# STRENGTH AND SLICE RANK OF FORMS ARE GENERICALLY EQUAL 

EDOARDO BALLICO, ARTHUR BIK, ALESSANDRO ONETO, AND EMANUELE VENTURA


#### Abstract

We prove that strength and slice rank of homogeneous polynomials of degree $d \geq 5$ over an algebraically closed field of characteristic zero coincide generically. To show this, we establish a conjecture of Catalisano, Geramita, Gimigliano, Harbourne, Migliore, Nagel and Shin concerning dimensions of secant varieties of the varieties of reducible homogeneous polynomials. These statements were already known in degrees $2 \leq d \leq 7$ and $d=9$.


## 1. Introduction

Ananyan and Hochster [AH20a] introduced the notion of strength of a polynomial to solve a famous conjecture by Stillman on the existence of a uniform bound, independent on the number of variables, for the projective dimension of a homogeneous ideal of a polynomial ring. Recently, polynomial strength and related questions have been intensively investigated [AH20b, BB+21, BV20, BDE19, DES17, ESS20, KZ18].

Let $\mathbb{k}$ be an algebraically closed field of characteristic zero, let $n \geq 1$ be an integer and let

$$
\mathcal{S}=\bigoplus_{d \geq 0} \mathcal{S}_{d}:=\mathbb{k}\left[x_{0}, \ldots, x_{n}\right]
$$

be the standard graded polynomial ring in $n+1$ variables over $\mathbb{k}$. So the elements of $\mathcal{S}_{d}$ are homogeneous polynomials, also called forms, of degree $d$. Fix an integer $d \geq 2$ and let $f \in \mathcal{S}_{d}$ be a degree- $d$ form.

Definition 1.1. The strength of $f$ is the minimal integer $r \geq 0$ for which there exists a decomposition

$$
f=g_{1} \cdot h_{1}+\ldots+g_{r} \cdot h_{r}
$$

where $g_{1}, h_{1}, \ldots, g_{r}, h_{r}$ are forms of positive degree. We denote it by $\operatorname{str}(f)$.

Computing the strength of a given polynomial is a very difficult task. Hence, a natural problem is to determine the strength of a general homogeneous polynomial. In [BO20], A.B. and A.O. noticed that a conjectural answer to this problem was implicitly given in [CG+19, Remark 7.7] where the authors study dimensions of secant varieties of the varieties of reducible forms. In particular, it was conjectured that the strength of a general form coincides with its slice rank; see [BO20, Conjecture 1.1]. Recall that the value of the slice rank of a general form is classically known; see Remark 1.5.

Definition 1.2. The slice rank of $f$ is the minimal integer $r \geq 0$ for which there exists a decomposition

$$
f=\ell_{1} \cdot h_{1}+\ldots+\ell_{r} \cdot h_{r}
$$

where $\ell_{1}, \ldots, \ell_{r}$ are linear forms and $h_{1}, \ldots, h_{r}$ are forms of degree $d-1$. We denote it by sl. rk $(f)$.

Conjecture 1.3 ([BO20, Conjecture 1.1]). The strength and the slice rank of a general form in $\mathcal{S}_{d}$ are equal.

So far, this conjecture has been established in the following cases: when the degree $d$ is larger than $\frac{3}{2} n+\frac{1}{2}$ [Sza96], when twice the general slice rank is at most $n+2$ [CCG08] and for $d \leq 7$ and $d=9$ [BO20].

The aim of this paper is to establish Conjecture 1.3, thereby determining the strength of a general form, by proving the stronger conjecture from $[\mathrm{CG}+19$, Remark 7.7$]$ which we also state below.

Geometric formulation of the problem. For an integer $1 \leq j \leq d / 2$, we consider the variety of forms with a degree-j factor $X_{j}:=\left\{[g \cdot h] \mid g \in \mathbb{P} \mathcal{S}_{j}, h \in \mathbb{P} \mathcal{S}_{d-j}\right\} \subseteq \mathbb{P} \mathcal{S}_{d}$. The union of these varieties is the variety of reducible forms $X_{\text {red }}:=\bigcup_{j=1}^{\lfloor d / 2\rfloor} X_{j}$. For an integer $r \geq 1$, the $r$ th secant variety of $X_{\mathrm{red}}$ is the Zariski-closure

$$
\sigma_{r}\left(X_{\mathrm{red}}\right):=\overline{\left\{[f] \in \mathbb{P} \mathcal{S}_{d} \mid f=f_{1}+\ldots+f_{r},\left[f_{1}\right], \ldots,\left[f_{r}\right] \in X_{\mathrm{red}}\right\}}
$$

of the union of all linear spaces spanned by $r$ points on $X_{\text {red }}$. Since $X_{\text {red }}$ is reducible, we can describe its $r$ th secant variety as

$$
\sigma_{r}\left(X_{\mathrm{red}}\right)=\bigcup_{1 \leq a_{1}, \ldots, a_{r} \leq\lfloor d / 2\rfloor} J_{a_{1}, \ldots, a_{r}}
$$

where

$$
J_{a_{1}, \ldots, a_{r}}:=J\left(X_{a_{1}}, \ldots, X_{a_{r}}\right)=\overline{\left\{[f] \in \mathbb{P} \mathcal{S}_{d} \mid f=f_{1}+\ldots+f_{r},\left[f_{1}\right] \in X_{a_{1}}, \ldots,\left[f_{r}\right] \in X_{a_{r}}\right\}}
$$

is the join of the varieties $X_{a_{1}}, \ldots, X_{a_{r}}$. Now, the general slice rank and strength are

$$
\text { sl. } \mathrm{rk}_{d, n}^{\circ}:=\min \left\{r \in \mathbb{Z}_{\geq 0} \mid \sigma_{r}\left(X_{1}\right)=\mathbb{P} \mathcal{S}_{d}\right\} \quad \text { and } \quad \operatorname{str}_{d, n}^{\circ}:=\min \left\{r \in \mathbb{Z}_{\geq 0} \mid \sigma_{r}\left(X_{\mathrm{red}}\right)=\mathbb{P} \mathcal{S}_{d}\right\}
$$

