), by writing the equation of motion as

$$\ddot{\phi} = m(\rho)^2 \phi - \frac{d}{d\rho}\left(\frac{3\phi}{\rho}\right) \qquad m(\rho)^2 = m^2 - \frac{3}{\rho^2}. \tag{3.24}$$

Defining

$$\frac{\partial V_1}{\partial \phi} \equiv m(\rho)^2 \phi, \qquad \frac{\partial V_2}{\partial \phi} \equiv -\frac{3\phi}{\rho}, \tag{3.25}$$

one gets

$$\ddot{\phi} = \frac{\partial V_1}{\partial \phi} + \frac{d}{d\rho}\left(\frac{\partial V_2}{\partial \phi}\right). \tag{3.26}$$

The resulting Taylor expansion around ρ^* is

$$\frac{\partial V_1}{\partial \phi} + \frac{d}{d\rho}\left(\frac{\partial V_2}{\partial \phi}\right) = m^2\phi_* + \sum_{n\geq 1}\left(\left(\frac{\partial V_1}{\partial \phi}\right)^{(n)} + \left(\frac{\partial V_2}{\partial \phi}\right)^{(n+1)}\right)\frac{(\rho-\rho^*)^n}{n!}. \tag{3.27}$$

For sufficiently large ρ^*, ρ [3], one gets

$$\sum_{n\geq 1}\left(\left(\frac{\partial V_1}{\partial \phi}\right)^{(n)} + \left(\frac{\partial V_2}{\partial \phi}\right)^{(n+1)}\right)(\rho-\rho^*)^n = \sum_{n\geq 1}\left(\frac{\partial V_1}{\partial \phi}\right)^{(n)}\rho^{*n} + \sum_{n\geq 2}\left(\frac{\partial V_2}{\partial \phi}\right)^{(n)}\rho^{*n-1} \tag{3.28}$$

apart from numerical factors. Eq.(3.28) is tantamount to the Taylor expansion for a theory with equation of motion

$$\ddot{\phi} = m(\rho)^2\phi \qquad m^2(\rho) = m^2 - \frac{3}{\rho^2} - \frac{3}{\rho^*\rho} \tag{3.29}$$

except that this has an additional first-order term. Defining

$$\frac{\partial V_3}{\partial \phi} = \frac{1}{\rho^*}\frac{\partial V_2}{\partial \phi}, \tag{3.30}$$

one gets

$$\frac{\partial^{2+i}(V_1+V_3)}{\partial\phi^2\partial\rho^i}(\phi_*,\rho^*)\rho^{*i} \approx \rho^{*-2} \qquad \text{for } i\geq 1, \tag{3.31}$$

and

$$\frac{\partial^{i+1}(V_1+V_3)}{\partial\phi\partial\rho^i}(\phi_*,\rho^*)\rho^{*i} \approx \rho^{*-2}\phi_* \qquad \text{for } i\geq 1. \tag{3.32}$$

This means that the time-dependent part of the potential is negligible at higher orders than the zeroth one. Then the equation of motion reduces to

$$\ddot{\phi} \approx m^2\phi \tag{3.33}$$

for large ρ^* near the bounce. A solution to Eq.(3.33) is

$$\phi(\rho) = \frac{\phi_*}{2}e^{m\rho^*}(e^{-m\rho} + e^{-2m\rho^*}e^{m\rho}) \tag{3.34}$$

where integration constants are chosen such that $\dot{\phi}(\rho^*) = 0$ and $\phi(\rho^*) = \phi_*$. Thus, taking the limit $\rho^* \to +\infty$, Eq.(3.34) reduces to

$$\phi(t) = C_0 e^{-m\rho}, \qquad \lim_{\substack{\phi_*\to 0\\ \rho^*\to+\infty}} e^{m\rho^*}\ddot{\phi}_* = m^2 C_0. \tag{3.35}$$

[3] Equivalently, $\rho - \rho^* = -A\rho^*$ with A some constant of order $\lesssim O(1)$.

Notice that

$$\lim_{\rho^* \to +\infty} \ddot{\phi}_* \rho^{*4} = 0, \tag{3.36}$$

and thus other terms in the Taylor expansion are important only at zeroth-order (at which they are anyway subdominant with respect to the mass term), consistently with Eq.(3.5). Moreover, self-interactions in $V(\phi)$ are smaller than the mass term at least by a factor e^{-mt} and thus they are negligible for $t \gg m^{-1}$. As a consequence, the exponential behavior becomes important only in a narrow region around the false vacuum, namely, when the mass term dominates. This result agrees with the ones in [59], in which is stated that massless fields dominate the bounce for $\mathcal{R}m \ll 1$, and the mass is important only for large values of ρ.

For ϕ as in Eq.(3.12), energy is approximately conserved, as

$$\frac{\dot{\phi}^2}{2} - V(\phi) = 0 \qquad \text{for } \phi(\rho) = C_0 e^{-m\rho} \text{ and } V(\phi) = \frac{m^2}{2}\phi^2. \tag{3.37}$$

One may think that, by adding a mass term to a scalar field theory with a negative quartic potential, the field would overshoot for all $\phi(0)$, as the energy loss is reduced with respect to the scale invariant solution Eq.(1.93). Actually, the field undershoots: the reason is that the scalar field decays as a power of ρ even when the mass term dominates over the quartic one in the potential, but their contribution to the equation of motion is negligibly small with respect to the friction term and $\ddot{\phi}$. Such mass term induces additional loss of energy in the system, and, as a result, the field cannot climb the hill to reach the false vacuum. In this way, one recovers the well-known fact that a massive scalar field theory with a quartic potential does not have a bounce.

3.1.3 Higher-order kinetic terms

Quantum tunnelling through an energy barrier has an exponentially small probability to occur in the semi-classical approximation. The smallness of some numbers, for example the ratio among the Higgs vacuum expectation value and the Planck mass, may be viewed in these terms (thus alleviating the hierarchy problem [134–138]), as

$$\langle \phi \rangle = \int \mathcal{D}\phi \, \phi \, e^{-S} \equiv M_P e^{-W}, \tag{3.38}$$

where W is the generating functional computed on the bounce. The Coleman-de Luccia instanton is found by solving the equations of motion of the original theory with a pointlike source

$$\phi(0) \equiv M_P \exp\left(M_P^{-1}\psi(0)\right) = M_P \exp\left(M_P^{-1} \int d^4x \delta(x) \psi(x)\right). \tag{3.39}$$

which generates a singular instanton at $\rho = 0$. The singularity drives W to infinity. It has been shown that a possible way to make it finite is to add higher order kinetic terms $(\partial\psi)^n$, $n > 2$, to the Lagrangian [134–138]. Another application of higher-derivative terms to solve the hierarchy problem is addressed in [148] regarding the agravity theory [149].

To find the asymptotic bounce, the right-hand side of the equation of motion in presence of higher-order kinetic terms

$$\ddot{\psi} + \frac{3\dot{\psi}}{\rho} = \frac{dV}{d\psi} + n(n-1)\ddot{\psi}\dot{\psi}^{n-2} + \frac{3n\dot{\psi}^{n-1}}{\rho} \tag{3.40}$$

needs to be Taylor expanded around the turning point ρ^*. Here $V(\psi)$ contains non-derivative terms that generate a potential barrier, through which the scalar field can tunnel. At zeroth-order the kinetic term gives a vanishing contribution, as $\dot{\phi}(\rho^*) = 0$. Moreover, $\ddot{\psi}\dot{\psi}^{n-2}$ appears in the Taylor expansion of ψ^{n-1} to n−th order, namely

$$\left(\psi^{n-1}\right)^{(n)} = n!\frac{(n-1)}{2}\ddot{\psi}\dot{\psi}^{n-2} + \dots \tag{3.41}$$

Thus, if the latter satisfies Eq.(3.5), also the former does, as the calculation reported in Appendices B-C, do not depend on incidental cancellations among the first term on the right-hand side and the ones in . . . . Then, the Taylor expansion of the n-order kinetic term can be traded with the one for $(\psi^{n-1})^{(n)}$. Moreover,

$$\sum_{i=1} \left(\psi^{n-1}\right)_*^{(n+i)} \frac{(\rho-\rho^*)^i}{i!} \underset{\rho-\rho^*\approx -A\rho^*}{=} \sum_{i=n+1} \left(\frac{\psi^{n-1}}{\rho^{*n}}\right)_*^{(i)} \frac{\rho^{*i}}{i!} \tag{3.42}$$

apart from numerical factors. So one finds that Eq.(3.42) is tantamount to the Taylor expansion of a theory with equation of motion

$$\ddot{\psi} + \frac{3\dot{\psi}}{\rho}\left(1 - \dot{\psi}^{n-2}\right) = \frac{dV}{d\psi} + \frac{\psi^{n-1}}{\rho^{*n}} \qquad n > 2 \tag{3.43}$$

apart that the latter has additional n orders. So, order in the Taylor expansion for a theory as in Eq.(3.43) is negligible with respect to the zeroth-one, then the same holds for the original one. The latter term satisfies Eq.(3.5) for $n > 2$, $j \geq 1$ and, thus, it does not affect significantly the Taylor expansion. Moreover,

$$\frac{3\dot{\psi}}{\rho}\left(1 - \dot{\psi}^{n-2}\right) \approx \frac{3\dot{\psi}}{\rho} \tag{3.44}$$

to lowest order near the bounce at large times. Then, massless ψ satisfy Eq.(3.9) for large t on the bounce if higher-order kinetic terms are added to the Lagrangian. If ψ is massive, then Eq.(3.12) holds instead.

3.1.4 Changing the number of spacetime dimensions

The results of the previous sections may be extended to a scalar field theory defined on a spacetime of arbitrary dimension $d + 1$ (d space dimensions). The $O(d + 1)$-symmetric action is (here Γ is the Euler Gamma Function)

$$S_E = \frac{2\pi^{(d+1)/2}}{\Gamma\left(\frac{d+1}{2}\right)} \int d\rho\, \rho^d \left(\frac{\dot{\phi}^2}{2} + V(\phi)\right). \tag{3.45}$$

with equation of motion

$$\ddot{\phi} + \frac{d}{\rho}\dot{\phi} = \frac{dV}{d\phi}. \tag{3.46}$$

The proofs in Appendices B-C do not depend on d, as it only amounts to a numerical factor in Eq.(3.46). Eq.(3.11) instead does, as it is computed on the asymptotic bounce in three dimensions. If Eq.(3.5) holds, the equation of motion has a d dependent solution that, in the limit of large ρ^*, gives

$$\lim_{\substack{\phi_*\to 0\\ \rho^*\to+\infty}} \dot{\phi}(\rho) = -\frac{C_0}{\rho^d}, \qquad \lim_{\substack{\phi_*\to 0\\ \rho^*\to+\infty}} \rho^{*d+1}\left(\frac{dV}{d\phi}\right)_* = (d+1)C_0. \tag{3.47}$$

Eq.(3.5) and Eq.(3.47) suggest that, for $d > 3$, the scalar field should be massless. In $d = 2$ instead they give

$$\left(\frac{d^2V}{d\phi^2}\right)_* \rho^{*2} \ll 1, \qquad 4C_0\left(\frac{d^3V}{d\phi^3}\right)_* \rho^* \ll 1, \qquad 16C_0^2\left(\frac{d^4V}{d\phi^4}\right)_* \ll 1, \tag{3.48}$$

so, in this case, also cubic terms are excluded. In $d = 1$ instead one has that $\phi(\rho)$ diverges as $\rho \to +\infty$ (as can be seen by integrating the first of Eq.s(3.47)) and thus higher-order terms in the Taylor expanded potential are important in determining the asymptotic bounce.

In all the cases analyzed above, massive scalar fields are excluded, as they violate Eq.(3.5), dominating the Taylor expansion at each order. The proof in Sect.3.1.2 is independent of the number of spacetime dimensions, as it only amounts to a numerical factor in Eq.(3.46). So, if the potential is dominated by the mass term near the false vacuum, the asymptotic bounce is given by Eq.(3.12).

3.2 Asymptotic bounce with Einstein-Hilbert gravity

The asymptotic bounce in four-dimensional single scalar field theories with Einstein-Hilbert gravity is similar to the ones found above. The Euclidean action in this case is given by Eq.(2.4) with line element Eq.(2.3) and equations of motion Eq.(2.7)-(2.8). As $V(\phi_{\text{fv}}) = 0$, the scale factor at large times on the bounce reads

$$\rho(t) = t + \text{higher orders} \tag{3.49}$$

which implies that, to lowest order,

$$\frac{\dot{\rho}}{\rho} \approx \frac{1}{t} \tag{3.50}$$

and thus Eq.(1.68) holds, with ρ replaced by t. This readily gives the asymptotic bounce for a massless scalar field interacting with Einstein-Hilbert gravity

$$\begin{aligned} \lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}} \dot\phi(t) &= -\frac{C_0}{t^3}\,, \qquad \lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}} t^{*4}\left(\frac{dV}{d\phi}\right)_* = 4C_0\,, \\ \lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}} \phi(t) &= \frac{C_0}{2t^2} \end{aligned} \tag{3.51}$$

and the one for massive scalar fields

$$\lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}} \phi(t) = C_0 e^{-mt}, \qquad \lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}} \ddot\phi_* e^{mt^*} = m^2 C_0. \tag{3.52}$$

Again, higher-order kinetic terms do not affect these results. Notice that the power law time dependence of massless scalar fields is consistent with the Higgs bounce one discussed in Sect.2.5. Moreover, plugging Eq.(3.51) (or Eq.(3.52)) and Eq.(3.50) in Eq.(2.8), one finds that $\dot\rho \approx 1$, proving that Condition 1. holds. Condition 2. is also satisfied, as the action Eq.(2.4), computed on the asymptotic bounce, is a convergent integral at $t \to +\infty$.

The discussion in an arbitrary number d of space dimensions is analogous. The line element in this case is

$$ds^2 = dt^2 + \rho(t)^2 d\Omega_d^2 \tag{3.53}$$

and the equation of motion for the scale factor is as in Eq.(2.8) for $d \geq 2$, with the factor 3 replaced with a $d-$dependent term. Thus, at large radii on the bounce,

$$\dot\rho^2 = 1 + O(t^{2-2d}) \qquad \text{for } \rho(t) \approx t,\ d \geq 2 \text{ and } \phi(t) \propto t^{-d+1} \tag{3.54}$$

for massless scalar fields and

$$\dot\rho^2 = 1 + O(t^2 e^{-2mt}) \qquad \text{for } \rho(t) \approx t \text{ and } \phi(t) \propto e^{-mt}. \tag{3.55}$$

which consistently gives $\dot\rho \approx 1$ for $t \to +\infty$, thus satisfying Condition 1. Moreover, the on-shell action is convergent as regards the upper bound of integration, thus verifying Condition 2.

3.2.1 Examples

In the previous sections, the asymptotic bounce analysis in four spacetime dimensions was used to verify the exponential radial dependence of massive scalar fields, already described in [59], and to determine the asymptotic bounce for massless ones, possibly with higher-order kinetic terms. This new result is now tested in two scalar field theories and Einstein-Hilbert gravity, by comparing the second of Eq.s(3.9) with a numerical evaluation. The scalar field potentials are[4]

[4] The mass unit is $G = (M_P^2 8\pi)^{-1} = 1$

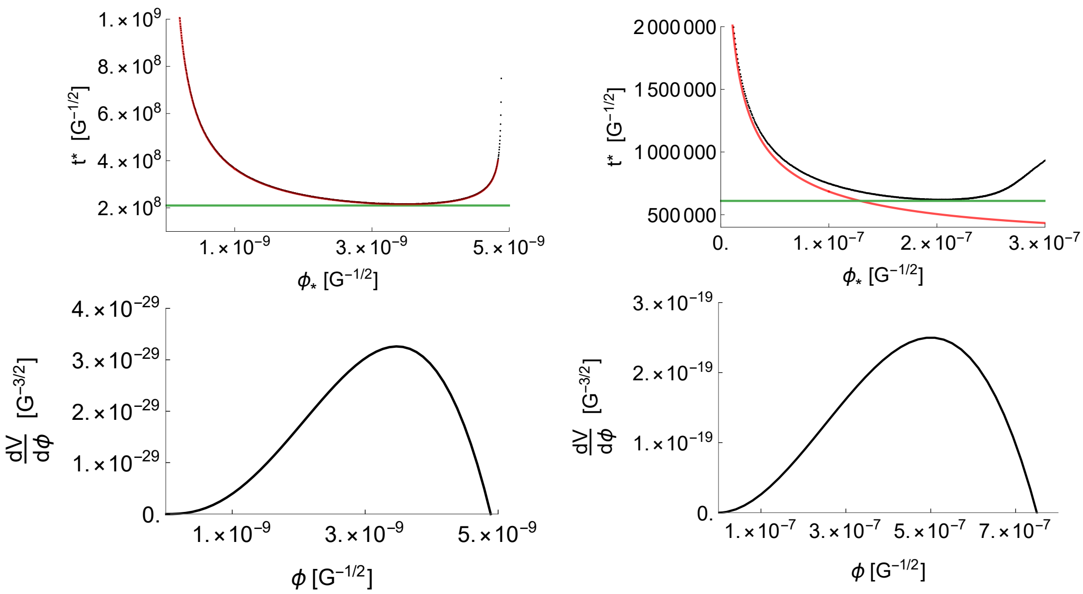

Figure 3.1: Top: t^* as a function of ϕ_* for potentials Eq.(1.90) (on the left) and Eq.(3.56) (on the right), in the vicinity of the bounce ($\phi_* \to 0$). Bottom: $V'(\phi)$ as a function of ϕ.

- the Higgs potential Eq.(1.90);
- a polynomial potential with vanishing quadratic term

$$V(\phi) = \alpha_1 \phi^5 + \alpha_2 \phi^4 + \alpha_3 \phi^3 \tag{3.56}$$

where $\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 10^{-6}$.

As reported in Fig.3.1 (on the bottom), they both have false vacuum states at $\phi_{\rm fv} = 0$. One can easily prove that Eq.(3.5) is satisfied for $\phi_* \to 0, t^* \to +\infty$ for all $\gamma, \beta, \lambda^*, \alpha_1$, α_2 and $\alpha_3 C_0 \ll 1$, and thus the asymptotic bounce should be as in Eq.s(3.9) if $C_o \ll 10^6$. The theoretical prediction for $t^*(\phi_*)$ (black line) is compared with a numerical evaluation (red line) in Fig.3.1 [5] . The two results agree in the $\phi_* \to 0$ limit, i.e. close to the bounce, for both potentials. The green line instead marks the position of the minimum of $t^*(\phi_*)$ in the numerical calculation, which corresponds to the maximum of $\frac{dV}{d\phi}$ in the Higgs case, as predicted by Eq.s(3.9). The polynomial potential, instead, reaches the right behaviour only for smaller ϕ_*.

