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The binomial decomposition of OWA functions, the 2-additive and 3-additive cases in n dimensions

Silvia Bortot[1,*], Ricardo Alberto Marques Pereira[1], Thuy Hong Nguyen[2]
[1] *Department of Economics and Management, University of Trento, Italy*
[2] *Department of Industrial Engineering, University of Trento, Italy*

In the context of the binomial decomposition of OWA functions, we investigate the constraints associated with the 2-additive and 3-additive cases in n dimensions. The 2-additive case depends on one coefficient whose feasible region does not depend on the dimension n. On the other hand, the feasible region of the 3-additive case depends on two coefficients and is explicitly dependent on the dimension n. This feasible region is a convex polygon with n vertices and n edges, which is strictly expanding in the dimension n. The orness of the OWA functions within the feasible region is linear in the two coefficients, and the vertices associated with maximum and minimum orness are identified. Finally, we discuss the 3-additive binomial decomposition in the asymptotic infinite dimensional limit.

1. INTRODUCTION

The ordered weighted averaging functions have the form $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}$ where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and $w_1, w_2, \ldots, w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. These functions have been introduced by Yager [34] and correspond to the Choquet integrals associated with symmetric capacities, see Fodor et al. [10]. The theory and applications of OWA functions are widely discussed in the literature, see for instance Yager and Kacprzyk [35], and Yager et al. [36].

In general, Choquet integration corresponds to a generalization of both weighted averaging (WA) and ordered weighted averaging (OWA), which remain as special cases. The seminal papers on Choquet integration are Choquet [8], Murofushi and Sugeno [31], Chateauneuf [7], Denneberg [9], Grabisch [13, 14], and Marichal [25]. For recent reviews of Choquet integration, on the other hand, see Grabisch and Labreuche [19, 20, 21], and Grabisch et al. [18].

The complex structure of Choquet capacities can be suitably described in the k-additivity framework introduced by Grabisch [15, 16], see also Calvo and De Baets [5], Kim and De Baets [24], and Miranda et al. [30]. The 2-additive case, in particular, has been examined by Miranda et al. [30], and Mayag et al. [27, 28]. Due to its low complexity and versatility, the 2-additive case is relevant in a variety of modelling contexts.

*Author to whom all corrispondence should be addressed; e-mail silvia.bortot@unitn.it

The characterization of symmetric Choquet integrals (OWA functions) has been studied by Fodor et al. [10], Calvo and De Baets [5], Kim and De Baets [24], and Miranda et al. [30]. It is shown, see Gajdos [12], that in the k-additive case the generating function of the OWA weights is polynomial of degree k, where the weights correspond to differences between consecutive generating function values and are therefore polynomial of degree $k-1$. In the symmetric 2-additive case, in particular, the generating function is quadratic and therefore the weights are linear (thus equidistant), as in the classical Gini welfare function.

In this paper we review the analysis of symmetric capacities in the Möbius representation framework and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [5], see also Bortot and Marques Pereira [4]. The binomial decomposition is formulated in terms of the functional basis of the binomial OWA functions, denoted C_j with $j = 1, \ldots, n$, which have $n - j + 1$ positive decreasing weights $w_1 > w_2 > \cdots > w_{n-j+1} > 0$ and $j-1$ null weights $w_{n-j+2} = \cdots = w_n = 0$.

The paper is organized as follows. In Section 2 we review the basic notions of weighted averaging and ordered weighted averaging functions in n dimensions. In Section 3 we present the basic definitions and results regarding Choquet capacities and integration, with reference to the Möbius representation framework. We consider the context of symmetric capacities and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [5], see also Bortot and Marques Pereira [4].

In Section 4 we examine the binomial decomposition of OWA functions focusing on the 2-additive and 3-additive cases. In particular, we investigate the parametric constraints associated with the 3-additive case in n dimensions. The resulting feasible region in two coefficients is a convex polygon with n vertices and n edges, which is strictly expanding in the dimension n. The orness of the OWA functions within the feasible region is linear in the two coefficients, and the vertices associated with maximum and minimum orness are identified. Finally, we discuss the 3-additive binomial decomposition in the asymptotic infinite dimensional limit. Section 5 contains some conclusive remarks.

2. AVERAGING FUNCTIONS: WA AND OWA FUNCTIONS

We begin by presenting notation and basic definitions regarding averaging functions on the domain $\mathbb{D}^n$, with $\mathbb{D} = \mathbb{R}$ and $n \geq 2$ throughout the text. Points in this domain are denoted by $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Comprehensive reviews of averaging functions can be found in Fodor and Roubens [11], Marichal [25], Marichal et al. [26], Calvo et al. [6], Beliakov et al. [1], Torra and Narukawa [33], Mesiar et al. [29], Grabisch et al. [22, 23], and Beliakov et al. [2].

Notation. Points in $\mathbb{D}^n$ are denoted $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $\mathbf{0} = (0, \ldots, 0)$, $\mathbf{1} = (1, \ldots, 1)$. Accordingly, for every $x \in \mathbb{D}$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i = 1, \ldots, n$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in \mathbb{D}^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1,\dots,n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)},\dots,x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\bar{x} = (x_1 + \cdots + x_n)/n$.

Definition 1 *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function.*

1. *A is* monotonic *if $\boldsymbol{x} \geq \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is* strictly monotonic *if $\boldsymbol{x} > \boldsymbol{y} \Rightarrow A(\boldsymbol{x}) > A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*

2. *A is* idempotent *if $A(x \cdot \mathbf{1}) = x$, for all $x \in \mathbb{D}$. On the other hand, A is* nilpotent *if $A(x \cdot \mathbf{1}) = 0$, for all $x \in \mathbb{D}$.*

3. *A is* symmetric *if $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, for any permutation σ on $\{1,\dots,n\}$ and all $\boldsymbol{x} \in \mathbb{D}^n$.*

4. *A is* invariant for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is* stable for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = A(\boldsymbol{x}) + t$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*

Definition 2 *A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is an* (n-ary) averaging function *if it is monotonic and idempotent. An averaging function is said to be* strict *if it is strictly monotonic. Note that monotonicity and idempotency implies that* $\min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq \max(\boldsymbol{x})$*, for all $\boldsymbol{x} \in \mathbb{D}^n$.*

Particular cases of averaging functions are weighted averaging (WA) functions, ordered weighted averaging (OWA) functions, and Choquet integrals, which contain the former as special cases.

Definition 3 *Given a weighting vector $\boldsymbol{w} = (w_1,\dots,w_n) \in [0,1]^n$, with $\sum_{i=1}^n w_i = 1$, the* Weighted Averaging (WA) function *associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_i. \tag{1}$$

Definition 4 *Given a weighting vector $\boldsymbol{w} = (w_1,\dots,w_n) \in [0,1]^n$, with $\sum_{i=1}^n w_i = 1$, the* Ordered Weighted Averaging (OWA) function *associated with $\boldsymbol{w}$ is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_{(i)}. \tag{2}$$

Weighted Averaging functions and Ordered Weighted Averaging functions are continuous and stable for translations. The traditional form of OWA functions as introduced by Yager [34] is as follows, $A(\boldsymbol{x}) = \sum_{i=1}^n \tilde{w}_i\, x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$. In [35, 36] the theory and applications of OWA functions are discussed in detail.

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DEFINITION 5 *Let A the OWA function associated with the weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$. The* orness *of A is defined as*

$$\text{Orness}\,(A) = \frac{1}{n-1} \sum_{i=1}^{n} (i-1)\, w_i\,. \tag{3}$$

The orness of A coincides with the value $A(\boldsymbol{x}_0)$, where $x_i^0 = (i-1)/(n-1)$,

$$\text{Orness}\,(A) = \frac{1-1}{n-1}\, w_1 + \frac{2-1}{n-1}\, w_2 + \cdots + \frac{(n-1)-1}{n-1}\, w_{n-1} + \frac{n-1}{n-1}\, w_n\,. \tag{4}$$

The orness takes values in $[0,1]$ interval, with zero orness corresponding to the weighting vector $\boldsymbol{w} = (1,0,\ldots,0) \in [0,1]^n$ and unit orness corresponding to the weighting vector $\boldsymbol{w} = (0,\ldots,0,1) \in [0,1]^n$.

The following result regards a form of dominance relation between weighting structures and OWA function values, see for instance Calvo and De Baets [5], and Bortot and Marques Pereira [4].

PROPOSITION 1 *Consider two OWA functions $A, B : \mathbb{D}^n \longrightarrow \mathbb{D}$ associated with weighting vectors $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$ and $\boldsymbol{v} = (v_1, \ldots, v_n) \in [0,1]^n$, respectively. It holds that $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$ if and only if*

$$\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i \qquad \textit{for} \quad k = 1, \ldots, n \tag{5}$$

where the case $k = n$ is an equality due to weight normalization.

The classical Gini welfare function A_G^c is an important example of the OWA averaging functions,

$$A_G^c(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{2(n-i)+1}{n^2}\, x_{(i)} \tag{6}$$

where the weights of A^c have unit sum, since $\sum_{i=1}^{n}(2(n-i)+1) = n^2$. The classical Gini inequality index G^c, related with the classical Gini welfare function by means of $G^c(\boldsymbol{x}) = \bar{x} - A_G^c(\boldsymbol{x})$, is given by

$$G^c(\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{n-2i+1}{n^2}\, x_{(i)} \tag{7}$$

where the coefficients of G^c have zero sum, since $\sum_{i=1}^{n}(n-2i+1) = 0$. Notice that G^c is not an OWA function, for further reading see for instance Bortot and Marques Pereira [4].