So Conjecture 1.3 is implied by the following stronger conjecture.
Conjecture $1.4\left(\left[C G+19\right.\right.$, Remark 7.7]). For each integer $r \geq 1$, we have $\operatorname{dim} \sigma_{r}\left(X_{\text {red }}\right)=\operatorname{dim} \sigma_{r}\left(X_{1}\right)$.

Remark 1.5. Recall that the value of the general slice rank is classically known as it equals the minimal codimension of a linear space contained in a general hypersurface. If $d \geq 3$, then we have

$$
\text { sl. } \mathrm{rk}_{d, n}^{\circ}:=\min \left\{r \in \mathbb{Z}_{\geq 0} \left\lvert\, r(n+1-r) \geq\binom{ n-r+d}{d}\right.\right\}
$$

and

$$
\operatorname{codim}_{\mathbb{P} \mathcal{S}_{d}} \sigma_{r}\left(X_{1}\right)=\binom{n-r+d}{d}-r(n+1-r)
$$

for all integers $1 \leq r<\mathrm{sl}^{2} . \mathrm{rk}_{d, n}^{\circ}$ by [Har92, Theorem 12.8]. Note that $\mathrm{sl} . \mathrm{rk}_{d, n}^{\circ} \leq n$. So we can (and often will) relax the assumption $r<\mathrm{sl} . \mathrm{rk}_{d, n}^{\circ}$ to $r<n$.

The classical approach to computing dimensions of secant and join varieties is via Terracini's Lemma [Ter11] which asserts that, if $Y_{1}, \ldots, Y_{r} \subseteq \mathbb{P}^{N}$ are projective varieties, $q_{1} \in Y_{1}, \ldots, q_{r} \in Y_{r}$ are general points and $p \in\left\langle q_{1}, \ldots, q_{r}\right\rangle$ is general, then

$$
T_{p} \sigma_{r} J\left(Y_{1}, \ldots, Y_{r}\right)=\left\langle T_{q_{1}} Y_{1}, \ldots, T_{q_{r}} Y_{r}\right\rangle ;
$$

see e.g. $[\mathrm{BC}+18$, Lemma 1] for a recent presentation. By direct computation, it is easy to observe that the tangent space to $X_{a}$ at a general point $[g \cdot h]$, with $\operatorname{deg}(g)=a$ and $\operatorname{deg}(h)=d-a$, is given by $\mathbb{P}(g, h)_{d}$ where $(g, h)_{d}:=(g, h) \cap \mathcal{S}_{d}$ is the degree- $d$ homogeneous part of the ideal generated by $g$ and $h$. Therefore

$$
\begin{equation*}
\operatorname{dim} J_{a_{1}, \ldots, a_{r}}=\operatorname{dim}\left(g_{1}, h_{1}, \ldots, g_{r}, h_{r}\right)_{d}-1 \tag{1}
\end{equation*}
$$

where $g_{i}, h_{i}$ are general forms with $\operatorname{deg}\left(g_{i}\right)=a_{i}$ and $\operatorname{deg}\left(h_{i}\right)=d-a_{i}$. The codimensions of the homogeneous parts of a homogeneous ideal are encoded in its Hilbert function, whose generating power series is called the Hilbert series. These are among the most studied algebraic invariants of a homogeneous ideal. The Hilbert series of an ideal generated by general forms is prescribed by Fröberg's famous conjecture; see [Frö85]. In [CCG08, Theorem 5.1], the authors used the known cases of Fröberg's conjecture to deduce the integers $d, n, r, a_{1}, \ldots, a_{r}$ with $2 r \leq n+2$ for which $J_{a_{1}, \ldots, a_{r}}=\mathbb{P} \mathcal{S}_{d}$. Similarly, in [CG+19, Theorem 7.4], the authors showed that Conjecture 1.4 holds if $2 r \leq n+1$. The strength of the general form corresponds to the minimal codimension of a complete intersection inside a general hypersurface. This is the perspective of [Sza96, Corollary A], where the author shows that Conjecture 1.3 holds if $d \geq \frac{3}{2} n+\frac{1}{2}$.
In [BO20], A.B. and A.O. proved the following results.
Theorem 1.6. Let $d \in\{3,4,5,6,7,9\}$ and $n, r \geq 1$ be integers such that $r<\mathrm{sl}^{2} \mathrm{rk}_{d, n}^{\circ}$. Then Conjecture 1.4 holds. Furthermore, unless $(d, n, r)=(4,3,2)$, the subvariety $\sigma_{r}\left(X_{1}\right)$ is the unique component of $\sigma_{r}\left(X_{\mathrm{red}}\right)$ of maximal dimension. If $(d, n, r)=(4,3,2)$, the codimensions of $\sigma_{r}\left(X_{1}\right), J\left(X_{1}, X_{2}\right)$ and $\sigma_{r}\left(X_{2}\right)$ each equal 1.

Corollary 1.7. When $d \leq 7$ and $d=9$, the general form of $\mathcal{S}_{d}$ has strength equal to its slice rank.

