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VARIATIONAL PROBLEMS FOR SUB–FINSLER METRICS IN CARNOT GROUPS AND INTEGRAL FUNCTIONALS DEPENDING ON VECTOR FIELDS

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Introduction

The main purpose of this dissertation is to present outcomes related to some differential structure's results and two kinds of variational problems. The latter share similar geometric analysis instruments, in spaces whose geometry is expressed in terms of a family of vector fields. The first part focuses on Finsler type distances properties in structures defined in terms of bracket generating vector fields, and in particular in Carnot groups. The second part describes properties of local functionals in more general structures, defined in terms of further degenerate families of vector fields.

The first chapter is devoted to the presentation and the analysis of some key results achieved by Venturini in an unpublished preprint [94] and by De Cecco and Palmieri in [44, 46–48]. In [94], the author studied a particular class of geodesic distance functions locally equivalent to the Euclidean one. Afterwards, he investigated the relationships between this class and the family of Borel measurable metrics locally equivalent to the Euclidean one, on a subdomain $\Omega \subset \mathbb{R}^n$. In the literature, this is a subclass of the so-called Finsler metrics, namely $F : \Omega \times \mathbb{R}^n \to [0, +\infty)$ such that $F(x, \cdot)$ is convex and positively 1-homogeneous on $\mathbb{R}^n$. Alongside, De Cecco and Palmieri rephrased the previous analysis in the setting of Lipschitz manifolds, which are a generalization of Riemannian manifolds and polyhedra, since they can show vertices, edges, conical points, even not isolated.

Once we have introduced the state of the art, our first aim is to extend the aforementioned analysis in the so-called Carnot groups, a particular class of Carnot–Carathéodory spaces which have been facing an increasing interest in the last century. For extensive sources of this topic we refer the reader to [3, 12, 16, 52, 69, 76, 79, 84, 85, 87].

More in general, a *sub-Finsler manifold* is given by the triple $(M, \Delta, \|\cdot\|)$ where M is a n-dimensional smooth manifold and Δ is a subbundle of the tangent bundle TM that satisfy the so called Hörmander (or equivalently bracket-generating) condition. If we equip Δ with a (possibly) continuously varying norm $\|\cdot\|$, it is natural to induce the so-called Carnot–Carathéodory distance d_{cc}, defined by minimizing only on those absolutely continuous paths whose tangent vector belongs to the fibers of Δ: the horizontal curves. We refer to $(M, \Delta, \|\cdot\|, d_{cc})$ as a *Carnot–Carathéodory space* and when the norm $\|\cdot\|$ comes from a scalar product, the quadruple is usually called *sub-Riemannian manifold*.

Sub-Finsler geometry is a rather new research fields and sub-Finsler structures $(\Delta, \|\cdot\|)$ have been deeply studied in the recent years, observing that, when they are induced by smooth norms, the theory is very similar to the sub-Riemannian case. On the other hand, the subject is wider when the norm is not supposed to be strictly convex nor smooth, even away from the origin. Indeed, in order to analyze the behaviour of geodesics or the applications in some optimal control problems, several works developed a sub-Finsler geometry on the Heisenberg group (cf. [9, 58]) and on nilpotent Lie groups of step bigger than 2 as the Engel and the Cartan groups, see e.g. [7, 8, 11, 19, 35, 76]. Moreover, sub-Finsler manifolds were also object of analysis in order to approach the Bernstein problem and the regularity of Lipschitz boundaries (see [67, 68]). Clearly, Riemannian and Finsler manifolds can be seen respectively as a special example of sub-Riemannian and sub-Finsler ones, when one considers $\Delta = TM$. Actually, the tangent metric space of a sub-Riemannian manifold is a Carnot group. This is a connected and simply connected Lie group whose associated Lie algebra admits a finite-step stratification, see e.g. [16, 52, 85, 89]. It possesses a very rich geometry and occupies a central position in the study of harmonic analysis and hypoelliptic partial differential equations. Moreover, Carnot groups arises in the CR geometric function theory (see [59]), quasiconformal mappings and recently in the applied sciences such as mechanical engineering and neurophysiology of the brain (see for instance [34, 75]).