3. THE BINOMIAL DECOMPOSITION OF OWA FUNCTIONS

In this section we present a brief review of the basic facts on Choquet integration, focusing on the Möbius representation framework. For recent reviews of Choquet integration see Grabisch and Labreuche [19, 20, 21], and Grabisch et al. [18] for the general case, Miranda et al. [30], and Mayag et al. [27, 28] for the 2-additive case in particular.

Consider a finite set of interacting elements $N = \{1, 2, \ldots, n\}$. Any subsets S, $T \subseteq N$ with cardinalities $0 \leq s, t \leq n$ are usually called coalitions. The concepts of capacity and Choquet integral in the definitions below are due to Choquet [8], Sugeno [32], Denneberg [9], and Grabisch [13, 14].

Definition 6 *A* capacity *on the set N is a set function $\mu : 2^N \longrightarrow [0, 1]$ satisfying*

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$ (boundary conditions)

(ii) $S \subseteq T \subseteq N \quad \Rightarrow \quad \mu(S) \leq \mu(T)$ (monotonicity conditions).

Definition 7 *Let μ be a capacity on N. The* Choquet integral *$\mathcal{C}_\mu : \mathbb{D}^n \longrightarrow \mathbb{D}$ with respect to μ is defined as*

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} \qquad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{D}^n \tag{8}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$. Moreover, $A_{(i)} = \{(i), \ldots, (n)\}$ and $A_{(n+1)} = \emptyset$.

Definition 8 *Let μ be a capacity on the set N. The* Möbius transform *$m_\mu : 2^N \longrightarrow \mathbb{R}$ associated with the capacity μ is defined as*

$$m_\mu(T) = \sum_{S \subseteq T} (-1)^{t-s} \mu(S) \qquad T \subseteq N \tag{9}$$

where s and t denote the cardinality of the coalitions S and T, respectively.

Conversely, given the Möbius transform m_μ, the associated capacity μ is obtained as

$$\mu(T) = \sum_{S \subseteq T} m_\mu(S) \qquad T \subseteq N\,. \tag{10}$$

In the Möbius representation, the boundary conditions take the form

$$m_\mu(\emptyset) = 0 \qquad \sum_{T \subseteq N} m_\mu(T) = 1 \tag{11}$$

and the monotonicity conditions can be expressed as follows: for each $i = 1, \ldots, n$ and each coalition $T \subseteq N \setminus \{i\}$, the monotonicity condition is written as

$$\sum_{S \subseteq T} m_\mu(S \cup \{i\}) \geq 0 \qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{12}$$

This form of the monotonicity conditions derives from the original monotonicity conditions in Definition 6, expressed as $\mu(T \cup \{i\}) - \mu(T) \geq 0$ for each $i \in N$ and $T \subseteq N \setminus \{i\}$.

Defining a capacity μ on a set N of n elements requires $2^n - 2$ real coefficients, corresponding to the capacity values $\mu(T)$ for $T \subseteq N$. In order to control exponential complexity, Grabisch [15] introduced the concept of k-additive capacities.

DEFINITION 9 *A capacity μ on the set N is said to be* k-additive *if its Möbius transform satisfies $m_\mu(T) = 0$ for all $T \subseteq N$ with $t > k$, and there exists at least one coalition $T \subseteq N$ with $t = k$ such that $m_\mu(T) \neq 0$.*

In the k-additive case, with $k = 1, \ldots, n$, the capacity μ is expressed as follows in terms of the Möbius transform m_μ,

$$\mu(T) = \sum_{S \subseteq T,\, s \leq k} m_\mu(S) \qquad\qquad T \subseteq N \tag{13}$$

and the boundary and monotonicity conditions (11) and (12) take the form

$$m_\mu(\emptyset) = 0 \qquad\qquad \sum_{T \subseteq N,\, t \leq k} m_\mu(T) = 1 \tag{14}$$

$$\sum_{S \subseteq T,\, s \leq k-1} m_\mu(S \cup \{i\}) \geq 0 \qquad\qquad T \subseteq N \setminus \{i\} \qquad i = 1, \ldots, n\,. \tag{15}$$

Finally, we examine the particular case of symmetric capacities and Choquet integrals, which play a crucial role in this paper.

DEFINITION 10 *A capacity μ is said to be* symmetric *if it depends only on the cardinality of the coalition considered, in which case we use the simplified notation*

$$\mu(T) = \mu(t) \qquad \textit{where} \qquad t = |T|\,. \tag{16}$$

Accordingly, for the Möbius transform m_μ associated with a symmetric capacity μ we use the notation

$$m_\mu(T) = m_\mu(t) \qquad \text{where} \qquad t = |T|\,. \tag{17}$$

In the symmetric case, the expression (10) for the capacity μ in terms of the Möbius transform m_μ reduces to

$$\mu(t) = \sum_{s=1}^{t} \binom{t}{s} m_\mu(s) \qquad\qquad t = 1, \ldots, n \tag{18}$$

and the boundary and monotonicity conditions (11) and (12) take the form

$$m_\mu(0) = 0 \qquad\qquad \sum_{s=1}^{n} \binom{n}{s} m_\mu(s) = 1 \tag{19}$$

$$\sum_{s=1}^{t} \binom{t-1}{s-1} m_\mu(s) \geq 0 \qquad t = 1, \ldots, n\,. \tag{20}$$

The monotonicity conditions correspond to $\mu(t) - \mu(t-1) \geq 0$ for $t = 1, \ldots, n$.

The Choquet integral (8) with respect to a symmetric capacity μ reduces to an Ordered Weighted Averaging (OWA) function, see Fodor et al. [10], and Yager [34],

$$\mathcal{C}_\mu(\boldsymbol{x}) = \sum_{i=1}^{n} [\mu(n-i+1) - \mu(n-i)] x_{(i)} = \sum_{i=1}^{n} w_i\, x_{(i)} = A(\boldsymbol{x}) \tag{21}$$

where the weights $w_i = \mu(n-i+1) - \mu(n-i)$ satisfy $w_i \geq 0$ for $i = 1, \ldots, n$ due to the monotonicity of the capacity μ, and $\sum_{i=1}^{n} w_i = 1$ due to the boundary conditions $\mu(0) = 0$ and $\mu(n) = 1$. Comprehensive reviews of OWA functions can be found in Yager and Kacprzyk [35] and Yager et al. [36].

The weighting structure of the OWA function (21) is of the general form $w_i = f(\frac{n-i+1}{n}) - f(\frac{n-i}{n})$ where f is a continuous and increasing function on the unit interval, with $f(0) = 0$ and $f(1) = 1$. Gajdos [12] shows that the OWA function A is associated with a k-additive capacity μ, with $k = 1, \ldots, n$, if and only if f is polynomial of order k. In fact, in (18), the k-additive case is obtained simply by taking $m_\mu(k+1) = \cdots = m_\mu(n) = 0$, and the binomial coefficient of the Möbius value $m_\mu(k)$ corresponds to $t(t-1)\ldots(t-k+1)/k!$, which is polynomial of order k in the coalition cardinality t.

We now consider OWA functions $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ and we recall the binomial decomposition of OWA functions due to Calvo and De Baets [5], with the addition of a uniqueness result, see also Bortot and Marques Pereira [4].

We begin by introducing the convenient notation

$$\alpha_j = \binom{n}{j} m_\mu(j) \qquad j = 1, \ldots, n\,. \tag{22}$$

In this notation the upper boundary condition (19) reduces to

$$\sum_{j=1}^{n} \alpha_j = 1 \tag{23}$$

and the monotonicity conditions (20) take the form

$$\sum_{j=1}^{i} \frac{\binom{i-1}{j-1}}{\binom{n}{j}}\, \alpha_j \geq 0 \qquad i = 1, \ldots, n\,. \tag{24}$$

DEFINITION 11 *The* binomial OWA functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, *with* $j = 1, \ldots, n$, *are defined as*

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}}\, x_{(i)} \qquad j = 1, \ldots, n \tag{25}$$

where the binomial weights w_{ji}, $i, j = 1, \ldots, n$ *are null when* $i + j > n + 1$ *according to the usual convention that* $\binom{p}{q} = 0$ *when* $p < q$, *with* $p, q = 0, 1, \ldots$

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Except for $C_1(\boldsymbol{x}) = \bar{x}$, the binomial OWA functions C_j, $j = 2, \ldots, n$ have an increasing number of null weights, in correspondence with $x_{(n-j+2)}, \ldots, x_{(n)}$. The weight normalization of the binomial OWA functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$, is due to the column-sum property of binomial coefficients,

$$\sum_{i=1}^{n} \binom{n-i}{j-1} = \sum_{i=0}^{n-1} \binom{i}{j-1} = \binom{n}{j} \qquad j = 1, \ldots, n\,. \tag{26}$$

Proposition 2 [Binomial decomposition] *Any OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \cdots + \alpha_n C_n(\boldsymbol{x}) \tag{27}$$

where the coefficients α_j, $j = 1, \ldots, n$ are subject to conditions (23) and (24). In the binomial decomposition the k-additive case, with $k = 1, \ldots, n$, is obtained simply by taking $\alpha_{k+1} = \cdots = \alpha_n = 0$.