The main results of this paper are the following complementing theorem and corollary.
Theorem 1.8. Let $d \geq 5$ and $n, r \geq 1$ be integers such that $r<\mathrm{sl}^{2} \mathrm{rk}_{d, n}^{\circ}$. Then Conjecture 1.4 holds. Furthermore, the subvariety $\sigma_{r}\left(X_{1}\right)$ is the unique component of $\sigma_{r}\left(X_{\mathrm{red}}\right)$ of maximal dimension.

Corollary 1.9. The general form of $\mathcal{S}_{d}$ has strength equal to its slice rank.
Structure of the paper. In Section 2, we find a numerical upper bound for the dimension of $J_{a_{1}, \ldots, a_{r}}$, which is an equality for $a_{1}=\ldots=a_{r}=1$ and $r<\mathrm{sl}^{2} . \mathrm{rk}_{d, n}^{\circ}$. In Section 3, we study this upper bound as $a_{1}, \ldots, a_{r}$ vary and prove the main result.

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## 2. An upper bound on the dimensions

Let $d, n \geq 2, r<n$ and $a_{1}, \ldots, a_{r} \leq d / 2$ be positive integers. We consider the subset

$$
J_{a_{1}, \ldots, a_{r}}^{\circ}:=\left\{f \in \mathbb{P} \mathcal{S}_{d} \mid f=\sum_{i=1}^{r} g_{i} \cdot h_{i}, \quad g_{i} \in \mathcal{S}_{a_{i}}, h_{i} \in \mathcal{S}_{d-a_{i}}, \quad\left(g_{1}, \ldots, g_{r}\right) \text { is a complete intersection }\right\}
$$

of $J_{a_{1}, \ldots, a_{r}}$. Let $\mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$ be the set of complete intersections in $\mathbb{P}^{n}$ of codimension $r$ defined by the intersection of hypersurfaces of degrees $a_{1}, \ldots, a_{r}$.

In order to give an upper bound on the dimension of $J_{a_{1}, \ldots, a_{r}}$, we first observe that the subset $J_{a_{1}, \ldots, a_{r}}^{\circ}$ is dense and then we bound the dimension of this subset by parametrizing it via the space of complete intersections $\mathrm{CI}_{n}\left(a_{1}, \ldots, a_{n}\right)$ whose dimension can be computed explicitely.

Lemma 2.1. The subset $J_{a_{1}, \ldots, a_{r}}^{\circ}$ is dense in $J_{a_{1}, \ldots, a_{r}}$.
Proof. Let $[f] \in \mathbb{P} \mathcal{S}_{d}$ be a form which admits a strength decomposition $f=\sum_{i=1}^{r} g_{i} h_{i}$ with $\operatorname{deg}\left(g_{i}\right)=a_{i}$. It is enough to show that $f \in \overline{J_{a_{1}, \ldots, a_{r}}^{\circ}}$. Consider general forms $\left(u_{1}, \ldots, u_{r}\right) \in \mathcal{S}_{a_{1}} \times \cdots \times \mathcal{S}_{a_{r}}$. By generality, since $r \leq n$, the $u_{i}$ 's form a regular sequence. Since being a regular sequence is an open condition in the Zariski topology, there exists an $\varepsilon>0$ such that

$$
\left(s u_{1}+g_{1}, \ldots, s u_{r}+g_{r}\right) \in \mathcal{S}_{a_{1}} \times \cdots \times \mathcal{S}_{a_{r}} \quad \text { is a regular sequence for all } s \in(0, \varepsilon] \cap \mathbb{Q} .
$$

For $s \in(0, \varepsilon] \cap \mathbb{Q}$, define $f_{s}:=\sum_{i=1}^{r}\left(g_{i}+s u_{i}\right) h_{i} \in J_{a_{1}, \ldots, a_{r}}^{\circ}$. Then $\lim _{s \rightarrow 0} f_{s}=f$ and hence $f \in \overline{J_{a_{1}, \ldots, a_{r}}^{\circ}}$.
Lemma 2.2. We have $\operatorname{dim} J_{a_{1}, \ldots, a_{r}} \leq \operatorname{dim} \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)+\binom{n+d}{d}-\operatorname{coeff} d\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)-1$.
Proof. If $I=\left(g_{1}, \ldots, g_{r}\right) \subseteq \mathcal{S}$ is an ideal defined by a regular sequence of degrees $a_{1}, \ldots, a_{r}$, then

$$
\operatorname{dim}(\mathcal{S} / I)_{d}=\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)
$$

Hence

$$
\operatorname{dim}\left(g_{1}, \ldots, g_{r}\right)_{d}=\binom{n+d}{d}-\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)=: N+1
$$

From Lemma 2.1, we derive that $\operatorname{dim} J_{a_{1}, \ldots, a_{r}}^{\circ}=\operatorname{dim} J_{a_{1}, \ldots, a_{r}}$. Now, let $E$ be the projective bundle on $\mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$ whose fiber at a point $Y \in \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$ is the projective space $\mathbb{P}\left(I_{Y}\right)_{d} \cong \mathbb{P}^{N}$. Then

$$
\operatorname{dim} E=\operatorname{dim} \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)+N
$$

We consider the morphism $E \longrightarrow J_{a_{1}, \ldots, a_{r}}^{\circ}$ given by $(Y, f) \mapsto f$. This map is surjective by definition of $E$ and $J_{a_{1}, \ldots, a_{r}}^{\circ}$. Thus

$$
\operatorname{dim} J_{a_{1}, \ldots, a_{r}}=\operatorname{dim} J_{a_{1}, \ldots, a_{r}}^{\circ} \leq \operatorname{dim} E=\operatorname{dim} \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)+N
$$

which gives the desired upper bound.

