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Some Optimization Problems in Electromagnetism

Ph.D. Thesis (34th cicle)

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Notation and symbols

Along this thesis we will use the following notation:

$\mathbb{R}^+$	the set $(0, \infty)$.
$\mathbb{R}_0^+$	the set $[0, +\infty)$.
$\boldsymbol{X}_N(\mathcal{O})$	the space $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap H(\mathrm{div}, \mathcal{O})$.
$\boldsymbol{X}_N^0(\mathcal{O})$	the space $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap H(\mathrm{div}{=}0, \mathcal{O})$.
$\boldsymbol{X}_T(\mathcal{O})$	the space $\boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \cap H_0(\mathrm{div}, \mathcal{O})$.
$\boldsymbol{X}_T^0(\mathcal{O})$	the space $\boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \cap H_0(\mathrm{div}{=}0, \mathcal{O})$.
$\boldsymbol{\mathcal{D}}(\Omega)$	the space of infinitely differentiable vector fields with compact support in Ω.
$\mathbf{curl}$	the (distributional) rotation operator.
$\mathbf{curl}_\tau$	the surface rotation operator.
div	the (distributional) divergence operator.
div_τ	the surface divergence operator.
$\mathcal{D}(\Omega)$	the space of infinitely differentiable functions with compact support in Ω.
D_x	the Jacobian matrix with respect to the variable x.
∇	the (distributional) gradient operator.
Im	imaginary part operator.
Re	real part operator.

Chapter 1

Introduction

The subject of this thesis is the study of some optimization problems in the context of the mathematical modelling of electromagnetic phenomena. Let us start by trying to answer the following question:

why electromagnetism?

It was November 2017 when my advisor Alberto Valli introduced me to the mathematical theory of Maxwell's equations and the research topics behind it. After that, I started diving into the subject and I was fascinated by several aspects. Despite Maxwell's equations being something everybody with a scientific education is somewhat familiar with, I could barely imagine the mathematical challenges that arise from a rigorous treatment of the theory. The spaces $H(\mathrm{div})$ and $\boldsymbol{H}(\mathbf{curl})$ are already more than a half a century old, yet the complete characterization of their trace spaces came only in 2002 [BCS02]. Differential constraints are frequently encountered, coming either from governing physical laws or playing the role of gauge conditions for potentials, and they make finite elements approximations more challenging. Those coming directly from the models, in turn, can be dealt with in several ways such as saddle point formulations with Lagrange multipliers, penalization techniques, or introducing new variables in the form of scalar or vector potentials. Coercivity estimates and Poincaré inequalities do hold, but a simultaneous control on the divergence and the rotation is required in addition to some geometrical hypothesis [Sch18]. Speaking of the latter, another peculiar feature of the topic - especially for eddy current problems - is the deep interplay between the geometry of the computational domain (and possibly of relevant subsets such as conductors or insulators) and the well-posedness of variational formulations which goes beyond the mere analytical regularity of boundaries, involving instead delicate aspects of algebraic topology that are sometimes overlooked. To name one, the role of harmonics fields in the modelling of the eddy current system and their precise characterization. Many of these are carefully addressed in the monograph [RV10] and in [Ghi10; GK04]. There is also a surprisingly large number of different resolution approaches that can be employed to tackle the same problem: depending on the nature of sources (e.g. voltage excitation or distributed current densities), on the geometry of computational domains and on the specific task to be achieved, one method may be preferable, even though in general each one has its advantages and drawbacks.

Different reduced models are known to derive from the full Maxwell system under additional assumptions, e.g. making hypotheses on the time dependence or getting rid of specific terms appearing in the equations. One of the most relevant is the eddy current approximation that arises by neglecting the famous displacement current, a term introduced by Maxwell himself that describes the propagation of electromagnetic waves. Indeed it is known that in many practical applications, the typical propagation speed of the wave is so large in comparison with the ratio between a reference length (e.g., the diameter of a device) and the time scale of the given physical system, that it can be thought as infinite, making diffusion the most significant phenomenon. Under the PDEs viewpoint, since one is disregarding the term containing the first time derivative of the electric induction, the underlying structure of the problem is *de facto* turned into parabolic from hyperbolic. When the time derivative of the magnetic induction is neglected too, one ends up with the so-called electro-magnetostatic model, where the diffusive process is also not taken into account and the mathematical structure is reduced to a div-curl system. This motivates why the eddy current model is sometimes referred to as the magneto-quasistatic approximation of the Maxwell system. In the time-harmonic framework[1], this amounts to say that the time propagation of the electromagnetic wave is considerably small with respect to the inverse of the angular frequency. The eddy current model has been shown to provide a consistent mathematical approximation of Maxwell's, in the sense that it can be viewed as either its low-frequency or low-permittivity limit; this can be understood as either the possibility to write asymptotic expansions [RV10, Chap. 2] and [BD19; Rod99], or in a suitable sense of operator limit [PP17].

Despite such approximation nature, there are a myriad of contexts where the eddy current model finds applications as the features of the physical system meet the mentioned requirements. Metallurgy has probably been one of the very first ones in the form of industrial melting procedures for metals: the core idea is to exploit the Joule effect by making an alternating current flow into a helical coil twisted around a conducting crucible. Detection of conductive objects [Jen+19] and defect detection in conductors is another relevant application, where flaws are recognized through the measurement of the variation of some key quantities. Interestingly enough, some diagnostic tools in medicine are also built upon the eddy current approximation. Electroencephalography (EEG) and magnetoencephalography (MEG) are two techniques used to map the human brain activity by recording respectively the electric and magnetic fields naturally

[1] Time-harmonic models are suitable whenever the physical quantities vary, or are assumed to vary, periodically in time. This typically happens when the source is prescribed as an alternating current of the form

$$\boldsymbol{J}_r(x,t) = \boldsymbol{J}^\star(x)\cos(\omega t + \phi) = \mathrm{Re}\left(\boldsymbol{J}_c(x)e^{i\omega t}\right).$$

Here $\omega \neq 0$ denotes the angular frequency, ϕ the phase angle, $\boldsymbol{J}^\star$ is real valued while $\boldsymbol{J}_c = \boldsymbol{J}^\star e^{i\phi}$ is complex-valued. In view of this assumptions, one seeks for electromagnetic fields $\boldsymbol{E}, \boldsymbol{H}$ featuring the same structure and this entails the disappearance of the time variable from the system at the price of having to deal with complex-valued vector fields.

arising; in this case, the employment of the eddy current model can be justified by their typical operating frequencies [Häm+93; DL19] (though it is not the most used model). It is noteworthy that in the mathematical theory of EEG, it is customary to consider sources represented by a finite sum of pointwise current dipoles: this is exactly the same ansatz used in Chapter 3. Similar considerations hold for the magnetic induction tomography (MIT), which is yet another medical, contactless imaging tool used to investigate the properties of biological tissues [Gri01]. The eddy current system is especially relevant in this thesis as up to two-thirds of it (namely Chapter 3 and Chapter 4) is based or related to it.

While most of the literature is focused on linear materials and hence on linear problems, nonlinearities also play a major role in electromagnetism. It is known, for example, that the physical properties of ferromagnetic and diamagnetic materials exhibit a strong dependence on external magnetic fields, giving rise to PDEs whose governing differential operator has a quasi-linear curl-curl structure [You13; BLS05; NT17]. This is precisely the type of operator that is considered in the first chapter, where the magnetic reluctivity is given as a scalar nonlinear function $\nu = \nu(x, |\boldsymbol{B}|)$ depending on the strength of the magnetic induction (or the curl of its vector potential). Highly non-linear systems featuring hysteresis loops are as well of interest and can lead to subtle mathematical models [FM89; FA93; Vis05].

Less famous, yet quite active in recent years is the study of Maxwell (quasi)variational inequalities (VIs), both of first and second kind. Inequalities in electromagnetism were probably first addressed by Duvaut and Lions [LD76], with the motivation of modelling the propagation of electromagnetic waves imposing, at the same time, a constraint on the strength of the electric field, resulting in a hyperbolic Maxwell obstacle problem under the mathematical viewpoint. Following this pioneering work, other authors contributed soon after [Mil77]. During the last twenty years the topic has become active again: in [BC00] second kind VIs in the time-harmonic setting are addressed, in [Pri96] a first kind parabolic VI in high-temperature superconductivity (HTS) is studied. The HTS framework appears to be fruitful in this respect as we can name the recent works of Yousept [You17; You20a; You21], and [WY19], where numerical schemes for the Bean's critical state model in HTS are analyzed. [You20b] provides a rather general well-posedness theory for hyperbolic Maxwell obstacle problems that find applications in electromagnetic shielding, a practice intended to block, redirect or reduce electromagnetic fields in a given region by means of suited conductive or magnetic barriers introduced to minimize interference. The first chapter of this thesis is strongly related to magnetic shielding and there can find applications.

The second major topic of this work is optimization, which blends with the described electromagnetic framework. It is needless to say how pivotal optimal control of PDEs and optimization in general has become in all sorts of engineering applications, and electromagnetism is no exception. In the works of Tröltzsch and Valli [TV15; TV16; TV18] different classes of optimal control problems are discussed for the eddy current system, both in the time-harmonic and transient regime. The case of full-Maxwell state equation is treated in [BY16]. Quasi-linear optimal control problems are addressed in [You13] and in [NT17], respectively in the static and time-dependent parabolic setting. The field of design optimization also finds fertile ground in electromagnetism: [Gan+15; GAL15; GS21] are all examples of topological sensitivity analysis being exploited to optimize the shape of electric motors and machines. We will be focusing on two types of optimal control problems and a topology optimization task. These fall into the class of infinite-dimensional optimization problems that cannot be solved directly apart from some special cases, therefore the determination of optimality conditions, adjoint equations and descent directions is of paramount importance as most numerical approximation algorithms rely upon them. Descent directions in the field of shape optimization are related to the concepts of shape topological and derivatives: while the shape derivative aims at measuring the sensitivity with respect to *regular* perturbations of the boundary [SZ92; DZ11], the topological derivative assesses the influence of creating small holes around certain points of the domain (or there changing the material properties). It has been observed that a combination of the two can lead to an improvement of level set methods seeking for optimal designs [HKO07; BHR04]. For what concerns optimal control problems discussed in Chapter 2 and Chapter 3, in the former we prove classical first order necessary and sufficient conditions involving Gâteaux derivatives, whereas in the latter a system of Clarke-stationarity type is achieved [Cla90]. This is a consequence of the approximation procedure and the dual formulation via Lagrange multipliers. Despite second order sufficient conditions have been investigated in various instances [CRT15; CT16; CT09] for scalar elliptic and parabolic operators, including optimal control tasks driven by obstacle problems [CW21], to the best of author's knowledge they haven't been studied in the $\boldsymbol{H}(\mathbf{curl})$ framework yet (even in the equality case). To conclude, in Chapter 4 we compute the first order topological derivative for a shape functional depending on the solution of a low-frequency electromagnetic problem which is closely related to the eddy current system, see Section 4.2 for details on this connection.

This thesis is articulated in three chapters. Despite the common thread being optimization in the context of Maxwell-related PDEs (and inequalities), they are somehow standalone and independent of each other. Chapter 2 contains the analysis of an optimal control problem governed by a quasi-linear magnetostatic obstacle problem, featuring first order (differential) state constraints. The same optimal control problem in the equality case was treated in [You13]. It is to be said that the transition to an obstacle problem is all but straightforward as the inequality structure makes the control-to-state mapping non-differentiable, and the presence of a first order state constraint in the obstacle set makes the derivation of a proper optimality system rather challenging. To the best of the author's knowledge, this represents the first scientific work where optimal control is addressed for a quasi-linear Maxwell VI, especially with state constraints involving the curl.

Following a concise preliminary discussion on $\boldsymbol{H}(\mathbf{curl})$ and $H(\mathrm{div})$ spaces and orthogonal decomposition results (that are also used in the other chapters), the first part of the chapter is dedicated to the existence theory for the state variational inequality. This is achieved utilizing a coupled relaxation-penalization technique: the penalization allows to obtain a sequence of approximating equations (VI_γ), while the relaxation to get rid of the explicit first order constraint with the introduction of an additional variable. Note that existence could be also proved directly with the theory of nonlinear monotone operators, but the approximation approach turns out to have several advantages. First, it naturally provides the so-called dual formulation of the state inequality (2.18), i.e. a mixed formulation where a Lagrange multiplier appears and is characterized via an inequality condition. Secondly, it serves as a basis for the subsequent study of optimality conditions as it features a solution mapping that is more regular, namely weakly Gâteaux differentiable. The optimal control analysis begins in Section 2.2; while the existence of optimal controls is shown in a standard way, the derivation of necessary optimality conditions demands a regularization of the optimization problem itself. For this, we make use of the regularized state equation, introduce an adjoint equation and state the corresponding optimality system (see Theorem 2.2.8) thanks to the improved regularity previously mentioned. Such an optimality system still depends on the approximation parameter γ: the ultimate task is to somehow pass to the limit to obtain optimality conditions (with a proper characterization of the involved Lagrange multipliers) for the original optimal control problem. In particular, we seek for orthogonality conditions and sign conditions that are analogous to those known for optimal control of classical H^1 obstacle problems (cf. [MP84, Thm. 3.2], [MRW15, Thm. 2]), except we have a nonzero obstacle function $d(\cdot)$ that is obviously expected to play a role.

The crucial ingredient to prove such conditions are uniform bounds for the Moreau-Yosida penalization term and its derivative, which respectively appear in the (regularized) state and adjoint equation. It is exactly in this instance that the presence of first order differential constraint heavily affects the analysis (especially for the adjoint equation), see Remark 2.2.13 for technical details. Theorem 2.2.9 is one of the main contributions of the chapter and contains an optimality system for each given optimal solution of the original problem (P_{sol}), as well as a sign condition for the adjoint multiplier (2.94). Similarly, using other technical assumptions we can show a first version of what we called orthogonality condition (2.116), see Corollary 2.2.12. Section 2.2.3 closes the chapter and is devoted to better characterizing the adjoint multiplier under mild geometrical assumptions (in fact, we just need that the obstacle set coincides with the computational domain and that it is simply connected). It is shown that taking a Helmholtz projection is enough to get one of the mentioned uniform bounds, and this turns out to be sufficient to deduce a satisfactory orthogonality condition (2.136) involving the obstacle: Theorem 2.2.15 shows the complete optimality system in this alternative setting and adds up to the most relevant contributions of the chapter. Most of the material is based on a preprint in collaboration with Irwin Yousept and Maurice Hensel (University of Duisburg-Essen), which is not yet submitted but very close to.

In Chapter 3, an optimal control problem for an $\boldsymbol{E}$-based formulation of the eddy current system with a pointwise source (Dirac measure) is discussed. The relevance of this type of problem is twofold: on the one hand, as already quoted, pointwise dipole sources are customary in applications related to medical imaging; on the other hand, one can think of it as a particular instance of a measure-valued optimal control problem, a topic that received considerable attention in the last decades due to its flexibility in applications and its intrinsic theoretical challenges. We refer e.g. to [CCK13; CK16; KPV14] for optimal control problems in measure spaces governed by parabolic equations, and to [CK12; PV13; CK19] for the elliptic case. Section 3.1.1 contains an overview of boundary value problems featuring measure sources and serves as a motivation for the apparent impossibility to treat the general case (namely, without assuming that the control measure is a Dirac mass) in the context of electromagnetism. Section 3.3 is dedicated to the study of the state equation. To this end we utilize a technique based on a spitting of the unknown, which involves the fundamental solution of a **curl curl** $-$ Id type operator. Even though this approach has already been applied to a related inverse problem [ACV11], our contribution lies in a deeper study of the solution mapping and in the need of keeping track of the control dependence throughout the analysis. In addition, we discuss how to deal with an alternative boundary condition (Section 3.3.3). If existence of optimal solutions comes with relative ease, some attention is required to set up an adjoint method that is consistent with the split structure of the state variable. This leads to the introduction of two adjoint states. In Theorem 3.4.5 we prove necessary and sufficient first order optimality conditions that are the main contribution of the chapter. The present results can be found in a published work of the author [Cas20].

Chapter 4 is the fruit of a collaboration between the author, Kevin Sturm (TU Wien) and Peter Gangl (TU Graz), which is very much still a work in progress. In particular, in this thesis we just compute the first order topological derivative for a low-frequency electromagnetic problem (which resembles a classical vector potential formulation for the eddy current system), but lay the groundwork for the derivation of higher-order ones and for the extension to the actual eddy current model. In Section 4.1 we provide a kind of new take on Beppo-Levi quotient spaces in $\mathbb{R}^3$ for the **curl** operator, which are needed to define appropriate *correctors*, also known as auxiliary fields. The very first one defined in Lemma 4.3.9 is already known in the literature[2], others, like (4.128), are not and indeed they turn out to be more delicate to analyze because non-solenoidal right hand sides entail the appearance of a gradient multiplier in strong interpretations, as shown in Lemma 4.6.2. A related novelty in this respect is Lemma 4.3.12, which appears to be an extension of results like [BS21, Lem. 3.7] (where the underlying context is elasticity) - used to deal with inhomogeneous Dirichlet conditions created by the addition of correctors - in the presence of a saddle point structure. Our analysis is performed within a Lagrangian framework that allows to consider at once the shape functional of interest and the differential problem it is constrained by. More precisely, the Lagrangian is given as the sum of the shape functional and the variational problem defining the state variable, and the adjoint (or averaged adjoint) is then defined by taking suitable partial derivatives (see Section 4.1.1). The Lagrangian framework is a customary approach both in shape and topology optimization [GS21; Stu15; Stu20; BS21; HLY15; GS22; IK08]; in Section 4.1.1 we review it and add some remarks concerning the adaptation to functional spaces of complex-valued functions, which are suited for the time-harmonic regime of the state equation. More specifically, we employ

[2] We refer to [AVV01, Sec. 5.1], but our approach is a bit different as we make use of a weak formulation in a quotient space, and *then* select a representative with suitable properties.

the so-called averaged adjoint method introduced by Sturm [Stu15]: as explained in [BS21], compared to other approaches, it has the advantage of shifting most of the work towards the asymptotic analysis of the averaged adjoint equation and of being applicable to a wide range of cost functions. It is known that the computation of topological derivatives requires pointwise evaluations of the state and adjoint field, and it is usual to assume that such regularity holds; under a summability assumption on the impressed current and using techniques of elliptic regularity theory, here we prove that those fields actually enjoy the necessary regularity. We refer respectively to Lemma 4.2.6 for the direct state and to Lemma 4.4.1 for the averaged adjoint. We stress another time that the first-order topological derivative can be computed with sole knowledge of the first order expansions provided by Theorem 4.3.14 and Theorem 4.4.5; the introduction of other auxiliary fields (cf. the boundary layer correctors (4.128) and (4.139)) and the derivation of improved expansions (cf. Theorem 4.6.8 and Theorem 4.6.10) is intended to be a starting point for a higher-order analysis eventually leading to higher-order derivatives, which is still a work in progress. The main problems to be addressed are the proper understanding of asymptotic expansions at infinity of the correctors, and the introduction of new ones; see also Section 4.7.

Chapter 2

Optimal control of a quasi-linear magnetostatic obstacle problem

This chapter is concerned with the discussion of an optimal control problem subject to a Maxwell quasi-linear variational inequality of the first kind, with first order state constraints. Researchers' interest in the analysis of electromagnetic obstacle problems has roots in the book of Duvat-Lions [LD76] and draws inspiration from the practice of electromagnetic shielding which aims to reduce, block or redirect electromagnetic fields in a specific region by means of suitable barriers (obstacles) of conductive or highly magnetic permeable materials. The mathematical theory addressing the well-posedness of Maxwell variational inequalities has been recently enriched with new contributions: along with an existence result for the general hyperbolic linear obstacle problem, Yousept [You20b] showed that under general assumptions, the electromagnetic fields respectively satisfy the Ampère-Maxwell and Faraday equation in the free-regions, i.e. the subdomains where no constraints are imposed on the fields. Another related contribution of Yousept is given by [You13], where a similar optimal control problem is addressed for the equality case and together with a finite element numerical analysis.

Despite the mentioned developments, the investigation of our optimal control problem introduces mayor theoretical difficulties, mostly due to two factors: the general lack of regularity of solution mappings associated with variational inequalities, and the presence of a first order state constraint of the form

$$|\,\mathbf{curl}\,\boldsymbol{H}(x)| \leq d(x) \quad \text{a.e. in } \omega,$$

where ω represents the obstacle region and $d\colon \omega \to \mathbb{R}^+$ the obstacle function. Let us briefly have a closer look at these two aspects in a simple H^1-scalar framework: it is clear that we cannot expect more regularity for a quasi-linear problem driven by a **curl curl** bilinear form.

If $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is a coercive bilinear form and $u \in H^{-1}(\Omega)$, a seminal result of Mignot [Mig76] states that the solution mapping $u \mapsto y(u)$ associated with the problem

$$y \in K = \{v \in H_0^1(\Omega) : v \geq 0 \ \text{ a.e. in } \Omega\}, \quad a(y, v-y) \geq \langle u, v-y \rangle \quad \forall v \in K,$$

in general merely admits a directional derivative at every point, which is characterized as the solution of another obstacle problem. The solution mapping turns out to be Gâteaux differentiable if and only if the critical cone enjoys a specific point-wise property. The regularity of such solution mappings is even less studied, and understood, if the obstacle features a first order constraint like $|\nabla v| \leq 1$: the problem was presented as open by Mignot and Puel in 1984 [MP84] and to our knowledge very few works addressing the topic are available in the literature besides [HS11; HMW12]; in particular, the present chapter appears to be the first contribution in the context of the mathematical modelling of Maxwell's equations. Our analysis will be confined to specific obstacle sets featuring first order constraints, for a more in-depth coverage of the theory of elliptic obstacle problems we refer the reader to the classic books of Kinderlehrer-Stampacchia and Troianello [KS00; Tro87].

As a consequence of the issues concerning differentiability, the employment of a standard adjoint calculus is usually not possible and different penalization techniques have been introduced in the literature to get a sequence of approximating variational equalities featuring smoother solution mappings. We mention for instance [MRW15], where the authors exploit a cubic version of the smoothed maximum function. In order to handle the first order state constraint, we employ a Moreau-Yosida type penalization technique combined with a relaxation. The former is based on known regularizations of the maximum function, the latter aims to reduce the constraint to zeroth order at the price of an additional variable. As explained in the sequel, we leave some freedom for the functional space $\boldsymbol{V}(\omega)$ corresponding to the relaxation variable:

$$\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega) \subseteq \boldsymbol{V}(\omega) \subseteq \boldsymbol{L}^2(\omega).$$

Let us mention that this does not affect the reliability of our approximation as Lemma 2.1.8 holds regardless of the choice, i.e. the solution of the approximating equality converges to the solution of state variational inequality. We also point out that the choice of $\boldsymbol{V}(\omega) = \boldsymbol{L}^2(\omega)$ has the advantage of allowing a simpler discretization of the corresponding nonlinearity by piecewise constant finite elements. Even tough our analysis could be carried out without the relaxation, we think that the reduction to zeroth order can be of interest for future research aiming to improve optimality conditions in the presence of first order constraints.

The main drawback of the penalization technique can be identified with the limiting analysis required for the optimality system: if on the one hand it is rather easy to prove that the state variational inequality is satisfied in the limit and the presence of an (approximating) adjoint is suitable for the implementation of numerical schemes, the limit of the Lagrange multiplier produced by the regularization term in the adjoint is hard to characterize effectively. In this respect, we present two possible characterizations (and corresponding optimality systems) of such multiplier that rely on different assumptions. Specifically, Corollary 2.2.12 is built upon the integral bound of Assumption **A1**; instead Theorem 2.2.15 only requires geometrical assumptions such as equality between the obstacle set and the computational domain, but the corresponding orthogonality condition (2.136) provides information not directly referred to the multiplier that appears in the equation (2.133). Such multiplier is in fact split into the sum of two quantities (2.137), one of which appears in the orthogonality condition. It is worth to underline that the optimal control problem under investigation features several theoretical difficulties, which justify the need of additional assumptions in order to derive optimality conditions that are usually to be expected. See also Remark 2.2.13 for a deeper digression on the topic and a comparison with existing literature.

Definition 2.0.1 (Weakly Lipschitz domains)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, i.e. it is open and connected. Let us define*

$$\mathcal{C} := (-1,1)^3, \quad \mathcal{C}_\pm := \{(x_1, x_2, x_3) \in \mathcal{C} \colon \pm x_3 > 0\}.$$

The set Ω is called weakly Lipschitz if its boundary $\partial\Omega$ is a Lipschitz submanifold. More precisely, if there exists a finite open covering $\{U_1, \dots, U_n\} \subset \mathbb{R}^3$ of $\partial\Omega$ and mappings $\boldsymbol{\varphi}_n \colon U_n \to \mathcal{C}$ such that for every $i = 1 \dots n$ it holds

- *$\boldsymbol{\varphi}_i$ is bijective, $\boldsymbol{\varphi}_i \in \boldsymbol{C}^{0,1}(U_i)$ and $\boldsymbol{\varphi}_i^{-1} \in \boldsymbol{C}^{0,1}(\mathcal{C})$,*
- *$\boldsymbol{\varphi}_i(U_i \cap \Omega) = \mathcal{C}_-$.*

According to the seminal Rademacher's theorem [EG15, Sec. 3.1.2, Thm. 2], the boundary of a weakly Lipschitz domain admits an outward normal vector $\boldsymbol{n} \in L^\infty(\partial\Omega)$. The geometrical framework for our analysis is described by the following

Assumption 2.0.2. *Let $\Omega \subset \mathbb{R}^3$ be an open and bounded domain with connected boundary, which we assume to possess the weak Lipschitz regularity. Further, let $\omega \subseteq \Omega$ denote the obstacle set which is assumed to be open, connected and weakly Lipschitz.*

Note that the weakly Lipschitz regularity is known to be the minimal assumption on $\partial\Omega$ to ensure the validity of the key compact embeddings (2.6), cf. [BPS16].

Let us introduce the convex set

$$\boldsymbol{K} := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) : |\,\mathbf{curl}\,\boldsymbol{v}| \leq d \text{ a.e. in } \omega\},$$

which bears the first-order type obstacle structure previously mentioned. The obstacle function $d \colon \omega \to \mathbb{R}^+$ is supposed to be of class $L^\infty(\omega)$. We are interested in the mathematical analysis of the following optimal control problem:

$$\begin{cases} \min\limits_{\boldsymbol{J},\boldsymbol{A}} \left(\frac{1}{2} \|\,\mathbf{curl}\,\boldsymbol{A} - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} \right) \text{ subject to} \\ \boldsymbol{A} \in \boldsymbol{K}, \quad \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}|)\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\,\mathrm{d}x \geq \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A})\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \end{cases} \tag{2.1}$$

where the control parameter β is positive and the state variable $\boldsymbol{A}$ is sought for in a space of divergence-free vector fields (cf. the last condition in (VI)). The precise definitions of the functional spaces for $\boldsymbol{A}, \boldsymbol{J}$ will be specified in the upcoming section. The variational inequality in (2.1) can be seen as a vector potential formulation for the magnetostatic obstacle problem in the presence of a nonlinear, scalar magnetic reluctivity ν, see [You13] for a more detailed treatment in the equality case. Note that $|\,\mathbf{curl}\,\boldsymbol{A}| = |\boldsymbol{B}|$ so that seeking for $\boldsymbol{A} \in \boldsymbol{K}$ amounts to require $|\boldsymbol{B}| \leq d(\cdot)$ almost everywhere, thereby imposing a constraint on the strength of the magnetic induction in the obstacle region ω. As previously quoted, it is exactly this feature that paves the way for applications, especially in the practice of magnetic shielding.

Here $\boldsymbol{B}_{\mathrm{d}} \in \boldsymbol{L}^2(\Omega)$ denotes the desired magnetic induction and the role of control variable is played by the impressed current density $\boldsymbol{J}\colon \Omega \to \mathbb{R}^3$, which is required to satisfy the physical condition

$$\operatorname{div} \boldsymbol{J} = 0 \quad \text{in } \Omega, \tag{2.2}$$

leading to $H(\text{div}{=}0, \Omega)$ as a natural space of admissible controls.

Remark 2.0.3. *Note that $\boldsymbol{J} \in H(\text{div}{=}0, \Omega)$ appears to be a generalization of the class of controls considered in [You13] for the same problem, but with equality state equation. Indeed there the author works with $\boldsymbol{J} \in H_0(\text{div}{=}0, \Omega_{\mathrm{c}})$ where $\Omega_{\mathrm{c}} \subset \Omega$ denotes a control domain which is open and weakly Lipschitz. In view of the boundary condition on $\partial\Omega_{\mathrm{c}}$, for all $\boldsymbol{J} \in H_0(\text{div}{=}0, \Omega_{\mathrm{c}})$ a simple extension to zero yields an element of $H(\text{div}{=}0, \Omega)$.*

The structure of the chapter is as follows. We begin with Section 2.1 which is dedicated to the development of a general existence theory for the state inequality in (2.1). To this end, in Section 2.1.2 we introduce and study a penalized version of the VI which takes the form of a sequence of equality problems. Its importance is twofold: on the one hand, its limiting analysis provides existence for the state variable (see Theorem 2.1.3); on the other hand, it turns out to be suited for the optimal control analysis as its solution mapping exhibits better differentiability properties. In Section 2.2, building upon the previous results we finally focus on the optimal control problem (2.1) and approximate it with a regularized version (P_γ). Section 2.2.2 and Section 2.2.3 cover the limiting analysis of the latter and lead to optimality systems for the original problem.

2.1 Existence theory with general source term

Before focusing on the optimal control problem, we shall present a general existence theory for the state equation appearing in (2.1) in strong form. Here general theory means that we prove well-posedness for sources $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, while the optimal control analysis is carried out for $\boldsymbol{J} \in H(\text{div}{=}0, \Omega)$ which is consistent with physical models. As shown in Lemma 2.2.3, it turns out that it is equivalent to seek for optimal solutions in $H(\text{div}{=}0, \Omega)$ or in $\boldsymbol{L}^2(\Omega)$, in this sense working with solenoidal controls causes no loss of generality. For $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, the inequality in (2.1) exhibits a gradient (Lagrange multiplier) term that would vanish in case of $\boldsymbol{J}$ being divergence free. We will show that such term can be seen as being produced by a Helmholtz decomposition of the source.

Despite our regularization procedure is in principle designed for the optimal control analysis, it turns out that it can be exploited to show existence for the state equation and the corresponding Lagrange multipliers at once. The penalization is therefore introduced and discussed immediately after the upcoming preliminaries, and then resumed in Section 2.2.

2.1.1 Preliminaries

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded domain. For what concerns the present chapter, we shall be interested in the real Hilbert spaces

$$\begin{aligned} \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\} \\ \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}),\ \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \text{ on } \partial\mathcal{O}\} \\ H(\mathrm{div}, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \mathrm{div}\, \boldsymbol{u} \in L^2(\mathcal{O})\} \\ H_0(\mathrm{div}, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \mathrm{div}\, \boldsymbol{u} \in L^2(\mathcal{O}),\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\}, \end{aligned} \tag{2.3}$$

where the divergence and **curl** operators, as well as the trace operators, have to be understood in the sense of distributions (see [Mon03; BCS02] for a detailed treatment of the trace spaces and their characterization). All the space are endowed with the natural inner products that make them Hilbert spaces. The subspaces of $\boldsymbol{L}^2$ of divergence-free vectors

$$\begin{aligned} H(\mathrm{div}{=}0, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \mathcal{O}\} \\ H_0(\mathrm{div}{=}0, \mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O}) : \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \mathcal{O},\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\mathcal{O}\} \end{aligned} \tag{2.4}$$

are also of interest. For the case $\mathcal{O} = \Omega$, we sometimes agree not to specify the domain when stating the introduced function spaces. In the given context let us also introduce the spaces

$$\begin{aligned} \boldsymbol{X}(\mathcal{O}) &= \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \cap H(\mathrm{div}, \mathcal{O}) \\ \boldsymbol{X}_N(\mathcal{O}) &= \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap H(\mathrm{div}, \mathcal{O}), \\ \boldsymbol{X}_T(\mathcal{O}) &= \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \cap H_0(\mathrm{div}, \mathcal{O}), \end{aligned}$$

and the corresponding versions with null divergence

$$\begin{aligned} \boldsymbol{X}_N^0(\mathcal{O}) &= \boldsymbol{H}_0(\mathbf{curl}, \mathcal{O}) \cap H(\mathrm{div}{=}0, \mathcal{O}), \\ \boldsymbol{X}_T^0(\mathcal{O}) &= \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) \cap H_0(\mathrm{div}{=}0, \mathcal{O}), \end{aligned} \tag{2.5}$$

which will be crucial for our analysis. Both $\boldsymbol{X}_N^0$ and $\boldsymbol{X}_T^0$ are endowed with the topology of $\boldsymbol{H}(\mathbf{curl})$ and their importance in the context of electromagnetic PDEs is well known: indeed if $\mathcal{O}$ is additionally assumed to be weakly Lipschitz, it has been proved (see [BPS16, Theorem 4.7]) that the embeddings

$$\boldsymbol{X}_N(\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O}) \quad \text{and} \quad \boldsymbol{X}_T(\mathcal{O}) \hookrightarrow \boldsymbol{L}^2(\mathcal{O}) \tag{2.6}$$

are compact. This combined with the divergence-free condition leads to Poincaré-Friedrichs inequalities of type (2.12) for the spaces $\boldsymbol{X}_N^0$ and $\boldsymbol{X}_T^0$.

Let us now briefly review some orthogonal decomposition results in $\boldsymbol{L}^2$ and $\boldsymbol{H}(\mathbf{curl})$. First of all we have the following elementary decompositions [DL99, pp. 313-314]

$$\begin{aligned} \boldsymbol{L}^2(\mathcal{O}) &= H_0(\mathrm{div}{=}0, \mathcal{O}) \oplus \nabla H^1(\mathcal{O}) \\ \boldsymbol{L}^2(\mathcal{O}) &= H(\mathrm{div}{=}0, \mathcal{O}) \oplus \nabla H_0^1(\mathcal{O}). \end{aligned} \tag{2.7}$$

Then, aiming to write the divergence-free part as the **curl** of a vector potential, one ends up with

$$\begin{aligned} \boldsymbol{L}^2(\mathcal{O}) &= \mathbf{curl}(\boldsymbol{X}_N^0(\mathcal{O})) \oplus \nabla H^1(\mathcal{O}) \oplus \mathcal{H}(m,\mathcal{O}), \\ \boldsymbol{L}^2(\mathcal{O}) &= \mathbf{curl}(\boldsymbol{X}_T^0(\mathcal{O})) \oplus \nabla H_0^1(\mathcal{O}) \oplus \mathcal{H}(e,\mathcal{O}), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \mathcal{H}(m,\mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\colon\ \mathbf{curl}\,\boldsymbol{u} = \mathbf{0},\ \operatorname{div}\boldsymbol{u} = 0,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\mathcal{O}\} \\ \mathcal{H}(e,\mathcal{O}) &= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\colon\ \mathbf{curl}\,\boldsymbol{u} = \mathbf{0},\ \operatorname{div}\boldsymbol{u} = 0,\ \boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \quad \text{on } \partial\mathcal{O}\} \end{aligned}$$

are respectively the so-called space of magnetic and electric harmonic fields. We refer the reader to [DL99, pp. 313-314] for details and other possible decompositions. It is known [RV10, Appx. A4] that both spaces are finite dimensional, in particular the dimension of $\mathcal{H}(e)$ is equal to the number of connected components of $\partial\mathcal{O}$ minus one, while the dimension of $\mathcal{H}(m)$ is equal to the first Betti number of $\mathcal{O}$. The following proposition summarizes some of the previous considerations and will be used several times.

Proposition 2.1.1. *Let $\mathcal{O} \subset \mathbb{R}^3$ be an open, bounded and weakly Lipschitz domain.*

- *It holds*

$$\begin{aligned} \boldsymbol{H}(\mathbf{curl},\mathcal{O}) &= (\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap H_0(\operatorname{div}{=}0,\mathcal{O})) \oplus \nabla H^1(\mathcal{O}), \\ \boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) &= (\boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \cap H(\operatorname{div}{=}0,\mathcal{O})) \oplus \nabla H_0^1(\mathcal{O}). \end{aligned} \tag{2.9}$$

- *If $\mathcal{O}$ is simply connected, then*

$$\boldsymbol{L}^2(\mathcal{O}) = \mathbf{curl}\,(\boldsymbol{H}_0(\mathbf{curl},\mathcal{O}) \cap H(\operatorname{div}{=}0,\mathcal{O})) \oplus \nabla H^1(\mathcal{O}). \tag{2.10}$$

- *If $\partial\mathcal{O}$ is connected, then*

$$\boldsymbol{L}^2(\mathcal{O}) = \mathbf{curl}\,(\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap H_0(\operatorname{div}{=}0,\mathcal{O})) \oplus \nabla H_0^1(\mathcal{O}). \tag{2.11}$$

All decompositions are orthogonal with respect to the $\boldsymbol{L}^2(\mathcal{O})$ inner product.

Proof. Let us first prove the first identity in (2.9), the second one can be proved analogously. The inclusion $(\boldsymbol{H}(\mathbf{curl},\mathcal{O}) \cap H_0(\operatorname{div}{=}0,\mathcal{O})) \oplus \nabla H^1(\mathcal{O}) \subset \boldsymbol{H}(\mathbf{curl},\mathcal{O})$ is obvious. For the other one, let $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\mathcal{O})$. We define $\eta \in H^1(\mathcal{O})$ as the solution of the Neumann problem

$$\begin{cases} \Delta\eta = \operatorname{div}\boldsymbol{u} & \text{in } \mathcal{O}, \\ \nabla\eta\cdot\boldsymbol{n} = \boldsymbol{u}\cdot\boldsymbol{n} & \text{on } \partial\mathcal{O}. \end{cases}$$

Then we have $\mathbf{0} = \mathbf{curl}\,\nabla\eta \in \boldsymbol{L}^2(\mathcal{O})$, $\operatorname{div}(\boldsymbol{u} - \nabla\eta) = 0$ by construction as well as $(\boldsymbol{u} - \nabla\eta)\cdot\boldsymbol{n} = 0$ on $\partial\mathcal{O}$, so that

$$\boldsymbol{u} = \underbrace{(\boldsymbol{u} - \nabla\eta)}_{\in H(\mathbf{curl},\mathcal{O})\cap H_0(\operatorname{div}=0,\mathcal{O})} + \underbrace{\nabla\eta}_{\in\nabla H^1(\mathcal{O})}$$

provides the desired decomposition. To conclude, (2.10) and (2.11) follow from (2.8) taking into account that $\mathcal{H}(e,\mathcal{O}) = \{\mathbf{0}\}$ if $\partial\mathcal{O}$ is connected, while $\mathcal{H}(m,\mathcal{O}) = \{\mathbf{0}\}$ if $\mathcal{O}$ is simply connected (see [Amr+98] or [RV10, Appx. A4]). □

Recalling that Ω features a connected boundary (cf. Assumption 2.0.2), owing to (2.6) and (2.11) the Poincaré-Friedrichs inequality

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \lesssim \|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_{L^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{X}_N^0 \tag{2.12}$$

holds true. Note that the latter inequality implies that $\|\operatorname{\mathbf{curl}} \cdot\|_{L^2(\Omega)}$ is an equivalent norm on $\boldsymbol{X}_N^0(\Omega)$. Finally $C > 0$ represents a generic constant whose value can vary from line to line. We close this section by presenting the basic assumption on the material parameter.

Assumption 2.1.2 (cf. [You13]). *We assume the magnetic reluctivity $\nu\colon \Omega \times \mathbb{R}_0^+ \to \mathbb{R}$ to be given as a Carathéodory function, i.e. for every $s \in \mathbb{R}_0^+$ the function $\nu(\cdot, s)$ is measurable while for almost every $x \in \Omega$ the function $\nu(x, \cdot)$ is continuous. Moreover, there are constants $\underline{\nu}, \overline{\nu} \in (0, \nu_0)$ such that*

$$\begin{aligned} \underline{\nu} \le \nu(x,s) \le \nu_0 \quad &\textit{for a.e. } x \in \Omega \textit{ and every } s \in \mathbb{R}_0^+, \\ (\nu(x,s)s - \nu(x,\hat{s})\hat{s})(s-\hat{s}) \ge \underline{\nu}(s-\hat{s})^2 \quad &\forall s, \hat{s} \in \mathbb{R}_0^+, \\ |\nu(x,s)s - \nu(x,\hat{s})\hat{s}| \le \overline{\nu}|s-\hat{s}| \quad &\forall s, \hat{s} \in \mathbb{R}_0^+ \end{aligned} \tag{2.13}$$

holds true. Here $\nu_0 > 0$ denotes the magnetic reluctivity in the vacuum.

In the magneto-static setting, the function $s \mapsto \nu(x,s)s$ describes the non-linear relation between $|\boldsymbol{B}|$ and $|\boldsymbol{H}|$ and it can be experimentally proven that such curve is supposed to be strictly monotone and bounded [BLS06, Sec. 3], [RKK02; BLS05]. In this sense Assumption 2.1.2 can be called a physical assumption. Later on we shall also require continuous differentiability, which appears to be reasonable.

Note that if the above assumptions are satisfied, then the following inequalities follow for a.e. $x \in \Omega$:

$$\begin{aligned} (\nu(x,|s|)s - \nu(x,|\hat{s}|)\hat{s}) \cdot (s-\hat{s}) \ge \underline{\nu}|s-\hat{s}|^2 \quad &\forall s, \hat{s} \in \mathbb{R}^3, \\ |\nu(x,|s|)s - \nu(x,|\hat{s}|)\hat{s}| \le L|s-\hat{s}| \quad &\forall s, \hat{s} \in \mathbb{R}^3 \end{aligned} \tag{2.14}$$

where $L = 2\nu_0 + \overline{\nu}$. A proof can be found in [You13, Lemma 2.2].

2.1.2 Weak formulation and penalization

Let us start with a weak formulation of the state inequality in (2.1). Given some $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, it reads

$$\begin{cases} \text{Find } (\boldsymbol{A}, \phi) \in \boldsymbol{K} \times H_0^1(\Omega) \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x + \int_\Omega \nabla\phi \cdot (\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x \\ \displaystyle\geq \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \\ \displaystyle\int_\Omega \boldsymbol{A} \cdot \nabla\psi\, \mathrm{d}x = 0 \quad \forall \psi \in H_0^1(\Omega). \end{cases} \tag{VI}$$

Here, the first order type obstacle set is given by

$$\boldsymbol{K} := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) : |\,\mathbf{curl}\, \boldsymbol{v}| \leq d \text{ a.e. in } \omega\}.$$

For the purpose of a shorter notation we introduce the operator

$$\boldsymbol{\Phi}\colon \boldsymbol{X}_N^0 \to (\boldsymbol{X}_N^0)^* \quad \text{via} \quad \langle \boldsymbol{\Phi}\boldsymbol{A}, \boldsymbol{v}\rangle := \int_\Omega \nu(\cdot, |\,\mathbf{curl}\, \boldsymbol{A}|)\, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \quad \forall \boldsymbol{A}, \boldsymbol{v} \in \boldsymbol{X}_N^0. \tag{2.15}$$

By the assumption on the nonlinearity ν (2.13) together with the Friedrichs type inequality (2.12), we immediately see that $\boldsymbol{\Phi}$ is strongly monotone, i.e. there exists a constant $C_\nu > 0$ only depending on Ω and $\underline{\nu}$ such that

$$\langle \boldsymbol{\Phi}\boldsymbol{A}_1 - \boldsymbol{\Phi}\boldsymbol{A}_2, \boldsymbol{A}_1 - \boldsymbol{A}_2\rangle \geq C_\nu \|\boldsymbol{A}_1 - \boldsymbol{A}_2\|^2_{\boldsymbol{H}(\mathbf{curl})} \quad \forall \boldsymbol{A}_1, \boldsymbol{A}_2 \in \boldsymbol{X}_N^0. \tag{2.16}$$

The main goal of the whole section is to prove the following

Theorem 2.1.3. *Let Assumption 2.1.2 and Assumption 2.0.2 be satisfied. Then for every $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, (VI) admits a unique solution $(\boldsymbol{A}, \phi)$, where $\nabla\phi$ is the $\boldsymbol{L}^2(\Omega)$-orthogonal projection of $\boldsymbol{J}$ onto $\nabla H_0^1(\Omega)$. In particular,*

$$\boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega) \implies \phi = 0. \tag{2.17}$$

Moreover, there exists a multiplier $\boldsymbol{m} \in \boldsymbol{H}(\mathbf{curl}, \omega)$ such that the solution $(\boldsymbol{A}, \phi)$ is characterized by the necessary and sufficient system

$$\begin{cases} \displaystyle\langle \boldsymbol{\Phi}\boldsymbol{A}, \boldsymbol{v}\rangle + \int_\Omega \nabla\phi \cdot \boldsymbol{v}\, \mathrm{d}x + \int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}), \\ \displaystyle\int_\Omega \boldsymbol{A} \cdot \nabla\psi\, \mathrm{d}x = 0 \quad \forall \psi \in H_0^1(\Omega), \\ \displaystyle\int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A})\, \mathrm{d}x \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \end{cases} \tag{2.18}$$

The multiplier $\boldsymbol{m}$ can be chosen in $\boldsymbol{H}(\mathbf{curl}, \omega) \cap H_0(\mathrm{div}{=}0, \omega) \cap \mathcal{H}(m, \omega)^\perp$ and there it is unique.

As previously announced, the proof of Theorem 2.1.3 is based on the resolution of (VI) in the simpler case of $\boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega)$, and this is done by means of a penalization-relaxation of the variational inequality. The existence of the Lagrange multipliers is then rather easy to get from the limiting analysis of the approximating equation.

In order to set up our approximation, we shall need to introduce a C^1-approximation of the max function: for $\gamma > 0$, we define the function $\max^\gamma(\cdot, 0)\colon \mathbb{R} \to \mathbb{R}$ via

$$x \mapsto \begin{cases} x - \gamma^{-1}, & x \geq 2\gamma^{-1} \\ \frac{\gamma}{4}x^2, & x \in (0, 2\gamma^{-1}) \\ 0, & x \leq 0. \end{cases} \tag{2.19}$$

Building upon the above real function, we devise a corresponding vector version which is tailored for our needs, in the sense that the resulting map can be exploited in a variational formulation as it turns out to be Lipschitz and monotone. Let $\gamma > 0$, we define

$$\boldsymbol{\theta}_\gamma \colon \omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, s) \mapsto \begin{cases} \max^\gamma(|s| - d(x), 0)\frac{s}{|s|}, & s \neq 0 \\ 0, & s = 0. \end{cases} \tag{2.20}$$

The following lemma illustrates some key properties of such vector version of (2.19).

Lemma 2.1.4. *The mapping $\boldsymbol{\theta}_\gamma$ is continuously differentiable with respect to the second variable, with derivative $\mathrm{D}_s\boldsymbol{\theta}_\gamma \colon \omega \times \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ given by*

$$(x, s) \mapsto \begin{cases} \frac{s \otimes s}{|s|^2} + \frac{|s| - d(x) - \gamma^{-1}}{|s|}\left(\mathrm{Id} - \frac{s \otimes s}{|s|^2}\right), & |s| \geq d(x) + 2\gamma^{-1} \\ \frac{\gamma}{2}(|s| - d(x))\frac{s \otimes s}{|s|^2} + \frac{\gamma(|s| - d(x))^2}{4|s|}\left(\mathrm{Id} - \frac{s \otimes s}{|s|^2}\right), & |s| \in (d(x), d(x) + 2\gamma^{-1}) \\ 0, & |s| \leq d(x). \end{cases} \tag{2.21}$$

Moreover, $\mathrm{D}_s\boldsymbol{\theta}_\gamma(x, s)$ is symmetric, positive semi-definite for all $s \in \mathbb{R}^3$ and almost every $x \in \omega$ and uniformly bounded in $\omega \times \mathbb{R}^3$. Finally, $\boldsymbol{\theta}_\gamma(x, \cdot)$ is monotone, Lipschitz-continuous for almost every $x \in \omega$ and it holds

$$|\boldsymbol{\theta}_\gamma(x, s) - \boldsymbol{\theta}_\infty(x, s)| \leq \frac{3}{\gamma} \quad \text{a.e. } x \in \omega, \ \forall s \in \mathbb{R}^3, \tag{2.22}$$

where $\boldsymbol{\theta}_\infty \colon \omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined via

$$\boldsymbol{\theta}_\infty(x, s) = \begin{cases} \max(|s| - d(x), 0)\frac{s}{|s|}, & s \neq 0 \\ 0, & s = 0 \end{cases} \tag{2.23}$$

for almost every $x \in \omega$.

Proof. It is apparent that $\boldsymbol{\theta}_\gamma$ is continuously differentiable at $s \neq 0$ since in such case it is the product and composition of C^1 mappings, and the same is true if $s = 0$ and $d(x) > 0$. If s and $d(x)$ are both zero, it suffices to check that $s \mapsto s|s|$ is continuously differentiable at the origin, which is easily verified. A direct computation shows that the Jacobian is given by (2.21). Note that for each $s \in \mathbb{R}^3$, the matrices

$$\frac{s \otimes s}{|s|^2} \quad \text{and} \quad \mathrm{Id} - \frac{s \otimes s}{|s|^2}$$

are (symmetric) projection matrices with spectrum $\{0, 1\}$ so that $\mathrm{D}_s\boldsymbol{\theta}_\gamma(x, s)$ is positive semi-definite for each $s \in \mathbb{R}^3$ and almost every $x \in \omega$, hence we can apply [RW98, Theorem 12.3] to conclude that $\boldsymbol{\theta}_\gamma$ is monotone.

For the uniform boundedness of $\mathrm{D}_s\boldsymbol{\theta}_\gamma$, we observe that

$$\frac{|s \otimes s|}{|s|^2} = 1 \quad \text{and} \quad \frac{|s| - d(x) - \gamma^{-1}}{|s|} \leq 1,$$

hence we find $|\mathrm{D}_s\boldsymbol{\theta}_\gamma(x, s)| \leq C$ for almost every $x \in \omega$ and all $s \in \mathbb{R}^3$. Combining [RW98, Theorem 9.2] and [RW98, Theorem 9.7], this also implies the Lipschitz continuity of $\boldsymbol{\theta}_\gamma$. To finish the proof we calculate

$$\begin{aligned} &|\boldsymbol{\theta}_\gamma(x, s) - \boldsymbol{\theta}_\infty(x, s)| \\ &\quad \leq \begin{cases} \Big||s| - d(x) - \gamma^{-1} - (|s| - d(x))\Big| = \gamma^{-1}, & \text{if } |s| \geq d(x) + 2\gamma^{-1} \\ \Big|\frac{\gamma}{4}(|s| - d(x))^2 - (|s| - d(x))\Big|, & \text{if } |s| \in (d(x), d(x) + 2\gamma^{-1}) \end{cases} \end{aligned}$$

which, along with bounding $|s| - d(x)$ by $2\gamma^{-1}$ from above, yields the desired estimate (2.22). ☐

We can now state the regularized version of (VI) in the presence of divergence-free source term. To this end, let us first introduce the space $\boldsymbol{V}(\omega)$ for the relaxation variable; it satisfies

$$\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega) \subseteq \boldsymbol{V}(\omega) \subseteq \boldsymbol{L}^2(\omega) \tag{2.24}$$

and it is supposed to be a closed subspace with respect to the $\boldsymbol{L}^2(\omega)$-topology.

Remark 2.1.5. *As it is not completely obvious, let us show that* $\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega)$ *is a closed subspace of* $\boldsymbol{L}^2(\omega)$*. Let* $\{\boldsymbol{a}_k\}_{k \geq 1} \subset \boldsymbol{H}(\mathbf{curl}, \omega)$ *be a sequence satisfying*

$$\mathbf{curl}\, \boldsymbol{a}_k \to \boldsymbol{b} \in \boldsymbol{L}^2(\Omega) \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } k \to \infty, \tag{2.25}$$

we wish to prove that $\boldsymbol{b} \in \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega)$*. Let us define* $\boldsymbol{b}_k = \mathbf{curl}\, \boldsymbol{a}_k$ *for every* $k \geq 1$*; as* $\operatorname{div} \boldsymbol{b}_k = 0$ *for all* k*, we readily see that* $\operatorname{div} \boldsymbol{b} = 0$ *in* ω *because the distributional divergence is preserved under weak limits. If* $\boldsymbol{h} \in \mathcal{H}(e, \omega)$*, we have for* $k \geq 1$

$$\int_\omega \boldsymbol{b}_k \cdot \boldsymbol{h} \,\mathrm{d}x = \int_\omega \mathbf{curl}\, \boldsymbol{a}_k \cdot \boldsymbol{h} \,\mathrm{d}x = \int_\omega \boldsymbol{a}_k \cdot \underbrace{\mathbf{curl}\, \boldsymbol{h}}_{=\mathbf{0}} \,\mathrm{d}x + \int_{\partial\omega} \underbrace{(\boldsymbol{h} \times \boldsymbol{n})}_{=\mathbf{0}} \cdot \boldsymbol{a}_k \,dS = 0,$$

which means $\boldsymbol{b}_k \perp^{\boldsymbol{L}^2(\omega)} \mathcal{H}(e,\omega)$ *for all* $k \geq 1$*. Since* $\boldsymbol{b}_k$ *converges strongly in* $\boldsymbol{L}^2(\omega)$ *to* $\boldsymbol{b}$*, the latter condition is preserved in the limit leading to*

$$\begin{cases} \operatorname{div} \boldsymbol{b} = 0 \\ \boldsymbol{b} \perp^{\boldsymbol{L}^2(\omega)} \mathcal{H}(e,\omega) \end{cases} \underbrace{\Longrightarrow}_{(2.8)} \boldsymbol{b} = \mathbf{curl}\, \boldsymbol{w}, \quad \boldsymbol{w} \in \boldsymbol{H}(\mathbf{curl},\omega).$$

The implication can be motivated as follows: by the second decomposition in (2.8)*, we write* $\boldsymbol{b} = \mathbf{curl}\, \boldsymbol{w} + \nabla\eta$ *for some* $\boldsymbol{w} \in \boldsymbol{X}_T^0(\omega)$ *and* $\eta \in \nabla H_0^1(\omega)$ *(note that no harmonic fields appear in view of the orthogonality condition); then we use the information on the divergence to deduce* $0 = \operatorname{div} \boldsymbol{b} = \Delta\eta$*, which in turn gives* $\eta = 0$*. This concludes the proof.*

Fix some $\boldsymbol{J} \in H(\text{div=0},\Omega)$ and for $\gamma > 0$ consider the problem

$$\begin{cases} \text{Find } (\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \text{ s.t.} \\ \langle \boldsymbol{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{v}\rangle + \gamma \int_\omega (\mathbf{curl}\, \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma) \cdot (\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{q})\, \mathrm{d}x + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot \boldsymbol{q}\, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v}\, \mathrm{d}x \\ \forall (\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega). \end{cases} \tag{VI$_\gamma$}$$

The following three lemmas provide existence and uniqueness for (VI$_\gamma$), a weak-to-strong continuity property for the corresponding solution mapping and its limiting analysis, namely we show that $\boldsymbol{A}_\gamma$ converges to the solution of the variational inequality (VI$_{\text{sol}}$).

Lemma 2.1.6. *Let Assumption* 2.1.2 *and Assumption* 2.0.2 *be satisfied. Then, for every* $\boldsymbol{J} \in H(\text{div=0},\Omega)$*, the regularized problem* (VI$_\gamma$) *admits a unique solution* $(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma)$*.*

Proof. We use the Browder-Minty theorem. To this end we define the operator

$$\boldsymbol{\mathcal{M}}_\gamma \colon \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \to \left(\boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)\right)^*$$

via

$$\left\langle \boldsymbol{\mathcal{M}}_\gamma(\boldsymbol{A}, \boldsymbol{p}), (\boldsymbol{v}, \boldsymbol{q})\right\rangle := \langle \boldsymbol{\Phi}\boldsymbol{A}, \boldsymbol{v}\rangle + \gamma \int_\omega (\mathbf{curl}\, \boldsymbol{A} - \boldsymbol{p}) \cdot (\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{q})\, \mathrm{d}x + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}) \cdot \boldsymbol{q}\, \mathrm{d}x \tag{2.26}$$

for $(\boldsymbol{A}, \boldsymbol{p}), (\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$. Note that given $a, b \in \mathbb{R}^3$ it holds by Young's Inequality that

$$\frac{1}{2}|a|^2 + |a - b|^2 \geq \frac{1}{3}|b|^2. \tag{2.27}$$

Let $(\boldsymbol{A}_1, \boldsymbol{p}_1), (\boldsymbol{A}_2, \boldsymbol{p}_2) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ be arbitrarily fixed. By construction of the operator, we then find

$$
\begin{aligned}
&\big\langle \boldsymbol{\mathcal{M}}_\gamma(\boldsymbol{A}_1, \boldsymbol{p}_1) - \boldsymbol{\mathcal{M}}_\gamma(\boldsymbol{A}_2, \boldsymbol{p}_2), (\boldsymbol{A}_1, \boldsymbol{p}_1) - (\boldsymbol{A}_2, \boldsymbol{p}_2) \big\rangle \\
&= \langle \boldsymbol{\Phi} \boldsymbol{A}_1 - \boldsymbol{\Phi} \boldsymbol{A}_2, \boldsymbol{A}_1 - \boldsymbol{A}_2 \rangle \\
&+ \gamma \| \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) - (\boldsymbol{p}_1 - \boldsymbol{p}_2) \|^2_{L^2(\omega)} + \gamma \int_\omega \underbrace{(\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_1) - \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_2)) \cdot (\boldsymbol{p}_1 - \boldsymbol{p}_2)}_{\geq 0} \, dx \\
&\underbrace{\geq}_{(2.16)} C_\nu \| \boldsymbol{A}_1 - \boldsymbol{A}_2 \|^2_{\boldsymbol{H}(\mathbf{curl})} + \gamma \| \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) - (\boldsymbol{p}_1 - \boldsymbol{p}_2) \|^2_{L^2(\omega)} \\
&\geq \min\{C_\nu, \gamma\} \left(\frac{1}{2} \| \boldsymbol{A}_1 - \boldsymbol{A}_2 \|^2_{\boldsymbol{H}(\mathbf{curl})} + \frac{1}{2} \| \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) \|^2_{\boldsymbol{L}^2(\omega)} \right) \\
&+ \min\{C_\nu, \gamma\} \left(\| \operatorname{\mathbf{curl}}(\boldsymbol{A}_1 - \boldsymbol{A}_2) - (\boldsymbol{p}_1 - \boldsymbol{p}_2) \|^2_{\boldsymbol{L}^2(\omega)} \right) \\
&\underbrace{\geq}_{(2.27)} \min\{C_\nu, \gamma\} \left[\frac{1}{2} \| \boldsymbol{A}_1 - \boldsymbol{A}_2 \|^2_{\boldsymbol{H}(\mathbf{curl})} + \frac{1}{3} \| \boldsymbol{p}_1 - \boldsymbol{p}_2 \|^2_{\boldsymbol{L}^2(\omega)} \right].
\end{aligned}
\tag{2.28}
$$

This shows $\boldsymbol{\mathcal{M}}_\gamma$ being strongly monotone. Hemicontinuity of $\boldsymbol{\mathcal{M}}_\gamma$ follows from (2.13) for the first term on the left-hand side of (2.26), from the bilinearity for the second term and from the Lipschitz-continuity of $\boldsymbol{\theta}_\gamma$ for the third term. This completes the proof with the usage of the Browder-Minty theorem [Lio69, Thm. V.15]. □

By means of Lemma 2.1.6, the canonical control-to-state mapping

$$
\boldsymbol{G}_\gamma \colon H(\mathrm{div}{=}0, \Omega) \to \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega), \quad \boldsymbol{J} \mapsto (\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \tag{2.29}
$$

is well-defined. In what follows, for $i \in \{1, 2\}$, we denote by $\pi_i \colon \mathbb{R}^2 \to \mathbb{R}$ the projection onto the i-th coordinate.

Lemma 2.1.7. *Let Assumption 2.1.2 and Assumption 2.0.2 be satisfied and let $\gamma > 0$ be fixed. Then, the control-to-state mapping $\boldsymbol{G}_\gamma$ is weak-strong continuous. In particular, the mapping $\pi_1 \circ \boldsymbol{G}_\gamma \colon H(\mathrm{div}{=}0, \Omega) \to \boldsymbol{X}_N^0$ is weak-strong continuous.*

Proof. Let $\{\boldsymbol{J}_\gamma^n\}_{n=1}^\infty \subset H(\mathrm{div}{=}0, \Omega)$ be a sequence satisfying

$$
\boldsymbol{J}_\gamma^n \rightharpoonup \boldsymbol{J}_\gamma \quad \text{weakly in } H(\mathrm{div}{=}0, \Omega) \quad \text{as } n \to \infty. \tag{2.30}
$$

For each $n \in \mathbb{N}$, let $\boldsymbol{A}_\gamma^n = (\pi_1 \circ \boldsymbol{G}_\gamma)(\boldsymbol{J}_\gamma^n) \in \boldsymbol{X}_N^0$ and $\boldsymbol{p}_\gamma^n = (\pi_2 \circ \boldsymbol{G}_\gamma)(\boldsymbol{J}_\gamma^n) \in \boldsymbol{V}(\omega)$; in view of an estimate like (2.28) and (VI_γ), it is readily seen that the sequence $\{(\boldsymbol{A}_\gamma^n, \boldsymbol{p}_\gamma^n)\}_{n=1}^\infty$ is bounded in $\boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ so that, up to extracting subsequences, we find $\boldsymbol{A}_\gamma \in \boldsymbol{X}_N^0$ and $\boldsymbol{p}_\gamma \in \boldsymbol{V}(\omega)$ satisfying:

$$
\begin{aligned}
(\boldsymbol{A}_\gamma^n, \boldsymbol{p}_\gamma^n) &\rightharpoonup (\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \qquad && \text{weakly in } \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \quad \text{as } n \to \infty, \\
\boldsymbol{A}_\gamma^n &\to \boldsymbol{A}_\gamma && \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } n \to \infty.
\end{aligned}
\tag{2.31}
$$

The second assertion is a consequence of the first one by means of the compact embedding (2.6). The strong monotonicity of the operator $\mathbf{\Phi}$, together with (VI_γ) tested with $(\boldsymbol{v},\boldsymbol{q}) := (\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma, \boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma) \in \boldsymbol{X}^0_N \times \boldsymbol{V}(\omega)$ leads to

$$\begin{aligned}
&C_\nu \|\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} \leq \langle \mathbf{\Phi}\boldsymbol{A}^n_\gamma - \mathbf{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\rangle \\
&= \langle \mathbf{\Phi}\boldsymbol{A}^n_\gamma, \boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\rangle - \langle \mathbf{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\rangle \\
&= -\gamma \int_\omega (\mathbf{curl}\,\boldsymbol{A}^n_\gamma - \boldsymbol{p}^n_\gamma)\cdot(\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma))\,\mathrm{d}x \\
&- \gamma\int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}^n_\gamma)\cdot(\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x + \int_\Omega \boldsymbol{J}^n_\gamma\cdot(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma)\,\mathrm{d}x - \langle \mathbf{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\rangle.
\end{aligned} \tag{2.32}$$

Next we shall focus on the first two terms on the right hand side of (2.32). The first one reads

$$\begin{aligned}
&-\gamma\int_\omega (\mathbf{curl}\,\boldsymbol{A}^n_\gamma - \boldsymbol{p}^n_\gamma)\cdot(\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma))\,\mathrm{d}x \\
&= -\gamma\int_\omega (\mathbf{curl}\,\boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma)\cdot(\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma))\,\mathrm{d}x \\
&- \gamma\|\,\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\|^2_{\boldsymbol{L}^2(\omega)},
\end{aligned} \tag{2.33}$$

while at the same time we have

$$\begin{aligned}
&-\frac{C_\nu}{2}\|\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} - \gamma\|\,\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\|^2_{\boldsymbol{L}^2(\omega)} \\
&\leq -\min\{C_\nu,\gamma\}\left[\frac{1}{2}\|\,\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma)\|^2_{\boldsymbol{L}^2(\omega)} + \|\,\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\|^2_{\boldsymbol{L}^2(\omega)}\right] \\
&\underbrace{\leq}_{(2.27)} -\min\{C_\nu,\gamma\}\frac{1}{3}\|\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma\|^2_{\boldsymbol{L}^2(\omega)}.
\end{aligned} \tag{2.34}$$

The second term reads

$$\begin{aligned}
&-\gamma\int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}^n_\gamma)\cdot(\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x \\
&= -\gamma\int_\omega (\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}^n_\gamma) - \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma))\cdot(\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x - \gamma\int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma)\cdot(\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x \\
&\leq -\gamma\int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma)\cdot(\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x
\end{aligned} \tag{2.35}$$

thanks to the monotonicity of $\boldsymbol{\theta}_\gamma$ (see Lemma 2.1.4). Summing up, (2.32), (2.33), (2.34) and (2.35) all together yield the estimate

$$\begin{aligned}
&\frac{C_\nu}{2}\|\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\|^2_{\boldsymbol{H}(\mathbf{curl})} + \min\{C_\nu,\gamma\}\frac{1}{3}\|\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma\|^2_{\boldsymbol{L}^2(\omega)} \\
&\leq -\gamma\int_\omega (\mathbf{curl}\,\boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma)\cdot(\mathbf{curl}(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma) - (\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma))\,\mathrm{d}x \\
&+ \int_\Omega \boldsymbol{J}^n_\gamma\cdot(\boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma)\,\mathrm{d}x - \langle \mathbf{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{A}^n_\gamma - \boldsymbol{A}_\gamma\rangle - \gamma\int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma)\cdot(\boldsymbol{p}^n_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x.
\end{aligned} \tag{2.36}$$

Now in view of (2.31) we can pass to the limit as $n\to\infty$ in (2.36) to get

$$(\boldsymbol{A}^n_\gamma, \boldsymbol{p}^n_\gamma) \to (\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \quad \text{strongly in } \boldsymbol{X}^0_N \times \boldsymbol{V}(\omega) \quad \text{as } n\to\infty. \tag{2.37}$$

We are left to prove that $(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) = \boldsymbol{G}_\gamma(\boldsymbol{J}_\gamma)$. Owing to the strong convergence (2.37), up to extracting subsequences, we deduce that

$$|\operatorname{\mathbf{curl}} \boldsymbol{A}^n_\gamma| \to |\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma| \quad \text{a.e. in } \Omega \quad \text{as } n \to \infty, \quad \boldsymbol{p}^n_\gamma \to \boldsymbol{p}_\gamma \quad \text{a.e. in } \omega \quad \text{as } n \to \infty, \tag{2.38}$$

which in turn, by the properties of ν and $\boldsymbol{\theta}_\gamma$, implies

$$\nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}^n_\gamma|) \to \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma|) \quad \text{a.e. in } \Omega \quad \text{as } n \to \infty \tag{2.39}$$

and

$$\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^n_\gamma) \to \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \quad \text{a.e. in } \omega \quad \text{as } n \to \infty. \tag{2.40}$$

As a consequence, in light of the Lipschitz continuity of $\boldsymbol{\theta}_\gamma$ and Assumption 2.1.2, Lebesgue's dominated convergence theorem gives that the term

$$\langle \boldsymbol{\Phi} \boldsymbol{A}^n_\gamma, \boldsymbol{v} \rangle + \gamma \int_\omega (\operatorname{\mathbf{curl}} \boldsymbol{A}^n_\gamma - \boldsymbol{p}^n_\gamma) \cdot (\operatorname{\mathbf{curl}} \boldsymbol{v} - \boldsymbol{q}) \,\mathrm{d}x + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^n_\gamma) \cdot \boldsymbol{q} \,\mathrm{d}x \tag{2.41}$$

converges to

$$\langle \boldsymbol{\Phi} \boldsymbol{A}_\gamma, \boldsymbol{v} \rangle + \gamma \int_\omega (\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma) \cdot (\operatorname{\mathbf{curl}} \boldsymbol{v} - \boldsymbol{q}) \,\mathrm{d}x + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot \boldsymbol{q} \,\mathrm{d}x$$

as $n \to \infty$ for all $(\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}^0_N \times \boldsymbol{V}(\omega)$. This concludes the proof since (2.41) is also equal to $\int_\Omega \boldsymbol{J}^n_\gamma \cdot \boldsymbol{v}$, which converges to $\int_\Omega \boldsymbol{J}_\gamma \cdot \boldsymbol{v}$ by assumption. □

Lemma 2.1.8. *Suppose that Assumption 2.1.2 holds together with Assumption 2.0.2. Let* $\{\boldsymbol{J}_\gamma\}_{\gamma>0} \subset H(\mathrm{div}{=}0, \Omega)$ *be a sequence satisfying* $\boldsymbol{J}_\gamma \rightharpoonup \boldsymbol{J}$ *weakly in* $H(\mathrm{div}{=}0, \Omega)$ *as* $\gamma \to \infty$ *and let* $(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma)$ *denote the unique solution of* (VI_γ) *featuring* $\boldsymbol{J}_\gamma$ *on the right hand side. Then there exists* $\boldsymbol{A} \in \boldsymbol{X}^0_N \cap \boldsymbol{K}$ *such that*

$$(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \to (\boldsymbol{A}, \operatorname{\mathbf{curl}} \boldsymbol{A}) \quad \textit{strongly in } \boldsymbol{X}^0_N \times \boldsymbol{V}(\omega) \quad \textit{as } \gamma \to \infty. \tag{2.42}$$

Moreover, $(\boldsymbol{A}, \boldsymbol{J}) \in \boldsymbol{X}^0_N \cap \boldsymbol{K} \times H(\mathrm{div}{=}0, \Omega)$ *satisfies the variational inequality*

$$\int_\Omega \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}|) \operatorname{\mathbf{curl}} \boldsymbol{A} \cdot \operatorname{\mathbf{curl}}(\boldsymbol{v} - \boldsymbol{A}) \,\mathrm{d}x \geq \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A}) \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0_N \cap \boldsymbol{K}. \tag{$\mathrm{VI_{sol}}$}$$

Remark 2.1.9. *As to avoid confusion, we underline that working with a weakly convergent sequence* $\{\boldsymbol{J}_\gamma\}_{\gamma>0}$ *on the right hand side of* (VI_γ) *is merely needed for the optimal control analysis carried out in Section 2.2. For what concerns existence theory for* (VI), *one may take it identically equal to* $\boldsymbol{J}$.