Example 1 Consider the case $n = 3$. Using the boundary condition $\alpha_1 + \alpha_2 + \alpha_3 = 1$ as in (23), we can write the monotonicity conditions (24) only in terms of α_2, α_3 as follows,

$$\begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{28}$$

and the corresponding feasible region is illustrated in Fig. 1.

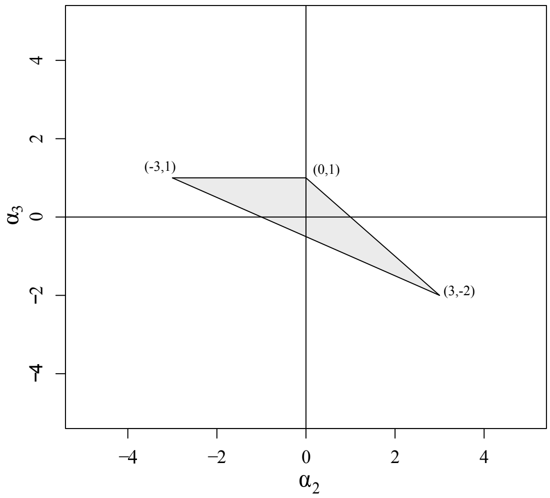

Figure 1: Feasible region associated with conditions (28).

The origin in Fig. 1 is associated with $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 0$, which corresponds in the binomial decomposition (27) to $A(\boldsymbol{x}) = C_1(\boldsymbol{x}) = \bar{x}$.

Given that the coefficients α_j, $j = 1, \ldots, n$ are constrained by the boundary and monotonicity conditions (23) and (24), the binomial decomposition (27) does not express a free (vector space) linear combination of the binomial OWA functions C_j, $j = 1, \ldots, n$, or even a simple convex combination of the binomial OWA functions, as the boundary condition $\alpha_1 + \cdots + \alpha_n = 1$ might suggest. In fact, the monotonicity conditions allow for negative α values, as illustrated by the feasible region in Fig. 1.

The following interesting result concerning the cumulative properties of binomial weights is due to Calvo and De Baets [5], see also Bortot and Marques Pereira [4].

PROPOSITION 3 *The binomial weights $w_{ji} \in [0, 1]$, with $i, j = 1, \ldots, n$, have the following cumulative property,*

$$\sum_{k=1}^{i} w_{j-1,k} \leq \sum_{k=1}^{i} w_{jk} \qquad i = 1, \ldots, n \quad j = 2, \ldots, n\,. \tag{29}$$

Given that binomial weights have the cumulative property (29), Proposition 1 implies that the binomial OWA functions C_j, $j = 1, \ldots, n$ satisfy the relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \cdots \geq C_n(\boldsymbol{x}) \geq 0$, for any $\boldsymbol{x} \in \mathbb{D}^n$.

PROPOSITION 4 *The orness of the binomial OWA functions C_j, with $j = 1, \ldots, n$, is given by*

$$\text{Orness}\,(C_j) = \frac{n-j}{(n-1)(j+1)} \qquad j = 1, \ldots, n\,. \tag{30}$$

Proof: From the definition of C_j (25) and the general definition of orness (3), we have

$$\text{Orness}\,(C_j) = C_j(\boldsymbol{x}_0) = \sum_{i=1}^{n} \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \, \frac{i-1}{n-1} \qquad j = 1, \ldots, n\,. \tag{31}$$

Using the formula

$$\sum_{i=1}^{n} \binom{n-i}{j-1} (i-1) = \binom{n}{j+1} \qquad j = 1, \ldots, n \tag{32}$$

and substituting in (31), we obtain

$$\text{Orness}\,(C_j) = \frac{1}{n-1} \frac{\binom{n}{j+1}}{\binom{n}{j}} = \frac{n-j}{(n-1)(j+1)} \qquad j = 1, \ldots, n\,. \tag{33}$$

Notice that the orness of the binomial OWA function is strictly decreasing with respect to $j = 1, \ldots, n$, from $\text{Orness}\,(C_1) = 1/2$ to $\text{Orness}\,(C_n) = 0$. $\square$

PROPOSITION 5 *In relation with the binomial decomposition, the orness of an OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is given by*

$$\text{Orness}\,(A) = \sum_{j=1}^{n} \frac{(n-j)}{(n-1)(j+1)} \, \alpha_j\,. \tag{34}$$

Proof: Considering the binomial decomposition as in Proposition 2,

$$\text{Orness}\,(A) = A(\boldsymbol{x}_0) = \sum_{j=1}^{n} \alpha_j\, C_j(\boldsymbol{x}_0) = \sum_{j=1}^{n} \frac{(n-j)}{(n-1)(j+1)}\,\alpha_j \tag{35}$$

where we have used that $C_j(\boldsymbol{x}_0) = \text{Orness}\,(C_j)$ as in Proposition 4. □

Summarizing, the binomial decomposition (27) holds for any OWA function A in terms of the binomial OWA functions C_j, $j = 1,\ldots,n$ and the corresponding coefficients α_j, $j = 1,\ldots,n$ subject to conditions (23) and (24).

Consider the binomial OWA functions C_j with $j = 1,\ldots,n$. The binomial weights w_{ji}, $i,j = 1,\ldots,n$ as in (25) have regularity properties which have interesting implications at the level of the functions C_j, $j = 1,\ldots,n$, see Bortot and Marques Pereira [4].

PROPOSITION 6 *The binomial weights $w_{ji} \in [0,1]$, with $i,j = 1,\ldots,n$, have the following properties,*

i.	*for* $j=1$	$1/n = w_{11} = w_{12} = \cdots = w_{1,n-1} = w_{1n}$
ii.	*for* $j=2$	$2/n = w_{21} > w_{22} > \cdots > w_{2,n-1} > w_{2n} = 0$
iii.	*for* $j=3,\ldots,n$	$j/n = w_{j1} > w_{j2} > \cdots > w_{j,n-j+2} = \cdots = w_{jn} = 0\,.$

Given that the binomial weights are non increasing, $w_{j1} \geq w_{j2} \geq \cdots \geq w_{jn}$ for $j = 1,\ldots,n$, the binomial OWA functions C_j, $j = 1,\ldots,n$ are Schur-concave, with strict Schur-concavity applying only to C_2. For this reason, the C_j, $j = 1,\ldots,n$ are also called *binomial welfare functions*, as particular instances of generalized Gini welfare functions, see Bortot and Marques Pereira [4].

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $C_2(\boldsymbol{x})$ has $n-1$ positive linearly decreasing weights and one null last weight. In terms of the classical Gini welfare function we have that

$$A^c(\boldsymbol{x}) = \frac{1}{n}\,C_1(\boldsymbol{x}) + \frac{n-1}{n}\,C_2(\boldsymbol{x}) \qquad\qquad G^c(\boldsymbol{x}) = \frac{n-1}{n}\,C_1(\boldsymbol{x}) - \frac{n-1}{n}\,C_2(\boldsymbol{x})\,. \tag{36}$$

The remaining $C_j(\boldsymbol{x})$, $j = 3,\ldots,n$, have $n-j+1$ positive non-linear decreasing weights and $j-1$ null last weights..

In dimensions $n = 2,3,4,5,6$ the weights $w_{ij} \in [0,1]$, $i,j = 1,\ldots,n$ of the binomial welfare functions C_j, $j = 1,\ldots,n$ are as follows,

$n=2$	$C_1:\ (\frac{1}{2},\frac{1}{2})$	$n=3$	$C_1:\ (\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$n=4$	$C_1:\ (\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4})$
	$C_2:\ (1,0)$		$C_2:\ (\frac{2}{3},\frac{1}{3},0)$		$C_2:\ (\frac{3}{6},\frac{2}{6},\frac{1}{6},0)$
			$C_3:\ (1,0,0)$		$C_3:\ (\frac{3}{4},\frac{1}{4},0,0)$
					$C_4:\ (1,,0,0)$

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$$
\begin{array}{lll}
n=5 & C_1: & (\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}) \\
 & C_2: & (\frac{4}{10},\frac{3}{10},\frac{2}{10},\frac{1}{10},0) \\
 & C_3: & (\frac{6}{10},\frac{3}{10},\frac{1}{10},0,0) \\
 & C_4: & (\frac{4}{5},\frac{1}{5},0,0,0) \\
 & C_5: & (1,0,0,0,0)
\end{array}
\qquad
\begin{array}{lll}
n=6 & C_1: & (\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}) \\
 & C_2: & (\frac{5}{15},\frac{4}{15},\frac{3}{15},\frac{2}{15},\frac{1}{15},0) \\
 & C_3: & (\frac{10}{20},\frac{6}{20},\frac{3}{20},\frac{1}{20},0,0) \\
 & C_4: & (\frac{10}{15},\frac{4}{15},\frac{1}{15},0,0,0) \\
 & C_5: & (\frac{5}{6},\frac{1}{6},0,0,0,0) \\
 & C_6: & (1,0,0,0,0,0)
\end{array}
$$

In the welfare context, the binomial welfare functions C_j, $j = 1, \ldots, n$ have null weights associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest individuals in the population, see Bortot and Marques Pereira [4].

4. THE BINOMIAL DECOMPOSITION: 2-ADDITIVE AND 3-ADDITIVE CASES

In this section we use the boundary condition (23) to write the binomial decomposition in Proposition 2 only in terms of $\alpha_2, \ldots, \alpha_n$, plus the corresponding OWA welfare functions $C_j(\boldsymbol{x})$, with $j = 2, \ldots, n$.