3.3 Decay from de Sitter space

The crucial difference among flat space and the de Sitter one is that the former is non-compact, while the latter is compact. As a result, the asymptotic bounce derived in Sect.s 3.1-3.2 is not quite informative in this case. As described in Chapter 2, boundary conditions on a fixed de Sitter background require the scalar field velocity to vanish at the edge of spacetime, which lies at $Ht = \pi$, leading to uptunneling

[5] C_0s are chosen as the ones determined numerically with the method described in Appendix D and reported in Table1.

as well as downtunneling solutions: the initial and final points of the evolution can be exchanged, and, thus, not much information may be gathered by analyzing boundary conditions [6]. It thus proves more useful to follow the line of reasoning outlined in Chapter 2 and look instead for relations among H and $V(\phi)$ constraining the Coleman-de Luccia instanton, which are the subject of the next section. As a starting point, a scalar field on a fixed de Sitter background is considered, and then results are extended to theories with an Einstein-Hilbert term in Sect.3.3.3. To investigate such bounds, $\phi(t)$ is mapped into another scalar field, $\psi(t)$, which is an instanton if the former is. One finds that a bounce does not exist if

$$V'(\phi) + 4H^2(\phi - \phi_{\text{top}}) \tag{3.58}$$

is monotonic throughout the potential barrier. This is the same as Eq.(2.20) for monotonically *increasing* Eq.(3.58), i.e.

$$V''(\phi) + 4H^2 > 0 \qquad \text{for } \phi \in [\phi_{\text{fv}}, \phi_{\text{tv}}]. \tag{3.59}$$

As increasing H makes

$$V''(\phi) + 4H^2 \tag{3.60}$$

larger, this puts an upper bound on H for a Coleman-de Luccia instanton to exist. For the same reasons, a monotonically *decreasing* Eq.(3.58), i.e.

$$V''(\phi) + 4H^2 < 0 \qquad \text{for } \phi \in [\phi_{\text{fv}}, \phi_{\text{tv}}], \tag{3.61}$$

puts a lower bound on H.
As described below, if the bound (which will be indicated as H_{bound} in the following) is determined by the equation

$$V''(\phi_{\text{top}}) = -4H^2_{\text{bound}} \tag{3.62}$$

then the static solution $\phi(t) = \phi_{\text{top}}$ is approached as $H \to H_{\text{bound}}$. In this way, one finds that potentials that softly break the non-monotonicity requirement also constrain the bounce by upper and lower bounds on H that are determined by Eq.(3.62).

Actually, the calculations reported in the next section are analogous if one just requires that Eq.(3.58) is of definite (and opposite) sign in the two regions $[\phi_{\text{fv}}, \phi_{\text{tv}}]$. Nonetheless, analogies with existing literature emerge if one additionally requires monotonicity. In the next Chapter, though, this aspect will turn out to be important in theories with an Einstein-Hilbert term and a non-minimal coupling. For the time being, though, the monotonicity requirement sufficies.

[6] The asymptotic bounce of a scalar field on a de Sitter background is given by

$$\phi(t) = \bar{\phi} + \frac{V'(\bar{\phi})}{4H^2}(Ht - \pi)^2 + \text{higher orders} \tag{3.57}$$

and it is computed by expanding Eq.(2.17) around the boundary $Ht = \pi$ with $\dot{\phi}(\pi) = 0$ and $\phi(\pi) = \bar{\phi}$. The bounce at small t corresponds to Eq.(3.57) with $\bar{\phi}$ exchanged with the initial condition $\phi(0)$ and $Ht - \pi \to Ht$.

3.3.1 Bounds on H

Consider the following change of variables [7]

$$\psi(Ht) = \frac{\phi(Ht) - \phi(\pi/2)}{\cos(Ht)} \tag{3.63}$$

that relates an instanton solution $\phi(t)$ to another one $\psi(t)$. The latter satisfies the following equation of motion

$$\cos(Ht)\ddot{\psi} + \left(\frac{3H}{\sin(Ht)} - 5H\sin(Ht)\right)\dot{\psi} = V' + 4H^2(\phi - \phi(\pi/2)). \tag{3.64}$$

There are two self-accelerating regions, one in $Ht \in [\arctan(\sqrt{3/2}), \pi/2]$ and one in $Ht \in [\pi - \arctan(\sqrt{3/2}), \pi]$. The number of turning points for ψ on the bounce (t^* such that $\dot{\psi}(t^*) = 0$) depends on $V(\phi)$, and it is preserved on the bounce as H changes, as, otherwise, infinite energy (i.e., non-instantonic) configurations would be crossed. As the spacetime background seen by ψ is approximately two copies of de Sitter space (one in $[0, \pi/2]$ and the other in $[\pi/2, \pi]$), the number of turning points in the ψ-instanton should be at most one for a Coleman-de Luccia (i.e. non-oscillating)) $\phi-$bounce.
One can easily see that, if Eq.(3.58) is monotonic throughout the potential barrier, there are no ψ-instantons with zero turning points. $\ddot{\psi}(0)$ and $\ddot{\psi}(\pi)$ read

$$\ddot{\psi}(0) = V'(\phi(0)) + 4H^2(\phi(0) - \phi(\pi/2)), \tag{3.65}$$
$$\ddot{\psi}(\pi) = -V'(\phi(\pi)) - 4H^2(\phi(\pi) - \phi(\pi/2)), \tag{3.66}$$

and thus

$$\begin{cases} \ddot{\psi}(\pi) > 0 & \text{for} \quad \phi(\pi/2) > \phi_{\text{top}} \\ \ddot{\psi}(0) > 0 & \text{for} \quad \phi(\pi/2) < \phi_{\text{top}} \end{cases} \tag{3.67}$$

if Eq.(3.58) is monotonically increasing (see Fig.3.3). If there are no turning points, this gives

$$\begin{cases} \dot{\psi}(Ht) > 0 & \text{for} \quad \phi(\pi/2) > \phi_{\text{top}} \\ \dot{\psi}(Ht) < 0 & \text{for} \quad \phi(\pi/2) < \phi_{\text{top}}. \end{cases} \tag{3.68}$$

for all t (see Fig.3.3). However, $\dot{\psi}(\pi/2)$ is given by

$$\dot{\psi}(\pi/2) = -\frac{\ddot{\phi}(\pi/2)}{2H} \tag{3.69}$$

contradicting Eq.(3.68). Analogously, one can prove that there are no $\psi-$instantons with zero turning points and monotonically decreasing Eq.(3.58). This leaves only ψ-instantons with one turning point. The acceleration at this point (which in the following is labelled as t^*) is given by

$$\ddot{\psi}(Ht^*) = \frac{1}{\cos(Ht^*)}\left(V'(\phi) + 4H^2(\phi(Ht^*) - \phi(\pi/2))\right) \tag{3.70}$$

[7]In this section fields are written as a function of Ht so that spacetime extends from 0 to π.

and should be negative if $\ddot{\psi}(0) > 0$, while positive if $\ddot{\psi}(0) < 0$ (see Fig.3.3). For simplicity, let us set $\ddot{\psi}(0) > 0$. This implies that either

$$\begin{aligned} V'(\phi(Ht^*)) + 4H^2\phi(Ht^*) < 4H^2\phi(\pi/2) \qquad &\text{for} \qquad Ht^* < \pi/2, \qquad \text{or} \\ V'(\phi(Ht^*)) + 4H^2\phi(Ht^*) > 4H^2\phi(\pi/2) \qquad &\text{for} \qquad Ht^* > \pi/2, \qquad \text{or} \\ V''(\phi(\pi/2)) + 4H^2 < 0 \qquad &\text{for} \qquad Ht^* = \pi/2. \end{aligned} \tag{3.71}$$

The latter in particular requires $\dot{\psi}(\pi/2) = 0$, i.e. $\phi(\pi/2) = \phi(Ht^*) = \phi_{\text{top}}$. As the bounce decreases monotonically in time, one has

$$\phi(Ht^*) > \phi(\pi/2) \qquad \text{for} \qquad Ht^* < \pi/2 \tag{3.72}$$

giving

$$V'(\phi(Ht^*)) < 0. \tag{3.73}$$

For the same reason, $\dot{\psi}(\pi/2)$ should be negative, giving $\phi(\pi/2) < \phi_{\text{top}}$ and thus

$$V'(\phi(Ht^*)) + 4H^2\phi(Ht^*) < 4H^2\phi_{\text{top}}. \tag{3.74}$$

Analogously, one gets

$$V'(\phi(Ht^*)) > 0, \qquad V'(\phi(Ht^*)) + 4H^2\phi(Ht^*) > 4H^2\phi_{\text{top}} \tag{3.75}$$

for $Ht^* > \pi/2$. If

$$V''(\phi) + 4H^2 > 0 \tag{3.76}$$

throughout the potential barrier, then

$$\begin{cases} V'(\phi) + 4H^2\phi > 4H^2\phi_{\text{top}} & \text{for} \qquad V'(\phi) < 0 \\ V'(\phi) + 4H^2\phi < 4H^2\phi_{\text{top}} & \text{for} \qquad V'(\phi) > 0 \end{cases} \tag{3.77}$$

which implies $\ddot{\psi}(0) > 0$ by Eq.(3.65) (see Fig.3.4), and thus $\ddot{\psi}(Ht^*) < 0$. However, Eq.(3.77) contradicts Eq.s(3.73)-(3.75) and thus there is no Coleman-de Luccia bounce. In the same way, if

$$V''(\phi) + 4H^2 < 0 \tag{3.78}$$

throughout the potential barrier, then $\ddot{\psi}(Ht^*) > 0$, which gives

$$\begin{cases} V'(\phi(Ht^*)) + 4H^2\phi(Ht^*) > 4H^2\phi_{\text{top}} & \text{for} \qquad V'(\phi(Ht^*)) < 0 \\ V'(\phi(Ht^*)) + 4H^2\phi(Ht^*) < 4H^2\phi_{\text{top}} & \text{for} \qquad V'(\phi(Ht^*)) > 0 \end{cases} \tag{3.79}$$

contradicting Eq.(3.78).

Consider now a more restrictive condition, that is H_{bound} is given by Eq.(3.62). This corresponds to having (see Fig.3.2)

$$\phi(Ht^*) \sim \phi_{\text{top}} \qquad \text{for} \qquad H \sim H_{\text{bound}}. \tag{3.80}$$

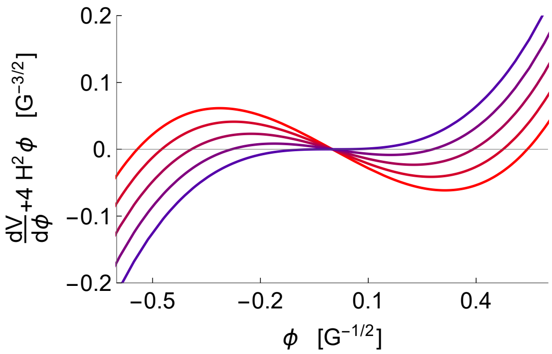

Figure 3.2: Eq.(3.58) with $V(\phi) = -0.5\phi^2 + 0.25\phi^4$ and H from $H = 0.4$ (red) to $H = H_{\text{bound}} = 0.5$ (blue). For $H \leq H_{\text{bound}}$ a small region around $\phi_{\text{top}} = 0$ opens up allowing for $\psi-$instantons with one turning point.

As H gets closer to H_{bound}, $\phi(Ht^*)$ approaches ϕ_{top} and, by regularity of Eq.(3.70), also $\phi(\pi/2)$ does. Then

$$\dot{\psi}(\pi/2) = 0 \qquad \ddot{\psi}(\pi/2) = 0 \tag{3.81}$$

for $H = H_{\text{bound}}$. Thus, in the limit that Eq.(3.62) holds, one finds that the only solution with a turning point is the static one $\phi(t) = \phi_{\text{top}}$.

Hitting a stationary point may also place a bound in theories that have non-monotonic Eq.(3.58) and H given by Eq.(3.62). If the monotonicity condition is softly broken [8], the static solution $\phi(t) = \phi_{\text{top}}$ should be reached for $Ht \sim \pi/2$. Requiring that $\dot{\psi}(\pi/2) = 0$, $\ddot{\psi}(\pi/2) = 0$, one finds again

$$\phi(\pi/2) \sim \phi_{\text{top}}, \qquad V''(\phi_{\text{top}}) + 4H_{\text{bound}}^2 \sim 0. \tag{3.82}$$

In this case there are both $\psi-$instantons with zero and one turning points for $H \approx H_{\text{bound}}$. If the monotonicity condition breaks for $\phi < \phi_{\text{top}}$ (see Fig.3.5, left), then one can easily show that $\dot{\psi}(\pi/2) > 0$ for both kinds of instanton, i.e. $\phi(\pi/2) > \phi_{\text{top}}$. If instead it breaks for $\phi > \phi_{\text{top}}$ (Fig.3.5, on the right), one has $\phi(\pi/2) < \phi_{\text{top}}$.
If the quartic derivative of $V(\phi)$ in ϕ is positive (negative) definite in $[\phi_{\text{fv}}, \phi_{\text{tv}}]$, then the monotonicity requirement is broken only on the right or on the left of ϕ_{top} (depending on other terms in the potential), for $H = H_{\text{bound}}$, as Eq.(3.60) has definite curvature. If such breaking is additionally small, then the bounce exists for $H < H_{\text{bound}}$ ($H > H_{\text{bound}}$) and one recovers the numerical evidence reported in Chapter 2.

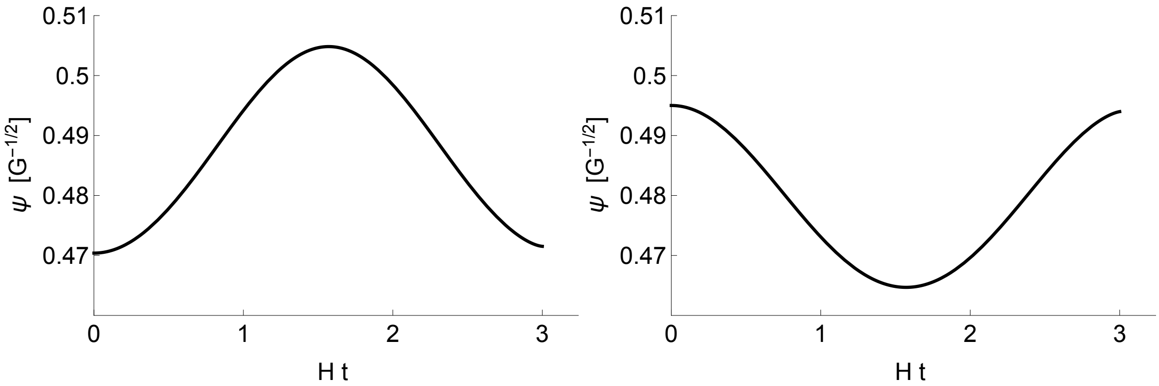

Figure 3.3: ψ-instanton with one turning point for $V(\phi) = -0.5\phi^2 + 0.25\phi^4$ (left), $V(\phi) = -0.5\phi^2 - 0.25\phi^4$ (right).

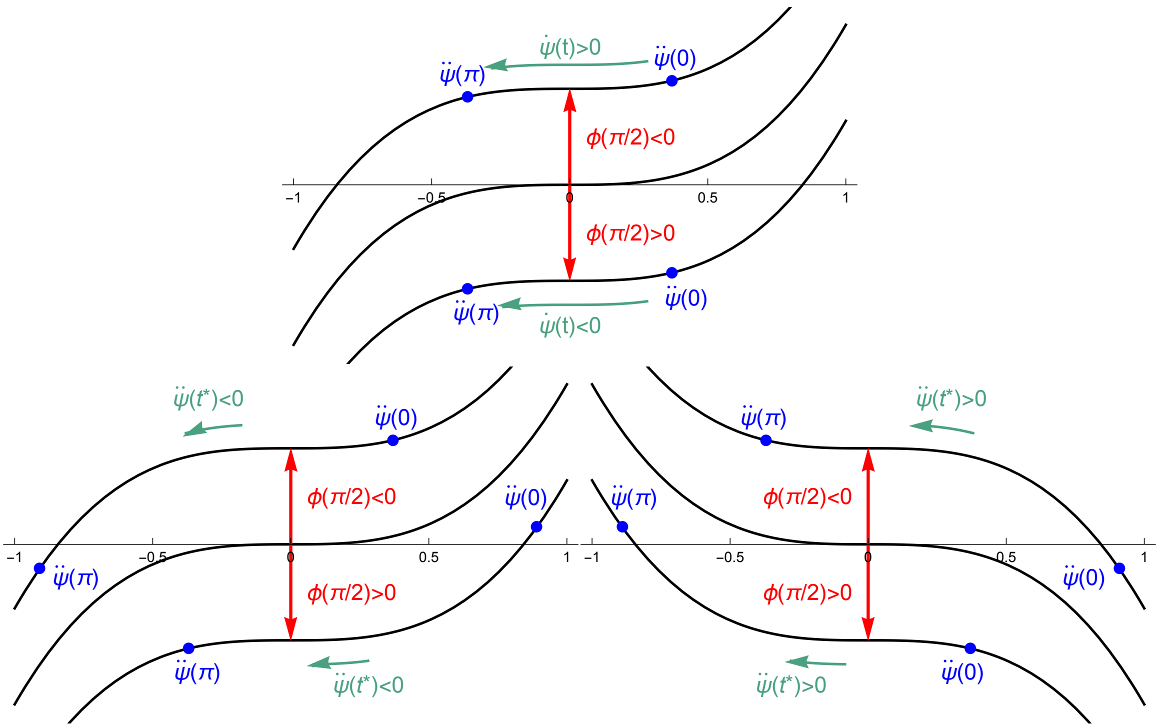

Figure 3.4: Eq.(3.58) as a function of ϕ (in black) with initial and final points for $\psi-$instantons without turning points (top) and with one turning point (bottom), with monotonically increasing (left) and monotonically decreasing (right) Eq.(3.58). Conditions on the instanton velocity and acceleration are reported in green.

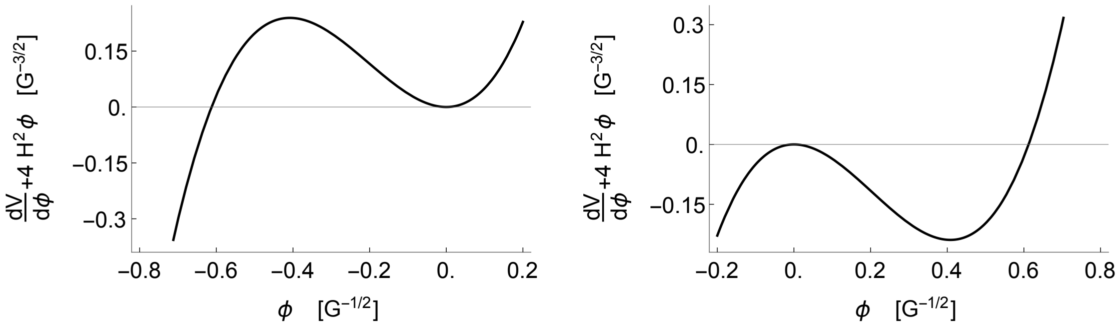

Figure 3.5: Eq.(3.58) for a polynomial potential Eq.(3.83) with $b = -\frac{1}{2\sqrt{6}}$ (left), $b = \frac{1}{2\sqrt{6}}$ (right) and $c = 1$.

3.3.2 Examples

To test this result, consider a scalar field theory with polynomial potential

$$V(\phi) = \frac{703}{100}\left(-\frac{\phi^2}{2} - b\phi^3 + c\frac{\phi^4}{4}\right) \tag{3.83}$$

which has $\phi_{top} = 0$ and $V''(\phi_{\text{top}}) = -7.03$. Eq.(3.58) is monotonic for $b \to 0$ and $4H^2 > 7.03$. Moreover, for $b > 0$, the monotonicity breaks on the right of ϕ_{top} and thus one has $\phi(\pi/2) < \phi_{\text{top}}$. Analogously, for $b < 0$ one has $\phi(\pi/2) > \phi_{\text{top}}$. The value of Eq.(3.62) as a function of $\phi(\pi/2)$ for various b, c is reported in Fig.3.6, on the left, and shows that:

- positive quartic powers give an upper bound (blue, red and black lines), while negative ones a lower bound (green line)
- positive cubic powers give $\phi(\pi/2) > 0$ (blue line), while negative ones give $\phi(\pi/2) < 0$ (green, red and black lines).
- the smaller is the cubic power, the closer is $\phi(\pi/2)$ to ϕ_{top}.