The problem we want to introduce concerns the class of geodesic distances $d : \Omega \times \Omega \to \mathbb{R}$

locally equivalent to the Carnot–Carathéodory distance, where Ω is a subdomain of a Carnot group $\mathbb{G}$. We will say that they belong to $\mathcal{D}_{cc}(\Omega)$ if there exist some $\alpha \geq 1$ such that

$$\frac{1}{\alpha} d_{cc}(x,y) \leq d(x,y) \leq \alpha d_{cc}(x,y) \qquad \forall x, y \in \Omega.$$

Setting the so-called horizontal bundle $H\mathbb{G}$, namely, the subbundle of the tangent bundle $T\mathbb{G}$ (see Section 2.1), it is quite natural to consider the family of maps $\varphi_d : H\mathbb{G} \to [0, +\infty)$. According to [46, 94], the latter is called the *metric derivative* associated to $d \in \mathcal{D}_{cc}(\Omega)$ by differentiation, in other words:

$$\varphi_d(x,v) \coloneqq \limsup_{t \searrow 0} \frac{d\big(x, x \cdot \delta_t \exp(\mathrm{d}_x \tau_{x^{-1}}[v])\big)}{t} \qquad \text{for every } (x,v) \in H\mathbb{G}.$$

Due to the notation introduced by Scott D. Pauls in [90], in the latter we denoted with $x \cdot \delta_t \exp(\mathrm{d}_x \tau_{x^{-1}}[v])$ the dilation curve starting from $x \in \mathbb{G}$ with direction given by the left translation at the identity of a horizontal vector on the fiber $H_x\mathbb{G}$ (see Subsection 2.1.1 and Definition 2.1.6). In particular, φ_d is the counterpart of the Finsler metric, with the difference that, fixing a point $x \in \Omega$, it is defined only on $H_x\mathbb{G}$, instead of all the tangent space at x. It turns out that the metric derivative is a *sub-Finsler convex metric*, namely

- $\varphi : H\mathbb{G} \to \mathbb{R}$ is Borel measurable, where $H\mathbb{G}$ is endowed with the product σ-algebra;
- $\varphi(x, \delta^*_\lambda v) = |\lambda| \varphi(x,v)$ for every $(x,v) \in H\mathbb{G}$ and $\lambda \in \mathbb{R}$;
- $\frac{1}{\alpha}\|v\|_x \leq \varphi(x,v) \leq \alpha \|v\|_x$ for every $(x,v) \in H\mathbb{G}$;
- $\varphi(x, v_1 + v_2) \leq \varphi(x, v_1) + \varphi(x, v_2)$ for every $x \in \mathbb{G}$ and $v_1, v_2 \in H_x\mathbb{G}$.

By $(\delta^*_\lambda)_{\lambda \in \mathbb{R}}$ we mean a family of dilations appropriately defined on $H_x\mathbb{G}$ (see Subsection 2.1.2) and by $\|\cdot\|_x$ we denote the norm induced by the sub-Riemannian structure (cf. Definition 2.1.3). Let us mention that, the assumption of convexity for φ involves only vectors on the first stratum. In general, in Carnot groups there where several definitions involving points, as the definition of horizontal convexity given by Danielli, Garofalo and Nhieu in [39] and, independently, by Lu, Manfredi and Stroffolini [73]. This being said, in Theorem 2.3.8

we will show that we can reconstruct the distance d by minimizing the length functional induced by the metric derivative φ_d. Furthermore, inspired by the proof contained in [28], in Section 4.4 we state the equivalence between the Γ-convergence of the so called length and energy functionals L_n and J_n associated to a sequence of distances $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}_{cc}(\Omega)$. Moreover, in Theorem 4.4.1 point (iv), we show an additional characterization when Ω is bounded. Let us recall that these kind of variational problems have been already studied in the literature, mostly for what concerns the homogenization of Riemannian and Finsler metrics, see e.g. [1, 5, 18, 42, 43].

