STRENGTH AND SLICE RANK OF FORMS ARE GENERICALLY EQUAL

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ABSTRACT. We prove that strength and slice rank of homogeneous polynomials of degree $d \ge 5$ over an algebraically closed field of characteristic zero coincide generically. To show this, we establish a conjecture of Catalisano, Geramita, Gimigliano, Harbourne, Migliore, Nagel and Shin concerning dimensions of secant varieties of the varieties of reducible homogeneous polynomials. These statements were already known in degrees $2 \le d \le 7$ and d = 9.

1. INTRODUCTION

Ananyan and Hochster [AH20a] introduced the notion of *strength* of a polynomial to solve a famous conjecture by Stillman on the existence of a uniform bound, independent on the number of variables, for the projective dimension of a homogeneous ideal of a polynomial ring. Recently, polynomial strength and related questions have been intensively investigated [AH20b, BB+21, BV20, BDE19, DES17, ESS20, KZ18].

Let k be an algebraically closed field of characteristic zero, let $n \ge 1$ be an integer and let

$$\mathcal{S} = \bigoplus_{d \ge 0} \mathcal{S}_d := \Bbbk[x_0, \dots, x_n]$$

be the standard graded polynomial ring in n + 1 variables over k. So the elements of S_d are homogeneous polynomials, also called *forms*, of degree d. Fix an integer $d \ge 2$ and let $f \in S_d$ be a degree-d form.

Definition 1.1. The strength of f is the minimal integer $r \ge 0$ for which there exists a decomposition

$$f = g_1 \cdot h_1 + \ldots + g_r \cdot h_r$$

where $g_1, h_1, \ldots, g_r, h_r$ are forms of positive degree. We denote it by str(f).

Computing the strength of a given polynomial is a very difficult task. Hence, a natural problem is to determine the strength of a general homogeneous polynomial. In [BO20], A.B. and A.O. noticed that a conjectural answer to this problem was implicitly given in [CG+19], Remark 7.7] where the authors study dimensions of secant varieties of the varieties of reducible forms. In particular, it was conjectured that the strength of a general form coincides with its slice rank; see [BO20, Conjecture 1.1]. Recall that the value of the slice rank of a general form is classically known; see Remark 1.5.

Definition 1.2. The *slice rank* of f is the minimal integer $r \ge 0$ for which there exists a decomposition

$$f = \ell_1 \cdot h_1 + \ldots + \ell_r \cdot h_r$$

where ℓ_1, \ldots, ℓ_r are linear forms and h_1, \ldots, h_r are forms of degree d-1. We denote it by sl. rk(f).

Conjecture 1.3 ([BO20, Conjecture 1.1]). The strength and the slice rank of a general form in S_d are equal.

So far, this conjecture has been established in the following cases: when the degree d is larger than $\frac{3}{2}n + \frac{1}{2}$ [Sza96], when twice the general slice rank is at most n + 2 [CCG08] and for $d \le 7$ and d = 9 [BO20].

The aim of this paper is to establish Conjecture 1.3, thereby determining the strength of a general form, by proving the stronger conjecture from [CG+19], Remark 7.7] which we also state below.

Geometric formulation of the problem. For an integer $1 \leq j \leq d/2$, we consider the variety of forms with a degree-j factor $X_j := \{[g \cdot h] \mid g \in \mathbb{PS}_j, h \in \mathbb{PS}_{d-j}\} \subseteq \mathbb{PS}_d$. The union of these varieties is the variety of reducible forms $X_{\text{red}} := \bigcup_{j=1}^{\lfloor d/2 \rfloor} X_j$. For an integer $r \geq 1$, the rth secant variety of X_{red} is the Zariski-closure

$$\sigma_r(X_{\text{red}}) := \overline{\{[f] \in \mathbb{P}\mathcal{S}_d \mid f = f_1 + \ldots + f_r, \ [f_1], \ldots, [f_r] \in X_{\text{red}}\}}$$

of the union of all linear spaces spanned by r points on X_{red} . Since X_{red} is reducible, we can describe its rth secant variety as

$$\sigma_r(X_{\rm red}) = \bigcup_{1 \le a_1, \dots, a_r \le \lfloor d/2 \rfloor} J_{a_1, \dots, a_r}$$

where

$$J_{a_1,\dots,a_r} := J(X_{a_1},\dots,X_{a_r}) = \overline{\{[f] \in \mathbb{P}S_d \mid f = f_1 + \dots + f_r, \ [f_1] \in X_{a_1},\dots,[f_r] \in X_{a_r}\}}$$

is the *join* of the varieties X_{a_1}, \ldots, X_{a_r} . Now, the general slice rank and strength are

sl. $\operatorname{rk}_{d,n}^{\circ} := \min\{r \in \mathbb{Z}_{\geq 0} \mid \sigma_r(X_1) = \mathbb{P}\mathcal{S}_d\}$ and $\operatorname{str}_{d,n}^{\circ} := \min\{r \in \mathbb{Z}_{\geq 0} \mid \sigma_r(X_{\operatorname{red}}) = \mathbb{P}\mathcal{S}_d\}.$

So Conjecture 1.3 is implied by the following stronger conjecture.