Proof. Let $\gamma > 0$. Testing (VI_γ) with $(\boldsymbol{v}, \boldsymbol{q}) = (\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma)$ leads to:

$$\langle \boldsymbol{\Phi} \boldsymbol{A}_\gamma, \boldsymbol{A}_\gamma \rangle + \gamma \| \operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma \|^2_{L^2(\omega)} + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot \boldsymbol{p}_\gamma \,\mathrm{d}x = \int_\Omega \boldsymbol{J}_\gamma \cdot \boldsymbol{A}_\gamma \,\mathrm{d}x. \tag{2.43}$$

Arguing as in (2.28), it is straightforward to see that (2.43) implies that the sequence $\{(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma)\}_{\gamma>0} \subset \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ is bounded and consequently there exists a subsequence, still denoted in the same way, and $(\boldsymbol{A}, \boldsymbol{p}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ such that

$$(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \rightharpoonup (\boldsymbol{A}, \boldsymbol{p}) \quad \text{weakly in } \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \quad \text{as } \gamma \to \infty. \tag{2.44}$$

By the compactness of the embedding (2.6), we also get that $\boldsymbol{A}_\gamma \to \boldsymbol{A}$ strongly in $\boldsymbol{L}^2(\Omega)$ as $\gamma \to \infty$. Next, we exploit the monotonicty of $\boldsymbol{\Phi}, \boldsymbol{\theta}_\gamma$ and divide both sides by γ in (2.43) to see that the boundedness of $\{\boldsymbol{A}_\gamma\}_{\gamma>0}, \{\boldsymbol{J}_\gamma\}_{\gamma>0}$ implies $\|\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma\|^2_{\boldsymbol{L}^2(\omega)} \to 0$ as $\gamma \to \infty$. Using the weak lower semicontinuity of $\|\cdot\|_{\boldsymbol{L}^2(\omega)}$, we thus obtain

$$0 = \liminf_{\gamma\to\infty} \|\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma\|_{\boldsymbol{L}^2(\omega)} \geq \|\operatorname{\mathbf{curl}} \boldsymbol{A} - \boldsymbol{p}\|_{\boldsymbol{L}^2(\omega)} \implies \operatorname{\mathbf{curl}} \boldsymbol{A} = \boldsymbol{p}. \tag{2.45}$$

Next we shall prove that $\boldsymbol{A} \in \boldsymbol{K}$, i.e. that $|\operatorname{\mathbf{curl}} \boldsymbol{A}| \leq d(\cdot)$ a.e. in ω. Proceeding in the same way as above, by (2.43), we also get

$$\int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot \boldsymbol{p}_\gamma \, \mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty, \tag{2.46}$$

while the choice $\boldsymbol{v} = \boldsymbol{0}$ in (VI_γ) together with (2.44) and $\lim_{\gamma\to\infty} \|\operatorname{\mathbf{curl}} \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma\|^2_{\boldsymbol{L}^2(\omega)} \to 0$ produces

$$\int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot \boldsymbol{q} \, \mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty \quad \forall \boldsymbol{q} \in \boldsymbol{V}(\omega). \tag{2.47}$$

As $\boldsymbol{\theta}_\gamma$ is monotone, for all $\boldsymbol{q} \in \boldsymbol{V}(\omega)$, it holds

$$\begin{aligned} 0 &\leq \int_\omega (\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) - \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{q})) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{q}) \, \mathrm{d}x \\ &= \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{q}) \, \mathrm{d}x + \int_\omega (\boldsymbol{\theta}_\infty(\cdot, \boldsymbol{q}) - \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{q})) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{q}) \, \mathrm{d}x \\ &\quad - \int_\omega \boldsymbol{\theta}_\infty(\cdot, \boldsymbol{q}) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{q}) \, \mathrm{d}x. \end{aligned} \tag{2.48}$$

Thanks to (2.46), (2.47) and the strong $\boldsymbol{L}^2(\omega)$ convergence of $\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{q})$ to $\boldsymbol{\theta}_\infty(\cdot, \boldsymbol{q})$ (see the uniform bound (2.22)), we can pass to the limit as $\gamma \to \infty$ in the above inequality to deduce

$$\int_\omega \boldsymbol{\theta}_\infty(\cdot, \boldsymbol{q}) \cdot (\boldsymbol{p} - \boldsymbol{q}) \, \mathrm{d}x \leq 0 \quad \forall \boldsymbol{q} \in \boldsymbol{V}(\omega).$$

Now we take $s \in (0,1)$, $\widetilde{\boldsymbol{q}} \in \boldsymbol{V}(\omega)$ arbitrary and set $\boldsymbol{q} = \boldsymbol{p} + s\widetilde{\boldsymbol{q}} \in \boldsymbol{V}(\omega)$ so that the last inequality entails

$$\int_\omega \boldsymbol{\theta}_\infty(\cdot, \boldsymbol{p} + s\widetilde{\boldsymbol{q}}) \cdot \widetilde{\boldsymbol{q}} \, \mathrm{d}x \leq 0 \quad \forall \widetilde{\boldsymbol{q}} \in \boldsymbol{V}(\omega). \tag{2.49}$$

By continuity, it follows that $\boldsymbol{\theta}_\infty(\cdot, \boldsymbol{p} + s\widetilde{\boldsymbol{q}}) \to \boldsymbol{\theta}_\infty(\cdot, \boldsymbol{p})$ almost everywhere in ω as $s \to 0$, moreover we have

$$\begin{aligned} |\boldsymbol{\theta}_\infty(x, \boldsymbol{p} + s\widetilde{\boldsymbol{q}})| &= |\max(|\boldsymbol{p} + s\widetilde{\boldsymbol{q}}| - d(x), 0)| \\ &\leq ||\boldsymbol{p} + s\widetilde{\boldsymbol{q}}| - d(x)| \leq |\boldsymbol{p}| + |\widetilde{\boldsymbol{q}}| + d(x) \quad \text{for a.e. } x \in \omega, \forall s \in (0,1) \end{aligned}$$

and therefore we can apply Lebesgue's dominated convergence theorem to pass to the limit in (2.49) as $s \to 0$. This implies

$$\int_\omega \boldsymbol{\theta}_\infty(\cdot,\boldsymbol{p}) \cdot \widetilde{\boldsymbol{q}} \,\mathrm{d}x \le 0 \quad \forall \widetilde{\boldsymbol{q}} \in \boldsymbol{V}(\omega) \implies \int_\omega \boldsymbol{\theta}_\infty(\cdot,\boldsymbol{p}) \cdot \widetilde{\boldsymbol{q}} \,\mathrm{d}x = 0 \quad \forall \widetilde{\boldsymbol{q}} \in \boldsymbol{V}(\omega),$$

and setting $\widetilde{\boldsymbol{q}} = \boldsymbol{p}$ in the last equation finally produces

$$0 = \int_\omega \boldsymbol{\theta}_\infty(\cdot,\boldsymbol{p}) \cdot \boldsymbol{p} \,\mathrm{d}x = \int_\omega \underbrace{\max(|\boldsymbol{p}| - d(x), 0)|\boldsymbol{p}|}_{\ge 0} \,\mathrm{d}x \implies \max(|\boldsymbol{p}| - d(x), 0)|\boldsymbol{p}| = 0 \tag{2.50}$$

for a.e. $x \in \omega$. It is now straightforward to see that (2.50) forces $|\boldsymbol{p}| \le d(\cdot)$ almost everywhere in ω and hence to conclude $\boldsymbol{A} \in \boldsymbol{K}$ taking (2.45) into account.

At this point we are able to show that the weak convergence (2.44) is actually strong. To this end, first we test (VI$_\gamma$) with $(\boldsymbol{v},\boldsymbol{q}) = (\boldsymbol{A}_\gamma - \boldsymbol{A}, \boldsymbol{p}_\gamma - \boldsymbol{p}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$:

$$\begin{aligned} \langle \boldsymbol{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{A}_\gamma - \boldsymbol{A}\rangle + \gamma \int_\omega (\mathbf{curl}\,\boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma) \cdot (\mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A}) - (\boldsymbol{p}_\gamma - \boldsymbol{p}))\,\mathrm{d}x \\ + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{p})\,\mathrm{d}x = \int_\Omega \boldsymbol{J}_\gamma \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\,\mathrm{d}x. \end{aligned} \tag{2.51}$$

At the same time, we have

$$\begin{aligned} &\int_\omega (\mathbf{curl}\,\boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma) \cdot (\mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A}) - (\boldsymbol{p}_\gamma - \boldsymbol{p}))\,\mathrm{d}x \\ &= \int_\omega (\mathbf{curl}\,\boldsymbol{A}_\gamma \underbrace{- \mathbf{curl}\,\boldsymbol{A} + \boldsymbol{p}}_{=\mathbf{0}} - \boldsymbol{p}_\gamma) \cdot (\mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A}) - (\boldsymbol{p}_\gamma - \boldsymbol{p}))\,\mathrm{d}x \\ &= \|\,\mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A}) - (\boldsymbol{p}_\gamma - \boldsymbol{p})\|^2_{\boldsymbol{L}^2(\omega)}, \end{aligned} \tag{2.52}$$

as well as

$$\begin{aligned} \langle \boldsymbol{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{A}_\gamma - \boldsymbol{A}\rangle &\ge C_\nu \|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{X}_N^0} + \langle \boldsymbol{\Phi}\boldsymbol{A}, \boldsymbol{A}_\gamma - \boldsymbol{A}\rangle, \\ \int_\omega \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{p})\,\mathrm{d}x &= \int_\omega (\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma) - \boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p})) \cdot (\boldsymbol{p}_\gamma - \boldsymbol{p})\,\mathrm{d}x \ge 0, \end{aligned} \tag{2.53}$$

where for the latter equality we used $\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}) = 0$ since $|\boldsymbol{p}(x)| \le d(x)$ for a.e. $x \in \omega$. Exploiting (2.52) and (2.53) in (2.51) yields

$$\begin{aligned} C_\nu \|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{X}_N^0} + \gamma \|\,\mathbf{curl}(\boldsymbol{A}_\gamma - \boldsymbol{A}) - (\boldsymbol{p}_\gamma - \boldsymbol{p})\|^2_{\boldsymbol{L}^2(\omega)} \\ \le \int_\Omega \boldsymbol{J}_\gamma \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\,\mathrm{d}x - \langle \boldsymbol{\Phi}\boldsymbol{A}, \boldsymbol{A}_\gamma - \boldsymbol{A}\rangle. \end{aligned}$$

Applying (2.27) in a similar fashion as in (2.28) to bound the left hand side from below, we get the estimate

$$\begin{aligned} \min\{C_\nu, \gamma\} \left[\frac{1}{2}\|\boldsymbol{A}_\gamma - \boldsymbol{A}\|^2_{\boldsymbol{H}(\mathbf{curl})} + \frac{1}{3}\|\boldsymbol{p}_\gamma - \boldsymbol{p}\|^2_{\boldsymbol{L}^2(\omega)}\right] \\ \le \int_\Omega \boldsymbol{J}_\gamma \cdot (\boldsymbol{A}_\gamma - \boldsymbol{A})\,\mathrm{d}x - \langle \boldsymbol{\Phi}\boldsymbol{A}, \boldsymbol{A}_\gamma - \boldsymbol{A}\rangle, \end{aligned}$$

from which it follows by (2.44) that

$$(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \to (\boldsymbol{A}, \boldsymbol{p}) = (\boldsymbol{A}, \mathbf{curl}\,\boldsymbol{A}) \quad \text{strongly in } \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \quad \text{as } \gamma \to \infty. \tag{2.54}$$

We are left to show that $\boldsymbol{A}$ satisfies ($\mathrm{VI}_{\mathrm{sol}}$). Let $\boldsymbol{v} \in \boldsymbol{X}_N^0 \cap \boldsymbol{K}$, from ($\mathrm{VI}_\gamma$) tested with the couple $(\boldsymbol{v} - \boldsymbol{A}_\gamma, \mathbf{curl}\,\boldsymbol{v} - \boldsymbol{p}_\gamma)$ we get

$$\begin{aligned}
&\int_\Omega \boldsymbol{J}_\gamma \cdot (\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x \\
&= \langle \boldsymbol{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{v} - \boldsymbol{A}_\gamma\rangle - \gamma \underbrace{\int_\omega (\mathbf{curl}\,\boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma) \cdot (\mathbf{curl}\,\boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma)\,\mathrm{d}x}_{\geq 0} \\
&+ \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \cdot (\mathbf{curl}\,\boldsymbol{v} - \boldsymbol{p}_\gamma)\,\mathrm{d}x \\
&\leq \langle \boldsymbol{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{v} - \boldsymbol{A}_\gamma\rangle + \gamma \int_\omega \big(\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) - \underbrace{\boldsymbol{\theta}_\gamma(\cdot, \mathbf{curl}\,\boldsymbol{v})}_{=\mathbf{0}}\big) \cdot (\mathbf{curl}\,\boldsymbol{v} - \boldsymbol{p}_\gamma)\,\mathrm{d}x \\
&\leq \langle \boldsymbol{\Phi}\boldsymbol{A}_\gamma, \boldsymbol{v} - \boldsymbol{A}_\gamma\rangle = \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}_\gamma|)\,\mathbf{curl}\,\boldsymbol{A}_\gamma \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_\gamma)\,\mathrm{d}x,
\end{aligned} \tag{2.55}$$

where we exploited the fact that $\boldsymbol{v} \in \boldsymbol{K}$ and $\boldsymbol{\theta}_\gamma$ being monotone. The passage to the limit as $\gamma \to \infty$ in (2.55) can be justified again by the Lebesgue dominated convergence theorem, taking into account Assumption 2.1.2 as well as the strong convergence of $\boldsymbol{A}_\gamma$ to $\boldsymbol{A}$ we have just proved, (2.54). This concludes the proof. □

2.1.3 Proof of Theorem 2.1.3

We now have all the tools to prove Theorem 2.1.3. In what follows we use the brackets $\langle\cdot,\cdot\rangle_\omega$ to indicate the duality pairing between the spaces $\boldsymbol{V}(\omega)$ and $\boldsymbol{V}(\omega)^*$, in order to distinguish it with the pairing between $\boldsymbol{X}_N^0$ and its dual, which we have been denoting by $\langle\cdot,\cdot\rangle$. The proof is divided into two steps.

Step 1: Divergence free source term. We start with a right-hand side $\widetilde{\boldsymbol{J}} \in H(\mathrm{div}{=}0, \Omega)$ and consider the variational inequality

$$\begin{cases} \text{Find } \widetilde{\boldsymbol{A}} \in \boldsymbol{X}_N^0 \cap \boldsymbol{K} \text{ s.t.} \\ \displaystyle\int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\widetilde{\boldsymbol{A}}|)\,\mathbf{curl}\,\widetilde{\boldsymbol{A}} \cdot \mathbf{curl}(\boldsymbol{v} - \widetilde{\boldsymbol{A}})\,\mathrm{d}x \geq \int_\Omega \widetilde{\boldsymbol{J}} \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{A}})\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0 \cap \boldsymbol{K}. \end{cases} \tag{2.56}$$

In view of Lemma 2.1.8, we can provide a solution to the above problem as a result of the limit of (VI_γ) (see ($\mathrm{VI}_{\mathrm{sol}}$) in particular), with the choice $\boldsymbol{V}(\omega) = \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \omega)$. Uniqueness is guaranteed by a standard energy argument.

Next we prove that $\widetilde{\boldsymbol{A}}$ can be characterized by means of a suitable system of type (2.18). To do so we exploit (VI_γ) once more. Since $\{\boldsymbol{A}_\gamma\}_{\gamma>0} \subset \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is uniformly bounded (see the proof of Lemma 2.1.8), it follows from (VI_γ) with $\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{v}$ that the sequence $\{\gamma\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma)\}_{\gamma>0}$ is bounded in $[\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \omega)]^*$ and therefore we find $\boldsymbol{\Psi} \in [\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \omega)]^*$ such that (up to subsequences)

$$\gamma\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma) \rightharpoonup \boldsymbol{\Psi} \quad \text{weakly in } [\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \omega)]^* \quad \text{as } \gamma \to \infty. \tag{2.57}$$

At the same time, in force of Riesz's representation theorem, there exists $\boldsymbol{m} \in \boldsymbol{H}(\mathbf{curl}, \omega)$ for which

$$\langle \boldsymbol{\Psi}, \mathbf{curl}\, \boldsymbol{v} \rangle_\omega = \int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \omega). \tag{2.58}$$

Combining (2.42), (2.57), (2.58) and $(\mathbf{curl}\, \boldsymbol{A}_\gamma - \boldsymbol{p}_\gamma) \to 0$ in $\boldsymbol{L}^2(\omega)$ we can pass to the limit as $\gamma \to \infty$ in (VI$_\gamma$) to get

$$\int_\Omega \nu(\cdot, |\, \mathbf{curl}\, \widetilde{\boldsymbol{A}}|)\, \mathbf{curl}\, \widetilde{\boldsymbol{A}} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x + \int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \widetilde{\boldsymbol{J}} \cdot \boldsymbol{v} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}).$$

Note that (VI$_\gamma$) features test functions $\boldsymbol{v} \in \boldsymbol{X}_N^0$, but an application of the Helmholtz decomposition (2.9) allows us to equivalently take $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$ since $\widetilde{\boldsymbol{J}}$ is divergence-free and only $\mathbf{curl}\, \boldsymbol{v}$ appears in the other terms.

Now we consider $\boldsymbol{v} \in \boldsymbol{K} \subset \boldsymbol{H}_0(\mathbf{curl})$ and test the last equation with $\boldsymbol{v} - \widetilde{\boldsymbol{A}}$, this yields

$$\begin{aligned} &\int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}(\boldsymbol{v} - \widetilde{\boldsymbol{A}}) \, \mathrm{d}x \\ &\quad = \int_\Omega \widetilde{\boldsymbol{J}} \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{A}}) \, \mathrm{d}x - \int_\Omega \nu(\cdot, |\, \mathbf{curl}\, \widetilde{\boldsymbol{A}}|)\, \mathbf{curl}\, \widetilde{\boldsymbol{A}} \cdot \mathbf{curl}(\boldsymbol{v} - \widetilde{\boldsymbol{A}}) \, \mathrm{d}x \underbrace{\leq}_{(2.56)} 0. \end{aligned}$$

To summarize, we proved that there is $\boldsymbol{m} \in \boldsymbol{H}(\mathbf{curl}, \omega)$ such that the solution $\widetilde{\boldsymbol{A}}$ of (2.56) satisfies

$$\begin{cases} \displaystyle\int_\Omega \nu(\cdot, |\, \mathbf{curl}\, \widetilde{\boldsymbol{A}}|)\, \mathbf{curl}\, \widetilde{\boldsymbol{A}} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x + \int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x = \int_\Omega \widetilde{\boldsymbol{J}} \cdot \boldsymbol{v} \, \mathrm{d}x \ \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \displaystyle\int_\omega \mathbf{curl}\, \boldsymbol{m} \cdot \mathbf{curl}(\boldsymbol{v} - \widetilde{\boldsymbol{A}}) \, \mathrm{d}x \leq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K} \\ \displaystyle\int_\Omega \widetilde{\boldsymbol{A}} \cdot \nabla \psi \, \mathrm{d}x = 0 \quad \forall \psi \in H_0^1(\Omega) \end{cases} \tag{2.59}$$

In remains to show that $\boldsymbol{m}$ can be chosen in $\boldsymbol{X}_T^0(\omega) \cap \mathcal{H}(m, \omega)^\perp$ in a unique way. Indeed it holds $\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega) = \mathbf{curl}(\boldsymbol{X}_T^0(\omega) \cap \mathcal{H}(m, \omega)^\perp)$ due to Helmholtz decomposition (2.9) and we would find again (2.58) but with $\boldsymbol{m} \in \boldsymbol{X}_T^0(\omega) \cap \mathcal{H}(m, \omega)^\perp$, as well as (2.59). If $\boldsymbol{m}' \in \boldsymbol{X}_T^0(\omega) \cap \mathcal{H}(m, \omega)^\perp$ is another multiplier that satisfies (2.59), we would get

$$\int_\omega \mathbf{curl}(\boldsymbol{m} - \boldsymbol{m}') \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \omega).$$

This implies $\|\, \mathbf{curl}(\boldsymbol{m} - \boldsymbol{m}')\|_{L^2(\omega)} = 0$ after testing with $\boldsymbol{v} = \boldsymbol{m} - \boldsymbol{m}'$ and finally $\boldsymbol{m} = \boldsymbol{m}'$ thanks to Poincaré-Friedrichs inequality (2.12), which also holds for vector fields in $\boldsymbol{X}_T^0(\omega) \cap \mathcal{H}(m, \omega)^\perp$ without further assumptions on ω.

Step 2: General source term. Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$. By the Helmholtz decomposition (2.7), we can uniquely decompose

$$\boldsymbol{J} = \widetilde{\boldsymbol{J}} + \nabla \phi, \tag{2.60}$$

where $\widetilde{\boldsymbol{J}} \in H(\mathrm{div}{=}0, \Omega)$ and $\phi \in H_0^1(\Omega)$. Then, following the previous step, there exists a unique $\widetilde{\boldsymbol{A}} \in \boldsymbol{X}_N^0 \cap \boldsymbol{K}$ with

$$\int_\Omega \nu(\cdot, |\, \mathbf{curl}\, \widetilde{\boldsymbol{A}}|)\, \mathbf{curl}\, \widetilde{\boldsymbol{A}} \cdot \mathbf{curl}(\boldsymbol{v} - \widetilde{\boldsymbol{A}}) \, \mathrm{d}x \geq \int_\Omega (\boldsymbol{J} - \nabla \phi) \cdot (\boldsymbol{v} - \widetilde{\boldsymbol{A}}) \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0 \cap \boldsymbol{K}.$$

Note that it is enough to take $v \in K$, as already observed. Therefore putting the gradient term on the left hand side yields

$$\int_\Omega \nu(\cdot, |\operatorname{\mathbf{curl}} \widetilde{A}|) \operatorname{\mathbf{curl}} \widetilde{A} \cdot \operatorname{\mathbf{curl}}(v - \widetilde{A}) \, dx + \int_\Omega \nabla\phi \cdot (v - \widetilde{A}) \, dx \geq \int_\Omega J \cdot (v - \widetilde{A}) \, dx \tag{2.61}$$

for all $v \in K$. We obtain that $(\widetilde{A}, \phi)$ is a solution to (VI) and characterization (2.18) readily follows from the one obtained in first step, that is (2.59). Uniqueness follows from the one of $\widetilde{A}$ and of the Helmholtz decomposition. This concludes the proof.

2.2 The optimal control problem

With the solution theory for (VI) at hand, let us now introduce the optimal control problem of focus. It consists of finding an optimal solution to

$$\begin{cases} \min\limits_{\substack{J \in L^2(\Omega) \\ A \in X_N^0}} \left(\frac{1}{2} \| \operatorname{\mathbf{curl}} A - B_\mathrm{d} \|^2_{L^2(\Omega)} + \frac{\beta}{2} \| J \|^2_{L^2(\Omega)} \right) \\ \text{subject to (VI).} \end{cases} \tag{P}$$

It turns out that the latter is equivalent to

$$\begin{cases} \min\limits_{\substack{J \in H(\mathrm{div}=0,\Omega) \\ A \in X_N^0}} \left(\frac{1}{2} \| \operatorname{\mathbf{curl}} A - B_\mathrm{d} \|^2_{L^2(\Omega)} + \frac{\beta}{2} \| J \|^2_{L^2(\Omega)} \right) \\ \text{subject to } (\mathrm{VI}_\mathrm{sol}), \end{cases} \tag{P_sol}$$

which is easier to handle thanks to the simpler structure of (VI_sol) in comparison with (VI). We will prove that the two problems are equivalent provided (P_sol) has at least a solution, hence we address this first.

In the context of the previous optimal control problem, it is useful to introduce the control-to-state mapping associated with (VI_sol),

$$G_\infty \colon H(\mathrm{div}=0, \Omega) \to X_N^0, \quad J \mapsto A. \tag{2.62}$$

Note that, due to Theorem 2.1.3 or Lemma 2.1.8, the mapping G_∞ is well-defined. The subscript notation is to remind the reader that the solution to (VI_sol) is obtained through an approximation procedure, in particular the conclusion of Lemma 2.1.8 can be restated in the language of solution mappings as follows. If $J_\gamma \rightharpoonup J$ weakly in $H(\mathrm{div}=0)$, then

$$(\pi_1 \circ G_\gamma)(J_\gamma) \to G_\infty(J) \quad \text{strongly in } X_N^0(\Omega) \quad \text{as } \gamma \to \infty. \tag{2.63}$$

In this sense, G_∞ can be thought as the limit of G_γ. The next lemma shows that G_∞ enjoys the same continuity property as G_γ.

Lemma 2.2.1. *Let Assumption 2.1.2 and Assumption 2.0.2 be satisfied. The mapping* $G_\infty \colon H(\mathrm{div}=0, \Omega) \to X_N^0$ *defined via* (2.62) *is weak-strong continuous.*

Proof. Let $\{\boldsymbol{J}_n\}_{n=1}^{\infty} \subset H(\mathrm{div}{=}0, \Omega)$ and $\boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega)$ be such that

$$\boldsymbol{J}_n \rightharpoonup \boldsymbol{J} \quad \text{weakly in } H(\mathrm{div}{=}0, \Omega) \quad \text{as } n \to \infty. \tag{2.64}$$

For each $n \in \mathbb{N}$, we denote by $\boldsymbol{A}_n = \boldsymbol{G}_\infty(\boldsymbol{J}_n) \in \boldsymbol{K}$ the solution to

$$\int_\Omega \nu(\cdot, |\mathbf{curl}\, \boldsymbol{A}_n|)\, \mathbf{curl}\, \boldsymbol{A}_n \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}_n)\, \mathrm{d}x \geq \int_\Omega \boldsymbol{J}_n \cdot (\boldsymbol{v} - \boldsymbol{A}_n)\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \tag{2.65}$$

and we denote by $\boldsymbol{A} = \boldsymbol{G}_\infty(\boldsymbol{J})$ the solution to (VI) corresponding to the control $\boldsymbol{J}$. Testing (VI) with $\boldsymbol{v} = \boldsymbol{A}_n$ and (2.65) with $\boldsymbol{v} = \boldsymbol{A}$ we get, after taking their sum, that

$$\langle \boldsymbol{\Phi} \boldsymbol{A} - \boldsymbol{\Phi} \boldsymbol{A}_n, \boldsymbol{A} - \boldsymbol{A}_n \rangle \leq \int_\Omega (\boldsymbol{J} - \boldsymbol{J}_n) \cdot (\boldsymbol{A} - \boldsymbol{A}_n)\, \mathrm{d}x. \tag{2.66}$$

In view of the strong monotonocity of $\boldsymbol{\Phi}$ (2.16), the above inequality implies that $\{\boldsymbol{A} - \boldsymbol{A}_n\}_{n=1}^{\infty} \subset \boldsymbol{X}_N^0$ is bounded, whence up to extracting subsequences we find $\widetilde{\boldsymbol{A}} \in \boldsymbol{X}_N^0$ satisfying

$$\boldsymbol{A} - \boldsymbol{A}_n \rightharpoonup \widetilde{\boldsymbol{A}} \quad \text{weakly in } \boldsymbol{X}_N^0 \quad \text{as } n \to \infty. \tag{2.67}$$

Thanks to the compact embedding (2.6), we observe that the above convergence is in fact strong in $\boldsymbol{L}^2(\Omega)$ so that passing to the limit in (2.66), in combination with the strong monotonicity of $\boldsymbol{\Phi}$, yields

$$\|\boldsymbol{A} - \boldsymbol{A}_n\|^2_{\boldsymbol{H}(\mathbf{curl})} \leq \int_\Omega (\boldsymbol{J} - \boldsymbol{J}_n) \cdot (\boldsymbol{A} - \boldsymbol{A}_n)\, \mathrm{d}x \to 0 \quad \text{as } n \to \infty.$$

This concludes the proof. □

Theorem 2.2.2. *Let Assumption 2.1.2 be satisfied. Then, there exists an optimal solution $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$ to the problem ($\mathrm{P}_{\mathrm{sol}}$).*

Proof. The proof is a simple consequence of Theorem 2.1.3 and Lemma 2.2.1 by standard arguments in optimal control. See for instance [Trö10, Sec. 4.4] concerning existence of optimal controls for nonlinear problems. □

As the solution mapping $\boldsymbol{G}_\infty$ is nonlinear, we underline that one cannot in general expect uniqueness for ($\mathrm{P}_{\mathrm{sol}}$).

Lemma 2.2.3. *The optimal control problems* (P) *and* ($\mathrm{P}_{\mathrm{sol}}$) *are equivalent.*

Proof. For the purpose of the proof, we shall introduce a slightly different notation. We rewrite problem (P) and ($\mathrm{P}_{\mathrm{sol}}$) respectively as

$$\begin{cases} \min\limits_{\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)} F(\boldsymbol{J}) \\ \text{subject to (VI)}, \end{cases} \qquad \begin{cases} \min\limits_{\boldsymbol{J} \in H(\mathrm{div}=0,\Omega)} F_{\mathrm{sol}}(\boldsymbol{J}) \\ \text{subject to (VI}_{\mathrm{sol}}), \end{cases}$$

where

$$F(\boldsymbol{J}) = \frac{1}{2}\|\operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}} - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{J} \in \boldsymbol{L}^2(\Omega),$$
$$F_{\mathrm{sol}}(\boldsymbol{J}) = \frac{1}{2}\|\operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}} - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega).$$

In particular, this means that $(\boldsymbol{A}_{\boldsymbol{J}}, \boldsymbol{J})$ satisfy (VI) and ($\mathrm{VI}_{\mathrm{sol}}$) respectively:

- if $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$,

$$\int_\Omega \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}}|) \operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}} \cdot \operatorname{\mathbf{curl}}(\boldsymbol{v} - \boldsymbol{A}_{\boldsymbol{J}}) \,\mathrm{d}x + \int_\Omega \nabla\phi \cdot (\boldsymbol{v} - \boldsymbol{A}_{\boldsymbol{J}}) \,\mathrm{d}x$$
$$\geq \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A}_{\boldsymbol{J}}) \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

  where $\nabla\phi$ denotes the $\boldsymbol{L}^2$-orthogonal projection of $\boldsymbol{J}$ onto $\nabla H^1_0(\Omega)$.

- If $\boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega)$, then

$$\int_\Omega \nu(\cdot, |\operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}}|) \operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}} \cdot \operatorname{\mathbf{curl}}(\boldsymbol{v} - \boldsymbol{A}) \,\mathrm{d}x \geq \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{v} - \boldsymbol{A}_{\boldsymbol{J}}) \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0_N \cap \boldsymbol{K}.$$

Note that F and F_{sol} actually have the same expression, the different name is to underline that F_{sol} acts on divergence-free controls and therefore refers to ($\mathrm{P}_{\mathrm{sol}}$).

Assume that $\boldsymbol{J}^\star_{\mathrm{sol}} \in H(\mathrm{div}{=}0, \Omega)$ is a solution of ($\mathrm{P}_{\mathrm{sol}}$). In particular

$$F_{\mathrm{sol}}(\boldsymbol{J}^\star_{\mathrm{sol}}) \leq F_{\mathrm{sol}}(\boldsymbol{J}) \qquad \forall \boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega). \tag{2.68}$$

and $(\boldsymbol{A}_{\boldsymbol{J}^\star_{\mathrm{sol}}}, \boldsymbol{J}^\star_{\mathrm{sol}}) \in \boldsymbol{X}^0_N \times \boldsymbol{L}^2(\Omega)$ satisfies (VI) with $\phi = 0$ (see (2.17)). Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, by (2.7) it can be decomposed as

$$\boldsymbol{J} = \boldsymbol{J}_{\mathrm{sol}} + \nabla\phi_{\boldsymbol{J}}, \quad \boldsymbol{J}_{\mathrm{sol}} \in H(\mathrm{div}{=}0, \Omega),\ \phi_{\boldsymbol{J}} \in H^1_0(\Omega).$$

Recall that, since the two terms of the decomposition are orthogonal in $\boldsymbol{L}^2(\Omega)$, it holds

$$\|\boldsymbol{J}\|^2_{\boldsymbol{L}^2(\Omega)} = \|\boldsymbol{J}_{\mathrm{sol}}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\phi_{\boldsymbol{J}}\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{2.69}$$

We can now estimate

$$F(\boldsymbol{J}^\star_{\mathrm{sol}}) = F_{\mathrm{sol}}(\boldsymbol{J}^\star_{\mathrm{sol}}) \underbrace{\leq}_{(2.68)} F_{\mathrm{sol}}(\boldsymbol{J}_{\mathrm{sol}}) = F(\boldsymbol{J}_{\mathrm{sol}}) = \frac{1}{2}\|\operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}_{\mathrm{sol}}} - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}_{\mathrm{sol}}\|^2_{\boldsymbol{L}^2(\Omega)}$$
$$\leq \frac{1}{2}\|\operatorname{\mathbf{curl}} \boldsymbol{A}_{\boldsymbol{J}_{\mathrm{sol}}} - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}_{\mathrm{sol}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\nabla\phi_{\boldsymbol{J}}\|^2_{\boldsymbol{L}^2(\Omega)} \underbrace{=}_{(2.69)} F(\boldsymbol{J}).$$

Since $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ was arbitrary, we conclude that $\boldsymbol{J}^\star_{\mathrm{sol}} \in \boldsymbol{L}^2(\Omega)$ is a minimizer of (P).

Now let $\boldsymbol{J}^\star \in \boldsymbol{L}^2(\Omega)$ be a solution of (P). First we shall show that it is divergence free, namely $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$. To this end, we decompose it as

$$\boldsymbol{J}^\star = \widetilde{\boldsymbol{J}}_{\mathrm{sol}} + \nabla\phi_{\boldsymbol{J}^\star}, \quad \widetilde{\boldsymbol{J}}_{\mathrm{sol}} \in H(\mathrm{div}{=}0, \Omega),\ \phi_{\boldsymbol{J}^\star} \in H^1_0(\Omega),\ \widetilde{\boldsymbol{J}}_{\mathrm{sol}} \perp^{L^2} \nabla\phi_{\boldsymbol{J}^\star}.$$

If $J^\star_{\mathrm{sol}} \in H(\mathrm{div}{=}0,\Omega)$ is a minimizer of ($\mathrm{P}_{\mathrm{sol}}$), we have

$$F_{\mathrm{sol}}(J^\star_{\mathrm{sol}}) \leq F_{\mathrm{sol}}(\widetilde{J}_{\mathrm{sol}}) = F(J^\star) - \frac{\beta}{2}\|\nabla\phi_{J^\star}\|_{L^2(\Omega)} \leq F(J^\star_{\mathrm{sol}}) - \frac{\beta}{2}\|\nabla\phi_{J^\star}\|_{L^2(\Omega)} = F_{\mathrm{sol}}(J^\star_{\mathrm{sol}}) - \frac{\beta}{2}\|\nabla\phi_{J^\star}\|_{L^2(\Omega)},$$

where for the first equality we used the orthogonality of the terms in the Helmholtz decomposition, and for the second inequality the fact that $J^\star_{\mathrm{sol}} \in L^2(\Omega)$ and $J^\star$ is optimal for (P). We deduce that $\nabla\phi_{J^\star} = 0$, which in turn implies $\phi_{J^\star} = 0$ due to $\phi_{J^\star} \in H^1_0(\Omega)$. In conclusion $J^\star = \widetilde{J}_{\mathrm{sol}}$ and we are left to prove that $J^\star$ is actually a minimizer of ($\mathrm{P}_{\mathrm{sol}}$). Indeed for every $J_{\mathrm{sol}} \in H(\mathrm{div}{=}0,\Omega)$,

$$F_{\mathrm{sol}}(J^\star) = F_{\mathrm{sol}}(\widetilde{J}_{\mathrm{sol}}) = F(\widetilde{J}_{\mathrm{sol}}) = F(J^\star) \leq F(J_{\mathrm{sol}}) = F_{\mathrm{sol}}(J_{\mathrm{sol}}).$$

This concludes the proof. □

As a consequence of Lemma 2.2.3, we can focus on ($\mathrm{P}_{\mathrm{sol}}$) for the rest of the chapter, having in mind that every result concerning optimal solutions of ($\mathrm{P}_{\mathrm{sol}}$) would hold for (P) too. This especially applies to optimality conditions, namely Theorem 2.2.9 and Theorem 2.2.15.

2.2.1 Necessary optimality conditions

This section is devoted to the establishment of an optimality system for ($\mathrm{P}_{\mathrm{sol}}$). Let us emphasize once again that the main difficulty is that the solution mapping G_∞ attached to ($\mathrm{VI}_{\mathrm{sol}}$) cannot be expected to possess sufficient differentiability properties. We recall that the loss of differentiability is mainly due to the inequality structure of ($\mathrm{VI}_{\mathrm{sol}}$): it is known that even for solution mappings arising from Poisson-type variational inequalities, merely directional differentiability can be achieved in general [Mig76] (or Gâteaux differentiability in a dense subset), making it difficult to efficiently employ the standard adjoint method to handle first order optimality conditions. To overcome this lack of regularity, we shall consider a smoothed version of ($\mathrm{P}_{\mathrm{sol}}$) built upon the approximation (VI_γ), as the latter features a (weakly) Gâteaux differentiable solution map (see Lemma 2.2.7). For such new optimal control problem, we prove well-posedness and optimality conditions by means of known techniques. The main difficulty is to then pass to the limit in the resulting system.

Due to lack of convexity, we cannot expect uniqueness for ($\mathrm{P}_{\mathrm{sol}}$) in general; on the other hand we wish to write necessary conditions for each solution of ($\mathrm{P}_{\mathrm{sol}}$). To this aim, if $J^\star$ is an optimal solution of ($\mathrm{P}_{\mathrm{sol}}$) provided by Theorem 2.2.2, using a trick of Barbu [Bar81] we define the regularized optimal control problem as

$$\begin{cases} \displaystyle\min_{\substack{J_\gamma \in H(\mathrm{div}=0,\Omega)\\ A_\gamma \in X^0_N}} \left(\frac{1}{2}\|\operatorname{\mathbf{curl}} A_\gamma - B_{\mathrm{d}}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|J_\gamma\|^2_{L^2(\Omega)} + \frac{\beta}{4}\|J_\gamma - J^\star\|^2_{L^2(\Omega)}\right) \\ \text{subject to } (\mathrm{VI}_\gamma). \end{cases} \tag{P_γ}$$

Lemma 2.2.4. *Let Assumption 2.1.2 be satisfied and let $\gamma > 0$. Then, there exists an optimal solution $\boldsymbol{J}_\gamma^\star \in H(\mathrm{div}{=}0, \Omega)$ to* (P_γ).

Proof. Combining Lemma 2.1.6 and Lemma 2.1.7, the proof follows by standard arguments of optimal control theory (see again [Trö10, Sec. 4.4]). □

For the ease of notation, we introduce a vector version of the nonlinearity ν by means of the mapping

$$\boldsymbol{\mathcal{F}}\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^3, \quad (x, s) \mapsto \nu(x, |s|)s, \tag{2.70}$$

for which we require the following regularity assumption to hold:

Assumption 2.2.5 (cf. [You13]). *For almost every $x \in \Omega$, the mappings $\nu(x, \cdot)\colon (0, \infty) \to \mathbb{R}$ and $\boldsymbol{\mathcal{F}}(x, \cdot)\colon \mathbb{R}^3 \to \mathbb{R}^3$ are continuously differentiable. Moreover, there is a constant $C > 0$ for which*

$$\left| \frac{\partial \boldsymbol{\mathcal{F}}_i}{\partial s_j}(x, s) \right| \leq C \quad \text{for a.e. } x \in \Omega,\ \forall s \in \mathbb{R}^3,$$

for all $i, j \in \{1, 2, 3\}$.

Moving towards the proof of differentiability for the solution mapping $\boldsymbol{G}_\gamma$, we introduce an auxiliary linear problem. Let $\overline{\boldsymbol{J}}, \boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega)$ and let $(\overline{\boldsymbol{A}}_\gamma, \overline{\boldsymbol{p}}_\gamma) = \boldsymbol{G}_\gamma(\overline{\boldsymbol{J}})$ be the corresponding state, it reads

$$\begin{cases} \text{Find } (\mathfrak{A}_\gamma, \mathfrak{p}_\gamma) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \text{ s.t.} \\ \displaystyle\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \overline{\boldsymbol{A}}_\gamma)\, \mathbf{curl}\, \mathfrak{A}_\gamma \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x + \gamma \int_\omega (\mathbf{curl}\, \mathfrak{A}_\gamma - \mathfrak{p}_\gamma) \cdot (\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{q}) \,\mathrm{d}x \\ \displaystyle+ \gamma \int_\omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \overline{\boldsymbol{p}}_\gamma) \mathfrak{p}_\gamma \cdot \boldsymbol{q} \,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \,\mathrm{d}x \quad \forall (\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega). \end{cases} \tag{$\mathrm{VI}_\gamma^{\mathrm{lin}}$}$$

Lemma 2.2.6. *Let Assumption 2.1.2, Assumption 2.2.5 and Assumption 2.0.2 be satisfied. For each $\overline{\boldsymbol{J}}, \boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega)$,* ($\mathrm{VI}_\gamma^{\mathrm{lin}}$) *admits a unique solution $(\mathfrak{A}_\gamma, \mathfrak{p}_\gamma)$.*

Proof. [You13, Proposition 3.7] provides us with

$$\mathrm{D}_s \boldsymbol{\mathcal{F}}(x, s) y \cdot y \geq \underline{\nu} |y| \quad \text{for a.e. } x \in \Omega, \quad \forall s, y \in \mathbb{R}^3, \tag{2.71}$$

with $\underline{\nu} > 0$ being the same as in Assumption 2.1.2. The bilinear form appearing on the left hand side of ($\mathrm{VI}_\gamma^{\mathrm{lin}}$) turns out to be coercive in $\boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ as a result of (2.71) in combination with (2.27), and the positive semi-definiteness of $\mathrm{D}_s \boldsymbol{\theta}_\gamma$ (see Lemma 2.1.4). The conclusion follows by the Lax-Milgram theorem. □

Lemma 2.2.7. *Under Assumption 2.1.2, Assumption 2.2.5 and Assumption 2.0.2, the control-to-state mapping* (2.29) *is weakly Gâteaux differentiable. For $\overline{\boldsymbol{J}}, \boldsymbol{J} \in H(\mathrm{div}{=}0, \Omega)$, the Gâteaux derivative $\boldsymbol{G}_\gamma'(\overline{\boldsymbol{J}})\boldsymbol{J}$ is given by the unique solution $(\mathfrak{A}_\gamma, \mathfrak{p}_\gamma)$ to* ($\mathrm{VI}_\gamma^{\mathrm{lin}}$)*. In particular, $\pi_1 \circ \boldsymbol{G}_\gamma\colon H(\mathrm{div}{=}0, \Omega) \to \boldsymbol{X}_N^0$ is also weakly Gâteaux differentiable with derivative $(\pi_1 \circ \boldsymbol{G}_\gamma)'(\overline{\boldsymbol{J}})\boldsymbol{J}$ given by $\mathfrak{A}_\gamma$.*

Proof. Let $\boldsymbol{J}, \overline{\boldsymbol{J}} \in H(\mathrm{div}{=}0, \Omega)$, $\gamma > 0$ and let $(\overline{\boldsymbol{A}}_\gamma, \overline{\boldsymbol{p}}_\gamma) = \boldsymbol{G}_\gamma(\overline{\boldsymbol{J}})$. For $\tau > 0$, we consider the solution $(\boldsymbol{A}_\gamma^\tau, \boldsymbol{p}_\gamma^\tau) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ to

$$\begin{aligned} \langle \boldsymbol{\Phi} \boldsymbol{A}_\gamma^\tau, \boldsymbol{v} \rangle + \gamma \int_\omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\tau - \boldsymbol{p}_\gamma^\tau) \cdot (\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{q}) \,\mathrm{d}x + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\tau) \cdot \boldsymbol{q} \,\mathrm{d}x \\ = \int_\Omega (\overline{\boldsymbol{J}} + \tau \boldsymbol{J}) \cdot \boldsymbol{v} \,\mathrm{d}x \end{aligned} \tag{2.72}$$

for all $(\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$. We now test both the above equation and the state equation (VI_γ) featuring $\overline{\boldsymbol{J}}$ on the right hand side with $(\boldsymbol{v}, \boldsymbol{q}) = (\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma, \boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma)$ and subtract them; we get

$$\begin{aligned} \langle \boldsymbol{\Phi} \boldsymbol{A}_\gamma^\tau - \boldsymbol{\Phi} \overline{\boldsymbol{A}}_\gamma, \boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma \rangle + \gamma \| \mathbf{curl}(\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma) - (\boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma) \|_{\boldsymbol{L}^2(\omega)}^2 \\ + \gamma \int_\omega (\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\tau) - \boldsymbol{\theta}_\gamma(\cdot, \overline{\boldsymbol{p}}_\gamma)) \cdot (\boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma) \,\mathrm{d}x = \tau \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma) \,\mathrm{d}x, \end{aligned}$$

which by the strong monotonocity of $\boldsymbol{\Phi}$ and the monotonicity of $\boldsymbol{\theta}_\gamma$ immediately implies the estimate

$$\begin{aligned} \frac{C_\nu}{2} \left(\| \boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma \|_{\boldsymbol{H}(\mathbf{curl})}^2 + \| \mathbf{curl}(\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma) \|_{\boldsymbol{L}^2(\omega)}^2 \right) \\ + \gamma \| \mathbf{curl}(\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma) - (\boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma) \|_{\boldsymbol{L}^2(\omega)}^2 \le \tau \int_\Omega \boldsymbol{J} \cdot (\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma) \,\mathrm{d}x. \end{aligned} \tag{2.73}$$

Applying (2.27) in the same fashion as in (2.28) and then dividing both sides by τ^2, we obtain

$$\left\| \frac{\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma}{\tau} \right\|_{\boldsymbol{H}(\mathbf{curl})} + \left\| \frac{\boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma}{\tau} \right\|_{\boldsymbol{L}^2(\omega)} \le C \quad \forall \tau > 0. \tag{2.74}$$

As a consequence of (2.74), up to extracting subsequences we find $(\boldsymbol{A}_\gamma, \boldsymbol{p}_\gamma) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$ such that

$$\left(\frac{\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma}{\tau}, \frac{\boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma}{\tau} \right) \rightharpoonup (\mathfrak{A}_\gamma, \mathfrak{p}_\gamma) \quad \text{weakly in } \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \quad \text{as } \tau \to 0. \tag{2.75}$$

Now let $(\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$. Subtracting ($\mathrm{VI}_\gamma$) featuring $\overline{\boldsymbol{J}}$ on the right hand side from (2.72), in view of (2.70), (2.15) it follows after diving by τ:

$$\begin{aligned} & \frac{1}{\tau} \int_\Omega (\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\tau) - \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \overline{\boldsymbol{A}}_\gamma)) \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \\ & + \frac{\gamma}{\tau} \int_\omega (\mathbf{curl}(\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma) - (\boldsymbol{p}_\gamma^\tau - \overline{\boldsymbol{p}}_\gamma)) \cdot (\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{q}) \,\mathrm{d}x \\ & + \frac{\gamma}{\tau} \int_\omega (\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\tau) - \boldsymbol{\theta}_\gamma(\cdot, \overline{\boldsymbol{p}}_\gamma)) \cdot \boldsymbol{q} \,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \boldsymbol{v} \,\mathrm{d}x. \end{aligned} \tag{2.76}$$

In view of Assumption 2.2.5, by the fundamental theorem of calculus we can rewrite the first integral in the above equation as

$$\begin{aligned}
&\frac{1}{\tau}\int_\Omega \left(\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau) - \boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\right)\cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \\
&= \frac{1}{\tau}\int_\Omega\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma + t(\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau - \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma))\cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_\Omega\int_0^1 \left[\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma + t\,\mathbf{curl}(\boldsymbol{A}_\gamma^\tau - \overline{\boldsymbol{A}}_\gamma)) - \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\right]\delta_{\mathbf{curl}}^{\tau,\gamma}(\boldsymbol{A})\cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}t\,\mathrm{d}x \\
&+ \int_\Omega\int_0^1 \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\delta_{\mathbf{curl}}^{\tau,\gamma}(\boldsymbol{A})\cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}t\,\mathrm{d}x,
\end{aligned} \tag{2.77}$$

where

$$\delta_{\mathbf{curl}}^{\tau,\gamma}(\boldsymbol{A}) := \frac{\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau - \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma}{\tau}$$

has been introduced for the purpose of a shorter notation. Combining (2.74) with the continuity and boundedness of $\mathrm{D}_s\boldsymbol{\mathcal{F}}$ (cf. Assumption 2.2.5), we readily observe that for all $t \in [0,1]$

$$\mathrm{D}_s\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma + t(\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau - \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)) - \mathrm{D}_s\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma) \to 0 \quad \text{as } \tau \to 0 \tag{2.78}$$

for almost every $x \in \Omega$, as well as,

$$\left|\mathrm{D}_s\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma + t(\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau - \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)) - \mathrm{D}_s\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\right| \leq C \quad \forall \tau > 0 \tag{2.79}$$

for almost every $x \in \Omega$. Therefore by (2.78), (2.79) and the Lebesgue dominated convergence theorem we obtain

$$\|\mathrm{D}_s\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma + t(\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau - \mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)) - \mathrm{D}_s\boldsymbol{\mathcal{F}}(x,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\|_{L^2(\Omega\times[0,1])} \to 0 \quad \text{as } \tau \to 0.$$

Combining the assertion above with (2.77), (2.75) we conclude that

$$\frac{1}{\tau}\int_\Omega \left(\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\tau) - \boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\right)\cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x$$

converges to

$$\int_\Omega\int_0^1 \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\,\mathbf{curl}\,\boldsymbol{\mathfrak{A}}_\gamma\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\overline{\boldsymbol{A}}_\gamma)\,\mathbf{curl}\,\boldsymbol{\mathfrak{A}}_\gamma\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x$$

as $\tau \to 0$. As $\boldsymbol{\theta}_\gamma$ is continuously differentiable (with respect to the second variable) with uniformly bounded derivative (see Lemma 2.1.4), arguing in the same way we also get

$$\int_\omega (\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma^\tau) - \boldsymbol{\theta}_\gamma(\cdot,\overline{\boldsymbol{p}}_\gamma))\cdot\boldsymbol{q}\,\mathrm{d}x \to \int_\omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\overline{\boldsymbol{p}}_\gamma)\boldsymbol{\mathfrak{p}}_\gamma\cdot\boldsymbol{q}\,\mathrm{d}x \quad \text{as } \tau \to 0 \tag{2.80}$$

for all $\boldsymbol{q} \in \boldsymbol{V}(\omega)$. Hence by (2.80) and (2.75) we can finally pass to the limit as $\tau \to 0$ in (2.76) to see that $(\boldsymbol{\mathfrak{A}}_\gamma, \boldsymbol{\mathfrak{p}}_\gamma)$ is a solution of $(\mathrm{VI}_\gamma^{\mathrm{lin}})$, which is uniquely solvable. This concludes the proof. □

It is now convenient to rewrite (P_γ) in the form of an optimization problem in a Hilbert space. To this end, if $\boldsymbol{J}^\star$ is an optimal solution of (VI) we introduce the reduced objective functional $f_\gamma \colon H(\mathrm{div}{=}0, \Omega) \to \mathbb{R}$ via

$$f_\gamma(\boldsymbol{J}) := \frac{1}{2}\| \mathbf{curl}((\pi_1 \circ \boldsymbol{G}_\gamma)(\boldsymbol{J})) - \boldsymbol{B}_\mathrm{d}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}\|^2_{L^2(\Omega)} + \frac{\beta}{4}\|\boldsymbol{J} - \boldsymbol{J}^\star\|^2_{L^2(\Omega)}. \tag{2.81}$$

It is readily seen that the optimal solutions to (P_γ) and ($\mathrm{P}_{\mathrm{sol}}$) are, respectively, characterized by

$$\boldsymbol{J}_\gamma \in \operatorname*{arg\,min}_{\boldsymbol{J}\in H(\mathrm{div}=0,\Omega)} f_\gamma(\boldsymbol{J}) \qquad \text{and} \qquad \boldsymbol{J} \in \operatorname*{arg\,min}_{\boldsymbol{J}\in H(\mathrm{div}=0,\Omega)} f_\infty(\boldsymbol{J}),$$

where $f_\infty \colon H(\mathrm{div}{=}0, \Omega) \to \boldsymbol{X}_N^0$ is given by

$$f_\infty(\boldsymbol{J}) := \frac{1}{2}\| \mathbf{curl}(\boldsymbol{G}_\infty(\boldsymbol{J})) - \boldsymbol{B}_\mathrm{d}\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}\|^2_{L^2(\Omega)}$$

and $\boldsymbol{G}_\infty$ is the solution mapping of (VI) defined in (2.62). As last step towards the derivation of an optimality system for (P_γ), we shall introduce an adjoint equation.

Fix $\gamma > 0$, let $\overline{\boldsymbol{J}} \in H(\mathrm{div}{=}0, \Omega)$ be given and let $(\overline{\boldsymbol{A}}_\gamma, \overline{\boldsymbol{p}}_\gamma) = \boldsymbol{G}_\gamma(\overline{\boldsymbol{J}}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)$, then the adjoint equation associated with $\overline{\boldsymbol{J}}$ reads:

$$\begin{cases} \text{Find } (\boldsymbol{Q}_\gamma, \boldsymbol{w}_\gamma) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega) \text{ s.t.} \\ \displaystyle\int_\Omega [\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \overline{\boldsymbol{A}}_\gamma)]^\mathrm{T}\, \mathbf{curl}\, \boldsymbol{Q}_\gamma \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x + \gamma \int_\omega (\mathbf{curl}\, \boldsymbol{Q}_\gamma - \boldsymbol{w}_\gamma) \cdot (\mathbf{curl}\, \boldsymbol{v} - \boldsymbol{q}) \,\mathrm{d}x \\ \displaystyle + \gamma \int_\omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \overline{\boldsymbol{p}}_\gamma)\boldsymbol{w}_\gamma \cdot \boldsymbol{q} \,\mathrm{d}x = \int_\Omega (\mathbf{curl}\, \overline{\boldsymbol{A}}_\gamma - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \quad \forall (\boldsymbol{v}, \boldsymbol{q}) \in \boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega). \end{cases} \tag{Adj_γ}$$

Note that the structure of (Adj_γ) is closely related to the one of ($\mathrm{VI}_\gamma^{\mathrm{lin}}$), since the only actual difference (besides the right-hand side) is the transpose taken on the matrix $\mathrm{D}_s\boldsymbol{\mathcal{F}}$ in the first term. In particular, the left hand side of (Adj_γ) also exhibits a coercive bilinear form (cf. the proof of Lemma 2.2.6) and therefore it follows again by the Lax-Milgram theorem that (Adj_γ) is well-posed for each choice of $\overline{\boldsymbol{J}}$.

We are ready to prove optimality conditions for (P_γ).

Theorem 2.2.8. *Let Assumption 2.1.2, Assumption 2.2.5 and Assumption 2.0.2 be satisfied. Let $\gamma > 0$ and $\boldsymbol{J}_\gamma^\star \in H(\mathrm{div}{=}0, \Omega)$ be an optimal control for (P_γ). Then, there exists a quartet*

$$(\boldsymbol{A}_\gamma^\star, \boldsymbol{p}_\gamma^\star, \boldsymbol{Q}_\gamma^\star, \boldsymbol{w}_\gamma^\star) \in \left(\boldsymbol{X}_N^0 \times \boldsymbol{V}(\omega)\right)^2 \tag{2.82}$$

such that:

$$\begin{cases} \langle \mathbf{\Phi} \mathbf{A}_\gamma^\star, \mathbf{v} \rangle + \gamma \int_\omega (\mathbf{curl}\, \mathbf{A}_\gamma^\star - \mathbf{p}_\gamma^\star) \cdot (\mathbf{curl}\, \mathbf{v} - \mathbf{q})\, \mathrm{d}x + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \mathbf{p}_\gamma^\star) \cdot \mathbf{q}\, \mathrm{d}x = \int_\Omega \mathbf{J}_\gamma^\star \cdot \mathbf{v}\, \mathrm{d}x \\ \forall (\mathbf{v}, \mathbf{q}) \in \mathbf{X}_N^0 \times \mathbf{V}(\omega) \end{cases}$$

$$\begin{cases} \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \mathbf{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \mathbf{Q}_\gamma^\star \cdot \mathbf{curl}\, \mathbf{v}\, \mathrm{d}x + \gamma \int_\omega (\mathbf{curl}\, \mathbf{Q}_\gamma^\star - \mathbf{w}_\gamma^\star) \cdot (\mathbf{curl}\, \mathbf{v} - \mathbf{q})\, \mathrm{d}x \\ + \gamma \int_\omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \mathbf{p}_\gamma^\star) \mathbf{w}_\gamma^\star \cdot \mathbf{q}\, \mathrm{d}x = \int_\Omega (\mathbf{curl}\, \mathbf{A}_\gamma^\star - \mathbf{B}_\mathrm{d}) \cdot \mathbf{curl}\, \mathbf{v}\, \mathrm{d}x \quad \forall (\mathbf{v}, \mathbf{q}) \in \mathbf{X}_N^0 \times \mathbf{V}(\omega) \end{cases}$$

$$\mathbf{J}_\gamma^\star = -\frac{2}{3} \beta^{-1} \mathbf{Q}_\gamma^\star + \frac{1}{3} \mathbf{J}^\star. \tag{2.83}$$

Proof. First of all, let us observe that thanks to Lemma 2.2.7, f_γ is Gâteaux differentiable with derivative at $\overline{\mathbf{J}} \in H(\mathrm{div}{=}0, \Omega)$ given by

$$f_\gamma'(\overline{\mathbf{J}})\mathbf{J} = \int_\Omega (\mathbf{curl}\, \overline{\mathbf{A}}_\gamma - \mathbf{B}_\mathrm{d}) \cdot \mathfrak{A}_\gamma\, \mathrm{d}x + \beta \int_\Omega \overline{\mathbf{J}} \cdot \mathbf{J}\, \mathrm{d}x + \frac{\beta}{2} \int_\Omega (\overline{\mathbf{J}} - \mathbf{J}^\star) \cdot \mathbf{J}\, \mathrm{d}x \tag{2.84}$$

for all $\mathbf{J} \in H_0(\mathrm{div}{=}0, \Omega)$, where $\overline{\mathbf{A}}_\gamma = (\pi_1 \circ \mathbf{G}_\gamma)(\overline{\mathbf{J}})$ and $(\mathfrak{A}_\gamma, \mathfrak{p}_\gamma)$ solves ($\mathrm{VI}_\gamma^{\mathrm{lin}}$). In addition, we can test the adjoint equation (Adj_γ) with the couple $(\mathfrak{A}_\gamma, \mathfrak{p}_\gamma)$ and the linearized problem ($\mathrm{VI}_\gamma^{\mathrm{lin}}$) with the adjoint state $(\mathbf{Q}_\gamma, \mathbf{w}_\gamma)$ to obtain

$$\int_\Omega (\mathbf{curl}\, \overline{\mathbf{A}}_\gamma - \mathbf{B}_\mathrm{d}) \cdot \mathfrak{A}_\gamma\, \mathrm{d}x = \int_\Omega \mathbf{J} \cdot \mathbf{Q}_\gamma\, \mathrm{d}x, \tag{2.85}$$

so that (2.84) simplifies to

$$f_\gamma'(\overline{\mathbf{J}})\mathbf{J} = \int_\Omega \left(\mathbf{Q}_\gamma + \beta \left(\frac{3}{2} \overline{\mathbf{J}} - \frac{1}{2} \mathbf{J}^\star \right) \right) \cdot \mathbf{J}\, \mathrm{d}x \quad \forall \mathbf{J} \in H(\mathrm{div}{=}0, \Omega). \tag{2.86}$$

Now let $\mathbf{J}_\gamma^\star \in H(\mathrm{div}{=}0, \Omega)$ be an optimal solution for (P_γ), the unique solvability of the state (Lemma 2.1.6) and adjoint equations immediately gives the first two assertions in (2.83).