The second purpose is to study a different application: the intrinsic analysis of sub-Finsler metrics, contained in the paper [54]. More precisely, under suitable regularity assumptions on the metric under consideration, we prove the following result (Theorem 2.7.1):

$$\varphi(x, \nabla_{\mathbb{G}} f(x)) = \mathrm{Lip}_{\delta_\varphi} f(x) \quad \text{for a.e. } x \in \mathbb{G}, \tag{0.0.1}$$

where $f : \Omega \subset \mathbb{G} \to \mathbb{R}$ is a Pansu-differentiable function (cf. Definition 2.1.15), δ_φ is the distance defined in (0.0.2) below, φ is a sub-Finsler convex metric, and the pointwise Lipschitz constant of f is given by

$$\mathrm{Lip}_{\delta_\varphi} f(x) = \limsup_{y\to x} \frac{|f(y) - f(x)|}{\delta_\varphi(x, y)} \quad \text{for every } x \in \mathbb{G}.$$

The equality (0.0.1) may be regarded as a generalization of a result achieved in [93], and further generalized by Chang Y. Guo in [70], to bounded Finsler metrics defined on open subsets of $\mathbb{R}^n$. Then, in order to prove (0.0.1), we crucially observe that the distance

$$\delta_\varphi(x, y) \coloneqq \sup\left\{|f(x) - f(y)| \,\middle|\, f\colon \mathbb{G} \to \mathbb{R} \text{ Lipschitz}, \ \|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\|_\infty \le 1\right\}, \tag{0.0.2}$$

introduced by De Cecco and Palmieri in [45, 46] for Finsler metrics, coincides with the intrinsic distance $d_{\varphi^\star}$, induced by the dual metric (cf. Definition 2.2.2). This happens, for instance, when we assume that the sub-Finsler metric φ is either lower semicontinuous or upper semicontinuous on the horizontal bundle (see Theorem 2.6.4 and Corollary 2.2.5). These results are obtained using methods of metric geometry and they are a generalization of the analogous statement in [48], due to De Cecco and Palmieri. The proof of Theorem 2.6.4 heavily relies on two results contained in [77] and appropriately specialized in our setting. The first one allows us to approximate an upper semicontinuous sub-Finsler metric with a family of Finsler metrics. The second result lets us approximate from below the intrinsic distance, induced by the sub-Finsler convex metric, with a family of induced Finsler distances. To conclude, we show that in many cases the distance δ_φ, albeit defined as a supremum among Lipschitz functions, is actually already determined by smooth functions (cf. Proposition 2.7.3). An important step in proving this fact is to approximate uniformly on compact sets any 1-Lipschitz function with a sequence of smooth 1-Lipschitz functions. Here, the key point is that the Lipschitz constant is preserved. Since this approximation result holds in much greater generality (for instance, on possibly rank-varying sub-Finsler manifold) and might be of independent interest, we will treat it in Subsection 2.7.1.

In the second part of the present Ph.D. Thesis, we focus on the analysis of the representation of local functionals, as integral functionals, of the form

$$F(u) = \int_\Omega f(x, u(x), Xu(x))\,\mathrm{d}x.$$

This point of view was recently introduced in the paper [60], where the authors started the analysis of variational functionals driven by a family of Lipschitz vector fields. This is an $m-$tuple $\mathbf{X} = (X_1, \ldots, X_m)$ of first-order differential operator X_j, with Lipschitz regular coefficients $c_{j,i}$ defined on a bounded open set $\Omega \subseteq \mathbb{R}^n$, namely

$$X_j(x) = \sum_{i=1}^{n} c_{j,i}(x)\partial_i \quad j = 1, \ldots, m.$$

Moreover we assume that the family $\mathbf{X}$ satisfies the structure assumption (LIC), which roughly means that $X_1(x), \ldots, X_m(x)$ are linearly independent for a.e. $x \in \Omega$ (cf. Definition 3.1.1). This condition is pretty general and encompasses many interesting example of sub-Riemannian manifolds, as the Carnot groups, when we consider the family of vector fields given by $X = \nabla_{\mathbb{G}}$, namely the *horizontal gradient*, properly defined in (2.1.7). This is the the main link between the analysis achieved in the first part of the Ph.D. Thesis and the present arguments.