Conjecture 1.4 ([CG+19, Remark 7.7]). For each integer $r \ge 1$, we have dim $\sigma_r(X_{red}) = \dim \sigma_r(X_1)$.

Remark 1.5. Recall that the value of the general slice rank is classically known as it equals the minimal codimension of a linear space contained in a general hypersurface. If $d \ge 3$, then we have

sl.
$$\operatorname{rk}_{d,n}^{\circ} := \min\left\{ r \in \mathbb{Z}_{\geq 0} \middle| r(n+1-r) \geq \binom{n-r+d}{d} \right\}$$

and

$$\operatorname{codim}_{\mathbb{P}S_d} \sigma_r(X_1) = \binom{n-r+d}{d} - r(n+1-r)$$

for all integers $1 \le r < \text{sl. rk}_{d,n}^{\circ}$ by [Har92, Theorem 12.8]. Note that $\text{sl. rk}_{d,n}^{\circ} \le n$. So we can (and often will) relax the assumption $r < \text{sl. rk}_{d,n}^{\circ}$ to r < n.

The classical approach to computing dimensions of secant and join varieties is via *Terracini's Lemma* [Ter11] which asserts that, if $Y_1, \ldots, Y_r \subseteq \mathbb{P}^N$ are projective varieties, $q_1 \in Y_1, \ldots, q_r \in Y_r$ are general points and $p \in \langle q_1, \ldots, q_r \rangle$ is general, then

$$T_p \sigma_r J(Y_1, \ldots, Y_r) = \langle T_{q_1} Y_1, \ldots, T_{q_r} Y_r \rangle;$$

see e.g. [BC+18, Lemma 1] for a recent presentation. By direct computation, it is easy to observe that the tangent space to X_a at a general point $[g \cdot h]$, with $\deg(g) = a$ and $\deg(h) = d - a$, is given by $\mathbb{P}(g,h)_d$ where $(g,h)_d := (g,h) \cap S_d$ is the degree-*d* homogeneous part of the ideal generated by *g* and *h*. Therefore

(1)
$$\dim J_{a_1,...,a_r} = \dim(g_1, h_1, \dots, g_r, h_r)_d - 1,$$

where g_i, h_i are general forms with $\deg(g_i) = a_i$ and $\deg(h_i) = d - a_i$. The codimensions of the homogeneous parts of a homogeneous ideal are encoded in its *Hilbert function*, whose generating power series is called the *Hilbert series*. These are among the most studied algebraic invariants of a homogeneous ideal. The Hilbert series of an ideal generated by general forms is prescribed by Fröberg's famous conjecture; see [Frö85]. In [CCG08, Theorem 5.1], the authors used the known cases of Fröberg's conjecture to deduce the integers $d, n, r, a_1, \ldots, a_r$ with $2r \le n+2$ for which $J_{a_1,\ldots,a_r} = \mathbb{P}S_d$. Similarly, in [CG+19, Theorem 7.4], the authors showed that Conjecture 1.4 holds if $2r \le n+1$. The strength of the general form corresponds to the minimal codimension of a complete intersection inside a general hypersurface. This is the perspective of [Sza96, Corollary A], where the author shows that Conjecture 1.3 holds if $d \ge \frac{3}{2}n + \frac{1}{2}$.

In [BO20], A.B. and A.O. proved the following results.

Theorem 1.6. Let $d \in \{3, 4, 5, 6, 7, 9\}$ and $n, r \ge 1$ be integers such that $r < \operatorname{sl.rk}_{d,n}^{\circ}$. Then Conjecture 1.4 holds. Furthermore, unless (d, n, r) = (4, 3, 2), the subvariety $\sigma_r(X_1)$ is the unique component of $\sigma_r(X_{\operatorname{red}})$ of maximal dimension. If (d, n, r) = (4, 3, 2), the codimensions of $\sigma_r(X_1)$, $J(X_1, X_2)$ and $\sigma_r(X_2)$ each equal 1.

Corollary 1.7. When $d \leq 7$ and d = 9, the general form of S_d has strength equal to its slice rank.

The main results of this paper are the following complementing theorem and corollary.

Theorem 1.8. Let $d \ge 5$ and $n, r \ge 1$ be integers such that $r < \text{sl.rk}_{d,n}^{\circ}$. Then Conjecture 1.4 holds. Furthermore, the subvariety $\sigma_r(X_1)$ is the unique component of $\sigma_r(X_{\text{red}})$ of maximal dimension.

Corollary 1.9. The general form of S_d has strength equal to its slice rank.

Structure of the paper. In Section 2, we find a numerical upper bound for the dimension of J_{a_1,\ldots,a_r} , which is an equality for $a_1 = \ldots = a_r = 1$ and $r < \text{sl.rk}_{d,n}^{\circ}$. In Section 3, we study this upper bound as a_1, \ldots, a_r vary and prove the main result.

Acknowledgements. E.B. is partially supported by MIUR and GNSAGA of INdAM (Italy). E.V. is supported by Vici Grant 639.033.514 of Jan Draisma from the Netherlands Organisation for Scientific Research.