Now, we compute the dimension of $\mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$.
Remark 2.3. The Hilbert polynomial $P_{a_{1}, \ldots, a_{r}}(t)$ of a complete intersection is uniquely determined by the degrees defining it since it is computed from the Koszul complex. In [Ser06, Section 4.6.1], it is shown that $\mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$ is parametrized by a Zariski-open subset of $\operatorname{Hilb}_{P_{a_{1}, \ldots, a_{r}}(t)}\left(\mathbb{P}^{n}\right)$. The latter is smooth at $[Y] \in \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$ and [Ser06, Theorem 4.3.5] yields

$$
T_{[Y]} \operatorname{Hilb}_{P_{a_{1}, \ldots, a_{r}}(t)}\left(\mathbb{P}^{n}\right)=H^{0}\left(N_{Y / \mathbb{P}^{n}}\right)
$$

So $\operatorname{dim} \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)=h^{0}\left(N_{Y / \mathbb{P}^{n}}\right)$, i.e., the dimension of the space of global sections of the normal bundle of $Y$.

Proposition 2.4. We have

$$
\operatorname{dim} \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)=\sum_{i=1}^{r} \operatorname{coeff}_{a_{i}}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)
$$

Proof. Let $Y \in \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)$ be a general point. By Remark 2.3, $\operatorname{dim} \mathrm{CI}_{n}\left(a_{1}, \ldots, a_{r}\right)=h^{0}\left(N_{Y / \mathbb{P}^{n}}\right)$. Since $Y$ is a complete intersection, its normal bundle is $N_{Y / \mathbb{P}^{n}}=\bigoplus_{i=1}^{r} \mathcal{O}_{Y}\left(a_{i}\right)$. Hence, the statement follows from the following equality:

$$
h^{0}\left(\mathcal{O}_{Y}\left(a_{i}\right)\right)=\operatorname{coeff}_{a_{i}}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)
$$

To see that this equality holds, first notice that $Y$ is projectively normal [Har77, Exercise II.8.4], because it is a smooth complete intersection, by the generality assumption. So, for all $k \geq 0$, the restriction map $H^{0}\left(\mathcal{O}_{\mathbb{P}^{n}}(k)\right) \rightarrow H^{0}\left(\mathcal{O}_{Y}(k)\right)$ is surjective. From the long exact sequence in cohomology of the short exact sequence

$$
0 \rightarrow \mathcal{I}_{Y}(k) \rightarrow \mathcal{O}_{\mathbb{P}^{n}}(k) \rightarrow \mathcal{O}_{Y}(k) \rightarrow 0
$$

one has $h^{1}\left(\mathcal{I}_{Y}(k)\right)=0$ for all $k \geq 0$. Since $\operatorname{HF}_{\mathcal{S} / I_{Y}}(d)=\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)$, where $I_{Y}$ is the homogeneous ideal of $Y$, the claimed equality follows.

Lemma 2.5. For integers $e \geq 0$ and $b_{1}, \ldots, b_{s} \geq 1$, we have the following identity:

$$
\operatorname{coeff}_{e}\left(\frac{\prod_{i=1}^{s}\left(1-t^{b_{i}}\right)}{(1-t)^{n+1}}\right)=\sum_{I \subseteq\{1, \ldots, s\}}(-1)^{\# I}\binom{n+e-\sum_{i \in I} b_{i}}{n}
$$

Here $\binom{a}{b}=0$ whenever $a<b$.
Proof. Left to the reader.

Theorem 2.6. Let $r<n$ and $a_{1}, \ldots, a_{r} \leq d / 2$ be positive integers and take $\ell_{d / 2}:=\#\left\{i \mid a_{i}=d / 2\right\}$. Then

$$
\operatorname{dim} J_{a_{1}, \ldots, a_{r}} \leq\binom{ n+d}{d}-\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)\left(1-t^{d-a_{i}}\right)}{(1-t)^{n+1}}\right)+\binom{\ell_{d / 2}}{2}-1
$$

When $d \geq 3, a_{1}=\ldots=a_{r}=1$ and $r<\mathrm{sl} . \mathrm{rk}_{d, n}^{\circ}$, equality holds.
Proof. First, we consider the case where $d \geq 3, a_{1}=\ldots=a_{r}=1$. In this case, by (1), it is enough to compute the codimension of $\left(\ell_{1}, \ldots, \ell_{r}, g_{1}, \ldots, g_{r}\right)_{d}$ which corresponds to

$$
\operatorname{dim} \mathcal{S}_{d} /\left(\ell_{1}, \ldots, \ell_{r}, g_{1}, \ldots, g_{r}\right)_{d}=\operatorname{dim} \mathcal{S}_{d}^{\prime} /\left(\overline{g_{1}}, \ldots, \overline{g_{r}}\right)_{d}
$$

where $\mathcal{S}^{\prime} \cong \mathcal{S} /\left(\ell_{1}, \ldots, \ell_{r}\right)$ is a polynomial ring in $n+1-r$ variables and $\overline{g_{i}}$ is the class of $g_{i}$ in $\mathcal{S}^{\prime}$. Since the $g_{i}$ are general of degree $d-1$, the latter dimension is obtained by [HL87, Theorem 1] which states that

$$
\operatorname{codim}_{\mathbb{P} \mathcal{S}_{d}} J_{a_{1}, \ldots, a_{r}}=\operatorname{coeff}_{d}\left(\frac{\left(1-t^{d-1}\right)^{r}}{(1-t)^{n+1-r}}\right)
$$

For the first statement, by Lemma 2.2 and Proposition 2.4, it is enough to prove that

$$
\begin{aligned}
\sum_{j=1}^{r} \operatorname{coeff}_{a_{j}}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right) & +\binom{n+d}{d}-\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)-1= \\
& \binom{n+d}{d}-\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)\left(1-t^{d-a_{i}}\right)}{(1-t)^{n+1}}\right)+\binom{\ell_{d / 2}}{2}-1
\end{aligned}
$$

or, equivalently, to prove that
(2) $\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)\left(1-t^{d-a_{i}}\right)}{(1-t)^{n+1}}\right)=\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)-\sum_{j=1}^{r} \operatorname{coeff}_{a_{j}}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\right)+\binom{\ell_{d / 2}}{2}$.