Proposition 7 *Any OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2 - \cdots - \alpha_n)\, \bar{x} + \alpha_2 C_2(\boldsymbol{x}) + \cdots + \alpha_n C_n(\boldsymbol{x}) \tag{37}$$

where the coefficients α_j, $j = 2, \ldots, n$ are subject to the boundary and monotonicity (BM) conditions

$$\sum_{j=2}^{n} \left[1 - n \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \right] \alpha_j \leq 1 \qquad i = 1, \ldots, n\,. \tag{38}$$

Proof: The expression of the binomial decomposition (37) is obtained directly from (27) in Proposition 2 by substituting for $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \cdots - \alpha_n$, as in the boundary condition (23).

Consider now the monotonicity conditions (24). Writing for $\alpha_1 = 1 - \alpha_2 - \alpha_3 - \cdots - \alpha_n$ we obtain

$$\frac{1}{n} + \left[\frac{\binom{i-1}{1}}{\binom{n}{2}} - \frac{1}{n} \right] \alpha_2 + \left[\frac{\binom{i-1}{2}}{\binom{n}{3}} - \frac{1}{n} \right] \alpha_3 + \cdots + \left[\frac{\binom{i-1}{i-1}}{\binom{n}{i}} - \frac{1}{n} \right] \alpha_i$$

$$- \frac{1}{n}(\alpha_{i+1} + \cdots + \alpha_n) \geq 0 \qquad i = 1, \ldots, n \tag{39}$$

which correspond to the following n combined boundary and monotonicity (BM) conditions in terms of the $n-1$ coefficients α_j, $j = 2, \ldots, n$,

$$\sum_{j=2}^{n} \left[1 - n \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \right] \alpha_j \leq 1 \qquad i = 1, \ldots, n\,. \tag{40}$$

The first and the last of these BM conditions are always of the form $\alpha_2 + \alpha_3 + \cdots + \alpha_n \leq 1$ and $\alpha_2 + 2\,\alpha_3 + \cdots + (n-1)\,\alpha_n \geq -1$, respectively. $\square$

In the binomial decomposition (37) the level of k-additivity of the OWA function A is controlled by the coefficients α_2, ..., α_n subject to the conditions (38). As k-additivity increases, the binomial decomposition of A includes an increasing number of binomial OWA functions.

4.1. The 2-additive case

We now examine the binomial decomposition of OWA functions (37) in the 2-additive case, focusing on the particular form of the BM conditions (38).

In the 2-additive case, with $n \geq 2$, the BM conditions (38) take the form

$$\left[1 - \frac{n\binom{i-1}{1}}{\binom{n}{2}}\right] \alpha_2 \leq 1 \qquad\qquad i = 1, \ldots, n\,. \tag{41}$$

These conditions can be written as $(n+1-2i)\,\alpha_2 \leq n-1$ for $i = 1, \ldots, n$, which reduce to

$$-1 \leq \alpha_2 \leq 1 \tag{42}$$

corresponding to the first and last of the n conditions (41), the others been dominated by these two. Notice that in the 2-additive case the BM conditions are independent of n.

Example 2 Consider the 2-additive case for $n = 3, 4, 5, 6$. We have the following BM conditions (41) in terms of the coefficient α_2,

$$n = 3 \begin{cases} \alpha_2 \leq 1 \\ 0 \leq 2 \\ \alpha_2 \geq -1 \end{cases} \qquad n = 4 \begin{cases} \alpha_2 \leq 1 \\ \alpha_2 \leq 3 \\ \alpha_2 \geq -3 \\ \alpha_2 \geq -1 \end{cases} \tag{43}$$

$$n = 5 \begin{cases} \alpha_2 \leq 1 \\ \alpha_2 \leq 2 \\ 0 \leq 2 \\ \alpha_2 \geq -2 \\ \alpha_2 \geq -1 \end{cases} \qquad n = 6 \begin{cases} \alpha_2 \leq 1 \\ 3\alpha_2 \leq 5 \\ \alpha_2 \leq 5 \\ \alpha_2 \geq -5 \\ 3\alpha_2 \geq -5 \\ \alpha_2 \geq -1 \end{cases} \tag{44}$$

Notice the invariance of the first and last BM conditions, the remaining being dominated by these two.

As an immediate consequence of Proposition 2 and Proposition 7, we have the following result.

PROPOSITION 8 *Any 2-additive OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2)\,\bar{x} + \alpha_2 C_2(\boldsymbol{x}) \tag{45}$$

where $C_2(\boldsymbol{x})$ is the binomial OWA function

$$C_2(\boldsymbol{x}) = \sum_{i=1}^{n} w_{2i}\, x_{(i)} = \sum_{i=1}^{n} \frac{2(n-i)}{n(n-1)}\, x_{(i)} \tag{46}$$

and the coefficient α_2 is subject to the BM conditions (41), which reduce to (42).

Given that C_2 is related with the classical Gini inequality index by means of

$$G^c(\boldsymbol{x}) = \frac{n-1}{n}\,\bar{x} - \frac{n-1}{n}\,C_2(\boldsymbol{x}) \tag{47}$$

we know from Proposition 8 that any 2-additive OWA function can be written as

$$A(\boldsymbol{x}) = \bar{x} - \frac{n}{n-1}\,\alpha_2 G^c(\boldsymbol{x}) \tag{48}$$

where α_2 is a free parameter subject to the conditions $-1 \leq \alpha_2 \leq 1$. The strict case $\alpha_2 > 0$ in (48) corresponds to the well-known Ben Porath and Gilboa's formula [3] for Weymark's generalized Gini welfare functions, with linearly decreasing (inequality averse) weight distributions, see also Grabisch [17].

In particular, with $\alpha_2 = (n-1)/n$ in (48), we obtain the classical Gini welfare function

$$A(\boldsymbol{x}) = A^c_G(\boldsymbol{x}) \qquad \alpha_2 = \frac{n-1}{n}\,. \tag{49}$$

Other interesting parametric choices for α_2 could be $\alpha_2 = (n-l)/n$ with $l = 0, 1, \ldots, n$. In the case $l = 0$ all the Choquet capacity structure lies in the non-additive Möbius values $m_\mu(2)$, the case $l = 1$ corresponds to the classical absolute Gini inequality index, and the remaining cases correspond to increasingly weak structure being associated with the values $m_\mu(2)$, towards the additive case $l = n$. In other words, the parametric choices associated with $l = 0, 1, \ldots, n$ correspond to an interpolation between $A(\boldsymbol{x}) = \bar{x} = C_1(\boldsymbol{x})$ (with $l = n$) and $A(\boldsymbol{x}) = C_2(\boldsymbol{x})$ (with $l = 0$) through the intermediate (with $l = 1$) case $A(\boldsymbol{x}) = A^c(\boldsymbol{x})$, the classical Gini welfare function.

Proposition 9 *Considering the binomial decomposition (45), the orness of the 2-additive OWA function associated with coefficient α_2 is given by*

$$\mathrm{Orness}\,(A) = \frac{1}{2} - \frac{1}{6}\,\frac{n+1}{n-1}\,\alpha_2 \tag{50}$$

where the coefficient α_2 is subject to the BM conditions (41), which reduce to (42).

Proof: It follows immediately from Proposition 5 and 2-additivity.

4.2. The 3-additive case

We now examine the binomial decomposition of OWA functions (37) in the 3-additive case, focusing on the particular form of the BM conditions (38).

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In the 3-additive case, with $n \geq 3$, the BM conditions (38) take the form

$$\left[1 - n\frac{\binom{i-1}{1}}{\binom{n}{2}}\right]\alpha_2 + \left[1 - n\frac{\binom{i-1}{2}}{\binom{n}{3}}\right]\alpha_3 \leq 1 \qquad i = 1, \ldots, n\,. \tag{51}$$

In contrast with the 2-additive case, notice that in the 3-additive case the BM conditions depend on n.

As an immediate consequence of Proposition 2 and Proposition 7, we have the following result.

PROPOSITION 10 *Any 3-additive OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ can be written uniquely as*

$$A(\boldsymbol{x}) = (1 - \alpha_2 - \alpha_3)\,\bar{x} + \alpha_2 C_2(\boldsymbol{x}) + \alpha_3 C_3(\boldsymbol{x}) \tag{52}$$

where $C_2(\boldsymbol{x})$ is as in (46), $C_3(\boldsymbol{x})$ is the binomial OWA function

$$C_3(\boldsymbol{x}) = \sum_{i=1}^{n} w_{3i}\, x_{(i)} = \sum_{i=1}^{n} \frac{3(n-i)(n-i-1)}{n(n-1)(n-2)}\, x_{(i)} \tag{53}$$

and the coefficients α_2 and α_3 are subject to the BM conditions (51).

The weight distribution of the OWA function A in (52) is thus given by

$$\begin{aligned} w_i &= (1 - \alpha_2 - \alpha_3)\, w_{1i} + \alpha_2\, w_{2i} + \alpha_3\, w_{3i} \\ &= (1 - \alpha_2 - \alpha_3)\, \frac{\binom{n-i}{0}}{\binom{n}{1}} + \alpha_2\, \frac{\binom{n-i}{1}}{\binom{n}{2}} + \alpha_3\, \frac{\binom{n-i}{2}}{\binom{n}{3}} \\ &= (1 - \alpha_2 - \alpha_3)\, \frac{1}{n} + \alpha_2\, \frac{2(n-i)}{n(n-1)} + \alpha_3\, \frac{3(n-i)(n-i-1)}{n(n-1)(n-2)} \end{aligned} \tag{54}$$

where the coefficients α_2 and α_3 are subject to the BM conditions (51).