By standard arguments, the necessary condition

$$f_\gamma'(\mathbf{J}_\gamma^\star)(\mathbf{J} - \mathbf{J}_\gamma^\star) \geq 0 \quad \forall \mathbf{J} \in H(\mathrm{div}{=}0, \Omega)$$

holds true and by (2.86) it is seen to be equivalent to

$$\int_\Omega \left(\beta^{-1} \mathbf{Q}_\gamma^\star + \frac{3}{2} \mathbf{J}_\gamma^\star - \frac{1}{2} \mathbf{J}^\star \right) \cdot (\mathbf{J} - \mathbf{J}_\gamma^\star)\, \mathrm{d}x \geq 0 \quad \forall \mathbf{J} \in H(\mathrm{div}{=}0, \Omega), \tag{2.87}$$

$(\mathbf{Q}_\gamma^\star, \mathbf{w}_\gamma^\star) \in \mathbf{X}_N^0 \times \mathbf{V}(\omega)$ being the unique solution of (Adj_γ) featuring $\mathbf{A}_\gamma^\star = (\pi_1 \circ \mathbf{G}_\gamma)(\mathbf{J}_\gamma^\star)$ on the right hand side in place of $\overline{\mathbf{A}}_\gamma$.

Due to $\boldsymbol{Q}_\gamma^\star \in \boldsymbol{X}_N^0(\Omega) \hookrightarrow H(\mathrm{div}{=}0,\Omega)$, we can choose

$$\boldsymbol{J} = -\beta^{-1}\boldsymbol{Q}_\gamma^\star - \frac{1}{2}(\boldsymbol{J}_\gamma^\star - \boldsymbol{J}^\star) \in H(\mathrm{div}{=}0,\Omega)$$

in (2.87). This forces

$$\beta^{-1}\boldsymbol{Q}_\gamma^\star + \frac{3}{2}\boldsymbol{J}_\gamma^\star - \frac{1}{2}\boldsymbol{J}^\star = \boldsymbol{0}$$

and hence shows the last condition in (2.83). □

2.2.2 Limiting Analysis of (P_γ)

This section is devoted to the derivation of a necessary optimality system for ($\mathrm{P}_{\mathrm{sol}}$) by means of a limit passage in (2.83). It is now useful to introduce some assumptions that will be used in the sequel in different instances.

A1 There exist $\alpha \in [0,1)$ and $\delta \in (0,1/2)$ such that

$$\gamma^{2+\alpha}\int_{\widetilde{\omega}_\gamma} |z_\gamma|^{1+2\alpha}\,\mathrm{d}x \le C \quad \forall\gamma > 0, \tag{2.88}$$

where:

$$z_\gamma = (|\boldsymbol{p}_\gamma^\star| - d)\frac{\boldsymbol{p}_\gamma^\star \cdot \boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}, \quad \widetilde{\omega}_\gamma = \{x \in \omega : |\boldsymbol{p}_\gamma^\star(x)| - d(x) \in (2\gamma^{-1/2-\delta},\infty)\}. \tag{2.89}$$

Recall that $(\boldsymbol{A}_\gamma^\star, \boldsymbol{p}_\gamma^\star)$ satisfies the first equation (2.83), i.e. the approximating state equation.

A2 The obstacle d is uniformly positive, namely there is $d_0 > 0$ such that

$$d(x) \ge d_0 > 0 \quad \text{for a.e. } x \in \omega. \tag{2.90}$$

Theorem 2.2.9. *Let Assumption 2.0.2, Assumption 2.1.2 and Assumption 2.2.5 be satisfied. Let $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0,\Omega)$ be an optimal solution of ($\mathrm{P}_{\mathrm{sol}}$). Then, there exist an optimal state $\boldsymbol{A}^\star \in \boldsymbol{X}_N^0$, an adjoint state $\boldsymbol{Q}^\star \in \boldsymbol{X}_N^0$ and unique multipliers $(\boldsymbol{m}^\star, \boldsymbol{n}^\star) \in (\boldsymbol{H}(\mathbf{curl},\omega) \cap H_0(\mathrm{div}{=}0,\omega) \cap \mathcal{H}(m,\omega)^\perp)^2$ such that it holds*

$$\begin{cases}
\displaystyle\int_\Omega \nu(\cdot,|\,\mathbf{curl}\,\boldsymbol{A}^\star|)\,\mathbf{curl}\,\boldsymbol{A}^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\omega \mathbf{curl}\,\boldsymbol{m}^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x & (2.91)\\
\displaystyle= \int_\Omega \boldsymbol{J}^\star\cdot\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{X}_N^0, & \\
\displaystyle\int_\omega \mathbf{curl}\,\boldsymbol{m}^\star\cdot\mathbf{curl}(\boldsymbol{v}-\boldsymbol{A}^\star)\,\mathrm{d}x \le 0 \quad \forall\boldsymbol{v}\in\boldsymbol{K}, & (2.92)\\
\displaystyle\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\omega \mathbf{curl}\,\boldsymbol{n}^\star\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x & (2.93)\\
\displaystyle= \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star - \boldsymbol{B}_{\mathrm{d}})\cdot\mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall\boldsymbol{v}\in\boldsymbol{X}_N^0, & \\
\displaystyle\int_\omega \mathbf{curl}\,\boldsymbol{n}^\star\cdot\mathbf{curl}\,\boldsymbol{Q}^\star\,\mathrm{d}x \ge 0, & (2.94)\\
\boldsymbol{J}^\star = -\beta^{-1}\boldsymbol{Q}^\star. & (2.95)
\end{cases}$$

The proof of Theorem 2.2.9 is based on the limiting analysis of the regularized problem (P_γ) and the corresponding optimality system (2.83). The following lemma states that each optimal solution of ($\mathrm{P}_{\mathrm{sol}}$) can be approximated by means of a sequence of solutions to (P_γ).

Lemma 2.2.10. *Let Assumption 2.1.2 and Assumption 2.0.2 be satisfied. Given an optimal control $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$ of ($\mathrm{P}_{\mathrm{sol}}$), there exists a sequence $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0} \subset H(\mathrm{div}{=}0, \Omega)$ of optimal solutions to (P_γ), such that*

$$\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star \quad \textit{strongly in } H(\mathrm{div}{=}0, \Omega) \quad \textit{as } \gamma \to \infty.$$

Proof. Given $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$, we know by Lemma 2.2.4 that for each $\gamma > 0$, there exists at least one optimal solution to (P_γ). Let us denote by $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0}$ a sequence of such solutions. Then, such sequence is bounded in $\boldsymbol{L}^2(\Omega)$. Indeed, let us estimate

$$\begin{aligned}
&\frac{\beta}{2}\|\boldsymbol{J}^\star_\gamma\|^2_{\boldsymbol{L}^2(\Omega)} \le f_\gamma(\boldsymbol{J}^\star_\gamma) \le f_\gamma(\boldsymbol{0}) \\
&= \frac{1}{2}\|\,\mathbf{curl}(\pi_1 \circ \boldsymbol{G}_\gamma)(\boldsymbol{0}) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{4}\|\boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)} \le C \quad \forall \gamma > 0,
\end{aligned}$$

where for the last inequality we have invoked (2.63) with $\boldsymbol{J}_\gamma = \boldsymbol{0}$ for all $\gamma > 0$. Thus, we are able to extract a subsequence, again denoted by the same symbol, such that

$$\boldsymbol{J}^\star_\gamma \rightharpoonup \overline{\boldsymbol{J}} \quad \text{weakly in } H(\mathrm{div}{=}0, \Omega) \quad \text{as } \gamma \to \infty \tag{2.96}$$

for some $\overline{\boldsymbol{J}} \in H(\mathrm{div}{=}0, \Omega)$. Next, we find

$$f_\gamma(\boldsymbol{J}^\star_\gamma) \le f_\gamma(\boldsymbol{J}^\star) = \frac{1}{2}\|\,\mathbf{curl}(\pi_1 \circ \boldsymbol{G}_\gamma)(\boldsymbol{J}^\star) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)},$$

as a result of which it holds by (2.63)

$$\limsup_{\gamma\to\infty} f_\gamma(\boldsymbol{J}^\star_\gamma) \le f_\infty(\boldsymbol{J}^\star). \tag{2.97}$$

On the other hand, taking into account (2.96) and combining it with (2.63), the limit inferior can be bounded from below as

$$\begin{aligned}
\liminf_{\gamma\to\infty} f_\gamma(\boldsymbol{J}^\star_\gamma) &\ge \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{G}_\infty(\overline{\boldsymbol{J}}) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{2}\|\overline{\boldsymbol{J}}\|^2_{\boldsymbol{L}^2(\Omega)} + \frac{\beta}{4}\|\overline{\boldsymbol{J}} - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)} \\
&= f_0(\overline{\boldsymbol{J}}) + \frac{\beta}{4}\|\overline{\boldsymbol{J}} - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)} \ge f_0(\boldsymbol{J}^\star) + \frac{\beta}{4}\|\overline{\boldsymbol{J}} - \boldsymbol{J}^\star\|^2_{\boldsymbol{L}^2(\Omega)}.
\end{aligned} \tag{2.98}$$

Now, combining (2.98) and (2.97), this leads to

$$\lim_{\gamma\to\infty} f_\gamma(\boldsymbol{J}^\star_\gamma) = f_\infty(\boldsymbol{J}^\star) \quad \text{and} \quad \overline{\boldsymbol{J}} = \boldsymbol{J}^\star. \tag{2.99}$$

Utilizing once again (2.63), this time with $\boldsymbol{J}_\gamma := \boldsymbol{J}^\star_\gamma \rightharpoonup \overline{\boldsymbol{J}} = \boldsymbol{J}^\star$, we have

$$\frac{1}{2}\|\,\mathbf{curl}(\pi_1 \circ \boldsymbol{G}_\gamma)(\boldsymbol{J}^\star_\gamma) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)} \to \frac{1}{2}\|\,\mathbf{curl}\,\boldsymbol{G}_\infty(\boldsymbol{J}^\star) - \boldsymbol{B}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(\Omega)},$$

from which, together with (2.99), it follows that

$$\frac{\beta}{2}\|\boldsymbol{J}^\star_\gamma\|^2_{L^2(\Omega)} + \frac{\beta}{4}\|\boldsymbol{J}^\star_\gamma - \boldsymbol{J}^\star\|^2_{L^2(\Omega)} \to \frac{\beta}{2}\|\boldsymbol{J}^\star\|^2_{L^2(\Omega)} \quad \text{as } \gamma \to \infty. \tag{2.100}$$

By computing explicitly the left hand side of (2.100) and recalling $\boldsymbol{J}^\star_\gamma \rightharpoonup \boldsymbol{J}^\star$ weakly in $H(\text{div}=0,\Omega)$ as $\gamma \to \infty$, we obtain

$$\frac{3\beta}{4}\lim_{\gamma\to\infty}\|\boldsymbol{J}^\star_\gamma\|^2_{L^2(\Omega)} \underbrace{=}_{(2.100)} \frac{\beta}{2}\|\boldsymbol{J}^\star\|^2_{L^2(\Omega)} - \frac{\beta}{4}\|\boldsymbol{J}^\star\|^2_{L^2(\Omega)} + \frac{\beta}{2}\lim_{\gamma\to\infty}\int_\Omega \boldsymbol{J}^\star_\gamma\cdot\boldsymbol{J}^\star\,\mathrm{d}x = \frac{3\beta}{4}\|\boldsymbol{J}^\star\|^2_{L^2(\Omega)},$$

which implies

$$\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star \quad \text{in } H(\text{div}=0,\Omega) \quad \text{as } \gamma\to\infty$$

in force of the weak convergence of $\boldsymbol{J}^\star_\gamma$ to $\boldsymbol{J}^\star$. This completes the proof. □

We are now in position to give a proof of the main result of the chapter, Theorem 2.2.9.

Proof of Theorem 2.2.9. Let $\boldsymbol{J}^\star \in H(\text{div}=0)$ be a solution of (P_{sol}). An application of Lemma 2.2.10 gives the existence of a sequence $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0}$ of solutions of (P_γ) such that $\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star$ in $H(\text{div}=0,\Omega)$ as $\gamma\to\infty$. By Lemma 2.1.8 applied to the same sequence (see also the first step of the proof of Theorem 2.1.3 and in particular (2.59)) with the choice $\boldsymbol{V}(\omega) = \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega)$ in (VI_γ), we readily obtain the existence of $\boldsymbol{A}^\star \in \boldsymbol{X}^0_N$ and $\boldsymbol{m}^\star \in \boldsymbol{H}(\mathbf{curl},\omega)$ for which (2.91) and (2.92) hold. Uniqueness of the state multiplier in $\boldsymbol{H}(\mathbf{curl},\omega)\cap H_0(\text{div}=0,\omega)\cap\mathcal{H}(m,\omega)^\perp$ can be motivated as done in the proof of Theorem 2.1.3.

We turn the attention to the adjoint equation (Adj_γ) with the specific choice $\boldsymbol{V}(\omega) = \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega)$, and test it with $(\boldsymbol{v},\boldsymbol{q}) = (\boldsymbol{Q}^\star_\gamma,\boldsymbol{w}^\star_\gamma)$. This yields

$$\begin{aligned}\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\cdot\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\,\mathrm{d}x + \gamma\|\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma - \boldsymbol{w}^\star_\gamma\|^2_{L^2(\omega)} \\ + \gamma\int_\omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma\cdot\boldsymbol{w}^\star_\gamma\,\mathrm{d}x = \int_\Omega(\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}})\cdot\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\,\mathrm{d}x.\end{aligned} \tag{2.101}$$

By virtue of [You13, Proposition 3.7], $\mathrm{D}_s\boldsymbol{\mathcal{F}}(x,s)$ is positive definite for almost every $x\in\Omega$ and all $s\in\mathbb{R}^3$; consequently, arguing as in the proof of Lemma 2.1.8 but with the adjoint equation one immediately gets the boundedness of $\{(\boldsymbol{Q}^\star_\gamma,\boldsymbol{w}^\star_\gamma)\}_{\gamma>0}$ in $\boldsymbol{X}^0_N\times\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega)$ (note that $\{\boldsymbol{A}^\star_\gamma\}_{\gamma>0}$ is already known to be uniformly bounded in $\boldsymbol{X}^0_N$, see Lemma 2.1.8). Hence there exists $(\boldsymbol{Q}^\star,\boldsymbol{w}^\star)\in\boldsymbol{X}^0_N\times\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega)$ such that (up to subsequences)

$$(\boldsymbol{Q}^\star_\gamma,\boldsymbol{w}^\star_\gamma)\rightharpoonup(\boldsymbol{Q}^\star,\boldsymbol{w}^\star) \quad \text{weakly in } \boldsymbol{X}^0_N\times\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega) \quad \text{as } \gamma\to\infty. \tag{2.102}$$

Going back to (2.101), we make use of the positive definiteness of $\mathrm{D}_s\boldsymbol{\mathcal{F}}$, $\mathrm{D}_s\boldsymbol{\theta}_\gamma$ and divide the equation by $\gamma>0$. As $\{\boldsymbol{A}^\star_\gamma\}_{\gamma>0}, \{\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ are bounded in $\boldsymbol{X}^0_N$, it follows

$$(\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma)|_\omega - \boldsymbol{w}^\star_\gamma \to 0 \quad \text{strongly in } \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega) \quad \text{as } \gamma\to\infty, \tag{2.103}$$

which in turn yields $\mathbf{curl}\, \boldsymbol{Q}^\star = \boldsymbol{w}^\star$ thanks to (2.102).

Next, owing again to the boundedness of $\{\boldsymbol{A}^\star_\gamma\}_{\gamma>0}, \{\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in $\boldsymbol{X}^0_N$, we can set $\boldsymbol{q} = \mathbf{curl}\, \boldsymbol{v}$ in (Adj$_\gamma$) to see that the sequence $\{\boldsymbol{\xi}_\gamma\}_{\gamma>0} = \{\gamma \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma\}_{\gamma>0} \subset [\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega)]^*$ is uniformly bounded, hence there exists $\boldsymbol{\xi} \in [\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega)]^*$ such that (up to extracting subsequences)

$$\boldsymbol{\xi}_\gamma \rightharpoonup \boldsymbol{\xi} \quad \text{weakly in } [\mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega)]^* \quad \text{as } \gamma \to \infty. \tag{2.104}$$

At the same time, by Riesz's representation theorem we find $\boldsymbol{n}^\star \in \boldsymbol{H}(\mathbf{curl}, \omega)$ for which

$$\langle \boldsymbol{\xi}, \mathbf{curl}\, \boldsymbol{v} \rangle_\omega = \int_\omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{v} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \omega), \tag{2.105}$$

and thus the validity of the limit adjoint equation (2.93) readily follows. Uniqueness for the state multiplier is already known from the proof of Theorem 2.1.3 and the one of the adjoint multiplier is obtained in a similar way. As $\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star$ in $H(\mathrm{div}{=}0, \Omega)$, (2.95) follows from the last condition in (2.83).

Let us now prove the sign condition (2.94). Another consequence of (2.101) is

$$\begin{aligned}
&\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \,\mathrm{d}x - \int_\Omega (\mathbf{curl}\, \boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \,\mathrm{d}x \\
&= -\gamma \| \,\mathbf{curl}\, \boldsymbol{Q}^\star_\gamma - \boldsymbol{w}^\star_\gamma \|^2_{L^2(\omega)} - \gamma \int_\omega \underbrace{\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}_{\geq 0} \,\mathrm{d}x \leq 0 \quad \forall \gamma > 0.
\end{aligned} \tag{2.106}$$

We shall need to estimate

$$\begin{aligned}
&\liminf_{\gamma>0} \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \,\mathrm{d}x \\
&\qquad = \liminf_{\gamma>0} \int_\Omega \underbrace{\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\, \mathbf{curl}(\boldsymbol{Q}^\star_\gamma - \boldsymbol{Q}^\star) \cdot \mathbf{curl}(\boldsymbol{Q}^\star_\gamma - \boldsymbol{Q}^\star)}_{\geq 0} \,\mathrm{d}x \\
&\qquad + 2 \liminf_{\gamma>0} \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
&\qquad - \liminf_{\gamma>0} \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
&\qquad \geq 2 \liminf_{\gamma>0} \int_\Omega \mathbf{curl}\, \boldsymbol{Q}^\star_\gamma \cdot \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x \\
&\qquad - \liminf_{\gamma>0} \int_\Omega \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star \,\mathrm{d}x.
\end{aligned} \tag{2.107}$$

By the strong convergence of $\mathbf{curl}\, \boldsymbol{A}^\star_\gamma$ to $\mathbf{curl}\, \boldsymbol{A}^\star$ and the continuity of $\mathrm{D}_s\boldsymbol{\mathcal{F}}$ (see Assumption 2.2.5), it holds (for a subsequence, still denoted in the same way)

$$\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star \to \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star)\, \mathbf{curl}\, \boldsymbol{Q}^\star \quad \text{a.e. in } \Omega \quad \text{as } \gamma \to \infty.$$

On the other hand, $\mathrm{D}_s\boldsymbol{\mathcal{F}}$ is also bounded whence

$$|\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}^\star_\gamma)\, \mathbf{curl}\, \boldsymbol{Q}^\star| \leq C |\,\mathbf{curl}\, \boldsymbol{Q}^\star| \in L^2(\Omega)$$

and we can apply Lebesgue's dominated convergence theorem to conclude

$$\mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)\,\mathbf{curl}\,\boldsymbol{Q}^\star \to \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)\,\mathbf{curl}\,\boldsymbol{Q}^\star \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty.$$

Using the latter convergence in (2.107) implies

$$\begin{aligned} \liminf_{\gamma>0} \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \,\mathrm{d}x \\ \geq \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star \cdot \mathbf{curl}\,\boldsymbol{Q}^\star \,\mathrm{d}x. \end{aligned} \tag{2.108}$$

Finally, in view of the limit adjoint equation (2.93) tested with $\boldsymbol{v} = \boldsymbol{Q}^\star \in \boldsymbol{X}^0_N$, it holds

$$\begin{aligned} &-\int_\omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{Q}^\star \,\mathrm{d}x \\ &= \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star \cdot \mathbf{curl}\,\boldsymbol{Q}^\star \,\mathrm{d}x - \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\,\boldsymbol{Q}^\star \,\mathrm{d}x \\ &\underbrace{\leq}_{(2.108)} \liminf_{\gamma>0} \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \,\mathrm{d}x \\ &- \liminf_{\gamma>0} \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \,\mathrm{d}x \underbrace{\leq}_{(2.106)} 0. \end{aligned}$$

This shows (2.94) and concludes the proof. □

Making use of the additional assumptions introduced at the beginning of the section, it is possible to find additional information on the limit multiplier corresponding to the adjoint variable. First we shall derive some estimates involving the fields $\boldsymbol{p}^\star_\gamma, \boldsymbol{w}^\star_\gamma$.

Lemma 2.2.11. *Let us suppose that Assumption A2 holds. Then there exist positive constants C_1, C_2, C_3 such that*

$$\gamma \int_{\{|\boldsymbol{p}^\star_\gamma| - d \geq 2\gamma^{-1}\}} (|\boldsymbol{p}^\star_\gamma| - d)\,\mathrm{d}x \leq C_1, \qquad \gamma \int_{\{|\boldsymbol{p}^\star_\gamma| - d \geq 2\gamma^{-1}\}} \left(\frac{\boldsymbol{p}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\right)^2 \mathrm{d}x \leq C_2 \quad \forall \gamma > 0 \tag{2.109}$$

and

$$\gamma^2 \int_{\{|\boldsymbol{p}^\star_\gamma| - d \in (0, 2\gamma^{-1})\}} (|\boldsymbol{p}^\star_\gamma| - d) \left(\frac{\boldsymbol{p}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\right)^2 \mathrm{d}x \leq C_3 \quad \forall \gamma > 0. \tag{2.110}$$

Proof. In order to derive the first bound, we test the state equation (VI_γ) with $(\boldsymbol{v}, \boldsymbol{q}) = (\boldsymbol{A}^\star_\gamma, \boldsymbol{p}^\star_\gamma)$. We obtain

$$\langle \boldsymbol{\Phi} \boldsymbol{A}^\star_\gamma, \boldsymbol{A}^\star_\gamma \rangle + \gamma \| \mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{p}^\star_\gamma \|^2_{\boldsymbol{L}^2(\omega)} + \gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma) \cdot \boldsymbol{p}^\star_\gamma \,\mathrm{d}x = \int_\Omega \boldsymbol{J}^\star_\gamma \cdot \boldsymbol{A}^\star_\gamma \,\mathrm{d}x. \tag{2.111}$$

Arguing as in the proofs of Lemma 2.2.10 and Lemma 2.1.8, we observe that the sequences $\{\boldsymbol{A}^\star_\gamma\}_{\gamma>0} \subset \boldsymbol{X}^0_N$ and $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0} \subset \boldsymbol{L}^2(\Omega)$ are bounded in the respective spaces so that exploiting the monotonicity of $\boldsymbol{\Phi}$, (2.111) implies

$$\gamma \int_\omega \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma) \cdot \boldsymbol{p}^\star_\gamma \,\mathrm{d}x \leq C \quad \forall \gamma > 0 \tag{2.112}$$

for some $C > 0$. Looking at the definition of $\boldsymbol{\theta}_\gamma$ (2.20), we rewrite the left-hand side of the above equation and bound it from below:

$$\begin{aligned}
&\frac{\gamma^2}{4}\int_{\{|\boldsymbol{p}_\gamma^\star|-d\in(0,2\gamma^{-1})\}} \underbrace{|\boldsymbol{p}_\gamma^\star|(|\boldsymbol{p}_\gamma^\star|-d)^2}_{\geq 0}\,\mathrm{d}x + \gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}} |\boldsymbol{p}_\gamma^\star|(|\boldsymbol{p}_\gamma^\star|-d-\gamma^{-1})\,\mathrm{d}x \\
&\geq \gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}} |\boldsymbol{p}_\gamma^\star|(|\boldsymbol{p}_\gamma^\star|-d-\gamma^{-1})\,\mathrm{d}x \\
&\geq \gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}} d(|\boldsymbol{p}_\gamma^\star|-d-\gamma^{-1})\,\mathrm{d}x \\
&= \gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}} \underbrace{d}_{\geq d_0>0}(|\boldsymbol{p}_\gamma^\star|-d)\,\mathrm{d}x - \int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}} d\,\mathrm{d}x \\
&\geq d_0\gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}} (|\boldsymbol{p}_\gamma^\star|-d)\,\mathrm{d}x - \|d\|_{L^1(\omega)},
\end{aligned} \tag{2.113}$$

where for the last inequality we exploited assumption (2.90), i.e. the uniform positivity of the obstacle d. Then, combining (2.112) with (2.113), the desired estimate follows.

To derive the second bound, we instead focus on the adjoint equation (Adj_γ). The latter tested with $(\boldsymbol{v},\boldsymbol{q}) = (\boldsymbol{Q}_\gamma^\star, \boldsymbol{w}_\gamma^\star)$ reads

$$\begin{aligned}
&\int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot,\mathbf{curl}\,\boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star\cdot\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star\,\mathrm{d}x + \gamma\|\,\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star - \boldsymbol{w}_\gamma^\star\|^2_{L^2(\omega)} \\
&\quad + \gamma\int_\omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star\,\mathrm{d}x = \int_\Omega(\mathbf{curl}\,\boldsymbol{A}_\gamma^\star - \boldsymbol{B}_{\mathrm{d}})\cdot\mathbf{curl}\,\boldsymbol{Q}_\gamma^\star\,\mathrm{d}x.
\end{aligned} \tag{2.114}$$

Combining (2.114), an application of [You13, Proposition 3.7] (i.e., the positive semi-definiteness of $\mathrm{D}_s\boldsymbol{\mathcal{F}}$), the boundedness of $\{\boldsymbol{A}_\gamma^\star\}_{\gamma>0}, \{\boldsymbol{Q}_\gamma^\star\}_{\gamma>0}$ in $\boldsymbol{X}_N^0$ (which is already known from the proof of Theorem 2.2.9) and the explicit expression of $\mathrm{D}_s\boldsymbol{\theta}_\gamma$ (2.21), we obtain

$$\begin{aligned}
&\gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\in(0,2\gamma^{-1})\}}\left[\frac{\gamma}{2}\underbrace{(|\boldsymbol{p}_\gamma^\star|-d)}_{\geq 0}\left(\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\right)^2 + \frac{\gamma(|\boldsymbol{p}_\gamma^\star|-d)^2}{4|\boldsymbol{p}_\gamma^\star|}\left(|\boldsymbol{w}_\gamma^\star|^2 - \left(\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\right)^2\right)\right] \\
&+\gamma\int_{\{|\boldsymbol{p}_\gamma^\star|-d\geq 2\gamma^{-1}\}}\left[\left(\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\right)^2 + \underbrace{\frac{|\boldsymbol{p}_\gamma^\star|-d-\gamma^{-1}}{|\boldsymbol{p}_\gamma^\star|}}_{\geq 0}\left(|\boldsymbol{w}_\gamma^\star|^2 - \left(\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\right)^2\right)\right]\mathrm{d}x \leq C_2
\end{aligned} \tag{2.115}$$

for some $C_2 > 0$ and all $\gamma > 0$. As

$$|\boldsymbol{w}_\gamma^\star|^2 - \left(\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\right)^2 \geq 0 \quad \text{a.e. in } \omega$$

by Cauchy-Schwartz inequality, the result follows. Finally, (2.110) is readily obtained from (2.115). □

Corollary 2.2.12. *Let Assumption 2.0.2, Assumption 2.1.2 and Assumption 2.2.5 be satisfied. Let* $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0,\Omega)$ *be an optimal solution of* ($\mathrm{P}_{\mathrm{sol}}$) *and let* $\boldsymbol{A}^\star \in \boldsymbol{X}_N^0(\Omega), \boldsymbol{Q}^\star \in \boldsymbol{X}_N^0(\Omega)$ *and* $\boldsymbol{n}^\star \in \boldsymbol{H}(\mathbf{curl},\omega)$ *denote the corresponding state, adjoint state and adjoint multiplier satisfying* (2.91), (2.93) *and* (2.92). *If Assumption* ***A1*** *and Assumption* ***A2*** *hold, we obtain*

$$\lim_{\gamma\to\infty}\int_\omega \boldsymbol{\xi}_\gamma \cdot d\frac{\boldsymbol{p}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\,\mathrm{d}x = \int_\omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{A}^\star\,\mathrm{d}x \tag{2.116}$$

where

$$\boldsymbol{\xi}_\gamma = \gamma \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star, \quad \gamma > 0.$$

Proof. We work in the setting of the proof of Theorem 2.2.9, without repeating the first part concerning the limiting analysis of the state and adjoint equations and related convergences. We focus on the orthogonality condition (2.116), under Assumption **A2** and Assumption **A1**. We aim to show that

$$\begin{aligned}
&\left|\int_\omega \boldsymbol{\xi}_\gamma\cdot(|\boldsymbol{p}_\gamma^\star|-d)\frac{\boldsymbol{p}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\,\mathrm{d}x\right| = \left|\gamma\int_\omega \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot,\boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star\cdot(|\boldsymbol{p}_\gamma^\star|-d)\frac{\boldsymbol{p}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\,\mathrm{d}x\right| \\
&\le \left|\int_{\{|\boldsymbol{p}_\gamma^\star|-d\in(0,2\gamma^{-1})\}} \frac{\gamma^2}{2}(|\boldsymbol{p}_\gamma^\star|-d)^2\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\,\mathrm{d}x\right| + \left|\int_{\{|\boldsymbol{p}_\gamma^\star|-d\ge 2\gamma^{-1}\}} \gamma(|\boldsymbol{p}_\gamma^\star|-d)\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\,\mathrm{d}x\right|
\end{aligned} \tag{2.117}$$

converges to zero as $\gamma\to\infty$ by working separately on the last two terms. This would in turn imply (2.116) thanks to $\boldsymbol{p}_\gamma^\star$ converging strongly to $\mathbf{curl}\,\boldsymbol{A}^\star$ in $\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl},\omega)$ and (2.105).

The first one is easily dealt with as by Cauchy-Schwartz inequality we have

$$\begin{aligned}
&\left|\int_{\{|\boldsymbol{p}_\gamma^\star|-d\in(0,2\gamma^{-1})\}} \frac{\gamma^2}{2}(|\boldsymbol{p}_\gamma^\star|-d)^2\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\,\mathrm{d}x\right| \\
&\le \frac{1}{2}\left(\int_{\{|\boldsymbol{p}_\gamma^\star|-d\in(0,2\gamma^{-1})\}} \gamma^2\underbrace{(|\boldsymbol{p}_\gamma^\star|-d)^3}_{\le 8\gamma^{-3}}\,\mathrm{d}x\right)^{\frac12} \\
&\cdot\left(\int_{\{|\boldsymbol{p}_\gamma^\star|-d\in(0,2\gamma^{-1})\}} \gamma^2(|\boldsymbol{p}_\gamma^\star|-d)\left(\frac{\boldsymbol{p}_\gamma^\star\cdot\boldsymbol{w}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}\right)^2\mathrm{d}x\right)^{\frac12} \underbrace{\le}_{(2.110)} C_3\sqrt{2}\gamma^{-\frac12}|\omega|^{\frac12}.
\end{aligned} \tag{2.118}$$

For what concerns the second one, we first introduce the set

$$U_\gamma := \{x\in\omega : |\boldsymbol{p}_\gamma^\star| - d(x) \ge 2\gamma^{-1}\}, \quad \gamma > 0$$

for the sake of readability and observe that (2.109) immediately gives (for all $\gamma > 1$)

$$C \ge \gamma\int_{U_\gamma}(|\boldsymbol{p}_\gamma^\star|-d)\,\mathrm{d}x \ge \gamma\int_{\widetilde{\omega}_\gamma}(|\boldsymbol{p}_\gamma^\star|-d)\,\mathrm{d}x \ge \gamma^{1/2-\delta}|\widetilde{\omega}_\gamma| \implies |\widetilde{\omega}_\gamma| \le C\gamma^{\delta-1/2} \tag{2.119}$$

as $\widetilde{\omega}_\gamma \subset U_\gamma$ for $\gamma > 1$, see (2.89). Using the notation of Assumption **A1** we have:

$$\begin{aligned}
&\left| \int_{U_\gamma} \gamma(|\boldsymbol{p}^\star_\gamma| - d)\frac{\boldsymbol{p}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\,\mathrm{d}x \right| \\
&\leq \left| \gamma \int_{U_\gamma\cap\{|\boldsymbol{p}^\star_\gamma|-d\leq 2\gamma^{-1/2-\delta}\}} (|\boldsymbol{p}^\star_\gamma| - d)\frac{\boldsymbol{p}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\,\mathrm{d}x \right| + \left| \gamma \int_{U_\gamma\cap\widetilde{\omega}_\gamma} z_\gamma\,\mathrm{d}x \right|,
\end{aligned} \tag{2.120}$$

and once more we aim to show that each term on the right hand side vanishes as $\gamma \to \infty$. The first one reads

$$\begin{aligned}
&\left| \gamma \int_{U_\gamma\cap\{|\boldsymbol{p}^\star_\gamma|-d\leq 2\gamma^{-1/2-\delta}\}} (|\boldsymbol{p}^\star_\gamma| - d)\frac{\boldsymbol{p}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\,\mathrm{d}x \right| \\
&\leq \gamma \left(\int_{U_\gamma\cap\{|\boldsymbol{p}^\star_\gamma|-d\leq 2\gamma^{-1/2-\delta}\}} (|\boldsymbol{p}^\star_\gamma| - d)^2\,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{U_\gamma\cap\{|\boldsymbol{p}^\star_\gamma|-d\leq 2\gamma^{-1/2-\delta}\}} \left(\frac{\boldsymbol{p}^\star_\gamma \cdot \boldsymbol{w}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\underbrace{\leq}_{(2.109)} C_2 \left(\gamma \int_{U_\gamma\cap\{|\boldsymbol{p}^\star_\gamma|-d\leq 2\gamma^{-1/2-\delta}\}} \gamma^{-1-2\delta}\,\mathrm{d}x \right)^{\frac{1}{2}} \leq C_2|\omega|\gamma^{-\delta} \to 0 \quad \text{as } \gamma \to \infty.
\end{aligned} \tag{2.121}$$

To conclude, the second one can be estimated as follows. By Hölder inequality,

$$\begin{aligned}
\left| \gamma \int_{U_\gamma\cap\widetilde{\omega}_\gamma} z_\gamma\,\mathrm{d}x \right| &\leq \gamma \left(\int_{U_\gamma\cap\widetilde{\omega}_\gamma} |z_\gamma|^{1+2\alpha}\,\mathrm{d}x \right)^{\frac{1}{1+2\alpha}} |\widetilde{\omega}_\gamma|^{\frac{2\alpha}{1+2\alpha}} \\
&\underbrace{\leq}_{(2.119)} C \left(\gamma \int_{\widetilde{\omega}_\gamma} |z_\gamma|^{1+2\alpha}\,\mathrm{d}x \right)^{\frac{1}{1+2\alpha}} \gamma^{\left(\delta-\frac{1}{2}\right)\frac{2\alpha}{1+2\alpha}+1-\frac{1}{1+2\alpha}} \\
&= C \left(\gamma^{2+\alpha} \int_{\widetilde{\omega}_\gamma} |z_\gamma|^{1+2\alpha} \right)^{\frac{1}{1+2\alpha}} \gamma^{-\frac{1+\alpha}{1+2\alpha}} \gamma^{\frac{(2\delta-1)2\alpha}{2(1+2\alpha)}+\frac{2\alpha}{1+2\alpha}} \\
&= C \left(\gamma^{2+\alpha} \int_{\widetilde{\omega}_\gamma} |z_\gamma|^{1+2\alpha} \right)^{\frac{1}{1+2\alpha}} \gamma^{\frac{-2(1+\alpha)+4\delta\alpha+2\alpha}{2(1+2\alpha)}} \underbrace{\leq}_{(2.88)} C\gamma^{\frac{-1+2\delta\alpha}{(1+2\alpha)}}.
\end{aligned}$$

As the latter exponent of γ is negative (see Assumption **A1**), (2.121), (2.120), (2.118) and (2.117) all together imply that

$$\int_\omega \boldsymbol{\xi}_\gamma \cdot (|\boldsymbol{p}^\star_\gamma| - d)\frac{\boldsymbol{p}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\,\mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty. \tag{2.122}$$

Summing up, we have

$$\int_\omega \boldsymbol{\xi}_\gamma \cdot (|\boldsymbol{p}^\star_\gamma| - d)\frac{\boldsymbol{p}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\,\mathrm{d}x = \int_\omega \boldsymbol{\xi}_\gamma \cdot \boldsymbol{p}^\star_\gamma\,\mathrm{d}x - \int_\omega \boldsymbol{\xi}_\gamma \cdot d\frac{\boldsymbol{p}^\star_\gamma}{|\boldsymbol{p}^\star_\gamma|}\,\mathrm{d}x,$$

as well as

$$\int_\omega \boldsymbol{\xi}_\gamma \cdot \boldsymbol{p}_\gamma^\star \, dx \to \int_\omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{A}^\star \, dx \quad \text{as } \gamma \to \infty$$

due to $\boldsymbol{p}_\gamma^\star \to \boldsymbol{p}^\star = \mathbf{curl}\, \boldsymbol{A}^\star$ strongly in $\boldsymbol{L}^2(\Omega)$ and (2.104), (2.105). This along with (2.122) proves (2.116) and concludes the proof. ☐

Remark 2.2.13. *Let us comment further on orthogonality condition* (2.116)*, which appears to be weaker then analogous orthogonality conditions that can be found in the literature, for instance [MRW15, Theorem 3.4] or [IK07, Theorem 1.1]. First of all we observe that the presence of the term* $\frac{\boldsymbol{p}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}$ *is ultimately related to the bilateral structure of our state constraint. On the technical level, the presence of the limit on the left hand side can be seen as either due to the lack of the regularity of* $\boldsymbol{\xi}_\gamma$ *or of* $d\frac{\boldsymbol{p}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|}$*: for example if the weak convergence* (2.104) *turned out to be strong, we would be able to rewrite* (2.116) *in the form*

$$\left\langle \boldsymbol{\xi}, d\frac{\mathbf{curl}\, \boldsymbol{A}^\star}{|\,\mathbf{curl}\, \boldsymbol{A}^\star|} - \mathbf{curl}\, \boldsymbol{A}^\star \right\rangle_\omega = 0, \tag{2.123}$$

whose multiplier-obstacle-optimal state structure (up to the normalization) resembles what is achieved in the cited references. Similar considerations would hold under (2.104) *together with*

$$d\frac{\boldsymbol{p}_\gamma^\star}{|\boldsymbol{p}_\gamma^\star|} \to d\frac{\mathbf{curl}\, \boldsymbol{A}^\star}{|\,\mathbf{curl}\, \boldsymbol{A}^\star|} \quad \text{strongly in } \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \omega),$$

which we cannot show as no information on the rotation or divergence of the limiting field is available, even if the obstacle is constant.

In order to further motivate how the penalization technique applied in our context affects the limiting analysis of (P_γ) *in terms of impossibility to get proper* $\boldsymbol{L}^p$ *bounds for* $\boldsymbol{\xi}_\gamma$*, we mention that such bounds are usually obtained either because the derivative of the penalization is an admissible test function (see [MP84, Theorem 3.2]), or through a smoothing of the sign function as in [IK07], none of which seems possible here.*

Finally, let us deepen the connection between the presence of a first order constraint and the lack of regularity by reviewing how the Moreau-Yosida penalization for bilateral pointwise obstacles like

$$\boldsymbol{K}_0 = \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \colon |\boldsymbol{v}| \le d \quad \text{a.e. in } \omega\},$$

is modified when considering a first order obstacle. Given $f\colon \boldsymbol{H}(\mathbf{curl}) \to \mathbb{R}$*, we do so by referring to a constrained minimization problem* $\min_{\boldsymbol{u}\in\boldsymbol{K}_0} f(\boldsymbol{u})$*, which is related to a variational inequality by means of differentiation*[1]*. In such context of nonlinear optimization, a widely*

[1]For instance, if f is convex and Gâteaux differentiable, $\boldsymbol{v}^\star \in \arg\min_{\boldsymbol{u}\in\boldsymbol{K}_0} f(\boldsymbol{u})$ is equivalent to the VI

$$f'(\boldsymbol{v}^\star)(\boldsymbol{v} - \boldsymbol{v}^\star) \ge 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}_0,$$

see [Trö10, Lem. 2.21].

employed technique is to introduce a Moreau-Yosida approximation of the indicator function of the convex set $\boldsymbol{K}_0$ to make the problem formally unconstrained. The resulting modified cost functional involves the mapping

$$\boldsymbol{L}^2(\Omega) \ni \boldsymbol{v} \mapsto \frac{1}{2}\|\max(|\boldsymbol{v}| - d, 0)\|^2_{L^2(\Omega)}, \tag{2.124}$$

whose derivative at a point $\boldsymbol{v}^\star \in \boldsymbol{L}^2(\Omega)$ is defined by the action

$$\boldsymbol{v} \mapsto \begin{cases} \langle \max(|\boldsymbol{v}^\star| - d, 0)\dfrac{\boldsymbol{v}^\star}{|\boldsymbol{v}^\star|}, \boldsymbol{v}\rangle_{L^2(\Omega)} & \text{if } \boldsymbol{v}^\star \neq 0, \\ 0 & \text{if } \boldsymbol{v}^\star = 0. \end{cases}$$

If the feasible set is instead

$$\boldsymbol{K}_1 = \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}) \colon |\,\mathbf{curl}\,\boldsymbol{v}| \leq d \quad \text{a.e. in } \omega\},$$

(2.124) *becomes*

$$\boldsymbol{H}_0(\mathbf{curl}) \ni \boldsymbol{v} \mapsto \frac{1}{2}\|\max(|\,\mathbf{curl}\,\boldsymbol{v}| - d, 0)\|^2_{L^2(\Omega)}.$$

Consequently the derivative reads

$$\boldsymbol{v} \mapsto \begin{cases} \langle \max(|\,\mathbf{curl}\,\boldsymbol{v}^\star| - d, 0)\dfrac{\mathbf{curl}\,\boldsymbol{v}^\star}{|\,\mathbf{curl}\,\boldsymbol{v}^\star|}, \mathbf{curl}\,\boldsymbol{v}\rangle_{L^2(\Omega)} & \text{if } \boldsymbol{v}^\star \neq 0, \\ 0 & \text{if } \boldsymbol{v}^\star = 0, \end{cases}$$

and it would appear in the corresponding variational inequality, with $\boldsymbol{v}$ playing the role of test function. The difference with the previous case is the presence of the **curl** *on the test function as well as the composition with the* **curl** *in the first term of the $\boldsymbol{L}^2$-pairing, making it apparent the loss of regularity to be expected from the term.*

2.2.3 An alternative multiplier characterization

Here we propose an alternative characterization of the limit Lagrange multiplier produced by the penalization term in the adjoint equation, namely $\boldsymbol{\xi}_\gamma = \gamma \mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma$. We shall make the assumption that $\omega = \Omega$, i.e. the obstacle set coincides with the whole computational domain. Without the need of Assumption **A1** but paying the price of a Helmoltz decomposition of $\boldsymbol{\xi}_\gamma$ - together with a further splitting by means of characteristic functions -, we are able to prove a key uniform $\boldsymbol{L}^2$-bound for its divergence-free part in the decomposition, which allows to obtain a characterization like (2.116), but without the limit in front.

Even though this improvement seemingly comes at little cost, one has to take into account the implicit loss of *interpretability* due to the fact that the divergence free part of a Helmoltz decomposition is equivalently obtained through an Hilbert projection, or solving a **curl curl** boundary value problem. We work in the exact same setting of Theorem 2.2.9, exception made for the assumption that $\omega = \Omega$ and it is simply connected. We provide the full optimality system for completeness.

Let us first introduce the Hilbert projection operator

$$\mathbb{P}_{\mathbf{curl}\, X_N^0}: \boldsymbol{L}^2(\Omega) \to \mathbf{curl}\, \boldsymbol{X}_N^0(\Omega)$$

induced by the Helmholtz decomposition (2.10) (recall that Ω is now also assumed to be simply connected). Due to the well-known vector calculus identity [RV10, Appx. A1]

$$(\mathbf{curl}\, \boldsymbol{a}) \cdot \boldsymbol{n} = \mathrm{div}_\tau(\boldsymbol{a} \times \boldsymbol{n}) \quad \text{on } \partial\Omega,$$

$\mathbb{P}_{\mathbf{curl}\, X_N^0}$ satisfies

$$\mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{u}) \in H_0(\mathrm{div}{=}0, \Omega) \quad \forall \boldsymbol{u} \in \boldsymbol{L}^2(\Omega).$$

Lemma 2.2.14. *Let Assumption 2.0.2, Assumption 2.1.2 and Assumption 2.2.5 be satisfied. Assume that $\omega = \Omega$ and that it is simply connected (in addition to have a connected boundary according to Assumption 2.0.2). Let $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$ be an optimal solution of ($\mathrm{P}_{\mathrm{sol}}$), $\{\boldsymbol{J}_\gamma^\star\}_{\gamma>0} \subset H(\mathrm{div}{=}0, \Omega)$ a sequence of solutions of (P_γ) provided by Lemma 2.2.10, $\{(\boldsymbol{A}_\gamma^\star, \boldsymbol{p}_\gamma^\star)\}_{\gamma>0}$ the corresponding sequence of direct states satisfying (VI_γ) and $\{(\boldsymbol{Q}_\gamma^\star, \boldsymbol{w}_\gamma^\star)\}_{\gamma>0}$ the corresponding sequence of adjoint states satisfying (Adj_γ) (with $\boldsymbol{A}_\gamma^\star$ in place of $\overline{\boldsymbol{A}}_\gamma$). Then, recalling the notation $\boldsymbol{\xi}_\gamma = \gamma \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star$, the sequences*

$$\{\mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{\xi}_\gamma)\}_{\gamma>0}, \quad \{\mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{\xi}_\gamma)\chi_{\{|\mathbf{curl}\, \boldsymbol{A}_\gamma^\star|>d\}}\}_{\gamma>0}, \quad \{\mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{\xi}_\gamma)\chi_{\{|\mathbf{curl}\, \boldsymbol{A}_\gamma^\star|\le d\}}\}_{\gamma>0}$$

are uniformly bounded in $\boldsymbol{L}^2(\Omega)$. Here $\chi_{\mathcal{O}}$ denotes the characteristic function of the set $\mathcal{O}$. In particular, up to extracting subsequences, there exist $\boldsymbol{\lambda}_{d_-}^\star, \boldsymbol{\lambda}_{d_+}^\star \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{n}^\star \in \boldsymbol{X}_N^0(\Omega)$ such that

$$\begin{aligned} \mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{\xi}_\gamma) &\rightharpoonup \mathbf{curl}\, \boldsymbol{n}^\star && \text{weakly in } \boldsymbol{L}^2(\Omega) && \text{as } \gamma \to \infty, \\ \mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{\xi}_\gamma)\chi_{\{|\mathbf{curl}\, \boldsymbol{A}_\gamma^\star|>d\}} &\rightharpoonup \boldsymbol{\lambda}_{d_+}^\star && \text{weakly in } \boldsymbol{L}^2(\Omega) && \text{as } \gamma \to \infty, \\ \mathbb{P}_{\mathbf{curl}\, X_N^0}(\boldsymbol{\xi}_\gamma)\chi_{\{|\mathbf{curl}\, \boldsymbol{A}_\gamma^\star|\le d\}} &\rightharpoonup \boldsymbol{\lambda}_{d_-}^\star && \text{weakly in } \boldsymbol{L}^2(\Omega) && \text{as } \gamma \to \infty. \end{aligned} \tag{2.125}$$

Moreover, it holds

$$\mathbf{curl}\, \boldsymbol{n}^\star = \boldsymbol{\lambda}_{d_-}^\star + \boldsymbol{\lambda}_{d_+}^\star. \tag{2.126}$$

Proof. For every $\gamma > 0$, thanks to (2.10) we decompose $\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star = \gamma^{-1}\boldsymbol{\xi}_\gamma \in \boldsymbol{L}^2(\Omega)$ as follows

$$\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star = \mathbb{P}_{\mathbf{curl}\, X_N^0}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star) + \nabla \psi_\gamma, \quad \psi_\gamma \in H^1(\Omega). \tag{2.127}$$

By definition, $\mathbb{P}_{\mathbf{curl}\, X_N^0}(\mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star)\boldsymbol{w}_\gamma^\star) = \mathbf{curl}\, \boldsymbol{g}_\gamma$ for some $\boldsymbol{g}_\gamma \in \boldsymbol{X}_N^0(\Omega)$ and we can test (Adj_γ) with $\boldsymbol{v} = \gamma \boldsymbol{g}_\gamma$, $\boldsymbol{q} = \mathbf{curl}\, \boldsymbol{v} = \gamma\, \mathbf{curl}\, \boldsymbol{g}_\gamma$. This yields

$$\gamma \int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{g}_\gamma \,\mathrm{d}x + \gamma^2 \int_\Omega (\mathbf{curl}\, \boldsymbol{g}_\gamma + \nabla \psi_\gamma) \cdot \mathbf{curl}\, \boldsymbol{g}_\gamma \,\mathrm{d}x$$
$$= \gamma \int_\Omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \boldsymbol{B}_\mathrm{d}) \cdot \mathbf{curl}\, \boldsymbol{g}_\gamma \,\mathrm{d}x.$$

As $\mathbf{curl}\,\nabla \equiv \mathbf{0}$ and $\boldsymbol{g}_\gamma \times \boldsymbol{n} = \mathbf{0}$ on $\partial\Omega$, the $\boldsymbol{L}^2(\Omega)$-inner product between $\mathbf{curl}\,\boldsymbol{g}_\gamma$ and $\nabla\psi_\gamma$ vanishes (due to an integration by parts for the $\mathbf{curl}$ operator, [RV10, Appx. A.1, Eq. A.6]) and we obtain

$$\gamma \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star_\gamma)^{\mathrm{T}}\, \mathbf{curl}\,\boldsymbol{Q}^\star_\gamma \cdot \mathbf{curl}\,\boldsymbol{g}_\gamma \,\mathrm{d}x + \gamma^2 \|\,\mathbf{curl}\,\boldsymbol{g}_\gamma\|^2_{\boldsymbol{L}^2(\Omega)} = \gamma \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star_\gamma - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\,\boldsymbol{g}_\gamma \,\mathrm{d}x.$$

Now it suffices to combine Assumption 2.2.5, the uniform boundedness of the sequences $\{\mathbf{curl}\,\boldsymbol{A}^\star_\gamma\}_{\gamma>0}$, $\{\mathbf{curl}\,\boldsymbol{Q}^\star_\gamma\}_{\gamma>0}$ in $\boldsymbol{L}^2(\Omega)$ (cf. the proof of Theorem 2.2.9) and an application of Hölder and Young inequalities to deduce that $\{\gamma\,\mathbf{curl}\,\boldsymbol{g}_\gamma\}_{\gamma>0} \subset \boldsymbol{L}^2(\Omega)$ is uniformly bounded. As a consequence, we find $\boldsymbol{\lambda}^\star \in \boldsymbol{L}(\Omega)$ for which (up to subsequences)

$$\gamma\,\mathbf{curl}\,\boldsymbol{g}_\gamma = \gamma\mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma) = \mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma) \rightharpoonup \boldsymbol{\lambda}^\star \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \tag{2.128}$$

as $\gamma \to \infty$. Additionally, since $\mathbf{curl}\,\boldsymbol{X}^0_N(\Omega)$ is weakly closed in $\boldsymbol{L}^2(\Omega)$ (the exact same argument of Remark 2.1.5 works under the hypothesis of weak convergence), there exists $\boldsymbol{n}^\star \in \boldsymbol{X}^0_N(\Omega)$ such that $\mathbf{curl}\,\boldsymbol{n}^\star = \boldsymbol{\lambda}^\star$. This proves the first assertion in (2.125).

Next we define the other adjoint multipliers that provide a decomposition of $\boldsymbol{\lambda}^\star$. Let us set for $\gamma > 0$

$$\begin{aligned} \boldsymbol{\lambda}_{\gamma,d_+} &:= \gamma\mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|>d\}} = \mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|>d\}}, \\ \boldsymbol{\lambda}_{\gamma,d_-} &:= \gamma\mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|\le d\}} = \mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma)\chi_{\{|\,\mathbf{curl}\,\boldsymbol{A}^\star_\gamma|\le d\}}. \end{aligned} \tag{2.129}$$

As

$$|\boldsymbol{\lambda}_{\gamma,d_+}| \le |\mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma)| \quad \text{and} \quad |\boldsymbol{\lambda}_{\gamma,d_-}| \le |\mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma)| \quad \text{a.e. in } \Omega$$

and since we just proved that the sequence $\{\mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma)\}_{\gamma>0}$ is bounded in $\boldsymbol{L}^2(\Omega)$, we deduce that $\{\boldsymbol{\lambda}_{\gamma,d_+}\}_{\gamma>0}$ and $\{\boldsymbol{\lambda}_{\gamma,d_-}\}_{\gamma>0}$ are uniformly bounded in $\boldsymbol{L}^2(\Omega)$ as well. Hence there exist $\boldsymbol{\lambda}^\star_{d_+}, \boldsymbol{\lambda}^\star_{d_+} \in \boldsymbol{L}^2(\Omega)$ such that (up to extracting subsequences)

$$\begin{aligned} \boldsymbol{\lambda}_{\gamma,d_+} &\rightharpoonup \boldsymbol{\lambda}^\star_{d_+} \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty, \\ \boldsymbol{\lambda}_{\gamma,d_-} &\rightharpoonup \boldsymbol{\lambda}^\star_{d_-} \quad \text{weakly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty. \end{aligned} \tag{2.130}$$

This completes the proof of (2.125). Additionally, due to

$$\boldsymbol{\lambda}_{\gamma,d_+} + \boldsymbol{\lambda}_{\gamma,d_-} = \mathbb{P}_{\mathbf{curl}\,\boldsymbol{X}^0_N}(\boldsymbol{\xi}_\gamma) \quad \forall \gamma > 0$$

and the weak convergence (2.128), it holds $\mathbf{curl}\,\boldsymbol{n}^\star = \boldsymbol{\lambda}^\star = \boldsymbol{\lambda}^\star_{d_+} + \boldsymbol{\lambda}^\star_{d_-}$ which proves (2.126) and concludes the proof. ☐

Theorem 2.2.15. *Let Assumption 2.0.2, Assumption 2.1.2 and Assumption 2.2.5 be satisfied. Assume that $\omega = \Omega$ and that it is simply connected (in addition to have a connected boundary according to Assumption 2.0.2). Let $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$ be an optimal solution of ($\mathrm{P}_{\mathrm{sol}}$). Then, there exist an optimal state $\boldsymbol{A}^\star \in \boldsymbol{X}^0_N(\Omega)$, an adjoint state $\boldsymbol{Q}^\star \in \boldsymbol{X}^0_N(\Omega)$, a unique state* *multiplier $\boldsymbol{m}^\star \in \boldsymbol{X}_N^0(\Omega)$ and adjoint multipliers $\boldsymbol{\lambda}^\star_{d_-}, \boldsymbol{\lambda}^\star_{d_+} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{n}^\star \in \boldsymbol{X}_N^0(\Omega)$ defined by Lemma 2.2.14 such that it holds*

$$\begin{cases} \int_\Omega \nu(\cdot, |\,\mathbf{curl}\,\boldsymbol{A}^\star|)\,\mathbf{curl}\,\boldsymbol{A}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{m}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x & (2.131)\\ = \int_\Omega \boldsymbol{J}^\star \cdot \boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0, & \\ \int_\Omega \mathbf{curl}\,\boldsymbol{m}^\star \cdot \mathbf{curl}(\boldsymbol{v} - \boldsymbol{A}^\star)\,\mathrm{d}x \le 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}, & (2.132)\\ \int_\Omega \mathrm{D}_s\boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\,\boldsymbol{A}^\star)^{\mathrm{T}}\,\mathbf{curl}\,\boldsymbol{Q}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x & (2.133)\\ = \int_\Omega (\mathbf{curl}\,\boldsymbol{A}^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\,\boldsymbol{v}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0, & \\ \int_\Omega \mathbf{curl}\,\boldsymbol{n}^\star \cdot \mathbf{curl}\,\boldsymbol{Q}^\star\,\mathrm{d}x \ge 0 & (2.134)\\ \boldsymbol{J}^\star = -\beta^{-1}\boldsymbol{Q}^\star, & (2.135)\\ \int_\Omega \boldsymbol{\lambda}^\star_{d_+} \cdot \left(d\frac{\mathbf{curl}\,\boldsymbol{A}^\star}{|\,\mathbf{curl}\,\boldsymbol{A}^\star|} - \mathbf{curl}\,\boldsymbol{A}^\star \right)\mathrm{d}x = 0. & (2.136) \end{cases}$$

Moreover, $\boldsymbol{\lambda}^\star_{d_-}, \boldsymbol{\lambda}^\star_{d_+}$ and $\boldsymbol{n}^\star$ are related by the identity

$$\mathbf{curl}\,\boldsymbol{n}^\star = \boldsymbol{\lambda}^\star_{d_-} + \boldsymbol{\lambda}^\star_{d_+}. \tag{2.137}$$

Proof. The proof is divided into three steps.

Step 1 (known information and limit state equation). Let $\boldsymbol{J}^\star \in H(\mathrm{div}{=}0, \Omega)$ be an optimal solution of ($\mathrm{P_{sol}}$). Repeating the first part of the proof of Theorem 2.2.9 (with the exception that $\omega = \Omega$), we find:

- A sequence $\{\boldsymbol{J}^\star_\gamma\}_{\gamma>0} \subset H(\mathrm{div}{=}0, \Omega)$ of solutions of (P_γ) satisfying

$$\boldsymbol{J}^\star_\gamma \to \boldsymbol{J}^\star \quad \text{strongly in } H(\mathrm{div}{=}0, \Omega) \quad \text{as } \gamma \to \infty.$$

- An optimal state $\boldsymbol{A}^\star \in \boldsymbol{X}_N^0$, a sequence $\{(\boldsymbol{A}^\star_\gamma, \boldsymbol{p}^\star_\gamma)\}_{\gamma>0} \subset \boldsymbol{X}_N^0 \times \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl})$ of solutions of (VI_γ) featuring $\boldsymbol{J}^\star_\gamma$ on the right hand side such that

$$(\boldsymbol{A}^\star_\gamma, \boldsymbol{p}^\star_\gamma) \to (\boldsymbol{A}^\star, \mathbf{curl}\,\boldsymbol{A}^\star) \quad \text{strongly in } \boldsymbol{X}_N^0 \times \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \Omega) \quad \text{as } \gamma \to \infty \tag{2.138}$$

  and a unique $\boldsymbol{m}^\star \in \boldsymbol{X}_N^0$ for which (2.131) and (2.132) hold. Indeed we stress that if $\Omega = \omega$, then the space $\boldsymbol{V}$ for the relaxation variable $\boldsymbol{q}$ can be taken as $\mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \Omega)$ and the choice $\boldsymbol{q} = \mathbf{curl}\,\boldsymbol{v}$ in (VI_γ), $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega)$, yields the existence of a unique state multiplier which lies in $\mathbf{curl}\,\boldsymbol{X}_N^0(\Omega)$: see the first step in the proof of Theorem 2.1.3, taking into account that $\mathcal{H}(m, \Omega) = \{0\}$ since Ω is now assumed to be simply connected.

- A uniformly bounded sequence of adjoint states $(\boldsymbol{Q}^\star_\gamma, \boldsymbol{w}^\star_\gamma) \subset \boldsymbol{X}_N^0 \times \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl}, \Omega)$ (which satisfy (2.83), or equivalently (Adj_γ) with $\boldsymbol{A}^\star_\gamma$ in place of $\overline{\boldsymbol{A}}_\gamma$) and an optimal adjoint state $(\boldsymbol{Q}^\star, \boldsymbol{w}^\star)$ for which

$$(\boldsymbol{Q}_\gamma^\star, \boldsymbol{w}_\gamma^\star) \rightharpoonup (\boldsymbol{Q}^\star, \boldsymbol{w}^\star) \quad \text{weakly in } \boldsymbol{X}_N^0 \times \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \Omega) \quad \text{as } \gamma \to \infty. \tag{2.139}$$

Taking into account the strong convergence of $\boldsymbol{J}_\gamma^\star$ to $\boldsymbol{J}^\star$ in $H(\mathrm{div}{=}0, \Omega)$, (2.135) follows from the last condition in (2.83).

Step 2 (adjoint multipliers, limit adjoint equation). The adjoint multipliers $\lambda_{d_-}^\star, \lambda_{d_+}^\star, \in L^2(\Omega)$, $\boldsymbol{n}^\star \in \boldsymbol{X}_N^0(\Omega)$ are defined through Lemma 2.2.14, in particular they satisfy relation (2.137).

Let $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega)$. Testing the adjoint equation in (2.83) with $(\boldsymbol{v}, \mathbf{curl}\, \boldsymbol{v}) \in \boldsymbol{X}_N^0 \times \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl}, \Omega)$ we get

$$\int_\Omega \mathrm{D}_s \boldsymbol{\mathcal{F}}(\cdot, \mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\mathrm{T}}\, \mathbf{curl}\, \boldsymbol{Q}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x + \gamma \int_\Omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star) \boldsymbol{w}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \\ = \int_\Omega (\mathbf{curl}\, \boldsymbol{A}_\gamma^\star - \boldsymbol{B}_{\mathrm{d}}) \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x.$$

In view of (2.139), (2.138) and Assumption 2.2.5, the first and the third integral in the above identity converge to their respective counterparts in (2.133). To deal with the remaining one, we write

$$\begin{aligned} &\gamma \int_\Omega \mathrm{D}_s \boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}_\gamma^\star) \boldsymbol{w}_\gamma^\star \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \\ &= \int_\Omega \boldsymbol{\xi}_\gamma \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \\ &\underbrace{=}_{(2.127)} \int_\Omega (\mathbb{P}_{\mathbf{curl}\, \boldsymbol{X}_N^0}(\boldsymbol{\xi}_\gamma) + \gamma \nabla \psi_\gamma) \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \\ &= \int_\Omega \mathbb{P}_{\mathbf{curl}\, \boldsymbol{X}_N^0}(\boldsymbol{\xi}_\gamma) \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \\ &\underbrace{\to}_{(2.125)} \int_\Omega \mathbf{curl}\, \boldsymbol{n}^\star \cdot \mathbf{curl}\, \boldsymbol{v}\, \mathrm{d}x \quad \text{as } \gamma \to \infty, \end{aligned}$$

where for the last equality we used $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl})$ and an integration by parts for the **curl** [RV10, Appx. A.1, Eq. A.6] to get rid of the gradient term. This proves (2.133).