2. An upper bound on the dimensions

Let $d, n \ge 2, r < n$ and $a_1, \ldots, a_r \le d/2$ be positive integers. We consider the subset

$$J_{a_1,\ldots,a_r}^{\circ} := \left\{ f \in \mathbb{P}\mathcal{S}_d \middle| f = \sum_{i=1}^r g_i \cdot h_i, \quad g_i \in \mathcal{S}_{a_i}, h_i \in \mathcal{S}_{d-a_i}, \quad (g_1,\ldots,g_r) \text{ is a complete intersection} \right\}$$

of J_{a_1,\ldots,a_r} . Let $\operatorname{CI}_n(a_1,\ldots,a_r)$ be the set of complete intersections in \mathbb{P}^n of codimension r defined by the intersection of hypersurfaces of degrees a_1,\ldots,a_r .

In order to give an upper bound on the dimension of $J_{a_1,...,a_r}$, we first observe that the subset $J_{a_1,...,a_r}^{\circ}$ is dense and then we bound the dimension of this subset by parametrizing it via the space of complete intersections $CI_n(a_1,...,a_n)$ whose dimension can be computed explicitly.

Lemma 2.1. The subset $J_{a_1,\ldots,a_r}^{\circ}$ is dense in J_{a_1,\ldots,a_r} .

Proof. Let $[f] \in \mathbb{P}S_d$ be a form which admits a strength decomposition $f = \sum_{i=1}^r g_i h_i$ with $\deg(g_i) = a_i$. It is enough to show that $f \in \overline{J_{a_1,\ldots,a_r}^{\circ}}$. Consider general forms $(u_1,\ldots,u_r) \in S_{a_1} \times \cdots \times S_{a_r}$. By generality, since $r \leq n$, the u_i 's form a regular sequence. Since being a regular sequence is an open condition in the Zariski topology, there exists an $\varepsilon > 0$ such that

 $(su_1 + g_1, \ldots, su_r + g_r) \in \mathcal{S}_{a_1} \times \cdots \times \mathcal{S}_{a_r}$ is a regular sequence for all $s \in (0, \varepsilon] \cap \mathbb{Q}$. For $s \in (0, \varepsilon] \cap \mathbb{Q}$, define $f_s := \sum_{i=1}^r (g_i + su_i)h_i \in J_{a_1, \ldots, a_r}^\circ$. Then $\lim_{s \to 0} f_s = f$ and hence $f \in \overline{J_{a_1, \ldots, a_r}^\circ}$. \Box

Lemma 2.2. We have dim
$$J_{a_1,...,a_r} \leq \dim \operatorname{CI}_n(a_1,\ldots,a_r) + \binom{n+d}{d} - \operatorname{coeff}_d\left(\frac{\prod_{i=1}^r (1-t^{a_i})}{(1-t)^{n+1}}\right) - 1.$$

Proof. If $I = (g_1, \ldots, g_r) \subseteq S$ is an ideal defined by a regular sequence of degrees a_1, \ldots, a_r , then

$$\dim (\mathcal{S}/I)_d = \operatorname{coeff}_d \left(\frac{\prod_{i=1}^r (1 - t^{a_i})}{(1 - t)^{n+1}} \right).$$

Hence

$$\dim (g_1, \dots, g_r)_d = \binom{n+d}{d} - \operatorname{coeff}_d \left(\frac{\prod_{i=1}^r (1-t^{a_i})}{(1-t)^{n+1}}\right) =: N+1$$

From Lemma 2.1, we derive that $\dim J^{\circ}_{a_1,\ldots,a_r} = \dim J_{a_1,\ldots,a_r}$. Now, let E be the projective bundle on $\operatorname{CI}_n(a_1,\ldots,a_r)$ whose fiber at a point $Y \in \operatorname{CI}_n(a_1,\ldots,a_r)$ is the projective space $\mathbb{P}(I_Y)_d \cong \mathbb{P}^N$. Then

$$\operatorname{im} E = \operatorname{dim} \operatorname{CI}_n(a_1, \ldots, a_r) + N.$$

We consider the morphism $E \longrightarrow J_{a_1,...,a_r}^{\circ}$ given by $(Y, f) \mapsto f$. This map is surjective by definition of E and $J_{a_1,...,a_r}^{\circ}$. Thus

$$\dim J_{a_1,\ldots,a_r} = \dim J_{a_1,\ldots,a_r}^{\circ} \le \dim E = \dim \operatorname{CI}_n(a_1,\ldots,a_r) + N,$$

which gives the desired upper bound.

Now, we compute the dimension of $CI_n(a_1,\ldots,a_r)$.

Remark 2.3. The Hilbert polynomial $P_{a_1,\ldots,a_r}(t)$ of a complete intersection is uniquely determined by the degrees defining it since it is computed from the Koszul complex. In [Ser06, Section 4.6.1], it is shown that $\operatorname{CI}_n(a_1,\ldots,a_r)$ is parametrized by a Zariski-open subset of $\operatorname{Hilb}_{P_{a_1,\ldots,a_r}(t)}(\mathbb{P}^n)$. The latter is smooth at $[Y] \in \operatorname{CI}_n(a_1,\ldots,a_r)$ and [Ser06, Theorem 4.3.5] yields

$$T_{[Y]}$$
 Hilb _{$P_{a_1,\ldots,a_r}(t)$} (\mathbb{P}^n) = $H^0(N_{Y/\mathbb{P}^n})$.