As further example, consider the Higgs potential Eq.(1.90) and compute $V''(\phi)$ as a function of ϕ. It vanishes at ϕ_{fv}, it gets positive in a small region with $\phi < \phi_{\text{top}}$ and it is negative for $\phi > \phi_{\text{top}}$, with $V''(\phi) \to -\infty$ for $\phi \to +\infty$. Then Eq.(3.60), as a function of ϕ, is always positive around the false vacuum; this region is larger the larger is H. Then, Eq.(3.58) is never monotonic. The quartic derivative instead is negative definite apart from a small region around the false vacuum of order $\phi \sim 10^{-9}$. The potential barrier is infinitely large, and so this may be regarded as a small monotonicity breaking for an otherwise negative definite $V''(\phi)$, for small enough H. Thus, H has a lower bound which is approximately determined by Eq.(3.62), i.e. $H^2_{\text{bound}} \sim 1.5 \times 10^{-20}\, G^{-1}$. The upper bound on H can only be found numerically, as there is no true vacuum state (see Fig.3.6). It is found to lie at $H^2 \sim 3.4 \times 10^{-20}\, G^{-1}$.

[8] This may be measured by comparing the range of scalar field values breaking the monotonicity condition with the width of the potential barrier $[\phi_{\text{fv}}, \phi_{\text{tv}}]$

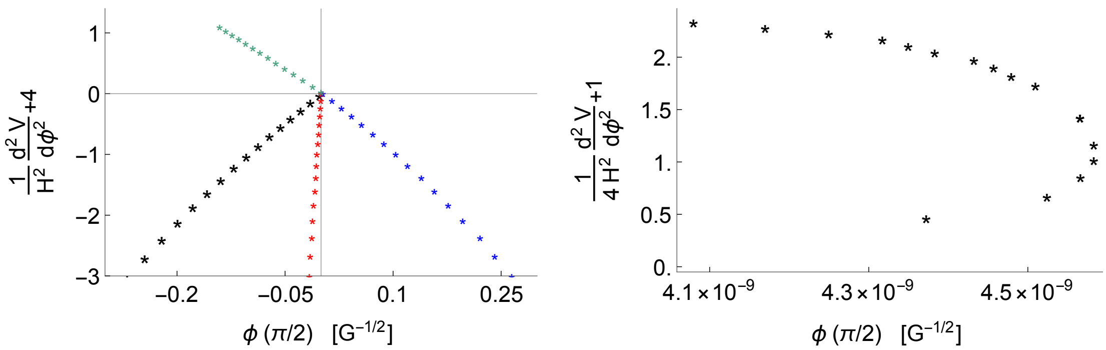

Figure 3.6: Top: Eq.(2.25) for a polynomial potential Eq.(3.83) for $b = \frac{1}{2\sqrt{6}}, c = 1$ (black),$b = \frac{1}{40\sqrt{6}}, c = 1$ (red), $b = -\frac{1}{2\sqrt{6}}, c = 1$ (blue), $b = \frac{1}{2\sqrt{6}}, c = -1$ (green). Bottom: Eq.(3.62) for the Higgs potential

3.3.3 Adding Einstein-Hilbert gravity

The bounds on H in scalar field theories with Einstein-Hilbert gravity are similar to the ones found in the previous section. In this case the action is Eq.(2.4) with $V(\phi_{\text{fv}}) > 0$ in order to have a de Sitter false vacuum state. In the asymptotic region on the bounce the scale factor is determined by Eq.(2.15). Again, $\dot{\phi}(\pi) = 0$, which allows having downtunneling as well as uptunneling solutions [9] and thus no obstructions to vacuum decay emerge from a spacetime boundary analysis.
To avoid any quantum gravity effects, all scales on the bounce should be (much) smaller than the Planck mass. By Eq.(2.8) one has then that

$$\rho(t) \approx \frac{1}{\bar{H}} \sin(\bar{H}t) \tag{3.84}$$

to lowest order at *all* times on the bounce. Under this approximation, bounds on H may be estimated analogously than in the case of a scalar field on a fixed de Sitter background, the only difference being that $\bar{H}$ is to be chosen among the ones satisfying Eq.(2.16). If no $\bar{H}$ satisfies the bound, than a bounce is excluded. Notice that this differs from Eq.(2.20) with H determined by

$$H^2 = \frac{V(\phi)}{3M_P^2} \tag{3.85}$$

as having $V(\phi) = 0$ in the potential barrier makes Eq.(2.20) unbounded from below. In the case of Higgs decay, one has $V(\phi_{\text{fv}}) = 0$ and thus de Sitter space emerges only by adding a cosmological constant in the action. Thus, the value of $\bar{H}$ is not constrained by the scalar field potential, and so bounds on it are the same as in Sect.3.3.

[9]Now though the equation of motion is invariant under $Ht \to \pi - Ht$ only in the neighbourhood of the initial and final points, and so the mapping is only approximate.

Chapter 4

Vacuum decay in quadratic gravity

Some theories of physical interest, such as modified gravity ones, have not been investigated much yet in the context of vacuum decay. As described in Chapter 2, the impact of a non-minimal coupling in the form $\xi\phi^2 R$ on Higgs decay has been explored in [48, 50, 99] and on a toy model in [49]. Quadratic gravity terms instead have been considered in [53, 150]. The purpose of this Chapter is to report a systematic study of this kind of theories, by focusing in particular on finding obstructions to the decay process when the false vacuum state has a flat or de Sitter geometry. Calculations and results reported in this Chapter concerning flat space false vacua are based on [132, 133], while the rest is original unpublished work. To do that, the results of Chapter 3 are generalized and extended to apply to these scenarios. A scalar field theory with Einstein-Hilbert gravity, a non-minimal coupling $\xi\phi^2 R$, and a quadratic Ricci scalar αR^2 is considered. These interactions are required by perturbative renormalizability in quantized field theories on a gravitational (classical) background [92–94], along with $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, which however vanish in $O(4)-$symmetric spacetimes. The non-minimal coupling is crucial in Higgs inflation (see [95, 96] and references therein) while the quadratic term allows to solve the strong coupling problem that affects it [151–153]. They appear also in Starobinsky [154] and scale-invariant inflation [155–158]. The line element is given by Eq.(2.3), and the Euclidean action is

$$S_E \;=\; 2\pi^2 \int_0^{+\infty} dt\rho(t)^3 \left(\frac{\dot{\phi}^2}{2} + V(\phi) - \frac{M_{\rm P}^2}{2} R - \frac{\xi}{2}\phi^2 R + \frac{\alpha}{36} R^2 \right)$$

with Ricci scalar given by Eq.(2.5). The scalar field equation of motion is

$$\ddot{\phi} + \frac{3\dot{\rho}\,\dot{\phi}}{\rho} = V'(\phi) - \xi\phi R. \tag{4.1}$$

while the $tt-$component of the Einstein equations is

$$\dot{\rho}^2 = 1 + \rho^2 \, \frac{\dfrac{\dot{\phi}^2}{2} - V(\phi) + \dfrac{\alpha}{36}R^2 + \left(\dfrac{\alpha}{3}\dot{R} - 6\xi\,\phi\,\dot{\phi}\right)\dfrac{\dot{\rho}}{\rho}}{3\left(M_{\rm P}^2 + \xi\phi^2 - \dfrac{\alpha}{9}R\right)}. \tag{4.2}$$

Taking the trace instead one gets

$$\alpha\left(\ddot{R}+\frac{3\dot{\rho}}{\rho}\dot{R}\right)=-3M_{\rm P}^2R-3\xi(1+6\xi)\phi^2R+3\dot{\phi}^2(1+6\xi)+12V(\phi)+18\xi\phi V'(\phi) \tag{4.3}$$

which highlights the presence of an additional scalar degree of freedom in the gravitational sector if $\alpha \neq 0$. The scalar field false vacuum value is set to zero, $\phi_{\rm fv}=0$, unless differently stated. First, in Sect.4.1, flat space false vacua are considered, by separately analyzing the effect of non-vanishing ξ, α and M_P on the asymptotic bounce of massless and massive fields. These results are then used to verify whether Conditions 1. and 2. of Chapter 3, which apply also in this case, are satisfied. While no obstructions to vacuum decay were found in theories with Einstein-Hilbert gravity, now the bounce is excluded for $\alpha \neq 0$, despite it may occur if it is massive and also $\xi = 0$, $\phi_{\rm fv} \neq 0$. Another argument, based on the scale factor behaviour on the bounce, allows to exclude vacuum decay in theories with $M_P = 0$, $\alpha = 0$, $\xi \neq 0$, with roughly $\phi_{\rm fv} \ll \ell$, where ℓ indicates the width of the potential barrier in the case of massless fields, while the scalar field mass otherwise. In Sect.4.2, bounds on the Hubble constant when the false vacuum is de Sitter are explored, setting $\alpha = 0$, $\xi \neq 0$ (Sect.4.2.1), $\alpha \neq 0$, $\xi = 0$ (Sect.4.2.2) and $\alpha \neq 0$, $\xi \neq 0$ (Sect.4.2.3), while the limit $M_P \to 0$ is discussed in each section. An additional argument, specific of modified gravity theories, allows to exclude a bounce if $\alpha \neq 0$, $M_P = 0$, $\xi = 0$.

4.1 Decay from flat space

4.1.1 Non-minimal coupling and Einstein-Hilbert gravity

The decay of scalar fields interacting with Einstein-Hilbert gravity and a non-minimal coupling has been already extensively analyzed in the literature, in particular as regards the Higgs field [48–50, 100]. No obstruction according to Conditions 1. and 2. has been found so far, and thus the same should hold for generic massless fields, as the asymptotic bounce is independent of the potential. Still, Eq.(3.5) may not be sufficient for Eq.(3.3) to hold. Moreover, the asymptotic bounce itself may differ from the $\xi = 0$ case. It is thus necessary to compute it from scratch for both massless and massive scalar fields.
As $\alpha = 0$, ϕ is the only propagating scalar degree of freedom, and Eq.(4.1) depends on gravity both through the friction term and the Ricci scalar. The former is approximately given by Eq.(3.50) near the bounce at large times. Using Eq.(4.3) in Eq.(4.1) and expanding around the false vacuum state one gets

$$\ddot{\phi}+\frac{3\dot{\phi}}{t}=\frac{dV}{d\phi}-\frac{\xi(1+6\xi)\phi\dot{\phi}^2}{M_{\rm P}^2} \tag{4.4}$$

to lowest order. Expanding Eq.(4.4) around the turning point t^*, and evaluating it for $t \leq t^*$, in the large t^* limit, allows determining under which conditions the zeroth-order term dominates the Taylor expansion, and thus when Eq.(3.9) holds. Barring numerical cancellations[1], one has

$$\sum_{n=0}^{+\infty}\left(\phi\dot{\phi}^2\right)_*\frac{(t-t^*)^n}{n!}\approx\sum_{n=2}^{+\infty}\left(\phi^3\right)_*\frac{(t-t^*)^{n-2}}{n!} \tag{4.5}$$

apart from numerical factors. This gives

$$\begin{aligned}\ddot{\phi}+\frac{3\dot{\phi}}{t} &\approx \sum_{n=0}^{+\infty}\left(\frac{dV}{d\phi}\right)_*\frac{(t-t^*)^n}{n!} \\ &- \frac{\xi(1+6\xi)}{M_{\rm P}^2}\sum_{n=2}^{+\infty}\left(\frac{\phi^3}{t^{*2}}\right)_*\frac{(t-t^*)^n}{n!}\end{aligned} \tag{4.6}$$

taking $t-t^*\approx -At^*$, where A is a constant of order $\lesssim O(1)$. As quartic interactions always satisfy Eq.(3.5), such term does not give an appreciable contribution to the Taylor expansion at large times on the bounce. Thus, if Eq.(3.11) hold, the potential can be safely approximated as Eq.(3.6), which in turn gives Eq.(3.9). If instead the scalar field is massive, it satisfies Eq.(3.12) at large times on the bounce. Taking Eq.(3.9) or Eq.(3.3) and $\rho(t)\approx t$ in Eq.(4.2) gives consistently $\dot{\rho}\approx 1$ and the Lagrangian decays sufficiently fast so that its integral Eq.(4.1) converges for $t\to\infty$. Thus, as expected, no obstructions to decay are found in this case.

The same calculations may be carried out in the $\phi_{\rm fv}\neq 0$ case and lead to analogous results, with

$$\begin{aligned}\lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}}\dot{\phi}(t) &= -\frac{C_0}{t^3}\,,\quad \lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}}\left(1-\frac{6\xi^2\phi_{\rm fv}^2}{M_{\rm P}^2+\xi(1+6\xi\phi_{\rm fv}^2)}\right)\left(\frac{dV}{d\phi}\right)_*=\frac{4C_0}{t^{*4}}\,, \\ \lim_{\substack{\phi_*\to 0\\ t^*\to+\infty}}\phi(t) &= \phi_{\rm fv}+\frac{C_0}{2t^2}\end{aligned} \tag{4.7}$$

in place of Eq.(3.51) if the scalar field is massless. If it is massive instead the exponential argument of Eq.(3.12) gets shifted as

$$m\to\left(1-\frac{6\xi^2\phi_{\rm fv}^2}{M_P^2+\xi(1+6\xi\phi_{\rm fv}^2)}\right)m.$$

4.1.2 Quadratic Ricci scalar and Einstein-Hilbert gravity

Setting $\xi=0$, $\alpha\neq 0$ the scalar field equation of motion depends on gravity only through the friction term, while the trace equation Eq.(4.3) determines the dynamics of R when subjected to a scalar field source, given by the trace of the stress-energy tensor. In this case, there are two propagating degrees of freedom, the scalar field and the Ricci scalar. To find the asympotic bounce, one uses the fact that the scalar

[1]Here the Taylor expansion of $\phi\dot{\phi}^2$ is traded with the one of ϕ^3, similarly to what was done in Sect.3.1.2

field equation of motion actually decouples near the bounce at all times. Moreover, if Eq. (4.3) holds at some time t, R may be determined by Eq. (4.3) setting

$$\frac{\dot{\rho}}{\rho} \approx \frac{1}{t} \tag{4.8}$$

in the second term on the left-hand side. In fact, the Ricci scalar depends only on deviations from the flat space solution (see Eq. (2.5)) and so, if small at some time, it should remain as such: in order to trigger large deviations from flat space in R, they should first appear in Eq. (4.8), but this means that higher-order deviations from flat space (in R), are determined by lower-order deviations (in the friction term), which is impossible. Thus, Eq. (3.50) may be safely taken to hold in the friction terms of Eq. (4.2) and Eq. (4.3) at all times. In this way one finds that Eq.(4.3) has a solution $R = R_{hom} + R_{ps}$ with

$$\begin{aligned} R_{\rm hom} &= \epsilon^2 C_1 \frac{J_1(\epsilon\, t')}{t}\,, \\ R_{\rm ps} &= -\epsilon^2 \frac{\pi}{2}\frac{J_1(\epsilon\, t')}{t}\int^t F(y) Y_1\left(\epsilon^3\, y'\right) y^2 dy \\ &+ \epsilon^2 \frac{\pi}{2}\frac{Y_1(\epsilon^3\, t')}{t}\int^t F(y) J_1\left(\epsilon\, y'\right) y^2 dy\,. \end{aligned} \tag{4.9}$$

Here, J, Y are Bessel function of the first kind,

$$A = \sqrt{\frac{3M_{\rm P}^2}{|\alpha|}}\,, \quad t' = At\,, \quad \begin{cases} \epsilon = 1 & \alpha > 0 \\ \epsilon = i & \alpha < 0\,, \end{cases} \tag{4.10}$$

C_1 is a constant, and the function $F(t)$ is given by

$$F(t) = \frac{3\dot{\phi}(t)^2}{\alpha} + \frac{12V(\phi(t))}{\alpha} \tag{4.11}$$

on the bounce at large times. Using Eq.(3.9), the Ricci scalar can be approximated as

$$R = \epsilon^2 (C_1 + \tilde{C}_1)\,\frac{J_1(\epsilon\, At)}{t} + \tilde{C}_2 \frac{Y_1(\epsilon^3\, At)}{t} + \text{higher orders} \tag{4.12}$$

where higher orders are computed by using the asymptotic forms of the Bessel functions J, Y for large arguments [159]

$$\begin{aligned} J_1^{\infty}(z) &= \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{3\pi}{4}\right) \\ Y_1^{\infty}(z) &= \sqrt{\frac{2}{\pi z}}\sin\left(z - \frac{3\pi}{4}\right) \end{aligned} \tag{4.13}$$

and are $O(t^{-6})$ if Eq.(3.11) holds, while $O(e^{-2mt})$ if the scalar field is massive. $\tilde{C}_1$ and $\tilde{C}_2$ are

$$\tilde{C}_1 = -\frac{\pi}{2}\epsilon^2 \int_0^{+\infty} F(y) Y_1\left(\epsilon^3\, y'\right) y^2 dy \tag{4.14}$$

$$\tilde{C}_2 = \frac{\pi}{2}\epsilon^2 \int_0^{+\infty} F(y) J_1\left(\epsilon\, y'\right) y^2 dy. \tag{4.15}$$

The Ricci scalar is dominated by the first two terms in Eq.(4.12) unless both $C_1+\tilde{C}_1$ and $\tilde{C}_2$ vanish. However, one can prove that $\tilde{C}_2$ is always non-vanishing. In fact,

$$\int_0^{+\infty} F(t)J_1(t')t^2dt \;\; < \;\; \int_0^{+\infty} t't^2F(t)dt \tag{4.16}$$

and the right-hand side is negative definite if

$$\int_0^{+\infty} t^3\left(\frac{\dot{\phi}^2}{2}+V(\phi)\right)dt < -\int_0^{+\infty} t^3V(\phi)dt, \tag{4.17}$$

that is, if the bounce action of a single scalar field theory is smaller than the one of a scalar field interacting with Einstein-Hilbert gravity, in the approximation that the gravitational backreaction is small. This inequality was proven in [101], thus implying that $\tilde{C}_2<0$ for $\alpha>0$ and $\tilde{C}_2>0$ for $\alpha<0$.

Finally, using Eq.(4.13) in Eq.(4.12) shows that R diverges for $t\to+\infty$ and $\alpha<0$, thus implying that Condition 1. is violated. For $\alpha>0$, instead, R undergoes damped oscillations around the fixed point $R=0$. They are, to leading order, the same of a free massive scalar field around its minimum, and thus they make the action undefined in the upper limit of integration, thereby violating Condition 2. Then the apparent metastability in the scalar field potential does not lead to decay if a R^2 term is present.