We now illustrate the 3-additive case for dimension $n = 3, 4, 5, 6$. The feasible regions in Fig. 2 refer to the binomial decomposition of 3-additive OWA functions in Proposition 10.

Example 3 Consider the 3-additive case for $n = 3, 4, 5, 6$. We have the following BM conditions (51) in terms of the two coefficients α_2 and α_3,

$$n = 3 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \qquad n = 4 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_2 + 3\alpha_3 \leq 3 \\ \alpha_2 \geq -3 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{55}$$

$$n = 5 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ \alpha_2 + 2\alpha_3 \leq 2 \\ \alpha_3 \leq 2 \\ \alpha_2 + \alpha_3 \geq -2 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \qquad n = 6 \begin{cases} \alpha_2 + \alpha_3 \leq 1 \\ 3\alpha_2 + 5\alpha_3 \leq 5 \\ 2\alpha_2 + 7\alpha_3 \leq 10 \\ 2\alpha_2 - \alpha_3 \geq -10 \\ 3\alpha_2 + 4\alpha_3 \geq -5 \\ \alpha_2 + 2\alpha_3 \geq -1 \end{cases} \tag{56}$$

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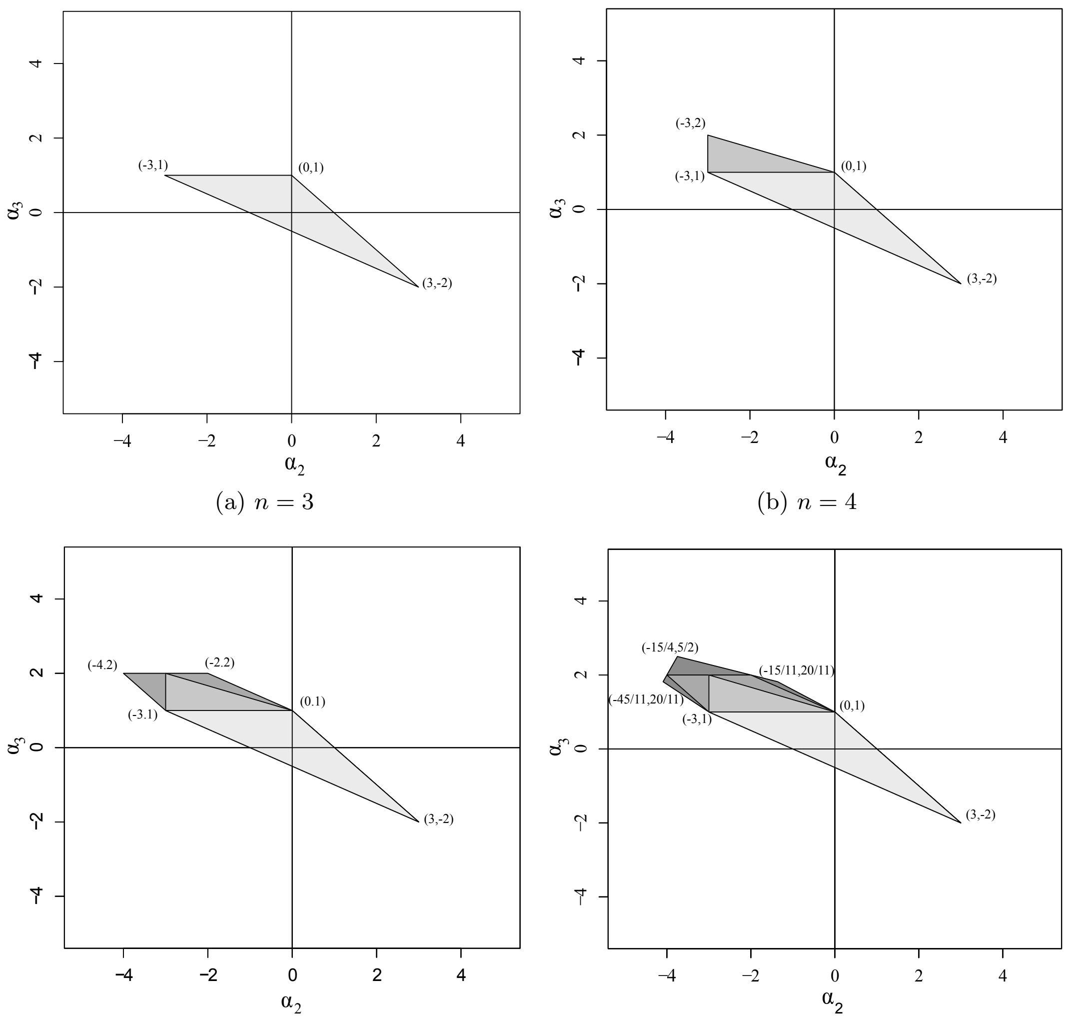
and the corresponding feasible regions are illustrated in Fig. 2. Notice the invariance of the first and last BM conditions, as explained in the final part of the proof of Proposition 10.



(a) $n = 3$ (b) $n = 4$

(c) $n = 5$ (d) $n = 6$

Figure 2: Feasible regions associated with conditions (55) and (56).

In relation to the binomial decomposition of OWA functions in the 3-additive case as illustrated in Fig. 2, we observe that the increasing dimension $n = 3, 4, 5, 6$ has the effect of extending the feasible region associated with the BM constraints. This effect emerges clearly when comparing the feasible regions in Fig. 2. Notice that the extension of the feasible region regards only its upper left portion, since the edges e_1 and e_n associated with the first and last BM conditions remain unchanged.

Proposition 11 *Considering the binomial decomposition (52), the orness of the*

3-additive OWA function A associated with coefficients α_2 and α_3 is given by

$$\text{Orness}\,(A) = \frac{1}{2} - \frac{1}{2}\,\frac{n+1}{n-1}\Big(\frac{1}{3}\,\alpha_2 + \frac{1}{2}\,\alpha_3\Big) \tag{57}$$

where the coefficients α_2 and α_3 are subject to the BM conditions (51).

Proof: It follows immediately from Proposition 5 and 3-additivity.

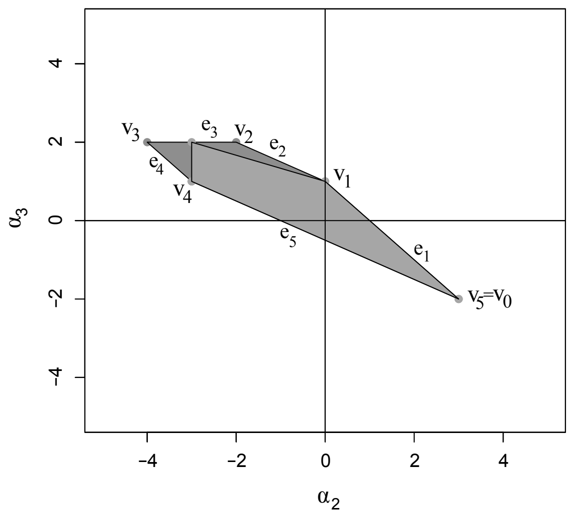


Figure 3: Vertices and edges of the feasible region in the case $n = 5$.

PROPOSITION 12 *Consider the feasible region associated with the 3-additive BM conditions (51) in dimension $n \geq 3$. The feasible region is convex and is delimited by n vertices v_i with $i = 1, \ldots, n$, and n edges e_i, with $i = 1, \ldots, n$ as illustrated in Fig. 3 in the particular case $n = 5$. The coordinates of vertices v_i, with $i = 1, \ldots, n-1$, are given by*

$$\alpha_2^{(i)} = -\frac{3(i-1)(n-1)}{n^2-1-3i(n-i)} \qquad \alpha_3^{(i)} = \frac{(n-1)(n-2)}{n^2-1-3i(n-i)} \qquad i = 1, \ldots, n-1 \tag{58}$$

and the coordinates of vertex v_n, which for convenience is also denoted v_0, are $(3, -2)$.

Proof: The feasible region is obtained as the intersection of n linear inequality constraints and thus it is convex. The coordinates of vertex $v_i = (\alpha_2^{(i)}, \alpha_3^{(i)})$, with $i = 1, \ldots, n-1$, are obtained by jointly solving the line equations associated with the BM conditions i and $i+1$ in (51), given by

$$\left[1 - n\,\frac{\binom{i-1}{1}}{\binom{n}{2}}\right]\alpha_2^{(i)} + \left[1 - n\,\frac{\binom{i-1}{2}}{\binom{n}{3}}\right]\alpha_3^{(i)} = 1 \tag{59}$$

$$\left[1 - n\,\frac{\binom{(i+1)-1}{1}}{\binom{n}{2}}\right]\alpha_2^{(i)} + \left[1 - n\,\frac{\binom{(i+1)-1}{2}}{\binom{n}{3}}\right]\alpha_3^{(i)} = 1 \tag{60}$$

which reduce to

$$\left[1 - 2\,\frac{(i-1)}{n-1}\right]\alpha_2^{(i)} + \left[1 - 3\,\frac{(i-1)(i-2)}{(n-1)(n-2)}\right]\alpha_3^{(i)} = 1 \tag{61}$$

$$\left[1 - 2\,\frac{i}{n-1}\right]\alpha_2^{(i)} + \left[1 - 3\,\frac{i(i-1)}{(n-1)(n-2)}\right]\alpha_3^{(i)} = 1\,. \tag{62}$$

Now subtracting the two equations above, we obtain $\alpha_2^{(i)} = -[3(i-1)/(n-2)]\,\alpha_3^{(i)}$ which yields (58). In particular, we obtain $v_1 = (0,1)$ and $v_{n-1} = (-3,1)$ in every dimension $n \geq 3$.