So dim $CI_n(a_1, \ldots, a_r) = h^0(N_{Y/\mathbb{P}^n})$, i.e., the dimension of the space of global sections of the normal bundle of Y.

Proposition 2.4. We have

dim CI_n(a₁,...,a_r) =
$$\sum_{i=1}^{r} \operatorname{coeff}_{a_i} \left(\frac{\prod_{i=1}^{r} (1-t^{a_i})}{(1-t)^{n+1}} \right).$$

Proof. Let $Y \in CI_n(a_1, \ldots, a_r)$ be a general point. By Remark 2.3, dim $CI_n(a_1, \ldots, a_r) = h^0(N_{Y/\mathbb{P}^n})$. Since Y is a complete intersection, its normal bundle is $N_{Y/\mathbb{P}^n} = \bigoplus_{i=1}^r \mathcal{O}_Y(a_i)$. Hence, the statement follows from the following equality:

$$h^{0}(\mathcal{O}_{Y}(a_{i})) = \operatorname{coeff}_{a_{i}}\left(\frac{\prod_{i=1}^{r}(1-t^{a_{i}})}{(1-t)^{n+1}}\right)$$

To see that this equality holds, first notice that Y is projectively normal [Har77, Exercise II.8.4], because it is a smooth complete intersection, by the generality assumption. So, for all $k \ge 0$, the restriction map $H^0(\mathcal{O}_{\mathbb{P}^n}(k)) \to H^0(\mathcal{O}_Y(k))$ is surjective. From the long exact sequence in cohomology of the short exact sequence

$$0 \to \mathcal{I}_Y(k) \to \mathcal{O}_{\mathbb{P}^n}(k) \to \mathcal{O}_Y(k) \to 0,$$

one has $h^1(\mathcal{I}_Y(k)) = 0$ for all $k \ge 0$. Since $\operatorname{HF}_{\mathcal{S}/I_Y}(d) = \operatorname{coeff}_d\left(\frac{\prod_{i=1}^r (1-t^{a_i})}{(1-t)^{n+1}}\right)$, where I_Y is the homogeneous ideal of Y, the claimed equality follows.

Lemma 2.5. For integers $e \ge 0$ and $b_1, \ldots, b_s \ge 1$, we have the following identity:

$$\operatorname{coeff}_e\left(\frac{\prod_{i=1}^s (1-t^{b_i})}{(1-t)^{n+1}}\right) = \sum_{I \subseteq \{1,\dots,s\}} (-1)^{\#I} \binom{n+e-\sum_{i \in I} b_i}{n}.$$

Here $\binom{a}{b} = 0$ whenever a < b.

Proof. Left to the reader.

Theorem 2.6. Let r < n and $a_1, \ldots, a_r \leq d/2$ be positive integers and take $\ell_{d/2} := \#\{i \mid a_i = d/2\}$. Then

$$\dim J_{a_1,\dots,a_r} \le \binom{n+d}{d} - \operatorname{coeff}_d \left(\frac{\prod_{i=1}^r (1-t^{a_i})(1-t^{d-a_i})}{(1-t)^{n+1}} \right) + \binom{\ell_{d/2}}{2} - 1.$$

When $d \ge 3$, $a_1 = \ldots = a_r = 1$ and $r < \text{sl. rk}_{d,n}^{\circ}$, equality holds.

Proof. First, we consider the case where $d \ge 3$, $a_1 = \ldots = a_r = 1$. In this case, by (1), it is enough to compute the codimension of $(\ell_1, \ldots, \ell_r, g_1, \ldots, g_r)_d$ which corresponds to

$$\dim \mathcal{S}_d/(\ell_1,\ldots,\ell_r,g_1,\ldots,g_r)_d = \dim \mathcal{S}'_d/(\overline{g_1},\ldots,\overline{g_r})_d$$

where $S' \cong S/(\ell_1, \ldots, \ell_r)$ is a polynomial ring in n + 1 - r variables and $\overline{g_i}$ is the class of g_i in S'. Since the g_i are general of degree d - 1, the latter dimension is obtained by [HL87, Theorem 1] which states that

$$\operatorname{codim}_{\mathbb{P}S_d} J_{a_1,\dots,a_r} = \operatorname{coeff}_d \left(\frac{(1-t^{d-1})^r}{(1-t)^{n+1-r}} \right).$$

For the first statement, by Lemma 2.2 and Proposition 2.4, it is enough to prove that

$$\sum_{j=1}^{r} \operatorname{coeff}_{a_{j}} \left(\frac{\prod_{i=1}^{r} (1-t^{a_{i}})}{(1-t)^{n+1}} \right) + \binom{n+d}{d} - \operatorname{coeff}_{d} \left(\frac{\prod_{i=1}^{r} (1-t^{a_{i}})}{(1-t)^{n+1}} \right) - 1 = \binom{n+d}{d} - \operatorname{coeff}_{d} \left(\frac{\prod_{i=1}^{r} (1-t^{a_{i}})(1-t^{d-a_{i}})}{(1-t)^{n+1}} \right) + \binom{\ell_{d/2}}{2} - 1$$

or, equivalently, to prove that

$$(2) \quad \operatorname{coeff}_d\left(\frac{\prod_{i=1}^r (1-t^{a_i})(1-t^{d-a_i})}{(1-t)^{n+1}}\right) = \operatorname{coeff}_d\left(\frac{\prod_{i=1}^r (1-t^{a_i})}{(1-t)^{n+1}}\right) - \sum_{j=1}^r \operatorname{coeff}_{a_j}\left(\frac{\prod_{i=1}^r (1-t^{a_i})}{(1-t)^{n+1}}\right) + \binom{\ell_{d/2}}{2}.$$