Step 3 (orthogonality condition). It remains to show (2.136). To this end, with the sequence $\{\boldsymbol{A}_\gamma^\star\}_{\gamma>0}$ at hand we first define for almost every $x \in \Omega$:

$$(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\leq d}(x) := \begin{cases} \mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x) & \text{if } |\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)| \leq d(x) \\ d(x) \dfrac{\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)}{|\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)|} & \text{if } |\mathbf{curl}\, \boldsymbol{A}_\gamma^\star(x)| > d(x), \end{cases}$$

and aim to prove that

$$(\mathbf{curl}\, \boldsymbol{A}_\gamma^\star)^{\leq d} \to \mathbf{curl}\, \boldsymbol{A}^\star \quad \text{strongly in } \boldsymbol{L}^2(\Omega) \quad \text{as } \gamma \to \infty. \tag{2.140}$$

We observe that

$$(\mathbf{curl}\, A^\star_\gamma)^{\le d}(\cdot) = \varrho(\cdot, \mathbf{curl}\, A^\star_\gamma(\cdot)),$$

where for almost every $x \in \Omega$, $\varrho\colon \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is defined as

$$\varrho(x,s) := s \min\left(1, \frac{d(x)}{|s|}\right) = \begin{cases} s & \text{if } |s| \le d(x), \\ d(x)\dfrac{s}{|s|} & \text{if } |s| > d(x). \end{cases} \tag{2.141}$$

Since ϱ is uniformly Lipschitz[2] in s (for almost every $x \in \Omega$), we have

$$\int_\Omega |(\mathbf{curl}\, A^\star_\gamma)^{\le d} - \mathbf{curl}\, A^\star|^2 \,\mathrm{d}x \underbrace{=}_{|\,\mathbf{curl}\, A^\star| \le d} \int_\Omega |(\mathbf{curl}\, A^\star_\gamma)^{\le d} - (\mathbf{curl}\, A^\star)^{\le d}|^2 \,\mathrm{d}x$$
$$= \int_\Omega |\varrho(\cdot, \mathbf{curl}\, A^\star_\gamma) - \varrho(\cdot, \mathbf{curl}\, A^\star)|^2 \,\mathrm{d}x \le \int_\Omega |\,\mathbf{curl}\, A^\star_\gamma - \mathbf{curl}\, A^\star|^2 \,\mathrm{d}x \to 0 \quad \text{as } \gamma \to \infty$$

due to the fact that $\mathbf{curl}\, A^\star_\gamma$ converges strongly to $\mathbf{curl}\, A^\star$ in $L^2(\Omega)$ as $\gamma \to \infty$ (see (2.138)). Therefore (2.140) is proved. Combining the latter with (2.125) and (2.137) gives

$$\int_\Omega \mathbb{P}_{\mathbf{curl}\, \mathbf{X}^0_N}(\xi_\gamma) \cdot (\mathbf{curl}\, A^\star_\gamma)^{\le d} \,\mathrm{d}x \to \int_\Omega \lambda^\star_{d_+} \cdot \mathbf{curl}\, A^\star \,\mathrm{d}x + \int_\Omega \lambda^\star_{d_-} \cdot \mathbf{curl}\, A^\star \,\mathrm{d}x \quad \text{as } \gamma \to \infty. \tag{2.142}$$

On the other hand, in view of the definition of $(\cdot)^{\le d}$, the left hand side of the latter equation can be rewritten as

$$\begin{aligned} &\int_\Omega \mathbb{P}_{\mathbf{curl}\, \mathbf{X}^0_N}(\xi_\gamma) \cdot (\mathbf{curl}\, A^\star_\gamma)^{\le d} \,\mathrm{d}x \\ &= \int_\Omega \mathbb{P}_{\mathbf{curl}\, \mathbf{X}^0_N}(\xi_\gamma) \chi_{\{|\,\mathbf{curl}\, A^\star_\gamma| > d\}} \cdot d \frac{\mathbf{curl}\, A^\star_\gamma}{|\,\mathbf{curl}\, A^\star_\gamma|} \,\mathrm{d}x \\ &+ \int_\Omega \mathbb{P}_{\mathbf{curl}\, \mathbf{X}^0_N}(\xi_\gamma) \chi_{\{|\,\mathbf{curl}\, A^\star_\gamma| \le d\}} \cdot \mathbf{curl}\, A^\star_\gamma \,\mathrm{d}x \\ &\underbrace{=}_{(2.129)} \int_\Omega \lambda_{\gamma,d_+} \cdot d \frac{\mathbf{curl}\, A^\star_\gamma}{|\,\mathbf{curl}\, A^\star_\gamma|} \,\mathrm{d}x + \int_\Omega \lambda_{\gamma,d_-} \cdot \mathbf{curl}\, A^\star_\gamma \,\mathrm{d}x \\ &\to \int_\Omega \lambda^\star_{d_+} \cdot d \frac{\mathbf{curl}\, A^\star}{|\,\mathbf{curl}\, A^\star|} \,\mathrm{d}x + \int_\Omega \lambda^\star_{d_-} \cdot \mathbf{curl}\, A^\star \,\mathrm{d}x \quad \text{as } \gamma \to \infty, \end{aligned} \tag{2.143}$$

where the last convergence follows again from $\mathbf{curl}\, A^\star_\gamma$ converging strongly to $\mathbf{curl}\, A^\star$ together with (2.130). Comparing (2.143) and (2.142) gives (2.136), and concludes the proof. □

[2]Indeed ϱ is readily seen to be continuously differentiable (with respect to s) in $\mathbb{R}^3 \setminus \{s \in \mathbb{R}^3 : |s| = d(x)\}$ for almost every $x \in \Omega$. Its Jacobian in the classical sense is given by

$$\mathrm{D}_s \varrho(x,s) = \begin{cases} \mathrm{Id} & \text{if } |s| < d(x) \\ \dfrac{d(x)}{|s|}\left(\mathrm{Id} - \dfrac{s \otimes s}{|s|^2}\right) & \text{if } |s| > d(x), \end{cases}$$

which is uniformly bounded by the constant 1 for all s and almost every $x \in \Omega$. As ϱ is continuous across $\{s \in \mathbb{R}^3 : |s| = d(x)\}$, this implies $\varrho(x,\cdot) \in W^{1,\infty}(\mathbb{R}^3)$ and thus Lipschitz continuity. Additionally, the global Lipschitz constant is bounded above by 1 independently of x.

2.3 Future research and open problems

Given the structure of (2.136) and considering the relation (2.137), the question naturally arises: are there conditions that guarantee $\boldsymbol{\lambda}^\star_{d_+} \neq \mathbf{0}$ and/or $\boldsymbol{\lambda}^\star_{d_-} = \mathbf{0}$? Observe that the latter information would imply $\boldsymbol{\lambda}^\star = \boldsymbol{\lambda}^\star_{d_+}$ and therefore a desirable clean correspondence between the adjoint multiplier appearing in the equation and the orthogonality condition. Unfortunately, at the moment both results appear to be out of reach mostly due to the presence of the Helmholtz projection operator (which, on the other hand, is necessary to get the crucial $\boldsymbol{L}^2$ bounds): for instance an explicit expression of $\mathrm{D}_s\boldsymbol{\theta}_\gamma(\cdot, \boldsymbol{p}^\star_\gamma)\boldsymbol{w}^\star_\gamma$ is available (see (2.21)), but once the projection is taken, nothing can be said in general concerning the pointwise properties of the resulting vector field. What is rather easy to see, on the contrary, is that in the presence of a trivial obstacle $d = +\infty$ (i.e. the problem is actually unconstrained) we would have $\boldsymbol{\lambda}^\star_{d_+} = \mathbf{0}$ and $\boldsymbol{\lambda}^\star = \boldsymbol{\lambda}^\star_{d_-}$; this suggests and strengthens the idea that $\boldsymbol{\lambda}^\star_{d_+}$ should carry information in the relevant case of non-trivial obstacle.

An interesting and challenging related problem would be to work with the same state inequality, but controlling the obstacle $d\colon \omega \to \mathbb{R}^+$. This kind of questions have been first addressed by Bergonioux et al [BL04] for scalar elliptic operators assuming H^2 regularity of the obstacle, and then by Ito and Kunisch with a generalized Moreau-Yosida regularization to tackle H^1 obstacles [IK07]. Taking into account the already demanding setting of first order constraints in the $\boldsymbol{H}(\mathbf{curl})$ vector framework, we think that it may be interesting to analyze an alternative, yet nontrivial variant which relies on the assumption that the obstacle is piecewise constant. Let $\{\Omega_i\}_{i=1}^n$ be an open partition of Ω and suppose that $d\colon \Omega \to \mathbb{R}^+_0$ satisfies

$$d(x) = d_i \in \mathbb{R}^+_0 \qquad \forall x \in \Omega_i,$$

so that the control variable actually reduces to $\boldsymbol{d} = (d_1, \dots, d_n) \in \mathbb{R}^n$. Despite being a finite dimensional control problem, the dependence on the solution of an infinite dimensional problem makes the derivation of optimality conditions harder than it seems. An example of a similar situation is exactly what is discussed in Chapter 3 of this thesis, which is based on [Cas20]. There the control variable is given by a dipole moment $p \in \mathbb{R}^3$ that provides excitation for a time harmonic eddy currents system.

A distributed boundary control problem may be also investigated in the future for the same state variational inequality: the literature concerning boundary control problems is rather rich and established [CGS93; CT94; LT91; LT89], but does not cover the electromagnetic framework yet. A first step in this direction would be to check whether suitable physical models featuring surface current densities (i.e., supported on the boundary) are available in the literature.

Chapter 3

Optimal control of the eddy current system with a dipole source

3.1 Introduction

There are several instances in applications where one might be interested in controlling a physical system by means of sources - controls - that have *small* support, i.e. occupy a *small* portion of the computational domain. Alternatively, it can be desirable to have knowledge of sub-domains where it is most efficient to put sensors or actuators with respect to some cost to be minimized.

Under the mathematical viewpoint, these tasks fall into the class of sparse optimal control problems or optimal control problems driven by pointwise sources, where the latter can be also seen as a subtopic of the former. Building upon the archetype paper of Stadler [Sta09], the theory of distributed (in the sense that they are a priori defined in the whole domain) sparse control problems allowed to consolidate the idea that the addition of a non-smooth L^1-control cost term in the objective functional entails sparsity properties of optimal solutions. For instance, it can be shown that if the control parameter appearing in front of the L^1-term is sufficiently large with respect to known data, then the optimal solution is even forced to be identically zero [Sta09, Lemma 3.1].

In recent times, the lack of reflexivity and compactness properties of the L^1-spaces have led to the study of optimal control problems in measure spaces like $\mathcal{M}(\Omega)$, the space of regular Borel measures, which exhibit better functional properties as well as similar sparsity features, in the sense of measures, of optimal solutions. Ultimately the point is that in addition to the functional-analytic advantages, one is sometimes able to infer *a posteriori* that optimal controls not only have *small* support, but are characterized by specials structures like

$$\boldsymbol{u} = \sum_{i=1}^{N} u_i \delta_{x_i}, \tag{3.1}$$

namely a linear combination of Dirac masses. Here u_i is, say, either a complex number or a time-dependent intensity $t \mapsto u_i(t)$. These are typical examples of singular elements in $\mathcal{M}(\Omega)$ (respectively in $L^2(I;\mathcal{M}(\Omega))$ in the time-dependent framework) that are usually of interest for modelling phenomena related to geology or acoustics: we refer to [Pie+20], where an inverse problem from point-wise measurements (state observations) is analyzed.

Sources (controls) of type (3.1) can be meaningful also for electromagnetic problems, where they represent electric dipoles located in the computational domain. We mention for example a work of Alonso Rodríguez et al. [ACV11] concerning inverse problems for the eddy current system, and some papers analyzing EEG models for the human brain [DL19; BF06]. Although the focus is not always towards an optimization problem, we stress the following fundamental difference: in the previous works the structural ansatz (3.1) is employed in the first place, while in [CCK13; CK19; KPV14; TVZ18] the authors consider general measure-valued controls and then possibly discuss if, and under which conditions, optimal solutions satisfy (3.1). Following [Cas20], in this chapter we stick to the first approach and study an optimal control problem for an $\boldsymbol{E}$-based formulation of the time-harmonic eddy current system, where the control variable is a single dipole source modelled through a Dirac delta located at some fixed point x_0. It is clear that the choice of a single dipole causes no loss of generality: since the problem is linear, extending of our results to the case (3.1) is straightforward. In the upcoming subsection we try to motivate why the measure-valued approach does not appear to be feasible in our context.

Despite the adopted simplifications, several mathematical difficulties are present, the most important being the fact that the state equation features a Maxwell type differential operator, with a Dirac distribution as right hand side. We propose an approach that seems new in the context of optimal control: the resolution of the differential problem is split into three steps, the first one being the determination of a fundamental solution to deal with the singularity at x_0 (this idea has been already used to tackle some inverse problems [Wol+07; ACV11]). After this, the specific structure of the eddy current problem leads to a state variable that is composed by two other terms: the sum of a vector field and a gradient, both defined as the solutions of suitable variational problems. The subsequent analysis inherits these features and thus two adjoint states, corresponding to two different *parts* of the state variable, need to be defined in order to apply the standard ajoint method and eventually derive first order optimality conditions. Additionally, the H^{-2} singularity produced by the fundamental solution calls for a modification of the quadratic tracking cost functional as it is not suitable for applications to approach a desired state in such a weak dual space: we choose to perform the optimization away from the point x_0 so that an L^2-norm can be employed, paying the price of neglecting the behaviour of the state near the point. We refer the reader to section Section 3.4 for further details and motivations. To the author's best knowledge the upcoming material, inspired from [Cas20], represents the first contribution towards the optimal control of electromagnetic fields in the presence of spike sources.

The structure of the chapter is as follows. In the upcoming Section 3.1.1 we briefly review some techniques to deal with PDEs featuring measure-valued sources; this serves as a motivation for the choice of working with a specific measure (i.e., the Dirac mass). Section 3.3 is dedicated to the study of the state equation, which is done by means of a splitting technique in combination with the employment of a fundamental solution. We also discuss (see Section 3.3.3) the problem in the presence of an alternative boundary condition, as well as more complicated geometries in Remark 3.3.5. Section 3.4 covers the discussion of the optimal control problem. We investigate in depth the dependence of the state variable on the control and introduce suitable adjoint equations to eventually derive first order conditions in Theorem 3.4.5.

3.1.1 On measure-valued sources in electromagnetism

As announced, the aim of this small section is to motivate why electromagnetic PDEs do not seem suitable state equations for measure-valued optimal control problems, i.e. PDE constrained optimization problems where optimal solutions are sought for in a space of measures. In other words, the upcoming considerations are intended to strengthen the choice of working with a fixed measure of the form (3.1). To this end, we shall briefly review some of the known basic techniques to deal with equations of type

$$\begin{cases} -\operatorname{div}(A\nabla y) = \mu & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

where Ω is a bounded Lipschitz domain of $\mathbb{R}^d$, A an elliptic matrix valued mapping and μ is a given regular Borel measure (that would play the role of control variable).

For what concerns the presentation we shall follow the classical $W^{1,p}$ theory in the scalar case and then point out what difficulties would arise in the $\boldsymbol{H}(\mathbf{curl})$ vector context. It will be apparent that suitable notions of weak solutions of (3.2) (at least the one in the sense of Stampacchia [MPS11, Sec. 2]) rely on a synergy between the concepts of duality and regularity, and the latter is well-known to be an issue for Maxwell's equation and especially for its eddy currents approximation [CDS03].

It is noteworthy that PDEs of the form (3.2) play an important role in the modern theory of optimal control: not only they represent the typical state equation for measure-valued optimal control problems, they also show up as adjoint equations in state-constrained optimization problems where it is useful to consider control-to-state mappings with values in the space of continuous functions. This naturally produces adjoint multipliers lying in measure spaces. The present material concerning $W^{1,p}$ theory for (3.2) draws inspiration from [MPS11].

To begin with, we make the following

Assumption 3.1.1.

- *$\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded Lipschitz domain.*
- *$A\colon \Omega \to \mathbb{R}^{3\times 3}$ has entries in $L^\infty(\Omega)$ and it is uniformly elliptic, that is there exists $c > 0$ such that*

$$A(x)z \cdot z \geq c|z|^2 \quad \text{a.e. } x \in \Omega,\ \forall z \in \mathbb{R}^d. \tag{3.3}$$

- *μ is a regular Borel measure, i.e. $\mu \in \mathcal{M}(\Omega)$.*
- *$p \in [2, +\infty)$ and $p > d$,*

and introduce the functional spaces that will use throughout the section. We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω, moreover

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$$

and

$$W_0^{1,p}(\Omega) := \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W^{1,p}}}, \quad W^{-1,p}(\Omega) := (W_0^{1,p'}(\Omega))^*$$

where $p' = \frac{p}{p-1}$ denotes the conjugate exponent corresponding to p. As a consequence of the restrictions on p in Assumption 3.1.1, $1 < p' < \frac{d}{d-1}$ and classical embeddings

$$E\colon W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega}), \quad E_0\colon W_0^{1,p}(\Omega) \hookrightarrow C_0(\overline{\Omega}) \tag{3.4}$$

are well-defined with dense ranges. Moreover, Riesz's representation theorem for measures provides us with the fundamental identification

$$\mathcal{M}(\Omega) \simeq (C_0(\overline{\Omega}))^*$$

and the corresponding dual pairing $\langle \mu, \eta \rangle_{(C_0(\overline{\Omega}))^* \times C_0(\overline{\Omega})} = \int_\Omega \eta \, d\mu(x)$.

Next we introduce the operator $\mathbf{\Theta}\colon W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ via

$$\langle \mathbf{\Theta} u, v \rangle_{W_0^{1,p'}} = \int_\Omega A \nabla u \cdot \nabla v \, dx \quad \forall (u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,p'}(\Omega), \tag{3.5}$$

whom with we now have all the ingredients to write a weak formulation for (3.2):

$$\begin{cases} \text{Find } y \in W_0^{1,p'}(\Omega) \quad \text{s.t.} \\ \langle \mathbf{\Theta} v, y \rangle_{W_0^{1,p'}} = \langle \mu, E_0 v \rangle_{C_0(\overline{\Omega})} \quad \forall v \in W_0^{1,p}(\Omega). \end{cases} \tag{3.6}$$

It is clear from the positive definiteness of A (3.3) that $\mathbf{\Theta}$ is at least injective for all $p \in [2, +\infty)$ and all $d \geq 1$; it is said to enjoy maximal regularity if it is an isomorphism between the corresponding space (for instance if $p = 2$ as a consequence of Lax-Milgram theorem). It turns out that the latter condition is in fact sufficient for (3.6) to be well-posed.

Theorem 3.1.2. *Let Assumption 3.1.1 hold and assume that $\mathbf{\Theta}$ is maximal regular, then (3.6) has a unique solution $y^\star \in W_0^{1,p'}(\Omega)$.*

A proof can be found in [MPS11]. We write one here as well to improve the understanding and the readability of the section.

Proof. We start by restating (3.6) in the equivalent dual form

$$\langle \mathbf{\Theta}^* y, v \rangle_{W_0^{1,p}} = \langle E_0^* \mu, v \rangle_{W_0^{1,p}} \quad \forall v \in W_0^{1,p}(\Omega), \tag{3.7}$$

where $E_0^* \colon C_0(\overline{\Omega})^* \simeq \mathcal{M}(\Omega) \hookrightarrow W^{-1,p'}(\Omega) = (W_0^{1,p}(\Omega))^*$ is the (continuous) dual emdedding of (3.4). Thanks to the maximal regularity of $\mathbf{\Theta}$ and the closed range theorem, $\mathbf{\Theta}^*$ is an isomorphism too; therefore $y^\star = (\mathbf{\Theta}^*)^{-1} E_0^* \mu$ is the unique solution of (3.7), and hence of (3.6). □

The importance of the interplay between duality and regularity is now more clear: the latter (in the sense of embedding into the space of continuous functions) is ultimately required to identify the measure μ with an element of $W^{-1,p'}$, while the former to have continuous invertibility of the adjoint, given the maximal regularity of $\mathbf{\Theta}$. In case of lack of maximal regularity, the proof of Theorem 3.1.2 cannot be carried out and one can either extend the space of test functions or restrict the space of solutions, leading to other (eventually equivalent) notions of weak solution. We refer to [MPS11] for a detailed discussion on this matter.

In the context of electromagnetic PDEs, regularity is what causes troubles as the counterpart of the embedding (3.4) is usually missing or *not natural*, making a formulation like (3.6) basically impossible to write. Indeed we have to work with the spaces of vector fields

$$\boldsymbol{W}^{0,p}(\mathbf{curl}, \Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) : \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p(\Omega)\}$$
$$W^{0,p}(\mathrm{div}, \Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^p(\Omega) : \mathrm{div}\, \boldsymbol{u} \in L^p(\Omega)\}$$

that provide control only on linear combinations of the first derivatives, which is not sufficient to guarantee continuity up to the boundary[1].

3.2 Preliminaries and geometrical assumptions

We shall now present our geometrical setting related to the eddy currents system, as well as the functional spaces required for our analysis. As we work with the eddy current system in the time-harmonic framework (the same is done in Chapter 4), we have to consider functional spaces of complex-valued vector fields. Given an open, bounded set $\mathcal{O} \subseteq \mathbb{R}^3$, let $L^2(\mathcal{O}), \boldsymbol{L}^2(\mathcal{O})$ respectively denote the space of all (equivalence classes of) $\mathbb{C}$-valued Lebesgue square-integrable functions and of $\mathbb{C}^3$-valued Lebesgue square-integrable vector fields.

[1] A sufficient condition is to simultaneously have $\boldsymbol{u}, \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p$, $\mathrm{div}\, \boldsymbol{u} \in L^p$ with $p > d$ and either $\boldsymbol{u} \times \boldsymbol{n}$ or $\boldsymbol{u} \cdot \boldsymbol{n}$ vanishing on the boundary; this implies $\boldsymbol{u} \in \boldsymbol{W}^{1,p}$ [AS13]. However these kind of spaces would not be suitable in practice as it is not natural to ask a priori for a global L^p control of the rotation and divergence at the same time. This is especially true for eddy currents models, where spaces featuring differential constraints that are imposed only in a portion of the domain are commonly used.

Let us further introduce the spaces

$$\begin{aligned} H^1(\mathcal{O}) &:= \{u \in L^2(\mathcal{O})\colon \nabla u \in \boldsymbol{L}^2(\mathcal{O})\} \\ \boldsymbol{H}(\mathbf{curl}, \mathcal{O}) &:= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\colon \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\} \\ H(\mathrm{div}, \mathcal{O}) &:= \{\boldsymbol{u} \in \boldsymbol{L}^2(\mathcal{O})\colon \mathrm{div}\, \boldsymbol{u} \in L^2(\mathcal{O}),\} \end{aligned} \tag{3.8}$$

which are endowed with their natural graph norms. Here the **curl**, ∇ and div operators are to be understood in the sense of distributions. Furthermore, let $\mathcal{D}(\mathcal{O})$ denote the space of infinitely differentiable $\mathbb{C}$-valued functions with compact support in $\mathcal{O}$, and let $\boldsymbol{\mathcal{D}}(\mathcal{O})$ be its vector counterpart.

The space $H_0^1(\mathcal{O})$ denotes closure of $\mathcal{D}(\mathcal{O})$ with respect to the $H^1(\mathcal{O})$-topology, similarly $\boldsymbol{H}_0(\mathbf{curl}, \mathcal{O})$ stands for the closure of $\boldsymbol{\mathcal{D}}(\mathcal{O})$ with respect to the $\boldsymbol{H}(\mathbf{curl}, \mathcal{O})$-topology. Analogously defined is the space $H_0(\mathrm{div}, \mathcal{O})$. With $H(\mathrm{div}{=}0, \mathcal{O})$ and $H_0(\mathrm{div}{=}0, \mathcal{O})$ we denote the kernels of the divergence in the respective spaces. We refer the reader to [RV10, Appendix A.1] and [Mon03; BCS02] for a detailed analysis of the trace spaces for $\boldsymbol{H}(\mathbf{curl}, \mathcal{O}), H(\mathrm{div}, \mathcal{O})$ and their characterization.

Assumption 3.2.1 (Geometry of the eddy currents model)**.** *The computational domain Ω is a bounded simply connected open set in $\mathbb{R}^3$ with Lipschitz boundary $\Gamma := \partial\Omega$. A non-empty open, connected subset $\Omega_C \subset \Omega$ denotes the conducting region and consequently $\Omega_I := \Omega \setminus \overline{\Omega_C}$ is the insulator, which is also assumed to be connected for simplicity; Ω_C is strictly contained in Ω in such a way that $\Gamma \cap \partial\Omega_C = \emptyset$ and it is assumed to be simply connected, implying that Ω_I is also simply connected. The set $\Gamma_C := \partial\Omega_I \cap \partial\Omega_C$ is the interface between the conductor and the insulator. We finally set $\Gamma_I := \partial\Omega_I = \Gamma \cup \Gamma_C$ and denote by $\boldsymbol{n}, \boldsymbol{n}_C$ and $\boldsymbol{n}_I$ respectively the unit outward normal vectors on Γ, Γ_C and Γ_I. For the sake of clarity we henceforth use the notation $\boldsymbol{H}_I := \boldsymbol{H}|_{\Omega_I}, \sigma_C := \sigma|_{\Omega_C}$ (and similar for other fields) to explicitly underline to which subdomain a certain scalar function, vector or matrix valued mapping is restricted.*

We will refer to the latter assumption for the rest of this chapter, if not otherwise specified. We close this section by stating the basic physical assumption for our analysis, namely the properties of the involved material parameters:

Assumption 3.2.2 (Material parameters)**.**

- *The magnetic reluctivity $\nu\colon \Omega \to \mathbb{R}^{3\times 3}$ (which is the inverse of the magnetic permeability μ), the electric conductivity $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}$ and the electric permittivity $\epsilon_I\colon \Omega_I \to \mathbb{R}^{3\times 3}$ all have entries in L^∞ of the corresponding spaces and are symmetric (valued). The conductivity identically vanishes in the insulator, that is $\sigma_I = 0$ almost everywhere in Ω_I. The matrices ν, σ_C and ϵ_I are uniformly positive definite in Ω, Ω_C and Ω_I respectively, in the sense of (3.3).*

- *Let $x_0 \in \Omega_C$. The parameters μ, σ satisfy a local homogeneity condition: there exists an open ball $B_r(x_0)$ with $B_r(x_0) \subset \Omega_C$ and two real constants $\nu_0 > 0, \sigma_0 > 0$ for which*

$$\nu(x) = \nu_0\, \mathrm{Id} \quad \text{and} \quad \sigma(x) = \sigma_0\, \mathrm{Id} \qquad \forall x \in B_r(x_0). \tag{3.9}$$

We stress that the latter assumption is not much restrictive in most instances because the location of the point (dipole) source x_0 is free to choose, and it seems reasonable to opt for a point that does not lie on the interface separating regions characterized by different materials properties.

3.3 Well-posedness of the state equation

This section is devoted to the mathematical analysis of the state equation. As anticipated, it is an $\boldsymbol{E}$-based formulation for the eddy current system with a dipole source in the form of a Dirac mass; it reads:

$$\begin{cases} \mathbf{curl}(\nu\,\mathbf{curl}\,\boldsymbol{E}) + i\omega\sigma\boldsymbol{E} = -i\omega p\delta_{x_0} & \text{in } \Omega \\ \mathrm{div}(\epsilon_I \boldsymbol{E}_I) = 0 & \text{in } \Omega_I \\ (\nu\,\mathbf{curl}\,\boldsymbol{E}_I) \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma \\ \epsilon_I \boldsymbol{E}_I \cdot \boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases} \tag{3.10}$$

where $p \in \mathbb{R}^3$, $\omega > 0$ and i denotes the imaginary unit. The point $x_0 \in \Omega_C$ is the same as in Assumption 3.2.2 so that (3.9) holds, and δ_{x_0} stands for the Dirac distribution centered at x_0. The third and fourth equations in (3.10) correspond to the choice of the so called magnetic boundary condition, reinterpreted after eliminating the magnetic field from the eddy current system [RV10, Section 1.3]. In Section 3.3.3 we discuss what adaptations are needed in order to handle another relevant boundary condition.

We point out that (3.10) is somehow already a simplified model, as the geometry provided by Assumption 3.2.1 entails that a couple of equations related to the topology of Ω_I can be a priori dropped, see [RV10, p. 22]. Prior to the control analysis, we need to address the well-posedness of problem (3.10). The first existence and uniqueness result for (3.10) can be found in [ACV11] in the context of inverse problems; in this sense, Theorem 3.3.8 is not original. However, here we are rather focused towards the analysis of a corresponding optimal control problem and thus we need to keep track of the dependence of the solution on the control p, see Remark 3.3.3 and (3.14). At the same time, we need to introduce and study some operators that will be involved in the adjoint calculus. None of these issues are addressed in [ACV11], making this section a new contribution to the study of (3.10) and its solution mapping $p \mapsto \boldsymbol{E}$.

3.3.1 The fundamental solution and the initial splitting

Thanks to the homogeneity assumption (3.9), we are able to employ a fundamental solution-based approach: ν and σ being constant in a neighbourhood of x_0 entails that locally we are dealing with a $\mathbf{curl}\,\mathbf{curl} - \mathrm{Id}$ type operator, whose fundamental solution is known.

Lemma 3.3.1. *Let $p \in \mathbb{R}^3$, $z = \sqrt{-i\omega\nu_0^{-1}\sigma_0}$ with $\mathrm{Re}\, z < 0$ and $q = -i\omega p\nu_0^{-1}$. Let*

$$\Phi_{x_0}(x) = \frac{\exp(iz|x - x_0|)}{4\pi|x - x_0|} \tag{3.11}$$

be the fundamental solution, up to translation in x_0, of the Helmholtz operator $-\Delta - z^2\,\mathrm{Id}$. Then the field

$$G = G_{x_0}(x) = q\Phi_{x_0}(x) + \frac{1}{z^2}(q\cdot\nabla)\nabla\Phi_{x_0}(x), \tag{3.12}$$

satisfies

$$\mathbf{curl}\,\mathbf{curl}\,G - z^2 G = q\delta_{x_0} \tag{3.13}$$

in the sense of distributions.

Proof. This result is adapted from one of Ammari et al. [BAF02]. As $\mathbf{curl}\,\nabla \equiv \mathbf{0}$ and

$$G = q\Phi_{x_0} + \frac{1}{z^2}(q\cdot\nabla)\nabla\Phi_{x_0} = q\Phi_{x_0} + \frac{1}{z^2}\nabla\,\mathrm{div}(q\Phi_{x_0}),$$

we readily have

$$\mathbf{curl}\,\mathbf{curl}\,G = \mathbf{curl}\,\mathbf{curl}(q\Phi_{x_0}) = -\boldsymbol{\Delta}(q\Phi_{x_0}) + \nabla\,\mathrm{div}(q\Phi_{x_0})$$

and in turn

$$\mathbf{curl}\,\mathbf{curl}\,G - z^2 G = -\boldsymbol{\Delta}(q\Phi_{x_0}) - z^2 q\Phi_{x_0}.$$

Note that we used the identity $\mathbf{curl}\,\mathbf{curl} = -\boldsymbol{\Delta} + \nabla\,\mathrm{div}$. The conclusion follows by noticing that

$$-\Delta\Phi_{x_0} = \delta_{x_0} + z^2\Phi_{x_0} \implies -\boldsymbol{\Delta}(q\Phi_{x_0}) = q\delta_{x_0} + z^2 q\Phi_{x_0}.$$

□

Let us observe that since $q \in \mathbb{C}^3$, we have

$$\begin{aligned} G &= q\Phi_{x_0} + \frac{1}{z^2}(q\cdot\nabla)\nabla\Phi \\ &= \mathrm{Id}(q\Phi_{x_0}) + \frac{1}{z^2}(\nabla^2\Phi_{x_0})q \\ &= [\mathrm{Id}\,\Phi_{x_0} + \nabla^2\Phi_{x_0}]q = Nq, \end{aligned} \tag{3.14}$$

where $N = N_{x_0}(x)$ is a symmetric matrix with entries in $H^{-2}(\Omega)$, as it inherits the type of singularity of $\Phi_{x_0}(\cdot)$ at $x = x_0$. However both Φ_{x_0} and G are smooth and bounded away from x_0.

In view of (3.9), we observe that formally

$$\nu_0\,\mathbf{curl}\,\mathbf{curl}\,E(x) + i\omega\sigma_0 E(x) = -i\omega p\delta_{x_0}(x), \quad x \in B_r(x_0),$$

which leads us to seek for a split solution of the form

$$E = G + M, \tag{3.15}$$

where $\boldsymbol{M}$ should carry information from outside the ball $B_r(x_0)$, through a suitably modified source. More precisely, let $\boldsymbol{J}\colon \Omega \to \mathbb{C}^3$ be defined via

$$\boldsymbol{J} = \begin{cases} \boldsymbol{0} & \text{in } B_r(x_0) \\ -\,\mathbf{curl}(\nu\,\mathbf{curl}\,\boldsymbol{G}) - i\omega\sigma\boldsymbol{G} & \text{in } B_r^c(x_0), \end{cases} \tag{3.16}$$

where $B^c(x_0)$ denotes the set $\Omega \setminus \overline{B_r(x_0)}$. It is apparent that $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ thanks to the regularity of $\boldsymbol{G}$ in $B_r^c(x_0)$.

We deduce that $\boldsymbol{M}$ should satisfy the strong problem

$$\begin{cases} \mathbf{curl}(\nu\,\mathbf{curl}\,\boldsymbol{M}) + i\omega\sigma\boldsymbol{M} = \boldsymbol{J} & \text{in } \Omega \\ \operatorname{div}(\epsilon_I\boldsymbol{M}) = -\operatorname{div}(\epsilon_I\boldsymbol{G}) & \text{in } \Omega_I \\ (\nu\,\mathbf{curl}\,\boldsymbol{M}) \times \boldsymbol{n} = -(\nu\,\mathbf{curl}\,\boldsymbol{G}) \times \boldsymbol{n} & \text{on } \Gamma \\ \epsilon_I\boldsymbol{M}\cdot\boldsymbol{n} = -\epsilon_I\boldsymbol{G}\cdot\boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{3.17}$$

Notice that (3.17) is a boundary value problem where $\boldsymbol{G}$ appears as a datum, but only in subsets of the domain where it is smooth. First we perform an homogenization which aims to modify the divergence constraint as well as the normal component condition in (3.17).

Proposition 3.3.2. *Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ be defined through (3.16). Let*

$$W := \{w \in H^1(\Omega_I) : w = 0 \text{ on } \Gamma_C\}, \tag{3.18}$$

$\eta_I \in W$ be the unique solution of

$$b(\eta_I, \xi) := \int_{\Omega_I} \epsilon_I \nabla\eta_I \cdot \nabla\overline{\xi}\,\mathrm{d}x = -\int_{\Omega_I} \epsilon_I \boldsymbol{G}\cdot\nabla\overline{\xi}\,\mathrm{d}x \qquad \forall \xi \in W, \tag{3.19}$$

and

$$\eta := \begin{cases} \eta_I & \text{in } \Omega_I \\ 0 & \text{in } \Omega_C \end{cases} \in H^1(\Omega). \tag{3.20}$$

If $\widehat{\boldsymbol{M}}$ is a solution of

$$\begin{cases} \mathbf{curl}(\nu\,\mathbf{curl}\,\widehat{\boldsymbol{M}}) + i\omega\sigma\widehat{\boldsymbol{M}} = \boldsymbol{J} & \text{in } \Omega \\ \operatorname{div}(\epsilon_I\widehat{\boldsymbol{M}}) = 0 & \text{in } \Omega_I \\ (\nu\,\mathbf{curl}\,\widehat{\boldsymbol{M}}) \times \boldsymbol{n} = -(\nu\,\mathbf{curl}\,\boldsymbol{G}) \times \boldsymbol{n} & \text{on } \Gamma \\ \epsilon_I\widehat{\boldsymbol{M}}\cdot\boldsymbol{n} = 0 & \text{on } \Gamma, \end{cases} \tag{3.21}$$

then the field

$$\boldsymbol{M} = \widehat{\boldsymbol{M}} + \nabla\eta \tag{3.22}$$

solves (3.17).

Proof. First, (3.19) is uniquely solvable by the Lax-Milgram theorem since $\boldsymbol{G}|_{\Omega_I} \in \boldsymbol{L}^2(\Omega_I)$ (recall that $x_0 \in \Omega_C$ and $\boldsymbol{G}$ is smooth away from such point). We notice that (3.19) can be interpreted in strong form as

$$\begin{cases} -\operatorname{div}(\epsilon_I \nabla \eta_I) = \operatorname{div}(\epsilon_I \boldsymbol{G}) & \text{in } \Omega_I \\ \eta_I = 0 & \text{on } \Gamma_C \\ \epsilon_I \nabla \eta_I \cdot \boldsymbol{n} = -\epsilon_I \boldsymbol{G} \cdot \boldsymbol{n} & \text{on } \Gamma. \end{cases} \tag{3.23}$$

The conclusion readily follows from (3.23) by noticing that $\mathbf{curl}(\nu\,\mathbf{curl}\,\nabla\eta) + i\omega\sigma\nabla\eta \equiv \mathbf{0}$ (here we use $\sigma|_{\Omega_I} = 0$ and (3.20)). □

Remark 3.3.3 (Dependence on p for η)**.** *In view of* (3.14)*, we have*

$$\boldsymbol{G} = Nq = (-i\omega\nu_0^{-1})Np =: Ap, \tag{3.24}$$

where A is a matrix valued mapping (with complex entries) depending on the fundamental solution Φ and on the point x_0. It follows by the linearity of problem (3.19) *that both*

$$\mathbb{R}^3 \ni p \mapsto \eta(p) \in H^1(\Omega) \quad \text{and} \quad \mathbb{R}^3 \ni p \mapsto (\nabla\eta)(p) \in \boldsymbol{L}^2(\Omega)$$

are real-linear mappings.

3.3.2 A weak formulation

A consequence of the preparatory splitting introduced in the previous subsection, in particular Proposition 3.3.2 and (3.15), is that once (3.21) is solved then

$$\boldsymbol{E} = \boldsymbol{G} + \widehat{\boldsymbol{M}} + \nabla\eta \tag{3.25}$$

provides us with a solution of the state equation (3.10). Let us therefore set a up a suitable variational formulation for (3.21); to this end we introduce the linear space

$$\boldsymbol{V} := \{\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \colon \operatorname{div}(\epsilon_I \boldsymbol{v}_I) = 0 \quad \text{in } \Omega_I, \quad \epsilon_I \boldsymbol{v} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma\}, \tag{3.26}$$

which turns out to be a Hilbert space if endowed with the standard $\boldsymbol{H}(\mathbf{curl})$ inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_V := \int_\Omega \boldsymbol{u} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_\Omega \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\, \mathrm{d}x. \tag{3.27}$$

As is not straightforward to see how a correct weak formulation for (3.21) looks like, we infer it by multiplying the first equation in (3.21) with $\overline{\boldsymbol{v}} \in \boldsymbol{V}$ and then integrating on Ω. We obtain:

$$\begin{aligned} \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x &= \int_\Omega \nu\,\mathbf{curl}\,\widehat{\boldsymbol{M}} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\, \mathrm{d}x - \int_\Gamma [(\nu\,\mathbf{curl}\,\widehat{\boldsymbol{M}}) \times \boldsymbol{n}] \cdot \overline{\boldsymbol{v}}\, \mathrm{d}S + i\omega \int_{\Omega_C} \sigma\widehat{\boldsymbol{M}} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\ &= \int_\Omega \nu\,\mathbf{curl}\,\widehat{\boldsymbol{M}} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\, \mathrm{d}x + i\omega \int_{\Omega_C} \sigma\widehat{\boldsymbol{M}} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_\Gamma [\nu\,\mathbf{curl}\,\boldsymbol{G} \times \boldsymbol{n}] \cdot \overline{\boldsymbol{v}}\, \mathrm{d}S. \end{aligned} \tag{3.28}$$

Next we expand the left hand side of the above equation:

$$
\begin{aligned}
\int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x \underbrace{=}_{(3.16)} &\int_{B^c_{x_0}} (-\operatorname{\mathbf{curl}}(\nu\operatorname{\mathbf{curl}}\boldsymbol{G}) - i\omega\sigma\boldsymbol{G})\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x\\
&= \int_\Omega [-\operatorname{\mathbf{curl}}((\nu-\nu_0)\operatorname{\mathbf{curl}}\boldsymbol{G})\cdot\overline{\boldsymbol{v}} - i\omega(\sigma-\sigma_0)\boldsymbol{G}\cdot\overline{\boldsymbol{v}}]\,\mathrm{d}x\\
&= \int_\Omega [-(\nu-\nu_0)\operatorname{\mathbf{curl}}\boldsymbol{G}\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{v}} - i\omega(\sigma-\sigma_0)\boldsymbol{G}\cdot\overline{\boldsymbol{v}}]\,\mathrm{d}x\\
&- \int_\Gamma \boldsymbol{n}\times((\nu-\nu_0)\operatorname{\mathbf{curl}}\boldsymbol{G})\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S\\
&= \int_\Omega [-(\nu-\nu_0)\operatorname{\mathbf{curl}}\boldsymbol{G}\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{v}} - i\omega(\sigma-\sigma_0)\boldsymbol{G}\cdot\overline{\boldsymbol{v}}]\,\mathrm{d}x\\
&+ \int_\Gamma (\boldsymbol{n}\times\nu_0\operatorname{\mathbf{curl}}\boldsymbol{G})\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S + \int_\Gamma (\nu\operatorname{\mathbf{curl}}\boldsymbol{G}\times\boldsymbol{n})\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S,
\end{aligned}
\tag{3.29}
$$

where abusing the notation ν_0 has been used in place of $\nu_0\,\mathrm{Id}_{\mathbb{R}^3}$ for the sake of readability.

Combining (3.29), (3.28) and the third condition in (3.21), we are led to the following formulation:

$$
\begin{cases}
\text{Find } \widehat{\boldsymbol{M}}\in\boldsymbol{V} \quad \text{s.t.}\\
a^+(\widehat{\boldsymbol{M}},\boldsymbol{v}) = \displaystyle\int_{B^c_{x_0}} [-(\nu-\nu_0)\operatorname{\mathbf{curl}}\boldsymbol{G}_p\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{v}} - i\omega(\sigma-\sigma_0)\boldsymbol{G}_p\cdot\overline{\boldsymbol{v}}]\,\mathrm{d}x\\
+ \displaystyle\int_\Gamma (\boldsymbol{n}\times\nu_0\operatorname{\mathbf{curl}}\boldsymbol{G}_p)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S \quad \forall\boldsymbol{v}\in\boldsymbol{V},
\end{cases}
\tag{3.30}
$$

where

$$
a^+(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \nu\operatorname{\mathbf{curl}}\boldsymbol{u}\cdot\operatorname{\mathbf{curl}}\overline{\boldsymbol{v}}\,\mathrm{d}x + i\omega\int_{\Omega_C}\sigma\boldsymbol{u}\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x, \quad (\boldsymbol{u},\boldsymbol{v})\in\boldsymbol{V}\times\boldsymbol{V}.
$$

The presence of the subscript in G_p in (3.30) is intended to underline the dependence of $\boldsymbol{G}$ on the control $p\in\mathbb{R}^3$, see also (3.24) and (3.14).

Addressing the well-posedness of (3.30) boils down to investigate the coercivity of the bilinear form $a^+\colon \boldsymbol{V}\times\boldsymbol{V}\to\mathbb{C}$, which is nontrivial. Indeed since vector fields in $\boldsymbol{V}$ are divergence-free just in Ω_I, no *pure* Poincaré-Friedrichs inequality (i.e., a bound of the $\boldsymbol{L}^2$-norm by means of the sole $\boldsymbol{L}^2$-norm of the **curl**) can be expected to hold in the whole Ω; at the same time the lower order term provides information only in the conductor. A possible workaround is to exploit the trace theorem on the common interface Γ_C together with the following Poincaré-Friedrichs type inequality in Ω_I, which can be found either in [RV10, Lemma 2.1], or in [FG97].

Lemma 3.3.4. *There is a constant $C>0$ such that*

$$
\begin{aligned}
\|\boldsymbol{w}_I\|_{L^2(\Omega_I)} \le C(&\|\operatorname{\mathbf{curl}}\boldsymbol{w}_I\|_{L^2(\Omega_I)} + \|\operatorname{div}(\epsilon_I\boldsymbol{w}_I)\|_{L^2(\Omega_I)}\\
&+ \|\boldsymbol{w}_I\times\boldsymbol{n}_I\|_{-\frac{1}{2},\operatorname{div}_\tau,\Gamma_C} + \|\epsilon_I\boldsymbol{w}_I\cdot\boldsymbol{n}\|_{-\frac{1}{2},\Gamma})
\end{aligned}
\tag{3.31}
$$

for all $\boldsymbol{w}_I \in \boldsymbol{H}(\mathbf{curl}, \Omega_I) \cap \boldsymbol{H}_{\epsilon_I}(\mathrm{div}, \Omega_I)$ *with* $\boldsymbol{w}_I \perp^{\epsilon_I} \mathcal{H}_{\epsilon_I}(\Gamma_C, \Gamma; \Omega_I)$. *Here*

$$\begin{aligned} \mathcal{H}_{\epsilon_I}(\Gamma_C, \Gamma; \Omega_I) = \{\boldsymbol{q}_I \in \boldsymbol{L}^2(\Omega_I) \colon \ \mathbf{curl}\, \boldsymbol{q}_I = \mathbf{0}, \quad \mathrm{div}(\epsilon_I \boldsymbol{q}_I) = 0, \\ \boldsymbol{q}_I \times \boldsymbol{n}_I = \mathbf{0} \text{ on } \Gamma_C, \quad \epsilon_I \boldsymbol{q}_I \cdot \boldsymbol{n}_I = 0 \text{ on } \Gamma\}, \end{aligned} \tag{3.32}$$

and $\perp^{\epsilon_I}$ *denotes orthogonality with respect to the* ϵ_I*-weighted* $\boldsymbol{L}^2(\Omega_I)$ *inner product, that is* $(\epsilon_I \cdot, \cdot)_{\boldsymbol{L}^2(\Omega_I)}$.

We shall briefly explain why the above lemma applies to functions in $\boldsymbol{V}$. Indeed $\boldsymbol{V} \subset \boldsymbol{H}(\mathbf{curl}; \Omega_I) \cap \boldsymbol{H}_{\epsilon_I}(\mathrm{div}; \Omega_I)$ in view of the divergence-free constraint. Moreover, it is known [RV10, Appx. A.4] that the space $\mathcal{H}_{\epsilon_I}(\Gamma_C, \Gamma; \Omega_I)$ has dimension equal to $p_{\Gamma_C} + n_\Gamma$, where the former denotes the number of connected components of Γ_C minus one and the latter the number of Γ-independent non-bounding cycles[2] in Ω_I. Both these numbers vanish under Assumption 3.2.1, in particular due to Γ_C being connected and Ω being simply connected[3].

Remark 3.3.5 (Γ_C not connected)**.** *For the sake of completeness, we briefly discuss how to proceed when* Γ_C *is not assumed to be connected*[4]*; in this case* $p_{\Gamma_C} \geq 1$*, then it is known [RV10, Appendix A.4] that* $\mathcal{H}_{\epsilon_I}(\Gamma_C, \Gamma; \Omega_I)$ *is spanned by* $\{\nabla w_i\}_{i=1 \dots p_{\Gamma_C}}$, $w_i \in H^1(\Omega_I)$ *being the solution of the mixed problem:*

$$\begin{cases} \mathrm{div}(\epsilon_I \nabla w_i) = 0 & \text{in } \Omega_I \\ \epsilon_I \nabla w_i \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \\ w_i = 0 & \text{on } \Gamma_C \setminus \Gamma_i \\ w_i = 1 & \text{on } \Gamma_i, \end{cases}$$

where $(\Gamma_i)_{i=1 \dots p_{\Gamma_C}}$ *denotes the i-th connected component. Fix any* $j \in \{1 \dots p_{\Gamma_C}\}$*; for each* $\boldsymbol{v} \in \boldsymbol{V}$*, we have:*

$$\begin{aligned} \int_{\Omega_I} \epsilon_I \boldsymbol{v}_I \cdot \nabla w_j \,\mathrm{d}x &= -\int_{\Omega_I} w_j \,\mathrm{div}(\epsilon_I \boldsymbol{v}_I) \,\mathrm{d}x + \int_{\partial\Omega_I} w_j \epsilon_I \boldsymbol{v}_I \cdot \boldsymbol{n} \,\mathrm{d}S \\ &= \int_{\Gamma} w_j \epsilon_I \boldsymbol{v}_I \cdot \boldsymbol{n} \,\mathrm{d}S + \sum_{i=1}^{p_{\Gamma_C}} \int_{\Gamma_i} w_j \epsilon_I \boldsymbol{v}_I \cdot \boldsymbol{n} \,\mathrm{d}S \\ &= \int_{\Gamma_j} \epsilon_I \boldsymbol{v}_I \cdot \boldsymbol{n} \,\mathrm{d}S, \end{aligned} \tag{3.33}$$

[2] More precisely, we say that a family $\mathcal{C}$ of disjoint cycles of Ω_I is formed by Γ-independent, non-bounding cycles if, for each non trivial subfamily $\mathcal{C}^* \subset \mathcal{C}$, the union of the cycles in $\mathcal{C}^*$ cannot be equal to $S \setminus \gamma$, where S denotes a surface contained in Ω_I and γ a union of cycles contained in Γ.

[3] The fact that the computational domain Ω is simply connected is sufficient to make n_Γ equal to zero. However, this may also happen when the topology of Ω is non-trivial. For a detailed discussion and examples we refer to [RV10, Section 1.4]

[4] Since Ω_I is assumed to be connected, this can only happen if Ω_C itself is a non-connected conductor, that is $\Omega_C = \coprod_i \Omega_C^{(i)}$ with $\Omega_C^{(i)}$ connected for each i. The presence of more conductors in a device is a situation that often arises in engineering applications. We refer the reader to [RV10] for an extensive discussion on the geometrical configurations that can arise in the context of the eddy currents problem.

since $\epsilon_I \boldsymbol{v}_I \cdot \boldsymbol{n} = 0$ on the external boundary Γ. Hence we see that it suffices to require the functions of $\boldsymbol{V}$ to satisfy the additional constraints

$$\int_{\Gamma_i} \epsilon_I \boldsymbol{v}_I \cdot \boldsymbol{n} = 0 \,\mathrm{d}S \qquad \forall i = 1 \ldots p_{\Gamma_C}$$

concerning the fluxes through each connected component of the interface Γ_C.

With this adjustment, $\boldsymbol{V}$ is yet again a Hilbert space equipped with the inner product (3.27) and its elements satisfy the orthogonality hypothesis of Lemma 3.3.4. This especially implies that Lemma 3.3.6 and Lemma 3.3.7 continue to hold true.

Lemma 3.3.6. *Let Assumption 3.2.2 hold. The sesquilinear forms*

$$\begin{aligned} a^+(\boldsymbol{w},\boldsymbol{v}) &= \int_\Omega \nu \,\mathbf{curl}\,\boldsymbol{w} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}} \,\mathrm{d}x + i\omega \int_{\Omega_C} \sigma \boldsymbol{w} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x, \\ a^-(\boldsymbol{w},\boldsymbol{v}) &= \int_\Omega \nu \,\mathbf{curl}\,\boldsymbol{w} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}} \,\mathrm{d}x - i\omega \int_{\Omega_C} \sigma \boldsymbol{w} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \end{aligned} \tag{3.34}$$

are continuous and (strongly) coercive in $\boldsymbol{V} \times \boldsymbol{V}$.

Proof. Continuity is obvious. For what concerns coercivity, as

$$a^+(\boldsymbol{w},\boldsymbol{v}) = \overline{a^-(\boldsymbol{v},\boldsymbol{w})} \quad \forall \boldsymbol{w},\boldsymbol{v} \in \boldsymbol{V} \tag{3.35}$$

is sufficies to prove the statement for a^+. For $\boldsymbol{v} \in \boldsymbol{V}$, we have by the positive definiteness of ν,σ:

$$\begin{aligned} |a^+(\boldsymbol{v},\boldsymbol{v})|^2 &= \left(\int_\Omega \nu \,\mathbf{curl}\,\boldsymbol{v} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}} \,\mathrm{d}x\right)^2 + \omega^2 \left(\int_{\Omega_C} \sigma \boldsymbol{v} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x\right)^2 \\ &\geq C(\|\,\mathbf{curl}\,\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\boldsymbol{v}_C\|^2_{L^2(\Omega_C)})^2. \end{aligned} \tag{3.36}$$

At the same time, Lemma 3.3.4 together with the continuity of the tangential trace yields

$$\begin{aligned} \|\boldsymbol{v}\|^2_{L^2(\Omega_I)} &\leq C(\|\,\mathbf{curl}\,\boldsymbol{v}_I\|_{L^2(\Omega_I)} + \|\boldsymbol{v}_I \times \boldsymbol{n}_I\|_{-\frac{1}{2},\mathrm{div}_\tau,\Gamma_C})^2 \\ &= C(\|\,\mathbf{curl}\,\boldsymbol{v}_I\|_{L^2(\Omega_I)} + \|\boldsymbol{v}_C \times \boldsymbol{n}_C\|_{-\frac{1}{2},\mathrm{div}_\tau,\Gamma_C})^2 \\ &\leq C(\|\,\mathbf{curl}\,\boldsymbol{v}_I\|^2_{L^2(\Omega_I)} + \|\boldsymbol{v}_C\|^2_{L^2(\Omega_C)} + \|\,\mathbf{curl}\,\boldsymbol{v}_C\|^2_{L^2(\Omega_C)}). \end{aligned} \tag{3.37}$$

The conclusion follows combining (3.36) and (3.37). □

Lemma 3.3.7. *Let Assumption 3.2.2 hold. Problem (3.30) has a unique solution $\widehat{\boldsymbol{M}} \in \boldsymbol{V}$.*

Proof. The mapping $\boldsymbol{\Phi}\colon \boldsymbol{V} \to \mathbb{C}$,

$$\begin{aligned} \boldsymbol{v} \mapsto \boldsymbol{\Phi}(\boldsymbol{v}) = &\int_{B^c_{x_0}} [-(\nu - \nu_0)\,\mathbf{curl}\,\boldsymbol{G}_p \cdot \mathbf{curl}\,\overline{\boldsymbol{v}} - i\omega(\sigma - \sigma_0)\boldsymbol{G}_p \cdot \overline{\boldsymbol{v}}] \,\mathrm{d}x \\ &+ \int_\Gamma (\boldsymbol{n} \times \nu_0 \,\mathbf{curl}\,\boldsymbol{G}_p) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}S, \end{aligned} \tag{3.38}$$

appearing on the right hand side of (3.30) is an antilinear form thanks to the continuity of the tangential trace in $\boldsymbol{H}(\mathbf{curl})$ [BCS02]. Therefore we can apply the Lax-Milgram theorem in force of Lemma 3.3.6. □

We are now ready to summarize the discussion on the solvability of the state equation (3.10).

Theorem 3.3.8. *Suppose that Assumption 3.2.2 holds. For every $p \in \mathbb{R}^3$, system (3.10) admits a solution $\boldsymbol{E} = \boldsymbol{E}_p \in \boldsymbol{H}^{-2}(\Omega)$ which can written in the form*

$$\boldsymbol{E} = \boldsymbol{G} + \widehat{\boldsymbol{M}} + \nabla\eta, \tag{3.39}$$

where $\widehat{\boldsymbol{M}} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ is the solution of (3.30), $\eta \in H^1(\Omega)$ solves (3.19) and $\boldsymbol{G}$ is the fundamental solution defined in (3.12). It is unique among all solutions $\hat{\boldsymbol{E}}$ for which $(\hat{\boldsymbol{E}} - \boldsymbol{G}) \in \boldsymbol{H}(\mathbf{curl}, \Omega)$.

Proof. Existence is a consequence of Proposition 3.3.2 and Lemma 3.3.7. Uniqueness follows by the one of problem (3.17) in $\boldsymbol{V} \hookrightarrow \boldsymbol{H}(\mathbf{curl}, \Omega)$. Indeed if $\boldsymbol{M}_1, \boldsymbol{M}_2 \in \boldsymbol{V}$ are both solutions of (3.17), then $\boldsymbol{M}_0 = \boldsymbol{M}_1 - \boldsymbol{M}_2$ solves the homogeneous system

$$\begin{cases} \mathbf{curl}(\nu\,\mathbf{curl}\,\boldsymbol{M}_0) + i\omega\sigma\boldsymbol{M}_0 = \mathbf{0} & \text{in} \quad \Omega \\ \operatorname{div}(\epsilon_I \boldsymbol{M}_0) = 0 & \text{in} \quad \Omega_I \\ (\nu\,\mathbf{curl}\,\boldsymbol{M}_0) \times \boldsymbol{n} = \mathbf{0} & \text{on} \quad \Gamma \\ \epsilon_I \boldsymbol{M}_0 \cdot \boldsymbol{n} = 0 & \text{on} \quad \Gamma. \end{cases} \tag{3.40}$$

Multiplying the first equation by $\overline{\boldsymbol{M}}_0 \in \boldsymbol{V}$ and integrating in Ω we obtain

$$\begin{aligned} 0 &= \int_\Omega \mathbf{curl}(\nu\,\mathbf{curl}\,\boldsymbol{M}_0) \cdot \overline{\boldsymbol{M}}_0 \,\mathrm{d}x + i\omega \int_\Omega \sigma\boldsymbol{M}_0 \cdot \overline{\boldsymbol{M}}_0 \,\mathrm{d}x \\ &= \int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{M}_0 \cdot \mathbf{curl}\,\overline{\boldsymbol{M}}_0 \,\mathrm{d}x + \int_\Gamma \underbrace{(\nu\,\mathbf{curl}\,\boldsymbol{M}_0 \times \boldsymbol{n})}_{=\mathbf{0}} \cdot \overline{\boldsymbol{M}}_0 \,\mathrm{d}S + i\omega \int_\Omega \sigma\boldsymbol{M}_0 \cdot \overline{\boldsymbol{M}}_0 \,\mathrm{d}x \\ &\geq C\|\boldsymbol{M}_0\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)}, \end{aligned}$$

where we used Lemma 3.3.6 for the last inequality. The claim follows. □

In view of Theorem 3.3.8, the solution mapping $\boldsymbol{S}\colon \mathbb{R}^3 \to \boldsymbol{H}^{-2}(\Omega)$ acting as

$$p \mapsto \boldsymbol{S}p = \boldsymbol{E}(p) \quad \text{s.t. } (\boldsymbol{E}(p) - \boldsymbol{G}_p) \in \boldsymbol{H}(\mathbf{curl}, \Omega) \tag{3.41}$$

is well defined and real-linear. In fact each term on the right hand side of (3.39) depends linearly on p: (3.14) and Remark 3.3.3 respectively prove the assertion for $\boldsymbol{G} = \boldsymbol{G}_p$ and $\nabla\eta = \nabla\eta_p$; the same is true for the mapping[5] $\boldsymbol{\Phi}$ defined in (3.38) and thus for $\widehat{\boldsymbol{M}}$.

[5] This is to be understood as $\boldsymbol{\Phi}_{\alpha p_1 + \beta p_2} = \alpha\boldsymbol{\Phi}_{p_1} + \beta\boldsymbol{\Phi}_{p_2}$ for all $\alpha, \beta \in \mathbb{R}$ and all $p_1, p_2 \in \mathbb{R}^3$, in the sense of linear mappings.

3.3.3 The electric boundary condition

As the boundary conditions appearing in (3.10) are not the only possible ones for an $\boldsymbol{E}$-based formulation of the eddy current problem, let us briefly explain how to modify our arguments if the so called electric boundary condition

$$\boldsymbol{E} \times \boldsymbol{n} = \boldsymbol{0} \qquad \text{on } \Gamma \tag{3.42}$$

is considered instead.

Again we look for a solution in the form $\boldsymbol{E} = \boldsymbol{G} + \widehat{\boldsymbol{M}} + \nabla \eta$, with the exception that now $\eta|_{\partial\Omega_I} = 0$ instead of the Neumann condition on Γ in (3.23) and we are left with the problem

$$\begin{cases} \mathbf{curl}(\nu\, \mathbf{curl}\, \widehat{\boldsymbol{M}}) + i\omega\sigma\widehat{\boldsymbol{M}} = \boldsymbol{J} & \text{in } \Omega \\ \operatorname{div}(\epsilon_I \widehat{\boldsymbol{M}}) = 0 & \text{in } \Omega_I \\ \widehat{\boldsymbol{M}} \times \boldsymbol{n} = -\boldsymbol{G} \times \boldsymbol{n} & \text{on } \Gamma \end{cases} \tag{3.43}$$

in place of (3.21).

We set

$$\boldsymbol{V}_0 := \{\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega) \colon \operatorname{div}(\epsilon_I \boldsymbol{u}_I) = 0 \text{ in } \Omega_I, \quad \boldsymbol{u}_I \times \boldsymbol{n} = \boldsymbol{0} \text{ on } \Gamma\}$$

and $\boldsymbol{\rho} = -\boldsymbol{G} \times \boldsymbol{n}$. Since the bilinear form a^+ is coercive in $\boldsymbol{V}_0$ (Lemma 3.3.6 applies to functions of $\boldsymbol{V}_0$ too), the resolution procedure becomes standard if we are able to find a suitable lifting of $\boldsymbol{\rho}$, that is a field $\boldsymbol{P} \in \boldsymbol{V}_0$ such that $\boldsymbol{P} \times \boldsymbol{n} = \boldsymbol{\rho}$ on Γ.

Let us consider the following $\mathbf{curl} - \operatorname{div}$ system for $\boldsymbol{P}_I \in \boldsymbol{H}(\mathbf{curl}; \Omega_I)$:

$$\begin{cases} \mathbf{curl}\, \boldsymbol{P}_I = \boldsymbol{\Psi} & \text{in } \Omega_I \\ \operatorname{div}(\epsilon_I \boldsymbol{P}_I) = 0 & \text{in } \Omega_I \\ \boldsymbol{P}_I \times \boldsymbol{n} = \boldsymbol{\rho} & \text{on } \Gamma \\ \boldsymbol{P}_I \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma_C \\ \displaystyle\int_\Gamma \boldsymbol{P}_I \cdot \boldsymbol{n}\, \mathrm{d}S = 0, \end{cases} \tag{3.44}$$

where $\boldsymbol{\Psi} = \nabla\phi$ and $\phi \in H^1(\Omega_I)$ satisfies

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega_I \\ \nabla\phi \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_C \\ \nabla\phi \cdot \boldsymbol{n} = \operatorname{div}_\tau \boldsymbol{\rho} & \text{on } \Gamma \\ \displaystyle\int_{\Omega_I} \phi\, \mathrm{d}x = 0 \end{cases} \tag{3.45}$$

In this way, we see that all compatibility conditions for the solvability of the $\mathbf{curl} - \operatorname{div}$ system [RBV19, Chap. 1, Sec. 2.1] are satisfied. In particular, they are also sufficient for existence and uniqueness.

Indeed the Neumann problem (3.45) is well-posed since $\int_\Gamma \mathrm{div}_\tau \boldsymbol{\rho} = -\int_\Gamma \boldsymbol{\rho} \cdot (\nabla_\tau 1) = 0$, while for (3.44) we have $\mathrm{div}\, \boldsymbol{\Psi} = \mathrm{div}\, \nabla\phi = 0$ in Ω_I and $\mathrm{div}_\tau \boldsymbol{\rho} = \nabla\phi \cdot \boldsymbol{n} = \boldsymbol{\Psi} \cdot \boldsymbol{n}$ on Γ by construction. Moreover the space of harmonic fields

$$\mathcal{H}(m;\Omega_I) := \{\boldsymbol{\rho} \in \boldsymbol{L}^2(\Omega_I) \colon \mathbf{curl}\, \boldsymbol{\rho} = \mathbf{0} \quad \text{in } \Omega_I,\ \mathrm{div}\, \boldsymbol{\rho} = 0 \quad \text{in } \Omega_I,\ \boldsymbol{\rho} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega_I\}$$

is trivial since Ω_I is simply connected [RV10, Appendix A.4]. Hence (3.44) has a unique solution and we can finally define

$$\boldsymbol{P} := \begin{cases} \boldsymbol{P}_I & \text{in } \Omega_I \\ \mathbf{0} & \text{in } \Omega_C \end{cases} \in \boldsymbol{V}_0,$$

which is the desired lifting.

3.4 The optimal control problem

Let us now discuss the optimal control problem; our analysis will be driven by the following task: suppose we want to approach two given desired electromagnetic field $\boldsymbol{E}_\mathrm{d}, \boldsymbol{H}_\mathrm{d} \in \boldsymbol{L}^2(\Omega)$ controlling the dipole intensity $p \in \mathbb{R}^3$ (its location has already been fixed in x_0). Since the solution $\boldsymbol{E}$ to (3.10) does not belong to $\boldsymbol{L}^2(\Omega)$ due to the singularity at $x = x_0$ inherited by the fundamental solution $\boldsymbol{G}$, we shall optimize the distance between the solution and the desired fields with respect to $\boldsymbol{L}^2(B^c_{x_0})$, where we recall that $B^c_{x_0} = \Omega \setminus \overline{B_r(x_0)}$. Although the eddy current state equation is driven by dipole source concentrated at x_0, the optimization problem will disregard the behaviour of the state variable close to the point x_0.

This may seem unreasonable at first sight, however, our resolution approach guarantees a priori the presence of a singularity of the same kind of $\boldsymbol{G}$ at $x = x_0$, therefore we focus the attention on the the state variable away from that point. In other words, we shall not be interested in a specific shape at the actuators, we aim at given fields in the complement of the actuators instead. This kind of approach is often seen in optimal control problems for PDEs where a control domain Ω_c and a disjoint state observation domain Ω_o are considered, see e.g. [CK12] or [PV13]. Roughly speaking, we are doing the same with the choices $\Omega_\mathrm{o} = B^c_{x_0}$ and $\Omega_\mathrm{c} = \{x_0\}$.

Let us introduce the set of admissible controls (dipoles)

$$\mathcal{P}_{ad} := \{p \in \mathbb{R}^3 : |(p)_i| \le p_{max},\ i = 1\ldots3\},$$

$0 < p_{max}$ being a bound for the maximal component-wise dipole intensity. Summing up, we are led to the following optimal control problem:

$$\begin{cases} \displaystyle\min_{\substack{p \in \mathcal{P}_{ad} \\ \boldsymbol{E} \in \boldsymbol{H}^{-2}(\Omega)}} F(\boldsymbol{E}, p) = \frac{\lambda_E}{2} \|\boldsymbol{E} - \boldsymbol{E}_\mathrm{d}\|^2_{\boldsymbol{L}^2(B^c_{x_0})} + \frac{\lambda_H}{2} \|\nu\, \mathbf{curl}\, \boldsymbol{E} - \boldsymbol{H}_\mathrm{d}\|^2_{\boldsymbol{L}^2(B^c_{x_0})} + \frac{\lambda}{2} |p|^2_{\mathbb{R}^3} \\ \text{subject to (3.10).} \end{cases} \tag{3.46}$$

The fact that $\boldsymbol{G}$ is smooth far from x_0 together with the assumption that $\boldsymbol{E}_{\mathrm{d}}, \boldsymbol{H}_{\mathrm{d}} \in \boldsymbol{L}^2(\Omega)$ ensure that both $(\boldsymbol{E} - \boldsymbol{E}_{\mathrm{d}})$ and $(\nu\, \mathbf{curl}\, \boldsymbol{E} - \boldsymbol{H}_{\mathrm{d}})$ lie in $\boldsymbol{L}^2(B^c_{x_0})$, making F well-defined on $\boldsymbol{H}^{-2}(\Omega) \times \mathcal{P}_{ad}$.

Before proceeding, we define the following reduced cost functional by composition with the control-to-state mapping (3.41):

$$
\begin{aligned}
f(p) &:= \frac{\lambda_E}{2}\|\boldsymbol{S}p - \boldsymbol{E}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(B^c_{x_0})} + \frac{\lambda_H}{2}\|\nu\, \mathbf{curl}(\boldsymbol{S}p) - \boldsymbol{H}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(B^c_{x_0})} + \frac{\lambda}{2}|p|^2_{\mathbb{R}^3} \\
&= \frac{\lambda_E}{2}\|\boldsymbol{E}_p - \boldsymbol{E}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(B^c_{x_0})} + \frac{\lambda_H}{2}\|\nu\, \mathbf{curl}(\boldsymbol{E}_p) - \boldsymbol{H}_{\mathrm{d}}\|^2_{\boldsymbol{L}^2(B^c_{x_0})} + \frac{\lambda}{2}|p|^2_{\mathbb{R}^3}.
\end{aligned}
\tag{3.47}
$$

Proposition 3.4.1. *Let Assumption 3.2.2 hold. There exists an optimal solution $p^\star \in \mathbb{R}^3$ to (3.46) satisfying*

$$p^\star \in \operatorname*{arg\,min}_{p \in \mathcal{P}_{ad}} f(p).$$

If $\lambda > 0$, it is unique.