4.1.3 Non-minimal coupling, quadratic gravity and Einstein-Hilbert gravity

Including a squared Ricci scalar results in a bounce action that is ill-defined for $\alpha>0$ due to the Einstein-Hilbert term. Thus, setting $\xi\neq 0$, along with $\alpha\neq 0$ and $M_{\rm P}^2\neq 0$ should not change much the situation. Eq.(4.1) and Eq.(4.3) are now coupled beyond the friction term so it is necessary first to disentangle them to read the scalar field asymptotic bounce and use it to find the Ricci scalar. To do that, use Eq.(4.1) in Eq.(4.3) to replace non-derivative terms in R. Solving Eq.(4.3) one then gets

$$R=C_1+\frac{C_2}{2t^2}+\int^t t'^{-3}\int^{t'} F(t'')t''^3dt'dt'' \tag{4.18}$$

with

$$\begin{aligned}\alpha F(t) \;&=\; 3\dot{\phi}^2(1+6\xi)+12V(\phi)+18\xi\phi V'(\phi)\\ &-\; 3\xi\left((1+6\xi)\phi+\frac{M_{\rm P}^2}{\phi}\right)\left(\ddot{\phi}+\frac{3\dot{\phi}}{t}-V'(\phi)\right).\end{aligned} \tag{4.19}$$

Defining $f(t)$ as

$$\int^t t'^{-3}\int^{t'} F(t'')t''^3dt'dt''\equiv f(t)F(t). \tag{4.20}$$

and replacing R in Eq.(4.1) with Eq.(4.18) one gets

$$\ddot{\phi}F_1(\phi,t) + \frac{3\dot{\phi}}{t}F_2(t) = V'(\phi)F_3(t) + 12\xi\phi V(\phi)f(t) \tag{4.21}$$

with

$$F_1(\phi,t) = 1 + \frac{3\xi}{\alpha}\phi f(t)\left((1+6\xi)\phi + \frac{M_{\rm P}^2}{\xi\phi}\right) \approx \frac{3}{\alpha}M_{\rm P}^2 f(t) + 1 \tag{4.22}$$

$$F_2(\phi,t) = 1 - \frac{3\xi}{\alpha}(1+6\xi)t\phi\dot{\phi}f(t) + \frac{3\xi}{\alpha}\phi f(t)\left((1+6\xi)\phi + \frac{M_{\rm P}^2}{\xi\phi}\right) \approx \frac{3}{\alpha}M_{\rm P}^2 f(t) + 1 \tag{4.23}$$

$$F_3(\phi,t) = 1 + \frac{18\xi^2}{\alpha}\phi^2 f(t) + \frac{3\xi}{\alpha}\phi f(t)\left((1+6\xi)\phi + \frac{M_{\rm P}^2}{\xi\phi}\right) \approx \frac{3}{\alpha}M_{\rm P}^2 f(t) + 1 \tag{4.24}$$

near the bounce at large times. Therefore, if $f(t) \neq -\frac{\alpha}{3M_P^2}$ the scalar field equation of motion may be approximated as

$$\ddot{\phi} + \frac{3\dot{\phi}}{t} = \frac{dV}{d\phi} \tag{4.25}$$

for small ϕ, as if the non-minimal coupling was negligible. Thus either Eq. (3.3) (if the scalar field is massless) or Eq. (3.12) (if the scalar field is massive) hold on the bounce at large times. Plugging these solutions in Eq. (4.3) and Taylor-expanding around the turning point t^* one finds that Eq. (3.21) (in which ϕ is to be replaced by R and ρ by t) holds for $R_* \gg t_*^{-6}$ ($R_* \gg e^{-2mt_*}$) in the massless (massive) case. This gives, near the bounce at large times,

$$\alpha\ddot{R} \approx -3M_P^2 R. \tag{4.26}$$

Thus, according to Eq.(3.35), R decreases exponentially in t

$$R \approx C_0 e^{-\sqrt{\frac{3}{\alpha}}M_P t}. \tag{4.27}$$

Anyway $t^{-6} \gg e^{-M_P t}$, $e^{-2mt} \gg e^{-M_P t}$ for $m \ll M_P$ and so Eq.(4.26) is not a consistent approximation of Eq. (4.3). This means that ϕ terms in Eq. (4.3), as well as the Einstein-Hilbert term, are important at each order in the Taylor expansion. Then R is given by Eq. (4.12), as $\phi^2 R$ in Eq. (4.3) is negligible with respect to R when ϕ is given by the asymptotic bounce. Constants $C_1 + \tilde{C}_1$ and C_2 determine the behaviour on and near the bounce. If they vanish, R is completely set by the scalar field and changing the initial condition $R(0)$ away from the bounce one still gives a bounce. Thus, this should correspond to the $\alpha = 0$ limit and, for $\alpha \neq 0$, one has $C_1 + \tilde{C}_1 \neq 0$ and $C_2 \neq 0$. If this is the case though, Eq.(4.25) is violated to lowest order for large t. This excludes a bounce for both massless and massive scalar fields[2].

[2]If instead $f(t) \approx -\frac{\alpha}{3\xi M_P^2}$ on the bounce at large times, R is given by Eq. (4.27) which implies again that Eq. (4.26) holds. This makes the approximation Eq.(4.25) again reliable, as ϕR gives a negligible contribution in the Taylor expansion. As a result, ϕ terms dominate the Einstein-Hilbert one for both massless and massive scalar field, which is in conflict with previous statements.

4.1.4 Non-minimal coupling

In the next sections, the asymptotic bounce in theories with scale invariant gravitational sector will be determined. Notice that the Planck mass appears in the denominator of the second term in Eq.(4.2), thus possibly spoiling the asymptotic behaviour of ρ and ϕ as $M_P \to 0$. Moreover, the Einstein-Hilbert term sources the oscillations in the Ricci scalar that forbid a bounce in the $\alpha \neq 0$, $M_P \neq 0$ case. Thus taking the scale invariant limit may lead to non-trivial results regarding the asymptotic bounce and, thus, obstructions to vacuum decay. Consider now $\alpha = 0$ besides $M_P = 0$. To have an analogous equation to Eq.(1.68), combine Eq.(4.3) with Eq.(4.1) to eliminate R. This gives

$$\ddot{u} + 3\frac{\dot{\rho}\,\dot{u}}{\rho} = \frac{dW}{du} \equiv \frac{4}{1+6\xi}\left(u\frac{dV}{du} - 2V(u)\right) , \tag{4.28}$$

where $u \equiv \phi^2$. This is the starting point for asymptotic bounce considerations in scalar field theories with a non-minimal coupling. In order to estimate the right-handside near the false vacuum state though, it is necessary to separately consider massless and massive fields.

Massless fields

As for Eq.(4.28), if u is massless with sufficiently small cubic interactions with respect to the potential $W(u)$, its asymptotic behaviour is as in Eq.(3.9), and thus

$$\phi(t) = \frac{\sqrt{C_0}}{t} . \tag{4.29}$$

Plugging this solution in Eq.(4.2) one finds an inconsistency in boundary conditions for gravity, as using $\rho(t) = t$ and Eq.(4.29) one gets in the large t limit

$$\dot{\rho}^2 = 3 + \frac{1}{3\xi} - \frac{V(\phi)t^4}{3\xi C_0}O\left(t^{-1}\right) \neq 1 \tag{4.30}$$

at leading order. Thus, there is no bounce if the false vacuum lives on flat space, unless also the scalar field sector is scale-invariant, with potential [3]

$$V = \frac{(6\xi+1)}{C_0}\phi^4. \tag{4.31}$$

This situation changes if the scalar field has a non-vanishing false vacuum value. In fact, if the potential is such that $V(\phi_{\text{fv}}) = 0$, $\phi_{\text{fv}} \neq 0$ and $V(u)$ satisfies Eq.(3.11), the asymptotic bounce is

$$\phi(t) \approx \sqrt{\phi_{\text{fv}}^2 + \frac{C_0}{2t^2}}. \tag{4.32}$$

[3] Our analysis actually excludes scale-invariant potentials as there is no undershoot/overshoot distinction in that case. The asymptotic bounce (and thus possible violations of Conditions 1. and 2.) should thus be found with other methods.

From the discussion above, one expects that Eq.(4.32) is satisfied only in a narrow region around $\phi_{\rm fv}$ (otherwise one finds again Eq.(4.30)), namely

$$\phi_{\rm fv}^2 \gg \frac{C_0}{2t^2} \tag{4.33}$$

and thus Eq.(4.32) may be replaced with

$$\phi(t) \approx \phi_{\rm fv} + \frac{C_0}{4\phi_{\rm fv} t^2} \, . \tag{4.34}$$

Using Eq.(4.34) and Eq.(3.50), Eq.(4.2) reads

$$\dot{\rho}(t) = 1 + O\left(t^{-4}\right) . \tag{4.35}$$

Eq. (4.33) implies that, if $\phi_{\rm fv}$ is much smaller than the potential barrier width ℓ [4] the scalar field has not yet reached the asymptotic bounce regime when $V(\phi) > 0$. This possibly makes $\dot{\rho}$ vanish, if additionally $V(\phi) \gg \dot{\phi}^2$, making ϕ self-accelerated. However, in order to avoid quantum gravity effects, all scales should be (much) smaller than the Planck mass, which gives, according to Eq. (4.2), $\dot{\rho} \approx 1$ on the bounce at *all* times. Imposing $\dot{\rho} = 0$ for $\phi \approx \phi_{\rm fv} + \ell$ and using $V(\phi) \gg \dot{\phi}^2$ gives

$$\rho_\ell^2 = \frac{3\xi(\phi_{\rm fv} + \ell)^2}{V(\phi_{\rm fv} + \ell)} . \tag{4.36}$$

This behaviour is avoided if $\rho_\ell > \bar{t}$, where $\bar{t}$ indicates the time by which the scalar field is given by the asymptotic bounce. Estimating $\bar{t}$ as

$$u_{\rm fv} \approx \frac{C_0}{2\bar{t}^2} \tag{4.37}$$

and C_0 as [5]

$$C_0 = -2\phi_{\rm fv}\bar{t}^3\sqrt{2V(B\phi_{\rm fv})} \tag{4.38}$$

where B is some constant of order unity, one finds the condition on $\phi_{\rm fv}$, ℓ and ξ in order for the bounce to exist

$$\frac{6\xi(\phi_{\rm fv} + \ell)^2 V(B\phi_{\rm fv})}{\phi_{\rm fv}^2 V(\phi_{\rm fv} + \ell)} + 1 < 0. \tag{4.39}$$

This condition is marginally satisfied for $\phi_{\rm fv} \approx \ell$ and $\xi \approx \frac{1}{24}$. Decreasing ξ shrinks the range of $\phi_{\rm fv}$ for which a bounce is forbidden. Moreover, Eq.s(4.37)-(4.38) underestimate the actual $\bar{t}$ thus having the same effect. This result is tested taking as

[4] The potential barrier width is defined here as the range of ϕ such that $V(\phi) > 0$.

[5] It corresponds to

$$\frac{\dot{\phi}^2}{2} = V(\phi)$$

at $\bar{t}$. On the asymptotic bounce one has $V(\phi) \gg \dot{\phi}^2$ for $\bar{t} \gg M_P^{-1}$, and thus this result in an underestimation of $\bar{t}$, whose effects are explained below.

$V(\phi)$ the Higgs potential Eq. (1.90) with $\phi \to \phi - \phi_{\text{fv}}$. The values of ϕ_{fv} for which $F(\phi_{\text{fv}}, \xi) = 0$, with $F(\phi_{\text{fv}}, \xi)$ defined as

$$F(\phi_{\text{fv}}, \xi) \equiv \frac{6\xi(\phi_{\text{fv}} + \ell)^2 V(B\phi_{\text{fv}})}{\phi_{\text{fv}}^2 V(\phi_{\text{fv}} + \ell)} + 1 \tag{4.40}$$

are computed numerically and reported in Fig.4.1 (on the right) as a function of ξ. The zeros of $F(\phi_{\text{fv}}, \xi)$ decrease for increasing ξ, and they lie approximately at $\phi_{\text{fv}} \approx \ell \approx 5 \times 10^{-9} G^{-1/2}$. The bounce is also computed numerically, varying $\phi_{\text{fv}} \in [10^{-8}, 10^{-3}]$ and $\xi \in [0.01, 10]$. The on-shell action is reported in Fig.4.1, on the left. It sharply increases for $\phi_{\text{fv}} \geq 10^{-8}$ and there is no bounce for lower ϕ_{fv}. The value of ϕ_{fv} for which the bounce disappears increases for increasing ξ. They are larger than the prediction reported in Fig.4.1 as $\bar{t}$ is actually an underestimation of the matching time.

These considerations suggest that a non-minimally coupled Higgs field has no bounce, since it has a vacuum expectation value at $v = 246\text{GeV} \approx 10^{-17} G^{1/2} \ll \ell$. Actually, v is generated by the interplay of a mass term and the quartic interaction: the Higgs mass affects the asymptotic bounce and in principle changes these results. Nonetheless, much of our reasoning is focused on the behavior near the potential barrier, which is unaffected by the mass term in a large range of field values. Still, the condition in Eq.(4.39) depends on the scalar field behavior near the false vacuum, and thus it should be reconsidered in the massive scalar field case. This will be addressed in the next section.

Massive fields

If the scalar field is massive, the right-hand side of Eq.(4.28) may be written as

$$\frac{dW}{du} = -\frac{2m^2}{6\xi + 1} u. \tag{4.41}$$

Thus, if $6\xi + 1 < 0$, the asymptotic bounce is

$$\phi(t) = C_0 e^{-C_1 t/2}, \qquad \text{where } C_1 = \sqrt{-\frac{2m^2}{6\xi + 1}}. \tag{4.42}$$

Plugging this solution in Eq.(4.2) one finds that there is an inconsistency in the boundary conditions for gravity, as using Eq.(3.50) and Eq.(3.12) gives

$$\dot{\rho}^2 = 3 + \left(\frac{m^2}{12\xi(1 + 6\xi)} - \frac{m^2}{6\xi} \right) t^2 + O(t) \tag{4.43}$$

which grows large at large t, thus violating Condition 1. This situation improves if the scalar field has a non-vanishing vacuum value. In fact, taking $V(\phi_{\text{fv}}) = 0$ and $\phi_{\text{fv}} \neq 0$, one gets

$$\left(u\frac{dV}{du} - 2V(u) \right) \approx \frac{m^2}{4}(u - u_{fv}) \tag{4.44}$$

where $u_{fv} = \phi_{\text{fv}}^2$. Thus

$$\phi(t) \approx \sqrt{u_{fv} + C_0 e^{-C_1 t}} \approx \sqrt{u_{fv}} + \frac{C_0}{2\sqrt{u_{fv}}} e^{-C_1 t} \qquad \text{with} \quad C_1 = \sqrt{\frac{m^2}{1+6\xi}} \tag{4.45}$$

for sufficiently large t. The transition to the asymptotic regime (say occuring at some $t = \bar{t}$) should be reached only in a narrow region around u_{fv},

$$\frac{C_0 e^{-C_1 \bar{t}}}{u_{fv}} \ll 1, \tag{4.46}$$

as otherwise the abovementioned inconsistencies in Eq.(4.2) occur. The values of C_0 and $\bar{t}$ are theory-dependent, and the combination

$$\ell \equiv C_0 e^{-C_1 \bar{t}} \tag{4.47}$$

marks the distance of ϕ from ϕ_{fv} at which the mass term dominates the potential. One gets

$$\dot{\rho}(t) = 1 + O\left(t^2 e^{-2C_1 t}\right) \tag{4.48}$$

for $\rho(t) = t$ and the scalar field given by the asymptotic bounce, thus satisfying Condition 1. If $\bar{t}$ is sufficiently small though, it might be the case that $\dot{\rho} = 0$ at finite times thus forbidding a bounce for some values of C_0, u_{fv}, ξ, m. To see this, take Eq.(4.2) and impose

$$\dot{\rho} > 0, \qquad \rho = t, \qquad \phi(t) \approx \sqrt{u_{fv}} + \frac{C_0}{2\sqrt{u_{fv}}} e^{-C_1 t}, \tag{4.49}$$

using

$$\frac{C_0 e^{-C_1 \bar{t}}}{u_{fv}} \ll 1. \tag{4.50}$$

The result is a function $F(t, C_0, C_1, u_{fv})$

$$F(t, C_0, C_1, u_{fv}) \equiv 1 + \frac{t^2}{6\xi u_{fv}} \left(\dot{\phi}^2 - 2V(\phi)\right) \tag{4.51}$$

whose zeros separate the region in which a bounce is allowed ($F(t, C_0, C_1, u_{fv}) > 0$) from the one in which it is not ($F(t, C_0, C_1, u_{fv}) < 0$). This holds for all t such that the asymptotic bounce is reached, and so it does also at $\bar{t}$. As F is a decreasing function of time, and $\bar{t}$ is roughly bound by C_1^{-1} from below [6], one has

$$F(\bar{t}, C_0, C_1, u_{fv}) \leq F(C_1^{-1}, C_0, C_1, u_{fv}). \tag{4.52}$$

[6] C_1^{-1} roughly marks the time at which the mass term dominates the potential and so the asymptotic bounce is reached.

The left-hand side may be rewritten as $F(x,\xi) = F(C_1^{-1}, C_0, u_{fv})$ where $x = \frac{C_0}{u_{fv}e}$ and $F(x,\xi)$ is given by

$$F(x,\xi) = \frac{-8(6\xi+1) + 8(6\xi+1)(x+1)^{3/2} - 3x(8\xi(x+3)+x+4)}{24\xi(x+1)} + 1. \tag{4.53}$$

The zeros of this function are reported in black in Fig.4.2, on the right, as a function of ξ. For xs below the curve the bounce is allowed, for values above it, it is forbidden. Zeros of $F(\bar{t}, C_0, C_1, u_{fv})$ lie at smaller values of x, due to Eq.(4.52), thus placing a more stringent bound than the one found above. To rephrase these results as a function of $\phi_{\rm fv}$ one needs to determine C_0, which depends on $V(\phi)$. Numerical results in a scalar field theory with non-minimal coupling and potential

$$V(\phi) = 1.5 \times 10^{-4}(\phi - \phi_{\rm fv})^2 + 0.07(\phi - \phi_{\rm fv})^3 + 2.25(\phi - \phi_{\rm fv})^4 \tag{4.54}$$

are reported in Fig.4.3, on the right. C_0 is estimated as the scalar field value at $t = C_1^{-1}$ such that the dominant (90%) contribution to the potential is given by the mass term, namely

$$\frac{2V(\phi)}{m^2(\phi - \phi_{\rm fv})^2} - 1 = 0.1 \qquad \text{with } \phi = \phi_{\rm fv} + \frac{C_0}{2\phi_{\rm fv}} e^{-1} \tag{4.55}$$

which gives $C_0 = 12 \times 10^{-4}\phi_{\rm fv}$. The value of $\phi_{\rm fv}$ beyond which a bounce was not detected numerically (red stars) lies well above the theoretical prediction (black line), thus suggesting that $\bar{t} \gg C_1^{-1}$. In Fig.4.1, on the left, the on-shell action as a function of ξ is reported, for $\phi_{\rm fv}$ in the interval $[0.01, 0.5]$. The bounce disappears at increasingly lower $\phi_{\rm fv}$ for growing values of ξ, in agreement with our estimates.

The relevant scale of the system in this case is set by $\ell = x\,\phi_{\rm fv}$. As a consequence, one finds that the scalar field false vacuum state is required to be bounded from below by ℓ, in order for a bounce to exist. In spite being formally similar to the result found in the massless case, it is physically is very different. First of all, such bound was derived imposing that the scalar field has not yet reached the asymptotic bounce, in contrast to what is described here. Moreover, if the scalar field mass is much smaller than the barrier width, the bound on $\phi_{\rm fv}$ given by the former is accordingly milder than the one given by the latter. This implies that the mass term in the Higgs potential is effectively negligible for our considerations as its mass is much smaller than the barrier width.