We analyze both sides of this equality. For the left hand side, we use Lemma 2.5 with e = d, s = 2r and $(b_i, b_{r+i}) = (a_i, d - a_i)$ for $i = 1, \ldots, r$. Since $a_i \leq d/2$ for all i, the summand corresponding to subset $I \subseteq \{1, \ldots, 2r\}$ is zero whenever the intersection $I \cap \{r+1, \ldots, 2r\}$ has more than two elements. The remaining summands correspond to subsets I such that $I \subseteq \{1, \ldots, r\}$, $I = I' \cup \{r+j\}$ for $I' \subseteq \{1, \ldots, r\}$ and $j \in \{1, \ldots, r\}$ or $I = I' \cup \{r+j, r+k\}$ for $I' \subseteq \{1, \ldots, r\}$ and distinct $j, k \in \{1, \ldots, r\}$. In the last case, the summand is zero unless $a_j = a_k = d/2$ and $I' = \emptyset$. So we get

$$\sum_{I \subseteq \{1,\dots,r\}} (-1)^{\#I} \binom{n+d-\sum_{i \in I} a_i}{n} + \sum_{j=1}^r \sum_{I' \subseteq \{1,\dots,r\}} (-1)^{\#I'+1} \binom{n+a_j-\sum_{i \in I'} a_i}{n} + \binom{\ell_{d/2}}{2}.$$

For the right hand side of (2), we use Lemma 2.5 with s = r and $b_i = a_i$ for i = 1, ..., r and varying e. We get

$$\sum_{\substack{I \subseteq \{1,\dots,r\}\\ n}} (-1)^{\#I} \binom{n+d-\sum_{i\in I} a_i}{n} - \sum_{j=1}^r \sum_{\substack{I \subseteq \{1,\dots,r\}\\ I \subseteq \{1,\dots,r\}}} (-1)^{\#I} \binom{n+a_j-\sum_{i\in I} a_i}{n} + \binom{\ell_{d/2}}{2}.$$

Hence (2) holds.

3. Numerical computations

Fix an integer $d \ge 5$. Let $n, r \ge 1$ and $1 \le a_1, \ldots, a_r \le d/2$ be integers such that $r < \text{sl. rk}_{d,n}^{\circ}$. Our goal is to prove that

 $\dim J_{a_1,\ldots,a_r} \le \dim \sigma_r(X_1)$

holds, and that we have equality if and only if $a_1 = \ldots = a_r = 1$. Write $\ell_j := \#\{i \in \{1, \ldots, r\} \mid a_i = j\}$ for all $j \in \mathbb{R}$. By Theorem 2.6, it suffices to prove that, for fixed n, r, the value of

$$F(a_1, \dots, a_r) := \operatorname{coeff}_d \left(\frac{\prod_{i=1}^r (1 - t^{a_i})(1 - t^{d-a_i})}{(1 - t)^{n+1}} \right) - \binom{\ell_{d/2}}{2} = \operatorname{coeff}_d \left(\frac{\prod_{i=1}^r (1 - t^{a_i})}{(1 - t)^{n+1}} \left(1 - \sum_{i=1}^r t^{d-a_i} \right) \right)$$

is minimal exactly when $a_1 = \ldots = a_r = 1$. We first prove that $F(a_1, \ldots, a_r)$ goes down when replacing all $a_i > 2$ by 2. Afterwards, we deal with the cases where $a_1, \ldots, a_r \in \{1, 2\}$. Take $\vartheta := \max\{a_1, \ldots, a_r\} \leq d/2$.

3.1. The case
$$\vartheta > 2$$
. Write $P_k := 1 + t + \ldots + t^k$ for $k \ge 0$ and $P_\infty := 1/(1-t)$.

Lemma 3.1. Let $s, \ell, k_1, \ldots, k_s \geq 0$ be integers. Then the coefficients of the power series

$$P_{\infty}^{\ell+1}P_{k_1}\cdots P_{k_s}$$

form a weakly increasing series.

Proof. We have

$$P_{\infty}^{\ell+1} = \sum_{k=0}^{\infty} \binom{\ell+k}{k} t^k$$

and so the lemma holds when s = 0. When f is a series whose coefficients increase weakly and $k \ge 0$ is an integer, then the same holds for the series fP_k . Hence the lemma holds for all s using induction.

We will often apply the next lemma with $g = P_a$ and $h = P_b$, where $a \ge b \ge 0$ are integers.