On the other hand, the coordinates of vertex $v_n = (\alpha_2^{(n)}, \alpha_3^{(n)})$ are obtained by jointly solving the line equations associated with the BM conditions $i = 1$ and $i = n$ in (51) which, in the 3-additive case, are the same in all dimension $n \geq 3$, as illustrated in (55)-(56). $\square$

Proposition 13 *Consider the feasible region associated with the 3-additive BM conditions (51) in dimension $n \geq 3$. The feasible region is strictly increasing with n, and the following holds:*

1. *The vertex v_i in dimension n, with $i = 2, \ldots, n-2$, lies on the edge e_{i+1} in dimension $n+1$, with $n \geq 4$.*

2. *The vertex v_i in dimension n, with $i = 2, \ldots, n-2$, is external to edge e_i in dimension $n-1$, with $n \geq 4$.*

Proof: We now prove each statement separately.

1. For instance, vertex v_2 in dimension $n = 4$ lies on the edge e_3 in dimension $n+1 = 5$, as illustrated in Fig. 3 . Consider the coordinates $\alpha_2^{(i)}, \alpha_3^{(i)}$ of vertex v_i in dimension n as in (58), with $i = 2, \ldots, n-2$. The fact that it lies on the edge e_{i+1} in dimension $n+1$ can be written as

$$\left[1 - (n+1)\frac{\binom{i}{1}}{\binom{n+1}{2}}\right]\alpha_2^{(i)} + \left[1 - (n+1)\frac{\binom{i}{2}}{\binom{n+1}{3}}\right]\alpha_3^{(i)} = 1 \tag{63}$$

   where we refer to BM condition $i+1$ in dimension $n+1$, see (51). This equation can be verified straightforwardly.

2. For instance, vertex v_2 in dimension $n = 5$ is external to edge e_2 in dimension $n-1 = 4$, as illustrated in Fig. 3. Consider the coordinates $\alpha_2^{(i)}, \alpha_3^{(i)}$ of vertex v_i in dimension n as in (58), with $i = 2, \ldots, n-2$. The fact that it is external to edge e_i in dimension $n-1$ can be written as

$$\left[1-(n-1)\frac{\binom{i-1}{1}}{\binom{n-1}{2}}\right]\alpha_2^{(i)}+\left[1-(n-1)\frac{\binom{i-1}{2}}{\binom{n-1}{3}}\right]\alpha_3^{(i)}>1 \tag{64}$$

where we refer to BM condition i in dimension $n-1$, see (51). This inequality reduces to

$$\frac{6(i-1)(n-i-1)}{(n-2)(n-3)(3i^2-3ni+n^2-1)}>0 \tag{65}$$

which holds since the lowest value of both $i-1$ and $n-i-1$ is 1 for $i = 2, \ldots, n-2$, and the lowest value of $(3i^2-3ni+n^2-1)$ is $(n^2-4)/4$ corresponding to $i = n/2$. Notice that here $n \geq 4$.

The fact that the feasible region is strictly increasing in n is a direct consequence of the two statements, particularly the latter. □

Proposition 14 *Consider the feasible region associated with the 3-additive BM conditions (51) in dimension $n \geq 3$. Given that the feasible region is convex and the orness is linear in the coefficients α_2 and α_3, the minimum and maximum orness values correspond to vertices of the feasible region. The vertex associated with minimum orness value is $m = v_i$ with $i = \text{floor}\,(h^-(n))$ or $i = \text{ceiling}\,(h^-(n))$, and the vertex associated with maximum orness value is $M = v_j$ with $j = \text{floor}\,(h^+(n))$ or $j = \text{ceiling}\,(h^+(n))$, where $h^-(3) = 1$ and $h^+(3) = 2$, and*

$$h^{\pm}(n)=\frac{3n\pm\sqrt{3(n^2-4)}}{6} \qquad n\geq 4\,. \tag{66}$$

In this way $1 \leq h^{\pm}(n) \leq n-1$ and therefore the vertices associated with minimum and maximum orness are among $v_1, \ldots, v_{n-1}$, with $n \geq 3$.

Proof: According to Proposition 11, the orness is linear in the coefficients α_2 and α_3,

$$\text{Orness}\,(A)=\frac{1}{2}-\frac{1}{2}\,\frac{n+1}{n-1}\Big(\frac{1}{3}\,\alpha_2+\frac{1}{2}\,\alpha_3\Big)\in[0,1] \tag{67}$$

and the orness level lines

$$2\alpha_2+3\alpha_3=6(1-2c)\,\frac{n-1}{n+1} \qquad c\in[0,1] \tag{68}$$

have slope $-2/3$ independent of n.
According to Proposition 12, the feasible region is convex and the orness of the 3-additive OWA function A associated with the vertex $v_i = (\alpha_2^{(i)}, \alpha_3^{(i)})$, with $i = 1, \ldots, n-1$, is given by

$$\text{Orness}\,(A)=\frac{1}{4}\,\frac{(6i^2+(2-4n)i+n^2-n-2)}{(3i^2-3ni+n^2-1)}=\text{orness}\,(i) \qquad i=1,\ldots,n-1 \tag{69}$$

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where the denominator $(3i^2 - 3ni + n^2 - 1)$ is always positive and its lowest value is $(n^2-4)/4$, corresponding to $i = n/2$.
Considering the continuous domain $i \in [1, n-1]$, the critical points of orness (i) correspond to the roots of

$$\text{orness}'(i) = -\frac{1}{4}\frac{(6i^2 - 6ni + n^2 + 2)}{(3i^2 - 3ni + n^2 - 1)^2}(n+1) = 0 \qquad i \in [1, n-1] \tag{70}$$

which are given by $i = h^{\pm}(n)$, where $h^{+}(3) = 2$, $h^{-}(3) = 1$ in the $n = 3$ case, and

$$h^{\pm}(n) = \frac{3n \pm \sqrt{3(n^2-4)}}{6} \qquad n \geq 4\,. \tag{71}$$

The two critical points $i = h^{\pm}(n)$ of orness (i) can be classified according to the second derivative orness$''(i)$, which is given by

$$\text{orness}''(i) = \frac{9}{2}\frac{(2i-n)(i^2 - ni + 1)}{(3i^2 - 3ni + n^2 - 1)^3}(n+1) \qquad i \in [1, n-1] \tag{72}$$

which takes the following values at the two critical points,

$$\text{orness}''(i = h^{+}(n)) = -2\sqrt{3}\,\frac{(n+1)}{(n^2-4)^{3/2}} < 0 \tag{73}$$

$$\text{orness}''(i = h^{-}(n)) = 2\sqrt{3}\,\frac{(n+1)}{(n^2-4)^{3/2}} > 0\,. \tag{74}$$

We therefore obtain that $i = h^{-}(n)$ corresponds to the minimum orness value, and $i = h^{+}(n)$ corresponds to the maximum orness value, in the continuous domain $i \in [1, n-1]$.

The graph of orness (i) on the extended continuous domain $i \in [0, n]$ is shown in Fig. 4 for dimensions $n = 3, 4, 5, 6$. As the dimensions n increases, we observe that the minimum orness value increases, whereas the maximum orness value decreases.

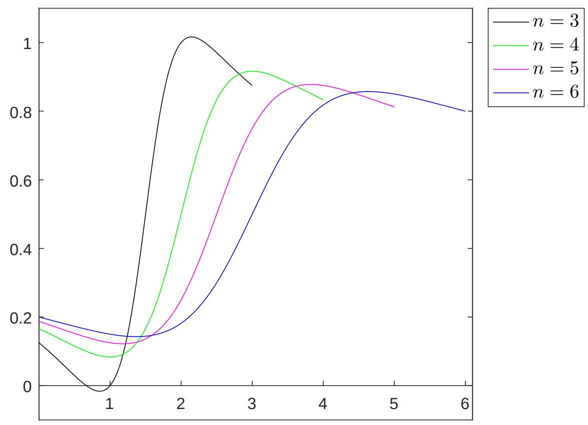

Figure 4: The orness of the 3-additive OWA function of dimensions $n = 3, 4, 5, 6$.

In the case $n = 3$ we observe a small range of the continuous variable in which the function orness (i) takes negative values, due to the sign of the numerator in (69). This small range is in any case outside the relevant continuous domain $i \in [1, n-1]$.

In relation with the feasible region associated with the 3-additive BM conditions (51) in dimension $n \geq 3$, the actual vertex associated with minimum orness value is thus $m = v_i$ with $i = \text{floor}\,(h^-(n))$ or $i = \text{ceiling}\,(h^-(n))$, and the actual vertex associated with maximum orness value is thus $M = v_j$ with $j = \text{floor}\,(h^+(n))$ or $j = \text{ceiling}\,(h^+(n))$.

The vertex $v_0 = v_n$ is in any case excluded because its orness $1/2$ is always intermediate between orness $(i = 1) = 1/2 - (n+1)/4(n-1)$ associated with v_1 and orness $(i = n-1) = 1/2 + (n+1)/4(n-1)$ associated with v_{n-1}. $\square$

4.3. The 3-additive asymptotic case

We now re-examine the binomial decomposition of OWA functions (37) in the 3-additive case, with the corresponding BM conditions (51) which explicitly depend on n.