We analyze both sides of this equality. For the left hand side, we use Lemma 2.5 with $e=d, s=2 r$ and $\left(b_{i}, b_{r+i}\right)=\left(a_{i}, d-a_{i}\right)$ for $i=1, \ldots, r$. Since $a_{i} \leq d / 2$ for all $i$, the summand corresponding to subset $I \subseteq\{1, \ldots, 2 r\}$ is zero whenever the intersection $I \cap\{r+1, \ldots, 2 r\}$ has more than two elements. The remaining summands correspond to subsets $I$ such that $I \subseteq\{1, \ldots, r\}, I=I^{\prime} \cup\{r+j\}$ for $I^{\prime} \subseteq\{1, \ldots, r\}$ and $j \in\{1, \ldots, r\}$ or $I=I^{\prime} \cup\{r+j, r+k\}$ for $I^{\prime} \subseteq\{1, \ldots, r\}$ and distinct $j, k \in\{1, \ldots, r\}$. In the last case, the summand is zero unless $a_{j}=a_{k}=d / 2$ and $I^{\prime}=\emptyset$. So we get

$$
\sum_{I \subseteq\{1, \ldots, r\}}(-1)^{\# I}\binom{n+d-\sum_{i \in I} a_{i}}{n}+\sum_{j=1}^{r} \sum_{I^{\prime} \subseteq\{1, \ldots, r\}}(-1)^{\# I^{\prime}+1}\binom{n+a_{j}-\sum_{i \in I^{\prime}} a_{i}}{n}+\binom{\ell_{d / 2}}{2} .
$$

For the right hand side of (2), we use Lemma 2.5 with $s=r$ and $b_{i}=a_{i}$ for $i=1, \ldots, r$ and varying $e$. We get

$$
\sum_{I \subseteq\{1, \ldots, r\}}(-1)^{\# I}\binom{n+d-\sum_{i \in I} a_{i}}{n}-\sum_{j=1}^{r} \sum_{I \subseteq\{1, \ldots, r\}}(-1)^{\# I}\binom{n+a_{j}-\sum_{i \in I} a_{i}}{n}+\binom{\ell_{d / 2}}{2}
$$

Hence (2) holds.

## 3. Numerical computations

Fix an integer $d \geq 5$. Let $n, r \geq 1$ and $1 \leq a_{1}, \ldots, a_{r} \leq d / 2$ be integers such that $r<\operatorname{sl}$. $\mathrm{rk}_{d, n}^{\circ}$. Our goal is to prove that

$$
\operatorname{dim} J_{a_{1}, \ldots, a_{r}} \leq \operatorname{dim} \sigma_{r}\left(X_{1}\right)
$$

holds, and that we have equality if and only if $a_{1}=\ldots=a_{r}=1$. Write $\ell_{j}:=\#\left\{i \in\{1, \ldots, r\} \mid a_{i}=j\right\}$ for all $j \in \mathbb{R}$. By Theorem 2.6, it suffices to prove that, for fixed $n, r$, the value of

$$
F\left(a_{1}, \ldots, a_{r}\right):=\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)\left(1-t^{d-a_{i}}\right)}{(1-t)^{n+1}}\right)-\binom{\ell_{d / 2}}{2}=\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)\right)
$$

is minimal exactly when $a_{1}=\ldots=a_{r}=1$. We first prove that $F\left(a_{1}, \ldots, a_{r}\right)$ goes down when replacing all $a_{i}>2$ by 2 . Afterwards, we deal with the cases where $a_{1}, \ldots, a_{r} \in\{1,2\}$. Take $\vartheta:=\max \left\{a_{1}, \ldots, a_{r}\right\} \leq d / 2$.
3.1. The case $\vartheta>2$. Write $P_{k}:=1+t+\ldots+t^{k}$ for $k \geq 0$ and $P_{\infty}:=1 /(1-t)$.

Lemma 3.1. Let $s, \ell, k_{1}, \ldots, k_{s} \geq 0$ be integers. Then the coefficients of the power series

$$
P_{\infty}^{\ell+1} P_{k_{1}} \cdots P_{k_{s}}
$$

form a weakly increasing series.
Proof. We have

$$
P_{\infty}^{\ell+1}=\sum_{k=0}^{\infty}\binom{\ell+k}{k} t^{k}
$$

and so the lemma holds when $s=0$. When $f$ is a series whose coefficients increase weakly and $k \geq 0$ is an integer, then the same holds for the series $f P_{k}$. Hence the lemma holds for all $s$ using induction.

We will often apply the next lemma with $g=P_{a}$ and $h=P_{b}$, where $a \geq b \geq 0$ are integers.
Lemma 3.2. Let $f, g, h$ be series whose coefficients are all nonnegative and suppose that $\operatorname{coeff}_{k}(g) \geq \operatorname{coeff}_{k}(h)$ for all $k \geq 0$. Then $\operatorname{coeff}_{k}(f g) \geq \operatorname{coeff}_{k}(f h)$ for all $k \geq 0$.