Proof. If $\lambda > 0$, thanks to the (real) linearity of $\boldsymbol{S}$ and the quadratic tracking structure, we obtain at once that f is weakly lower semi-continuous and strictly convex. This together with the fact that $\mathcal{P}_{ad}$ is compact entails by standard arguments [Trö10, Section 2.5] the existence and uniqueness of an optimal solution $p^\star \in \mathcal{P}_{ad}$. If $\lambda = 0$, we still have existence but uniqueness is no longer guaranteed. □

3.4.1 First order conditions

By theorem Theorem 3.3.8, we know that to each control $p \in \mathcal{P}_{ad}$ there corresponds a unique state

$$\boldsymbol{E}_p = \widehat{\boldsymbol{M}}_p + \nabla \eta_p + \boldsymbol{G}_p.$$

Prior to deriving and discussing necessary and sufficient conditions for optimality, we need to further clarify the dependence of $\boldsymbol{E}$ on p, which in turn depends on the one of $\widehat{\boldsymbol{M}}, \nabla\eta$ (for $\boldsymbol{G}$ we can refer to (3.14)).

To begin with, we shall investigate in details how the right hand side of (3.19) depends linearly (with respect to real numbers) on the control p: this will be pivotal for deriving optimality conditions. We subsequently perform a similar computation for the right hand side of (3.30).

For what concerns the variational problem for η, we have:

$$
\begin{aligned}
-\int_{\Omega_I} \epsilon_I \boldsymbol{G}_p \cdot \nabla\overline{\xi}\, \mathrm{d}x &= -\int_{\Omega_I} \epsilon_I A p \cdot \nabla\overline{\xi}\, \mathrm{d}x \\
&= -\int_{\Omega_I} p \cdot A^{\mathrm{T}}(\epsilon_I \nabla\overline{\xi})\, \mathrm{d}x = p \cdot \left(\int_{\Omega_I} -A^{\mathrm{T}}(\epsilon_I \nabla\overline{\xi})\, \mathrm{d}x\right),
\end{aligned}
\tag{3.48}
$$

and consequently we can introduce the mapping $\widetilde{\mathcal{G}}\colon H^1(\Omega_I) \to \mathbb{R}^3$ via

$$\widetilde{\mathcal{G}}(\xi) = \int_{\Omega_I} -A^{\mathrm{T}}(\epsilon_I \nabla\overline{\xi})\, \mathrm{d}x, \qquad \xi \in H^1(\Omega_I). \tag{3.49}$$

We stress one more time that the matrix A has entries in $L^\infty(B^c_{x_0})$ (see (3.14) and (3.24)), making (3.49) well defined.

Next we work on the right hand side of the variational problem for $\widehat{\boldsymbol{M}}$ (3.30). It reads

$$
\begin{aligned}
&\int_{B^c_{x_0}} [-(\nu-\nu_0)\,\mathbf{curl}\,\boldsymbol{G}_p\cdot\mathbf{curl}\,\overline{\boldsymbol{v}} - i\omega(\sigma-\sigma_0)\boldsymbol{G}_p\cdot\overline{\boldsymbol{v}}]\,\mathrm{d}x \\
&\quad + \int_\Gamma (\boldsymbol{n}\times\nu_0\,\mathbf{curl}\,\boldsymbol{G}_p)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S \\
&\quad = \int_{B^c_{x_0}} [-(\nu-\nu_0)\,\mathbf{curl}(Ap)\cdot\mathbf{curl}\,\overline{\boldsymbol{v}} - i\omega(\sigma-\sigma_0)Ap\cdot\overline{\boldsymbol{v}}]\,\mathrm{d}x \\
&\quad + \int_\Gamma (\boldsymbol{n}\times\nu_0\,\mathbf{curl}(Ap))\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S \\
&\quad = \int_{B^c_{x_0}} [-(\nu-\nu_0)\sum_{j=1}^3(\mathbf{curl}\,A^{(j)}p_j\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}) - i\omega p\cdot A^{\mathrm{T}}(\sigma-\sigma_0)\overline{\boldsymbol{v}}]\,\mathrm{d}x \\
&\quad + \int_\Gamma \boldsymbol{n}\times\nu_0\sum_{j=1}^3\mathbf{curl}\,A^{(j)}p_j\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S \\
&\quad = \sum_{j=1}^3 p_j\left(-\int_{B^c_{x_0}}[(\nu-\nu_0)\,\mathbf{curl}\,A^{(j)}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x - \int_{B^c_{x_0}} i\omega\cdot A^{(j)}[(\sigma-\sigma_0)\overline{\boldsymbol{v}}]\,\mathrm{d}x\right. \\
&\quad \left. + \int_\Gamma [\boldsymbol{n}\times\nu_0\,\mathbf{curl}\,A^{(j)}]\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S\right),
\end{aligned}
$$

where $A^{(j)}$ denotes the j-th column of the matrix A. We can thus define the mapping $\mathcal{G}\colon \boldsymbol{H}(\mathbf{curl},\Omega)\to\mathbb{R}^3$ component-wise via

$$
\begin{aligned}
(\mathcal{G}(\boldsymbol{v}))_j = -\int_{B^c_{x_0}}[(\nu-\nu_0)\,\mathbf{curl}\,A^{(j)}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x - \int_{B^c_{x_0}} i\omega p\cdot A^{(j)}[(\sigma-\sigma_0)\overline{\boldsymbol{v}}]\,\mathrm{d}x \\
+ \int_\Gamma [\boldsymbol{n}\times\nu_0\,\mathbf{curl}\,A^{(j)}]\cdot\overline{\boldsymbol{v}}\,\mathrm{d}S.
\end{aligned}
\tag{3.50}
$$

Now we can exploit the operators $\mathcal{G},\widetilde{\mathcal{G}}$ to better characterize $\eta_p,\widehat{\boldsymbol{M}}_p$: given $p\in\mathbb{R}^3$, problems (3.19) and (3.30) can be respectively restated as

$$
\begin{cases}
\text{Find } \eta_p\in W \quad \text{s.t.} \\
b(\eta_p,\xi) = \widetilde{\mathcal{G}}(\xi)\cdot p \quad \forall\xi\in W,
\end{cases}
\tag{3.51}
$$

and

$$
\begin{cases}
\text{Find } \widehat{\boldsymbol{M}}_p\in\boldsymbol{V} \quad \text{s.t.} \\
a^+(\widehat{\boldsymbol{M}}_p,\boldsymbol{v}) = \mathcal{G}(\boldsymbol{v})\cdot p \quad \forall\boldsymbol{v}\in\boldsymbol{V}.
\end{cases}
\tag{3.52}
$$

Lemma 3.4.2. *Let Assumption 3.2.2 hold. The cost functional $f\colon\mathbb{R}^3\to\mathbb{R}$ defined in (3.47) is Fréchet differentiable. Its directional (Gâteaux) derivative at $\hat{p}\in\mathbb{R}^3$ in the direction $p\in\mathbb{R}^3$ is* *given by*

$$f'(\hat{p})p$$
$$= \operatorname{Re}\left\{\lambda_E \int_{B_{x_0}^c} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \overline{\boldsymbol{E}}_p \,\mathrm{d}x + \lambda_H \int_{B_{x_0}^c} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu \,\mathbf{curl}\, \overline{\boldsymbol{E}}_p \,\mathrm{d}x\right\} + \lambda \hat{p} \cdot p, \tag{3.53}$$

where $\boldsymbol{E}_p, \boldsymbol{E}_{\hat{p}}$ are the states associated with $p, \hat{p}$ respectively.

Proof. To prove the Fréchet differentiability, we define the mappings

$$\boldsymbol{S}^i : \mathbb{R}^3 \to \boldsymbol{H}(\mathbf{curl}, B_{x_0}^c), \quad \boldsymbol{S}^r : \mathbb{R}^3 \to \boldsymbol{H}(\mathbf{curl}, B_{x_0}^c)$$

via $\boldsymbol{S}^i(p) = \operatorname{Im}(\boldsymbol{S}p)$ and $\boldsymbol{S}^r(p) = \operatorname{Re}(\boldsymbol{S}p)$. Note that even if the notation was not changed, $\boldsymbol{H}(\mathbf{curl}, B_{x_0}^c)$ here denotes the $\mathbb{R}^3$-valued version; the same can be said for $\boldsymbol{L}^2$ in the equation below.

Splitting the real and imaginary parts, we can rewrite (3.47) as the sum of two purely real contributions:

$$f(p) = \frac{\lambda_E}{2}\|\boldsymbol{S}^r p - \operatorname{Re}(\boldsymbol{E}_{\mathrm{d}})\|^2_{\boldsymbol{L}^2(B_{x_0}^c)} + \frac{\lambda_H}{2}\|\nu \,\mathbf{curl}(\boldsymbol{S}^r p) - \operatorname{Re}(\boldsymbol{H}_{\mathrm{d}})\|^2_{\boldsymbol{L}^2(B_{x_0}^c)}$$
$$+ \frac{\lambda_E}{2}\|\boldsymbol{S}^i p - \operatorname{Im}(\boldsymbol{E}_{\mathrm{d}})\|^2_{\boldsymbol{L}^2(B_{x_0}^c)} + \frac{\lambda_H}{2}\|\nu \,\mathbf{curl}(\boldsymbol{S}^i p) - \operatorname{Im}(\boldsymbol{H}_{\mathrm{d}})\|^2_{\boldsymbol{L}^2(B_{x_0}^c)} + \frac{\lambda}{2}|p|^2_{\mathbb{R}^3}.$$

As both $\boldsymbol{S}^i$ and $\boldsymbol{S}^r$ are (real) linear and bounded, the chain rule gives that f is Fréchet differentiable between the corresponding spaces.

To conclude, we compute the directional (Gateaux) derivative at each $\hat{p} \in \mathbb{R}^3$:

$$\frac{f(\hat{p} + tp) - f(\hat{p})}{t}$$
$$= \lambda_E t \int_{B_{x_0}^c} |\boldsymbol{E}_p|^2 \,\mathrm{d}x + \lambda_E \operatorname{Re}\int_{B_{x_0}^c} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \overline{\boldsymbol{E}}_p \,\mathrm{d}x + t\lambda_H \int_{B_{x_0}^c} |\nu \,\mathbf{curl}\, \boldsymbol{E}_p|^2 \,\mathrm{d}x$$
$$+ \lambda_H \operatorname{Re}\left\{\int_{B_{x_0}^c} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu \,\mathbf{curl}\, \overline{\boldsymbol{E}}_p \,\mathrm{d}x\right\} + t\lambda|p|^2 + \lambda \hat{p} \cdot p,$$

therefore

$$\lim_{t \to 0^+} \frac{f(\hat{p} + tp) - f(\hat{p})}{t}$$
$$= \operatorname{Re}\left\{\lambda_E \operatorname{Re}\int_{B_{x_0}^c} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \overline{\boldsymbol{E}}_p \,\mathrm{d}x + \lambda_H \int_{B_{x_0}^c} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu \,\mathbf{curl}\, \overline{\boldsymbol{E}}_p \,\mathrm{d}x\right\} + \lambda \hat{p} \cdot p$$

□

As previously announced, we shall define two adjoint states, a vector one and a scalar one, which respectively correspond to $\widehat{\boldsymbol{M}}$ and η in (3.25).

Definition 3.4.3. *Let $\hat{p} \in \mathbb{R}^3$ be a given control with associated state $\boldsymbol{E} = \boldsymbol{E}_{\hat{p}}$. The problem to find $(\boldsymbol{Q}, \psi) \in \boldsymbol{V} \times W$ such that*

$$\begin{cases} a^-(\boldsymbol{Q}, \boldsymbol{v}) = \lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x + \lambda_H \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu \,\mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \\ b(\psi, \xi) = \lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \nabla \overline{\xi} \,\mathrm{d}x \quad \forall \xi \in W \end{cases} \tag{3.54}$$

is called adjoint equation of the control problem (3.46). Note that $b(\cdot, \cdot)$ is the Hermitian form appearing in the weak formulation for η (3.19) and $a^-(\cdot, \cdot)$ is the conjugate of the sesquilinear form $a^+(\cdot, \cdot)$. It is defined in (3.34) and it appears in the weak formulation for $\widehat{\boldsymbol{M}}$ (3.30).

Theorem 3.4.4. *Let Assumption 3.2.2 be satisfied. For every given control $\hat{p} \in \mathcal{P}_{ad}$, the adjoint system (3.54) has a unique solution $(\boldsymbol{Q}_{\hat{p}}, \psi_{\hat{p}}) = (\hat{\boldsymbol{Q}}, \hat{\psi}) \in \boldsymbol{V} \times W$.*

Proof. In view of Lemma 3.3.6, the sesquilinear forms appearing on the left hand side are coercive in the corresponding spaces. Additionally, the right hand sides are bounded anti-linear forms on the same spaces. The claim follows by the Lax-Milgram theorem. □

Let $p, \hat{p} \in \mathcal{P}_{ad}$. Testing the weak formulations (3.54) with respectively $\boldsymbol{Q}_{p-\hat{p}} \in \boldsymbol{V}$ and $\eta_{p-\hat{p}} \in W$ and summing the two equations, we get

$$\begin{aligned} a^-(\hat{\boldsymbol{Q}}, \boldsymbol{Q}_{p-\hat{p}}) &+ b(\hat{\psi}, \eta_{p-\hat{p}}) \\ &= \lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \overline{\boldsymbol{Q}}_{p-\hat{p}} \,\mathrm{d}x + \lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot \nabla \overline{\eta}_{p-\hat{p}} \,\mathrm{d}x \\ &\qquad + \lambda_H \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu \,\mathbf{curl}\, \overline{\boldsymbol{Q}}_{p-\hat{p}} \,\mathrm{d}x \\ &= \lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot (\overline{\boldsymbol{Q}}_{p-\hat{p}} + \nabla \overline{\eta}_{p-\hat{p}}) \,\mathrm{d}x \\ &\qquad + \lambda_H \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu (\mathbf{curl}\, \overline{\boldsymbol{E}}_{p-\hat{p}} - \mathbf{curl}\, \overline{A}(p - \hat{p})) \,\mathrm{d}x \\ &= \lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_{\mathrm{d}}) \cdot [\overline{\boldsymbol{E}}_{p-\hat{p}} - \overline{A}(p - \hat{p})] \,\mathrm{d}x \\ &\qquad + \lambda_H \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_{\mathrm{d}}) \cdot \nu (\mathbf{curl}\, \overline{\boldsymbol{E}}_{p-\hat{p}} - \mathbf{curl}\, \overline{A}(p - \hat{p})) \,\mathrm{d}x. \end{aligned} \tag{3.55}$$

On the other hand, the sesquilinear forms a^+, a^- satisfy (3.35) while $b(\cdot, \cdot)$ is Hermitian, so that rearranging the terms in (3.55) produces:

$$\begin{aligned}
&\lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d}) \cdot \overline{\boldsymbol{E}}_{p-\hat{p}} \,\mathrm{d}x + \lambda_H \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_\mathrm{d}) \cdot \nu \,\mathbf{curl}\, \overline{\boldsymbol{E}}_{p-\hat{p}} \,\mathrm{d}x \\
&= \overline{[\mathcal{G}(\hat{\boldsymbol{Q}}) + \widetilde{\mathcal{G}}(\hat{\psi})]} \cdot (p - \hat{p}) + \lambda_E \int_{B^c_{x_0}} [(\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d}) \cdot \overline{A}(p - \hat{p})] \,\mathrm{d}x \\
&+ \lambda_H \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_\mathrm{d}) \cdot \nu \,\mathbf{curl}\, \overline{A}(p - \hat{p}) \,\mathrm{d}x,
\end{aligned} \tag{3.56}$$

where we also used (3.51) and (3.52). The above expression is not yet completely satisfying as the free control p still appears implicitly in the right hand side of (3.56), which in turn shows up in the expression of the directional derivative (3.53). Nevertheless, we can still make use of the adjoint states to overcome this problem. Indeed we have:

$$\begin{aligned}
\int_{B^c_{x_0}} [(\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d}) \cdot \overline{A}(p - \hat{p})] \,\mathrm{d}x &= \int_{B^c_{x_0}} \overline{A}^\mathrm{T} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d}) \cdot (p - \hat{p}) \,\mathrm{d}x \\
&= \left(\int_{B^c_{x_0}} \overline{A}^\mathrm{T} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d}) \,\mathrm{d}x \right) \cdot (p - \hat{p}) \\
&= \sum_{i=1}^{3} (p - \hat{p})_i \int_{B^c_{x_0}} \sum_{j=1}^{n} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d})_j (\overline{A}^\mathrm{T})_{ij} \,\mathrm{d}x \\
&= \sum_{i=1}^{3} (p - \hat{p})_i \int_{B^c_{x_0}} (\boldsymbol{E}_{\hat{p}} - \boldsymbol{E}_\mathrm{d}) \cdot \overline{A}^{(i)} \,\mathrm{d}x,
\end{aligned} \tag{3.57}$$

where $\overline{A}^{(i)}$ denotes the i-th column of the matrix $\overline{A}$, the conjugate of A. Similarly, for the last term in (3.56) we can write

$$\begin{aligned}
&\int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_\mathrm{d}) \cdot \nu \,\mathbf{curl}\, \overline{A}(p - \hat{p}) \,\mathrm{d}x \\
&\qquad = \int_{B^c_{x_0}} \left((\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_\mathrm{d}) \cdot \nu \sum_{i=1}^{3} (p - \hat{p})_i \,\mathbf{curl}\, \overline{A}^{(i)} \right) \mathrm{d}x \\
&\qquad = \sum_{i=1}^{3} (p - \hat{p})_i \int_{B^c_{x_0}} (\nu \,\mathbf{curl}\, \boldsymbol{E}_{\hat{p}} - \boldsymbol{H}_\mathrm{d}) \cdot \nu \,\mathbf{curl}\, \overline{A}^{(i)} \,\mathrm{d}x.
\end{aligned} \tag{3.58}$$

In the first equality in (3.58), we used the fact that

$$\mathbf{curl}(A\mathbf{q}) = \sum_{k=1}^{3} q_k \,\mathbf{curl}\, A^{(k)},$$

where $\mathbf{q}$ is a fixed vector of $\mathbb{R}^3$ and $A^{(k)}$ denotes the k-th column of the matrix $A = A(x)$. In fact, using the Levi-Civita symbol, the left hand side can be rewritten as:

$$\mathbf{curl}(A\mathbf{q}) = \partial_i (A_{jl} q_l) \epsilon_{ijk} \mathbf{e}_k = [q_l \partial_i A_{jl} + \partial_i q_l A_{jl}] \epsilon_{ijk} \mathbf{e}_k = q_l \partial_i A_{jl} \epsilon_{ijk} \mathbf{e}_k.$$

The right hand side is instead equal to

$$\sum_{l=1}^{3} q_l \,\mathbf{curl}\, A^{(l)} = q_l \,\mathbf{curl}\, A^{(l)} = q_l \partial_i A_j^{(l)} \epsilon_{ijk} \mathbf{e}_k,$$

on the other hand, $A_j^{(l)}$ is the *j-th* component of the column vector $A^{(l)}$, namely A_{jl}.

As final step towards optimality conditions, we shall suitably modify the vector field $A^{(i)}$, whose components $A_j^{(i)}$ are given by

$$A_j^{(i)} = -i\omega \nu_0^{-1} [\Phi_{x_0} \delta_{ij} + \partial_i \partial_j \Phi_{x_0}],$$

in order to get an element of $\boldsymbol{V}$. As a result, we will obtain a feasible test function for (3.54). To this end we first consider an $\boldsymbol{H}(\mathbf{curl}, \Omega)$-extension $\mathcal{A}^{(i)}$ of $A^{(i)}$, namely $\mathcal{A}^{(i)} \in \boldsymbol{H}(\mathbf{curl}, \Omega)$ and $\mathcal{A}^{(i)}|_{B_{x_0}^c} = A^{(i)}|_{B_{x_0}^c}$. Subsequently, for each $j = 1, \ldots 3$, let $u_j \in H^1(\Omega_I)$ be the solution of the following problem:

$$\begin{cases} \operatorname{div}(\epsilon_I \nabla u_j) = \operatorname{div}(\epsilon_I A^{(j)}) & \text{in } \Omega_I \\ \epsilon_I \nabla u_j \cdot \boldsymbol{n} = \epsilon_I A^{(j)} \cdot \boldsymbol{n} & \text{on } \Gamma \\ u_j = 0 & \text{on } \Gamma_C, \end{cases} \tag{3.59}$$

and set

$$\widetilde{u}_j := \begin{cases} u_j & \text{in } \Omega_I \\ 0 & \text{in } \Omega_C. \end{cases}$$

Then by construction

$$\mathcal{A}^{(j)} - \nabla \widetilde{u}_j \in \boldsymbol{V}$$

for each $j = 1, \ldots 3$, so that $\mathcal{A}^{(j)} - \nabla \widetilde{u}_j$ is an admissible test function for the first equation in (3.54).

We now have all the tools to derive first order conditions for (3.46).

Theorem 3.4.5. *Suppose that Assumption 3.2.2 holds. Let $p^\star \in \mathcal{P}_{ad}$ be an optimal control for (3.46) and let $(\boldsymbol{Q}^\star, \psi^\star) \in (\boldsymbol{V} \times W)$ be the adjoint state characterized by (3.54). Moreover, we define*

$$\boldsymbol{a}^-(\boldsymbol{Q}^\star, \boldsymbol{\mathcal{A}}) := \begin{pmatrix} a^-(\boldsymbol{Q}^\star, \mathcal{A}^{(1)} - \nabla \widetilde{u}_1) \\ a^-(\boldsymbol{Q}^\star, \mathcal{A}^{(2)} - \nabla \widetilde{u}_2) \\ a^-(\boldsymbol{Q}^\star, \mathcal{A}^{(3)} - \nabla \widetilde{u}_3) \end{pmatrix}$$

and

$$\boldsymbol{b}(\psi^\star, \widetilde{u}) := \begin{pmatrix} b(\psi^\star, \widetilde{u}_1) \\ b(\psi^\star, \widetilde{u}_2) \\ b(\psi^\star, \widetilde{u}_3) \end{pmatrix}.$$

where for each $j \in \{1,2,3\}$, $\mathcal{A}^{(j)}$ is an $\boldsymbol{H}(\mathbf{curl},\Omega)$ extension of $A^{(j)} \in \boldsymbol{H}(\mathbf{curl}, B^c_{x_0})$ and $\widetilde{u}_j$ is defined through (3.59). The the following inequality holds true:

$$\operatorname{Re}\left(\overline{\mathcal{G}(\boldsymbol{Q}^\star)+\widetilde{\mathcal{G}}(\psi^\star)} + \boldsymbol{a}^-(\boldsymbol{Q}^\star,\boldsymbol{\mathcal{A}}) + \boldsymbol{b}(\psi^\star,\widetilde{u}) + \lambda p^\star\right)\cdot(p-p^\star) \geq 0 \quad \forall p \in \mathcal{P}_{ad}, \tag{3.60}$$

where the operators $\mathcal{G},\widetilde{\mathcal{G}}$ are defined in (3.49), (3.50). Conversely, if (3.60) holds for some $p^\star$ and $\lambda > 0$, then $p^\star$ is the optimal solution of (3.46).

Proof. It is well known that for an optimal control $p^\star$, the inequality

$$f'(p^\star)(p-p^\star) \geq 0 \quad \forall p \in \mathcal{P}_{ad} \tag{3.61}$$

holds. If $\lambda > 0$, the strict convexity of the objective functional implies that such variational inequality is both necessary and sufficient. It is enough to show that (3.61) is actually equivalent to (3.60).

In force of (3.53), we already know that the derivative of the cost functional reads:

$$\begin{aligned} f'(p^\star)(p-p^\star) = \operatorname{Re}&\left\{\lambda_E \int_{B^c_{x_0}} (\boldsymbol{E}_{p^\star} - \boldsymbol{E}_{\mathrm{d}})\cdot \overline{\boldsymbol{E}}_{p-p^\star}\,\mathrm{d}x\right\} \\ &+ \operatorname{Re}\left\{\lambda_H \int_{B^c_{x_0}} (\nu\,\mathbf{curl}\,\boldsymbol{E}_{p^\star} - \boldsymbol{H}_{\mathrm{d}})\cdot \nu\,\mathbf{curl}\,\overline{\boldsymbol{E}}_{p-p^\star}\,\mathrm{d}x\right\} \\ &+ \lambda p^\star\cdot(p-p^\star). \end{aligned} \tag{3.62}$$

In view of, (3.56), (3.57) and (3.58), we see that the first two terms in the above expression are equal to (disregarding the real part operator upfront):

$$\begin{aligned} &[\overline{\mathcal{G}(\boldsymbol{Q}^\star)+\widetilde{\mathcal{G}}(\psi^\star)}]\cdot(p-p^\star) + \sum_{i=1}^{3}(p-p^\star)_i\left(\lambda_E\int_{B^c_{x_0}}(\boldsymbol{E}_{p^\star}-\boldsymbol{E}_{\mathrm{d}})\cdot\overline{A}^{(i)}\,\mathrm{d}x\right) \\ &+\sum_{i=1}^{3}(p-p^\star)_i\left(\nu_H\int_{B^c_{x_0}}(\nu\,\mathbf{curl}\,\boldsymbol{E}_{p^\star}-\boldsymbol{H}_{\mathrm{d}})\cdot\nu\,\mathbf{curl}\,\overline{A}^{(i)}\,\mathrm{d}x\right). \end{aligned} \tag{3.63}$$

On the other hand, for each $i \in \{1,2,3\}$ we have by (3.54)

$$\begin{aligned} &a^-(\boldsymbol{Q}^\star, \mathcal{A}^{(i)} - \nabla\widetilde{u}_i) \\ &= \lambda_E\int_{B^c_{x_0}}(\boldsymbol{E}_{p^\star}-\boldsymbol{E}_{\mathrm{d}})\cdot(\overline{\mathcal{A}^{(i)}-\nabla\widetilde{u}_i})\,\mathrm{d}x + \lambda_H\int_{B^c_{x_0}}(\nu\,\mathbf{curl}\,\boldsymbol{E}_{p^\star}-\boldsymbol{H}_{\mathrm{d}})\cdot\nu\,\mathbf{curl}\,\overline{\mathcal{A}}^{(i)}\,\mathrm{d}x \\ &= \lambda_E\int_{B^c_{x_0}}(\boldsymbol{E}_{p^\star}-\boldsymbol{E}_{\mathrm{d}})\cdot\overline{A}^{(i)}\,\mathrm{d}x + \lambda_H\int_{B^c_{x_0}}(\nu\,\mathbf{curl}\,\boldsymbol{E}_{p^\star}-\boldsymbol{H}_{\mathrm{d}})\cdot\nu\,\mathbf{curl}\,\overline{A}^{(i)}\,\mathrm{d}x \\ &\underbrace{-\lambda_E\int_{B^c_{x_0}}(\boldsymbol{E}_{p^\star}-\boldsymbol{E}_{\mathrm{d}})\cdot\nabla\overline{\widetilde{u}}_i\,\mathrm{d}x}_{=b(\psi^\star,\widetilde{u}_i)} \end{aligned}$$

since $A^{(i)}|_{B^c_{x_0}} = \mathcal{A}^{(i)}|_{B^c_{x_0}}$ by construction. The latter computation together with (3.63) gives the desired result. □

Chapter 4

Topological asymptotic expansion for a problem in low-frequency electromagnetism

The topological derivative method was first introduced in [EKS94] and then rigorously justified some years later in [SZ99]. The idea is to assess the sensitivity of a given shape functional, which depends on the solution of a partial differential equation, with respect to topological perturbations of the latter. One usually calls a topological perturbation either the change of the material coefficients (and/or of the source term) in the neighbourhood of a given point, or the removal of such neighbourhood resulting in a change of the domain (i.e. creation of a hole) where the PDE is solved. It is clear that in this case, an additional boundary condition has to be prescribed on the boundary of the hole.

In order to make the concept of topological perturbation more precise, let us consider a bounded domain $\Omega \subset \mathbb{R}^3$ and a differential operator $\mathcal{A}$. If V is a suitable functional space, the solution $u = u_0 \in V$ of

$$\begin{cases} \mathcal{A}u = f & \text{in } \Omega \\ \text{b.c.} & \text{on } \partial\Omega \end{cases}$$

is called *unperturbed* or *direct* state. Now let $\omega \subset \mathbb{R}^3$ be a bounded domain satisfying $\mathbf{0} \in \omega$, $z \in \Omega$ and $\varepsilon > 0$ small enough to guarantee $\omega_\varepsilon := z + \varepsilon\omega \subset \Omega$. The *perturbed* state $u_\varepsilon \in V_\varepsilon$ is defined as the solution of

$$\begin{cases} \mathcal{A}_\varepsilon u_\varepsilon = f_\varepsilon & \text{in } \Omega \\ \text{b.c.} & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\mathcal{A}_\varepsilon$ is a perturbation of the initial operator $\mathcal{A}$ with respect to ω_ε (for instance, in the context of electromagnetism we may think of a change of the conductivity or permittivity matrix in ω_ε) and f_ε a modification of the source term.

The other possibility is to define $u_\varepsilon \in V_\varepsilon$ by means of

$$\begin{cases} \mathcal{A}u_\varepsilon = f & \text{in } \Omega_\varepsilon \\ \text{b.c.} & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{4.2}$$

where $\Omega_\varepsilon := \Omega \setminus \overline{\omega_\varepsilon}$, hence the PDE is now solved in a punctured version of Ω.

In this context, let $\mathcal{J} : V_\varepsilon \to \mathbb{R}$ be a given cost functional (e.g., the energy functional associated with the space V_ε). We say that $\mathcal{J}_\varepsilon$ admits a first order topological asymptotic expansion with respect to the inclusion ω at the point z, if there exist $l_1 : [0, +\infty) \to [0, +\infty)$ with $l_1(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and $d\mathcal{J}_z(\Omega, \omega) \in \mathbb{R}$ such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) + l_1(\varepsilon) d\mathcal{J}_z(\Omega, \omega) + o(l_1(\varepsilon)) \tag{4.3}$$

for sufficiently small values of $\varepsilon > 0$. If (4.3) holds, the scalar $d\mathcal{J}_z(\Omega, \omega)$ is called first order topological derivative of $\mathcal{J}$ at z, and the mapping $z \mapsto \mathcal{J}_z(\Omega, \omega)$ topological derivative or gradient. Note the latter not only depends on the point $z \in \Omega$, but also on the differential operator $\mathcal{A}$, the domain Ω, the inclusion ω and the choice of l_1, which is usually taken as $l_1(\varepsilon) = |\omega_\varepsilon| = \varepsilon^3 |\omega|$. In a similar fashion we say that $\mathcal{J}$ admits a second order expansion if, in addition to $d\mathcal{J}_z(\Omega, \omega)$, there exists $d^2\mathcal{J}_z(\Omega, \omega) \in \mathbb{R}$ such that

$$\mathcal{J}_\varepsilon(u_\varepsilon) = \mathcal{J}_0(u_0) + l_1(\varepsilon) d\mathcal{J}_z(\Omega, \omega) + l_2(\varepsilon) d^2\mathcal{J}_z(\Omega, \omega) + o(l_2(\varepsilon)) \tag{4.4}$$

for sufficiently small $\varepsilon > 0$. Here l_2 is a non-negative real function satisfying $l_2(\varepsilon) = o(l_1(\varepsilon))$ for $\varepsilon \to 0^+$. Following the same ideas, one can define higher order topological expansions and derivatives.

The importance of (4.3) in applications is that it can be used as a basis for numerical algorithms aiming to optimize the shape/topology of a region. For instance if u_ε is given through (4.2), it follows from (4.3) that the sign and the magnitude of $d\mathcal{J}_z(\Omega, \omega)$ provides information on *where to put holes* in order to minimize $\mathcal{J}$, or analogously $d\mathcal{J}_z(\Omega, \omega) \geq 0$ provides an optimality condition (cf. [BM91] in the context of shape optimization for the Laplace equation). The utility of topological derivatives is not confined to design optimization and ranges through image processing and reconstruction [ABM07; Bel+08; HL09], inverse problems and object detection [NSŻ19; DA15]. In force of these motivations, topological asymptotic expansions have been derived for a variety of PDEs: linear elasticity [GGM01], the Helmholtz equation [SAM03], the Stokes and Navier-Stokes problem [HM04; Ams05] and Maxwell's equations [AVV01; MPS05]. Despite higher order topological expansions being overall less investigated, they also have been derived in different contexts [HL08; BC17] and it has been shown that they can contribute to improve the efficacy and accuracy of iterative methods [Bon08; Bon11]. This observation adds up to the motivations for the study of higher order topological derivatives for the eddy current system and related models, to which this chapter is a contribution.

The problem of computing topological derivatives naturally arises and many authors have contributed in this direction to establish methods whose efficiency can vary depending on the context. In principle, to derive topological expansions it suffices to have knowledge of the asymptotic behaviour (with suitable accuracy depending on the desired order) of the perturbed state equation and of the cost functional. In most cases, the expansion of the latter is obtained with the help of an adjoint equation due to the advantage that this can have on the numerical level.

Two well known computational methods are the method of Amstutz [Ams03] and the averaged adjoint method introduced by Sturm [Stu15]. Both rely on the introduction of an adjoint equation: the difference is that Amstutz's adjoint only depends on the unperturbed state variable, while the averaged adjoint variable also depends on the perturbed state (see (4.95)). Consequently, the asymptotic analysis of the averaged adjoint equation is generally more involved, yet the method is applicable to a wider range of cost functions and immediately provides a formula for the topogical gradients once the asymptotics are known. While first order derivatives can be[1] computed with the sole knowledge of the state and adjoint variable, higher order derivatives are known to require the introduction of corrector equations in the form of exterior and/or transmission problems.

To the author's best knowledge, no works are available in the literature concerning the topological expansion for the eddy current system or a low-frequency Maxwell problem like the one discussed here: in [MPS05] the first order topological expansion is computed for the linear homogeneous Maxwell problem. There the physical parameters are assumed to be strictly positive allowing for weak formulations in $\boldsymbol{H}(\mathbf{curl})$, without the need of divergence constraints on the test functions, which is exactly what gives rise to saddle point structures (for example (4.30)). In addition to that, aiming to develop the theory for higher order derivatives, we are led to introduce corrector equations in the whole space featuring the **curl curl** operator and non-divergence free sources, see e.g. (4.128). Besides requiring a suitable functional framework (which is developed in Section 4.1), this also has the consequence of producing gradient multipliers supported in the whole $\mathbb{R}^3$ that need to be dealt with in order to get the necessary asymptotic expansions. It is well-known that writing topological derivatives in explicit form demands for pointwise evaluations of the unperturbed state and adjoint variable, and respective rotations (gradients in the H^1 setting): another contribution of this chapter is to show that under reasonable assumptions on the impressed current - which is not subject of the topological perturbation -, those fields actually enjoy the required regularity. The employed techniques are classical from elliptic regularity theory, but the results are a pertinent addition in the given context.

The concept of topological derivative is placed in the broader context of design and shape optimization and it is customary employed in numerical algorithms to allow topology changes. As it turns out to be quite challenging (and technical) to deal with on the analytical level, it is worth mentioning that a more constructive approach based on the so-called level-set method is possible. It was mainly developed by Jouve and Allaire [AJT02; AJT04] and to some extent it allows to handle topology without explicitly requiring the computation of a topological gradient[2]. Yet another alternative approach by Allaire is built upon the classical homogenization method: it is explained in the monograph [All01] together with related numerical schemes. These are be valuable options to explore and compare with the topological derivative method in our context, but in this thesis we confine ourselves to the derivation of a topological expansion,

[1] At least in the linear case and for symmetric inclusions such as balls or ellipses.

[2] However, the choice of topology for the initial design guess becomes a critical issue.

leaving such investigations for future research.

The structure of the chapter is as follows. In Section 4.1.1 we review the Lagrangian framework on which our analysis is built, with some remarks and adaptations tailored for the complex setting. Right after (Section 4.1) we introduce the functional spaces needed for the corrector equations and the asymptotic analysis, providing an alternative perspective on Beppo-Levi spaces for the **curl** operator. Section 4.2 is dedicated to the explanation of the low-frequency electromagnetic problem that plays the role of unperturbed state equation, and its connections with the classical eddy current system. We show that our model can be seen as a simplified version of a vector potential formulation for the latter and prove well-posedness, a saddle point interpretation and a regularity result. In Section 4.3 we discuss the topological perturbation, introduce the perturbed state equation and analyze it: by means of an auxiliary vector field (*corrector*), we formulate one of main theorems that provides the asymptotics for the state variable, namely Theorem 4.3.14. Section 4.4 features a similar structure but for the averaged adjoint equation. In Section 4.5 we compute the first order topological derivative utilizing the previous results on the state and averaged adjoint variables. To conclude, Section 4.6 is devoted to the derivation of improved asymptotic expansions, which is done through the introduction of two further correctors.

4.1 Preliminaries and functional spaces

4.1.1 The Lagrangian approach in the complex framework

We will perform the computation of topological derivatives using the so called averaged adjoint method in a Lagrangian framework. It was first introduced by Sturm [Stu15] in the context of shape optimization and it ultimately allows to obtain the asymptotic expansion of a cost functional which depends on the solution of a given variational problem. Later the method has been applied to compute topological derivatives [Stu20; BS21]. The theory was developed for real functional spaces and thus an adaption to the complex framework is necessary for our application.

Let $\boldsymbol{X}$ be a Hilbert space of complex-valued functions and let $\varepsilon_0 > 0$. For all $\varepsilon \in [0, \varepsilon_0]$, we call state equation the problem to find $\boldsymbol{u}_\varepsilon \in \boldsymbol{X}$ such that

$$a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) = f_\varepsilon(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}, \tag{4.5}$$

where $a_\varepsilon \colon \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{C}$ is a sesquilinear form and $f_\varepsilon \colon \boldsymbol{X} \to \mathbb{C}$ an antilinear form. We henceforth assume that the abstract state equation admits a unique solution for each $\varepsilon \in [0, \varepsilon_0]$ and we call $\boldsymbol{u}_0$ unperturbed state and $\boldsymbol{u}_\varepsilon$ perturbed state variable. We consider a cost functional $\mathcal{J}_\varepsilon \colon \boldsymbol{X} \to \mathbb{R}$ which we assume to be differentiable at $\boldsymbol{u}_0 \in \boldsymbol{X}$ and we shall be interested in the asymptotic expansion at $\varepsilon = 0$ of

$$j \colon [0, \varepsilon_0] \to \mathbb{R}, \quad \varepsilon \mapsto j(\varepsilon) = \mathcal{J}_\varepsilon(\boldsymbol{u}_\varepsilon).$$

To this aim, we introduce the Lagrangian $\mathcal{L} \colon [0, \varepsilon_0] \times \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ via

$$\mathcal{L}(\varepsilon, \boldsymbol{u}, \boldsymbol{q}) = \mathcal{J}_\varepsilon(\boldsymbol{u}) + \mathrm{Re}\left\{a_\varepsilon(\boldsymbol{u}, \boldsymbol{q}) - f_\varepsilon(\boldsymbol{q})\right\}, \quad \boldsymbol{u}, \boldsymbol{q} \in \boldsymbol{X}. \tag{4.6}$$

Notice that working with the abstract state equation (cf. [Stu15])

$$\partial_q \mathcal{L}(\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{q})(\boldsymbol{w}) = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{X}, \tag{4.7}$$

- used to deal with more general Lagrangians -, we would have obtained

$$\operatorname{Re} a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{q}) = \operatorname{Re} f_\varepsilon(\boldsymbol{q}) \quad \forall \boldsymbol{q} \in \boldsymbol{X} \tag{4.8}$$

in place of (4.5). Keep in mind that for consistency, all the directional derivatives here appearing have to be taken with respect to real variations only. Equation (4.8) appears hard to deal with on the theoretical level as, for instance, $\boldsymbol{v} \mapsto \operatorname{Re} f_\varepsilon(\boldsymbol{v})$ is not an antilinear form on $\boldsymbol{V}$, but merely complex additive and real linear.

However, the choice of $i\boldsymbol{q}$ as test function shows that (4.8) is actually equivalent to the same variational problem without real parts, i.e. (4.5), which is solvable via classical techniques. Indeed, for each $\boldsymbol{v} \in \boldsymbol{X}$ we can set $\boldsymbol{q} = i\boldsymbol{v}$ in (4.8) to obtain

$$\operatorname{Re} a_\varepsilon(\boldsymbol{u}_\varepsilon, i\boldsymbol{v}) = -\operatorname{Re} i a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) = \operatorname{Re} f_\varepsilon(i\boldsymbol{v}) = -\operatorname{Re} i f_\varepsilon(\boldsymbol{v}),$$

which implies $\operatorname{Im} a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{v}) = \operatorname{Im} f_\varepsilon(\boldsymbol{v})$ and therefore (4.5) (using again (4.8)). For the sake of completeness, we mention that similar considerations have been made in the context of shape sensitivity for an inverse problem in [HLY15], where the authors introduce a similar Lagrangian featuring the real part operator.

In order to introduce the notion of averaged adjoint variable, we require that for all $(\varepsilon, \boldsymbol{\phi}, \boldsymbol{q}) \in [0, \varepsilon_0] \times \boldsymbol{X} \times \boldsymbol{X}$ the mapping

$$[0,1] \ni t \mapsto \partial_{\boldsymbol{u}} \mathcal{L}(\varepsilon, t\boldsymbol{u}_\varepsilon + (1-t)\boldsymbol{u}_0, \boldsymbol{q})(\boldsymbol{\phi})$$

belongs to $C^1([0,1])$. Here $\boldsymbol{u}_0 \in \boldsymbol{X}$ denotes the solution to the state equation (4.5) corresponding to the choice $\varepsilon = 0$.

Definition 4.1.1. *For each $\varepsilon \in [0, \varepsilon_0]$, we call abstract averaged adjoint equation the variational problem to find $\boldsymbol{q}_\varepsilon \in \boldsymbol{X}$ such that*

$$\int_0^1 \partial_{\boldsymbol{u}} \mathcal{L}(\varepsilon, t\boldsymbol{u}_\varepsilon + (1-t)\boldsymbol{u}_0, \boldsymbol{q}_\varepsilon)(\boldsymbol{v}) \, \mathrm{d}t = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{X}. \tag{4.9}$$

As it is enough for our application, we assume that for each $\varepsilon \in [0, \varepsilon_0]$ the latter problem is uniquely solvable too.

Theorem 4.1.2 (cf. [Stu15]). *Let $l_1(\cdot)$ be a scalar, non negative function with $l_1(\varepsilon) \to 0$ as $\varepsilon \to 0$. If the limits*

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_\varepsilon) - \mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_0)}{l_1(\varepsilon)} &=: \mathcal{R}^{(1)}(\boldsymbol{u}_0, \boldsymbol{q}_0), \\ \lim_{\varepsilon \to 0^+} \frac{\mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_0) - \mathcal{L}(0, \boldsymbol{u}_0, \boldsymbol{q}_0)}{l_1(\varepsilon)} &=: \partial_{l_1} \mathcal{L}(0, \boldsymbol{u}_0, \boldsymbol{q}_0) \end{aligned} \tag{4.10}$$

exist, then it holds

$$j(\varepsilon) = j(0) + l_1(\varepsilon)(\mathcal{R}^{(1)}(\boldsymbol{u}_0, \boldsymbol{q}_0) + \partial_{l_1} \mathcal{L}(0, \boldsymbol{u}_0, \boldsymbol{q}_0)) + o(l_1(\varepsilon)) \quad \text{as } \varepsilon \to 0. \tag{4.11}$$

Proof. First we note that testing (4.9) with $\boldsymbol{v} = \boldsymbol{u}_\varepsilon - \boldsymbol{u}_0 \in \boldsymbol{X}$ implies (for all $\varepsilon \in [0, \varepsilon_0]$)

$$0 = \int_0^1 \frac{d}{dt}\mathcal{L}(\varepsilon, \boldsymbol{u}_0 + t(\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0), \boldsymbol{q}_\varepsilon))\, dt \implies \mathcal{L}(\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{q}_\varepsilon) = \mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_\varepsilon). \tag{4.12}$$

At the same time, using the state equation (4.5) we have

$$\mathcal{L}(\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{q}) = \mathcal{J}_\varepsilon(\boldsymbol{u}_\varepsilon) + \mathrm{Re}\big(\underbrace{a_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{q}) - f_\varepsilon(\boldsymbol{q})}_{=0}\big) = \mathcal{J}_\varepsilon(\boldsymbol{u}_\varepsilon) \quad \forall \boldsymbol{q} \in \boldsymbol{X}, \forall \varepsilon \geq 0. \tag{4.13}$$

With this in mind, we can expand

$$\begin{aligned} j(\varepsilon) - j(0) &= \mathcal{J}_\varepsilon(\boldsymbol{u}_\varepsilon) - \mathcal{J}_0(\boldsymbol{u}_0) \\ &\underbrace{=}_{(4.13)} \mathcal{L}(\varepsilon, \boldsymbol{u}_\varepsilon, \boldsymbol{q}_\varepsilon) - \mathcal{L}(\boldsymbol{0}, \boldsymbol{u}_0, \boldsymbol{q}_0) \\ &\underbrace{=}_{(4.12)} \mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_\varepsilon) - \mathcal{L}(\boldsymbol{0}, \boldsymbol{u}_0, \boldsymbol{q}_0) \\ &= \mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_\varepsilon) - \mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_0) + \mathcal{L}(\varepsilon, \boldsymbol{u}_0, \boldsymbol{q}_0) - \mathcal{L}(\boldsymbol{0}, \boldsymbol{u}_0, \boldsymbol{q}_0); \end{aligned}$$

the conclusion follows dividing both sides by $l_1(\varepsilon) > 0$ with $\varepsilon > 0$ and passing to the limit as $\varepsilon \to 0$. □

4.1.2 Beppo-Levi type spaces in electromagnetism

Throughout the chapter we will work with complex-valued functions and corresponding functional spaces, if not otherwise specified. The asymptotic analyses we will perform require function spaces of vector fields defined in the whole $\mathbb{R}^3$ with merely square integrable **curl**. This turns out to be a common feature in the context of topological derivatives as in several related works (e.g. [Stu20; BS21; PS20]), the classical Beppo-Levi space (and variants)

$$\dot{BL}(\mathbb{R}^3) = \{u \in H^1_{loc}(\mathbb{R}^3) : \nabla u \in \boldsymbol{L}^2(\mathbb{R}^3)\}/\mathbb{C} \tag{4.14}$$

is systematically used.

In the **curl** setting, the counterpart of (4.14) is not unambiguously defined: in [GS21], as well as in [ABM06], the authors provide a construction by completion. Starting from the set of divergence-free test vectors

$$\boldsymbol{\mathcal{D}}_{\mathrm{div}}(\mathbb{R}^3) = \{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3),\ \mathrm{div}\,\boldsymbol{\psi} = 0 \quad \text{in } \mathbb{R}^3\}$$

endowed with the inner product $(\boldsymbol{u}, \boldsymbol{v})_{\mathbf{curl}} = \int_{\mathbb{R}^3} \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}$, they both consider its Hilbert completion. Despite being straightforward, this approach has the drawback that the resulting space inherits an abstract nature and therefore may not be ideal for practical use.

We propose an alternative approach which can be summarized as follows: a *natural* **curl** version of the Beppo-Levi space is introduced by suitably ruling out gradients, then it is shown that is always possible to uniquely find a divergence-free representative in each equivalence class. It is worth pointing out that the transition from the gradient operator to the curl operator is non-trivial, since having the L^2-control on the sole curl is significantly weaker than having it on the gradient[3].

Let us define

$$\boldsymbol{W}(\mathbb{R}^3) := \{\boldsymbol{u} \in \boldsymbol{L}^2_{loc}(\mathbb{R}^3),\ \mathbf{curl}\,\boldsymbol{u} \in \boldsymbol{L}^2(\mathbb{R}^3)\}, \qquad \dot{\boldsymbol{W}}(\mathbb{R}^3) := \boldsymbol{W}(\mathbb{R}^3)/\nabla H^1_{loc}(\mathbb{R}^3). \tag{4.15}$$

Lemma 4.1.3. *The space* $(\dot{\boldsymbol{W}}(\mathbb{R}^3), (\cdot,\cdot)_{\dot{W}})$ *with*

$$([\boldsymbol{u}],[\boldsymbol{v}])_{\dot{W}} := \int_{\mathbb{R}^3} \mathbf{curl}\,\boldsymbol{u} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x = (\boldsymbol{u},\boldsymbol{v})_W \tag{4.16}$$

is a complex Hilbert space.

Proof. By the classical theory of seminormed spaces [Rud73, p. 33], it suffices to check that the kernel of the seminorm induced by (4.16) on $\boldsymbol{W}(\mathbb{R}^3)$ is exactly $\nabla H^1_{loc}(\mathbb{R}^3)$, and that $(\boldsymbol{W}(\mathbb{R}^3), (\cdot,\cdot)_W)$ is sequentially complete, i.e. Cauchy sequences have a limit, which is not necessarily unique.

The inclusion $\nabla H^1_{loc} \subset \mathrm{Ker}\,|\cdot|_W$ is trivial; for the reverse one it suffices to combine the Poincaré lemma for currents [Dem97, p. 20] together with the fact that for $T \in \mathcal{D}'(\mathbb{R}^3)$, $\nabla T \in \boldsymbol{L}^2_{loc}(\mathbb{R}^3)$ implies $T \in L^2_{loc}(\mathbb{R}^3)$ (see for instance [SM13]). To show sequential completeness, note that if $(\boldsymbol{u}_n)_n \subset \boldsymbol{W}(\mathbb{R}^3)$ is a Cauchy sequence, then $(\mathbf{curl}\,\boldsymbol{u}_n)_n$ is a Cauchy sequence in $\boldsymbol{L}^2(\mathbb{R}^3)$ and thus we find $\boldsymbol{f} \in \boldsymbol{L}^2(\mathbb{R}^3)$ such that $\mathbf{curl}\,\boldsymbol{u}_n \to \boldsymbol{f}$ in $\boldsymbol{L}^2(\mathbb{R}^3)$. In view of [BKK17, Theorem 1], the global div-curl problem

$$\begin{cases} \mathbf{curl}\,\boldsymbol{w} = \boldsymbol{f} & \text{in } \mathbb{R}^3, \\ \operatorname{div} \boldsymbol{w} = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{4.17}$$

has a unique solution $\boldsymbol{w} \in \{\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}' \colon \nabla \boldsymbol{\eta} \in \boldsymbol{L}^2(\mathbb{R}^3)\}$, which implies $\boldsymbol{w} \in \boldsymbol{L}^2_{loc}(\mathbb{R}^3)$ (again [SM13]). Summing up, $\boldsymbol{u}_n \to \boldsymbol{w}$ in $\boldsymbol{W}(\mathbb{R}^3)$ and the proof is complete. □

[3] It is known (see e.g. [SM13]) that distributions with square integrable gradient are L^2_{loc}-regular. This is not true in general for the **curl** (nor for the divergence). For instance, consider the Dirac distribution $\delta_0 \in \mathcal{D}'(\mathbb{R}^3)$; its gradient $\boldsymbol{T} = \nabla\delta_0 \colon \boldsymbol{\mathcal{D}}(\mathbb{R}^3) \to \mathbb{R}$ is defined via

$$\langle \boldsymbol{T}, \boldsymbol{\varphi} \rangle = \langle \nabla\delta_0, \boldsymbol{\varphi} \rangle = -\sum_{i=1}^{3} \langle \delta_0, \partial_{x_i}\varphi_i \rangle = -(\operatorname{div}\boldsymbol{\varphi})(\mathbf{0}) \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$$

and is clearly not represented by any $\boldsymbol{L}^1_{loc}$ vector (just like the Dirac mass itself). On the other hand, its distributional rotation $\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{T}$ is regular since it is the zero distribution:

$$\langle \boldsymbol{H}, \boldsymbol{\varphi} \rangle = \langle \mathbf{curl}\,\boldsymbol{T}, \boldsymbol{\varphi} \rangle = \langle \boldsymbol{T}, \mathbf{curl}\,\boldsymbol{\varphi} \rangle = \langle \nabla\delta_0, \mathbf{curl}\,\boldsymbol{\varphi} \rangle = -\operatorname{div}(\mathbf{curl}\,\boldsymbol{\varphi})(\mathbf{0}) = \mathbf{0} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3).$$

Lemma 4.1.4. *Each class* $[\boldsymbol{u}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$ *has a unique divergence-free representative* $\widetilde{\boldsymbol{u}} \in [\boldsymbol{u}] \cap \boldsymbol{L}^2_\rho(\mathbb{R}^3)$*, where*

$$\boldsymbol{L}^2_\rho(\mathbb{R}^3) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^1_{loc}(\mathbb{R}^3) : \boldsymbol{v}\rho = \frac{\boldsymbol{v}}{(1+|x|^2)^{1/2}} \in \boldsymbol{L}^2(\mathbb{R}^3) \right\}. \tag{4.18}$$

Proof. Choose any representative $\boldsymbol{u} \in [\boldsymbol{u}]$, then $\mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^2(\mathbb{R}^3)$ is independent of the specific choice (of the representative) and we can solve (4.17) with $\boldsymbol{f} = \mathbf{curl}\, \boldsymbol{u}$. This provides us with $\widetilde{\boldsymbol{u}} \in \{\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}' : \nabla \boldsymbol{\eta} \in \boldsymbol{L}^2\} \hookrightarrow \boldsymbol{L}^2_{loc}(\mathbb{R}^3)$ with $\operatorname{div} \widetilde{\boldsymbol{u}} = 0$, which belongs to $[\boldsymbol{u}]$ as well. The existence proof is complete recalling that up to an additive constant, $\widetilde{\boldsymbol{u}} \in \boldsymbol{L}^2_\rho(\mathbb{R}^3)$ too[4]. It remains to check uniqueness. If $\widetilde{\boldsymbol{a}}$ in another representative with the same properties, $\widetilde{\boldsymbol{w}} = \widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{a}}$ is rotation-free and divergence-free in the sense of distributions whence

$$-\boldsymbol{\Delta} \widetilde{\boldsymbol{w}} = (-\,\mathbf{curl}\,\mathbf{curl} + \nabla \operatorname{div}) \widetilde{\boldsymbol{w}} = \boldsymbol{0} \implies \Delta \widetilde{w}_i = 0, \quad i = 1,2,3.$$

By Weyl's lemma, $\widetilde{w}_i$ is smooth and harmonic. For $r > 0$ and all $y \in \mathbb{R}^3$, by the mean value property and the fact that $\widetilde{w}_i \in L^2_\rho(\mathbb{R}^3)$, it follows

$$\begin{aligned} |\widetilde{w}_i(y)| &\leq \frac{1}{|B_r(y)|} \int_{B_r(y)} |\widetilde{w}_i| \,\mathrm{d}x \leq \frac{1}{|B_r(y)|} \left(\int_{B_r(y)} \widetilde{w}_i^2 \rho^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_r(y)} \rho^{-2} \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\sim \frac{c}{r^3} \left(\int_{B_r(y)} (1 + |x|^2) \,\mathrm{d}x \right)^{\frac{1}{2}} \sim c r^{\frac{5}{2}-3} \quad \text{as } r \to \infty, \end{aligned}$$

which shows $\widetilde{w}_i = 0$ and therefore $\widetilde{\boldsymbol{w}} = \boldsymbol{0}$. □

The following Helmholtz decomposition result in the whole space is a consequence of the functional theory just developed.

Lemma 4.1.5. *Let* $\boldsymbol{u} \in \boldsymbol{L}^2(\mathbb{R}^3)$. *There exists a unique couple* $([\boldsymbol{p}], [\psi]) \in \dot{\boldsymbol{W}}(\mathbb{R}^3) \times \dot{BL}(\mathbb{R}^3)$ *such that*

$$\boldsymbol{u} = \mathbf{curl}\, \boldsymbol{p} + \nabla \psi \quad \text{in } \mathbb{R}^3. \tag{4.19}$$

Moreover, $[\psi] = [0]$ *if* $\operatorname{div} \boldsymbol{u} = 0$ *and* $[\boldsymbol{p}] = [\boldsymbol{0}]$ *if* $\mathbf{curl}\, \boldsymbol{u} = \boldsymbol{0}$.

Proof. First we show how to determine $[\psi], [\boldsymbol{p}]$ given $\boldsymbol{u} \in \boldsymbol{L}^2(\mathbb{R}^3)$. The variational problem

$$[\psi] \in \dot{BL}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \nabla[\psi] \cdot \nabla \overline{v} \,\mathrm{d}x = \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \nabla \overline{v} \,\mathrm{d}x \quad \forall v \in \dot{BL}(\mathbb{R}^3) \tag{4.20}$$

is uniquely solvable by Riesz's representation theorem in $\dot{BL}(\mathbb{R}^3)$, and reads $\Delta\psi = \operatorname{div} \boldsymbol{u}$ in $\mathbb{R}^3$ in strong form.

[4]This can be seen by combining the pseudo-Poicaré inequality $\|\phi\|_{L^2_\rho(\mathbb{R}^3)} \leq C \|\nabla \phi\|_{L^2(\mathbb{R}^3)}$ $\forall \phi \in \mathcal{D}(\mathbb{R}^3)$, which in turn is a consequence of the classical Hardy inequality [Eva10, Subsec. 5.8.4] or [RS19, p. 71], with the fact that $\dot{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\{v \in \mathcal{D}' : \nabla v \in \boldsymbol{L}^2(\mathbb{R}^3)\}/\mathbb{C}$.

In a similar way, we solve

$$[\boldsymbol{p}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \mathbf{curl}[\boldsymbol{p}] \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x = \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \dot{\boldsymbol{W}}(\mathbb{R}^3), \tag{4.21}$$

which is equivalent to $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{p} = \mathbf{curl}\,\boldsymbol{u}$ in the sense of distributions in $\mathbb{R}^3$. Next we define $\boldsymbol{w} := \boldsymbol{u} - \mathbf{curl}\,\boldsymbol{p} - \nabla\psi$. By construction $\boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^2_\rho(\mathbb{R}^3)$ and

$$\begin{cases} \mathbf{curl}\,\boldsymbol{w} = \mathbf{curl}\,\boldsymbol{u} - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{p} - \mathbf{curl}\,\nabla\psi = \mathbf{0}, \\ \operatorname{div}\boldsymbol{w} = \operatorname{div}\boldsymbol{u} - \operatorname{div}\mathbf{curl}\,\boldsymbol{p} - \Delta\psi = 0. \end{cases}$$

We deduce that $\boldsymbol{w}$ is vector harmonic in the whole space and lies in $\boldsymbol{L}^2_\rho(\mathbb{R}^3)$, which implies $\boldsymbol{w} = \mathbf{0}$ (this is shown in the proof of Lemma 4.1.4) and thus the validity of (4.19). □

Remark 4.1.6. *Notice that in light of Lemma 4.1.4, for each $[\boldsymbol{p}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$, we can uniquely choose a representative $\boldsymbol{p}^\star \in [\boldsymbol{p}]$ which is divergence free and belongs to $\boldsymbol{L}^2_\rho(\mathbb{R}^3)$.*

4.2 Model problem and unperturbed state equation

As our analysis is built upon a model which is closely related to a vector potential formulation for the eddy current system, we will work in the following geometrical framework, which is similar to the one described in Assumption 3.2.1.

Assumption 4.2.1 (Geometry)**.** *The computational domain Ω is assumed to be simply connected and with connected boundary $\Gamma := \partial\Omega$. An open, connected subset $\Omega_C \subset \Omega$ denotes the conducting region and consequently $\Omega_I := \Omega \setminus \overline{\Omega_C}$ is the insulator, which is also assumed to be connected for simplicity; Ω_C is strictly contained in Ω in such a way that $\partial\Omega \cap \partial\Omega_C = \emptyset$ and it is assumed to be simply connected, implying that Ω_I is also simply connected. The set $\Gamma_C := \partial\Omega_I \cap \partial\Omega_C$ is the interface between the conductor and the insulator.*

The time-harmonic eddy current system with electric boundary condition reads:

$$\begin{cases} \mathbf{curl}\,\boldsymbol{H} = \sigma\boldsymbol{E} + \boldsymbol{J} & \text{in } \Omega \\ \mathbf{curl}\,\boldsymbol{E} = -i\omega\mu\boldsymbol{H} & \text{in } \Omega \\ \operatorname{div}(\epsilon\boldsymbol{E}|_{\Omega_I}) = 0 & \text{in } \Omega_I \\ \boldsymbol{E} \times \boldsymbol{n} = \mathbf{0} & \text{on } \Gamma, \\ \int_\Gamma \boldsymbol{E}|_{\Omega_I} \cdot \boldsymbol{n}\, \mathrm{d}S = 0 \end{cases} \tag{4.22}$$

where i denotes the imaginary unit and the last condition is needed for having a uniquely defined solution $\boldsymbol{E}$ in Ω_I. The Gauss' law for the magnetic field $\operatorname{div}(\mu\boldsymbol{H}) = 0$ in Ω has been dropped since it is a consequence of the second equation in (4.22). Here $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ and

$$\operatorname{div}\boldsymbol{J}|_{\Omega_I} = 0 \quad \text{in } \Omega_I \quad \text{and} \quad \int_\Gamma \boldsymbol{J}|_{\Omega_I} \cdot \boldsymbol{n}\, \mathrm{d}S = 0 \tag{4.23}$$

due to (4.22) and the fact that σ vanishes in Ω_I.

For what concerns the physical parameters, the electric conductivity σ is assumed to be a real positive constant in Ω_C and identically vanishing in Ω_I, while the magnetic reluctivity $\nu = \mu^{-1}$ (the inverse of the magnetic permeability) is piece-wise constant and positive in the whole domain. In other words,

$$\sigma = \sigma_0 \chi_{\Omega_C}, \quad \nu = \nu_1 \chi_{\Omega_C} + \nu_2 \chi_{\Omega_I}, \quad \sigma_0, \nu_1, \nu_2 \in \mathbb{R}^+, \tag{4.24}$$

where $\chi_{\mathcal{O}}$ denotes the characteristic function of a domain $\mathcal{O}$. The quantities ν_1, ν_2 respectively represent the magnetic reluctivity of Ω_C, Ω_I and σ_0 the conductivity of Ω_C. The physical parameters will be subject of the topological perturbation, see (4.62).

Considering that $\operatorname{div}(\mu \boldsymbol{H}) = 0$ in Ω and

$$\mu \boldsymbol{H} \cdot \boldsymbol{n} = i\omega^{-1} \operatorname{\mathbf{curl}} \boldsymbol{E} \cdot \boldsymbol{n} = i\omega^{-1} \operatorname{div}_\tau (\boldsymbol{E} \times \boldsymbol{n}) = \boldsymbol{0} \quad \text{on} \quad \Gamma,$$

we look for a global magnetic vector potential $\boldsymbol{A}$ such that:

$$\mu \boldsymbol{H} = \nu^{-1} \boldsymbol{H} = \operatorname{\mathbf{curl}} \boldsymbol{A}. \tag{4.25}$$

Let us now consider the spaces

$$\boldsymbol{X}_N = \boldsymbol{X}_N(\Omega) = \boldsymbol{H}_0(\operatorname{\mathbf{curl}}, \Omega) \cap H(\operatorname{div}, \Omega)$$

and

$$H^1_\sharp(\Omega_C) = \left\{ Q_C \in H^1(\Omega_C) : \int_{\Omega_C} Q_C \, \mathrm{d}x = 0 \right\}.$$

As described in [RV10, Chap. 6], in the present geometric configuration and for the electric boundary condition $\boldsymbol{E} \times \boldsymbol{n} = 0$, the eddy current problem in terms of a vector magnetic potential $\boldsymbol{A} \in \boldsymbol{X}_N$ and a scalar electric potential $V_C \in H^1_\sharp(\Omega_C)$ can be written as

$$\begin{cases} \displaystyle\int_\Omega (\nu \operatorname{\mathbf{curl}} \boldsymbol{A} \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{w}} + \nu_* \operatorname{div} \boldsymbol{A} \operatorname{div} \overline{\boldsymbol{w}}) \, \mathrm{d}x + \int_{\Omega_C} \sigma (i\omega \boldsymbol{A} + \nabla V_C) \cdot \overline{\boldsymbol{w}} \, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{w}} \, \mathrm{d}x, \\ \displaystyle\int_{\Omega_C} \sigma (i\omega \boldsymbol{A} + \nabla V_C) \cdot \nabla \overline{Q_C} \, \mathrm{d}x = \int_{\Omega_C} \boldsymbol{J}|_{\Omega_C} \cdot \nabla \overline{Q_C} \, \mathrm{d}x + \int_{\Gamma_C} \boldsymbol{J}|_{\Omega_I} \cdot \boldsymbol{n}_I \overline{Q_C} \, \mathrm{d}S \end{cases} \tag{EC}$$

for all $(\boldsymbol{w}, Q_C) \in \boldsymbol{X}_N(\Omega) \times H^1_\sharp(\Omega_C)$, where $\nu_* > 0$ is an arbitrarily chosen penalization constant.