We begin by illustrating the 3-additive case for increasing dimension $n = 3, \ldots, 12$. The feasible regions in Fig. 5 refer to the binomial decomposition of 3-additive OWA functions in Proposition 10.

It is clear in Fig. 5 that the progressive development of the border of the expanding feasible region with increasing n suggests an asymptotic curve form for the upper-left polygonal border. This is in fact the subject of the following proposition.

Proposition 15 *Consider the feasible region associated with the 3-additive BM conditions (51) in dimension $n \geq 3$. The asymptotic form of the feasible region in the infinite dimensional limit is given by the parametric curve*

$$x(t) = -\frac{3t}{3t^2 - 3t + 1} \qquad y(t) = \frac{1}{3t^2 - 3t + 1} \tag{75}$$

with parameter $t \in [0, 1]$, as illustrated in Fig. 6. The five important points indicated in Fig. 6 are

- *point P_1 is associated with the maximum value of $x(t) + y(t)$, for $t = 0$,*
- *point P_2 is associated with the maximum value of $y(t)$, for $t = 1/2$,*
- *point P_3 is associated with the minimum value of $x(t)$, for $t = 1/\sqrt{3}$,*
- *point P_4 is associated with the minimum value of $x(t) + y(t)$, for $t = 2/3$,*
- *point P_5 is associated with $t = 1$.*

Proof: The parametric coordinate functions $x(t)$ and $y(t)$ are obtained from (58) by substituting $i = 1 + t(n-2)$ and taking the asymptotic limit $n \to \infty$. In this way the parametric curve starts with vertex v_1 associated with $t = 0$ and ends with vertex

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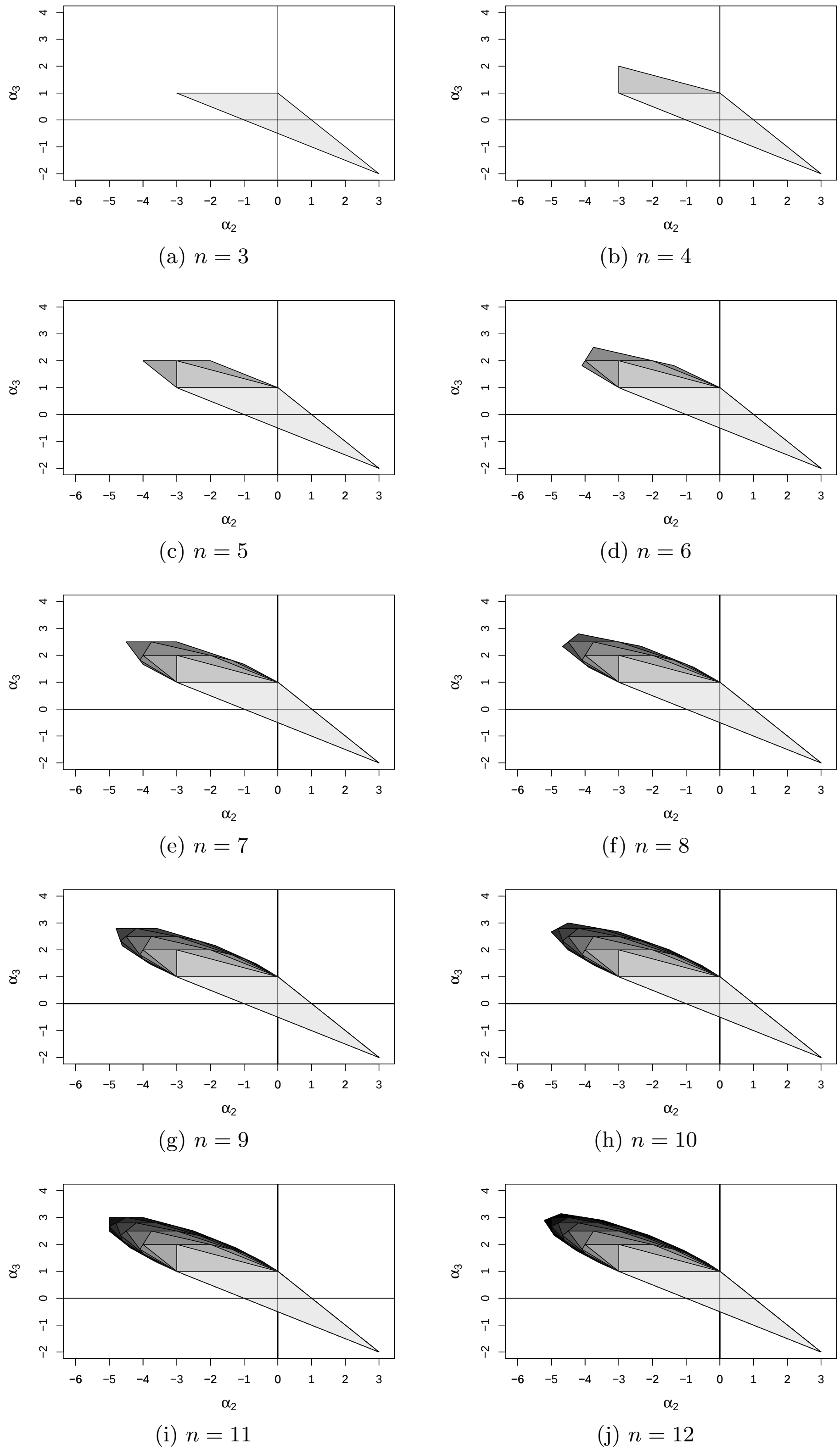

(a) $n = 3$ (b) $n = 4$ (c) $n = 5$ (d) $n = 6$ (e) $n = 7$ (f) $n = 8$ (g) $n = 9$ (h) $n = 10$ (i) $n = 11$ (j) $n = 12$

Figure 5: Feasible regions associated with conditions (51).

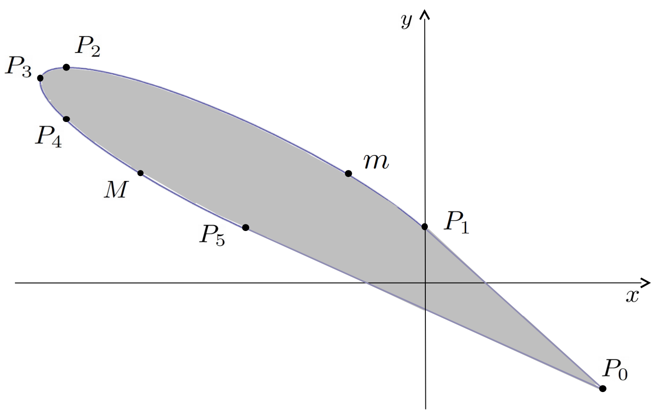

Figure 6: Asymptotic form of the feasible region.

v_{n-1} associated with $t = 1$. The points P_1, P_2, P_3, P_4 correspond to the critical points of the parametric functions $x(t)$, $y(t)$, and $x(t) + y(t)$, as indicated below

$$x'(t) = \frac{9t^2 - 3}{(3t^2 - 3t + 1)^2} \qquad y'(t) = \frac{3 - 6t}{(3t^2 - 3t + 1)^2} \tag{76}$$

$$x'(t) + y'(t) = \frac{3t(3t - 2)}{(3t^2 - 3t + 1)^2} \,. \tag{77}$$

It follows that the point the point P_1 has coordinates $(x(t = 0), y(t = 0)) = (0, 1)$, the point P_2 has coordinates $(x(t = 1/2), y(t = 1/2)) = (-6, 4)$, the point P_3 has coordinates $(x(t = 1/\sqrt{3}), y(t = 1/\sqrt{3})) = (-3 - 2\sqrt{3}, 2 + \sqrt{3})$, and the point P_4 has coordinates $(x(t = 2/3), y(t = 2/3)) = (-6, 3)$. Finally, the points P_4 has coordinates $(x(t = 1), y(t = 1)) = (-3, 1)$. $\square$

In the asymptotic infinite dimensional limit, each point (x, y) in the feasible region depicted in Fig. 6 corresponds to an OWA function $A = (1 - x - y)\, C_1 + x\, C_2 + y\, C_3$ whose weight density is expressed by

$$\omega(\gamma) = (1 - x - y) + 2x\,(1 - \gamma) + 3y\,(1 - \gamma)^2 \qquad \gamma \in [0, 1] \tag{78}$$

which is obtained from (54) as follows: the equation is multiplied by n, the left hand side nw_i is substituted by $\omega(\gamma)$ with $\gamma \in [0, 1]$, and the right hand side is transformed as

- x, y substitute α_2, α_3,
- $i = 1, \ldots, n$ is substituted by $1 + \gamma(n - 1) \in [1, n]$ with $\gamma \in [0, 1]$,
- the asymptotic limit $n \to \infty$ is taken.

Naturally, the definite integral of the weight density $\omega(\gamma)$ over $\gamma \in [0, 1]$ has value 1 for any point (x, y) in the feasible region.

In Fig. 7 we depict the weight densities of the OWA functions corresponding to the important points $P_0, P_1, \ldots, P_5$ on the border of the asymptotic feasible region as in Fig. 6.