However, the integral representation problem has a very long history and it exhibits a natural application when dealing with relaxed functionals and related Γ-limits in a suitable topology. In the Euclidean setting it is now very well understood and we refer the interested reader to the papers [4, 24–27] for a complete overview of the subject.

A crucial result concerning our analysis is given by [81, Theorem 3.12]. Denoting with $\mathcal{A}$ the class of all open subsets of Ω, the authors found conditions such that a local functional $F:\, L^p(\Omega)\times\mathcal{A}\to[0,\infty]$ can be represented as

$$F(u,A)=\int_A f(x,Xu(x))\,\mathrm{d}x \tag{0.0.3}$$

for a suitable Carathéodory function $f:\Omega\times\mathbb{R}^m\to[0,\infty)$, for any $A\subseteq\Omega$ open and $u\in L^p(\Omega)$ s.t. $u\big|_A\in W^{1,p}_{X,\mathrm{loc}}(A)$ (cf. Definition 3.1.2 and [59]). We also point out that functional (0.0.3) was studied in [60] as far as its relaxation and in connection with the well-known Meyers–Serrin approximation theorem in $W^{1,p}_X(\Omega)$. Inspired also by the results proved in [25, 26], the aim of Chapter 3 is to extend the results achieved in [81], when one drops the assumption of translations-invariance on the functional F.

More precisely, we will find some sufficient and necessary conditions under which a local functional $F:W^{1,p}_{X,\mathrm{loc}}(\Omega)\times\mathcal{A}\longrightarrow[0,+\infty]$ admits an integral representation of the form

$$F(u,A)=\int_A f(x,u(x),Xu(x))\,\mathrm{d}x\quad\forall\,u\in W^{1,p}_{X,\mathrm{loc}}(\Omega),\,\forall\,A\in\mathcal{A}, \tag{0.0.4}$$

for a suitable Carathéodory function $f:\Omega\times\mathbb{R}\times\mathbb{R}^m\to[0,\infty)$. Due to the lack of translations invariance, in this new framework the dependence of the integrand with respect to the function u is expected. Moreover, if F is defined on $L^p_{\mathrm{loc}}(\Omega)\times\mathcal{A}$, under reasonable improvements of some of the assumptions, it is easy to extend the integral representation obtaining that

$$F(u,A)=\int_A f(x,u(x),Xu(x))\,\mathrm{d}x\quad\forall\,A\in\mathcal{A},\,\forall u\in L^p_{\mathrm{loc}}(\Omega)\text{ s.t. }u|_A\in W^{1,p}_{X,\mathrm{loc}}(A).$$

The main goal of Chapter 3 is to present the representation formula, contained in the paper [55], for the following three different classes of functionals:

(a) convex functionals (Theorem 3.2.1);

(b) $W^{1,\infty}$ weakly*- seq. l.s.c. functionals (Theorem 3.3.2);

(c) none of the above conditions (Theorem 3.4.1).

For dealing with the situation (c), we exploit some continuity condition introduced in [26], known as *weak* and *strong condition* (ω) (see Definition 3.2.4). An obstacle is that, no analogue of approximation results, by a reasonable notion of piecewise X-affine function, holds in general (see [81, Section 2.3] and Remark 3.1.7). To overcome this difficulty we rely on the method employed in [81]. This consists in firstly applying one of the classical results for Sobolev spaces in [25, 26] to the functional, obtaining an integral representation with respect to a "Euclidean" Lagrangian f_e of the form

$$F(u,A)=\int_A f_e(x,u(x),Du(x))\mathrm{d}x \quad \forall u\in W^{1,p}_{\mathrm{loc}}(\Omega),\, \forall A\in\mathcal{A}.$$

Then, it is necessary to find sufficient conditions on f_e that guarantee the existence of a "non Euclidean" Lagrangian f such that

$$\int_A f_e(x,u(x),Du(x))\mathrm{d}x=\int_A f(x,u(x),Xu(x))\mathrm{d}x \quad \forall A\in\mathcal{A},\, \forall u\in C^\infty(A). \tag{0.0.5}$$