Besides the relation $\operatorname{\mathbf{curl}} \boldsymbol{A} = \mu \boldsymbol{H}$ in Ω, which defines the magnetic field $\boldsymbol{H}$, from the knowledge of $\boldsymbol{A}$ and V_C the electric field $\boldsymbol{E}$ in Ω_C is recovered by setting

$$\boldsymbol{E}_{|\Omega_C} = -i\omega \boldsymbol{A}_{|\Omega_C} - \nabla V_C \quad \text{in } \Omega_C. \tag{4.26}$$

It is worth noting that, as a consequence of the conditions $\operatorname{div} \boldsymbol{J}_{|\Omega_I} = 0$ in Ω_I and $\int_\Gamma \boldsymbol{J}_{|\Omega_I} \cdot \boldsymbol{n}_I = 0$, which imply $\int_{\Gamma_C} \boldsymbol{J}_{|\Omega_I} \cdot \boldsymbol{n}_I = 0$, (EC) is also satisfied for all $Q_C \in H^1(\Omega_C)$, without the average-free constraint.

Following the arguments presented in [RV10, Chap. 6], it is easy to prove that a solution to problem (EC) satisfies $\operatorname{div} \boldsymbol{A} = 0$ in Ω, therefore the couple $(\boldsymbol{A}, V_C) \in \boldsymbol{X}_N^0 \times H_\sharp^1(\Omega_C)$ is a solution of the following problem

$$\begin{cases} \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{w}} \,\mathrm{d}x + \int_{\Omega_C} \sigma(i\omega \boldsymbol{A} + \nabla V_C) \cdot \overline{\boldsymbol{w}} \,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{w}} \,\mathrm{d}x, \\ \int_{\Omega_C} \sigma(i\omega \boldsymbol{A} + \nabla V_C) \cdot \nabla \overline{Q_C} \,\mathrm{d}x = \int_{\Omega_C} \boldsymbol{J}|_{\Omega_C} \cdot \nabla \overline{Q_C} \,\mathrm{d}x + \int_{\Gamma_C} \boldsymbol{J}|_{\Omega_I} \cdot \boldsymbol{n}_I \overline{Q_C} \,\mathrm{d}S \end{cases} \quad (\mathrm{EC_{sol}})$$

for all $(\boldsymbol{v}, Q_C) \in \boldsymbol{X}_N^0(\Omega) \times H_\sharp^1(\Omega_C)$ (or equivalently $Q_C \in H^1(\Omega_C)$), where $\boldsymbol{X}_N^0$ has been defined in (2.5). Adapting the proof in [RV10, Sec. 6.1.2], using (2.12) we can show that ($\mathrm{EC_{sol}}$) is associated with a coercive sesquilinear form, and therefore has a unique solution by the Lax-Milgram lemma. Hence problems (EC) and ($\mathrm{EC_{sol}}$) are equivalent.

In order to simplify the picture, for $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ let us now consider the following problem

$$\begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{X}_N^0 \text{ s.t.} \\ a(\boldsymbol{A}, \boldsymbol{v}) := \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\omega \int_\Omega \sigma \boldsymbol{A} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0. \end{cases} \quad (4.27)$$

Let us recall that the space

$$\boldsymbol{X}_N^0 = \boldsymbol{X}_N^0(\Omega) = \{\boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) : \operatorname{div} \boldsymbol{A} = 0 \text{ in } \Omega\},$$

which has been introduced in (2.5)), is a closed subspace of $\boldsymbol{H}(\mathbf{curl}, \Omega)$ and it turns out to be a Hilbert space endowed with the canonical norm $\|\cdot\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}$. Additionally, since Ω features a connected boundary, we know by (2.12) that there is a constant $C > 0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \leq C \|\mathbf{curl}\, \boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} \quad \forall \boldsymbol{u} \in \boldsymbol{X}_N^0, \quad (4.28)$$

making the norm $\|\mathbf{curl} \cdot\|_{\boldsymbol{L}^2(\Omega)}$ equivalent to the $\boldsymbol{H}(\mathbf{curl})$ one.

The following existence result is standard:

Proposition 4.2.2. *Let Assumption 4.2.1 and (4.24) hold, and $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$. Then (4.27) has a unique solution $\boldsymbol{A} \in \boldsymbol{X}_N^0$.*

Proof. In view of (4.24) and (4.28), the sesquilinear form $a\colon \boldsymbol{X}_N^0 \times \boldsymbol{X}_N^0 \to \mathbb{C}$ is readily seen to be continuous and coercive. The conclusion follows by the Lax-Milgram lemma. □

Remark 4.2.3. *We point out that Proposition 4.2.2 remains valid in the extreme cases $\sigma = \sigma_0 > 0$ in Ω (hence $\Omega_I = \emptyset$) and $\sigma_0 = 0$ (hence $\sigma = 0$ in Ω and $\Omega_C = \emptyset$), that will be relevant in Section 4.3.4 as a consequence of Assumption 4.3.13. Indeed, if $\Omega_I = \emptyset$ the bilinear form reads*

$$a(\boldsymbol{A}, \boldsymbol{v}) = \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\omega \int_\Omega \sigma_0 \boldsymbol{A} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x,$$

which is a fortiori coercive. If $\Omega_C = \emptyset$, it results in

$$a(\boldsymbol{A}, \boldsymbol{v}) = \int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x,$$

which is still coercive in $\boldsymbol{X}_N^0$ thanks to Poincaré-Friedrichs inequality (4.28).

Finally, it is also possible to work in the following "dual" geometrical framework: the insulator Ω_I is a bounded and connected domain which is strictly contained in Ω, i.e. $\overline{\Omega_I} \subsetneq \Omega$, and the conductor Ω_C is defined as $\Omega_C := \Omega \setminus \overline{\Omega_I}$. This will be important in Section 4.3.4 as the case $\Omega_I = \emptyset$ corresponds to such scenario (the inclusion ω_ε is an insulating domain surrounded by the conductor).

It is worth to derive a saddle point formulation which is equivalent to (4.27). The usefulness of such formulation featuring a gradient Lagrange multiplier will be apparent in later sections: the presence of divergence free constraints in the functional spaces, which are pivotal to ensure well-posedness of **curl curl** driven problems, entails the appearance of non-trivial gradients in the strong interpretation of such problems (unless the source term lies in $H(\mathrm{div}{=}0)$ and the bilinear form turns out to be invariant under the addition of gradients). In this sense, bringing to light the saddle point structure of the unperturbed state equation appears to be the right set up for the forthcoming analysis.

The classical theory of elliptic boundary valued problems gives the following result

Lemma 4.2.4. *Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ and let $\boldsymbol{A} \in \boldsymbol{X}_N^0$ be the solution of (4.27). There exists a unique solution $p \in H_0^1(\Omega)$ to*

$$\int_\Omega \nabla p \cdot \nabla \overline{\psi} \, \mathrm{d}x = -i\omega \int_\Omega \sigma \boldsymbol{A} \cdot \nabla \overline{\psi} \, \mathrm{d}x + \int_\Omega \boldsymbol{J} \cdot \nabla \overline{\psi} \, \mathrm{d}x \quad \forall \psi \in H_0^1(\Omega). \tag{4.29}$$

Lemma 4.2.5. *Let $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$, $\boldsymbol{A} \in \boldsymbol{X}_N^0$ be the solution of (4.27) and $p \in H_0^1(\Omega)$ the solution of (4.29). The couple $(\boldsymbol{A}, p)$ is a solution to the saddle-point problem*

$$\begin{cases} \displaystyle\int_\Omega (\nu \, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{w}} + i\omega\sigma \boldsymbol{A} \cdot \overline{\boldsymbol{w}}) \, \mathrm{d}x + \int_\Omega \nabla p \cdot \overline{\boldsymbol{w}} \, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{w}} \, \mathrm{d}x & \forall \boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}) \\ \displaystyle\int_\Omega \boldsymbol{A} \cdot \nabla \overline{\eta} \, \mathrm{d}x = 0 & \forall \eta \in H_0^1(\Omega). \end{cases} \tag{4.30}$$

The converse is also true: if $(\boldsymbol{A}, p)$ is a solution of (4.30), then p satisfies (4.29) and $\boldsymbol{A}$ satisfies (4.27).

Proof. The second equation in (4.30) is immediately implied by $\boldsymbol{A} \in \boldsymbol{X}_N^0$. Let now $\boldsymbol{w} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$; by the Helmholtz decomposition (2.9), it can be written as

$$\boldsymbol{w} = \boldsymbol{v} + \nabla \xi, \quad \text{with } \boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega), \ \xi \in H_0^1(\Omega).$$

It holds

$$\int_\Omega \nu \, \mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + i\omega \int_\Omega \sigma \boldsymbol{A} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x,$$

which in turn implies

$$\begin{aligned}\int_\Omega \nu\,\mathbf{curl}\,\boldsymbol{A}\cdot\mathbf{curl}(\underbrace{\overline{\boldsymbol{v}+\nabla\xi}}_{=\overline{\boldsymbol{w}}})\,\mathrm{d}x + i\omega\int_\Omega \sigma\boldsymbol{A}\cdot(\overline{\boldsymbol{v}+\nabla\xi})\,\mathrm{d}x - i\omega\int_\Omega \sigma\boldsymbol{A}\cdot\nabla\overline{\xi}\,\mathrm{d}x \\ = \int_\Omega \boldsymbol{J}\cdot(\overline{\boldsymbol{v}+\nabla\xi})\,\mathrm{d}x - \int_\Omega \boldsymbol{J}\cdot\nabla\overline{\xi}\,\mathrm{d}x\end{aligned} \tag{4.31}$$

in view of $\mathbf{curl}\,\nabla \equiv \mathbf{0}$. At the same time, we observe that

$$\begin{aligned}-i\omega\int_\Omega \sigma\boldsymbol{A}\cdot\nabla\overline{\xi}\,\mathrm{d}x + \int_\Omega \boldsymbol{J}\cdot\nabla\overline{\xi}\,\mathrm{d}x \underbrace{=}_{(4.29)} \int_\Omega \nabla p\cdot\nabla\overline{\xi}\,\mathrm{d}x \\ \underbrace{=}_{\mathrm{div}\,\boldsymbol{v}=0} \int_\Omega \nabla p\cdot\nabla\overline{\xi}\,\mathrm{d}x + \int_\Omega \nabla p\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x = \int_\Omega \nabla p\cdot\overline{\boldsymbol{w}}\,\mathrm{d}x.\end{aligned} \tag{4.32}$$

Putting together (4.31) and (4.32) gives the first equation in (4.30). Viceversa, if $(\boldsymbol{A}, p)$ is a solution of (4.30), then (4.27) is readily satisfied since $\boldsymbol{v}$ is divergence-free. (4.29) is easily obtained by taking $\boldsymbol{w} = \nabla\eta$ in (4.30) and using $\mathbf{curl}\,\nabla \equiv \mathbf{0}$. This concludes the proof. □

By choosing a test function $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}(\Omega)$, it is easy to show that the strong (distributional) form of (4.30) reads

$$\mathbf{curl}(\nu\,\mathbf{curl}\,\boldsymbol{A}) + i\omega\sigma\boldsymbol{A} + \nabla p = \boldsymbol{J} \quad \text{in } \Omega. \tag{4.33}$$

Thus setting $\boldsymbol{H} = \nu\,\mathbf{curl}\,\boldsymbol{A}$ would require the condition $\nabla p_{|\Omega_I} = \mathbf{0}$ (or, equivalently, $p_{|\Omega_I} = 0$) in Ω_I, in accordance with $\sigma|_{\Omega_I} = 0$ and (4.22). This is not true in general, even if we assume the stronger condition $\mathrm{div}\,\boldsymbol{J} = 0$ in Ω.

Let us prove the latter statement. First note that if the solution to (4.29) vanishes in one of the two subdomains Ω_C, Ω_I, then the same is true in the other part. In fact, we have

$$\Delta p_{|\Omega_I} = \mathrm{div}\,\boldsymbol{J}_{|\Omega_I} = 0, \quad \Delta p_{|\Omega_C} = \mathrm{div}\,\boldsymbol{J}_{|\Omega_C} - i\omega\sigma_0\,\mathrm{div}\,\boldsymbol{A}_{|\Omega_C} = 0$$

and $p_{|\Omega_I} = p_{|\Omega_C}$ on Γ_C, thus a vanishing interface datum on Γ_C would entail $p = 0$ in the whole Ω. Since the other interface condition, coming from the fact that $\mathrm{div}(i\omega\sigma\boldsymbol{A} + \nabla p) = \mathrm{div}\,\boldsymbol{J} = 0$, reads

$$\nabla p_{|\Omega_I}\cdot\boldsymbol{n}_I + \nabla p_{|\Omega_C}\cdot\boldsymbol{n}_C = -i\omega\sigma_0\boldsymbol{A}_{|\Omega_C}\cdot\boldsymbol{n}_C \quad \text{on } \Gamma_C,$$

we see that the information that p vanishes in one of the two subdomains is equivalent to $\boldsymbol{A}_{|\Omega_C}\cdot\boldsymbol{n}_C = 0$ on Γ_C. Indeed this last condition yields $\nabla p_{|\Omega_I}\cdot\boldsymbol{n}_I + \nabla p_{|\Omega_C}\cdot\boldsymbol{n}_C = 0$ on Γ_C, thus $\Delta p = 0$ in the whole Ω.

Summing up, the result $p = 0$ in Ω is achieved if and only if it is known that $\boldsymbol{A}_{|\Omega_C}\cdot\boldsymbol{n}_C = 0$ on Γ_C. Note also that the condition $\boldsymbol{A}_{|\Omega_C}\cdot\boldsymbol{n}_C = 0$ on Γ_C, giving $p = 0$ in Ω, would permit us to set not only $\boldsymbol{H} = \nu\,\mathbf{curl}\,\boldsymbol{A}$ in Ω, but also $\boldsymbol{E}_{|\Omega_C} = -i\omega\boldsymbol{A}_{|\Omega_C}$ in Ω_C, resulting in the validity of the Ampère equation (the first in (4.22)). However, there is no reason that induces to think that the solution $\boldsymbol{A}$ to (4.27) enjoys this property. In conclusion, what we have seen is that the approach just presented does not give the correct solution of the eddy current problem: instead of solving (4.27) (or its saddle-point variant (4.30)), we should solve ($\mathrm{EC}_{\mathrm{sol}}$). In other words, we should replace the sesquilinear form

$$\begin{aligned} a(\boldsymbol{A}, \boldsymbol{v}) &:= \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\omega \int_\Omega \sigma \boldsymbol{A} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\ &= \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\omega^{-1} \int_\Omega i\omega\sigma \boldsymbol{A} \cdot (-i\omega \overline{\boldsymbol{v}}) \,\mathrm{d}x \end{aligned}$$

with

$$\begin{aligned} &\mathcal{A}[(\boldsymbol{A}, V_C), (\boldsymbol{v}, Q_C)] \\ &\quad = \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\omega^{-1} \int_{\Omega_C} \sigma(i\omega \boldsymbol{A} + \nabla V_C) \cdot (-i\omega \overline{\boldsymbol{v}} + \nabla \overline{Q_C}) \,\mathrm{d}x. \end{aligned}$$

However, as a first step towards the analysis of the topological asymptotic expansion for the eddy current problem, in the remainder of this chapter we will focus on problem (4.27) (or its saddle-point version (4.30)): let us call it a *low-frequency electromagnetic problem.*

Starting from corrector equations, the asymptotic analysis requires point-wise evaluations of the state variable $\boldsymbol{A}_0$ and its rotation $\mathbf{curl}\, \boldsymbol{A}_0$ and therefore we shall prove a regularity result for the unperturbed state equation (4.27). Some results in this direction are present in [HLY15] for an $\boldsymbol{E}$-based formulation of the eddy current problem, and provide interior regularity for the electric field in the conducting domain. Since our setting is slightly different, we are also interested in the insulator and need continuity of the **curl** too, we formulate our own version and provide a proof.

Recall that the Hölder space $C^{0,\lambda}(\overline{\Omega})$, $\lambda \in (0, 1]$, consists of bounded continuous functions $f\colon \Omega \to \mathbb{C}$ such that

$$|f|_{C^{0,\lambda}(\Omega)} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\lambda} < +\infty$$

and it is a Banach space if equipped with the norm $\|\cdot\|_{C^0(\Omega)} + |\cdot|_{C^{0,\lambda}(\Omega)}$. If $\lambda = 1$, such functions are called Lipschitz continuous. The space $C^{k,\lambda}(\overline{\Omega})$ denotes the space of functions that are bounded together with their derivatives up to order $k \in \mathbb{N}$ and such that the k-th derivatives belong to $C^{0,\lambda}(\Omega)$. Analogously defined are the vector versions $\boldsymbol{C}^{0,\lambda}(\overline{\Omega}), \boldsymbol{C}^{k,\lambda}(\overline{\Omega})$. The linear space of functions $f\colon \Omega \to \mathbb{C}$ admitting partial derivatives up to order k, such that the k-th derivatives belong to $C^{0,\lambda}(K)$ for any compact set $K \subset \Omega$, is denoted by $C^{k,\lambda}_{loc}(\Omega)$ and it is a locally convex topological vector space; $\boldsymbol{C}^{k,\lambda}_{loc}(\Omega)$ is defined accordingly.

Finally, we shall need the space

$$\boldsymbol{W}^{0,p}_{loc}(\mathbf{curl}, \Omega) = \{\boldsymbol{u} \in \boldsymbol{L}^p_{loc}(\Omega)\colon\ \mathbf{curl}\, \boldsymbol{u} \in \boldsymbol{L}^p_{loc}(\Omega)\}, \quad p \in [1, \infty); \tag{4.34}$$

note that according to the definition, $\boldsymbol{W}^{0,2}_{loc}(\mathbf{curl}, \Omega) = \boldsymbol{H}_{loc}(\mathbf{curl}, \Omega)$.

Lemma 4.2.6. *Let $A_0 \in X_N^0$ be the unique solution of (4.27).*

- *If*
$$J \in W_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_I) \quad \text{for some } \alpha > 0, \tag{4.35}$$
then
$$A_0 \in C_{loc}^{1,\gamma_1}(\Omega_I), \quad \mathbf{curl}\, A_0 \in C_{loc}^{1,\delta_1}(\Omega_I) \quad \text{for some } \gamma_1, \delta_1 \in (0,1). \tag{4.36}$$
In particular, A_0 and $\mathbf{curl}\, A_0$ are locally Lipschitz continuous in Ω_I.

- *If*
$$J \in W_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_C) \quad \text{for some } \alpha > 0, \tag{4.37}$$
then
$$A_0 \in C_{loc}^{1,\gamma_2}(\Omega_C), \quad \mathbf{curl}\, A_0 \in C_{loc}^{1,\delta_2}(\Omega_C) \quad \text{for some } \gamma_2, \delta_2 \in (0,1). \tag{4.38}$$
In particular, A_0 and $\mathbf{curl}\, A_0$ are locally Lipschitz continuous in Ω_C.

Remark 4.2.7. *Note that (4.35) and (4.37) are weaker than assuming $J \in W_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega)$. In particular, the latter would imply that $J \times n$ does not jump across the interface $\partial\Omega_C$, a request that may be restrictive in some (rare) instance.*

Proof. We work with the saddle point formulation (4.30) because it features a larger space of test functions.

Regularity in the insulator. First we focus on the multiplier p: in view of (4.29), it satisfies in strong form
$$\Delta p|_{\Omega_I} = \operatorname{div} J|_{\Omega_I} \quad \text{in } \Omega_I. \tag{4.39}$$
As a consequence, the field $j^\star := J - \nabla p$ has the property
$$\begin{cases} \operatorname{div} j^\star|_{\Omega_I} = 0 & \text{in } \Omega_I, \\ \mathbf{curl}\, j^\star|_{\Omega_I} = \mathbf{curl}\, J|_{\Omega_I} \underbrace{\in}_{(4.35)} L_{loc}^{3+\alpha}(\Omega_I) \hookrightarrow L_{loc}^2(\Omega_I), \end{cases} \tag{4.40}$$
which in turn implies $j^\star|_{\Omega_I} \in H_{loc}(\mathbf{curl}, \Omega_I) \cap H_{loc}(\operatorname{div}, \Omega_I)$. It follows[5] $j^\star|_{\Omega_I} \in H_{loc}^1(\Omega_I)$ and finally
$$j^\star|_{\Omega_I} = (J - \nabla p)|_{\Omega_I} \in L_{loc}^6(\Omega_I) \hookrightarrow L_{loc}^{3+\alpha}(\Omega_I) \tag{4.41}$$
by Sobolev embedding. Let $\eta \in \mathcal{D}(\Omega_I)$ and test (4.30) with the extension
$$\tilde{\eta} = \begin{cases} \eta & \text{in } \Omega_I \\ 0 & \text{in } \Omega_C \end{cases} \in \mathcal{D}(\Omega) \subset H_0(\mathbf{curl}, \Omega).$$

[5] This implication is a simple consequence of the embedding
$$H_0(\mathbf{curl}, D) \cap H(\operatorname{div}, D) \hookrightarrow H^1(D),$$
(which holds if the domain D is of class $C^{1,1}$ or if it is convex, cf. [Cos90; Amr+98]) and a cut-off argument. See also the last part of the proof of Theorem 4.3.14, which is related.

This yields

$$\int_{\Omega_I} \nu_2 \,\mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{\eta}} \,\mathrm{d}x = \int_{\Omega_I} (\boldsymbol{J} - \nabla p)|_{\Omega_I} \cdot \overline{\boldsymbol{\eta}} \,\mathrm{d}x \quad \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}(\Omega_I),$$

from which we readily get

$$-\nu_2 \boldsymbol{\Delta} \boldsymbol{A}_0|_{\Omega_I} \underbrace{=}_{\operatorname{div} \boldsymbol{A}_0 = 0} \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_I} = \boldsymbol{j}^\star|_{\Omega_I} \underbrace{\in}_{(4.41)} \boldsymbol{L}^{3+\alpha}_{loc}(\Omega_I). \tag{4.42}$$

By classical interior elliptic regularity theory [Jos07, Thm. 9.2.2] applied componentwise it follows $\boldsymbol{A}_0|_{\Omega_I} \in \boldsymbol{W}^{2,3+\alpha}_{loc}(\Omega_I)$ and the latter embeds into $\boldsymbol{C}^{1,\gamma_1}_{loc}(\Omega_I)$ for some $\gamma_1 \in (0,1)$, leading to (4.36) (the assertion for $\boldsymbol{A}_0$).

Taking the **curl** of (4.42) gives

$$-\nu_2 \boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_I}) = \mathbf{curl}\, \boldsymbol{J}|_{\Omega_I} \underbrace{\in}_{(4.35)} \boldsymbol{L}^{3+\alpha}_{loc}(\Omega_I)$$

and therefore

$$\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_I} \in \boldsymbol{W}^{2,3+\alpha}_{loc}(\Omega_I) \hookrightarrow \boldsymbol{C}^{1,\delta_1}_{loc}(\Omega_I),$$

for some $\delta_1 \in (0,1)$, which completes the proof of (4.36).

Regularity in the conductor. Once again we start from the multiplier p. We have

$$\Delta p|_{\Omega_C} = -i\omega\sigma_0 \operatorname{div} \boldsymbol{A}_0|_{\Omega_C} + \operatorname{div} \boldsymbol{J}|_{\Omega_C} \underbrace{=}_{\operatorname{div} \boldsymbol{A}_0 = 0} \operatorname{div} \boldsymbol{J}|_{\Omega_C} \quad \text{in } \Omega_C,$$

hence arguing as in the first part of the proof we get

$$\boldsymbol{j}^\star|_{\Omega_C} = (\boldsymbol{J} - \nabla p)|_{\Omega_C} \in \boldsymbol{L}^6_{loc}(\Omega_C) \hookrightarrow \boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C). \tag{4.43}$$

Next, given any $\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}(\Omega_C)$ we test (4.30) with

$$\widetilde{\boldsymbol{\eta}} = \begin{cases} \boldsymbol{\eta} & \text{in } \Omega_C \\ \boldsymbol{0} & \text{in } \Omega_I \end{cases} \in \boldsymbol{\mathcal{D}}(\Omega).$$

This produces

$$\int_{\Omega_C} \nu_1 \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{\eta}} \,\mathrm{d}x + i\omega \int_{\Omega_C} \sigma_0 \boldsymbol{A} \cdot \overline{\boldsymbol{\eta}} \,\mathrm{d}x + \int_{\Omega_C} \nabla p \cdot \overline{\boldsymbol{\eta}} \,\mathrm{d}x = \int_{\Omega_C} \boldsymbol{J}|_{\Omega_C} \cdot \overline{\boldsymbol{\eta}} \,\mathrm{d}x,$$

from which it follows

$$-\nu_1 \boldsymbol{\Delta} \boldsymbol{A}_0|_{\Omega_C} \underbrace{=}_{\operatorname{div} \boldsymbol{A}_0 = 0} \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_C} = (\boldsymbol{j}^\star - i\omega\sigma_0 \boldsymbol{A}_0)|_{\Omega_C} \quad \text{in } \Omega_C. \tag{4.44}$$

Now $\boldsymbol{A}_0 \in \boldsymbol{X}^0_N(\Omega)$ implies[5] $\boldsymbol{A}_0 \in \boldsymbol{H}^1_{loc}(\Omega) \hookrightarrow \boldsymbol{L}^6_{loc}(\Omega)$, hence a combination of (4.43) and (4.44) furnishes $\boldsymbol{\Delta} \boldsymbol{A}_0|_{\Omega_C} \in \boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C)$. The $\boldsymbol{C}^{1,\gamma_2}(\Omega_C)$ regularity of $\boldsymbol{A}_0$ readily follows, giving (4.38) (the statement for $\boldsymbol{A}_0$).

To conclude, we take the **curl** of (4.44):

$$-\nu_1 \boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_C}) = \mathbf{curl}\, \boldsymbol{j}^\star|_{\Omega_C} - i\omega\sigma_0 \,\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_C} = \mathbf{curl}\, \boldsymbol{J}|_{\Omega_C} - i\omega\sigma_0 \,\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_C}$$

in Ω_C. We already know that $\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_C}$ is locally Hölder continuous, therefore $\boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{A}_0|_{\Omega_C}) \in \boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C)$ (using (4.37) as well) and we can conclude as in the first part. This shows the assertions for $\mathbf{curl}\, \boldsymbol{A}_0$ in (4.38) and completes the proof. □

4.3 Asymptotic analysis of the state equation

As to avoid confusion with the topological inclusion later introduced we henceforth assume, without loss of generality, that the angular frequency ω is equal to 1. This clearly has no impact on our analysis.

4.3.1 Some technical results and scaled inequalities

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\Gamma = \partial\Omega$. For a given $z \in \Omega$ and $\varepsilon > 0$ we consider the affine transformation

$$T_\varepsilon \colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T_\varepsilon(x) = z + \varepsilon x. \tag{4.45}$$

We denote by Ω_ε the set $T_\varepsilon^{-1}(\Omega)$ and accordingly define $\Gamma_\varepsilon := T_\varepsilon^{-1}(\Gamma)$. The purpose of this subsection is to analyze the behaviour of some L^p and Sobolev norms (and related inequalities) with respect to the transformation T_ε and its inverse T_ε^{-1}. We are especially interested in the scaling behaviour in dependence of the parameter $\varepsilon > 0$, because it is of paramount importance for the derivation of asymptotic expansions. Let us introduce for $\varepsilon > 0$ the scaled norms

$$\begin{aligned} \|\boldsymbol{u}\|_{\varepsilon,\mathbf{curl}} &:= \varepsilon\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \|\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}, && \boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon) \\ \|\phi\|_{\varepsilon,\nabla} &:= \varepsilon\|\phi\|_{L^2(\Omega_\varepsilon)} + \|\nabla\phi\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} && \phi \in H^1(\Omega_\varepsilon). \end{aligned} \tag{4.46}$$

Note that T_ε is a diffeomorphism, hence a bi-Lipschitz mapping and therefore it holds $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\Omega)$ if and only $\boldsymbol{u} \circ T_\varepsilon \in \boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)$. Analogously $\phi \in H^1(\Omega)$ if and only if $\phi \circ T_\varepsilon \in H^1(\Omega_\varepsilon)$.

Lemma 4.3.1.

- *For all* $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega_\varepsilon),\ p \in [1,\infty)$, *(the same holds in the scalar case),*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega_\varepsilon)} = \varepsilon^{-\frac{3}{p}}\|\boldsymbol{u} \circ T_\varepsilon^{-1}\|_{\boldsymbol{L}^p(\Omega)}. \tag{4.47}$$

- *For all* $\boldsymbol{u} \in \boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)$ *and all* $\phi \in H^1(\Omega_\varepsilon)$,

$$\|\,\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} = \varepsilon^{-\frac{1}{2}}\|\,\mathbf{curl}(\boldsymbol{u} \circ T_\varepsilon^{-1})\|_{\boldsymbol{L}^2(\Omega)}, \quad \|\nabla\phi\|_{L^2(\Omega_\varepsilon)} = \varepsilon^{-\frac{1}{2}}\|\nabla(\phi \circ T_\varepsilon^{-1})\|_{L^2(\Omega)} \tag{4.48}$$

as well as

$$\|\boldsymbol{u}\|_{\varepsilon,\mathbf{curl}} = \varepsilon^{-\frac{1}{2}}\|\boldsymbol{u} \circ T_\varepsilon^{-1}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}, \quad \|\phi\|_{\varepsilon,\nabla} = \varepsilon^{-\frac{1}{2}}\|\phi \circ T_\varepsilon^{-1}\|_{H^1(\Omega)}. \tag{4.49}$$

Proof. We employ a change of variable with T_ε^{-1}, as $|\det(\mathrm{D}_x T_\varepsilon^{-1})| = \varepsilon^{-3}$ (here D_x denotes the Jacobian matrix with respect to the variable x) we obtain

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega_\varepsilon)} = \left(\int_{T_\varepsilon^{-1}(\Omega)} |\boldsymbol{u}|^p \,\mathrm{d}x\right)^{\frac{1}{p}} = \left(\varepsilon^{-3}\int_\Omega |\boldsymbol{u} \circ T_\varepsilon^{-1}|^p \,\mathrm{d}x\right)^{\frac{1}{p}},$$

which is (4.47). We now prove (4.48) for the **curl** operator, the same argument works for the gradient.

$$\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} = \left(\int_{T_\varepsilon^{-1}(\Omega)} |\operatorname{\mathbf{curl}} \boldsymbol{u}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \left(\varepsilon^{-3}\int_{\Omega} |(\operatorname{\mathbf{curl}} \boldsymbol{u})\circ T_\varepsilon^{-1}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$$
$$= \left(\varepsilon^{-3}\varepsilon^{2}\int_{\Omega} |\operatorname{\mathbf{curl}}(\boldsymbol{u}\circ T_\varepsilon^{-1})|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} = \varepsilon^{-\frac{1}{2}}\|\operatorname{\mathbf{curl}}(\boldsymbol{u}\circ T_\varepsilon^{-1})\|_{L^2(\Omega)},$$

where we used the same change of variables as before, and $(\operatorname{\mathbf{curl}} \boldsymbol{u})\circ T_\varepsilon^{-1} = \varepsilon \operatorname{\mathbf{curl}}(\boldsymbol{u}\circ T_\varepsilon^{-1})$. To conclude, (4.49) follows from (4.46) together with the previous steps. □

The following result is also useful: it provides a quantitative estimate (with respect to ε) for the boundary norm of a vector field on the inflated domain, having knowledge of its decay behaviour at infinity.

Lemma 4.3.2. *Let $\boldsymbol{v} \in \boldsymbol{L}^2_{loc}(\mathbb{R}^3)$ satisfy*

$$|\boldsymbol{v}(x)| = \frac{a}{|x|^r} + \mathcal{O}\left(\frac{1}{|x|^{r+1}}\right) \quad \text{as } |x| \to \infty \tag{4.50}$$

for some $a, r \in \mathbb{R}^+$. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary Γ and let $z \in \Omega$. There exists a constant $C > 0$ such that (for sufficiently small ε)

$$\|\boldsymbol{v}\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^{r-1}. \tag{4.51}$$

Proof. Let $\varepsilon > 0$ be sufficiently small so that the term $1/|x|^r$ dominates the remainder in (4.50). We have

$$\|\boldsymbol{v}\|^2_{L^2(\Gamma_\varepsilon)} = \int_{\Gamma_\varepsilon} |\boldsymbol{v}|^2 \, dS \leq a^{2r}\int_{\Gamma_\varepsilon}\left(\frac{1}{|x|}\right)^{2r} dS$$
$$\leq a^{2r}\int_{\Gamma_\varepsilon}\left(\frac{\varepsilon}{\min_{x\in\Gamma}|x-z|}\right)^{2r} dS \leq C|\Gamma_\varepsilon|\varepsilon^{2r} = C\varepsilon^{2r-2},$$

and the conclusion follows. □

We close the section by discussing the scaling of a Friedrichs inequality for the **curl** and some extension operators.

Lemma 4.3.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $z \in \Omega$ and let T_ε denote the transformation* (4.45).

- *There exists a continuous extension operator $Z_{\Gamma_\varepsilon}\colon H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ which satisfies*

$$\|Z_{\Gamma_\varepsilon}(h)\|_{\varepsilon,\nabla} \leq C(\varepsilon^{\frac{1}{2}}\|h\|_{L^2(\Gamma_\varepsilon)} + |h|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}) \quad \forall h \in H^{\frac{1}{2}}(\Gamma_\varepsilon) \tag{4.52}$$

for some $C > 0$ not depending on ε. We recall that the seminorm $|\cdot|_{H^{\frac{1}{2}}}$ is defined as

$$|h|_{H^{\frac{1}{2}}(\Gamma)} = \left(\int_\Gamma\int_\Gamma \frac{|h(x)-h(y)|^2}{|x-y|^3}\, dS_x\, dS_y\right)^{\frac{1}{2}}.$$

- *If Ω additionally has a connected boundary, there is a constant $C > 0$ not depending on ε such that*

$$\varepsilon \|\boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} \leq C(\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_{L^2(\Omega_\varepsilon)} + \|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega_\varepsilon)}) \quad \forall \boldsymbol{u} \in X_N(\Omega_\varepsilon), \tag{4.53}$$

where we remind that $X_N(\Omega_\varepsilon) = \boldsymbol{H}_0(\mathbf{curl}, \Omega_\varepsilon) \cap H(\operatorname{div}, \Omega_\varepsilon)$.

Proof. The existence of a continuous extension operator $Z_\Gamma \colon H^{\frac{1}{2}}(\Gamma) \to H^1(\Omega)$ is well-known (see e.g. [Wlo87]), in particular it satisfies

$$\|Z_\Gamma h\|_{H^1(\Omega)} \leq C\|h\|_{H^{\frac{1}{2}}(\Gamma)} = C(\|h\|^2_{L^2(\Gamma)} + |h|^2_{H^{\frac{1}{2}}(\Gamma)})^{\frac{1}{2}} \quad \forall h \in H^{\frac{1}{2}}(\Gamma). \tag{4.54}$$

With such operator at hand, we define $Z_{\Gamma_\varepsilon} \colon H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon)$ via

$$Z_{\Gamma_\varepsilon}(g) = Z_\Gamma(g \circ T_\varepsilon^{-1}) \circ T_\varepsilon \quad \forall g \in H^{\frac{1}{2}}(\Gamma_\varepsilon),$$

which is readily seen to be well defined and continuous.

To check the validity of (4.52), we fix $h \in H^{\frac{1}{2}}(\Gamma_\varepsilon)$ and write

$$\begin{aligned}
\|Z_{\Gamma_\varepsilon}(h)\|_{\varepsilon,\nabla} &= \varepsilon\|Z_\Gamma(h \circ T_\varepsilon^{-1}) \circ T_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \|\nabla((Z_\Gamma(h \circ T_\varepsilon^{-1}) \circ T_\varepsilon))\|_{L^2(\Omega_\varepsilon)} \\
&= \varepsilon^{-\frac{1}{2}}(\|Z_\Gamma(h \circ T_\varepsilon^{-1})\|_{L^2(\Omega)} + \|\nabla Z_\Gamma(h \circ T_\varepsilon^{-1})\|_{L^2(\Omega)}) \\
&\underbrace{\leq}_{(4.54)} C\varepsilon^{-\frac{1}{2}}\|h \circ T_\varepsilon^{-1}\|_{H^{\frac{1}{2}}(\Gamma)} \\
&= C\varepsilon^{-\frac{1}{2}}\left(\int_\Gamma |h \circ T_\varepsilon^{-1}|^2 \,\mathrm{d}x + \int_\Gamma\int_\Gamma \frac{|(h \circ T_\varepsilon^{-1})(x) - (h \circ T_\varepsilon^{-1})(y)|^2}{|x-y|^3}\, dS_x\, dS_y\right)^{\frac{1}{2}} \\
&= C\varepsilon^{-\frac{1}{2}}\left(\varepsilon^2\int_{\Gamma_\varepsilon} |h|^2 \,\mathrm{d}x + \varepsilon^4\int_{\Gamma_\varepsilon}\int_{\Gamma_\varepsilon} \frac{|h(x) - h(y)|^2}{|T_\varepsilon(x) - T_\varepsilon(y)|^3}\, dS_x\, dS_y\right)^{\frac{1}{2}} \\
&= C\left(\varepsilon\int_{\Gamma_\varepsilon} |h|^2 \,\mathrm{d}x + \int_{\Gamma_\varepsilon}\int_{\Gamma_\varepsilon} \frac{|h(x) - h(y)|^2}{|x-y|^3}\, dS_x\, dS_y\right)^{\frac{1}{2}} \\
&\leq C\varepsilon^{\frac{1}{2}}\|h\|_{L^2(\Gamma_\varepsilon)} + |h|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}.
\end{aligned}$$

If Ω has a connected boundary in addition to being weakly Lipschitz, we know that it holds

$$\|\boldsymbol{u}\|_{L^2(\Omega)} \leq C(\|\operatorname{\mathbf{curl}} \boldsymbol{u}\|_{L^2(\Omega)} + \|\operatorname{div} \boldsymbol{u}\|_{L^2(\Omega)}) \quad \forall \boldsymbol{u} \in X_N(\Omega)$$

for some $C > 0$ depending only on the fixed domain Ω [Sch18]. Therefore (4.53) follows again by a change variables, taking into account the norms scaling proved in Lemma 4.3.1 (it is clear that the divergence operator has the same behaviour as **curl** and ∇, with respect to the transformation T_ε). □

We recall the definitions of the trace spaces

$$H^{-\frac{1}{2}}(\mathrm{div}_\tau,\Gamma) = \{\boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma):\ \boldsymbol{v}\cdot\boldsymbol{n} = 0,\ \ \mathrm{div}_\tau \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\},$$
$$H^{-\frac{1}{2}}(\mathbf{curl}_\tau,\Gamma) = \{\boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma):\ \boldsymbol{v}\cdot\boldsymbol{n} = 0,\ \ \mathbf{curl}_\tau \boldsymbol{v} \in H^{-\frac{1}{2}}(\Gamma)\},$$

where $\boldsymbol{n}$ denotes the outward unit vector normal to Γ. Both spaces are endowed with the natural graph norm and it can be shown that they are in duality [BCS02].

Lemma 4.3.4. *(Tangential trace extension) There exists a linear and continuous mapping*

$$E_\Gamma\colon H^{-\frac{1}{2}}(\mathrm{div}_\tau,\Gamma) \to \boldsymbol{H}(\mathbf{curl},\Omega) \cap H(\mathrm{div}{=}0,\Omega)$$

such that

$$\boldsymbol{g} \mapsto E_\Gamma \boldsymbol{g}, \quad \begin{cases} \mathbf{curl}\,\mathbf{curl}(E_\Gamma\boldsymbol{g}) + (E_\Gamma\boldsymbol{g}) = \mathbf{0} & \text{in } \Omega, \\ (E_\Gamma\boldsymbol{g}) \times \boldsymbol{n} = \boldsymbol{g} & \text{on } \Gamma. \end{cases} \tag{4.55}$$

Proof. It is known [BCS02] that the trace mapping $\gamma_t\colon \boldsymbol{H}(\mathbf{curl},\Omega) \to H^{-\frac{1}{2}}(\mathbf{curl}_\tau,\Gamma)$ acting as $\gamma_t(\boldsymbol{v}) = \boldsymbol{n}\times\boldsymbol{v}\times\boldsymbol{n}$ is linear, continuous and surjective. Moreover

$$\|\boldsymbol{s}\|_{H^{-\frac{1}{2}}(\mathbf{curl}_\tau,\Gamma)} = \inf_{\substack{\boldsymbol{u}\in\boldsymbol{H}(\mathbf{curl},\Omega)\\ \gamma_t(\boldsymbol{u})=\boldsymbol{s}}} \|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \tag{4.56}$$

is an equivalent norm on $H^{-\frac{1}{2}}(\mathbf{curl}_\tau,\Gamma)$. It is readily verified that the unique element $\boldsymbol{u}^\star \in \boldsymbol{H}(\mathbf{curl},\Omega)$ realizing the minimum in (4.56) satisfies

$$\int_\Omega \mathbf{curl}\,\boldsymbol{u}^\star \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_\Omega \boldsymbol{u}^\star\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl},\Omega),$$

as well as

$$\begin{cases} \gamma_t(\boldsymbol{u}^\star) = \boldsymbol{s} & \text{on } \Gamma \\ \mathrm{div}\,\boldsymbol{u}^\star = 0 & \text{in } \Omega. \end{cases}$$

If $\boldsymbol{g} \in H^{-\frac{1}{2}}(\mathrm{div}_\tau,\Gamma)$, then $\boldsymbol{g}\times\boldsymbol{n} \in H^{-\frac{1}{2}}(\mathbf{curl}_\tau,\Gamma)$ and we can take $\boldsymbol{s} = \boldsymbol{g}\times\boldsymbol{n}$. Moreover, $\|\boldsymbol{g}\|_{H^{-\frac{1}{2}}_{\mathrm{div}_\tau}} = \|\boldsymbol{g}\times\boldsymbol{n}\|_{H^{-\frac{1}{2}}_{\mathbf{curl}_\tau}}$ ([BCS02]). The boundary condition reads $\gamma_t(\boldsymbol{u}^\star) = \boldsymbol{n}\times\boldsymbol{u}^\star\times\boldsymbol{n} = \boldsymbol{g}\times\boldsymbol{n}$ which implies $\boldsymbol{u}^\star\times\boldsymbol{n} = \boldsymbol{g}$ on Γ, so that $E_\Gamma(\boldsymbol{g}) := \boldsymbol{u}^\star$ defines the desired continuous extension map. □

Let us now define

$$\boldsymbol{L}^2_\tau(\Gamma) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Gamma):\ \boldsymbol{v}\cdot\boldsymbol{n} = 0\}$$

and subsequently

$$\boldsymbol{V}_\tau(\Gamma) := \{\boldsymbol{v} \in \boldsymbol{L}^2_\tau(\Gamma):\ \mathrm{div}_\tau \boldsymbol{v} \in L^2(\Gamma)\} = H(\mathrm{div}_\tau,\Gamma), \tag{4.57}$$

which is equipped with the natural graph norm

$$\|\boldsymbol{g}\|_{\boldsymbol{V}_\tau(\Gamma)} := (\|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Gamma)} + \|\,\mathrm{div}_\tau\,\boldsymbol{g}\|^2_{L^2(\Gamma)})^{\frac{1}{2}}.$$

Note that the continuous embedding

$$\boldsymbol{V}_\tau(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\operatorname{div}_\tau, \Gamma) \tag{4.58}$$

holds true.

Lemma 4.3.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain let T_ε denote the transformation (4.45). There exists a continuous extension operator $E_{\Gamma_\varepsilon}\colon \boldsymbol{V}_\tau(\Gamma_\varepsilon) \to \boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon) \cap H(\operatorname{div}{=}0, \Omega_\varepsilon)$ which satisfies*

$$\|E_{\Gamma_\varepsilon}(\boldsymbol{h})\|_{\varepsilon,\mathbf{curl}} \leq C(\varepsilon^{\frac{1}{2}}\|\boldsymbol{h}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{h}\|_{L^2(\Gamma_\varepsilon)}) \quad \forall \boldsymbol{h} \in \boldsymbol{V}_\tau(\Gamma_\varepsilon) \tag{4.59}$$

for some $C > 0$ not depending on ε.

Proof. The existence of a continuous lifting $E_\Gamma\colon H^{-\frac{1}{2}}(\operatorname{div}_\tau, \Gamma) \to \boldsymbol{H}(\mathbf{curl}, \Omega)$ is known from Lemma 4.3.4; in view of (4.58), the restriction of E_Γ to $\boldsymbol{V}_\tau(\Gamma)$ is also continuous, resulting in the following inequality:

$$\|E_\Gamma(\boldsymbol{g})\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\|\boldsymbol{g}\|_{\boldsymbol{V}_\tau(\Gamma)} = C(\|\boldsymbol{g}\|^2_{\boldsymbol{L}^2(\Gamma)} + \|\operatorname{div}_\tau \boldsymbol{g}\|^2_{L^2(\Gamma)})^{\frac{1}{2}} \quad \forall \boldsymbol{g} \in \boldsymbol{V}_\tau(\Gamma). \tag{4.60}$$

We can now define $E_{\Gamma_\varepsilon}\colon \boldsymbol{V}_\tau(\Gamma_\varepsilon) \to \boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon)$ via

$$E_{\Gamma_\varepsilon}(\boldsymbol{g}) = E_\Gamma(\boldsymbol{g} \circ T_\varepsilon^{-1}) \circ T_\varepsilon \quad \forall \boldsymbol{g} \in \boldsymbol{V}_\tau(\Gamma_\varepsilon),$$

which is readily seen to be well defined and continuous. For each $\boldsymbol{h} \in \boldsymbol{V}_\tau(\Gamma_\varepsilon)$, it holds

$$\begin{aligned}
\|E_{\Gamma_\varepsilon}(\boldsymbol{h})\|_{\varepsilon,\mathbf{curl}} &= \varepsilon\|E_\Gamma(\boldsymbol{h} \circ T_\varepsilon^{-1}) \circ T_\varepsilon\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \|\mathbf{curl}((E_\Gamma(\boldsymbol{h} \circ T_\varepsilon^{-1}) \circ T_\varepsilon))\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \\
&= \varepsilon^{-\frac{1}{2}}(\|E_\Gamma(\boldsymbol{h} \circ T_\varepsilon^{-1})\|_{\boldsymbol{L}^2(\Omega)} + \|\mathbf{curl}\, E_\Gamma(\boldsymbol{h} \circ T_\varepsilon^{-1})\|_{\boldsymbol{L}^2(\Omega)}) \\
&\underbrace{\leq}_{(4.60)} C\varepsilon^{-\frac{1}{2}}\|\boldsymbol{h} \circ T_\varepsilon^{-1}\|_{\boldsymbol{V}_\tau(\Gamma)} \\
&= C\varepsilon^{-\frac{1}{2}}(\|\boldsymbol{h} \circ T_\varepsilon^{-1}\|^2_{\boldsymbol{L}^2(\Gamma)} + \|\operatorname{div}_\tau(\boldsymbol{h} \circ T_\varepsilon^{-1})\|^2_{L^2(\Gamma)})^{\frac{1}{2}} \\
&= C\varepsilon^{-\frac{1}{2}}(\varepsilon^2\|\boldsymbol{h}\|^2_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-2}\|(\operatorname{div}_\tau \boldsymbol{h}) \circ T_\varepsilon^{-1}\|^2_{L^2(\Gamma)})^{\frac{1}{2}} \\
&= C\varepsilon^{-\frac{1}{2}}(\varepsilon^2\|\boldsymbol{h}\|^2_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-2}\varepsilon^2\|\operatorname{div}_\tau \boldsymbol{h}\|^2_{L^2(\Gamma_\varepsilon)})^{\frac{1}{2}} \\
&\leq C\varepsilon^{-\frac{1}{2}}(\varepsilon\|\boldsymbol{h}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \|\operatorname{div}_\tau \boldsymbol{h}\|_{L^2(\Gamma_\varepsilon)}) \\
&= C(\varepsilon^{\frac{1}{2}}\|\boldsymbol{h}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{h}\|_{L^2(\Gamma_\varepsilon)})
\end{aligned}$$

for some $C > 0$ independent of ε. □

4.3.2 Topological perturbation and the Lagrangian

Let $\omega \subset \mathbb{R}^3$ be a bounded and connected domain containing the origin. For $z \in \Omega \setminus \partial\Omega_C$ and $\varepsilon > 0$, we set $\omega_\varepsilon = z + \varepsilon\omega = T_\varepsilon(\omega)$ and

$$\Omega_{C,\varepsilon} = \begin{cases} \Omega_C \cup \omega_\varepsilon & \text{if } z \in \Omega_I \\ \Omega_C \setminus \omega_\varepsilon & \text{if } z \in \Omega_C, \end{cases} \tag{4.61}$$

and finally

$$\begin{aligned}\sigma_\varepsilon(x) &= \chi_{\Omega_{C,\varepsilon}}(x)\sigma_0.\\ \nu_\varepsilon(x) &= \chi_{\Omega_{C,\varepsilon}}(x)\nu_1 + \chi_{\Omega\setminus\Omega_{C,\varepsilon}}(x)\nu_2.\end{aligned} \tag{4.62}$$

Note that if $z \in \Omega_I$, the perturbation of σ physically means that a *small* piece of conductor replaces a portion of insulating material.

Let us now specialize the Lagrangian framework explained in Section 4.1.1 to our model problem. In view of the state equation (4.27), we shall build our analysis upon the Lagrangian $\mathcal{L}\colon [0,1] \times \boldsymbol{X}_N^0 \times \boldsymbol{X}_N^0 \to \mathbb{R}$ defined as

$$\begin{aligned}\mathcal{L}(\varepsilon, \boldsymbol{A}, \boldsymbol{q}) := \frac{\lambda_1}{2}\int_{\Omega_{C,\varepsilon}} |\boldsymbol{A}|^2 \,\mathrm{d}x + \frac{\lambda_2}{2}\int_\Omega \nu_\varepsilon |\operatorname{\mathbf{curl}} \boldsymbol{A}|^2 \,\mathrm{d}x \\ + \operatorname{Re}\left(\int_\Omega \nu_\varepsilon \operatorname{\mathbf{curl}} \boldsymbol{A} \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{q}} \,\mathrm{d}x + i\int_\Omega \sigma_\varepsilon \boldsymbol{A} \cdot \overline{\boldsymbol{q}} \,\mathrm{d}x - \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{q}} \,\mathrm{d}x\right),\end{aligned} \tag{4.63}$$

where the perturbed conducting domain $\Omega_{C,\varepsilon}$ is defined in (4.61), and the corresponding coefficients $\nu_\varepsilon, \sigma_\varepsilon$ in (4.62). It is readily verified that the abstract definition of state equation Section 4.1.1 applied to (4.63) produces (4.27), keeping in mind that the real parts can be safely neglected as previously observed in Section 4.1.1 (see the lines right after (4.8)). In particular, the *perturbed* state equation corresponding to our model reads

$$\begin{cases}\text{Find } \boldsymbol{A}_\varepsilon \in \boldsymbol{X}_N^0 \text{ s.t.}\\ \displaystyle\int_\Omega \nu_\varepsilon \operatorname{\mathbf{curl}} \boldsymbol{A}_\varepsilon \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{v}} \,\mathrm{d}x + i\int_\Omega \sigma_\varepsilon \boldsymbol{A}_\varepsilon \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0.\end{cases} \tag{4.64}$$

Proposition 4.3.6. *Let Assumption 4.2.1 be satisfied, $\boldsymbol{J} \in \boldsymbol{L}^2(\Omega)$ and let the perturbed physical parameters be defined by (4.62) and (4.61). Then for all $\varepsilon > 0$, (4.64) admits a unique solution $\boldsymbol{A}_\varepsilon \in \boldsymbol{X}_N^0$.*

Proof. In view of (4.28) and (4.62) (and σ_0, ν_1, ν_2 being positive), the left hand side of (4.64) can be interpreted as a sesquilinear form on $\boldsymbol{X}_N^0 \times \boldsymbol{X}_N^0$ which is continuous and coercive (uniformly in $\varepsilon > 0$) regardless of the choice $z \in \Omega_I$ or $z \in \Omega_C$. Indeed, it holds

$$\sigma_\varepsilon(x) \geq 0, \quad 0 < \min(\nu_1, \nu_2) \leq \nu_\varepsilon(x) \leq \max(\nu_1, \nu_2) \quad \forall x \in \Omega, \forall \varepsilon > 0$$

in both cases. The conclusion follows by the Lax-Milgram lemma. □

Remark 4.3.7. *Note that Proposition 4.3.6 remains valid in the extreme cases $\Omega_I = \emptyset$ and $\Omega_C = \emptyset$, that will be of interest in Section 4.3.4 as a consequence of Assumption 4.3.13. Indeed, if $\Omega_I = \emptyset$ then $\Omega = \Omega_C$, $\Omega_{C,\varepsilon} = \Omega \setminus \omega_\varepsilon$ and*

$$\nu_\varepsilon = \nu_1 \chi_{\Omega\setminus\omega_\varepsilon} + \nu_2 \chi_{\omega_\varepsilon}, \quad \sigma_\varepsilon = \sigma_0 \chi_{\Omega\setminus\omega_\varepsilon}.$$

Therefore $\nu_\varepsilon(x) > 0, \sigma_\varepsilon(x) \geq 0$ for all $x \in \Omega$ and the bilinear form induced by the left hand side of (4.64) is still uniformly coercive (and continuous) in $\boldsymbol{X}_N^0$ thanks to Poincaré-Friedrichs inequality (4.28). If instead $\Omega_C = \emptyset$, we have

$$\nu_\varepsilon = \nu_1 \chi_{\omega_\varepsilon} + \nu_2 \chi_{\Omega\setminus\omega_\varepsilon}, \quad \sigma_\varepsilon = \sigma_0 \chi_{\omega_\varepsilon}$$

and the same considerations hold.

We remind that it is possible to work in the "dual" geometrical framework: the insulator Ω_I is a bounded and connected domain which is strictly contained in Ω, i.e. $\overline{\Omega_I} \subsetneq \Omega$, and the conductor Ω_C is defined as $\Omega_C := \Omega \setminus \overline{\Omega_I}$. This will be important in Section 4.3.4 as the case $\Omega_I = \emptyset$ corresponds to this scenario.

Remark 4.3.8. *Here we can see the benefit of a vector potential formulation featuring a global divergence-free constraint, which is not affected by the topological perturbation. An **E**-based weak formulation of the eddy current system relies upon the space (compare with (3.26), where the boundary condition is different)*

$$\boldsymbol{V} = \{\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) : \operatorname{div} \boldsymbol{u}_I = 0 \quad \text{in } \Omega_I\};$$

it is apparent that the topological perturbation of the conducting domain (4.61) also entails a modification of the insulator, resulting in the ε-dependent space

$$\boldsymbol{V}_\varepsilon = \{\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) : \operatorname{div} \boldsymbol{u}_I = 0 \quad \text{in } \Omega_{I,\varepsilon}\}, \quad \Omega_{I,\varepsilon} = \Omega \setminus \Omega_{C,\varepsilon}.$$

More specifically, let us assume that $z \in \Omega_I$ so that according to (4.61), the perturbed insulator is

$$\Omega_{I,\varepsilon} = \Omega \setminus \Omega_{C,\varepsilon} = \Omega \setminus (\Omega_C \cup \omega_\varepsilon),$$

whose boundary has three connected components: $\partial\Omega, \partial\omega_\varepsilon$ and $\partial\Omega_C = \Gamma_C$. If this is the case, the space of harmonic fields $\mathcal{H}(e, \Omega_{I,\varepsilon})$ is non trivial and in order to ensure the validity of a Poincaré-Friedrichs inequality (and hence coercivity of the sesquilinear form), one has to suitably modify the space as

$$\widetilde{\boldsymbol{V}}_\varepsilon = \left\{\boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) : \operatorname{div} \boldsymbol{u}_I = 0 \quad \text{in } \Omega_{I,\varepsilon}, \quad \int_{\partial\omega_\varepsilon} \boldsymbol{u} \cdot \boldsymbol{n}\, dS = \int_{\Gamma_C} \boldsymbol{u} \cdot \boldsymbol{n}\, dS = 0\right\}.$$

However, the Poincaré-Friedrichs constant will in general depend on the domain $\Omega_{I,\varepsilon}$ (or $\Omega_{C,\varepsilon}$), and therefore on $\varepsilon > 0$, unless one can find a bi-Lipschitz transformation mapping $\Omega_{I,\varepsilon}$ to a reference domain, which appears to be challenging. Note instead that if we work in the space $\boldsymbol{X}_N^0(\Omega)$, the higher order term in (4.64) is alone sufficient to guarantee coercivity. This is not true if the divergence-free constraint is imposed in a subset of Ω.

4.3.3 Corrector field for the state problem

We shall now introduce an auxiliary vector field - *corrector* - that will appear in the asymptotic expansion of the state variable. The classical approach to **curl curl** driven variational problems in $\mathbb{R}^3$ consists in finding suitably constrained functional spaces to obtain coercivity [Hip02; AVV01]. At least for what concerns the first corrector $\boldsymbol{U}^{(1)}$, we choose instead to work in the natural **curl** counterpart of the quotient Beppo-Levi space $\dot{B}L(\mathbb{R}^3) = \{u \in L^2_{loc}(\mathbb{R}^3) \colon \nabla u \in \boldsymbol{L}^2(\mathbb{R}^3)\}/\mathbb{C}$. Existence will immediately follow from the Hilbert space structure, and in a subsequent step we show that it is possible to extract a specific divergence-free representative. To do so we rely upon the theory developed in Section 4.1. Our approach is equivalent to the known one as the selection criterion for the representative turns out to exploit the same constraints (null divergence and weighted $\boldsymbol{L}^2$-summability), which are seemingly encoded in the quotient structure. On the other hand, we think that our construction provides a cleaner extension to the electromagnetic framework of the classical Beppo-Levi spaces.

Let us introduce the material parameters associated with the reference inclusion ω, which are a version of the perturbed physical parameters defined in the whole space. If $z \in \Omega_C$, they read

$$\nu_\omega(x) := \begin{cases} \nu_1 & x \in \mathbb{R}^3 \setminus \overline{\omega} \\ \nu_2 & x \in \omega \end{cases} \quad \text{and} \quad \sigma_\omega(x) := \begin{cases} \sigma_0 & x \in \mathbb{R}^3 \setminus \overline{\omega} \\ 0 & x \in \omega; \end{cases} \tag{4.65}$$

if $z \in \Omega_I$ we instead define

$$\widehat{\nu}_\omega(x) := \begin{cases} \nu_2 & x \in \mathbb{R}^3 \setminus \overline{\omega} \\ \nu_1 & x \in \omega \end{cases} \quad \text{and} \quad \widehat{\sigma}_\omega(x) := \begin{cases} 0 & x \in \mathbb{R}^3 \setminus \overline{\omega} \\ \sigma_0 & x \in \omega. \end{cases} \tag{4.66}$$

In principle, according to whether $z \in \Omega_C$ or $z \in \Omega_I$, we have to use one definition or the other and make suitable changes of sign (e.g., a minus on the right hand of (4.67) if $z \in \Omega_I$) in the corrector equations we are about to introduce. However, we choose to focus the main presentation on the case $z \in \Omega_C$, hence to work with (4.65). As mentioned, the other case would require to use $\widehat{\nu}_\omega, \widehat{\sigma}_\omega$ and some minor sign adjustments that would also affect the expression of the topological derivative (but only with a change of sign), see Theorem 4.5.1. Some results for $z \in \Omega_I$ are collected in Appendix A for completeness.

Note that we have

$$0 < \min(\nu_1, \nu_2) \leq \nu_\omega(x) \leq \max(\nu_1, \nu_2), \quad 0 \leq \sigma_\omega(x) \leq \sigma_0 \quad \forall x \in \mathbb{R}^3.$$

Lemma 4.3.9. *Let $\boldsymbol{A}_0 \in \boldsymbol{X}_N^0$ be the unique solution of (4.27). The equation*

$$\int_{\mathbb{R}^3} \nu_\omega \, \mathbf{curl}[\boldsymbol{U}^{(1)}] \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \dot{\boldsymbol{W}}(\mathbb{R}^3) \tag{4.67}$$

admits a unique solution $[\boldsymbol{U}^{(1)}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$. Moreover, there exists a unique divergence-free representative $\boldsymbol{U}^{(1)} \in [\boldsymbol{U}^{(1)}] \cap \boldsymbol{L}^2_\rho(\mathbb{R}^3)$ which satisfies

$$\int_{\mathbb{R}^3} \nu_\omega \, \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{W}(\mathbb{R}^3), \tag{4.68}$$

and enjoys the asymptotic behaviour at infinity

$$|\boldsymbol{U}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{4.69}$$

Note that the pointwise evaluation on the right hand side of (4.67) is meaningful thanks to Lemma 4.2.6.

Proof. In view of Lemma 4.1.3, the left hand side is equivalent to an inner product of $\dot{\boldsymbol{W}}(\mathbb{R}^3)$ and the right hand side induces a bounded linear functional on the same space (by extending the constant vector $\mathbf{curl}\, A_0(z)$ to zero outside ω): well-posedness follows by Riesz's representation theorem in $\dot{\boldsymbol{W}}(\mathbb{R}^3)$. The existence of a unique divergence-free representative is provided by Lemma 4.1.4, and characterization (4.68) readily follows from (4.67).

Now let us consider the space

$$\boldsymbol{S} = \{\boldsymbol{w} : \boldsymbol{w} \in \boldsymbol{L}^2_\rho(\mathbb{R}^3),\ \mathbf{curl}\, \boldsymbol{w} \in \boldsymbol{L}^2(\mathbb{R}^3),\ \operatorname{div} \boldsymbol{w} = 0 \ \text{ in } \mathbb{R}^3\}. \tag{4.70}$$

It is clear that $\boldsymbol{U}^{(1)} \in \boldsymbol{S}$ and that (4.68) holds in particular for all $\boldsymbol{v} \in \boldsymbol{S} \hookrightarrow \boldsymbol{W}(\mathbb{R}^3)$. According to (4.65) and an integration by parts, this means

$$\begin{aligned}\nu_2 \int_\omega \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + \nu_1 \int_{\mathbb{R}^3 \setminus \overline{\omega}} \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\ = (\nu_1 - \nu_2) \int_{\partial\omega} (\mathbf{curl}\, A_0(z) \times \boldsymbol{n}) \cdot \overline{\boldsymbol{v}}\, dS \quad \forall \boldsymbol{v} \in \boldsymbol{S},\end{aligned}$$

where $\boldsymbol{n}$ is the unit normal vector on $\partial\omega$, pointing outward ω. We can now apply [AVV01, Lemma 4] to obtain that $|\boldsymbol{U}^{(1)}| = \mathcal{O}(1/|x|)$ as $|x| \to \infty$, which can be improved to (4.69), see again [AVV01, pp. 785-786]. □

4.3.4 Basic first order expansion

To begin with, we prove a standard result concerning the ε-rate of the difference between the perturbed and unperturbed state in the $\boldsymbol{H}(\mathbf{curl})$ norm.

Lemma 4.3.10. *Assume that $z \in \Omega_C$ (see Lemma A.0.2 for the case $z \in \Omega_I$) and $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$ for some $\alpha > 0$. Let $\boldsymbol{A}_\varepsilon$ be the solution of (4.64) and $\boldsymbol{A}_0$ the solution of (4.27). There is constant $C > 0$ independent of ε such that*

$$\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\varepsilon^{\frac{3}{2}} \tag{4.71}$$

for sufficiently small values of $\varepsilon > 0$.

Proof. We first subtract (4.27) from (4.64) to get

$$\begin{aligned}\int_\Omega \nu_\varepsilon\, \mathbf{curl}(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + i \int_\Omega \sigma_\varepsilon (\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\ = i\sigma_0 \int_{\omega_\varepsilon} \boldsymbol{A}_0 \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + (\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0_N.\end{aligned} \tag{4.72}$$

We now make the choice $\boldsymbol{v} = \boldsymbol{A}_\varepsilon - \boldsymbol{A}_0 \in \boldsymbol{X}^0_N$ in (4.72), which gives

$$\begin{aligned}\int_\Omega \nu_\varepsilon |\, \mathbf{curl}(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0)|^2\, \mathrm{d}x + i \int_\Omega \sigma_\varepsilon |\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0|^2\, \mathrm{d}x \\ = i\sigma_0 \int_{\omega_\varepsilon} \boldsymbol{A}_0 \cdot (\overline{\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0})\, \mathrm{d}x + (\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0)}\, \mathrm{d}x.\end{aligned}$$

Next we use the Poincaré-Friedrichs inequality (4.28) and $\sigma_\varepsilon \geq 0$ on the left hand side and Hölder's inequality on the right hand side, then thanks to Lemma 4.2.6 we find $\bar{\varepsilon} > 0$ for which $B_{\bar{\varepsilon}}(z) \subset \Omega_I$ (or Ω_C), $\boldsymbol{A}_0, \mathbf{curl}\,\boldsymbol{A}_0 \in \boldsymbol{C}^0(\overline{B_{\bar{\varepsilon}}(z)})$ and $\omega_\varepsilon \subset B_{\bar{\varepsilon}}(z)$ whenever $\varepsilon \in [0, \bar{\varepsilon}]$. This implies the existence of a constant $C > 0$ independent of ε such that

$$\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\|\boldsymbol{A}_0\|_{\boldsymbol{L}^\infty(B_{\bar{\varepsilon}}(z))}\sqrt{|\omega_\varepsilon|}\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|_{\boldsymbol{L}^2(\Omega)}$$
$$+ C\|\,\mathbf{curl}\,\boldsymbol{A}_0\|_{\boldsymbol{L}^\infty(B_{\bar{\varepsilon}}(z))}\sqrt{|\omega_\varepsilon|}\|\,\mathbf{curl}(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0)\|_{\boldsymbol{L}^2(\Omega)} \underbrace{\leq}_{|\omega_\varepsilon| = \varepsilon^3|\omega|} C\varepsilon^{\frac{3}{2}}\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}.$$

This concludes the proof. □

Definition 4.3.11. *Let $T_\varepsilon\colon \mathbb{R}^3 \to \mathbb{R}^3$ denote the transformation*

$$x \mapsto T_\varepsilon(x) = z + \varepsilon x = x_\varepsilon.$$

For almost every $x \in T_\varepsilon^{-1}(\Omega) = \Omega_\varepsilon$, we define the first variation of the state $\boldsymbol{A}_\varepsilon$ by

$$\boldsymbol{U}_\varepsilon^{(1)}(x) := \left(\frac{\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0}{\varepsilon}\right) \circ T_\varepsilon(x), \quad \varepsilon > 0. \tag{4.73}$$

The field $\boldsymbol{U}^{(1)}$ defined in Lemma 4.3.9 aims to approximate $\boldsymbol{U}_\varepsilon^{(1)}$.