4.1.5 Quadratic gravity

As already discussed in Sect.4.1.2, theories with quadratic gravitational terms have an additional propagating degree of freedom, whose asymptotic bounce is difficult to determine by means of the Taylor expansion. Nonetheless, one can find the

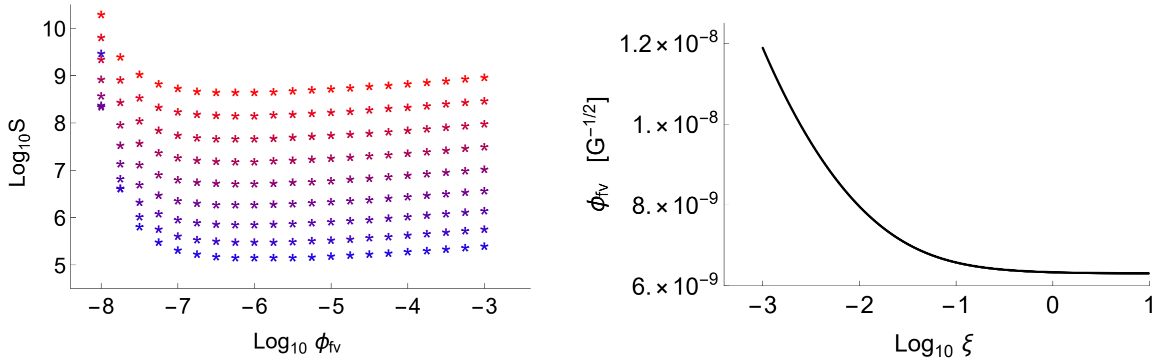

Figure 4.1: Left: numerical bounce action as a function of $\phi_{\rm fv}$ for a scalar field theory with potential Eq.(1.90) and a non-minimal coupling to gravity. The non-minimal coupling is changed from $\xi = 0.01$ (red) to $\xi = 10$ (blue). The action sharply increases for $\phi_{\rm fv} \approx 10^{-8}$. The bounce disappears for lower values of $\phi_{\rm fv}$. Right: zeros of $F(\phi_{\rm fv}, \xi)$ as a function of ξ.

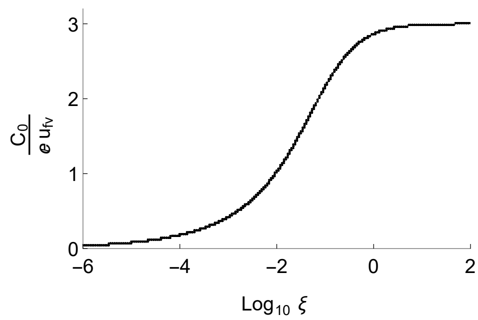

Figure 4.2: Zeros of $F(x, \xi)$ as a function of ξ.

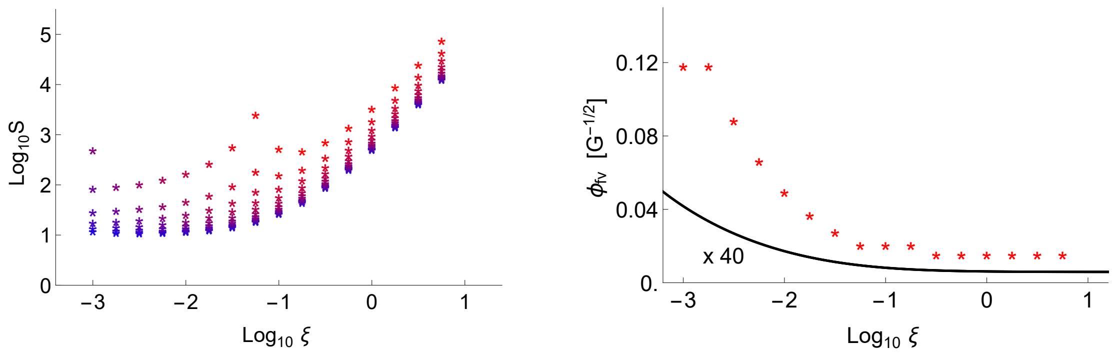

Figure 4.3: Left: numerical bounce action as a function of ξ for a scalar field theory with potential Eq.(4.54) and a non-minimal coupling to gravity. $\phi_{\rm fv}$ is changed from $\phi_{\rm fv} = 0.5$ (blue) to $\phi_{\rm fv} = 0.016$ (red). The bounce disappears for increasingly lower values of $\phi_{\rm fv}$ at higher ξ. Right: zeros of $F(\phi_{\rm fv}, \xi)$ (magnified by a factor 40) as a function of ξ for the potential Eq.(4.54) (black line) compared with the numerical values (red stars).

asymptotic bounce of R solving Eq.(4.3) with

$$\frac{\dot{\rho}}{\rho} \approx \frac{1}{t}, \tag{4.56}$$

and replacing the scalar field with its asymptotic bounce. Setting $M_P = 0$, $\xi = 0$, $\alpha \neq 0$, one finds

$$R(t) \quad = \quad C_1 + \frac{C_2}{2t^2} + \text{higher orders}$$

where C_1 and C_2 are real constants, and higher-orders are $O(t^{-4})$ if Eq.(3.11) holds and $O(e^{-2mt})$ if the scalar field is massive. Plugging Eq.(4.57), Eq.(3.50) and Eq.(3.9) in Eq.(4.2) one gets $\dot{\rho} \neq 1$ at large times on the bounce, violating then Condition 1. This result is independent of the values of C_1 and C_2. Thus, there is no bounce for scale-invariant gravity with a quadratic Ricci scalar and a flat spacetime in the false vacuum state.

4.1.6 Non-minimal coupling and quadratic gravity

Consider now adding a non-minimal coupling and thus setting $\xi \neq 0$, $\alpha \neq 0$ and $M_P = 0$. Analogous calculations to the ones reported in Sect.4.1.3 allows determining the asymptotic bounce. In order to have a finite bounce action, $f(t)$ now should satisfy

$$\lim_{t \to +\infty} \phi^2 f(t) = 0 \tag{4.57}$$

as $F(t)$ is monotonically decreasing at large times on the bounce

$$\begin{aligned} f(t) \quad &= \quad \frac{\int_t^{+\infty} t'^{-3} \int_{t'}^{+\infty} t''^3 F(t'')}{F(t)} \\ &\leq \quad \int_t^{+\infty} t'^{-3} \int_{t'}^{+\infty} t''^3 = \frac{t^2}{8}. \end{aligned} \tag{4.58}$$

Then

$$F_1(\phi, t) \approx F_2(\phi, t) \approx F_3(\phi, t) \approx 1 \tag{4.59}$$

so again

$$\ddot{\phi} + \frac{3\dot{\phi}}{t} = \frac{dV}{d\phi} \tag{4.60}$$

for small ϕ. Then ϕ satisfies Eq.(3.9) if massless, (or Eq.(3.12) if massive), at large times on the bounce. This gives (from Eq.(4.18))

$$R = \frac{C_1}{2t^2} + \text{higher orders} \tag{4.61}$$

where higher orders are as in the previous section. Plugging these solutions in Eq.(4.2) one finds that Condition 1. is violated. Setting $\phi_{\text{fv}} \neq 0$ amounts to adding a linear non-minimal coupling to gravity, ϕR, and having an Einstein-Hilbert-like term on the bounce at large times. Now, though, the effective Planck mass $\sqrt{\xi}\phi_{\rm fv}$ might be of the same order of m and a bounce in principle cannot be excluded in the massive case. In the case of Higgs decay, computing R according to Eq.(4.18) with $M_P^2 = 0$ at large times on the bounce and for $t \ll m^{-1}$ (when Eq.(3.9) holds) one finds

$$R \approx \frac{C_1}{2t^2} + \frac{3(1+6\xi)C_0^2}{8\alpha t^4} \tag{4.62}$$

leading to a logarithmic (quadratic) divergence in the action for $C_1 = 0$ ($C_1 \neq 0$), driven by the non-minimal coupling term $\phi_{\rm fv}^2 R$. As $\phi_{\rm fv}$ may be of order m though, the action is not necessarily large at $t \approx m^{-1}$ and thus in principle a bounce with small on-shell action cannot be excluded.

4.2 Decay from de Sitter space

4.2.1 Non-minimal coupling

While flat space decay is described by a scalar field trajectory that probes $\phi_{\rm fv}$ at $t \to +\infty$, the de Sitter one requires to reach some $\bar{\phi}$ with $\ddot{\phi} > 0$ and $\dot{\phi} = 0$ at the boundary of spacetime $t = \pi\bar{H}^{-1}$. These conditions for $\xi = 0$ and $\alpha = 0$ read respectively

$$V'(\bar{\phi}) > 0, \qquad V(\bar{\phi}) > 0 \tag{4.63}$$

and are derived by setting the right-handside of Eq.(4.1) and $\bar{R} \equiv 12\bar{H}^2$ in Eq.(4.3) to be positive at $t = \pi\bar{H}^{-1}$. If $\xi \neq 0$, $\alpha = 0$ these conditions are instead

$$V'(\bar{\phi}) - \xi\bar{\phi}\bar{R} > 0 \qquad \bar{R} = \frac{12V(\bar{\phi}) + 18\xi\bar{\phi}V'(\bar{\phi})}{3M_P^2 + 3\xi(1+6\xi)\bar{\phi}^2} > 0. \tag{4.64}$$

Adding a non-minimal coupling thus changes the scalar field values at which $\bar{\phi}$ may be located according to the sign and magnitude of ξ and $\bar{R}$. If $M_P = 0$, also other conditions should be imposed on the theory. In particular, $\xi > 0$ is required in order to have a positive definite effective squared Planck mass $\xi\phi^2$ and $\phi \neq 0$ on the bounce, so that the solution to Eq.(4.2) is regular.
Besides these conditions, there are also additional bounds on H that result from a generalization of the calculations reported in Sect.3.3. To do that, the same approximations considered in Sect.3.3.3 are adopted. In fact, using Eq.(3.84), one has that

$$V''(\phi) + 4(1-3\xi)\bar{H}^2 \tag{4.65}$$

replaces

$$V''(\phi) + 4\bar{H}^2 \tag{4.66}$$

setting bounds on $\bar{H}$. As a result, thus, $H_{\rm bound}$ is also a function of ξ, and according to Eq.(4.65), it gets larger for increasing ξ. In order to determine when a bounce is allowed, one should consider all possible values of $\bar{H}$, and compare them to the bounds predicted by the discussion in Sect.3.3, with Eq.(2.22) replaced by Eq.(4.65). Notice that, for $\xi \geq 1/3$, the function

$$V'(\phi) + 4\bar{H}^2(\phi - \phi_{\text{top}})(1 - 3\xi) \tag{4.67}$$

is positive definite for $\phi < \phi_{\text{top}}$ while negative for $\phi > \phi_{\text{top}}$, independently of $\bar{H}$. From the discussion in Sect.3.3, one has that no bounce exists for such values ξ.

The theoretical bound on $\bar{H}$ is compared with the numerical one in two toy models in Fig.4.4 and Fig.4.5 ($\xi > 0$ on the left, $\xi < 0$ on the right), with

$$V(\phi) = V_0 + \frac{703}{100}\phi^2\left(-\frac{\phi^2}{2} - \phi^3 + \frac{\phi^4}{4}\right) \tag{4.68}$$

and

$$V(\phi) = V_0 + \frac{703}{100}\left(-\frac{\phi^2}{2} - \frac{\phi^3}{2\sqrt{6}} + \frac{\phi^4}{4}\right) \tag{4.69}$$

respectively. Both potentials have $\phi_{\text{top}} = 0$ and $\phi_{\text{fv}} < 0$. The former has

$$V''(\phi_{\text{top}}) = 0, \tag{4.70}$$

and thus the upper bound on $\bar{H}$ is determined by the requirement that

$$V''(\phi) + 4(1 - 3\xi)\bar{H}^2 < 0 \tag{4.71}$$

somewhere in the potential barrier. The latter has

$$V''(\phi_{\text{top}}) \neq 0 \tag{4.72}$$

and the upper bound is set as

$$V''(\phi_{\text{top}}) + 4(1 - 3\xi)H^2_{\text{bound}} = 0. \tag{4.73}$$

The red stars and blue squares represent the numerical upper bound, i.e. the value of $\bar{H}$ above which no bounce was found numerically for $\phi < 0$ and $\phi > 0$ respectively. The black line represents the theoretical bound on $\bar{H}$ for the quadratic potential, while the requirement

$$V''(\phi) + 4(1 - 3\xi)\bar{H}^2 < 0 \qquad \text{for} \qquad \phi \in [\phi_{\text{fv}}, \phi_{\text{top}}] \tag{4.74}$$

in the quartic case, as considering the whole potential barrier set the bound at too high values of V_0 to be properly represented in the plot. The dashed line marks the value $\xi = 1/3$. The expected behaviour is, roughly, that increasing ξ softens the bound on $\bar{H}$, and this is indeed observed in Fig.4.4-4.5. $\bar{H}$ is driven to $+\infty$ as $\xi = 1/3$ is approached in both cases. If ξ is negative, the bounce disappears for $\xi < -0.1$ in the quadratic case, as, for large enough $|\xi|$ and $\xi < 0$, the first of

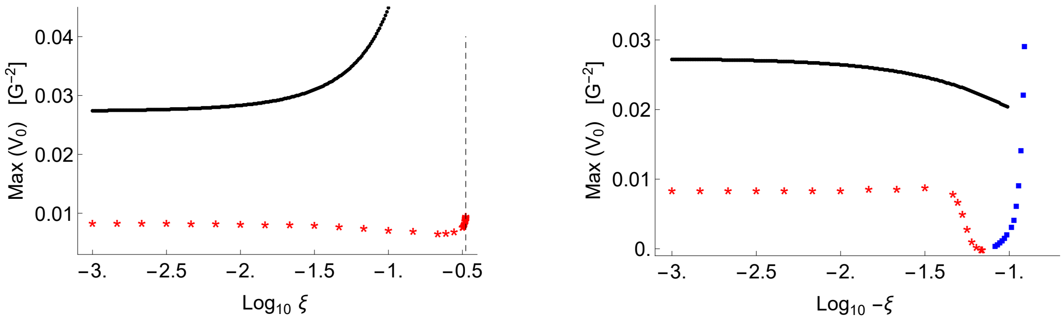

Figure 4.4: Theoretical (black line) and numerical (red stars, blue squares) bound on V_0, for $\xi > 0$ (left) and $\xi < 0$ (right). The potential is Eq.(4.68).

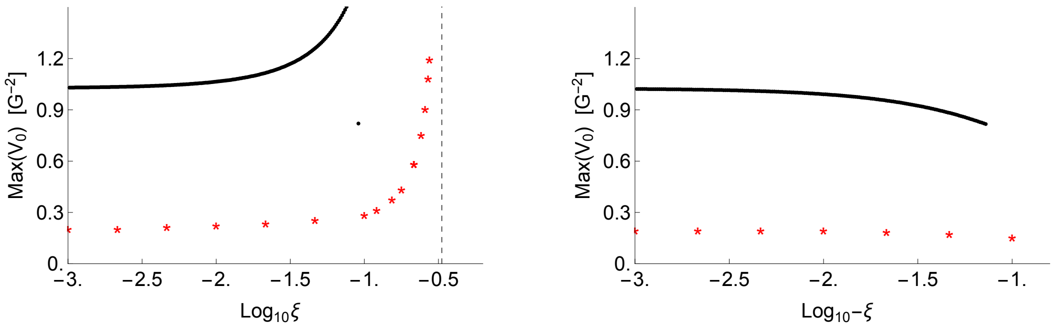

Figure 4.5: Theoretical (black line) and numerical (red stars) bound on V_0, for $\xi > 0$ (left) and $\xi < 0$ (right). The potential is Eq.(4.69).

Eq.(4.64) is always violated for $\phi < \phi_{\rm top}$. In the quartic case instead $\bar{\phi}$ migrates to the region $\phi > \phi_{\rm top}$, in which the first of Eq.(4.64) is satisfied for $\bar{R}$ such that

$$\lim_{\phi \to 0} \frac{V(\phi) - V_0}{\xi \bar{R}} < 1. \tag{4.75}$$

In this case, thus, the bounce survives for larger values of ξ.

As further example consider the Higgs field with an Einstein-Hilbert term, a cosmological constant $V(\phi_{\rm fv})$ and a non-minimal coupling. The first of Eq.(4.64) is satisfied for some ϕ and inflationary values of $\bar{H}$, ($\bar{H} > 10^{-6}$ in units $G = 1$) if $\xi < 10^{-9}$. If $\xi > 0$, the non-minimal coupling leaves $H_{\rm bound}$ almost untouched, as, for such $\bar{H}$ ($\bar{H} \sim 10^{-10}$), the non-minimal coupling is very small with respect to other scales in the system. Nonetheless, the upper bound may only be predicted numerically, as there is no true vacuum state, and thus a bounce cannot be in principle excluded. For $\xi < 0$, instead, the region available for $\bar{\phi}$ grows as ξ gets smaller, and thus a bounce is in principle allowed for all values of ξ, with $H_{\rm bound}$ getting smaller as ξ decreases. $H_{\rm bound}$ would cross the current cosmological constant value at extremely large values of $|\xi|$ ($\xi \sim -10^{100}$) for which $V'(\phi)$ is positive definite for $\phi \leq M_P$. Thus Higgs decay with non-minimal coupling is excluded in the current Universe and in the semi-classical approximation.

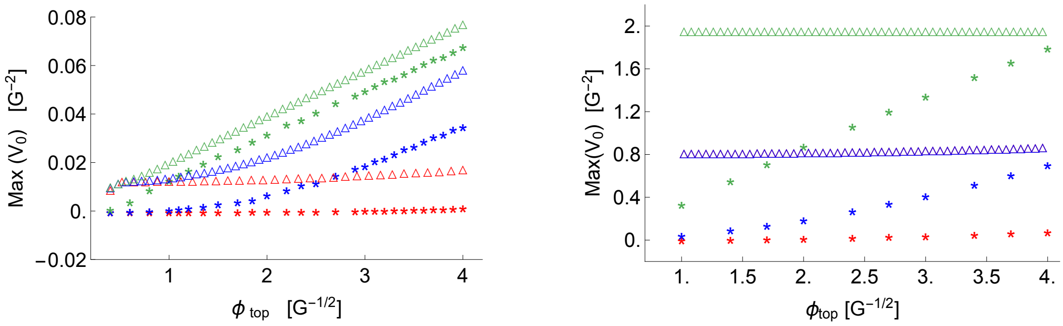

Figure 4.6: Left: Bound on V_0 as a function of $\phi_{\rm top}$ for $\log\xi = -3, -2, -1$ (red, blue and green respectively). The potential is Eq.(4.68) with shifted false vacuum value $\phi \to \phi - \phi_{\rm fv}$ so that $\phi_{\rm fv} > 0$. Stars represent the numerical bound, which is determined by the highest value of V_0 for which a bounce was detected numerically. Triangles represent the bound Eq.(4.73). Right: same as on the left, in a theory with potential Eq.(4.71) with shifted false vacuum value $\phi \to \phi - \phi_{\rm fv}$ so that $\phi_{\rm fv} > 0$. Triangles represent the theoretical bound Eq.(4.73).