Indeed, by Proposition 3.1.21 we can crucially exploits the convexity of $f_e(x,u,\cdot)$ in order to guarantee (0.0.5). This result shown in [81] and the same ideas can be adapted to the cases (a) and (b), for which the convexity of $f_e(x,u,\cdot)$ is granted. On the contrary, due to the weaker assumptions on the class of functionals, case (c) is more demanding and requires a further step. In other words, we have to extend the equality (0.0.5) to the whole space $W^{1,p}_{X,\mathrm{loc}}(\Omega)$. Neverthless, we will show that the convexity of $f_e(x,u,\cdot)$ is not a necessary assumption in order to obtain (0.0.5). Hence, we need to find a more suitable notion of convexity that is less demanding. For this reason, in Definition 3.3.7 we define the weaker concept of *X-convexity*, which strongly depends on the chosen family of vector fields. In Proposition 3.3.8, we prove that under a classical growth assumption on the functional, this new condition is equivalent to (0.0.5). Finally, by slightly modifying a zig-zag argument contained in [26, Lemma 2.11], we show that X-convexity is a consequence of a reasonable lower semicontinuity assumption (cf. Lemma 3.3.9). Finally, in order to give a complete characterization of the classes of functionals studied, we will also exhibit that, in each of the theorems (a), (b) and (c), the hypotheses are also necessary.

On the other hand, the previous representation results are crucial in order to move the focus on a strongly related problem, the Γ-compactness analysis. Starting from the seminal works by E. De Giorgi and T. Franzoni (see [49, 50]), the study of Γ-convergence has pervaded the evolution of modern calculus of variations, and has developed in several different directions, exhibiting important applications to many branches of calculus of variations, such as homogenization, minimal surfaces and partial differential equations. For an exhaustive introduction to this topic, we refer to the monographs [17, 18, 38].

Mostly, G. Buttazzo and G. Dal Maso have investigated Γ-convergence in the classical framework of Lebesgue spaces, Sobolev spaces and BV spaces, see e.g. [25,27,37]. In the last years the authors of [81,82] started the investigation of the $\Gamma-$convergence of translations-invariant local functionals, depending on suitable families of Lipschitz vector fields. Their strategy strongly relies on the representation of translations-invariant local functional as an integral functional (0.0.3). This tool is clearly a generalization of the Euclidean result in [27] and it allows the authors to achieve a Γ-compactness property. Let us notice that similar results have been proved in [83], under stronger conditions on the family $\mathbf{X}$. Moreover, Γ-convergence for functionals in (0.0.3) has been also developed in the framework of Dirichlet forms (cf. [63,86]), when the X-gradient satisfies the Hörmander condition (see, for instance [14] and references therein). Since that and other important works (see for instance [64]), the possibility to extend classical results to this new framework has been extensively studied in many papers. Indeed, many homogenization problems have been solved firstly in the Heisenberg group (cf. [51]) and then in Carnot groups (cf. [13, 62, 83]).

The main purpose in Chapter 4 is to generalize the Γ-compactness result, presented in [81], to three different classes $\mathfrak{J}$ of integral functionals (see [56]). Clearly, these are not translations-invariant functionals and they can be defined both on $L^p(\Omega)$ and on $W^{1,p}_X(\Omega)$. To be more precise, in Theorem 4.2.7 we show a $\Gamma(L^p)$-compactness result, under standard boundedness and coercivity requirements, for a class of non-negative convex integral functionals. After that, in Theorem 4.3.12 we exhibit a $\Gamma(W_X^{1,p})$-compactness result under standard boundedness assumption for the same class of non-negative convex integral functionals. To conclude, in Theorem 4.3.15 we present a $\Gamma(W_X^{1,p})$-compactness result for a class of possibly non-convex integral functionals which uniformly satisfies a suitable condition, inspired by the classical notion introduced in [26]: the *strong condition* (ωX) (see Definition 4.3.2). Differently from L^p, we point out that in $W_X^{1,p}$ setting, no coercivity assumption is required and we can also treat the situation with the exponent $p = 1$. In order to achieve these three main results, the general strategy we want to adopt is standard and consists of two main steps:

1. Given a sequence $(F_h)_h$ in an appropriate class of integral functional $\mathfrak{I}$, find a subsequence $(F_{h_k})_k$ and a local functional F such that

$$F(\cdot, A) = \Gamma - \lim_{k\to\infty} F_{h_k}(\cdot, A) \quad \text{for any } A \in \mathcal{A},$$

and moreover show that such an F satisfies some structural properties.