The following Lemma is the key tool which allows us to deal with inhomogeneous Dirichlet conditions in the presence of a saddle point structure.

Lemma 4.3.12. *Let $\Omega_\varepsilon = T_\varepsilon^{-1}(\Omega)$ and $\Gamma_\varepsilon = T_\varepsilon^{-1}(\Gamma)$. Let $(\boldsymbol{g}_\varepsilon, h_\varepsilon) \in \boldsymbol{V}_\tau(\Gamma_\varepsilon) \times H^{\frac{1}{2}}(\Gamma_\varepsilon)$ and $\boldsymbol{\Phi}_\varepsilon \in (\boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon), \|\cdot\|_{\varepsilon,\mathbf{curl}})^*$ be defined for sufficiently small values of $\varepsilon > 0$. Recall that $\boldsymbol{V}_\tau(\Gamma_\varepsilon) = \{\boldsymbol{v} \in \boldsymbol{L}^2_\tau(\Gamma_\varepsilon) : \mathrm{div}_\tau\,\boldsymbol{v} \in L^2(\Gamma_\varepsilon)\}$, see (4.57). Assume further that $(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) \in \boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon) \cap H(\mathrm{div}{=}0, \Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ satisfy*

$$\int_{\Omega_\varepsilon} \nu_\omega\,\mathbf{curl}\,\boldsymbol{w}_\varepsilon \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + i\int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 \boldsymbol{w}_\varepsilon \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + \int_{\Omega_\varepsilon} \varepsilon\nabla\phi_\varepsilon \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x = \langle \boldsymbol{\Phi}_\varepsilon, \boldsymbol{v}\rangle_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)} \tag{4.74}$$

for all $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$, as well as

$$\begin{cases} \boldsymbol{w}_\varepsilon \times \boldsymbol{n} = \boldsymbol{g}_\varepsilon & \text{on } \Gamma_\varepsilon \\ \phi_\varepsilon = h_\varepsilon & \text{on } \Gamma_\varepsilon. \end{cases}$$

Then there is a constant $C > 0$, independent of ε, such that

$$\|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} \leq C(\|\boldsymbol{\Phi}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)}$$
$$+ \varepsilon^{-\frac{1}{2}}\|\,\mathrm{div}_\tau\,\boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} + \varepsilon^{\frac{1}{2}}\|h_\varepsilon\|_{L^2(\Gamma_\varepsilon)} + |h_\varepsilon|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}). \tag{4.75}$$

The norm $\|\cdot\|_{\varepsilon,\mathbf{curl}}$ is defined in (4.46).

Proof. Let us consider the continuous lifting operators

$$E_{\Gamma_\varepsilon}\colon \boldsymbol{V}_\tau(\Gamma_\varepsilon) \to \boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)\cap H(\mathrm{div}{=}0,\Omega_\varepsilon), \quad Z_{\Gamma_\varepsilon}\colon H^{\frac{1}{2}}(\Gamma_\varepsilon) \to H^1(\Omega_\varepsilon),$$

that are respectively defined in Lemma 4.3.5 and Lemma 4.3.3.

Now let

$$\widetilde{\boldsymbol{w}}_\varepsilon := \boldsymbol{w}_\varepsilon - E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon) \quad \text{and} \quad \widetilde{\phi}_\varepsilon := \phi_\varepsilon - Z_{\Gamma_\varepsilon}(h_\varepsilon);$$

by construction we have $\widetilde{\boldsymbol{w}}_\varepsilon \times \boldsymbol{n} = \boldsymbol{0}$ and $\widetilde{\phi}_\varepsilon|_{\Gamma_\varepsilon} = 0$ on Γ_ε and by manipulating (4.74) it follows

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}\, \widetilde{\boldsymbol{w}}_\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 \widetilde{\boldsymbol{w}}_\varepsilon \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x + \underbrace{\int_{\Omega_\varepsilon} \varepsilon \nabla(\widetilde{\phi}_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x}_{=0} \\
&\quad = \langle \boldsymbol{\Phi}_\varepsilon, \boldsymbol{v}\rangle - \int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}(E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&\quad - i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - \int_{\Omega_\varepsilon} \varepsilon \nabla Z_{\Gamma_\varepsilon}(h_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon).
\end{aligned} \tag{4.76}$$

Note that the third integral vanishes due to $\widetilde{\phi}_\varepsilon \in H_0^1(\Omega_\varepsilon)$ in combination with $\boldsymbol{v} \in H(\mathrm{div}{=}0,\Omega_\varepsilon)$. As $\widetilde{\boldsymbol{w}}_\varepsilon \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$ by construction, we can test (4.76) with it. This yields

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}\, \widetilde{\boldsymbol{w}}_\varepsilon \cdot \mathbf{curl}\, \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 \widetilde{\boldsymbol{w}}_\varepsilon \cdot \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x \\
&= \langle \boldsymbol{\Phi}_\varepsilon, \widetilde{\boldsymbol{w}}_\varepsilon\rangle - \int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}(E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)) \cdot \mathbf{curl}\, \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x - i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon) \cdot \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x \\
&- \int_{\Omega_\varepsilon} \varepsilon \nabla(Z_{\Gamma_\varepsilon}(h_\varepsilon)) \cdot \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x.
\end{aligned}$$

The left hand side can be estimated as

$$\left| \int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}\, \widetilde{\boldsymbol{w}}_\varepsilon \cdot \mathbf{curl}\, \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 \widetilde{\boldsymbol{w}}_\varepsilon \cdot \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x \right| \geq C \|\widetilde{\boldsymbol{w}}_\varepsilon\|^2_{\varepsilon,\mathbf{curl}} \tag{4.77}$$

thanks to an application of the scaled Poincaré-Friedrichs inequality (4.53) (and $\mathrm{div}\, \widetilde{\boldsymbol{w}}_\varepsilon = 0$), and the fact that $\nu_\omega(x) \geq \min(\nu_1, \nu_2) > 0$ and $\sigma_\omega \geq 0$ in $\mathbb{R}^3$.

For the right hand side we have $|\langle \boldsymbol{\Phi}_\varepsilon, \widetilde{\boldsymbol{w}}_\varepsilon\rangle| \leq \|\boldsymbol{\Phi}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} \|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\varepsilon,\mathbf{curl}}$ by hypothesis and taking into account $\nu_\omega(x) \leq \max(\nu_1, \nu_2)$ and $\sigma_\omega(x) \leq \sigma_0$,

$$\begin{aligned}
&\left| \int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}(E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)) \cdot \mathbf{curl}\, \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon) \cdot \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x + \int_{\Omega_\varepsilon} \varepsilon \nabla Z_{\Gamma_\varepsilon}(h_\varepsilon) \cdot \overline{\widetilde{\boldsymbol{w}}_\varepsilon}\, \mathrm{d}x \right| \\
&\leq C(\|\mathbf{curl}(E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon))\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \|\mathbf{curl}\, \widetilde{\boldsymbol{w}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \varepsilon^2 \|E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}) \\
&+ C\varepsilon \|\nabla Z_{\Gamma_\varepsilon}(h_\varepsilon)\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \\
&\leq C \|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\varepsilon,\mathbf{curl}} (\|\mathbf{curl}(E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon))\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \varepsilon \|E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \|Z_{\Gamma_\varepsilon}(h_\varepsilon)\|_{\varepsilon,\nabla}) \\
&= C \|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\varepsilon,\mathbf{curl}} (\|E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)\|_{\varepsilon,\mathbf{curl}} + \|Z_{\Gamma_\varepsilon}(h_\varepsilon)\|_{\varepsilon,\nabla}).
\end{aligned} \tag{4.78}$$

In force of (4.52) and (4.59), it holds

$$\|Z_{\Gamma_\varepsilon}(h_\varepsilon)\|_{\varepsilon,\nabla} \leq C(\varepsilon^{\frac{1}{2}}\|h_\varepsilon\|_{L^2(\Omega_\varepsilon)} + |h_\varepsilon|_{H^{1/2}(\Gamma_\varepsilon)}) \tag{4.79}$$

as well as

$$\|E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)\|_{\varepsilon,\mathbf{curl}} \leq C(\varepsilon^{\frac{1}{2}}\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)}). \tag{4.80}$$

Putting together (4.77), (4.78), (4.80) and (4.79) furnishes

$$\begin{aligned}\|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\varepsilon,\mathbf{curl}} \leq C(\|\boldsymbol{\Phi}\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \\ + \varepsilon^{\frac{1}{2}}\|h_\varepsilon\|_{L^2(\Omega_\varepsilon)} + |h_\varepsilon|_{H^{1/2}(\Gamma_\varepsilon)}).\end{aligned}$$

To conclude, we observe that

$$\|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} = \|\widetilde{\boldsymbol{w}}_\varepsilon + E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)\|_{\varepsilon,\mathbf{curl}} \leq \|\widetilde{\boldsymbol{w}}_\varepsilon\|_{\varepsilon,\mathbf{curl}} + \|E_{\Gamma_\varepsilon}(\boldsymbol{g}_\varepsilon)\|_{\varepsilon,\mathbf{curl}}$$

and apply (4.80) yet another time. This completes the proof. □

As is usually done in the context of topological derivatives, we shall derive the asymptotic expansion of the state and averaged adjoint variables under the following additional

Assumption 4.3.13.

- *If $z \in \Omega_C$, then $\Omega_I = \emptyset$. In other words, prior to the perturbation it holds $\Omega_C = \Omega$ and $\Omega_I = \emptyset$, while after the topological perturbation is performed $\Omega_C = \Omega \setminus \overline{\omega_\varepsilon}$ and $\Omega_I = \omega_\varepsilon$.*
- *If $z \in \Omega_I$, then $\Omega_C = \emptyset$. In other words, prior to the perturbation it holds $\Omega_I = \Omega$ and $\Omega_C = \emptyset$, while after the topological perturbation is performed $\Omega_C = \omega_\varepsilon$ and $\Omega_I = \Omega \setminus \overline{\omega_\varepsilon}$.*

Working with Assumption 4.3.13 allows us to present less technical and better understandable proofs of Theorem 4.3.14, Theorem 4.4.5 and Theorem 4.6.8. The essential feature that distinguishes our framework from classical scalar elliptic problems can be identified with the presence of divergence free-constraints in the functional spaces, which in turn entail an underlying structure of saddle point problems and the appearance of gradient multipliers (cf. Lemma 4.6.2, Remark 4.6.3). This is not affected by the geometrical simplification given by Assumption 4.3.13. At the same time, Remark 4.2.3 and Remark 4.3.7 show that previously stated existence theorems for $\boldsymbol{A}_0, \boldsymbol{A}_\varepsilon$ remain valid under Assumption 4.3.13. The same can be said for the averaged adjoint, see Remark 4.4.2.

The following results asserts the $\boldsymbol{U}^{(1)}$ indeed provides an approximation of $\boldsymbol{U}_\varepsilon^{(1)}$. The theorem is stated and proved for $z \in \Omega_C$, see Theorem A.0.4 for $z \in \Omega_I$.

Theorem 4.3.14. *Assume $z \in \Omega_C$ and let Assumption 4.2.1 and Assumption 4.3.13 hold. Let $\boldsymbol{U}_\varepsilon^{(1)}$ be defined by (4.73) and $\boldsymbol{U}^{(1)}$ through Lemma 4.3.9. If $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_C)$ for some $\alpha > 0$, there is a constant $C > 0$ not depending on $\varepsilon > 0$ for which*

$$\|\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon \tag{4.81}$$

for sufficiently small ε.

Proof. We observe that

$$\begin{aligned}(\nu_\varepsilon \circ T_\varepsilon)(x) &= ((\nu_1 \chi_{\Omega_{C,\varepsilon}} + \nu_2 \chi_{\Omega \setminus \Omega_{C,\varepsilon}}) \circ T_\varepsilon)(x) \\ &= ((\nu_2 \chi_{\omega_\varepsilon} + \nu_1 \chi_{\Omega_C \setminus \omega_\varepsilon} + \nu_2 \chi_{\Omega_I}) \circ T_\varepsilon)(x) \\ &= (\nu_2 \chi_\omega + \nu_1 \chi_{T_\varepsilon^{-1}(\Omega) \setminus \omega})(x) = \nu_\omega|_{\Omega_\varepsilon}(x),\end{aligned}$$

where ν_ω is defined is (4.65) and Assumption 4.3.13 has been used. In a similar way we have $(\sigma_\varepsilon \circ T_\varepsilon)(x) = \sigma_\omega|_{\Omega_\varepsilon}(x)$. Keeping this in mind, the change of variables $x \mapsto T_\varepsilon(x)$ in (4.72) gives the following equation for $\boldsymbol{U}_\varepsilon^{(1)}$:

$$\begin{aligned}&\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\ =& i\sigma_0 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x + (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{A}_0)(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x\end{aligned} \tag{4.82}$$

for all $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$. For the sake of brevity we sometimes use the notation $x_\varepsilon = z + \varepsilon x = T_\varepsilon(x)$, $x \in \mathbb{R}^3$.

Next we note that if $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$, then the extension

$$\boldsymbol{v}^\star := \begin{cases} \boldsymbol{v} & \text{in } \Omega_\varepsilon \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases}$$

belongs to $\boldsymbol{W}(\mathbb{R}^3)$ and therefore it is a feasible test function for (4.68). This yields

$$\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x = (\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x, \tag{4.83}$$

so that we can now subtract (4.83) from (4.82) to obtain:

$$\begin{aligned}&\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\ =& - i\sigma_0 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x + (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \\ &+ i\sigma_0 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x.\end{aligned} \tag{4.84}$$

We are in position to apply Lemma 4.3.12 to equation (4.84) with the choices

$$(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) = (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}, 0), \quad (\boldsymbol{g}_\varepsilon, h_\varepsilon) = (\boldsymbol{w}_\varepsilon|_{\Gamma_\varepsilon} \times \boldsymbol{n}, 0)$$

and $\mathbf{\Phi}_\varepsilon$ equal to the mapping induced by the whole right hand side of (4.84), which we denote by $\hat{\mathbf{\Phi}}_\varepsilon$. On the boundary Γ_ε it holds

$$\boldsymbol{g}_\varepsilon = \boldsymbol{w}_\varepsilon \times \boldsymbol{n} = (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \times \boldsymbol{n} = -\boldsymbol{U}^{(1)} \times \boldsymbol{n} \tag{4.85}$$

and we stress that $\boldsymbol{g}_\varepsilon$ belongs to $\boldsymbol{V}_\tau(\Gamma_\varepsilon)$ since $\boldsymbol{U}^{(1)}$ is harmonic in the exterior of ω (see (4.68)). Therefore by Lemma 4.3.12, the following inequality holds true:

$$\begin{aligned}\|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} &\leq C(\|\hat{\mathbf{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{U}^{(1)} \times \boldsymbol{n}\|_{L^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau(\boldsymbol{U}^{(1)} \times \boldsymbol{n})\|_{L^2(\Gamma_\varepsilon)}) \\ &= C(\|\hat{\mathbf{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{U}^{(1)}\|_{L^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\mathbf{curl}\,\boldsymbol{U}^{(1)} \cdot \boldsymbol{n}\|_{L^2(\Gamma_\varepsilon)}) \\ &= C(\|\hat{\mathbf{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{U}^{(1)}\|_{L^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\mathbf{curl}\,\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)}),\end{aligned} \tag{4.86}$$

and it remains to estimate each term appearing on the right hand side.

For what concerns the boundary norms, we can exploit Lemma 4.3.2 combined with (4.69) to get

$$\varepsilon^{\frac{1}{2}}\|\boldsymbol{U}^{(1)}\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}} \tag{4.87}$$

for all $\varepsilon > 0$ sufficiently small. Moreover, thanks to the decay $|\mathbf{curl}\,\boldsymbol{U}^{(1)}(x)| = \mathcal{O}(1/|x|^3)$ as $|x| \to \infty$ ([AVV01, p. 786]), Lemma 4.3.2 also entails

$$\|\mathbf{curl}\,\boldsymbol{U}^{(1)}\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^2 \implies \varepsilon^{-\frac{1}{2}}\|\mathbf{curl}\,\boldsymbol{U}^{(1)}\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}}. \tag{4.88}$$

We are left to estimate the dual norm of $\hat{\mathbf{\Phi}}_\varepsilon$, which can be done by working separately on each term in (4.84). For the first one we have

$$\begin{aligned}\left|-i\sigma_0 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{U}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x\right| &\leq \varepsilon\sigma_0\|\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon\setminus\omega)}\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \\ &\leq \varepsilon\sigma_0\|\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}.\end{aligned} \tag{4.89}$$

Here we used the information $\boldsymbol{U}^{(1)} \in \boldsymbol{L}^2(\mathbb{R}^3)$, which follows from the decay property (4.69).

For the second one, under the hypothesis that $\boldsymbol{J} \in \boldsymbol{W}^{1,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$, we have by Lemma 4.2.6 that $\mathbf{curl}\,\boldsymbol{A}_0$ is locally Lipschitz continuous whence

$$\begin{aligned}&\left|(\nu_1 - \nu_2)\int_\omega (\mathbf{curl}\,\boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{A}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x\right| \\ &\leq C\|\mathbf{curl}\,\boldsymbol{A}_0(z+\varepsilon x) - \mathbf{curl}\,\boldsymbol{A}_0(z)\|_{\boldsymbol{L}^2(\omega)}\|\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\omega)} \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}.\end{aligned} \tag{4.90}$$

For the third term we need to argue as follows. By Hölder inequality,

$$\begin{aligned}\left|i\sigma_0\int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x\right| &\leq C\varepsilon\|\boldsymbol{A}_0(x_\varepsilon)\|_{\boldsymbol{L}^{\frac{6}{5}}(\omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^6(\omega)} \\ &\leq C\varepsilon\|\boldsymbol{v}\|_{\boldsymbol{L}^6(\omega)} \underbrace{=}_{(4.47)} C\varepsilon^{\frac{1}{2}}\|\boldsymbol{v} \circ T_\varepsilon^{-1}\|_{\boldsymbol{L}^6(\omega_\varepsilon)},\end{aligned}$$

where we used $\|\boldsymbol{A}_0\|_{\boldsymbol{L}^\infty(\omega_\varepsilon)} \leq C$ for all $\varepsilon > 0$, which follows from the continuity of $\boldsymbol{A}_0$. Setting $\boldsymbol{v}^\star = \boldsymbol{v} \circ T_\varepsilon^{-1}$, we are left to estimate $\|\boldsymbol{v}^\star\|_{\boldsymbol{L}^6(\omega_\varepsilon)}$.

Let K denote an open set such that $\overline{K}$ is strictly contained in Ω_C and $\omega_\varepsilon \subset K$ for all ε sufficiently small; we have $\|\boldsymbol{v}^\star\|_{\boldsymbol{L}^6(\omega_\varepsilon)} \leq \|\boldsymbol{v}^\star\|_{\boldsymbol{L}^6(K)}$. Now let $K^\star$ be an open with smooth boundary satisfying $\overline{K} \subsetneq K^\star$ as well as $\overline{K^\star} \subsetneq \Omega_C$. Let $\phi \in \mathcal{D}(K^\star)$ be a cut-off function such that $\phi = 1$ in K; since $\boldsymbol{v}^\star \in \boldsymbol{H}(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$, then $\boldsymbol{v}^\star \in \boldsymbol{H}(\mathbf{curl}, \boldsymbol{K}^\star) \cap H(\mathrm{div}, K^\star)$ whence

$$\phi \boldsymbol{v}^\star \in \boldsymbol{H}_0(\mathbf{curl}, \boldsymbol{K}^\star) \cap H(\mathrm{div}, K^\star) \implies \phi \boldsymbol{v}^\star \in \boldsymbol{H}^1(K^\star). \tag{4.91}$$

The last implication is a consequence of the continuous embedding

$$\boldsymbol{X}_N(D) = \boldsymbol{H}_0(\mathbf{curl}, D) \cap H(\mathrm{div}, D) \hookrightarrow \boldsymbol{H}^1(D),$$

which is known to hold if the domain D is of class $C^{1,1}$ (or if it is convex), see [Cos90; Amr+98]. By virtue of (4.91), $(\phi \boldsymbol{v}^\star)|_K = \boldsymbol{v}^\star \in \boldsymbol{H}^1(K)$ and especially

$$\|\boldsymbol{v}^\star\|_{\boldsymbol{H}^1(K)} \leq C \|\boldsymbol{v}^\star\|_{\boldsymbol{H}(\mathbf{curl}, K^\star) \cap H(\mathrm{div}, K^\star)} \leq C \|\boldsymbol{v}^\star\|_{\boldsymbol{H}(\mathbf{curl}, \Omega) \cap H(\mathrm{div}, \Omega)} \underbrace{=}_{\mathrm{div}\, \boldsymbol{v}^\star = 0} C \|\boldsymbol{v}^\star\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)}. \tag{4.92}$$

Combining (4.92) with Sobolev inequality [Eva10, p. 279], we obtain

$$\|\boldsymbol{v}^\star\|_{\boldsymbol{L}^6(\omega_\varepsilon)} \leq \|\boldsymbol{v}^\star\|_{\boldsymbol{L}^6(K)} \leq C \|\boldsymbol{v}^\star\|_{\boldsymbol{H}^1(K)} \leq C \|\boldsymbol{v}^\star\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} \underbrace{=}_{(4.49)} \varepsilon^{\frac{1}{2}} \|\boldsymbol{v}\|_{\varepsilon, \mathbf{curl}},$$

from which we conclude

$$\left| i\sigma_0 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \right| \leq C\varepsilon \|\boldsymbol{v}\|_{\varepsilon, \mathbf{curl}}. \tag{4.93}$$

Summing up, the latter equation, (4.86), (4.89), (4.90), (4.88) and (4.87) all together imply that

$$\|\boldsymbol{w}_\varepsilon\|_{\varepsilon, \mathbf{curl}} \leq C\varepsilon$$

for some $C > 0$ not depending on ε, and ε sufficiently small. This concludes the proof. □

Remark 4.3.15. *Recalling the definition of $\boldsymbol{U}_\varepsilon^{(1)}$ (4.73), the conclusion of the theorem can be restated as*

$$\left\| \frac{\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0}{\varepsilon} \circ T_\varepsilon - \boldsymbol{U}^{(1)} \right\|_{\varepsilon, \mathbf{curl}} \leq C\varepsilon.$$

Expanding the terms and changing variables, in view of Lemma 4.3.1 this implies

$$\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0 - \varepsilon \boldsymbol{U}^{(1)} \circ T_\varepsilon^{-1}\|_{\boldsymbol{H}(\mathbf{curl}, \Omega)} \leq C\varepsilon^{\frac{5}{2}}, \tag{4.94}$$

which can be seen as a first order expansion of $\boldsymbol{A}_\varepsilon$ in $\boldsymbol{H}(\mathbf{curl})$, making use of $\boldsymbol{U}^{(1)}$.

4.4 Averaged adjoint equation and its asymptotic analysis

According to (4.9), (4.63) and taking into account the remarks we made on complex Lagrangians, we state the averaged adjoint equation corresponding to our model. Given $\varepsilon > 0$, it is the problem

$$\begin{cases} \text{Find } \boldsymbol{q}_\varepsilon \in \boldsymbol{X}_N^0 \text{ s.t.} \\ \displaystyle\int_\Omega \nu_\varepsilon \,\mathbf{curl}\, \boldsymbol{q}_\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_\Omega \sigma_\varepsilon \boldsymbol{q}_\varepsilon \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x = -\lambda_1 \int_{\Omega_{C,\varepsilon}} (\boldsymbol{A}_\varepsilon + \boldsymbol{A}_0) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\ \displaystyle -\lambda_2 \int_\Omega \nu_\varepsilon \,\mathbf{curl}(\boldsymbol{A}_\varepsilon + \boldsymbol{A}_0) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \qquad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0, \end{cases} \tag{4.95}$$

where $\boldsymbol{A}_\varepsilon$ is the solution of (4.64), $\boldsymbol{A}_0$ is the solution of the same problem with $\varepsilon = 0$ (i.e., (4.27)) and $\Omega_{C,\varepsilon}$ denotes the perturbed conductor (4.61). In the unperturbed setting $\varepsilon = 0$, it reads

$$\begin{cases} \text{Find } \boldsymbol{q}_0 \in \boldsymbol{X}_N^0 \text{ s.t.} \\ \displaystyle\int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{q}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_\Omega \sigma \boldsymbol{q}_0 \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x = -2\lambda_1 \int_{\Omega_C} \boldsymbol{A}_0 \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\ \displaystyle -2\lambda_2 \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0. \end{cases} \tag{4.96}$$

In the following lemma we discuss the well-posedness and the regularity properties of (4.95) and (4.96).

Lemma 4.4.1. *Problem (4.95) has a unique solution $\boldsymbol{q}_\varepsilon \in \boldsymbol{X}_N^0$ for all $\varepsilon > 0$ sufficiently small and problem (4.96) admits a unique solution $\boldsymbol{q}_0 \in \boldsymbol{X}_N^0$. Additionally, we have the following regularity result:*

- *If*
$$\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_I) \quad \text{for some } \alpha > 0,$$
then
$$\boldsymbol{q}_0 \in \boldsymbol{C}_{loc}^{1,\beta_1}(\Omega_I), \quad \mathbf{curl}\, \boldsymbol{q}_0 \in \boldsymbol{C}_{loc}^{1,\kappa_1}(\Omega_I) \quad \text{for some } \beta_1, \kappa_1 \in (0,1). \tag{4.97}$$
In particular, $\boldsymbol{q}_0$ and $\mathbf{curl}\, \boldsymbol{q}_0$ are locally Lipschitz continuous in Ω_I.
- *If*
$$\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_C) \quad \text{for some } \alpha > 0,$$
then
$$\boldsymbol{q}_0 \in \boldsymbol{C}_{loc}^{1,\beta_2}(\Omega_C), \quad \mathbf{curl}\, \boldsymbol{q}_0 \in \boldsymbol{C}_{loc}^{1,\kappa_2}(\Omega_C) \quad \text{for some } \beta_2, \kappa_2 \in (0,1). \tag{4.98}$$
In particular, $\boldsymbol{q}_0$ and $\mathbf{curl}\, \boldsymbol{q}_0$ are locally Lipschitz continuous in Ω_C.

The space $\boldsymbol{W}_{loc}^{0,p}(\mathbf{curl}, \Omega)$ is defined in (4.34).

Proof. As the left hand side of (4.95) features the same sesquilinear form of the state equation (4.64) and the right hand side induces an antilinear, bounded functional on $\boldsymbol{X}_N^0$, existence and uniqueness readily follow by an application of Lax-Milgram lemma. The same argument can be used for (4.96).

For the regularity statements we focus on $\boldsymbol{q}_0$, hence on equation (4.96). Arguing as in Lemma 4.2.5, it can be proved that there exists a multiplier $\nabla m \in \nabla H_0^1(\Omega)$ for which the saddle point formulation

$$\int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{q}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_\Omega \sigma \boldsymbol{q}_0 \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x + \int_\Omega \nabla m \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x$$
$$= -2\lambda_1 \int_{\Omega_C} \boldsymbol{A}_0 \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - 2\lambda_2 \int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega) \tag{4.99}$$

holds true.

Regularity in the insulator. Let $\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}(\Omega_I)$. Testing (4.99) with

$$\widetilde{\boldsymbol{\eta}} = \begin{cases} \boldsymbol{\eta} & \text{in } \Omega_I \\ \boldsymbol{0} & \text{in } \Omega_C \end{cases} \in \boldsymbol{\mathcal{D}}(\Omega)$$

(keeping in mind that $\sigma = 0$ in Ω_I) we obtain

$$\int_{\Omega_I} \nu_2 \,\mathbf{curl}(\boldsymbol{q}_0 + 2\lambda_2 \boldsymbol{A}_0) \cdot \mathbf{curl}\, \overline{\boldsymbol{\eta}} \,\mathrm{d}x + \int_{\Omega_I} \nabla m \cdot \overline{\boldsymbol{\eta}} \,\mathrm{d}x = 0 \quad \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}(\Omega_I),$$

whence

$$\nu_2 \,\mathbf{curl}\,\mathbf{curl}(\boldsymbol{q}_0 + 2\lambda_2 \boldsymbol{A}_0)|_{\Omega_I} + \nabla m|_{\Omega_I} = \boldsymbol{0} \implies -\nu_2 \boldsymbol{\Delta} \boldsymbol{q}_0|_{\Omega_I} = 2\lambda_2 \nu_2 \boldsymbol{\Delta} \boldsymbol{A}_0|_{\Omega_I} - \nabla m|_{\Omega_I} \tag{4.100}$$

in Ω_I. Note that we used the information $\operatorname{div} \boldsymbol{q}_0 = \operatorname{div} \boldsymbol{A}_0 = 0$ in Ω. Making use of (4.42), we can further expand (4.100) as

$$-\nu_2 \boldsymbol{\Delta} \boldsymbol{q}_0|_{\Omega_I} = -2\lambda_2 \boldsymbol{j}^\star|_{\Omega_I} - \nabla m|_{\Omega_I} \quad \text{in } \Omega_I, \tag{4.101}$$

where $\boldsymbol{j}^\star \in \boldsymbol{L}_{loc}^{3+\alpha}(\Omega_I)$ and $\operatorname{div} \boldsymbol{j}^\star = 0$ (see (4.40)). Taking the divergence of both sides in the last equation, we see that $m|_{\Omega_I} \in H^1(\Omega_I)$ is harmonic and hence smooth by Weyl's lemma [Jos07, Cor. 1.2.1]. Consequently, $\boldsymbol{\Delta} \boldsymbol{q}_0|_{\Omega_I} \in \boldsymbol{L}_{loc}^{3+\alpha}(\Omega_I)$ which gives $\boldsymbol{q}_0|_{\Omega_I} \in \boldsymbol{C}_{loc}^{1,\beta_1}(\Omega_I)$ for some $\beta_1 \in (0,1)$ thanks to classical elliptic regularity theory [Jos07, Chap. 9] and Sobolev-Morrey embeddings. This proves the assertion for $\boldsymbol{q}_0$ in (4.97). To conclude, we take the **curl** of both sides in (4.101); this yields

$$-\nu_2 \boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{q}_0)|_{\Omega_I} = -2\lambda_2 \,\mathbf{curl}\, \boldsymbol{j}^\star|_{\Omega_I} \underbrace{=}_{(4.40)} -2\lambda_2 \,\mathbf{curl}\, \boldsymbol{J}|_{\Omega_I} \in \boldsymbol{L}_{loc}^{3+\alpha}(\Omega_I),$$

which completes the proof of (4.97) by the same regularity arguments as above.

Regularity in the conductor. We argue as before and given $\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}(\Omega_C)$, we test (4.96) with

$$\widetilde{\boldsymbol{\eta}} = \begin{cases} \boldsymbol{\eta} & \text{in } \Omega_C \\ \mathbf{0} & \text{in } \Omega_I \end{cases} \in \boldsymbol{\mathcal{D}}(\Omega).$$

This gives

$$-\nu_1 \boldsymbol{\Delta} \boldsymbol{q}_0|_{\Omega_C} = (-i\sigma_0 \boldsymbol{q}_0 - 2\lambda_1 \boldsymbol{A}_0 + 2\lambda_2 \nu_1 \boldsymbol{\Delta} \boldsymbol{A}_0 - \nabla m)|_{\Omega_C} \quad \text{in } \Omega_C \tag{4.102}$$

and taking the divergence of both sides, we deduce that $m|_{\Omega_C}$ is also harmonic, hence smooth. From Lemma 4.2.6, we already know that $\boldsymbol{\Delta} \boldsymbol{A}_0|_{\Omega_C}$ belongs to $\boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C)$ (see (4.44) and the lines below it) and $\boldsymbol{A}_0$ is (locally) continuous; the fact that $\boldsymbol{q}_0|_{\Omega_C} \in \boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C)$ is implied by $\boldsymbol{q}_0 \in \boldsymbol{X}^0_N$ (see again the proof of Lemma 4.2.6). Summing up, the whole right hand side of (4.102) lies in $\boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C)$ with $\alpha > 0$ so that $\boldsymbol{q}_0 \in \boldsymbol{C}^{1,\beta_2}_{loc}(\Omega_C)$ and the first part of (4.98) is proved.

Lastly, we take the **curl** of both sides in (4.102) to obtain

$$-\nu_1 \boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{q}_0|_{\Omega_C}) = (-i\sigma_0\, \mathbf{curl}\, \boldsymbol{q}_0 - 2\lambda_1\, \mathbf{curl}\, \boldsymbol{A}_0 + 2\lambda_2 \nu_1 \boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{A}_0))|_{\Omega_C}.$$

Now $\mathbf{curl}\, \boldsymbol{q}_0$ is locally Hölder continuous as $\boldsymbol{q}_0 \in \boldsymbol{C}^{1,\beta_2}_{loc}(\Omega_C)$, $\boldsymbol{A}_0$ is continuous and $\boldsymbol{\Delta}(\mathbf{curl}\, \boldsymbol{A}_0)$ lies in $\boldsymbol{L}^{3+\alpha}_{loc}(\Omega_C)$ in the given hypotheses by virtue of Lemma 4.2.6 (see the last part of the proof). This proves the remaining assertion in (4.98) and completes the proof. ☐

Remark 4.4.2. *We remind that Lemma 4.4.1 (the statements concerning existence and uniqueness for (4.95) and (4.96)) holds true even if $\Omega_C = \emptyset$ or $\Omega_I = \emptyset$, i.e. in the geometrical situation described by Assumption 4.3.13. Since the bilinear form is the same appearing in the state equations (4.64), (4.27), the arguments of Remark 4.2.3 and Remark 4.3.7 are also pertinent here. Moreover, the right hand side of (4.95) (and (4.96)) can be seen as a continuous antilinear form on $\boldsymbol{X}^0_N$ regardless of Ω_C or Ω_I being empty.*

Lemma 4.4.3. *Assume that $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$ for some $\alpha > 0$ and $z \in \Omega_C$ (see Lemma A.0.5 for the case $z \in \Omega_I$). There is a constant $C > 0$ not depending on ε such that*

$$\|\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\varepsilon^{\frac{3}{2}} \tag{4.103}$$

for sufficiently small $\varepsilon > 0$.

Proof. We start subtracting (4.96) from (4.95), this yields:

$$\begin{aligned}
&\int_\Omega \nu_\varepsilon\, \mathbf{curl}(\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + i \int_\Omega \sigma_\varepsilon (\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&\quad = (\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{q}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + i\sigma_0 \int_{\omega_\varepsilon} \boldsymbol{q}_0 \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - \lambda_1 \int_{\Omega_{C,\varepsilon}} (\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&\quad + 2\lambda_1 \int_{\omega_\varepsilon} \boldsymbol{A}_0 \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - \lambda_2 \int_\Omega \nu_\varepsilon\, \mathbf{curl}(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&\quad + 2\lambda_2(\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x
\end{aligned} \tag{4.104}$$

for all $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega)$. Next we test (4.104) with $\boldsymbol{v} = \boldsymbol{q}_\varepsilon - \boldsymbol{q}_0$. Exploiting the continuity of $\boldsymbol{A}_0, \boldsymbol{q}_0$ and of $\mathbf{curl}\, \boldsymbol{A}_0, \mathbf{curl}\, \boldsymbol{q}_0$ (Lemma 4.4.1, Lemma 4.2.6) and what we already know on the scaling behaviour of $\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|_{\boldsymbol{H}(\mathbf{curl})}$ from (4.71), we can mimic the estimates performed in the proof of Lemma 4.3.10. This leads to (4.103) and concludes the proof. □

4.4.1 Corrector and asymptotic analysis for the adjoint problem

In this section we introduce another auxiliary vector field, which is related to the averaged adjoint variable. The considerations made in Section 4.3.3 remain valid, i.e. we have $z \in \Omega_C$ in mind. For the case $z \in \Omega_I$ we refer to Appendix A.

Lemma 4.4.4. *Let $\boldsymbol{U}^{(1)}$ be defined through Lemma 4.3.9. There exists a unique solution $[\boldsymbol{Q}^{(1)}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$ to*

$$\int_{\mathbb{R}^3} \nu_\omega\, \mathbf{curl}[\boldsymbol{Q}^{(1)}] \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(z) + 2\lambda_2\, \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\ - \lambda_2 \int_{\mathbb{R}^3} \nu_\omega\, \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \dot{\boldsymbol{W}}(\mathbb{R}^3). \tag{4.105}$$

Moreover, there is a unique divergence-free representative $\boldsymbol{Q}^{(1)} \in [\boldsymbol{Q}^{(1)}] \cap \boldsymbol{S}$ (the space $\boldsymbol{S}$ is defined in (4.70)) that satisfies

$$\int_{\mathbb{R}^3} \nu_\omega\, \mathbf{curl}\, \boldsymbol{Q}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(z) + 2\lambda_2\, \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\ - \lambda_2 \int_{\mathbb{R}^3} \nu_\omega\, \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{W}(\mathbb{R}^3) \tag{4.106}$$

and enjoys the decay behaviour at infinity

$$|\boldsymbol{Q}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty. \tag{4.107}$$

Proof. Existence and uniqueness of $[\boldsymbol{Q}^{(1)}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$ readily follow by Riesz's representation theorem in $\dot{\boldsymbol{W}}$, see Lemma 4.1.3. The existence of a unique divergence free representative $\boldsymbol{Q}^{(1)} \in [\boldsymbol{Q}^{(1)}] \cap \boldsymbol{S}$ for which (4.106) holds can be argued as in Lemma 4.3.9, making use of the properties of the space $\dot{\boldsymbol{W}}$ stated in Lemma 4.1.4.

It remains to show (4.107). By construction, the field $\boldsymbol{Q}^\star := \boldsymbol{Q}^{(1)} + \lambda_2 \boldsymbol{U}^{(1)} \in \boldsymbol{S}$ satisfies

$$\int_{\mathbb{R}^3} \nu_\omega\, \mathbf{curl}\, \boldsymbol{Q}^\star \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(z) + 2\lambda_2\, \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \ \ \forall \boldsymbol{v} \in \boldsymbol{S}.$$

Therefore it can be shown (see [AVV01, pp. 785-786] and [AVV01, Lemma 4]) that

$$|\boldsymbol{Q}^\star(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty \underbrace{\Longrightarrow}_{(4.69)} |\boldsymbol{Q}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty.$$

□

Let us now go back to the convergence analysis concerning $\boldsymbol{Q}_\varepsilon^{(1)}$. Following the theory presented for the state equation, we define the first variation of the averaged adjoint $\boldsymbol{q}_\varepsilon$; for almost every $x \in T_\varepsilon^{-1}(\Omega) = \Omega_\varepsilon$ it reads

$$\boldsymbol{Q}_\varepsilon^{(1)}(x) := \left(\frac{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0}{\varepsilon}\right) \circ T_\varepsilon(x), \quad \varepsilon > 0. \tag{4.108}$$

Theorem 4.4.5. *Assume $z \in \Omega_C$ and let Assumption 4.3.13 hold. Let $\boldsymbol{Q}_\varepsilon^{(1)}$ be given by (4.108) and $\boldsymbol{Q}^{(1)}$ be defined via (4.106). If $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_C)$ ($\alpha > 0$), there exists a constant $C > 0$ not depending on ε such that*

$$\|\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}\|_{\varepsilon,\mathbf{curl}} \le C\varepsilon. \tag{4.109}$$

We refer to Theorem A.0.7 for the case $z \in \Omega_I$.

Proof. Changing variables in (4.104) gives the following equation for $\boldsymbol{Q}_\varepsilon^{(1)}$:

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{Q}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_{\Omega_\varepsilon} \varepsilon^2 \sigma_\omega \boldsymbol{Q}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad = (\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{q}_0(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\sigma_0 \int_\omega \varepsilon \boldsymbol{q}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - \lambda_1 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad + 2\lambda_1 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - \lambda_2 \int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad + 2\lambda_2 (\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon).
\end{aligned} \tag{4.110}$$

As $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$, the extension

$$\boldsymbol{v}^\star := \begin{cases} \boldsymbol{v} & \text{in } \Omega_\varepsilon \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases}$$

belongs to $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \hookrightarrow \boldsymbol{W}(\mathbb{R}^3)$ and therefore it can be chosen as test function in (4.106). This furnishes

$$\begin{aligned}
\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{Q}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x &= (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(z) + 2\lambda_2 \, \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad - \lambda_2 \int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x.
\end{aligned} \tag{4.111}$$

Subtracting (4.111) from (4.110) gives

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}(\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + i\int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x \\
&= -i\sigma_0 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{Q}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x - \lambda_1 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + i\sigma_0 \int_\omega \varepsilon \boldsymbol{q}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x \\
&+ 2\lambda_1 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon)\cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + (\nu_1 - \nu_2)\int_\omega (\mathbf{curl}\,\boldsymbol{q}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{q}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&+ 2\lambda_2(\nu_1-\nu_2)\int_\omega (\mathbf{curl}\,\boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{A}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&+ \lambda_2 \left(-\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\,\boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\,\boldsymbol{U}^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \right).
\end{aligned} \tag{4.112}$$

Now we apply Lemma 4.3.12 to (4.112) with the choices $\boldsymbol{\Phi}_\varepsilon = \boldsymbol{\Theta}_\varepsilon \in (\boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon), \|\cdot\|_{\varepsilon,\mathbf{curl}})^*$, $(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) = (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}, 0)$ and $(\boldsymbol{g}_\varepsilon, h_\varepsilon) = (\boldsymbol{w}_\varepsilon \times \boldsymbol{n}, 0)$. Here $\boldsymbol{\Theta}_\varepsilon$ is a shortcut for the whole right hand side of (4.112). It follows that

$$\|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} \le C(\|\boldsymbol{\Theta}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}} \|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)}), \tag{4.113}$$

and it remains to estimate each term on the right hand side of the above equation.

We start with the boundary norms. Likewise (4.85), we have

$$\boldsymbol{g}_\varepsilon = \boldsymbol{w}_\varepsilon \times \boldsymbol{n} \underbrace{=}_{\boldsymbol{Q}_\varepsilon^{(1)} \times \boldsymbol{n} = \boldsymbol{0}} -\boldsymbol{Q}^{(1)} \times \boldsymbol{n} \quad \text{on } \Gamma_\varepsilon,$$

and

$$\begin{aligned}
\varepsilon^{\frac{1}{2}} \|\boldsymbol{Q}^{(1)} \times \boldsymbol{n}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} &= \varepsilon^{\frac{1}{2}} \|\boldsymbol{Q}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \le C\varepsilon^{\frac{3}{2}}, \\
\varepsilon^{-\frac{1}{2}} \|\operatorname{div}_\tau(\boldsymbol{Q}^{(1)} \times \boldsymbol{n})\|_{L^2(\Gamma_\varepsilon)} &= \varepsilon^{-\frac{1}{2}} \|\mathbf{curl}\,\boldsymbol{Q}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \le C\varepsilon^{\frac{3}{2}}
\end{aligned} \tag{4.114}$$

thanks to Lemma 4.3.2 and the fact that $\boldsymbol{Q}^{(1)}$ and $\boldsymbol{U}^{(1)}$ share the same asymptotic behaviour at infinity (see also the proof of Theorem 4.3.14).

Next we turn our attention to $\|\boldsymbol{\Theta}_\varepsilon\|$; first we make the splitting

$$\boldsymbol{\Theta}_\varepsilon(\boldsymbol{v}) = \boldsymbol{\Theta}_\varepsilon^{(1)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(2)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(3)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(4)}(\boldsymbol{v}),$$

where

$$\begin{aligned}
\boldsymbol{\Theta}_\varepsilon^{(1)}(\boldsymbol{v}) &= -i\sigma_0 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{Q}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x - \lambda_1 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x, \\
\boldsymbol{\Theta}_\varepsilon^{(2)}(\boldsymbol{v}) &= i\sigma_0 \int_\omega \varepsilon \boldsymbol{q}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + 2\lambda_1 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x, \\
\boldsymbol{\Theta}_\varepsilon^{(3)}(\boldsymbol{v}) &= (\nu_1 - \nu_2)\int_\omega (\mathbf{curl}\,\boldsymbol{q}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{q}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&\quad + 2\lambda_2(\nu_1-\nu_2)\int_\omega (\mathbf{curl}\,\boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{A}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x, \\
\boldsymbol{\Theta}_\varepsilon^{(4)}(\boldsymbol{v}) &= \lambda_2 \left(-\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\,\boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\,\boldsymbol{U}^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \right).
\end{aligned} \tag{4.115}$$

By Hölder and triangle inequality and keeping in mind $\boldsymbol{Q}^{(1)} \in \boldsymbol{L}^2(\mathbb{R}^3)$ (which follows from the decay (4.107)),

$$|\Theta_\varepsilon^{(1)}(\boldsymbol{v})| \leq C(\varepsilon\|\boldsymbol{Q}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \varepsilon\|\boldsymbol{U}_\varepsilon^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)})\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}$$
$$\leq C(\varepsilon + \varepsilon\|\boldsymbol{U}_\varepsilon^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}) \leq C\varepsilon(1 + \|\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}) \underbrace{\leq}_{(4.81)} C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}. \tag{4.116}$$

For the second one we get

$$|\Theta_\varepsilon^{(2)}(\boldsymbol{v})| \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \tag{4.117}$$

as we can repeat the arguments used for (4.93) in the proof of Theorem 4.3.14.

Since $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_C)$, we know that $\mathbf{curl}\, \boldsymbol{q}_0, \mathbf{curl}\, \boldsymbol{A}_0$ are Lipschitz continuous in a neighbourhood of $z \in \Omega_C$ and therefore (compare with the proof for the state equation, (4.90))

$$|\Theta_\varepsilon^{(3)}(\boldsymbol{v})| \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}. \tag{4.118}$$

Finally, by Hölder inequality we obtain

$$|\Theta_\varepsilon^{(4)}(\boldsymbol{v})| = \left|\lambda_2 \int_{\Omega_\varepsilon} \nu_\omega\, \mathbf{curl}(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x\right|$$
$$\leq C\|\,\mathbf{curl}(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)})\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}\|\,\mathbf{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \underbrace{\leq}_{(4.81)} C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}. \tag{4.119}$$

To conclude, (4.116), (4.117), (4.118) and (4.119) all together imply

$$\|\Theta_\varepsilon\|_{(\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon),\|\cdot\|_{\varepsilon,\mathbf{curl}})^*} \leq C\varepsilon,$$

and the above estimate in combination with (4.113) and (4.114) concludes the proof. □

4.5 First order topological derivative

In this section we aim to compute the first order topological derivative corresponding to the shape functional

$$\mathcal{J}(\Omega_C) = J(\Omega_C, \boldsymbol{A}) = \frac{\lambda_1}{2}\int_{\Omega_C} |\boldsymbol{A}|^2\, \mathrm{d}x + \frac{\lambda_2}{2}\int_{\Omega} \nu_{\Omega_C}|\,\mathbf{curl}\,\boldsymbol{A}|^2\, \mathrm{d}x, \tag{4.120}$$

subject to

$$\boldsymbol{A} \in \boldsymbol{X}_N^0, \quad \int_\Omega \nu_{\Omega_C}\, \mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\, \mathrm{d}x + i\int_\Omega \sigma_{\Omega_C}\boldsymbol{A} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x = \int_\Omega \boldsymbol{J} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0. \tag{4.121}$$

The new notation $\nu_{\Omega_C}, \sigma_{\Omega_C}$ is to stress the fact that we consider the conductor Ω_C as the main subject of the topological perturbation. Given $\varepsilon > 0$, we remind that the conducting domain is perturbed in the following way: for $z \in \Omega \setminus \partial\Omega_C$ we set $\omega_\varepsilon = z + \varepsilon\omega$ and

$$\Omega_{C,\varepsilon} = \begin{cases} \Omega_C \cup \omega_\varepsilon & \text{if } z \in \Omega_I \\ \Omega_C \setminus \omega_\varepsilon & \text{if } z \in \Omega_C, \end{cases}$$

so that the perturbed parameters read

$$\sigma_{\Omega_{C,\varepsilon}} = \sigma_\varepsilon(x) = \chi_{\Omega_{C,\varepsilon}}(x)\sigma_0,$$
$$\nu_{\Omega_{C,\varepsilon}} = \nu_\varepsilon(x) = \chi_{\Omega_{C,\varepsilon}}(x)\nu_1 + \chi_{\Omega\setminus\Omega_{C,\varepsilon}}(x)\nu_2.$$

It is also useful to recall the Lagrangian function introduced in Section 4.3.2:

$$\begin{aligned}\mathcal{L}(\varepsilon, \boldsymbol{A}, \boldsymbol{q}) \\ = J(\Omega_{C,\varepsilon}, \boldsymbol{A}) + \mathrm{Re}\left(\int_\Omega \nu_{\Omega_{C,\varepsilon}} \,\mathbf{curl}\, \boldsymbol{A} \cdot \mathbf{curl}\, \overline{\boldsymbol{q}}\, \mathrm{d}x + i\int_\Omega \sigma_{\Omega_{C,\varepsilon}} \boldsymbol{A}\cdot\overline{\boldsymbol{q}}\,\mathrm{d}x - \int_\Omega \boldsymbol{J}\cdot\overline{\boldsymbol{q}}\,\mathrm{d}x\right),\end{aligned} \tag{4.122}$$

where

$$\mathcal{J}(\Omega_{C,\varepsilon}) = J(\Omega_{C,\varepsilon}, \boldsymbol{A}) = \frac{\lambda_1}{2}\int_{\Omega_{C,\varepsilon}} |\boldsymbol{A}|^2\,\mathrm{d}x + \frac{\lambda_2}{2}\int_\Omega \nu_{\Omega_{C,\varepsilon}} |\,\mathbf{curl}\,\boldsymbol{A}|^2\,\mathrm{d}x.$$

By definition, the first order topological derivative is given by

$$d\mathcal{J}(\Omega_C, \omega) = \lim_{\varepsilon\to 0^+} \frac{\mathcal{J}(\Omega_{C,\varepsilon}) - \mathcal{J}(\Omega_C)}{|\omega_\varepsilon|}, \tag{4.123}$$

or equivalently characterized through the validity of the expansion

$$\mathcal{J}(\Omega_{C,\varepsilon}) = \mathcal{J}(\Omega_C) + d\mathcal{J}(\Omega_C,\omega)|\omega_\varepsilon| + o(|\omega_\varepsilon|) \quad \text{as } \varepsilon \to 0^+.$$

Theorem 4.5.1. *Assume that, for some $\alpha > 0$, $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$ if $z \in \Omega_C$ (resp. $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_I)$ if $z \in \Omega_I$). The first order topological derivative associated with the shape functional (4.120) subject to the low-frequency electromagnetic problem (4.121) is given by*

$$\begin{aligned}&d\mathcal{J}(\Omega_C, \omega) = \\ &-\tau_z\Big(\frac{\lambda_1}{2}|\boldsymbol{A}_0(z)|^2 + \frac{\lambda_2}{2}(\nu_1-\nu_2)|\,\mathbf{curl}\,\boldsymbol{A}_0(z)|^2 + \mathrm{Re}\{(\nu_1-\nu_2)\,\mathbf{curl}\,\boldsymbol{A}_0(z)\cdot\mathbf{curl}\,\overline{\boldsymbol{q}}_0(z) \\ &+ i\sigma_0\boldsymbol{A}_0(z)\cdot\overline{\boldsymbol{q}}_0(z)\}\Big) - \tau_z\frac{1}{|\omega|}(\nu_1-\nu_2)\,\mathrm{Re}\left(\int_\omega \mathbf{curl}\,\boldsymbol{A}_0(z)\cdot\mathbf{curl}\,\overline{\boldsymbol{Q}}^{(1)}\,\mathrm{d}x\right),\end{aligned} \tag{4.124}$$

where $\boldsymbol{A}_0, \boldsymbol{q}_0 \in \boldsymbol{X}^0_N$ are the solutions of (4.27) and (4.95) corresponding to $\varepsilon = 0$,

$$\tau_z = \begin{cases} +1 & \text{if } z \in \Omega_C \\ -1 & \text{if } z \in \Omega_I \end{cases}$$

and $\boldsymbol{Q}^{(1)}$ is defined via (4.106) if $z \in \Omega_C$, via (A.11) if $z \in \Omega_I$.

Proof. It suffices to apply Theorem 4.1.2 to the Lagrangian function (4.122), with the choice $l(\varepsilon) = |\omega_\varepsilon|$. In particular, it follows from (4.11) that

$$d\mathcal{J}(\Omega_C,\omega) = \lim_{\varepsilon\to0^+}\frac{\mathcal{L}(\varepsilon,\boldsymbol{A}_0,\boldsymbol{q}_0) - \mathcal{L}(0,\boldsymbol{A}_0,\boldsymbol{q}_0)}{|\omega_\varepsilon|} + \lim_{\varepsilon\to0^+}\frac{\mathcal{L}(\varepsilon,\boldsymbol{A}_0,\boldsymbol{q}_0) - \mathcal{L}(0,\boldsymbol{A}_0,\boldsymbol{q}_0)}{|\omega_\varepsilon|} \tag{4.125}$$

provided both limit exist and are finite. According to (4.61) and (4.62), we first observe that

$$\nu_\varepsilon(x) - \nu(x) = (\chi_{\Omega_{C,\varepsilon}}(x) - \chi_{\Omega_C}(x))\nu_1 + (\chi_{\Omega\setminus\Omega_{C,\varepsilon}}(x) - \chi_{\Omega\setminus\Omega_C}(x))\nu_2$$
$$= -\tau_z(\nu_1 - \nu_2)\chi_{\omega_\varepsilon}(x) \quad \forall x \in \Omega,$$

and

$$\sigma_\varepsilon(x) - \sigma(x) = \sigma_0(\chi_{\Omega_{C,\varepsilon}}(x) - \chi_{\Omega_C}(x)) = -\tau_z\sigma_0\chi_{\omega_\varepsilon}(x) \quad \forall x \in \Omega.$$

For what concerns the first limit, we have

$$\begin{aligned}
&\mathcal{L}(\varepsilon, \boldsymbol{A}_0, \boldsymbol{q}_0) - \mathcal{L}(0, \boldsymbol{A}_0, \boldsymbol{q}_0)\\
&= -\tau_z \left(\frac{\lambda_1}{2} \int_{\omega_\varepsilon} |\boldsymbol{A}_0|^2 \,\mathrm{d}x + \frac{\lambda_2}{2}(\nu_1 - \nu_2) \int_{\omega_\varepsilon} |\,\mathbf{curl}\, \boldsymbol{A}_0|^2 \,\mathrm{d}x \right)\\
&- \tau_z \operatorname{Re}\left((\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{q}}_0 \,\mathrm{d}x + i \int_{\omega_\varepsilon} \sigma_0 \boldsymbol{A}_0 \cdot \overline{\boldsymbol{q}}_0 \,\mathrm{d}x \right)\\
&= -\varepsilon^3\tau_z \left(\frac{\lambda_1}{2} \int_{\omega} |\boldsymbol{A}_0(z+\varepsilon x)|^2 \,\mathrm{d}x + \frac{\lambda_2}{2}(\nu_1 - \nu_2) \int_{\omega} |\,\mathbf{curl}\, \boldsymbol{A}_0(z+\varepsilon x)|^2 \,\mathrm{d}x \right)\\
&- \varepsilon^3\tau_z \operatorname{Re}\left((\nu_1 - \nu_2) \int_{\omega} \mathbf{curl}\, \boldsymbol{A}_0(z+\varepsilon x) \cdot \mathbf{curl}\, \overline{\boldsymbol{q}}_0(z+\varepsilon x) \,\mathrm{d}x \right)\\
&- \varepsilon^3\tau_z \operatorname{Re}\left(i \int_{\omega} \sigma_0 \boldsymbol{A}_0(z+\varepsilon x) \cdot \overline{\boldsymbol{q}}_0(z+\varepsilon x) \,\mathrm{d}x \right).
\end{aligned}$$

Exploiting the continuity of $\boldsymbol{A}_0, \mathbf{curl}\, \boldsymbol{A}_0, \boldsymbol{q}_0, \mathbf{curl}\, \boldsymbol{q}_0$ at z provided by Lemma 4.4.1 and Lemma 4.2.6, as well as $|\omega_\varepsilon| = \varepsilon^3|\omega|$, we deduce

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \frac{\mathcal{L}(\varepsilon, \boldsymbol{A}_0, \boldsymbol{q}_0) - \mathcal{L}(0, \boldsymbol{A}_0, \boldsymbol{q}_0)}{|\omega_\varepsilon|}\\
&= -\tau_z\Big(\frac{\lambda_1}{2}|\boldsymbol{A}_0(z)|^2 + \frac{\lambda_2}{2}(\nu_1 - \nu_2)|\,\mathbf{curl}\, \boldsymbol{A}_0(z)|^2 + \operatorname{Re}\{(\nu_1 - \nu_2)\,\mathbf{curl}\, \boldsymbol{A}_0(z) \cdot \mathbf{curl}\, \overline{\boldsymbol{q}}_0(z)\\
&+ i\sigma_0 \boldsymbol{A}_0(z) \cdot \overline{\boldsymbol{q}}_0(z)\}\Big).
\end{aligned} \tag{4.126}$$

Note that the local summability assumption $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$ is enough to guarantee continuity of $\boldsymbol{A}_0, \mathbf{curl}\, \boldsymbol{A}_0, \boldsymbol{q}_0, \mathbf{curl}\, \boldsymbol{q}_0$ at $z \in \Omega_C$ (cf. Lemma 4.2.6 and Lemma 4.4.1), which in turn suffices to achieve (4.126). The same can be said for $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_I)$ if $z \in \Omega_I$.

To compute the second limit, we write

$$
\begin{aligned}
&\mathcal{L}(\varepsilon, \boldsymbol{A}_0, \boldsymbol{q}_\varepsilon) - \mathcal{L}(\varepsilon, \boldsymbol{A}_0, \boldsymbol{q}_0) = \\
&\operatorname{Re}\left(\int_\Omega \nu_\varepsilon \,\mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x + i\int_\Omega \sigma_\varepsilon \boldsymbol{A}_0 \cdot (\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x - \int_\Omega \boldsymbol{J}\cdot(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x\right) \\
&= \operatorname{Re}\left(-\tau_z(\nu_1-\nu_2)\int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x - i\tau_z\sigma_0\int_{\omega_\varepsilon} \boldsymbol{A}_0\cdot(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x\right) \\
&+ \operatorname{Re}\left(\underbrace{\int_\Omega \nu \,\mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x + i\int_\Omega \sigma \boldsymbol{A}_0 \cdot (\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x - \int_\Omega \boldsymbol{J}\cdot(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x}_{=0 \text{ due to } (4.27)}\right) \\
&= \operatorname{Re}\left(-\tau_z(\nu_1-\nu_2)\int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x - i\tau_z\sigma_0\int_{\omega_\varepsilon} \boldsymbol{A}_0\cdot(\overline{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0})\,\mathrm{d}x\right) \\
&= \varepsilon^3 \operatorname{Re}\left(-\tau_z(\nu_1-\nu_2)\int_\omega (\mathbf{curl}\, \boldsymbol{A}_0)\circ T_\varepsilon \cdot \varepsilon\varepsilon^{-1}\,\mathbf{curl}\left(\overline{\frac{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0}{\varepsilon}}\circ T_\varepsilon\right)\,\mathrm{d}x\right) \\
&+ \varepsilon^3 \operatorname{Re}\left(-i\tau_z\sigma_0\int_\omega \boldsymbol{A}_0\circ T_\varepsilon \cdot \varepsilon\left(\overline{\frac{\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0}{\varepsilon}}\right)\mathrm{d}x\right) \\
&\underbrace{=}_{(4.108)} -\varepsilon^3\tau_z \operatorname{Re}\left((\nu_1-\nu_2)\int_\omega (\mathbf{curl}\, \boldsymbol{A}_0)\circ T_\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{Q}}_\varepsilon^{(1)}\,\mathrm{d}x + i\sigma_0\int_\omega \boldsymbol{A}_0\circ T_\varepsilon\cdot \varepsilon\overline{\boldsymbol{Q}}_\varepsilon^{(1)}\,\mathrm{d}x\right).
\end{aligned}
$$

Since $|\omega_\varepsilon| = \varepsilon^3|\omega|$, we are left to evaluate the limit as $\varepsilon \to 0^+$ of

$$
-\tau_z\frac{1}{|\omega|}\operatorname{Re}\left((\nu_1-\nu_2)\int_\omega (\mathbf{curl}\, \boldsymbol{A}_0)\circ T_\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{Q}}_\varepsilon^{(1)}\,\mathrm{d}x + i\sigma_0\int_\omega \boldsymbol{A}_0\circ T_\varepsilon\cdot \varepsilon\overline{\boldsymbol{Q}}_\varepsilon^{(1)}\,\mathrm{d}x\right).
$$

In view of (4.109) (respectively (A.13) if $z \in \Omega_I$), we have that $\mathbf{curl}\, \boldsymbol{Q}_\varepsilon^{(1)}$ convergence strongly to $\mathbf{curl}\, \boldsymbol{Q}^{(1)}$ in $\boldsymbol{L}^2(\omega)$, whence

$$
\int_\omega (\mathbf{curl}\, \boldsymbol{A}_0)\circ T_\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{Q}}_\varepsilon^{(1)}\,\mathrm{d}x \to \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z)\cdot \mathbf{curl}\, \overline{\boldsymbol{Q}}^{(1)}\,\mathrm{d}x \quad \text{as } \varepsilon \to 0^+.
$$

Next we aim to show that

$$
\varepsilon\boldsymbol{Q}_\varepsilon^{(1)} \to \boldsymbol{0} \quad \text{strongly in } \boldsymbol{L}^2(\omega) \quad \text{as } \varepsilon \to 0^+. \tag{4.127}
$$

To this end, let us write

$$
\begin{aligned}
\varepsilon\|\boldsymbol{Q}_\varepsilon^{(1)}\|_{L^2(\Omega_\varepsilon)} &\le \varepsilon\|\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}\|_{L^2(\Omega_\varepsilon)} + \varepsilon\|\boldsymbol{Q}^{(1)}\|_{L^2(\Omega_\varepsilon)} \\
&\underbrace{\le}_{(4.109)} C\varepsilon + \varepsilon\|\boldsymbol{Q}^{(1)}\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon(1+\|\boldsymbol{Q}^{(1)}\|_{L^2(\mathbb{R}^3)}) \underbrace{\le}_{(4.107)} C\varepsilon
\end{aligned}
$$

if $z \in \Omega_C$. If $z \in \Omega_I$, we can carry out the same estimate but with (A.13) in place of (4.109). We also rely on the fact that $\boldsymbol{Q}^{(1)}$ enjoys the same decay at infinity regardless of $z \in \Omega_I$ or $z \in \Omega_C$, see Lemma A.0.6. Summing up, we have shown that

$$
\frac{\mathcal{L}(\varepsilon, \boldsymbol{A}_0, \boldsymbol{q}_\varepsilon) - \mathcal{L}(\varepsilon, \boldsymbol{A}_0, \boldsymbol{q}_0)}{|\omega_\varepsilon|} \to -\tau_z\frac{1}{|\omega|}(\nu_1-\nu_2)\operatorname{Re}\left(\int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z)\cdot \mathbf{curl}\, \overline{\boldsymbol{Q}}^{(1)}\,\mathrm{d}x\right)
$$

as $\varepsilon \to 0^+$, which concludes the proof along with (4.125) and (4.126). □

4.6 Improved asymptotic expansions

In the context of topological derivatives, it is customary to introduce corrector fields in order to improve the scaling behaviour of certain terms and derive higher order expansions [BS21; Stu20]. In this section we introduce two more corrector fields, one for the state and one for the averaged adjoint, that allow to derive a second-order version of (4.94). Despite Theorem 4.6.8 and Theorem 4.6.10 being preliminary results, such new correctors can lay the groundwork for the introduction and analysis of similar fields that are needed to obtain higher-order topological derivatives for our problem (or related problems, e.g. the eddy current system). We develop the theory having $z \in \Omega_C$ in mind; once again, the case $z \in \Omega_I$ can be carried out in a similar way, but we don't explicitly state the corresponding results for the sake of brevity, and because of the work in progress nature of the section.

4.6.1 New corrector equations

Lemma 4.6.1. *Let $A_0 \in X_N^0$ be the unique solution of (4.27). The variational problem*

$$\int_{\mathbb{R}^3} \nu_\omega \, \mathbf{curl}\, \widehat{U}^{(1)} \cdot \mathbf{curl}\, \overline{v} \,\mathrm{d}x = i\sigma_0 \int_\omega A_0(z) \cdot \overline{v} \,\mathrm{d}x \quad \forall v \in S \tag{4.128}$$

has a unique solution $\widehat{U}^{(1)} \in S$, where the latter is defined in (4.70).