Consider now briefly the limit $M_P \to 0$. As stated above, one needs $\xi > 0$ and $\phi \neq 0$ on the bounce, besides Eq.s(4.64). Moreover, as the bounds described in Sect.3.3 are independent of M_P, similar results to the ones just described in the $M_P \neq 0$ case should apply (see Fig.4.6 for an example).
Notice that, for sufficiently small values of the Hubble constant, the bounds in parameter space described in Sect.4.1.4 apply also to these theories. The typical time scale of de Sitter space is $\pi(2\bar{H})^{-1}$ and should be compared withρ_ℓ if the scalar field is massless, while $C_1^{-1} = (1+6\xi)^{1/2} m^{-1}$ if massive. Setting

$$\rho_\ell \ll \frac{\pi}{2\bar{H}} \qquad \pi C_1 \gg 2\bar{H}, \tag{4.76}$$

and using Eq.(4.36), Eq.(4.45) one finds that the Hubble constant is sufficiently small if

$$\bar{H}^2 \ll \frac{\pi^2 V(\phi_{\rm fv}+\ell)}{6\xi(\phi_{\rm fv}+\ell)^2} \qquad 2\bar{H}\sqrt{1+6\xi} \ll m\pi. \tag{4.77}$$

4.2.2 Quadratic gravity

Quadratic gravitational terms have important consequences on bounds on the Hubble constant that are mainly related to the Ricci scalar being a propagating degree of freedom. First of all, R needs to satisfy similar boundary conditions as the scalar field, namely[7]

$$\dot{R}(0) = 0 \qquad \dot{R}(\pi) = 0. \tag{4.79}$$

[7] The asymptotic behaviour of R may be computed analosously as the scalar field one

$$R = \bar{R} + \frac{(3M_P^2\bar{R} - 12V(\bar{\phi}))}{8\alpha\bar{H}^2}(\bar{H}t - \pi)^2 + \text{higher orders.} \tag{4.78}$$

In contrast to the flat space case, thus, the on-shell action is well behaved near the boundary in theories with $\xi = 0$, $\alpha \neq 0$. Nonetheless, the smaller is the cosmological constant, the more difficult is to make predictions on the on-shell action, due to the oscillations described in Sect.4.2.2 and occurring for $t \ll \pi(2\bar{H})^{-1}$. This makes a calculation of the tunneling exponent for Higgs decay in the current universe particularly difficult, even numerically. Moreover, it is questionable whether such bounce would contribute to the false vacuum decay, due to the high number of oscillations in R, for the same reasons concerning the oscillating instantons described in Chapter 2.
Notice also that, in this case, $\bar{R}$ is not fully determined by $\bar{\phi}$, and that the scalar field equation of motion depends on $\rho(t)$ only through the friction term which is, to lowest order, determined by Eq.(2.15). Thus, a bound on $\bar{H}$ should be computed as in Sect.3.3 and compared to *all* real values of (constant) $\bar{H}$.
Moreover, the Ricci scalar should be sufficiently constant before approaching $t = \pi\bar{H}^{-1}$. This implies that it should stay close to a fixed point of Eq.(4.3). On the other hand, propagating scalar degrees of freedom do not reach fixed points of the equations of motion at the boundary. So, in order to have a bounce, one should require that the velocity of the Ricci scalar is small

$$\frac{|\bar{H}\dot{R}|}{\bar{R}^2} \ll 1. \tag{4.80}$$

$\bar{R}$ is determined as (see Eq.(4.2))

$$-\frac{\bar{R}}{12} = \frac{-36V(\bar{\phi}) + \alpha\bar{R}^2}{12(9M_P^2 - \alpha\bar{R})} \tag{4.81}$$

where $\bar{H}\dot{R}$ was neglected with respect to $\bar{R}^2$ and $\dot{\phi}^2 \ll V(\phi)$ sufficiently close to the boundary. Eq.(4.81) gives

$$\bar{R} = \frac{4V(\bar{\phi})}{M_P^2} \tag{4.82}$$

which is also the fixed point of Eq.(4.3). Estimating the magnitude of $\dot{R}$ and $\ddot{R}$ from Eq.(4.3) one has that small deviations from the fixed point give small velocities, namely

$$\alpha\bar{H}\dot{R} \sim 4AV(\bar{\phi}) \qquad \text{for} \qquad \bar{R} \sim 4AV(\bar{\phi})M_P^{-2} \tag{4.83}$$

where $A \lesssim O(1)$. Then

$$\frac{\bar{H}\dot{R}}{\bar{R}^2} = \frac{AM_p^4}{4\alpha V(\bar{\phi})} \tag{4.84}$$

and thus it depends, as expected, on the magnitude of deviations from the fixed point. By setting $M_P^2 = 0$ instead Eq.(4.81) turns into the requirement $V(\bar{\phi}) = 0$, contradicting that, in general, $V(\bar{\phi}) > 0$. Thus a bounce does not exist if $\alpha \neq 0$, $\xi = 0$ and $M_P = 0$. In Fig.4.2.2 the bounce for R (black line) in a theory with potential

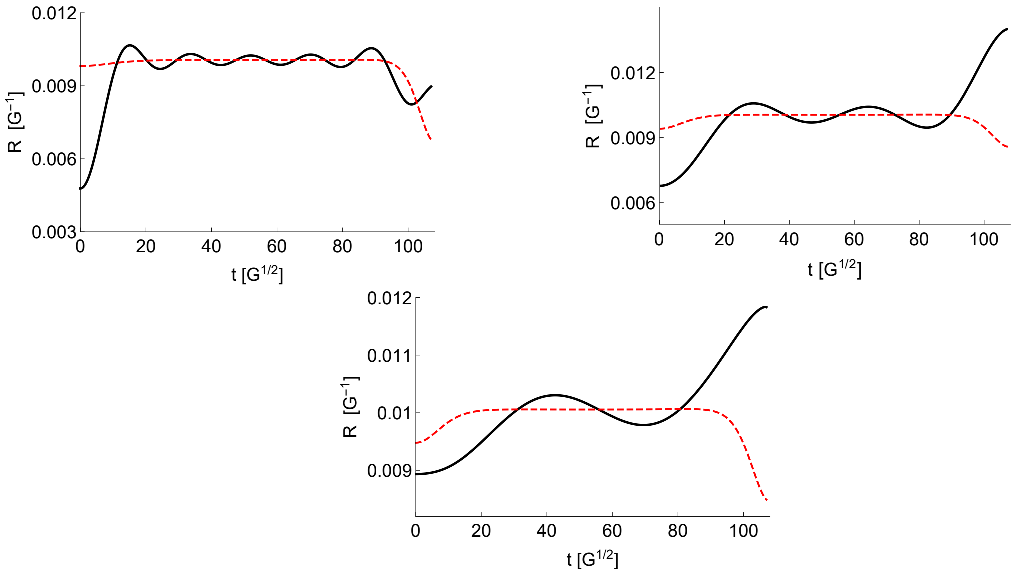

Figure 4.7: Bounce of the Ricci scalar for a scalar field theory with potential Eq.(4.68) with an Einstein-Hilbert term and a quadratic Ricci scalar for $\alpha = 1, 3.6, 8$ respectively on the top left, right and on the bottom. One can notice that R is closed to the fixed point for $\bar{H}t \geq \pi/2$, and that it slightly deviates when $t \approx \pi \bar{H}^{-1}$ is approached.

Eq.(4.68) is reported as a function of t for $\alpha = 1, 3.6$ and 8. This is compared to the fixed point (red line)

$$R = \frac{4V(\phi) - \dot{\phi}^2}{M_P^2}. \tag{4.85}$$

Near the boundary

$$\frac{|\bar{H}\dot{R}|}{\bar{R}^2} \sim 0.1, 0.06, 0.02 \ll 1 \tag{4.86}$$

respectively. No bounce with $M_P^2 = 0$ was found.
In principle, one may also try to impose a similar bound than the ones described in Sect.3.3 on $\bar{R}$, setting Eq.(2.15) in the friction term of Eq.(4.3), and treating it as any scalar degree of freedom. The equations of motion though are entangled through the trace of the stress-energy tensor, which makes a clean analysis of an upper bound for $\bar{H}$ complicated. Notice though that the curvature $-3M_P^2\alpha^{-1}$ decreases for increasing α, and thus the bounce disappears for sufficiently large α, supposing constant $\bar{H}$ (whose value is for example set by the physics of the false vacuum state) and $V(\bar{\phi})$. In the above mentioned numerical example a bounce was found up to $\alpha \sim 8$.

4.2.3 Non-minimal coupling and quadratic Ricci scalar

As described in Sect.4.2.1, the non-minimal coupling constrains the bounce by shifting the bound on $\bar{H}$ behaving as a mass for the scalar field. When quadratic terms are added instead, the Ricci scalar velocity should be very small near the boundary. If both $\xi \neq 0$ and $\alpha \neq 0$, the bounds on $\bar{H}$ carry over directly from the $\alpha = 0$ case, apart from the fact that $\bar{R}$ is a free parameter, and not completely fixed by the scalar field. The value of H_{bound} should be the same anyway, as αR^2 does not participate to the scalar field equation of motion. The bound on $\dot{R}$ instead may be significantly modified, especially in the $M_P = 0$ limit, in which there is no bounce if $\xi = 0$. Imposing Eq.(4.81) in this case gives

$$\bar{R} = \frac{4V(\bar{\phi})}{M_P^2 + \xi\bar{\phi}^2}. \tag{4.87}$$

Plugging this result in non-derivative terms in R in Eq.(4.3) and using it to estimate $\bar{H}\dot{R}$ one gets

$$|\bar{H}\dot{R}| \approx \left| \frac{6\xi\bar{\phi}}{\alpha} \left(V'(\bar{\phi}) - \frac{4\xi\bar{\phi}V(\bar{\phi})}{M_P^2 + \xi\bar{\phi}^2} \right) \right| \tag{4.88}$$

setting both terms on the right-hand side to be separately small according to Eq.(4.80) implies

$$\left|\frac{\xi}{\alpha}\right| \ll \left| \frac{8V(\bar{\phi})^2}{3\bar{\phi}V'(\bar{\phi})(M_P^2 + \xi\bar{\phi}^2)^2} \right| \qquad |\xi\bar{\phi}^2| \ll \frac{2\alpha V(\bar{\phi})}{3(M_P^2 + \xi\bar{\phi}^2)} \tag{4.89}$$

If $M_P = 0$ instead

$$\bar{R} = \frac{4V(\bar{\phi})}{\xi\bar{\phi}^2}. \tag{4.90}$$

In the semi-classical approximation one needs $\xi\phi_{\text{fv}}^2 M_P^2 \gg V(\bar{\phi})$, and thus Eq.(4.80) gives

$$\left|\frac{\xi}{\alpha}\right| \ll \frac{V(\bar{\phi})}{\xi^2\bar{\phi}^4} \ll 1 \qquad \text{and} \qquad \left|\frac{\xi}{\alpha}\right| \ll \left| \frac{V(\bar{\phi})^2}{\xi^2\bar{\phi}^5 V'(\bar{\phi})} \right| \tag{4.91}$$

The bound Eq.(4.89) was computed for a scalar field theory with potential Eq.(4.68) and $\alpha = 1$. Using Eq.(4.80) one finds $|\xi| \ll 0.1$, while a bounce was found numerically up to $\xi \sim 0.02$, for positive ξ. Moreover,

$$|\bar{H}\dot{R}| \sim 0.15\bar{R}^2 \tag{4.92}$$

and $|\bar{H}\dot{R}| \sim 1.3 \times 10^{-5}$. Using Eq.(4.88) one gets

$$|\bar{H}\dot{R}| \sim 2 \times 10^{-5} \tag{4.93}$$

agreeing within a multiplication constants of $O(1)$ with the numerical calculation. No bounce instead was found in the $M_P = 0$ limit.

Conclusion

This thesis aims at exploring the effect of modified gravity on vacuum decay in cosmological contexts. Much of our intuition of the vacuum decay process comes from quantum tunneling calculations, in which one measures the probability that a particle propagates through a potential barrier, which depends on its shape. Even if the particle is in a false vacuum state such amplitude is non-vanishing, and the probability to detect it outside the well grows exponentially in time, with decay rate correspondingly small. This formalism was extended to field theory by Coleman and Callan in the '70s and it relies on the Euclidean path integral formulation. The decay rate is determined as a combination of the Euclidean action computed on a trajectory with specific boundary conditions, the Coleman-de Luccia instanton (also known as bounce), and its fluctuations around it. The former, in particular, determines the tunneling exponent, i.e. the argument of the exponential in the decay rate formula, thus controlling strong enhancements or suppressions of the false vacuum lifetime. As the equations of motion are highly non-linear, the Coleman-de Luccia instanton is usually found with numerical methods, despite some analytical approximations exist. The path integral formulation of the problem allows also easily including gravitational effects. This result is particularly important for cosmological calculations, in which one computes the decay rate of a scalar field theory interacting with gravity. In this context, the bounce boundary conditions (and thus the decay rate) depend also on the false vacuum geometry, which may be flat, open (Anti-de Sitter) or closed (de Sitter). They correspond respectively to a vanishing, negative or positive cosmological constant. The latter is particularly important as it is a good description for our Universe in its current state, as well as during an early inflationary phase. In this way, vacuum stability calculations allow to rule out cosmological models and particle physics theories that are inconsistent with the anthropic principle. The Higgs field is particularly important in this context: despite it is stable on scales of order T_U within the Standard Model, unknown high energy physics may change it by several orders of magnitude. Modified gravity terms may be important at such energies, in particular the ones that are required to make the gravitational sector perturbative renormalizable, such as a non-minimal coupling and a quadratic Ricci scalar. Moreover, the false vacuum lifetime is usually computed in the approximation of a false vacuum state with flat geometry, while our current Universe is de Sitter. The reason is that, given that a bounce exist, its value is largely independent of the false vacuum geometry, if the cosmological constant is sufficiently small. Nonetheless, boundary conditions do depend on it, and thus the existence of a bounce is not a given fact. It is thus important to account for both gravitational effects and appropriate boundary conditions when computing the decay rate.

In this thesis a systematic study of vacuum decay in modified gravity theories is proposed. This topic is addressed by determining existence conditions for the Coleman-de Luccia instanton in gravitational settings, focusing in particular on real scalar fields interacting with Einstein-Hilbert gravity, non-minimal coupling $\xi\phi^2 R$ and a quadratic Ricci scalar αR^2. Flat and de Sitter false vacuum states were considered, due to their important cosmological implications related to such spacetimes. The behaviour of the bounce at the boundary of spacetime (the asymptotic bounce) was determined, in order to verify whether the proper boundary conditions are met, and if the on-shell action is well-defined and finite. This method proves particularly useful in calculations in which spacetime is non-compact, and so it was used in the flat false vacuum case. One finds that the asymptotic bounce is independent of the scalar field potential apart from the smallest powers in the scalar field, i.e. the mass term and cubic self-interactions. Massive scalar fields in particular decrease exponentially, while massless field decrease as an inverse squared power. Higher-order kinetic terms do not affect the bounce, and similar calculations may be carried out in a spacetimes of various dimensions. While no obstructions to vacuum decay are found in scalar field theories with Einstein-Hilbert gravity and a non-minimal coupling, the bounce is excluded in scale invariant theories with $\alpha \neq 0$, despite it may occur if the scalar field is massive and also $\xi \neq 0$. A similar argument, based on the scale factor behaviour on the bounce, allows to exclude vacuum decay in theories a flat false vacuum state and $M_P = 0$, $\alpha = 0$, $\xi \neq 0$, with $\phi_{\rm fv} \ll \ell$, where ℓ indicates the width of the potential barrier in the case of massless fields, while the scalar field mass instead. This excludes Higgs decay in theories with a non-minimal coupling, if the mass term at low energies is taken into account.
If the false vacuum spacetime is compact, instead, the asymptotic bounce does not constrain vacuum decay. A different condition, resulting in a bound on H for a bounce to exist, was thereby considered. It was found that, by using a change of field variables in an appropriate class of potentials, numerical and analytical results in the literature may be partially reproduced. In the Higgs field case, this results in a lower bound on H, lying at $H \sim 10^{-10}$ in Planck mass units, while an upper bound may only be determined numerically. This result may be easily extended to scalar field theories with Einstein-Hilbert gravity, as long as the semi-classical approximation holds, i.e. all scales are much smaller than the Planck mass. In the same approximations, one finds that a non-minimal coupling behaves as a scalar field mass when this bound is extended to modified gravity theories. For $\xi > 1/3$ this excludes a bounce for all Hubble constant values. Another condition on decay from de Sitter space derives from the requirement of having sufficiently constant H when gravity has a propagating degree of freedom. It was found that this forbids a bound if $M_P = 0, \alpha \neq 0, \xi = 0$.

Appendix

A Bounce action from a perturbative expansion

Some concerns have been raised in the literature [50, 91] as regards the use of a perturbative expansion to determine the bounce action. To find Eq. (2.51) one expands the scalar field and the scale factor around the bounce in flat space as

$$\phi(t) = \phi_0(t) + \kappa\phi_1(t) \tag{A.1}$$

$$\rho(t) = t + \kappa\rho_1(t). \tag{A.2}$$

Using the equations of motion, the action is expanded in the same way and determined as a function of the bounce radius $\mathcal{R}$. Actually, it has been shown [50] that ϕ_1 does not satisfy the proper boundary conditions and thus the perturbative expansion is unreliable. There might be, however, an alternative interpretation of the calculation that makes the final result justified. Consider two theories that have a bounce, and one of them is the vanishing coupling limit of the other (in this case it is the single scalar field theory arising from the $M_P\mathcal{R} \to +\infty$ limit of the same theory with Einstein-Hilbert gravity). By continuity, the action, as the coupling is turned on, it changes by a small amount, for sufficiently small values of the coupling. Moreover, if the gravitational backreaction is small, one might choose an *off-shell* profile with (approximately) the same shape as $\phi_0(t)$, as a function of an arbitrary parameter $\mathcal{R}$, and use it to determine $\rho_1(t)$ and thus S_E, which is to be minimized as a function of $\mathcal{R}$. In this way, one avoids to use the perturbation equation, and uses the flat space solution only as a field profile to keep the action functional finite. To keep the approximation under control is instead trickier. While the natural choice might be requiring $M_P\mathcal{R} \gg 1$ one can see from the potential scaling argument in Sec.2.5 that $\mathcal{R}$ can be arbitrarily small, increasing λ accordingly. In this case, the relevant parameter is thus $M_P\mathcal{R}\sqrt{\lambda}$. The detailed generalization to theories with multiple couplings λ_i seems to be more involved, despite one may guess that

$$M_P\mathcal{R}\sqrt{\lambda_i} \gg 1 \tag{A.3}$$

keeps the gravitational backreaction small.

B Coefficients f_{*n}

Here, coefficients f_{*n} of Eq. (3.4) are computed. Radial derivatives of arbitrary order are denoted by an index (n), while radial derivatives of first and second order are denoted by one dot or two dots respectively. Derivatives of the potential with respect to the scalar field of order i are indicated as $\frac{d^iV}{d\phi^i}$. As f_{*n}s are computed at ρ^* all quantities are implicitly evaluated at the turning point (recall that $\dot{\phi}_* = 0$). Using Eq. (1.68) one gets

$$\frac{d^iV}{d\phi^i}^{(n+1)} = \left(\frac{d^{i+1}V}{d\phi^{i+1}}\dot{\phi}\right)^{(n)}, \qquad \phi^{(n)} = \left(\frac{d^2V}{d\phi^2}\dot{\phi}\right)^{(n-3)} + \sum_{i=2}^{n-1} B_i \frac{\phi^{(i)}}{\rho^{n-i}} \tag{B.1}$$

where B_is are numerical factors, whose value is not relevant for the following discussion.