2. Choose a suitable subclass $\mathfrak{J} \subseteq \mathfrak{I}$ and show that, whenever $(F_h)_h$ belongs to $\mathfrak{J}$, then F belongs to $\mathfrak{J}$.

In the case of $\Gamma(L^p)$-convergence the approach is classic, since we can achieve the first step exploiting some basic results contained in [38] and applying some properties of the X-gradient. Moreover, the second step is based on the integral representation result (a) for convex local functionals, introduced in Chapter 3. Actually, one has to verify that the abstract $\Gamma(L^p)$-limit F satisfies all the assumptions relative to the class $\mathfrak{J}$.

The situation is more delicate when we perform the Γ-limit with respect to the strong topology of $W_X^{1,p}(\Omega)$. In order to achieve the first step, we introduce a suitable notion of *uniform fundamental estimate*, inspired by the classical one contained in [38] and modifying some arguments (see Definition 4.3.4). The latter allows us to drop the hypothesis of coercivity, and to mimicking the results performed in the L^p case, adapting them to this new framework.

On the other hand, the second step relies on the possibility to exploit a slight variant of the two integral representation results (a) and (c), performed respectively in Theorem 3.2.1 and Theorem 3.4.1. Indeed, they allows us to represent the $\Gamma(W^{1,p}_X)-$limit in an integral form. To this aim, we will show that the strong condition (ωX) is a necessary assumption in order to get one of the hypothesis of the latter theorem. Furthermore, this new condition behaves well with respect to the passage to the Γ-limit, provided we perform this operation with respect to the strong topology of $W^{1,p}_X(\Omega)$. To conclude, we point out that some of the results achieved in the non-Euclidean framework were unsolved, or untreated, even in the classical Sobolev space. In particular, in Subsection 4.3.3 we list some remarks and problems that are still opens.

Chapter 1

Metrics and Distances from $\mathbb{R}^n$ to Lipschitz Manifolds

We begin by presenting the objects of main interest for our future analysis, that is Borel measurable metrics and associated geodesic distances that satisfy reasonable bounds from above and below. Moreover, we show the connection between the classical analysis in metric spaces and the important tool of the metric derivative. Their properties are presented and proved in the subsequent sections and most of them are essentially known in literature. The main references used are the works by De Cecco and Palmieri [44, 46–48], an unpublished preprint by Venturini [94] and some chapters of the Ph.D. Thesis of Davini [42]. All this material has been reorganized and presented in convenient form for later use. We want also to underline that some proofs have been slightly simplified or modified for an easier understanding.

1.1 Notation and Metric spaces

Definition 1.1.1. *Let X be a non-empty set. We say that $d : X \times X \to [0, +\infty)$ is a* distance *if it satisfies the following conditions:*

- $d(x,x) = 0$ *for every* $x \in X$;
- $d(x,y) = d(y,x)$ *for every* $x, y \in X$;
- $d(x,y) \leq d(x,z) + d(z,y)$ *for every* $x, y, z \in X$.

The set X, endowed with the topology induced by d, will be called metric space*, and will be denoted by the couple (X,d).*

If the second assumption is not satisfied, we say that d is a *pseudodistance* on X.