Theorem 3.3. Assume that $a_{r}=\vartheta>2$. Then $F\left(a_{1}, \ldots, a_{r}\right)>F\left(a_{1}, \ldots, a_{r-1}, a_{r}-1\right)$.
Proof. Take

$$
f:=\frac{\prod_{i=1}^{r-1}\left(1-t^{a_{i}}\right)}{(1-t)^{n}}
$$

Then we have

$$
F\left(a_{1}, \ldots, a_{r}\right)=\operatorname{coeff}_{d}\left(\frac{\prod_{i=1}^{r}\left(1-t^{a_{i}}\right)}{(1-t)^{n+1}}\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)\right)=\operatorname{coeff}_{d}\left(f P_{\vartheta-1}\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)\right)
$$

and similarly

$$
F\left(a_{1}, \ldots, a_{r-1}, a_{r}-1\right)=\operatorname{coeff}_{d}\left(f P_{\vartheta-2}\left(\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)+t^{d-\vartheta}(1-t)\right)\right)
$$

We need to show that the difference

$$
\operatorname{coeff}_{d}\left(f P_{\vartheta-1}\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)\right)-\operatorname{coeff}_{d}\left(f P_{\vartheta-2}\left(\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)+t^{d-\vartheta}(1-t)\right)\right)
$$

is positive. This difference equals

$$
\begin{aligned}
\operatorname{coeff}_{d}\left(f\left(t^{\vartheta-1}\left(1-\sum_{i=1}^{r} t^{d-a_{i}}\right)-t^{d-\vartheta}+t^{d-1}\right)\right) & =\operatorname{coeff}_{d-\vartheta+1}(f)-\ell_{\vartheta-1}-\left(\ell_{\vartheta}-1\right) \operatorname{coeff}_{1}(f)-\operatorname{coeff}_{\vartheta}(f) \\
& =\operatorname{coeff}_{d-\vartheta+1}\left(f\left(1-t^{d-2 \vartheta+1}\right)\right)-\ell_{\vartheta-1}-\left(\ell_{\vartheta}-1\right)\left(n-\ell_{1}\right)
\end{aligned}
$$

Take

$$
g:=P_{\infty}^{n-r} P_{d-2 \vartheta} \prod_{i=1}^{r-1} P_{a_{i}-1}=\frac{\prod_{i=1}^{r-1}\left(1-t^{a_{i}}\right)}{(1-t)^{n}}\left(1-t^{d-2 \vartheta+1}\right)=f\left(1-t^{d-2 \vartheta+1}\right)
$$

By Lemma 3.1, the coefficients of $g$ are weakly increasing. So

$$
\operatorname{coeff}_{d-\vartheta+1}(g) \geq \operatorname{coeff}_{\vartheta+1}(g)
$$



$$
\begin{aligned}
\operatorname{coeff}_{\vartheta+1}(g) & \geq \operatorname{coeff}_{\vartheta+1}\left(P_{\infty}^{n-r} P_{d-2 \vartheta} P_{1}^{r-\ell_{1}-\ell_{\vartheta}} P_{\vartheta-1}^{\ell_{\vartheta}-1}\right) \\
& \geq \operatorname{coeff}_{\vartheta+1}\left(P_{\infty} P_{1}^{m-\ell_{\vartheta}-1} P_{\vartheta-1}^{\ell_{\vartheta}-1}\right) \\
& =\operatorname{coeff}_{\vartheta+1}\left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1}\left(1-t^{\vartheta}\right)^{\ell_{\vartheta}-1}\right) \\
& =\operatorname{coeff}_{\vartheta+1}\left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1}\right)-\left(\ell_{\vartheta}-1\right)(m-1) \\
& \geq \operatorname{coeff}_{4}\left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1}\right)-\left(\ell_{\vartheta}-1\right)(m-1)
\end{aligned}
$$

So it suffices to prove that

$$
\begin{equation*}
\operatorname{coeff}_{4}\left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1}\right)>\ell_{\vartheta-1}+\left(\ell_{\vartheta}-1\right)(2 m-1) \tag{3}
\end{equation*}
$$

for all $\ell_{\vartheta-1} \geq 0, \ell_{\vartheta} \geq 1$ and $m>\ell_{\vartheta-1}+\ell_{\vartheta}$. Note that

$$
\ell_{\vartheta-1}+\left(\ell_{\vartheta}-1\right)(2 m-1) \leq\left(m-\ell_{\vartheta}-1\right)+\left(\ell_{\vartheta}-1\right)(2 m-1)=2 \ell_{\vartheta}(m-1)-m
$$

We have

$$
\operatorname{coeff}_{4}\left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1}\right) \geq \operatorname{coeff}_{4}\left(P_{\infty} P_{1}^{m-2}\right)=\sum_{k=0}^{4}\binom{m-2}{k}
$$

which is strictly greater than $2(m-1)(m-1)-m \geq 2 \ell_{\vartheta}(m-1)-m$ for $m \geq 10$. This leaves the case $m \leq 9$, where we verified that (3) holds by computer. This finishes the proof.