Lemma 3.2. Let f, g, h be series whose coefficients are all nonnegative and suppose that $\operatorname{coeff}_k(g) \ge \operatorname{coeff}_k(h)$ for all $k \ge 0$. Then $\operatorname{coeff}_k(fg) \ge \operatorname{coeff}_k(fh)$ for all $k \ge 0$. **Theorem 3.3.** Assume that $a_r = \vartheta > 2$. Then $F(a_1, ..., a_r) > F(a_1, ..., a_{r-1}, a_r - 1)$.

Proof. Take

$$f := \frac{\prod_{i=1}^{r-1} (1 - t^{a_i})}{(1 - t)^n}.$$

Then we have

$$F(a_1, \dots, a_r) = \operatorname{coeff}_d \left(\frac{\prod_{i=1}^r (1 - t^{a_i})}{(1 - t)^{n+1}} \left(1 - \sum_{i=1}^r t^{d-a_i} \right) \right) = \operatorname{coeff}_d \left(f P_{\vartheta - 1} \left(1 - \sum_{i=1}^r t^{d-a_i} \right) \right)$$

and similarly

$$F(a_1,\ldots,a_{r-1},a_r-1) = \operatorname{coeff}_d \left(fP_{\vartheta-2} \left(\left(1 - \sum_{i=1}^r t^{d-a_i} \right) + t^{d-\vartheta} (1-t) \right) \right) \right).$$

We need to show that the difference

$$\operatorname{coeff}_d\left(fP_{\vartheta-1}\left(1-\sum_{i=1}^r t^{d-a_i}\right)\right) - \operatorname{coeff}_d\left(fP_{\vartheta-2}\left(\left(1-\sum_{i=1}^r t^{d-a_i}\right) + t^{d-\vartheta}(1-t)\right)\right)$$

This difference equals

is positive. This difference equals

$$\operatorname{coeff}_d \left(f\left(t^{\vartheta-1} \left(1 - \sum_{i=1}^r t^{d-a_i} \right) - t^{d-\vartheta} + t^{d-1} \right) \right) = \operatorname{coeff}_{d-\vartheta+1}(f) - \ell_{\vartheta-1} - (\ell_\vartheta - 1) \operatorname{coeff}_1(f) - \operatorname{coeff}_\vartheta(f) \\ = \operatorname{coeff}_{d-\vartheta+1}(f(1 - t^{d-2\vartheta+1})) - \ell_{\vartheta-1} - (\ell_\vartheta - 1)(n - \ell_1).$$

Take

$$g := P_{\infty}^{n-r} P_{d-2\vartheta} \prod_{i=1}^{r-1} P_{a_i-1} = \frac{\prod_{i=1}^{r-1} (1-t^{a_i})}{(1-t)^n} (1-t^{d-2\vartheta+1}) = f(1-t^{d-2\vartheta+1}).$$

By Lemma 3.1, the coefficients of g are weakly increasing. So

 $\operatorname{coeff}_{d-\vartheta+1}(g) \ge \operatorname{coeff}_{\vartheta+1}(g).$

Write $m = n - \ell_1$. As $\ell_1 + \ldots + \ell_\vartheta = r < \text{sl. rk}_{d,n}^\circ \le n$, we have $m > \ell_2 + \ldots + \ell_\vartheta$. Note that

$$\operatorname{coeff}_{\vartheta+1}(g) \geq \operatorname{coeff}_{\vartheta+1} \left(P_{\infty}^{n-r} P_{d-2\vartheta} P_{1}^{r-\ell_{1}-\ell_{\vartheta}} P_{\vartheta-1}^{\ell_{\vartheta}-1} \right)$$
$$\geq \operatorname{coeff}_{\vartheta+1} \left(P_{\infty} P_{1}^{m-\ell_{\vartheta}-1} P_{\vartheta-1}^{\ell_{\vartheta}-1} \right)$$
$$= \operatorname{coeff}_{\vartheta+1} \left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1} (1-t^{\vartheta})^{\ell_{\vartheta}-1} \right)$$
$$= \operatorname{coeff}_{\vartheta+1} \left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1} \right) - (\ell_{\vartheta}-1)(m-1)$$
$$\geq \operatorname{coeff}_{4} \left(P_{\infty}^{\ell_{\vartheta}} P_{1}^{m-\ell_{\vartheta}-1} \right) - (\ell_{\vartheta}-1)(m-1)$$

So it suffices to prove that

 ℓ_{ϑ}

(3)
$$\operatorname{coeff}_4\left(P_{\infty}^{\ell_{\vartheta}}P_1^{m-\ell_{\vartheta}-1}\right) > \ell_{\vartheta-1} + (\ell_{\vartheta}-1)(2m-1)$$

for all $\ell_{\vartheta-1} \ge 0$, $\ell_{\vartheta} \ge 1$ and $m > \ell_{\vartheta-1} + \ell_{\vartheta}$. Note that

$$_{-1} + (\ell_{\vartheta} - 1)(2m - 1) \le (m - \ell_{\vartheta} - 1) + (\ell_{\vartheta} - 1)(2m - 1) = 2\ell_{\vartheta}(m - 1) - m.$$

We have

$$\operatorname{coeff}_4\left(P_{\infty}^{\ell_{\vartheta}}P_1^{m-\ell_{\vartheta}-1}\right) \ge \operatorname{coeff}_4\left(P_{\infty}P_1^{m-2}\right) = \sum_{k=0}^4 \binom{m-2}{k}$$

which is strictly greater than $2(m-1)(m-1) - m \ge 2\ell_{\vartheta}(m-1) - m$ for $m \ge 10$. This leaves the case $m \le 9$, where we verified that (3) holds by computer. This finishes the proof.