Using the first equation in Eq.(B.1), the (n+1)-th derivative of V' may be written as

$$\begin{aligned}\frac{dV}{d\phi}^{(n+1)} &= \\ &= \frac{d^2V}{d\phi^2}\phi^{(n+1)} + \dots + \left(\frac{d^2V}{d\phi^2}\right)^{(n-1)}\ddot{\phi} = \ddot{\phi}\left(\left(\frac{d^3V}{d\phi^3}\right)^{(n-3)}\ddot{\phi} + \dots + \frac{d^3V}{d\phi^3}\phi^{(n-1)}\right) + \\ &+ \phi^{(3)}\left(\left(\frac{d^3V}{d\phi^3}\right)^{(n-4)}\ddot{\phi} + \dots + \frac{d^3V}{d\phi^3}\phi^{(n-2)}\right) + \dots + \frac{d^2V}{d\phi^2}\phi^{(n+1)}\end{aligned} \tag{B.2}$$

which can be further expanded using again Eq.(B.1). Then one gets

$$\begin{aligned}\frac{dV}{d\phi}^{(n+1)} &= \frac{d^2V}{d\phi^2}\phi^{(n+1)} + \frac{d^3V}{d\phi^3}(\ddot{\phi}\,\phi^{(n-1)} + \phi^{(3)}\phi^{(n-2)} + \dots + \phi^{(n+1)/2}\phi^{(n+1)/2}) + \\ &+ \frac{d^4V}{d\phi^4}(\ddot{\phi}^2\phi^{(n-3)} + \phi^{(3)}\ddot{\phi}\phi^{(n-4)} + \dots + \phi^{(n+1)/3}\phi^{(n+1)/3}\phi^{(n+1)/3}) + \dots .\end{aligned} \tag{B.3}$$

Each $\frac{d^iV}{d\phi^i}$ in Eq. (B.3) is multiplied by $i-1$ terms that are derivatives of $\ddot{\phi}$. They are of order $n+5-2i$ or lower, and thus these terms are non-vanishing only if $n+5-2i > 1$. So, the highest-order derivative $\frac{d^{\bar{\imath}}V}{d\phi^{\bar{\imath}}}$ that appears in Eq.(B.3) is the one satisfying $n+5-2\,\bar{\imath} = 3$ for even n and $n+5-2\,\bar{\imath} = 2$ for odd n. Expanding radial derivatives of $\ddot{\phi}$ in Eq.(B.3) using Eq.s(B.1), $V'^{(n+1)}$ may be expressed in terms of derivatives of the potential with respect to the scalar field, $\ddot{\phi} = V'_*$ and ρ^* only. From Eq.(B.3) one finds that the highest-order derivative (the $\bar{\imath}$-th term) is multiplied only by radial derivatives of the scalar field of order 2 or 3 and thus it contributes as

$$\frac{d^{\bar{\imath}}V}{d\phi^{\bar{\imath}}}\frac{\ddot{\phi}^{\bar{\imath}-1}}{\rho} \quad \text{even } n, \qquad \frac{d^{\bar{\imath}}V}{d\phi^{\bar{\imath}}}\ddot{\phi}^{\bar{\imath}-1} \quad \text{odd } n \tag{B.4}$$

to f_{*n}.

The second-highest derivative $\bar{\imath}-1$ is multiplied by radial derivatives of the scalar field of order 2, 3, 4, 5. Using Eq.(B.1), derivatives of order 4 and 5 may be expressed in terms of lower derivatives. As can be seen from Eq.(B.1), this results in an additional V'' contribution (numerical coefficients are omitted for simplicity)

$$\frac{d^{\bar{\imath}-1}V}{d\phi^{\bar{\imath}-1}}\frac{\ddot{\phi}^{\bar{\imath}-2}}{\rho^3}\left(1+A_1\frac{d^2V}{d\phi^2}\rho^2\right) \quad \text{even } n, \tag{B.5}$$

$$\frac{d^{\bar{\imath}-1}V}{d\phi^{\bar{\imath}-1}}\frac{\ddot{\phi}^{\bar{\imath}-1}}{\rho^2}\left(1+A_2\frac{d^2V}{d\phi^2}\rho^2\right) \quad \text{odd } n.$$

The third-highest derivative $\bar{\imath}-2$ is multiplied by radial derivatives of the scalar field of order 2, 3, 4, 5, 6, 7. Using Eq.(B.1), to express derivatives of order 4, 5, 6, 7 in terms of lower order ones, one finds additional contributions with respect to the previous case

$$\frac{d^{\bar{\imath}-2}V}{d\phi^{\bar{\imath}-2}}\frac{\ddot{\phi}^{\bar{\imath}-3}}{\rho^5}\left(1+A_3\frac{d^2V}{d\phi^2}\rho^2+A_4\left(\frac{d^2V}{d\phi^2}\rho^2\right)^2+A_5\frac{d^3V}{d\phi^3}\rho^4\ddot{\phi}\right) \quad \text{even } n, \tag{B.6}$$

$$\frac{d^{\bar{\imath}-2}V}{d\phi^{\bar{\imath}-2}}\frac{\ddot{\phi}^{\bar{\imath}-3}}{\rho^4}\left(1+A_6\frac{d^2V}{d\phi^2}\rho^2+A_7\left(\frac{d^2V}{d\phi^2}\rho^2\right)^2+A_8\frac{d^3V}{d\phi^3}\rho^4\ddot{\phi}\right) \quad \text{odd } n.$$

In general, the $\bar{\imath}-i$th term has contributions from terms in Eq.(B.3) that are multiplied with a radial derivative of the scalar field of order $n-2i+3$ or higher. In this way, the dependence of the $\bar{\imath}-i$th term on $\ddot{\phi}$ and $\frac{d^jV}{d\phi^j}$ can be fully determined. The dependence on ρ can be fixed by dimensional consistency. In particular, in each $\bar{\imath}-i$-th term, these contributions appear always in the combination $\frac{d^jV}{d\phi^j}\ddot{\phi}^{j-2}\rho^{2j-2}$ with $j\geq 2$.
Consider

$$\lim_{t^*\to+\infty}\ddot{\phi}_*\rho^{4*}=0. \tag{B.7}$$

If all derivatives of the potential in the scalar field are finite for $\phi\to 0$, then

$$\frac{d^jV}{d\phi^j}\ddot{\phi}_*^{\,j-2}\rho^{*2j-2}\ll 1 \qquad \text{for } j>2 \text{ and large } \rho^*. \tag{B.8}$$

If also $\left(\frac{d^2V}{d\phi^2}\right)_*\rho^{*2}\ll 1$, then

$$\left(\frac{dV}{d\phi}^{(n+1)}\right)_*\approx\sum_{i=0}^{i=\bar{\imath}-2}\tilde{A}_i\left(\frac{dV^{\bar{\imath}-i}}{d\phi^{\bar{\imath}-i}}\right)_*\frac{\ddot{\phi}_*^{\bar{\imath}-i-1}}{\rho^{*2i+1}} \quad \text{even } n, \tag{B.9}$$

$$\left(\frac{dV}{d\phi}^{(n+1)}\right)_*\approx\sum_{i=0}^{i=\bar{\imath}-2}\tilde{A}_i\left(\frac{dV^{\bar{\imath}-i}}{d\phi^{\bar{\imath}-i}}\right)_*\frac{\ddot{\phi}_*^{\bar{\imath}-i-1}}{\rho^{*2i}} \quad \text{odd } n,$$

and the sum in Eq.(3.4) is negligible with respect to V'_* for large ρ^* if Eq.(B.8) holds.

C Coefficients f_{*n} for radius-dependent potentials

Here cefficients f_{*n} of Eq.(3.18) are computed. Partial derivatives of order i are indicated as ∂^i. The equations are implicitly evaluated at ρ^* (such that $\dot{\phi}(\rho^*) = 0$) and field value ϕ_*. Using the equation of motion for the scalar field one gets

$$\frac{\partial^i V}{\partial\phi^i}^{(n+1)} = \left(\frac{\partial^{i+1}V}{\partial\phi^{i+1}}\dot{\phi} + \frac{\partial^{i+1}V}{\partial\rho\partial\phi^i}\right)^{(n)}, \qquad \phi^{(n)} = \left(\frac{\partial^2 V}{\partial\phi^2}\dot{\phi} + \frac{\partial^2 V}{\partial\phi\partial\rho}\right)^{(n-3)} + \sum_{i=2}^{n-1}\frac{B_i\phi^{(i)}}{\rho^{n-i}} \tag{C.1}$$

where B_is are numerical factors, whose value is not relevant for the following discussion. As the explicit radial dependence of the potential makes the calculation more involved with respect to the radius independent case, numerical coefficients are omitted for simplicity in the following. Using the first equation in Eq.s(C.1), one finds

$$\begin{aligned}\frac{\partial V}{\partial\phi}^{(n+1)} =& \frac{\partial^2 V}{\partial\phi^2}\phi^{(n+1)} + \dots + \left(\frac{\partial^2 V}{\partial\phi^2}\right)^{(n-1)}\ddot{\phi} + \frac{\partial^2 V}{\partial\phi\partial t}^{(n)} = \ddot{\phi}\left(\left(\frac{\partial^3 V}{\partial\phi^3}\right)^{(n-3)}\ddot{\phi} + \dots + \right.\\ &\left. + \frac{\partial^3 V}{\partial\phi^3}\phi^{(n-1)}\right) + \phi^{(3)}\left(\left(\frac{\partial^3 V}{\partial\phi^3}\right)^{(n-4)}\ddot{\phi} + \dots + \frac{\partial^3 V}{\partial\phi^3}\phi^{(n-2)}\right) + \dots + \\ &+ \frac{\partial^2 V}{\partial\phi^2}\phi^{(n+1)} + \frac{\partial}{\partial t}\left(\frac{\partial^2 V}{\partial\phi^2}\phi^{(n)} + \dots + \frac{\partial^2 V}{\partial\phi^2}^{(n-2)}\ddot{\phi} + \frac{\partial^2 V}{\partial\phi\partial t}^{(n-1)}\right) + \\ &+ \sum_{j=1}^{n-1}\frac{\partial^2 V}{\partial\rho\partial\phi^2}^{(n-j-1)}\phi^{(j+1)},\end{aligned} \tag{C.2}$$

which can be further expanded using again Eq.s(C.1). Then

$$\begin{aligned}\frac{\partial V}{\partial\phi}^{(n+1)} =& \\ =& \frac{\partial V}{\partial\phi\partial\rho^{n+1}} + \sum_j \frac{\partial^2 V}{\partial\phi^2\partial\rho^j}\phi^{(n+1-j)} + \sum_j \frac{\partial^3 V}{\partial\phi^3\partial\rho^j}(\ddot{\phi}\,\phi^{(n-1-j)} + \phi^{(3)}\phi^{(n-2-j)} + \\ &+ \dots + \phi^{(n+1)/2}\phi^{(n+1)/2-j}) + \sum_j \frac{\partial^4 V}{\partial\phi^4\partial\rho^j}(\ddot{\phi}^2\phi^{(n-3-j)} + \phi^{(3)}\ddot{\phi}\phi^{(n-4-j)} + \dots + \\ &+ \phi^{(n+1)/3}\phi^{(n+1)/3}\phi^{(n+1)/3-j}) + \dots .\end{aligned} \tag{C.3}$$

where sums on j run from $j = 0$ to some upper limit, for which derivatives of the scalar field are of order two. The result is similar to the radius independent case: each $\frac{\partial^{i+j}V}{\partial\phi^i\partial\rho^j}$ in Eq.(C.3) is multiplied by $i-1$ terms that are derivatives of $\ddot{\phi}$. They are of order $n + 5 - 2i$ or lower, and thus these terms are non-vanishing only if $n + 5 - 2i > 1$. So, the highest-order derivative $\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}}}$ that appears in Eq.(C.3) is the one satisfying $n + 5 - 2\,\bar{\imath} = 3$ for even n and $n + 5 - 2\,\bar{\imath} = 2$ for odd n. The difference with respect to the radius independent case is that now the potential contains also partial derivatives in ρ and they are multiplied by $i-1$ terms that are derivatives of $\ddot{\phi}$, the only exception being radial derivatives of $\frac{\partial V}{\partial \phi}$. For example, the sixth-order radial derivative $\frac{\partial V}{\partial \phi}^{(6)}$ $(n=5)$ is expanded in terms of $\frac{\partial^2 V}{\partial \phi^2}$, $\frac{\partial^3 V}{\partial \phi^3}$ and $\frac{\partial^4 V}{\partial \phi^4}$ as:

$$\begin{aligned}\frac{\partial V}{\partial \phi}^{(6)} =& \frac{\partial^2 V}{\partial \phi^2}\phi^{(6)} + \frac{\partial^2 V}{\partial \phi^2 \partial \rho}\phi^{(5)} + \frac{\partial^2 V}{\partial \phi^2 \partial \rho^2}\phi^{(4)} + \frac{\partial^2 V}{\partial \phi^2 \partial \rho^3}\phi^{(3)} + \frac{\partial^2 V}{\partial \phi^2 \partial \rho^4}\phi^{(2)} + \\ &+ \frac{\partial^3 V}{\partial \phi^3}(\ddot{\phi}\,\phi^{(4)} + \phi^{(3)}\phi^{(3)}) + \frac{\partial^3 V}{\partial \phi^3 \partial \rho}\ddot{\phi}\,\phi^{(3)} + \frac{\partial^3 V}{\partial \phi^3 \partial \rho^2}\ddot{\phi}^2 + \frac{\partial^4 V}{\partial \phi^4}\ddot{\phi}^3 + \frac{\partial V}{\partial \phi \partial \rho^6}.\end{aligned} \tag{C.4}$$

Expanding radial derivatives of $\ddot{\phi}$ in Eq.s(C.1) using Eq.(C.3) one gets

$$\begin{aligned}\phi^{(n+1)} =& \sum_j \frac{\partial^2 V}{\partial \phi^2 \partial \rho^j}\phi^{(n-1-j)} + \sum_j \frac{\partial^3 V}{\partial \phi^3 \partial \rho^j}(\phi^{(n-3-j)}\ddot{\phi} + \phi^{(n-4-j)}\phi^{(3)} + \dots) + \\ &+ \sum_{i=2}^{n} B_i \frac{\phi^{(i)}}{\rho^{n-i+1}} + \frac{\partial V}{\partial \phi \partial \rho^{n-1}}.\end{aligned} \tag{C.5}$$

As a result, using Eq.(C.5), Eq.(C.3) may be expressed in terms of partial derivatives of the potential with respect to the scalar field and ρ, $\ddot{\phi}$ and the radius only. Partial derivatives in ρ should be compensated with appropriate powers of ρ with respect to the $j=0$ term. Moreover, radial derivatives of $\frac{\partial V}{\partial \phi}$ compensate for some factors $\ddot{\phi}\rho^{-j}$ with respect to the radius independent case, as can be seen from Eq.(C.5). There are also terms $\frac{\partial^{i+j} V}{\partial \phi^i \partial \rho^j}\rho^j$ added to $\frac{\partial^i V}{\partial \phi^i}$ and $\frac{\partial V}{\partial \phi \partial \rho^j}$ to $\ddot{\phi}\rho^{-j}$. All the ones that are multiplied by a negative or vanishing power of ρ contribute. It is easier to see how this works with some examples. The highest-order derivative (the $\bar{\imath}$-th term) is multiplied only by radial derivatives of the scalar field of order 2 or 3 and thus it is

$$\frac{\partial^{\bar{\imath}} V}{\partial \phi^{\bar{\imath}}}\ddot{\phi}^{\bar{\imath}-1} \quad \text{odd } n, \tag{C.6}$$

$$\frac{\ddot{\phi}^{\bar{\imath}-1}}{\rho}\left(\frac{\partial^{\bar{\imath}} V}{\partial \phi^{\bar{\imath}}} + \frac{\partial^{\bar{\imath}} V}{\partial \phi^{\bar{\imath}} \partial \rho}\rho\right) + \ddot{\phi}^{\bar{\imath}-2}\frac{\partial V}{\partial \phi \partial \rho}\frac{\partial^{\bar{\imath}} V}{\partial \phi^{\bar{\imath}}} \quad \text{even } n. \tag{C.7}$$

Two additional terms appear with respect to the radius-dependent case, one that replaces $\ddot{\phi}\rho^{-1}$, and a radial derivative which is compensated by an additional power of ρ.

The second-highest derivative $\bar{\imath}-1$ is multiplied by radial derivatives of the scalar field of order 2, 3, 4, 5. Using Eq.(C.5), derivatives of order 4 and 5 may be expressed in terms of lower order ones. One gets

$$\text{odd } n \qquad \frac{\ddot{\phi}^{\bar{\imath}-2}}{\rho^2}\left[\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\left(1+\frac{\partial^2 V}{\partial\phi^2}\rho^2\right)+\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}-1}\partial\rho}\rho+\frac{\partial^{\bar{\imath}+1}V}{\partial\phi^{\bar{\imath}-1}\partial\rho^2}\rho^2\right]+$$
$$+\frac{\ddot{\phi}^{\bar{\imath}-3}}{\rho}\left(\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\frac{\partial V}{\partial\phi\partial\rho}+\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\frac{\partial V}{\partial\phi\partial\rho^2}\rho+\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}-1}\partial\rho}\frac{\partial V}{\partial\phi\partial\rho}\rho\right)+ \tag{C.8}$$
$$+\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\ddot{\phi}^{\bar{\imath}-4}\frac{\partial V}{\partial\phi\partial\rho}\frac{\partial V}{\partial\phi\partial\rho}$$

$$\text{even } n \qquad \frac{\ddot{\phi}^{\bar{\imath}-2}}{\rho^3}\left[\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\left(1+\frac{\partial^2 V}{\partial\phi^2}\rho^2+\frac{\partial^3 V}{\partial\phi^2\partial\rho}\rho^3\right)+\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}}\partial\rho}\rho\right.$$
$$\left.\left(1+\frac{\partial^2 V}{\partial\phi^2}\rho^2\right)+\frac{\partial^{\bar{\imath}+1}V}{\partial\phi^{\bar{\imath}-1}\partial\rho^2}\rho^2+\frac{\partial^{\bar{\imath}+2}V}{\partial\phi^{\bar{\imath}-1}\partial\rho^3}\rho^3\right]+$$
$$+\frac{\ddot{\phi}^{\bar{\imath}-3}}{\rho^2}\left[\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\frac{\partial V}{\partial\phi\partial\rho}\left(1+\frac{\partial^2 V}{\partial\phi^2}\rho^2\right)+\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\frac{\partial V}{\partial\phi\partial\rho^2}\rho+\right.$$
$$\left.+\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}-1}\partial\rho}\frac{\partial V}{\partial\phi\partial\rho}\rho+\frac{\partial^{\bar{\imath}+1}V}{\partial\phi^{\bar{\imath}-1}\partial\rho^2}\frac{\partial V}{\partial\phi d\rho}+\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}-1}\partial\rho}\frac{\partial V}{\partial\phi\partial\rho^2}+\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\frac{\partial V}{\partial\phi\partial\rho^3}\right]+$$
$$+\frac{\ddot{\phi}^{\bar{\imath}-4}}{\rho}\left[\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}}\frac{\partial V}{\partial\phi\partial\rho}\frac{\partial V}{\partial\phi\partial\rho}+\frac{\partial^{\bar{\imath}}V}{\partial\phi^{\bar{\imath}-1}\partial\rho}\frac{\partial V}{\partial\phi\partial\rho}\frac{\partial V}{\partial\phi\partial\rho}\rho+\right.$$
$$\left.\frac{\partial^{\bar{\imath}-1}V}{\partial\phi^{\bar{\imath}-1}\partial\rho}\frac{\partial V}{\partial\phi\partial\rho^2}\frac{\partial V}{\partial\phi\partial\rho}\rho\right] \tag{C.9}$$

The first terms for even and odd n are the same as in the radius independent case, and other contributions appear with additional radial derivatives, following the rules described above. Terms that appeared as

$$\frac{\partial^i V}{\partial\phi^i}\rho^{2i-2}\ddot{\phi}^{i-2} \tag{C.10}$$

in the radius independent case are now

$$\frac{\partial^i V}{\partial\phi^i\partial\rho^j}\rho^{2i-2+j}\ddot{\phi}^{i-2}. \tag{C.11}$$

If they are finite in the large ρ^* limit one gets, apart from numerical factors,

$$\frac{\partial V}{\partial\phi}^{(n+1)}=$$
$$=\frac{\partial^{n+1}V}{\partial\phi\partial t^{n+1}}+\sum_{i=0}^{i=\bar{\imath}-2}\sum_{m=0}^{\bar{\imath}-i-1}\sum_{j_0,\ldots j_m}\sum_{n=0}^{2i+1}\frac{\partial V^{\bar{\imath}-i}}{\partial\phi^{\bar{\imath}-i}\partial\rho^{j_0}}\frac{\ddot{\phi}^{\bar{\imath}-i-1-m}}{\rho^{2i+1-n}}\frac{\partial V}{\partial\phi\partial\rho^{j_1}}\times\cdots\times\frac{\partial V}{\partial\phi\partial\rho^{j_m}} \tag{C.12}$$

where $j_0+\cdots+j_m=n$, while in the radius independent case one gets

$$\frac{\partial V}{\partial\phi}^{(n+1)}=\sum_{i=0}^{i=\bar{\imath}-2}\frac{\partial V^{\bar{\imath}-i}}{\partial\phi^{\bar{\imath}-i}}\frac{\ddot{\phi}^{\bar{\imath}-i-1}}{\rho^{2i+1}} \tag{C.13}$$

which is negligible with respect to the zeroth-order term under conditions described in Sect.3.1.1. Imposing also that

$$\frac{\partial^i V}{\partial\phi^i\partial\rho^j}\rho^j \ll \frac{\partial^i V}{\partial\phi^i}, \tag{C.14}$$

and that

$$\frac{\partial^j V}{\partial\phi\partial\rho^j}\rho^j \ll \ddot{\phi}, \tag{C.15}$$

besides Eq.(3.11), one has that Eq.(3.9) is satisfied in the radius-dependent case.