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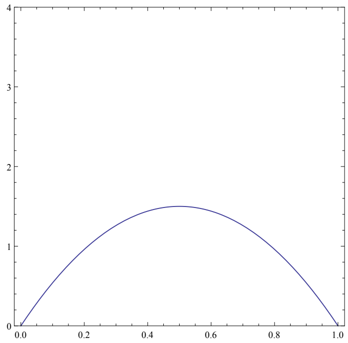
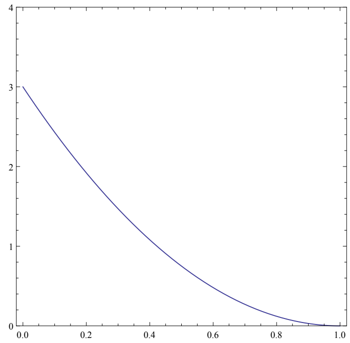
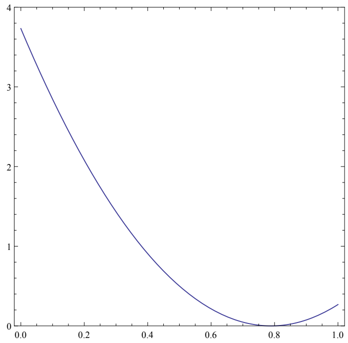


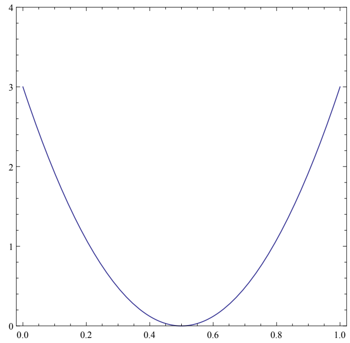
(a) Point $P_0(3, -2)$



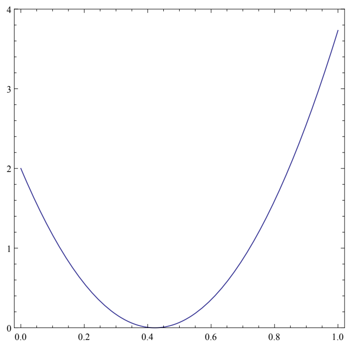
(b) Point $P_1(0, 1)$



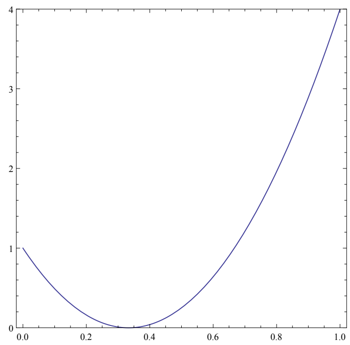
(c) Point $m(-3 + \sqrt{3}, 2)$



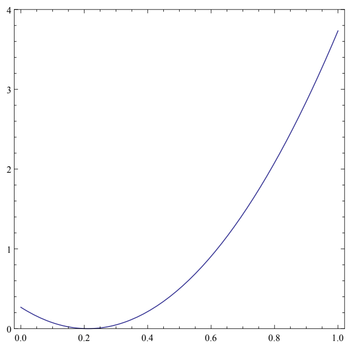
(d) Point $P_2(-6, 4)$



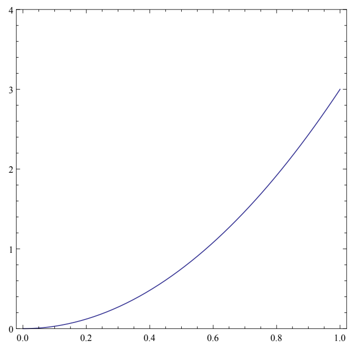
(e) Point $P_3(-3 + 2\sqrt{3}, 2 + \sqrt{3})$



(f) Point $P_4(-6, 3)$



(g) Point $M(-3 - \sqrt{3}, 2)$



(h) Point $P_5(-3, 1)$

Figure 7: Weight density of important OWA functions.

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PROPOSITION 16 *Consider the feasible region associated with the* 3*-additive BM conditions (51) in dimension* $n \geq 3$*. In the asymptotic infinite dimensional limit, the orness of the* 3*-additive OWA function associated with a point* (x, y) *in the feasible region is given by*

$$\text{Orness}\,(A) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\,x + \frac{1}{2}\,y\right). \tag{79}$$

The orness is a linear function of the coordinates x, y *and therefore the minimum and maximum orness values are associated with points in the border of the feasible region. In the asymptotic case, in particular, the points associated with minimum and maximum orness are* $m = (-3+\sqrt{3}, 2)$ *and* $M = (-3-\sqrt{3}, 2)$*, as indicated in Fig. 6. The corresponding orness values are*

$$\min = (3-\sqrt{3})/6 \approx 0.211325 \qquad \text{Max} = (3+\sqrt{3})/6 \approx 0.788675\,. \tag{80}$$

Proof: According to Proposition 11, in the asymptotic limit $n \to \infty$ the level lines of the orness function are

$$\text{Orness}\,(A(x,y)) = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\,x + \frac{1}{2}\,y\right) = c \qquad c \in [0,1] \tag{81}$$

$$2x + 3y = 6(1-2c) \qquad c \in [0,1] \tag{82}$$

with slope $-2/3$ independent of n, as illustrated in Fig. 8. The points m and M along the border of the feasible region corresponding to the minimum and maximum orness values are those for which the slope of the parametric curve $(x(t), y(t))$ coincides with the slope of the level curve,

$$\frac{y'(t)}{x'(t)} = -\frac{1-2t}{1-3t^2} = -\frac{2}{3} \tag{83}$$

assuming $t \neq 1/\sqrt{3}$, the parameter value associated with the point P_3. The two solutions of (83), associated with the points m and M, are $t = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{3}}\right)$. □

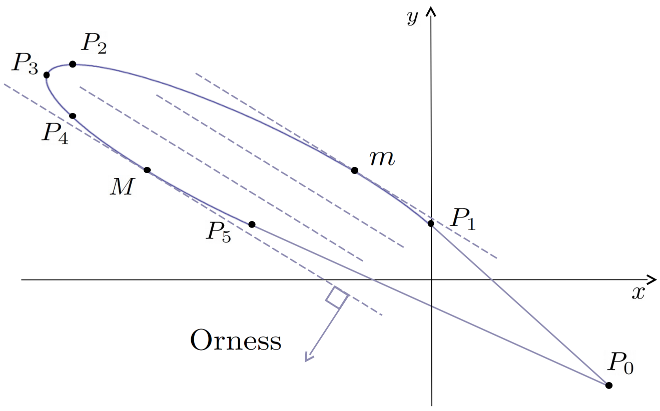

Figure 8: Asymptotic form of the feasible region, the orness level lines.

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Moreover, along the parametric curve $(x(t), y(t))$ with $t \in [0,1]$, from point P_1 to point P_5, the orness as in (79) varies as indicated in Fig. 9. Notice that the values of the parameter t associated with the points m and M correspond precisely with the associated minimum and maximum orness values. Point P_2 is associated with parameter $t = 1/2$ and orness value $1/2$.

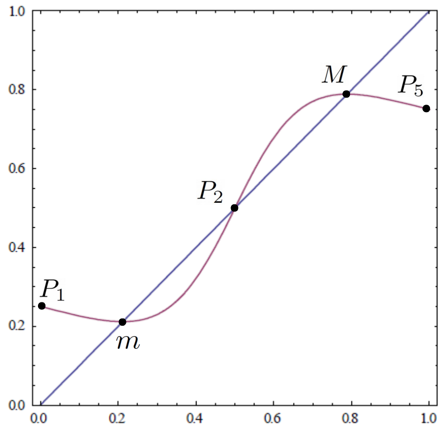

Figure 9: The orness values along the upper left border of the asymptotic feasible region.

The orness (79) of the 3-additive OWA functions associated with the important points indicated in Fig. 8 is as follows: $P_0 = (3, -2)$ with orness $1/2$, $P_1 = (0, 1)$ with orness $1/4$, $m = (-3+\sqrt{3}, 2)$ with orness $1/2-\sqrt{3}/6$, $P_2 = (-6, 4)$ with orness $1/2$, $P_3 = (-3-2\sqrt{3}, 2+\sqrt{3})$ with orness $1/2+\sqrt{3}/12$, $P_4 = (-6, 3)$ with orness $3/4$, $M = (-3-\sqrt{3}, 2)$ with orness $1/2+\sqrt{3}/6$, and $P_5 = (-3, 1)$ with orness $3/4$.

6. CONCLUSIONS

In the context of the binomial decomposition of OWA functions in terms of the binomial OWA functions C_j, with $j = 1, \ldots, n$, we have investigated the constraints associated with the 2-additive and 3-additive cases in n dimensions. We have described the corresponding forms of the feasible regions, in terms of the coefficient α_2 in the 2-additive case, and in terms of the coefficients α_2, α_3 in the 3-additive case. In the 2-additive case we have shown that the feasible region does not depend on the dimension n.

In the 3-additive case, on the other hand, the feasible region expands with the increasing dimension n. The orness of the OWA functions within the feasible region is linear in the coefficients α_2 and α_3, and the vertices associated with maximum and minimum orness have been identified.

Interestingly, whereas the triangular lower-right border of the feasible region remains unchanged, the polygonal upper-left border of the feasible region in the asymptotic infinite dimensional limit tends to a smooth curve with parameter $t \in [0,1]$. We have derived the parametric equation of this asymptotic curve and we have examined in detail the orness values along the curve, establishing the maximum and minimum orness points along the asymptotic curve.

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