By Theorem 3.3, it suffices to focus on the cases where $a_{1}, \ldots, a_{r} \in\{1,2\}$. In these cases, we will regard $F\left(a_{1}, \ldots, a_{r}\right)$ as a function $A_{\ell_{1}, \ell_{2}}$ (defined below) depending only on $\ell_{1}$ and $\ell_{2}$.
3.2. The case $\vartheta=2$. Recall that $d \geq 5$. We define

$$
\begin{aligned}
& A_{\ell_{1}, \ell_{2}}:=\operatorname{coeff}_{d}\left(\frac{(1-t)^{\ell_{1}}\left(1-t^{2}\right)^{\ell_{2}}}{(1-t)^{n+1}}\left(1-\ell_{1} t^{d-1}-\ell_{2} t^{d-2}\right)\right) \text { for } \ell_{1}, \ell_{2} \geq 0 \\
& B_{\ell_{1}, \ell_{2}}:=A_{\ell_{1}-1, \ell_{2}+1}-A_{\ell_{1}, \ell_{2}} \text { for } \ell_{1} \geq 1 \text { and } \ell_{2} \geq 0 \\
& C_{\ell_{1}, \ell_{2}}:=B_{\ell_{1}-1, \ell_{2}+1}-B_{\ell_{1}, \ell_{2}} \text { for } \ell_{1} \geq 2 \text { and } \ell_{2} \geq 0 \\
& D_{\ell_{1}, \ell_{2}}:=C_{\ell_{1}-1, \ell_{2}+1}-C_{\ell_{1}, \ell_{2}} \text { for } \ell_{1} \geq 3 \text { and } \ell_{2} \geq 0 \text { and } \\
& E_{\ell_{1}, \ell_{2}}:=D_{\ell_{1}-1, \ell_{2}+1}-D_{\ell_{1}, \ell_{2}} \text { for } \ell_{1} \geq 4 \text { and } \ell_{2} \geq 0
\end{aligned}
$$

The goal of this subsection is to prove the following theorem.
Theorem 3.4. We have $A_{\ell_{1}, \ell_{2}}>A_{\ell_{1}+\ell_{2}, 0}$ for all integers $\ell_{1} \geq 0$ and $\ell_{2} \geq 1$ such that $\ell_{1}+\ell_{2}<\mathrm{sl} . \mathrm{rk}_{d, n}^{\circ}$.
We write $m=n-\ell_{1}$ and we assume that $\ell_{1}+\ell_{2}<n$. So $\ell_{2}<m$. In particular, we have $m \geq 1$.
Lemma 3.5. Let $\ell_{1}, \ell_{2} \geq 0$ be integers such that $\ell_{1}+\ell_{2}<n$.
(a) We have

$$
A_{\ell_{1}, \ell_{2}}=\operatorname{coeff}_{d}\left(P_{\infty}^{m+1-\ell_{2}} P_{1}^{\ell_{2}}\right)-\ell_{2}\binom{m+2}{2}-\ell_{1}(m+1)+\ell_{2}^{2}
$$

(b) When $\ell_{1} \geq 1$, we have

$$
B_{\ell_{1}, \ell_{2}}=\operatorname{coeff}_{d-1}\left(P_{\infty}^{m+1-\ell_{2}} P_{1}^{\ell_{2}}\right)-\binom{m+2}{2}-\ell_{2} m-\ell_{1}+1
$$

(c) When $\ell_{1} \geq 2$, we have

$$
C_{\ell_{1}, \ell_{2}}=\operatorname{coeff}_{d-2}\left(P_{\infty}^{m+1-\ell_{2}} P_{1}^{\ell_{2}}\right)-2(m+1)-\ell_{2} .
$$

(d) When $\ell_{1} \geq 3$, we have

$$
D_{\ell_{1}, \ell_{2}}=\operatorname{coeff}_{d-3}\left(P_{\infty}^{m+1-\ell_{2}} P_{1}^{\ell_{2}}\right)-3
$$

(e) When $\ell_{1} \geq 4$, we have

$$
E_{\ell_{1}, \ell_{2}}=\operatorname{coeff}_{d-4}\left(P_{\infty}^{m+1-\ell_{2}} P_{1}^{\ell_{2}}\right)
$$

Proof. These calculations are straightforward.
Lemma 3.6. Let $\ell_{1} \geq 1$ and $\ell_{2} \geq 0$ be integers such that $\ell_{1}+\ell_{2}<n$.
(a) When $\ell_{1}<\mathrm{sl} . \mathrm{rk}_{d, n}^{\circ}$, we have $B_{\ell_{1}, 0}>0$.
(b) When $\ell_{1} \geq 2$, we have $C_{\ell_{1}, \ell_{2}} \geq 0$.
(c) When $\ell_{1} \geq 3$, we have $D_{\ell_{1}, \ell_{2}} \geq 0$.
(d) When $\ell_{1} \geq 4$, we have $E_{\ell_{1}, \ell_{2}} \geq 2$.

Proof. We prove the parts of the lemma in reverse order.
(d). We have $E_{\ell_{1}, \ell_{2}}=\operatorname{coeff}_{d-4}\left(P_{\infty}^{m+1-\ell_{2}} P_{1}^{\ell_{2}}\right) \geq \operatorname{coeff}_{1}\left(P_{1}^{m+1}\right)=m+1 \geq 2$.
(c). By (d), we have $D_{\ell_{1}, \ell_{2}} \geq D_{\ell_{1}+\ell_{2}, 0}$. So we may assume that $\ell_{2}=0$. Now, we have

$$
D_{\ell_{1}, 0}=\operatorname{coeff}_{d-3}\left(P_{\infty}^{m+1}\right)-3=\binom{m+d-3}{d-3}-3 \geq\binom{ 1+d-3}{d-3}-3=(d-2)-3 \geq 0
$$