D Numerical implications of the asymptotic bounce

Having the asymptotic bounce at disposal allows improving existing numerical methods and introducing some new others. In this section, the possible implications of the asymptotic bounce for the shooting method are considered, and an alternative numerical method to find the bounce is discussed.

A cut-off for the shooting method

In the shooting method one determines the bounce numerically as the trajectory bracketed between undershoots and overshoots. This allows determining it with arbitrary precision, computational limits aside. In general, using this method implies a large range of integration, as the bounce initial condition $\phi(0)$ can be large ($O(0.1M_P)$) and/or the friction term can be very effective in slowing down the scalar field. Moreover, one should compute the bounce with sufficient precision to get a good estimate of the on-shell action S_E. The Lagrangian must be integrated up to a cut-off, which should be carefully chosen. Knowing the asymptotic bounce allows for a different method to find the (numerical) on-shell action instead of truncation by matching the numerical bounce with the asymptotic one at some $\bar{t}$. The action may be computed as

$$S_E = S_{C,1} + S_{C,2} \tag{D.1}$$

with

$$S_{C,1} \equiv 2\pi^2 \int_0^{\bar{t}} \sqrt{\bar{g}}\mathcal{L}(\bar{\phi},\bar{R})dt \tag{D.2}$$

$$S_{C,2} \equiv 2\pi^2 \int_{\bar{t}}^{\infty} t^3\mathcal{L}(\phi_{+\infty},R_{+\infty})dt \tag{D.3}$$

where $\bar{R}$ and $\bar{\phi}$ are determined numerically and $\phi_{+\infty}$, $R_{+\infty}$ are respectively the scalar field and the Ricci scalar as given by the asymptotic bounce. The asymptotic bounce of a massless scalar fields with small cubic self-interaction and Einstein-Hilbert gravity is

$$\phi_{+\infty}(t) = \frac{C}{2t^2} \tag{D.4}$$

	ϕ_0	$\frac{\lvert\Delta S\rvert}{S_{sm}}$	S	C_0	$\bar{t}$	t^*	$\frac{(S_J - S_E)}{S_J}$
Higgs	0.071	10^{-4}	2063.3	17.6×10^3	10^3	10^{10}	$\times$
Polynomial $\alpha_3 = 10^{-6}$	24×10^{-4}	10^{-4}	6.5961	1617	10^4	10^6	$\times$
Polynomial $\alpha_3 = 10^{-5}$	76×10^{-4}	10^{-5}	6.6293	537	10^3	10^6	$\times$
Polynomial $\alpha_3 = 10^{-4}$	23×10^{-3}	10^{-3}	6.7442	172	10^3	10^5	$\times$
Polynomial $\alpha_3 = 10^{-3}$	68×10^{-3}	10^{-3}	7.1529	59.0	10^2	10^4	$\times$
Polynomial $\alpha_3 = 10^{-2}$	18×10^{-2}	10^{-3}	9.0185	23.3	10^2	10^4	$\times$
Higgs + $0.1\phi^2 R/2$	154×10^{-3}	10^{-5}	2049	8160	10^5	10^8	5×10^{-5}
Higgs + $\phi^2 R/2$	17.7×10^{-3}	$\times 10^{-5}$	2094	7.04×10^5	10^5	10^{10}	3×10^{-3}
Higgs + $10\phi^2 R/2$	2.00×10^{-3}	10^{-3}	2140	6.42×10^5	10^6	10^{10}	3×10^{-2}

Table 1: On-shell action computed with the asymptotic bounce cut-off with the shooting method (sm in the Table) one. The initial condition ϕ_0, the on-shell action computed with the asymptotic bounce cut-off S and its relative deviation with respect to the shooting method result are reported, along with C_0 and the order of magnitude of $\bar{t}$ and t^*.

	t_C	$t_{C,sm}$
Higgs	374s	134s
Polynomial $\alpha_3 = 10^{-6}$	350s	93s
Polynomial $\alpha_3 = 10^{-5}$	277s	102s
Polynomial $\alpha_3 = 10^{-4}$	276s	76s
Polynomial $\alpha_3 = 10^{-3}$	331s	113s
Polynomial $\alpha_3 = 10^{-2}$	346s	211s*

Table 2: Computational time needed to find the bounce by minimization with the shooting method, in theories with a scalar field with potentials Eq.(1.90) and Eq.(3.56) and Einstein-Hilbert gravity.

and $R_{+\infty}(t)$ is given by Eq.(2.5) and Eq.(2.8) with $\phi(t) = \phi_{+\infty}(t)$. The closer is the trajectory to the bounce, the closer is C to C_0. C and $\bar{t}$ may be determined by continuity as

$$\bar{t} + 2\frac{\phi(\bar{t})}{\dot{\phi}(\bar{t})} = 0 \tag{D.5}$$

and $C = -\bar{t}^3\,\dot{\phi}(\bar{t})$. The qualitative description of overshoot trajectories given in Chapter 1 suggests that, at some finite $\bar{t}$, they satisfy Eq.(D.5), and then

$$\phi(t) < \phi_{+\infty}(t) \qquad \text{for } t > \bar{t}. \tag{D.6}$$

Instead undershoot trajectories near the bounce have

$$\phi(t) > \phi_{+\infty}(t) \tag{D.7}$$

at sufficiently large times, and thus there may be no $\bar{t}$ for which Eq.(D.5) holds. Then, $\bar{t}$ may be determined as the point of closest approach

$$3 - 2\frac{\phi(\bar{t})\ddot{\phi}(\bar{t})}{\dot{\phi}(\bar{t})^2} = 0. \tag{D.8}$$

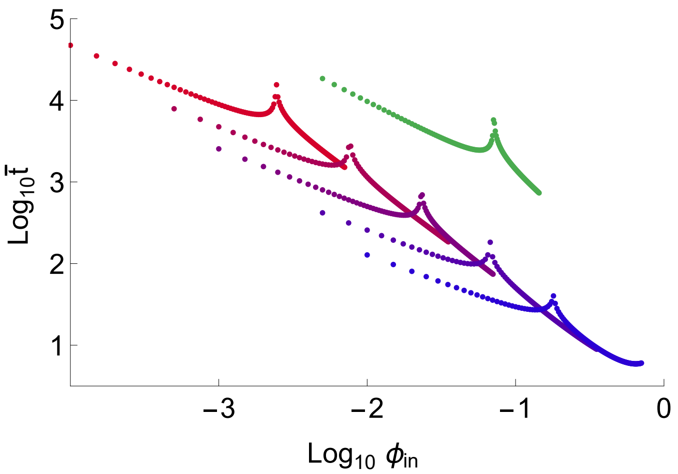

Figure 8: Matching time $\bar{t}$ as a function of the initial condition ϕ_{in} for a single-scalar field theory with Einstein-Hilbert gravity. The maximum marks the bounce initial condition ϕ_0. Plots from red to blue corresponds to theories with potential Eq.(3.56) (from $\alpha_3 = 10^{-6}$ to $\alpha_3 = 10^{-2}$), the green one for the Higgs theory with Einstein-Hilbert gravity (the potential is Eq.(1.90)).

The matching time as defined here separates the bounce-like behaviour of undershots and overshots from the region in which they part, and thus it should get infinitely large on the bounce. Nonetheless, $\bar{t}$ is always finite off the bounce, and in particular $\bar{t} \leq t^*$.

The on-shell action S_C with the asymptotic bounce cut-off and the one found with the shooting method S_{sm} in scalar field theories with Einstein-Hilbert gravity and a non-minimal coupling are compared in Table1. Moreover, $\bar{t} \ll t^*$ and $\bar{t}$ has a maximum on the bounce as a function of ϕ_{in} (Fig.8). There is a small relative deviation among S_C and S_{sm} and calculations in the Jordan and in the Einstein frame are in good agreement. In the Higgs case, C_0 roughly corresponds to the one derived from minimization of the on-shell action, with a small backreaction [101]

$$C_0 = 2 \lim_{t \to +\infty} h(t) t^2 = \frac{4\sqrt{2}}{|\lambda|} \mathcal{R} = 17.4 \times 10^5 \quad \text{with} \quad \mathcal{R} = 350.$$

An alternative numerical method

The discussion above suggests that for every trajectory with initial conditions

$$\phi(0) = \phi_{in} \qquad \dot{\phi}(0) = 0 \tag{D.9}$$

that in general has infinite on-shell action, there is another one that is on-shell only for $t < \bar{t}$, and that has finite S_C. S_C should have a stationary point on the bounce, as Eq.(3.3) holds on-shell. One can show that S_C has a saddle point there (see Appendix E) and thus it is not suitable for minimization to find the bounce. Instead, by slightly changing this functional, one can turn the saddle point into a minimum (or a maximum), at least in the case of single-scalar field theories with Einstein-Hilbert gravity and a non-minimal coupling. The full calculation is reported in Appendix E. The new functional is given by

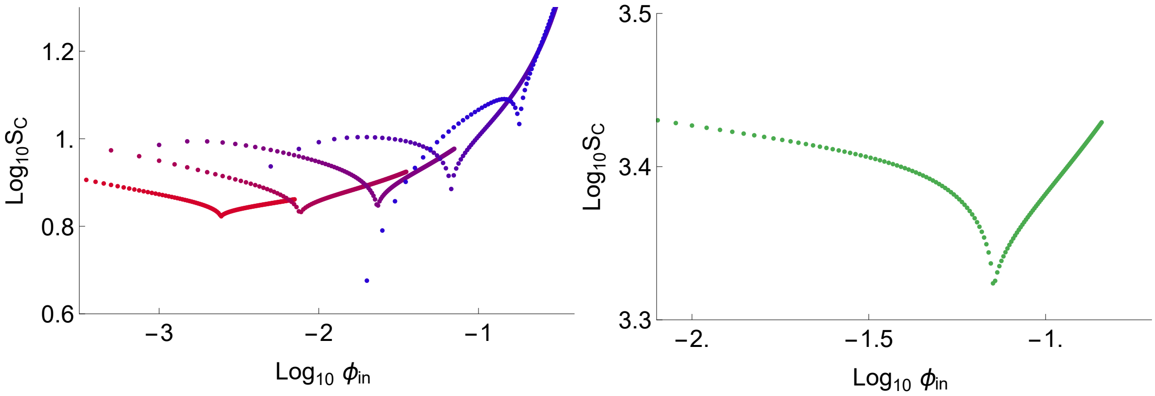

Figure 9: S_C as a function of ϕ_{in} for single scalar field theories with potentials Eq.(3.56) (on the left) and Eq.(1.90) (on the right) and Einstein-Hilbert gravity. The coupling α_3 is varied from 10^{-6} (red) and 10^{-2} (blue).

$$S_{C,1} \equiv 2\pi^2 \int_0^{\bar{t}} \bar{\rho}(t)^3 \mathcal{L}(\bar{\phi}, \bar{R}) dt \tag{D.10}$$
$$S_{C,2} \equiv 2\pi^2 \int_{\bar{t}}^{\infty} t^3 \left(\frac{C^2}{2t^6} + V\left(\frac{C}{2t^2}\right)\right) dt$$

and it has a minimum on the bounce for $\xi \geq -\frac{1}{6}$ and a maximum otherwise (see Fig.9-10) The computational time is of the same order of magnitude of the shooting method one though (see Table2) which gives no clear advantage in using this method over the standard shooting method.

E S_C has a saddle point on the bounce

S_C, as defined in Eq.(D.1), has a saddle point on the bounce, which can be turned into a minimum or maximum by slightly changing $S_{C,2}$. One has

$$\frac{dS_C}{dC} = 2\pi^2\, \bar{t}^3 \left(V(\phi(\bar{t})) - V\left(\frac{C}{2\bar{t}^2}\right)\right) + \pi^2 \frac{C}{\bar{t}^2} + \pi^2 \int_{\bar{t}}^{+\infty} \frac{dV}{d\phi}\left(\frac{C}{2t^2}\right) t\, dt + B_\phi + B_g. \tag{E.1}$$

Here B_ϕ, B_g are boundary terms for the scalar field and for gravity, that appear by using the equations of motion in the variation of the first term of Eq.(D.1). There is one for the scalar field

$$B_\phi = \bar{t}^3 \dot{\phi}(\bar{t}) \frac{\delta\phi}{\delta C}(\bar{t}) = -\frac{\pi^2 C}{\bar{t}^2}, \tag{E.2}$$

while the gravitational one can be computed from the Hawking-Gibbons-York boundary term [161, 162] evaluated at $t = \bar{t}$

$$\delta S_{GHY} = \oint_{\partial V} d^3x \epsilon \sqrt{|h|} n_\alpha V^\alpha \tag{E.3}$$

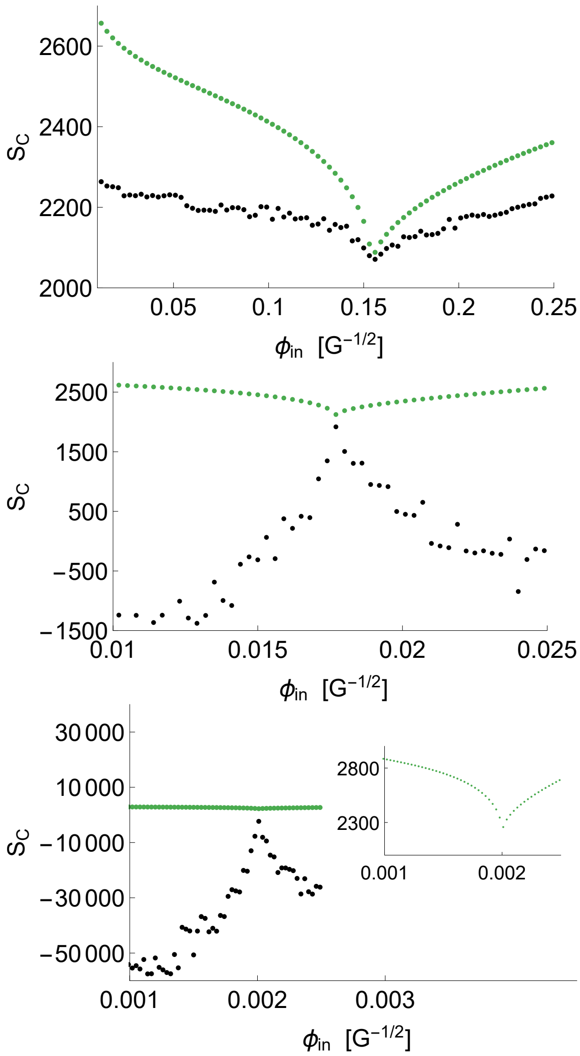

Figure 10: S_C as a function of ϕ_{in} for a single scalar field with potential Eq.(1.90), Einstein-Hilbert gravity and a non-minimal coupling, in the Jordan frame (in black) and in the Einstein one (in green) (top: $\xi = 0.1$, center: $\xi = 1$, bottom $\xi = 10$.)

with

$$V^\alpha = g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu} + g^{\alpha\mu}\delta\Gamma^\nu_{\mu\nu} \tag{E.4}$$

$$\delta\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}(\partial_\beta\delta g_{\gamma\mu} + \partial_\gamma\delta g_{\beta\mu} - \partial_\mu\delta g_{\gamma\beta}) + \frac{1}{2}\delta g^{\alpha\mu}(\partial_\beta g_{\gamma\mu} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\gamma\beta})$$

and $\delta g_{\alpha\beta}$ is the variation of the metric, that has inverse $\delta g^{\alpha\beta} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\mu\nu}$. Moreover, n_α is the unit normal to ∂V and h is the determinant of $h_{\alpha\beta}$, the induced metric on the boundary. ϵ is $+1$ if ∂V is timelike, -1 if it is spacelike. Choosing a timelike future-oriented one-form $n_\alpha = (1,0,0,0)$ one gets

$$B_g = \frac{\pi}{16}\bar{t}^3\left(2g^{\alpha\beta}\delta\dot{g}_{\alpha\beta} + \delta g^{\alpha\beta}\dot{g}_{\alpha\beta}\right) = 0 \tag{E.5}$$

Thus the first term in Eq.(E.1) dominates and it gives

$$\frac{dS_C}{dC} \approx 0 \qquad \frac{d^2S_C}{dC^2} \approx 0. \tag{E.6}$$

One can turn the saddle point into a minimum or a maximum in the case of a single-scalar field with Einstein-Hilbert gravity and a non-minimal coupling. To do that, S_C may be redefined as

$$S_C \equiv S_{C,1} + S_{C,2} \tag{E.7}$$

where

$$S_{C,1} \equiv 2\pi^2\int_0^{\bar{t}} \bar{\rho}^3\left(\frac{\dot{\bar{\phi}}^2}{2} + V(\bar{\phi}) - \frac{M_{\rm P}^2}{2}\bar{R} - \frac{\xi}{2}\bar{\phi}^2\bar{R}\right)dt \tag{E.8}$$

$$S_{C,2} \equiv 2\pi^2\int_{\bar{t}}^{\infty} t^3\left(\frac{C^2}{2t^6} + V\left(\frac{C}{2t^2}\right)\right)dt.$$

Now

$$\frac{dS_C}{dC} = -2\pi^2\frac{M_{\rm P}^2}{2}R\,\bar{t}^3\frac{d\bar{t}}{dC} - 2\pi^2\frac{\xi}{2}R\,\phi^2\bar{t}^3\frac{d\bar{t}}{dC} + \pi^2\int_{\bar{t}}^{+\infty}\frac{dV}{d\phi}\left(\frac{C}{2t^2}\right)t\,dt. \tag{E.9}$$

To determine $\bar{t}(C)$ the bounce velocity is matched with the numerical estimate

$$-\frac{C_0}{(\bar{t}+\delta)^3} = -\frac{C}{\bar{t}^3} \tag{E.10}$$

where δ is a real number that satisfies $\rho(t) - t \approx \delta$ at large times on the bounce. It gives

$$C = C_0\left(1 - \frac{3\delta}{\bar{t}}\right) + \text{h.o.} \tag{E.11}$$

where higher orders are suppressed at large $\bar{t}$ and for $\delta \neq 0$. Thus

$$\frac{d\bar{t}}{dC} = \frac{\bar{t}^2}{3C_0\delta} \tag{E.12}$$

to lowest order. Eq.(E.9) then gives

$$\frac{dS_C}{dC} \approx -\frac{\pi}{8}R(\bar{t})\,\bar{t}^3\frac{d\bar{t}}{dC} \approx -\frac{C_0\pi^2}{3\delta\bar{t}} = -\frac{C_0(C_0-C)\pi^2}{9\delta^2}(1+6\xi) \qquad \frac{d^2S_C}{dC^2} \approx \frac{\pi^2C_0}{9\delta^2}(1+6\xi) \tag{E.13}$$

S_C has a minimum on the bounce for $\xi > -\frac{1}{6}$ and a maximum otherwise.

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