By Theorem 3.3, it suffices to focus on the cases where $a_1, \ldots, a_r \in \{1, 2\}$. In these cases, we will regard $F(a_1, \ldots, a_r)$ as a function A_{ℓ_1, ℓ_2} (defined below) depending only on ℓ_1 and ℓ_2 .

3.2. The case $\vartheta = 2$. Recall that $d \ge 5$. We define

$$\begin{aligned} A_{\ell_1,\ell_2} &:= \operatorname{coeff}_d \left(\frac{(1-t)^{\ell_1} (1-t^2)^{\ell_2}}{(1-t)^{n+1}} \left(1-\ell_1 t^{d-1} - \ell_2 t^{d-2} \right) \right) & \text{for } \ell_1, \ell_2 \ge 0, \\ B_{\ell_1,\ell_2} &:= A_{\ell_1-1,\ell_2+1} - A_{\ell_1,\ell_2} & \text{for } \ell_1 \ge 1 \text{ and } \ell_2 \ge 0, \\ C_{\ell_1,\ell_2} &:= B_{\ell_1-1,\ell_2+1} - B_{\ell_1,\ell_2} & \text{for } \ell_1 \ge 2 \text{ and } \ell_2 \ge 0, \\ D_{\ell_1,\ell_2} &:= C_{\ell_1-1,\ell_2+1} - C_{\ell_1,\ell_2} & \text{for } \ell_1 \ge 3 \text{ and } \ell_2 \ge 0 \text{ and} \\ E_{\ell_1,\ell_2} &:= D_{\ell_1-1,\ell_2+1} - D_{\ell_1,\ell_2} & \text{for } \ell_1 \ge 4 \text{ and } \ell_2 \ge 0. \end{aligned}$$

The goal of this subsection is to prove the following theorem.

Theorem 3.4. We have $A_{\ell_1,\ell_2} > A_{\ell_1+\ell_2,0}$ for all integers $\ell_1 \ge 0$ and $\ell_2 \ge 1$ such that $\ell_1 + \ell_2 < \text{sl.rk}_{d,n}^{\circ}$.

We write $m = n - \ell_1$ and we assume that $\ell_1 + \ell_2 < n$. So $\ell_2 < m$. In particular, we have $m \ge 1$.

Lemma 3.5. Let $\ell_1, \ell_2 \ge 0$ be integers such that $\ell_1 + \ell_2 < n$.

(a) We have

$$A_{\ell_1,\ell_2} = \operatorname{coeff}_d(P_{\infty}^{m+1-\ell_2}P_1^{\ell_2}) - \ell_2\binom{m+2}{2} - \ell_1(m+1) + \ell_2^2$$

(b) When $\ell_1 \geq 1$, we have

$$B_{\ell_1,\ell_2} = \operatorname{coeff}_{d-1}(P_{\infty}^{m+1-\ell_2}P_1^{\ell_2}) - \binom{m+2}{2} - \ell_2 m - \ell_1 + 1.$$

(c) When $\ell_1 \geq 2$, we have

$$C_{\ell_1,\ell_2} = \operatorname{coeff}_{d-2}(P_{\infty}^{m+1-\ell_2}P_1^{\ell_2}) - 2(m+1) - \ell_2.$$

(d) When $\ell_1 \geq 3$, we have

$$D_{\ell_1,\ell_2} = \text{coeff}_{d-3}(P_{\infty}^{m+1-\ell_2}P_1^{\ell_2}) - 3.$$

(e) When $\ell_1 \geq 4$, we have

$$E_{\ell_1,\ell_2} = \operatorname{coeff}_{d-4}(P_{\infty}^{m+1-\ell_2}P_1^{\ell_2}).$$

Proof. These calculations are straightforward.

Lemma 3.6. Let $\ell_1 \ge 1$ and $\ell_2 \ge 0$ be integers such that $\ell_1 + \ell_2 < n$.

- (a) When $\ell_1 < \text{sl. rk}^{\circ}_{d,n}$, we have $B_{\ell_1,0} > 0$.
- (b) When $\ell_1 \geq 2$, we have $C_{\ell_1,\ell_2} \geq 0$.
- (c) When $\ell_1 \ge 3$, we have $D_{\ell_1,\ell_2} \ge 0$.
- (d) When $\ell_1 \ge 4$, we have $E_{\ell_1,\ell_2} \ge 2$.

Proof. We prove the parts of the lemma in reverse order.