(b). By (c), we have $C_{\ell_{1}, \ell_{2}} \geq C_{\ell_{1}+\ell_{2}, 0}$. So we may assume that $\ell_{2}=0$. Now, we have

$$
(m+1) \geq 2 \text { and } \frac{(m+d-2) \cdots(m+2)}{(d-2)!}-2 \geq \frac{(1+d-2) \cdots(1+2)}{(d-2)!}-2=\frac{d-1}{2}-2 \geq 0
$$

and so
$C_{\ell_{1}, 0}=\operatorname{coeff}_{d-2}\left(P_{\infty}^{m+1}\right)-2(m+1)=\binom{m+d-2}{d-2}-2(m+1)=(m+1)\left(\frac{(m+d-2) \cdots(m+2)}{(d-2)!}-2\right) \geq 0$.
(a). By (b), $B \ell_{\ell_{1}, \ell_{2}} \geq B_{\ell_{1}+\ell_{2}, 0}$. So we may assume $\ell_{2}=0$. Since $\ell_{1}<\mathrm{sl}^{\text {r }} \mathrm{rk}_{d, n}^{\circ}$, we have $\ell_{1}(m+1)<\binom{m+d}{d}$. So $d!\ell_{1}<(m+d) \cdots(m+2)$. We get

$$
\begin{aligned}
d!B_{\ell_{1}, 0} & =d!\left(\operatorname{coeff}_{d-1}\left(P_{\infty}^{m+1}\right)-\binom{m+2}{2}-\ell_{1}+1\right) \\
& =d!\left(\binom{m+d-1}{d-1}-\frac{m(m+3)}{2}\right)-d!\ell_{1} \\
& >d!\left(\binom{m+d-1}{d-1}-\frac{m(m+3)}{2}\right)-(m+d) \cdots(m+2) \\
& =d(m+d-1) \cdots(m+1)-\frac{d!}{2} m(m+3)-(m+d) \cdots(m+2) \\
& =(m+d-1) \cdots(m+2)(d(m+1)-(m+d))-\frac{d!}{2} m(m+3) \\
& =(m+d-1) \cdots(m+2)(d-1) m-\frac{d!}{2} m(m+3) \\
& =m\left((m+d-1) \cdots(m+2)(d-1)-\frac{d!}{2}(m+3)\right) .
\end{aligned}
$$

So it suffices to prove that

$$
c_{0}+c_{1} m+\ldots+c_{d-2} m^{d-2}:=(m+d-1) \cdots(m+2)(d-1)-\frac{d!}{2}(m+3) \geq 0
$$

We have

$$
\begin{aligned}
c_{1} & =(d-1) \operatorname{coeff}_{1}((m+d-1) \cdots(m+2))-\frac{d!}{2} \\
& =(d-1) \sum_{i=2}^{d-1} \frac{(d-1)!}{i}-\frac{d!}{2} \\
& =(d-1)!\left(\sum_{i=2}^{d-1} \frac{d-1}{i}-\frac{d}{2}\right) \\
& \geq(d-1)!\left(\frac{d-1}{2}+\frac{d-1}{d-1}-\frac{d}{2}\right)>0
\end{aligned}
$$

and $c_{i}>0$ for $i=2, \ldots, d-3$. Hence

$$
\begin{aligned}
c_{0}+c_{1} m+\ldots+c_{d-2} m^{d-2} & \geq c_{0}+c_{1}+\ldots+c_{d-2} \\
& =(1+d-1) \cdots(1+2)(d-1)-\frac{d!}{2}(1+3) \\
& =\frac{d!}{2}(d-1)-\frac{d!}{2} \cdot 4=\frac{d!}{2}(d-5) \geq 0 .
\end{aligned}
$$

This finishes the proof.
Theorem 3.4 now follows easily.
Proof of Theorem 3.4. By parts (a) and (b) of Lemma 3.6, we have

$$
A_{\ell_{1}, \ell_{2}}-A_{\ell_{1}+1, \ell_{2}-1}=B_{\ell_{1}+1, \ell_{2}-1} \geq B_{\ell_{1}+\ell_{2}, 0}>0 .
$$

Repeating this, we find that

$$
A_{\ell_{1}, \ell_{2}}>A_{\ell_{1}+1, \ell_{2}-1}>\cdots>A_{\ell_{1}+\ell_{2}, 0}
$$

as desired.

### 3.3. The conclusion of the proof.

Proof of Theorem 1.8. Let $d \geq 5, n, r \geq 1$ and $1 \leq a_{1}, \ldots, a_{r} \leq d / 2$ be integers such that $r<\mathrm{sl}^{2} \mathrm{rk}_{d, n}^{\circ}$. We need to show that

$$
\operatorname{dim} J_{a_{1}, \ldots, a_{r}} \leq \operatorname{dim} \sigma_{r}\left(X_{1}\right)
$$

holds, and that we have equality only for $a_{1}=\ldots=a_{r}=1$. By Theorem 2.6, it suffices to prove that

$$
F\left(a_{1}, \ldots, a_{r}\right)
$$

is minimal exactly when $a_{1}=\ldots=a_{r}=1$. By Theorem 3.3, it suffices to do this in the case where $a_{1}, \ldots, a_{r} \in\{1,2\}$. Here, we have $F\left(a_{1}, \ldots, a_{r}\right)=A_{\ell_{1}, \ell_{2}}$ and so the statement holds by Theorem 3.4.

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(E. Ballico, A. Oneto) Universita di Trento, Via Sommarive, $14-38123$ Povo (Trento), Italy

Email address: edoardo.ballico@unitn.it, alessandro.oneto@unitn.it
(A. Bik) MPI for Mathematics in the Sciences, Leipzig, Germany

Email address: arthur.bik@mis.mpg.de
(E. Ventura) Universität Bern, Mathematisches Institut, Sidlerstrasse 5, 3012 Bern, Switzerland

Email address: emanueleventura.sw@gmail.com, emanuele.ventura@math.unibe.ch