Proof. In force of [AVV01, Appx. 2, Lemma 12], there is a constant $C > 0$ such that

$$\|v\|_{L^2_\rho(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \frac{|v(x)|^2}{1+|x|^2} \,\mathrm{d}x \right)^{\frac{1}{2}} \le C \| \mathbf{curl}\, v \|_{L^2(\mathbb{R}^3)} \quad \forall v \in S \tag{4.129}$$

so that $\| \mathbf{curl}(\cdot) \|_{L^2(\mathbb{R}^3)}$ is an equivalent norm on S. As the left hand side of (4.128) is an inner product in S, to get existence and uniqueness it suffices to show that the right hand side can be seen as a bounded linear functional. Indeed let $R > 0$ be such that $\omega \subset B_R(0)$, for all $v \in S$ we have

$$\begin{aligned} \left| \int_\omega A_0(z) \cdot \overline{v} \,\mathrm{d}x \right| &\le C \left(\int_{B_R} |v(x)|^2 \,\mathrm{d}x \right)^{\frac{1}{2}} \\ &\le C \left(\int_{B_R} (1+R^2) \frac{|v(x)|^2}{1+|x|^2} \,\mathrm{d}x \right)^{\frac{1}{2}} \le C \|v\|_{L^2_\rho(\mathbb{R}^3)}, \end{aligned}$$

and the conclusion follows by exploiting (4.129). □

Next we provide a saddle point interpretation of (4.128) which will be useful for the improved asymptotic analysis.

Lemma 4.6.2. *Let $\widehat{U}^{(1)}$ be the solution to (4.128) and let $[\xi] \in \dot{BL}(\mathbb{R}^3)$ be the solution of*

$$\int_{\mathbb{R}^3} \nabla[\xi] \cdot \nabla \overline{\varphi} \,\mathrm{d}x = i\sigma_0 \int_\omega A_0(z) \cdot \nabla \overline{\varphi} \,\mathrm{d}x \quad \forall \varphi \in \dot{BL}(\mathbb{R}^3). \tag{4.130}$$

Then

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} \nu_\omega \, \mathbf{curl}\, \widehat{\boldsymbol{U}}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + \int_{\mathbb{R}^3} \nabla[\xi] \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x = i\sigma_0 \int_\omega \boldsymbol{A}_0(z) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x & \forall \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \\ \displaystyle\int_{\mathbb{R}^3} \widehat{\boldsymbol{U}}^{(1)} \cdot \nabla \overline{\varphi} \, \mathrm{d}x = 0 & \forall \varphi \in \dot{BL}(\mathbb{R}^3). \end{cases} \tag{4.131}$$

Additionally, there exists a representative $\xi \in [\xi]$ that enjoys the following decay behaviour at infinity:

$$\xi(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{and} \quad \nabla\xi(x) = \mathcal{O}\left(\frac{1}{|x|^3}\right) \quad \text{as } |x| \to \infty. \tag{4.132}$$

Proof. Let

$$\boldsymbol{a}^\star = \begin{cases} \boldsymbol{A}_0(z) & \text{in } \omega \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \overline{\omega}. \end{cases}$$

Applying decomposition (4.19) to $i\sigma_0 \boldsymbol{a}^\star \in \boldsymbol{L}^2(\mathbb{R}^3)$, we find $([\boldsymbol{p}], [\xi]) \in \dot{\boldsymbol{W}}(\mathbb{R}^3) \times \dot{BL}(\mathbb{R}^3)$ such that

$$i\sigma_0 \boldsymbol{a}^\star = \mathbf{curl}\, \boldsymbol{p} + \nabla\xi \quad \text{in } \mathbb{R}^3, \tag{4.133}$$

with ξ being precisely characterized by (4.130). Let $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$. We can find a unique $s \in \dot{BL}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla s \cdot \nabla \overline{\eta} \, \mathrm{d}x = \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla \overline{\eta} \, \mathrm{d}x \quad \forall \eta \in \dot{BL}(\mathbb{R}^3) \iff \Delta s = \operatorname{div} \boldsymbol{v} \quad \text{in } \mathbb{R}^3$$

and observe that $\widetilde{\boldsymbol{v}} = (\boldsymbol{v} - \nabla s) \in \boldsymbol{S}$ by construction, making it an admissible test function for (4.128). Indeed, $\mathbf{curl}\, \widetilde{\boldsymbol{v}} = \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{L}^2(\mathbb{R}^3)$, $\operatorname{div} \widetilde{\boldsymbol{v}} = \operatorname{div} \boldsymbol{v} - \operatorname{div} \nabla s = 0$ and $\widetilde{\boldsymbol{v}} \in \boldsymbol{L}^2(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^2_\rho(\mathbb{R}^3)$. With the choice of $\widetilde{\boldsymbol{v}}$ as test function, the right hand side of (4.128) reads

$$\begin{aligned} i\sigma_0 \int_{\mathbb{R}^3} \boldsymbol{a}^\star \cdot \widetilde{\boldsymbol{v}} \, \mathrm{d}x &= \int_{\mathbb{R}^3} \mathbf{curl}\, \boldsymbol{p} \cdot \widetilde{\boldsymbol{v}} \, \mathrm{d}x + \int_{\mathbb{R}^3} \nabla\xi \cdot \widetilde{\boldsymbol{v}} \, \mathrm{d}x \underbrace{=}_{\operatorname{div} \widetilde{\boldsymbol{v}} = 0} \int_{\mathbb{R}^3} \mathbf{curl}\, \boldsymbol{p} \cdot \widetilde{\boldsymbol{v}} \, \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \mathbf{curl}\, \boldsymbol{p} \cdot (\boldsymbol{v} - \nabla s) \, \mathrm{d}x \underbrace{=}_{\operatorname{div} \mathbf{curl}\, \boldsymbol{p} = 0} \int_{\mathbb{R}^3} \mathbf{curl}\, \boldsymbol{p} \cdot \boldsymbol{v} \, \mathrm{d}x. \end{aligned} \tag{4.134}$$

Note that we have used the implication

$$\boldsymbol{u} \in \boldsymbol{L}^2(\mathbb{R}^3), \ \operatorname{div} \boldsymbol{u} = 0 \implies \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \nabla \eta \, \mathrm{d}x = 0 \quad \forall \eta \in \dot{BL}(\mathbb{R}^3), \tag{4.135}$$

whose validity follows from

$$0 = \langle \operatorname{div} \boldsymbol{u}, \varphi \rangle = \int_{\mathbb{R}^3} \boldsymbol{u} \cdot \nabla \varphi \, \mathrm{d}x \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3),$$

and then arguing by density (of $\dot{\mathcal{D}}(\mathbb{R}^3)$ in $\dot{BL}(\mathbb{R}^3)$).

As

$$\int_{\mathbb{R}^3} \nu_\omega \, \mathbf{curl}\, \widehat{\boldsymbol{U}}^{(1)} \cdot \mathbf{curl}\, \widetilde{\boldsymbol{v}} \, \mathrm{d}x = \int_{\mathbb{R}^3} \nu_\omega \, \mathbf{curl}\, \widehat{\boldsymbol{U}}^{(1)} \cdot \mathbf{curl}\, \boldsymbol{v} \, \mathrm{d}x,$$

the first equation in (4.131) follows combining (4.133) and (4.134). The second condition in (4.131) follows from div $\widehat{\boldsymbol{U}}^{(1)} = 0$ in $\mathbb{R}^3$ and the mentioned density of test functions $\dot{\mathcal{D}}(\mathbb{R}^3)$ in $\dot{BL}(\mathbb{R}^3)$.

To show the decay behaviour (4.132), let us first observe that there is a representative $\xi \in [\xi]$ satisfying the strong formulation

$$\begin{cases} \Delta \xi = 0 & \text{in } \omega \cup (\mathbb{R}^3 \setminus \overline{\omega}) \\ \xi^+ = \xi^- & \text{on } \partial\omega \\ (\nabla \xi \cdot \boldsymbol{n})^+ - (\nabla \xi \cdot \boldsymbol{n})^- = i\sigma_0 \boldsymbol{A}_0(z) \cdot \boldsymbol{n} & \text{on } \partial\omega, \end{cases} \tag{4.136}$$

where we recall that $(\cdot)^\pm$ means that the corresponding trace is taken referring to the domain ω or ω^c (respectively the plus sign for ω, the minus for ω^c).

By virtue of (4.136), we have the representation formula

$$\xi(y) = \int_{\partial\omega} \frac{\partial}{\partial \boldsymbol{n}_x} G(x,y) \xi(x) \, dS_x + \int_{\mathbb{R}^3 \setminus \overline{\omega}} \nabla_x G(x,y) \cdot \nabla \xi(x) \, \mathrm{d}x, \quad y \in \mathbb{R}^3 \setminus \overline{\omega}, \tag{4.137}$$

where $G(x,y) = \dfrac{1}{4\pi |x-y|}$ denotes the fundamental solution of the Laplacian in $\mathbb{R}^3$. Using (4.136), we can further expand as

$$\begin{aligned} \xi(y) &= \int_{\partial\omega} \frac{\partial}{\partial \boldsymbol{n}_x} G(x,y) \xi(x) \, dS_x + \int_{\mathbb{R}^3 \setminus \overline{\omega}} \nabla_x G(x,y) \cdot \nabla \xi(x) \, \mathrm{d}x \\ &\quad + \int_\omega \nabla_x G(x,y) \cdot \nabla \xi(x) \, \mathrm{d}x - \int_\omega \nabla_x G(x,y) \cdot \nabla \xi(x) \, \mathrm{d}x \\ &= \int_{\partial\omega} G(x,y) (i\sigma_0 \boldsymbol{A}_0(z) \cdot \boldsymbol{n}) \, dS_x + \int_{\partial\omega} \frac{\partial}{\partial \boldsymbol{n}_x} G(x,y) \xi(x) \, dS_x \\ &\quad - \int_\omega \nabla_x G(x,y) \cdot \nabla \xi(x) \, \mathrm{d}x. \end{aligned}$$

In view of [Amm08, Lemma 3.1.3], we know that

$$\int_{\partial\omega} \frac{\partial}{\partial \boldsymbol{n}_x} G(x,y) \xi(x) \, dS_x = \mathcal{O}\left(\frac{1}{|y|^2}\right) \quad \text{as } |y| \to \infty,$$

as well as

$$\int_{\partial\omega} G(x,y) (i\sigma_0 \boldsymbol{A}_0(z) \cdot \boldsymbol{n}) \, dS_x = \mathcal{O}\left(\frac{1}{|y|^2}\right) \quad \text{as } |y| \to \infty$$

due to $\int_{\partial\omega} (i\sigma_0 \boldsymbol{A}_0(z) \cdot \boldsymbol{n}) \, dS_x = 0$. The statement for $\nabla\xi$ can be proved in similar way, starting from taking the gradient of (4.137). By classical arguments, the gradient can be put under the integral signs, resulting in a representation formula for $\nabla \xi(y)$. This implies (4.132) and concludes the proof. □

Remark 4.6.3. *The strong interpretation of* (4.128) *reads:*

$$\begin{cases} \mathbf{curl}(\nu_\omega\,\mathbf{curl}\,\widehat{\boldsymbol{U}}^{(1)}) + \nabla\xi = i\sigma_0 \boldsymbol{a}^\star & \text{in } \mathbb{R}^3 \\ \operatorname{div}\widehat{\boldsymbol{U}}^{(1)} = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{4.138}$$

The proof follows the exact same lines of the one of Lemma 4.6.2, but starting with a smooth compactly supported vector field $\boldsymbol{\eta} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ *in place of* $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$.

Lemma 4.6.4. *There exists a unique solution* $\widehat{\boldsymbol{Q}}^{(1)} \in \boldsymbol{S}$ *to*

$$\int_{\mathbb{R}^3} \nu_\omega\,\mathbf{curl}\,\widehat{\boldsymbol{Q}}^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x = i\sigma_0 \int_\omega \boldsymbol{q}_0(z)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + 2\lambda_1 \int_\omega \boldsymbol{A}_0(z)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{S}. \tag{4.139}$$

Proof. We omit the proof as it follows the exact same lines of the one of Lemma 4.6.1, given the apparent similarity in structure between (4.139) and (4.128). □

Lemma 4.6.5. *Let* $\widehat{\boldsymbol{Q}}^{(1)}$ *be the solution to* (4.139) *and let* $\psi \in \dot{B}L(\mathbb{R}^3)$ *be the solution of*

$$\int_{\mathbb{R}^3} \nabla[\psi]\cdot\nabla\overline{\varphi}\,\mathrm{d}x = i\sigma_0 \int_\omega \boldsymbol{q}_0(z)\cdot\nabla\overline{\varphi}\,\mathrm{d}x + 2\lambda_1 \int_\omega \boldsymbol{A}_0(z)\cdot\nabla\overline{\varphi}\,\mathrm{d}x \quad \forall\varphi \in \dot{B}L(\mathbb{R}^3). \tag{4.140}$$

Then

$$\int_{\mathbb{R}^3} \nu_\omega\,\mathbf{curl}\,\widehat{\boldsymbol{Q}}^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_{\mathbb{R}^3} \nabla[\psi]\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x = i\sigma_0 \int_\omega \boldsymbol{q}_0(z)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x + 2\lambda_1 \int_\omega \boldsymbol{A}_0(z)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x \tag{4.141}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$. *Moreover,*

$$\int_{\mathbb{R}^3} \widehat{\boldsymbol{Q}}^{(1)}\cdot\nabla\overline{\varphi}\,\mathrm{d}x = 0 \quad \forall\varphi \in \dot{B}L(\mathbb{R}^3) \tag{4.142}$$

and there exists a representative $\psi \in [\psi]$ *that enjoys the following decay behaviour at infinity:*

$$\psi(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{and} \quad |\nabla\psi(x)| = \mathcal{O}\left(\frac{1}{|x|^3}\right) \quad \text{as } |x| \to \infty. \tag{4.143}$$

Proof. We omit the proof as it is almost identical to the one of Lemma 4.6.2. □

4.6.2 Improved asymptotic expansions

Before proceeding, we shall need to assume that $\widehat{\boldsymbol{U}}^{(1)}, \widehat{\boldsymbol{Q}}^{(1)}$ have a certain decay at infinity.

Assumption 4.6.6. *Let* $\widehat{\boldsymbol{U}}^{(1)}$ *be defined by Lemma 4.6.1 and* $\widehat{\boldsymbol{Q}}^{(1)}$ *by Lemma 4.6.4. There exist* $\delta_1, \delta_2 \in (0, \frac{1}{2})$ *such that*

$$|\widehat{\boldsymbol{U}}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^{1-\delta_1}}\right), \qquad |\widehat{\boldsymbol{Q}}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^{1-\delta_2}}\right) \quad \text{as } |x| \to \infty, \tag{4.144}$$

and

$$|\,\mathbf{curl}\,\widehat{\boldsymbol{U}}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^{2-\delta_1}}\right), \qquad |\,\mathbf{curl}\,\widehat{\boldsymbol{Q}}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^{2-\delta_2}}\right) \quad \text{as } |x| \to \infty. \tag{4.145}$$

Remark 4.6.7. *Let us motivate why Assumption 4.6.6 appears to be a reasonable hypothesis in the given context. In view of (4.65) and the divergence-free condition, the field $\widehat{\boldsymbol{U}}^{(1)}$ satisfies*

$$-\nu_1 \boldsymbol{\Delta} \widehat{\boldsymbol{U}}^{(1)} = -\nabla \xi \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega},$$

and we know that $\nabla \xi$ has the decay $\mathcal{O}(1/|x|^3)$ as $|x| \to \infty$ (see Lemma 4.6.2). Using spherical coordinates, it is easy to see that a solution of $\Delta f = 1/|x|^3$ (vanishing at infinity) has a decay behaviour of order

$$\mathcal{O}\left(\frac{\log |x|}{|x|}\right).$$

Since $\log |x|/|x|$ is eventually smaller than $1/|x|^\delta$ for $|x| \to \infty$ and all $\delta > 0$, assuming (4.144) and thus (4.145) seems consistent.

Theorem 4.6.8. *Let Assumption 4.6.6 and Assumption 4.3.13 hold. Let $\boldsymbol{U}_\varepsilon^{(1)}$ be defined by (4.73), $\boldsymbol{U}^{(1)}$ through Lemma 4.3.9 and $\widehat{\boldsymbol{U}}^{(1)}$ by means of Lemma 4.6.1. If $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$ ($\alpha > 0$), there is a constant $C > 0$ not depending on $\varepsilon > 0$ for which*

$$\|\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)} - \varepsilon \widehat{\boldsymbol{U}}^{(1)}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon \tag{4.146}$$

for sufficiently small ε.

Proof. The proof follows the structure of the one of Theorem 4.3.14, and it is carried out in the case $z \in \Omega_C$.

Let $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$; we recall that

$$\boldsymbol{v}^\star := \begin{cases} \boldsymbol{v} & \text{in } \Omega_\varepsilon \\ \mathbf{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases} \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$$

and that the equation satisfied by the difference $\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}$ is (see (4.84)) :

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \\ &= - i\sigma_0 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x + (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \\ &\quad + i\sigma_0 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon). \end{aligned} \tag{4.147}$$

This time, before applying Lemma 4.3.12 we make use of the other corrector $\widehat{\boldsymbol{U}}^{(1)}$. Employing the test function $\boldsymbol{v}^\star \in \boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$ in the first equation of (4.131) implies

$$\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \widehat{\boldsymbol{U}}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + \int_{\Omega_\varepsilon} \nabla \xi \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x = i\sigma_0 \int_\omega \boldsymbol{A}_0(z) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x,$$

and then we subtract the latter equation multiplied by ε from (4.147) to obtain

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)} - \varepsilon\widehat{\boldsymbol{U}}^{(1)}) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + i\int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)} - \varepsilon\widehat{\boldsymbol{U}}^{(1)}) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x \\
&- \int_{\Omega_\varepsilon} \varepsilon\nabla\xi \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x = -i\sigma_0 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{U}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x - i\sigma_0\int_{\Omega_\varepsilon\setminus\omega} \varepsilon^3\widehat{\boldsymbol{U}}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x \\
&+ i\sigma_0 \int_\omega \varepsilon(\boldsymbol{A}_0(x_\varepsilon) - \boldsymbol{A}_0(z)) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + (\nu_1 - \nu_2)\int_\omega (\mathbf{curl}\,\boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{A}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x.
\end{aligned} \tag{4.148}$$

We are in position to apply Lemma 4.3.12 to equation (4.148) with the choices

$$(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) = (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)} - \varepsilon\widehat{\boldsymbol{U}}^{(1)}, \xi|_{\Omega_\varepsilon}), \quad (\boldsymbol{g}_\varepsilon, h_\varepsilon) = (\boldsymbol{w}_\varepsilon|_{\Gamma_\varepsilon} \times \boldsymbol{n}, \xi|_{\Gamma_\varepsilon})$$

and $\boldsymbol{\Phi}_\varepsilon$ equal to the mapping induced by the whole right hand side, which we denote by $\hat{\boldsymbol{\Phi}}_\varepsilon$. Note that in contrast to Theorem 4.3.14, a gradient term shows up in (4.148) so that $\phi_\varepsilon \neq 0, h_\varepsilon \neq 0$. Therefore by Lemma 4.3.12, the following inequality holds true:

$$\begin{aligned}
\|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} \le C(\|\hat{\boldsymbol{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \\
+ \varepsilon^{\frac{1}{2}}\|\xi\|_{L^2(\Gamma_\varepsilon)} + |\xi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}),
\end{aligned} \tag{4.149}$$

and it remains to estimate each term appearing on the right hand side.

Boundary norms. It holds

$$\begin{aligned}
\boldsymbol{w}_\varepsilon \times \boldsymbol{n} &= (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)} - \varepsilon\widehat{\boldsymbol{U}}^{(1)}) \times \boldsymbol{n} \\
&\underbrace{=}_{\boldsymbol{U}_\varepsilon^{(1)}\times\boldsymbol{n}=\boldsymbol{0}} -\boldsymbol{U}^{(1)} \times \boldsymbol{n} - \varepsilon\widehat{\boldsymbol{U}}^{(1)} \times \boldsymbol{n},
\end{aligned} \tag{4.150}$$

so that

$$\begin{aligned}
\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} &\le \|\boldsymbol{U}^{(1)} \times \boldsymbol{n}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon\|\widehat{\boldsymbol{U}}^{(1)} \times \boldsymbol{n}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \\
&= \|\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon\|\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)},
\end{aligned} \tag{4.151}$$

and

$$\begin{aligned}
\|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} &\le \|\operatorname{div}_\tau(\boldsymbol{U}^{(1)} \times \boldsymbol{n})\|_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\operatorname{div}_\tau(\widehat{\boldsymbol{U}}^{(1)} \times \boldsymbol{n})\|_{L^2(\Gamma_\varepsilon)} \\
&= \|\mathbf{curl}\,\boldsymbol{U}^{(1)} \cdot \boldsymbol{n}\|_{L^2(\Gamma_\varepsilon)} + \varepsilon\|\mathbf{curl}\,\widehat{\boldsymbol{U}}^{(1)} \cdot \boldsymbol{n}\|_{L^2(\Gamma_\varepsilon)} \\
&= \|\mathbf{curl}\,\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon\|\mathbf{curl}\,\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)}.
\end{aligned} \tag{4.152}$$

From the proof of Theorem 4.3.14, we already know that

$$\|\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \le C\varepsilon, \quad \|\mathbf{curl}\,\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \le C\varepsilon^2 \tag{4.153}$$

for sufficiently small $\varepsilon > 0$, with $C > 0$ not depending on ε.

In view of (4.144) and (4.145), Lemma 4.3.2 guarantees

$$\varepsilon\|\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \le C\varepsilon^{1-\delta_1}, \quad \varepsilon\|\mathbf{curl}\,\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \le C\varepsilon^{2-\delta_1}. \tag{4.154}$$

Recalling (4.151) and the first inequality in (4.153), this implies

$$\varepsilon^{\frac{1}{2}} \|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \| \operatorname{div}_\tau \boldsymbol{g}_\varepsilon \|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}-\delta_1} \tag{4.155}$$

for $\varepsilon > 0$ sufficiently small and $C > 0$ independent of ε.

According to [BS21, Lemma 3.4] (that provides the ε-scaling for $|v|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}$, given the decay behaviour of v) and (4.132), it holds

$$\|\xi\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon \quad \text{and} \quad |\xi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}}$$

whence

$$\varepsilon^{\frac{1}{2}} \|\xi\|_{L^2(\Gamma_\varepsilon)} + |\xi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}}. \tag{4.156}$$

Dual norm of $\hat{\boldsymbol{\Phi}}_\varepsilon$. We only estimate the terms that did not appear in (4.84). We start with

$$\left| i\sigma_0 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^3 \widehat{\boldsymbol{U}}^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \right| \leq \varepsilon^2 \sigma_0 \|\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon \setminus \omega)} \|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \tag{4.157}$$

and now focus on $\varepsilon^2 \|\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon \setminus \omega)}$. Recalling that $\Omega_\varepsilon = \varepsilon^{-1}(\Omega - z)$, as a consequence of the decay property (4.144) we find $\varepsilon_0 > 0$ such that

$$x \in \Omega_\varepsilon \setminus \Omega_{\varepsilon_0}, \; \varepsilon < \varepsilon_0 \implies |\widehat{\boldsymbol{U}}^{(1)}(x)| \leq \frac{C}{|x|^{1-\delta_1}}.$$

Therefore we can write

$$\begin{aligned}
\varepsilon^2 \|\widehat{\boldsymbol{U}}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon \setminus \omega)} \underbrace{=}_{\varepsilon < \varepsilon_0} & \varepsilon^2 \left(\int_{\Omega_\varepsilon \setminus \Omega_{\varepsilon_0}} |\widehat{\boldsymbol{U}}^{(1)}(x)|^2 \, \mathrm{d}x + \int_{\Omega_{\varepsilon_0} \setminus \omega} |\widehat{\boldsymbol{U}}^{(1)}(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\
& \leq C\varepsilon^2 \left(\int_{\Omega_\varepsilon \setminus \Omega_{\varepsilon_0}} \frac{1}{|x|^{2-2\delta_1}} \, \mathrm{d}x + 1 \right)^{\frac{1}{2}} \\
& \leq C\varepsilon^2 \left(\int_c^{\varepsilon^{-1} \frac{\operatorname{diam} \Omega}{2}} \frac{r^2}{r^{2-2\delta_1}} \, \mathrm{d}r + 1 \right)^{\frac{1}{2}} \\
& \leq C\varepsilon^2 (\varepsilon^{-\frac{2\delta_1 + 1}{2}} + 1) \leq C\varepsilon^{\frac{3}{2}-\delta_1},
\end{aligned}$$

where we used the fact that $\Omega_\varepsilon, \Omega_{\varepsilon_0}$ are bounded, hence we can fit $\Omega_\varepsilon \setminus \Omega_{\varepsilon_0}$ in a three dimensional annulus of radii $\operatorname{diam}(\Omega_\varepsilon)/2 = \varepsilon^{-1} \operatorname{diam}(\Omega)/2$ and $c > 0, c \in \mathbb{R}$. The above estimate combined with (4.157) gives

$$\left| i\sigma_0 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^3 \widehat{\boldsymbol{U}}^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \right| \leq C\varepsilon^{\frac{3}{2}-\delta_1} \|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \tag{4.158}$$

for sufficiently small ε. Since $\boldsymbol{A}_0$ is Lipschitz in a neighbourhood of the point z (see Lemma 4.2.6),

$$\left| i\sigma_0 \int_\omega \varepsilon (\boldsymbol{A}_0(x_\varepsilon) - \boldsymbol{A}_0(z)) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \right| \leq C\varepsilon \|\boldsymbol{A}_0(x_\varepsilon) - \boldsymbol{A}_0(z)\|_{\boldsymbol{L}^2(\omega)} \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\omega)} \leq C\varepsilon \|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}. \tag{4.159}$$

The conclusion follows by (4.149), (4.155), (4.156), (4.158) and (4.159) all together. Note that the leading power of ε turns out to be still equal to one thanks to $\delta_1 \in (0, \frac{1}{2})$, which implies $\frac{3}{2} - \delta_1 > 1$. □

Corollary 4.6.9. *Inequality (4.146) is equivalent to*

$$\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0 - \varepsilon \boldsymbol{U}^{(1)} \circ T_\varepsilon^{-1} - \varepsilon^2 \widehat{\boldsymbol{U}}^{(1)} \circ T_\varepsilon^{-1}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\varepsilon^{\frac{5}{2}}, \tag{4.160}$$

which can be seen as a second order expansion of $\boldsymbol{A}_\varepsilon$ in $\boldsymbol{H}(\mathbf{curl})$, featuring the correctors $\boldsymbol{U}^{(1)}, \widehat{\boldsymbol{U}}^{(1)}$.

Proof. Recalling the definition of $\boldsymbol{U}_\varepsilon^{(1)}$ (4.73) and multiplying by $\varepsilon > 0$, we rewrite (4.146) as

$$\|(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \circ T_\varepsilon - \varepsilon \boldsymbol{U}^{(1)} - \varepsilon^2 \widehat{\boldsymbol{U}}^{(1)}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon^2,$$

□

the conclusion follows applying (4.49).

A similar result can be stated for the averaged adjoint field.

Theorem 4.6.10. *Let Assumption 4.6.6 and Assumption 4.3.13 hold. Let $\boldsymbol{Q}^{(1)}$ be defined via (4.106) and $\widehat{\boldsymbol{Q}}^{(1)}$ through (4.139). If $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_C)$ ($\alpha > 0$), there is a constant $C > 0$ not depending on ε for which*

$$\|\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)} - \varepsilon \widehat{\boldsymbol{Q}}^{(1)}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon \tag{4.161}$$

for sufficiently small $\varepsilon > 0$.

Proof. We follow the lines of the proof of Theorem 4.4.5. The equation satisfied by $\boldsymbol{Q}_\varepsilon^{(1)}$ reads

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\, \boldsymbol{Q}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_{\Omega_\varepsilon} \varepsilon^2 \sigma_\omega \boldsymbol{Q}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad = (\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{q}_0(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i\sigma_0 \int_\omega \varepsilon \boldsymbol{q}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - \lambda_1 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad + 2\lambda_1 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - \lambda_2 \int_{\Omega_\varepsilon} \nu_\omega \,\mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad + 2\lambda_2(\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon).
\end{aligned} \tag{4.162}$$

If $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$, the extension

$$\boldsymbol{v}^\star := \begin{cases} \boldsymbol{v} & \text{in } \Omega_\varepsilon \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases}$$

belongs to $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \hookrightarrow \boldsymbol{W}(\mathbb{R}^3)$ and therefore it can be chosen as test function in (4.106). This yields

$$\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{Q}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(z) + 2\lambda_2 \, \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \\ - \lambda_2 \int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x. \tag{4.163}$$

At the same time, $\boldsymbol{v}^\star$ is also admissible as test function for the first equation in (4.141) producing

$$\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \widehat{\boldsymbol{Q}}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + \int_{\Omega_\varepsilon} \nabla \psi \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x = i\sigma_0 \int_\omega \boldsymbol{q}_0(z) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x + 2\lambda_1 \int_\omega \boldsymbol{A}_0(z) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x. \tag{4.164}$$

Subtracting first (4.163) and then (4.164) multiplied by $\varepsilon > 0$ from (4.162) gives

$$\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}(\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)} - \varepsilon \widehat{\boldsymbol{Q}}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)} - \varepsilon \widehat{\boldsymbol{Q}}^{(1)}) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \\ - \int_{\Omega_\varepsilon} \varepsilon \nabla \psi \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x = -i\sigma_0 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{Q}^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x - i\sigma_0 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^3 \widehat{\boldsymbol{Q}}^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \\ -\lambda_1 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x + i\sigma_0 \int_\omega \varepsilon (\boldsymbol{q}_0(x_\varepsilon) - \boldsymbol{q}_0(z)) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x + 2\lambda_1 \int_\omega \varepsilon (\boldsymbol{A}_0(x_\varepsilon) - \boldsymbol{A}_0(z)) \, \mathrm{d}x \\ + (\nu_1 - \nu_2) \left(\int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{q}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \right) \\ + 2\lambda_2 (\nu_1 - \nu_2) \left(\int_\omega (\mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \right) \\ + \lambda_2 \left(- \int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + \int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \right),$$

which we rewrite as

$$\int_{\Omega_\varepsilon} \nu_\omega \, \mathbf{curl}(\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)} - \varepsilon \widehat{\boldsymbol{Q}}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + i \int_{\Omega_\varepsilon} \sigma_\omega \varepsilon^2 (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)} - \varepsilon \widehat{\boldsymbol{Q}}^{(1)}) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \\ - \int_{\Omega_\varepsilon} \varepsilon \nabla \psi \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x = \boldsymbol{\Theta}_\varepsilon(\boldsymbol{v}), \tag{4.165}$$

where $\boldsymbol{\Theta}_\varepsilon(\boldsymbol{v})$ is a shortcut for the whole right hand side.

We are now in position to apply Lemma 4.3.12 to (4.165) with the choices $\boldsymbol{\Phi}_\varepsilon = \boldsymbol{\Theta}_\varepsilon \in (\boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon), \| \cdot \|_{\varepsilon, \mathbf{curl}})^*$, $(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) = (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)} - \varepsilon \widehat{\boldsymbol{Q}}^{(1)}, \psi|_{\Omega_\varepsilon})$ and $(\boldsymbol{g}_\varepsilon, h_\varepsilon) = (\boldsymbol{w}_\varepsilon \times \boldsymbol{n}, \psi|_{\Gamma_\varepsilon})$. It follows

$$\|\boldsymbol{w}_\varepsilon\|_{\varepsilon, \mathbf{curl}} \leq C(\|\boldsymbol{\Theta}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}} \|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \| \operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \\ + \varepsilon^{\frac{1}{2}} \|\psi\|_{L^2(\Gamma_\varepsilon)} + |\psi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}), \tag{4.166}$$

and it remains to estimate each term on the right hand side of the above equation. *Boundary norms.*

Considering that the couples $(\boldsymbol{U}^{(1)}, \boldsymbol{Q}^{(1)})$, (ξ, ψ) and $(\widehat{\boldsymbol{U}}^{(1)}, \widehat{\boldsymbol{Q}}^{(1)})$ share the same asymptotic behaviour at infinity (see (4.107), (4.69), (4.143), (4.132), (4.144)), we can mimic the arguments used in the proof of Theorem 4.6.8 to obtain

$$\varepsilon^{\frac{1}{2}}\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}-\delta_2}, \quad \varepsilon^{\frac{1}{2}}\|\psi\|_{L^2(\Gamma_\varepsilon)} + |\psi|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)} \leq C\varepsilon^{\frac{3}{2}}. \tag{4.167}$$

Dual norm of $\boldsymbol{\Theta}_\varepsilon$. First we make the splitting

$$\boldsymbol{\Theta}_\varepsilon(\boldsymbol{v}) = \boldsymbol{\Theta}_\varepsilon^{(1)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(2)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(3)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(4)}(\boldsymbol{v}) + \boldsymbol{\Theta}_\varepsilon^{(5)}(\boldsymbol{v}),$$

where

$$\begin{aligned}
\boldsymbol{\Theta}_\varepsilon^{(1)}(\boldsymbol{v}) &= -i\sigma_0 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{Q}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x - i\sigma_0 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^3 \widehat{\boldsymbol{Q}}^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x - \lambda_1 \int_{\Omega_\varepsilon\setminus\omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x,\\
\boldsymbol{\Theta}_\varepsilon^{(2)}(\boldsymbol{v}) &= i\sigma_0 \int_\omega \varepsilon(\boldsymbol{q}_0(x_\varepsilon) - \boldsymbol{q}_0(z)) \cdot \overline{\boldsymbol{v}}\,\mathrm{d}x + 2\lambda_1 \int_\omega \varepsilon(\boldsymbol{A}_0(x_\varepsilon) - \boldsymbol{A}_0(z))\,\mathrm{d}x,\\
\boldsymbol{\Theta}_\varepsilon^{(3)}(\boldsymbol{v}) &= (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\,\boldsymbol{q}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{q}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x\\
&\quad + (\nu_1 - \nu_2)2\lambda_2 \int_\omega (\mathbf{curl}\,\boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\,\boldsymbol{A}_0(z)) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x,\\
\boldsymbol{\Theta}_\varepsilon^{(4)}(\boldsymbol{v}) &= \lambda_2 \left(-\int_{\Omega_\varepsilon} \nu_\omega\,\mathbf{curl}\,\boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + \int_{\Omega_\varepsilon} \nu_\omega\,\mathbf{curl}\,\boldsymbol{U}^{(1)} \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \right).
\end{aligned} \tag{4.168}$$

We provide explicit estimates only for the terms that either do not appear or are different from those in (4.115). By Hölder and triangle inequality,

$$\begin{aligned}
|\boldsymbol{\Theta}_\varepsilon^{(1)}(\boldsymbol{v})| &\leq C(\varepsilon\|\boldsymbol{Q}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} + \varepsilon^2\|\widehat{\boldsymbol{Q}}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon\setminus\omega)} + \varepsilon\|\boldsymbol{U}_\varepsilon^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)})\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}\\
&\leq C(\varepsilon + \varepsilon^{\frac{3}{2}-\delta_2} + \varepsilon\|\boldsymbol{U}_\varepsilon^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}) \leq C\varepsilon(1 + \|\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon)}) \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}.
\end{aligned} \tag{4.169}$$

Here we used $\boldsymbol{Q}^{(1)} \in \boldsymbol{L}^2(\mathbb{R}^3)$, which follows from (4.107), $\varepsilon^2\|\widehat{\boldsymbol{Q}}^{(1)}\|_{\boldsymbol{L}^2(\Omega_\varepsilon\setminus\omega)} \leq C\varepsilon^{\frac{3}{2}-\delta_2}$ which can be obtained with the same argument employed for (4.158), and $\delta_2 \in (0, \frac{1}{2})$.

As $\boldsymbol{A}_0, \boldsymbol{q}_0$ are continuous at $x = z$ (cf. Lemma 4.2.6, Lemma 4.4.1),

$$|\boldsymbol{\Theta}_\varepsilon^{(2)}(\boldsymbol{v})| \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}}. \tag{4.170}$$

We already know from (4.118), (4.119) that

$$|\boldsymbol{\Theta}_\varepsilon^{(4)}(\boldsymbol{v})| \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \tag{4.171}$$

as well as

$$|\boldsymbol{\Theta}_\varepsilon^{(3)}(\boldsymbol{v})| \leq C\varepsilon\|\boldsymbol{v}\|_{\varepsilon,\mathbf{curl}} \tag{4.172}$$

provided $\mathbf{curl}\,\boldsymbol{q}_0, \mathbf{curl}\,\boldsymbol{A}_0$ are Lipschitz continuous in a neighbourhood of z (and this is guaranteed under the hypothesis $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_C)$ for some $\alpha > 0$). Summing up, it holds

$$\|\boldsymbol{\Theta}_\varepsilon\|_{(\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon),\|\cdot\|_{\varepsilon,\mathbf{curl}})^*} \leq C\varepsilon,$$

and this estimate in combination with (4.166), (4.167) concludes the proof.

□

Corollary 4.6.11. *Inequality* (4.161) *is equivalent to*

$$\|\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0 - \varepsilon \boldsymbol{Q}^{(1)} \circ T_\varepsilon^{-1} - \varepsilon^2 \widehat{\boldsymbol{Q}}^{(1)} \circ T_\varepsilon^{-1}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C\varepsilon^{\frac{5}{2}}, \tag{4.173}$$

which can be seen as a second order expansion of $\boldsymbol{q}_\varepsilon$ *in* $\boldsymbol{H}(\mathbf{curl})$*.*

Proof. Recalling the definition of $\boldsymbol{Q}_\varepsilon^{(1)}$ (4.108) and multiplying by $\varepsilon > 0$, we rewrite (4.161) as

$$\|(\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0) \circ T_\varepsilon - \varepsilon \boldsymbol{Q}^{(1)} - \varepsilon^2 \widehat{\boldsymbol{Q}}^{(1)}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon^2,$$

□

the conclusion follows by an application of (4.49).

4.7 Future research and open problems

As announced in the main introduction, this chapter is still a work in progress as a standalone scientific work, therefore it's easy to identify potential improvements. First, remove Assumption 4.6.6, which means to understand the decay behaviour of the correctors $\widehat{\boldsymbol{U}}^{(1)}, \widehat{\boldsymbol{Q}}^{(1)}$. In addition, deriving complete asymptotic expansions like

$$\widehat{\boldsymbol{U}}^{(1)}(x) = \boldsymbol{R}^{(1)}(x) + \mathcal{O}\left(\frac{1}{|x|^a}\right) \quad \text{as } |x| \to \infty,$$

with $a > 0$ and $\boldsymbol{R}^{(1)}$ (or at least its absolute value) to be explicitly determined, is needed to define the so-called *regular correctors* that are in turn required to compute higher-order derivatives [BGS21; BS21]. In particular, the field $\boldsymbol{R}^{(1)}$ would appear as boundary datum in the variational problem defining the regular corrector. The main difficulty here is that $\widehat{\boldsymbol{U}}^{(1)}$ is characterized by the strong formulation

$$\begin{cases} \mathbf{curl}(\nu_\omega\, \mathbf{curl}\, \widehat{\boldsymbol{U}}^{(1)}) + \nabla\xi = \boldsymbol{f} & \text{in } \mathbb{R}^3 \\ \operatorname{div} \widehat{\boldsymbol{U}}^{(1)} = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

where

$$\boldsymbol{f} = \begin{cases} \boldsymbol{f}_0 & \text{in } \omega, \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \end{cases} \in \boldsymbol{L}^2(\mathbb{R}^3).$$

Using the definition of ν_ω (4.65) and the divergence-free condition, we can restate the system as

$$\begin{cases} -\nu_1 \boldsymbol{\Delta} \widehat{\boldsymbol{U}}^{(1)} = -\nabla\xi & \text{in } \mathbb{R}^3 \setminus \overline{\omega} \\ -\nu_2 \boldsymbol{\Delta} \widehat{\boldsymbol{U}}^{(1)} = \boldsymbol{f}_0 - \nabla\xi & \text{in } \omega, \end{cases} \tag{4.174}$$

which has the form of a pseudo-transmission problem (with jump conditions on the boundary to be precisely determined) for the vector Laplacian. Moreover, $\nabla\xi$ is supported in the whole space and merely its decay behaviour at infinity is known (cf. (4.132)), making the derivation of a complete asymptotic quite challenging, even with a representation formula involving potentials at hand. Alternatively, taking the **curl** of both sides in the first equation in (4.174) one can see that $\mathbf{curl}\, \widehat{\boldsymbol{U}}^{(1)}$ is vector harmonic in the exterior domain $\mathbb{R}^3 \setminus \overline{\omega}$, but this seems too little to derive a complete asymptotic. Similar considerations hold for $\widehat{\boldsymbol{Q}}^{(1)}$.

The computation of higher order topological derivatives also demands for the introduction and the investigation of new *boundary layer corrector* equations. For example, a field $\boldsymbol{U}^{(2)} \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$ satisfying (D_x denotes the Jacobian matrix with respect to the variable x)

$$\int_{\mathbb{R}^3} \nu_\omega \,\mathbf{curl}[\boldsymbol{U}^{(2)}] \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x = (\nu_1 - \nu_2) \int_\omega ((\mathrm{D}_x \,\mathbf{curl}\, \boldsymbol{A}_0)(z))x \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$$

needs to be introduced and its asymptotic expansion at infinity to be understood. In this respect, notice that the proof of the decay properties of $\boldsymbol{U}^{(1)}$ relies (see Lemma 4.3.9) on the right hand side showing a constant vector multiplied by the rotation of the test function, which is not true in this case.

In [GS21] the authors compute the topological derivative for a quasi-linear magnetostatic problem proposing a projection trick to avoid the usage of fundamental solutions and potentials, it would be interesting to extend the work of this chapter to a non-linear low-frequency electromagnetic problem, or the eddy current model.

Appendix A

Appendix: some results for $z \in \Omega_I$

The presentation of Chapter 4 is mainly focused on the case $z \in \Omega_C$, i.e. when a *small* piece of conductor $\omega_\varepsilon = z + \varepsilon\omega$ (ω is a bounded connected domain which contains the origin) is removed from the interior of Ω_C. The case $z \in \Omega_I$ instead corresponds to the creation of a *small* conductor inside the original insulator, and can be also of interest in applications. The purpose of this section is to shortly show what modifications are needed to derive all the results for $z \in \Omega_I$: we collect the definitions of *new* correctors and restate the main results, omitting all the technical details that can be already found in Chapter 4. Another motivation for keeping the presentation concise is that most changes boil down to a change of sign. As the existence results (Proposition 4.3.6 for the perturbed state and Lemma 4.4.1 for the adjoint) are not affected[1] by the choice of z, we start from corrector equations, that is roughly from the beginning of Section 4.3.3.

In this case, the material parameters associated with the reference inclusion ω read

$$\widehat{\nu}_\omega(x) := \begin{cases} \nu_2 & x \in \mathbb{R}^3 \setminus \overline{\omega} \\ \nu_1 & x \in \omega \end{cases} \quad \text{and} \quad \widehat{\sigma}_\omega(x) := \begin{cases} 0 & x \in \mathbb{R}^3 \setminus \overline{\omega} \\ \sigma_0 & x \in \omega. \end{cases}$$

Note that we have

$$0 < \min(\nu_1, \nu_2) \le \widehat{\nu}_\omega(x) \le \max(\nu_1, \nu_2), \quad 0 \le \widehat{\sigma}_\omega(x) \le \sigma_0 \quad \forall x \in \mathbb{R}^3$$

as for $\nu_\omega, \sigma_\omega$ (see (4.65)).

Lemma A.0.1. *Let $\boldsymbol{A}_0 \in \boldsymbol{X}_N^0$ be the unique solution of* (4.27). *The equation*

$$\int_{\mathbb{R}^3} \widehat{\nu}_\omega \,\mathbf{curl}[\boldsymbol{U}^{(1)}] \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x = -(\nu_1 - \nu_2) \int_\omega \mathbf{curl}\,\boldsymbol{A}_0(z) \cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \dot{\boldsymbol{W}}(\mathbb{R}^3) \tag{A.1}$$

[1] With this sentence we are just stating that the proofs of existence and uniqueness do not change depending on whether $z \in \Omega_C$ or $z \in \Omega_I$, because the material parameters $\nu_\varepsilon, \sigma_\varepsilon$ satisfy $0 < \nu_\varepsilon(x) \le \max(\nu_1, \nu_2)$ and $\sigma_\varepsilon(x) \ge 0$ for all $x \in \Omega$ in both cases. This doesn't mean the fields themselves are not affected, on the contrary, for a given $\varepsilon > 0$, $\boldsymbol{A}_\varepsilon$ (and $\boldsymbol{q}_\varepsilon$) will in general vary along with z.

admits a unique solution $[\boldsymbol{U}^{(1)}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$. Moreover, there exists a unique divergence-free representative $\boldsymbol{U}^{(1)} \in [\boldsymbol{U}^{(1)}] \cap \boldsymbol{L}^2_\rho(\mathbb{R}^3)$ which satisfies

$$\int_{\mathbb{R}^3} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x = -(\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{W}(\mathbb{R}^3), \tag{A.2}$$

and enjoys the asymptotic behaviour at infinity

$$|\boldsymbol{U}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty.$$

Proof. The parameter $\widehat{\nu}_\omega$ has the same properties as ν_ω and the right hand side of (A.1) only differs from (4.67) by a change of sign. The proof is therefore identical to the one of Lemma 4.3.9. □

Lemma A.0.2. *Assume $z \in \Omega_I$ and $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_I)$ for some $\alpha > 0$. Let $\boldsymbol{A}_\varepsilon$ be the solution of (4.64) and $\boldsymbol{A}_0$ the solution of (4.27). There is constant $C > 0$ independent of ε such that*

$$\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\varepsilon^{\frac{3}{2}} \tag{A.3}$$

for sufficiently small values of $\varepsilon > 0$.

Proof. We subtract (4.27) from (4.64) to get

$$\begin{aligned}\int_\Omega \nu_\varepsilon \,\mathbf{curl}(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_\Omega \sigma_\varepsilon (\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\ = -i\sigma_0 \int_{\omega_\varepsilon} \boldsymbol{A}_0 \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - (\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}^0_N.\end{aligned} \tag{A.4}$$

Testing with $\boldsymbol{v} = \boldsymbol{A}_\varepsilon - \boldsymbol{A}_0 \in \boldsymbol{X}^0_N$ in (A.4) gives

$$\begin{aligned}\int_\Omega \nu_\varepsilon |\,\mathbf{curl}(\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0)|^2 \,\mathrm{d}x + i \int_\Omega \sigma_\varepsilon |\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0|^2 \,\mathrm{d}x \\ = -i\sigma_0 \int_{\omega_\varepsilon} \boldsymbol{A}_0 \cdot (\overline{\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0}) \,\mathrm{d}x - (\nu_1 - \nu_2) \int_{\omega_\varepsilon} \mathbf{curl}\, \boldsymbol{A}_0 \cdot \mathbf{curl}\, (\overline{\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0}) \,\mathrm{d}x.\end{aligned}$$

The proof now proceeds in the exact same way as in Lemma 4.3.10, since the changes of sign would not affect the norm estimates. □

The following result is the counterpart of Lemma 4.3.12 for the case $z \in \Omega_I$.

Lemma A.0.3. *Let $\Omega_\varepsilon = T_\varepsilon^{-1}(\Omega)$ and $\Gamma_\varepsilon = T_\varepsilon^{-1}(\Gamma)$. Let $(\boldsymbol{g}_\varepsilon, h_\varepsilon) \in \boldsymbol{V}_\tau(\Gamma_\varepsilon) \times H^{\frac{1}{2}}(\Gamma_\varepsilon)$ and $\boldsymbol{\Phi}_\varepsilon \in (\boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon), \|\cdot\|_{\varepsilon,\mathbf{curl}})^*$ be defined for sufficiently small values of $\varepsilon > 0$. Recall that $\boldsymbol{V}_\tau(\Gamma_\varepsilon) = \{\boldsymbol{v} \in \boldsymbol{L}^2_\tau(\Gamma_\varepsilon) : \mathrm{div}_\tau \boldsymbol{v} \in L^2(\Gamma_\varepsilon)\}$, see (4.57). Assume further that $(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) \in \boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon) \cap H(\mathrm{div}{=}0, \Omega_\varepsilon) \times H^1(\Omega_\varepsilon)$ satisfy*

$$\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{w}_\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_{\Omega_\varepsilon} \widehat{\sigma}_\omega \varepsilon^2 \boldsymbol{w}_\varepsilon \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x + \int_{\Omega_\varepsilon} \varepsilon \nabla \phi_\varepsilon \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x = \langle \boldsymbol{\Phi}_\varepsilon, \boldsymbol{v} \rangle_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)}$$

for all $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$*, as well as*

$$\begin{cases} \boldsymbol{w}_\varepsilon \times \boldsymbol{n} = \boldsymbol{g}_\varepsilon & \text{on } \Gamma_\varepsilon \\ \phi_\varepsilon = h_\varepsilon & \text{on } \Gamma_\varepsilon. \end{cases}$$

Then there is a constant $C > 0$*, independent of* ε*, such that*

$$\begin{aligned} \|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} \leq C(&\|\boldsymbol{\Phi}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}}\|\boldsymbol{g}_\varepsilon\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} \\ &+ \varepsilon^{-\frac{1}{2}}\|\operatorname{div}_\tau \boldsymbol{g}_\varepsilon\|_{L^2(\Gamma_\varepsilon)} + \varepsilon^{\frac{1}{2}}\|h_\varepsilon\|_{L^2(\Gamma_\varepsilon)} + |h_\varepsilon|_{H^{\frac{1}{2}}(\Gamma_\varepsilon)}). \end{aligned}$$

Proof. The parameters $\widehat{\nu}_\omega, \widehat{\sigma}_\omega$ have the same properties as $\nu_\omega, \sigma_\omega$ in the sense that

$$0 < \min(\nu_1, \nu_2) \leq \widehat{\nu}_\omega(x) \leq \max(\nu_1, \nu_2), \quad 0 \leq \widehat{\sigma}_\omega(x) \leq \sigma_0 \quad \forall x \in \mathbb{R}^3.$$

Hence the proof is carried out as for Lemma 4.3.12. □

Theorem A.0.4. *Let Assumption 4.2.1 and Assumption 4.3.13 hold, and* $z \in \Omega_I$*. Let* $\boldsymbol{U}_\varepsilon^{(1)}$ *be defined by (4.73) and* $\boldsymbol{U}^{(1)}$ *through Lemma A.0.1. If* $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_I)$ *for some* $\alpha > 0$*, then there is a constant* $C > 0$ *not depending on* $\varepsilon > 0$ *for which*

$$\|\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}\|_{\varepsilon,\mathbf{curl}} \leq C\varepsilon \tag{A.5}$$

for sufficiently small ε*.*

Proof. We observe that

$$\begin{aligned} (\nu_\varepsilon \circ T_\varepsilon)(x) &= ((\nu_1 \chi_{\Omega_{C,\varepsilon}} + \nu_2 \chi_{\Omega \setminus \Omega_{C,\varepsilon}}) \circ T_\varepsilon)(x) \\ &= ((\nu_1 \chi_{\Omega_C \cup \omega_\varepsilon} + \nu_2 \chi_{\Omega_I \setminus \omega_\varepsilon}) \circ T_\varepsilon)(x) \\ &= (\nu_1 \chi_\omega + \nu_2 \chi_{T_\varepsilon^{-1}(\Omega) \setminus \omega})(x) = \widehat{\nu}_\omega|_{\Omega_\varepsilon}(x), \end{aligned}$$

In a similar fashion we have $(\sigma_\varepsilon \circ T_\varepsilon)(x) = \widehat{\sigma}_\omega|_{\Omega_\varepsilon}(x)$. Keeping this in mind, the change of variables $x \mapsto T_\varepsilon(x)$ in (A.4) gives the following equation for $\boldsymbol{U}_\varepsilon^{(1)}$:

$$\begin{aligned} &\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \, \mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x + i \int_{\Omega_\varepsilon} \widehat{\sigma}_\omega \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x \\ &= - i\sigma_0 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \, \mathrm{d}x - (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{A}_0)(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \, \mathrm{d}x \end{aligned} \tag{A.6}$$

for all $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$.

Next we note that if $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$, then the extension

$$\boldsymbol{v}^\star := \begin{cases} \boldsymbol{v} & \text{in } \Omega_\varepsilon \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases}$$

belongs to $\boldsymbol{W}(\mathbb{R}^3)$ and therefore it is a feasible test function for (A.2). This yields

$$\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x = -(\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(z) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x, \tag{A.7}$$

so that we can now subtract (A.7) from (A.6) to obtain:

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}(\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x + i \int_{\Omega_\varepsilon} \widehat{\sigma}_\omega \varepsilon^2 (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&= - i\sigma_0 \int_\omega \varepsilon^2 \boldsymbol{U}^{(1)} \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x - (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}} \,\mathrm{d}x \\
&\quad - i\sigma_0 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}} \,\mathrm{d}x.
\end{aligned} \tag{A.8}$$

We now apply Lemma A.0.3 to equation (A.8), with the choices

$$(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) = (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}, 0), \quad (\boldsymbol{g}_\varepsilon, h_\varepsilon) = (\boldsymbol{w}_\varepsilon|_{\Gamma_\varepsilon} \times \boldsymbol{n}, 0)$$

and $\boldsymbol{\Phi}_\varepsilon$ equal to the mapping induced by the whole right hand side of (A.8), which we denote by $\hat{\boldsymbol{\Phi}}_\varepsilon$. On the boundary Γ_ε it holds

$$\boldsymbol{g}_\varepsilon = \boldsymbol{w}_\varepsilon \times \boldsymbol{n} = (\boldsymbol{U}_\varepsilon^{(1)} - \boldsymbol{U}^{(1)}) \times \boldsymbol{n} = -\boldsymbol{U}^{(1)} \times \boldsymbol{n}$$

and we stress that $\boldsymbol{g}_\varepsilon$ belongs to $\boldsymbol{V}_\tau(\Gamma_\varepsilon)$ since $\boldsymbol{U}^{(1)}$ is harmonic in the exterior of ω (see (A.2)). Therefore by Lemma 4.3.12 the following inequality holds true:

$$\begin{aligned}
\|\boldsymbol{w}_\varepsilon\|_{\varepsilon,\mathbf{curl}} &\le C(\|\hat{\boldsymbol{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}} \|\boldsymbol{U}^{(1)} \times \boldsymbol{n}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \|\operatorname{div}_\tau(\boldsymbol{U}^{(1)} \times \boldsymbol{n})\|_{L^2(\Gamma_\varepsilon)}) \\
&= C(\|\hat{\boldsymbol{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}} \|\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \|\mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \boldsymbol{n}\|_{L^2(\Gamma_\varepsilon)}) \\
&= C(\|\hat{\boldsymbol{\Phi}}_\varepsilon\|_{\boldsymbol{H}(\mathbf{curl},\Omega_\varepsilon)^*} + \varepsilon^{\frac{1}{2}} \|\boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)} + \varepsilon^{-\frac{1}{2}} \|\mathbf{curl}\, \boldsymbol{U}^{(1)}\|_{\boldsymbol{L}^2(\Gamma_\varepsilon)}).
\end{aligned}$$

Comparing (A.8) and (4.84) and considering that the decay behaviour of $\boldsymbol{U}^{(1)}$ does not change if $z \in \Omega_I$ (cf. Lemma A.0.1 and Lemma 4.3.9), the proof now follows the exact same lines of the one of Theorem 4.3.14. □

Next we focus on the averaged adjoint equation.

Lemma A.0.5. *Assume $z \in \Omega_I$ and $\boldsymbol{J} \in \boldsymbol{W}^{0,3+\alpha}_{loc}(\mathbf{curl}, \Omega_I)$ for some $\alpha > 0$. Let $\boldsymbol{q}_\varepsilon, \boldsymbol{q}_0$ denote the solution to (4.95) respectively for $\varepsilon > 0$ and $\varepsilon = 0$. There is a constant $C > 0$ not depending on ε such that*

$$\|\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \le C\varepsilon^{\frac{3}{2}} \tag{A.9}$$

for sufficiently small $\varepsilon > 0$.

Proof. We subtract (4.96) from (4.95), this yields:

$$\begin{aligned}
&\int_\Omega \nu_\varepsilon \,\mathbf{curl}(\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0)\cdot \mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x + i\int_\Omega \sigma_\varepsilon(\boldsymbol{q}_\varepsilon - \boldsymbol{q}_0)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&\quad = -(\nu_1-\nu_2)\int_{\omega_\varepsilon} \mathbf{curl}\,\boldsymbol{q}_0\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x - i\sigma_0\int_{\omega_\varepsilon}\boldsymbol{q}_0\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x - \alpha_1\int_{\Omega_{C,\varepsilon}}(\boldsymbol{A}_\varepsilon-\boldsymbol{A}_0)\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&\quad + 2\alpha_1\int_{\omega_\varepsilon}\boldsymbol{A}_0\cdot\overline{\boldsymbol{v}}\,\mathrm{d}x - \alpha_2\int_\Omega \nu_\varepsilon\,\mathbf{curl}(\boldsymbol{A}_\varepsilon-\boldsymbol{A}_0)\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&\quad - 2\alpha_2(\nu_1-\nu_2)\int_{\omega_\varepsilon}\mathbf{curl}\,\boldsymbol{A}_0\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in \boldsymbol{X}_N^0(\Omega).
\end{aligned} \tag{A.10}$$

Next we test (A.10) with $\boldsymbol{v} = \boldsymbol{q}_\varepsilon - \boldsymbol{q}_0$. Exploiting the continuity of $\boldsymbol{A}_0, \boldsymbol{q}_0$ and of $\mathbf{curl}\,\boldsymbol{A}_0, \mathbf{curl}\,\boldsymbol{q}_0$ (Lemma 4.4.1, Lemma 4.2.6) and what we already know on the scaling behaviour of $\|\boldsymbol{A}_\varepsilon - \boldsymbol{A}_0\|_{\boldsymbol{H}(\mathbf{curl})}$ from (A.3), we can mimic the estimates performed in the proof of Lemma 4.3.10. □

Lemma A.0.6. *Let $\boldsymbol{U}^{(1)}$ be defined through Lemma A.0.1. There exists a unique solution $[\boldsymbol{Q}^{(1)}] \in \dot{\boldsymbol{W}}(\mathbb{R}^3)$ to*

$$\begin{aligned}
\int_{\mathbb{R}^3}\widehat{\nu}_\omega\,\mathbf{curl}[\boldsymbol{Q}^{(1)}]\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x &= -(\nu_1-\nu_2)\int_\omega(\mathbf{curl}\,\boldsymbol{q}_0(z)+2\alpha_2\,\mathbf{curl}\,\boldsymbol{A}_0(z))\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&\quad - \alpha_2\int_{\mathbb{R}^3}\widehat{\nu}_\omega\,\mathbf{curl}\,\boldsymbol{U}^{(1)}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\dot{\boldsymbol{W}}(\mathbb{R}^3).
\end{aligned}$$

Moreover, there is a unique divergence-free representative $\boldsymbol{Q}^{(1)} \in [\boldsymbol{Q}^{(1)}] \cap \boldsymbol{S}$ (the space $\boldsymbol{S}$ is defined in (4.70)) that satisfies

$$\begin{aligned}
\int_{\mathbb{R}^3}\widehat{\nu}_\omega\,\mathbf{curl}\,\boldsymbol{Q}^{(1)}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x &= -(\nu_1-\nu_2)\int_\omega(\mathbf{curl}\,\boldsymbol{q}_0(z)+2\alpha_2\,\mathbf{curl}\,\boldsymbol{A}_0(z))\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \\
&\quad - \alpha_2\int_{\mathbb{R}^3}\widehat{\nu}_\omega\,\mathbf{curl}\,\boldsymbol{U}^{(1)}\cdot\mathbf{curl}\,\overline{\boldsymbol{v}}\,\mathrm{d}x \quad \forall \boldsymbol{v}\in\boldsymbol{W}(\mathbb{R}^3)
\end{aligned} \tag{A.11}$$

and enjoys the decay behaviour at infinity

$$|\boldsymbol{Q}^{(1)}(x)| = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x|\to\infty. \tag{A.12}$$

Proof. We omit the proof as it is almost identical to the one of Lemma 4.4.4. In particular, note that $\widehat{\nu}_\omega$ is strictly positive in $\mathbb{R}^3$ like ν_ω and that the right hand sides of (A.11) and (4.128) only differ by a sign (and the fact that $\boldsymbol{U}^{(1)}$ is technically different as it is now given by (A.7)). □

Theorem A.0.7. *Let Assumption 4.2.1 and Assumption 4.3.13 hold and assume $z \in \Omega_I$. Let $\boldsymbol{Q}_\varepsilon^{(1)}$ be given by (4.108) and $\boldsymbol{Q}^{(1)}$ be defined via (A.11). If $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_I)$ ($\alpha > 0$), there exists a constant $C > 0$ not depending on ε such that*

$$\|\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}\|_{\varepsilon,\mathbf{curl}} \le C\varepsilon \tag{A.13}$$

for sufficiently small $\varepsilon > 0$.

Proof. Changing variables in (A.10) gives the following equation for $\boldsymbol{Q}_\varepsilon^{(1)}$:

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{Q}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + i \int_{\Omega_\varepsilon} \varepsilon^2 \widehat{\sigma}_\omega \boldsymbol{Q}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&= -(\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{q}_0(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x - i\sigma_0 \int_\omega \varepsilon \boldsymbol{q}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - \alpha_1 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&+ 2\alpha_1 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - \alpha_2 \int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&- 2\alpha_2(\nu_1 - \nu_2) \int_\omega \mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon).
\end{aligned} \tag{A.14}$$

As $\boldsymbol{v} \in \boldsymbol{X}_N^0(\Omega_\varepsilon)$, the extension

$$\boldsymbol{v}^\star := \begin{cases} \boldsymbol{v} & \text{in } \Omega_\varepsilon \\ \boldsymbol{0} & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_\varepsilon} \end{cases}$$

belongs to $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3) \hookrightarrow \boldsymbol{W}(\mathbb{R}^3)$ and therefore it can be chosen as test function in (4.106). This yields

$$\begin{aligned}
\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{Q}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x &= -(\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(z) + 2\alpha_2 \,\mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&\quad - \alpha_2 \int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x.
\end{aligned} \tag{A.15}$$

Subtracting (A.15) from (A.14) yields

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}(\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + i \int_{\Omega_\varepsilon} \widehat{\sigma}_\omega \varepsilon^2 (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&= -i\sigma_0 \int_\omega \varepsilon^2 \boldsymbol{Q}^{(1)} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - \alpha_1 \int_{\Omega_\varepsilon \setminus \omega} \varepsilon^2 \boldsymbol{U}_\varepsilon^{(1)} \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - i\sigma_0 \int_\omega \varepsilon \boldsymbol{q}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&+ 2\alpha_1 \int_\omega \varepsilon \boldsymbol{A}_0(x_\varepsilon) \cdot \overline{\boldsymbol{v}}\, \mathrm{d}x - (\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{q}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{q}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&- 2\alpha_2(\nu_1 - \nu_2) \int_\omega (\mathbf{curl}\, \boldsymbol{A}_0(x_\varepsilon) - \mathbf{curl}\, \boldsymbol{A}_0(z)) \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \\
&+ \alpha_2 \left(- \int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{U}_\varepsilon^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x + \int_{\Omega_\varepsilon} \widehat{\nu}_\omega \,\mathbf{curl}\, \boldsymbol{U}^{(1)} \cdot \mathbf{curl}\, \overline{\boldsymbol{v}}\, \mathrm{d}x \right).
\end{aligned} \tag{A.16}$$

Next we apply Lemma A.0.3 to (A.16) with the choices $\boldsymbol{\Phi}_\varepsilon = \boldsymbol{\Theta}_\varepsilon \in (\boldsymbol{H}(\mathbf{curl}, \Omega_\varepsilon), \|\cdot\|_{\varepsilon,\mathbf{curl}})^*$, $(\boldsymbol{w}_\varepsilon, \phi_\varepsilon) = (\boldsymbol{Q}_\varepsilon^{(1)} - \boldsymbol{Q}^{(1)}, 0)$ and $(\boldsymbol{g}_\varepsilon, h_\varepsilon) = (\boldsymbol{w}_\varepsilon \times \boldsymbol{n}, 0)$, where $\boldsymbol{\Theta}_\varepsilon$ is a shortcut for the whole right hand side of (A.16).

The proof now follows the exact same lines as the one of Theorem 4.4.5, since the asymptotic decay of $\boldsymbol{U}^{(1)}$ and $\boldsymbol{Q}^{(1)}$ doesn't change depending on $z \in \Omega_I$ or $z \in \Omega_C$. Additionally, $\boldsymbol{J} \in \boldsymbol{W}_{loc}^{0,3+\alpha}(\mathbf{curl}, \Omega_I)$ guarantees that $\boldsymbol{A}_0, \boldsymbol{q}_0, \mathbf{curl}\, \boldsymbol{A}_0, \mathbf{curl}\, \boldsymbol{q}_0$ are locally Lipschitz continuous in a neighbourhood of $z \in \Omega_I$, as shown in Lemma 4.2.6 and Lemma 4.4.1. □

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