(d). We have $E_{\ell_1,\ell_2} = \operatorname{coeff}_{d-4}(P_{\infty}^{m+1-\ell_2}P_1^{\ell_2}) \ge \operatorname{coeff}_1(P_1^{m+1}) = m+1 \ge 2.$

(c). By (d), we have $D_{\ell_1,\ell_2} \ge D_{\ell_1+\ell_2,0}$. So we may assume that $\ell_2 = 0$. Now, we have

$$D_{\ell_1,0} = \operatorname{coeff}_{d-3}(P_{\infty}^{m+1}) - 3 = \binom{m+d-3}{d-3} - 3 \ge \binom{1+d-3}{d-3} - 3 = (d-2) - 3 \ge 0.$$

(b). By (c), we have $C_{\ell_1,\ell_2} \ge C_{\ell_1+\ell_2,0}$. So we may assume that $\ell_2 = 0$. Now, we have

$$(m+1) \ge 2$$
 and $\frac{(m+d-2)\cdots(m+2)}{(d-2)!} - 2 \ge \frac{(1+d-2)\cdots(1+2)}{(d-2)!} - 2 = \frac{d-1}{2} - 2 \ge 0$

and so

$$C_{\ell_1,0} = \operatorname{coeff}_{d-2}(P_{\infty}^{m+1}) - 2(m+1) = \binom{m+d-2}{d-2} - 2(m+1) = (m+1)\left(\frac{(m+d-2)\cdots(m+2)}{(d-2)!} - 2\right) \ge 0.$$

(a). By (b), $B_{\ell_1,\ell_2} \ge B_{\ell_1+\ell_2,0}$. So we may assume $\ell_2 = 0$. Since $\ell_1 < \text{sl.rk}_{d,n}^\circ$, we have $\ell_1(m+1) < \binom{m+d}{d}$. So $d!\ell_1 < (m+d)\cdots(m+2)$. We get

$$d!B_{\ell_{1},0} = d! \left(\operatorname{coeff}_{d-1}(P_{\infty}^{m+1}) - \binom{m+2}{2} - \ell_{1} + 1 \right)$$

$$= d! \left(\binom{m+d-1}{d-1} - \frac{m(m+3)}{2} \right) - d!\ell_{1}$$

$$> d! \left(\binom{m+d-1}{d-1} - \frac{m(m+3)}{2} \right) - (m+d)\cdots(m+2)$$

$$= d(m+d-1)\cdots(m+1) - \frac{d!}{2}m(m+3) - (m+d)\cdots(m+2)$$

$$= (m+d-1)\cdots(m+2) \left(d(m+1) - (m+d) \right) - \frac{d!}{2}m(m+3)$$

$$= (m+d-1)\cdots(m+2)(d-1)m - \frac{d!}{2}m(m+3)$$

$$= m \left((m+d-1)\cdots(m+2)(d-1) - \frac{d!}{2}(m+3) \right).$$

So it suffices to prove that

$$c_0 + c_1 m + \ldots + c_{d-2} m^{d-2} := (m+d-1)\cdots(m+2)(d-1) - \frac{d!}{2}(m+3) \ge 0$$

We have

$$c_{1} = (d-1) \operatorname{coeff}_{1} \left((m+d-1)\cdots(m+2) \right) - \frac{d!}{2}$$
$$= (d-1) \sum_{i=2}^{d-1} \frac{(d-1)!}{i} - \frac{d!}{2}$$
$$= (d-1)! \left(\sum_{i=2}^{d-1} \frac{d-1}{i} - \frac{d}{2} \right)$$
$$\geq (d-1)! \left(\frac{d-1}{2} + \frac{d-1}{d-1} - \frac{d}{2} \right) > 0$$

and $c_i > 0$ for $i = 2, \ldots, d - 3$. Hence

$$c_0 + c_1 m + \ldots + c_{d-2} m^{d-2} \ge c_0 + c_1 + \ldots + c_{d-2}$$

= $(1 + d - 1) \cdots (1 + 2)(d - 1) - \frac{d!}{2}(1 + 3)$
= $\frac{d!}{2}(d - 1) - \frac{d!}{2} \cdot 4 = \frac{d!}{2}(d - 5) \ge 0.$

This finishes the proof.

Theorem 3.4 now follows easily.

$$A_{\ell_1,\ell_2} - A_{\ell_1+1,\ell_2-1} = B_{\ell_1+1,\ell_2-1} \ge B_{\ell_1+\ell_2,0} > 0.$$

Repeating this, we find that

$$A_{\ell_1,\ell_2} > A_{\ell_1+1,\ell_2-1} > \dots > A_{\ell_1+\ell_2,0}$$

as desired.

3.3. The conclusion of the proof.

Proof of Theorem 1.8. Let $d \ge 5$, $n, r \ge 1$ and $1 \le a_1, \ldots, a_r \le d/2$ be integers such that $r < \text{sl. rk}_{d,n}^{\circ}$. We need to show that

 $\dim J_{a_1,\ldots,a_r} \le \dim \sigma_r(X_1)$

holds, and that we have equality only for $a_1 = \ldots = a_r = 1$. By Theorem 2.6, it suffices to prove that

$$F(a_1,\ldots,a_r)$$

is minimal exactly when $a_1 = \ldots = a_r = 1$. By Theorem 3.3, it suffices to do this in the case where $a_1, \ldots, a_r \in \{1, 2\}$. Here, we have $F(a_1, \ldots, a_r) = A_{\ell_1, \ell_2}$ and so the statement holds by Theorem 3.4. \Box

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