Notation. We denote by $\mathrm{Lip}([0,1],X)$, or by $\mathrm{Lip}(X)$, the family of all Lipschitz curves $\gamma : [0,1] \to X$ joining two fixed points x and y in X. This means that there exists a finite constant L such that $d(\gamma(t),\gamma(s)) \leq L|t-s|$, for every $s,t \in [0,1]$. To make it easier, in the sequel we will not specify the end-points x and y. We equip $\mathrm{Lip}([0,1],X)$ with the metric given by the uniform convergence with respect to d, namely

$$\gamma_n \to \gamma \ \text{ in } \ \mathrm{Lip}(X) \quad \text{whenever} \quad \sup_{t\in[0,1]} d(\gamma_n(t),\gamma(t)) \to 0 \ \text{ as } \ n \to \infty.$$

We denote by $\mathrm{LIP}_d(X)$ the family of real-valued Lipschitz functions on X or $\Omega \subset X$ with respect to d (we omit it when the ambient space is $\mathbb{R}^n$). If $f \in \mathrm{LIP}(X)$ we denote by Df the Euclidean gradient, meaning as a row vector. For any $f \in \mathrm{LIP}_d(X)$, we write $\mathrm{Lip}_d f\colon X \to [0,+\infty)$ to mean the *pointwise Lipschitz constant* of f, which is defined by

$$\mathrm{Lip}_d f(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{d(x,y)} \quad \text{for every } x \in X.$$

■

Definition 1.1.2. *Let (X,d) be a metric space. Then we define the* classical length functional *of a Lipschitz curve $\gamma : [0,1] \to X$ as:*

$$L_d(\gamma) \coloneqq \sup\Big\{\sum_{i=1}^{k-1} d(\gamma(t_{i+1}),\gamma(t_i)) \ : \ 0 \leq t_1 < \ldots < t_k \leq 1, \ k \in \mathbb{N}\Big\}. \tag{1.1.1}$$

We will say that $d\colon X \times X \to [0,+\infty)$ is a geodesic distance *on X if*

$$d(x,y) = \inf\{L_d(\gamma) \ : \ \gamma \in \mathrm{Lip}(X)\} \quad \textit{for every } x,y \in X. \tag{1.1.2}$$

In particular, a metric space (X,d) such that d is geodesic is called a *length space*. We want to point out that the length L_d is lower semicontinuous with respect to the uniform and pointwise convergence of curves.

Theorem 1.1.3 (Busemann)**.** *Let d be a pseudodistance and let us assume that every closed ball in (X,d) is compact. Let $x,y \in X$, then the minimum problem*

$$\min\big\{L_d(\gamma)\ :\ \gamma \in \mathrm{Lip}([0,1],X)\big\}$$

admits a solution, provided the family $\mathrm{Lip}([0,1],X) \neq \emptyset$*. In particular, if d is a geodesic distance, there exists a curve $\gamma \in \mathrm{Lip}([0,1],X)$ such that $L_d(\gamma) = d(x,y)$.*

Remark 1.1.4. Moreover, if $\gamma : [0,1] \to X$ is a Lipschitz curve, then the *classical metric derivative*

$$|\dot\gamma(t)|_d := \lim_{s\to 0} \frac{d(\gamma(t+s),\gamma(t))}{|s|} \quad \text{exists for a.e. } t \in [0,1]. \tag{1.1.3}$$

The existence of the limit is a general fact that holds in any metric space (see [6,22]). Indeed, $|\dot\gamma(t)|_d$ is a measurable function and it holds that

$$L_d(\gamma) = \int_0^1 |\dot\gamma(t)|_d \,\mathrm{d}t. \tag{1.1.4}$$

A proof of the previous result is showed, for instance, in [6]. Anyway, we want to present a simpler proof for $\Omega \subset \mathbb{R}^n$ (see Theorem 1.2.13) and for Carnot groups (see Theorem 2.3.8), even if (1.1.4) clearly holds in both settings.

Notation. From now on, $X = \Omega$ will be an open and connected subset of $\mathbb{R}^n$. For any $u,v \in \mathbb{R}^n$, we denote by $\langle u,v\rangle$ the Euclidean scalar product, and by $|v|$ the induced norm. Given $x \in \mathbb{R}^n$ and $r > 0$ we set the usual ball $B_r(x) := \{y \in \mathbb{R}^n\ :\ |x-y| < r\}$ and the $n-1$-dimensional sphere $\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n\ :\ |v| = 1\}$. We denote by $\mathcal{L}^n$ the n-th dimensional Lebesgue measure, and for any set $E \subseteq \Omega$ we write $|E| := \mathcal{L}^n(E)$. We usually omit the variable of integration when writing an integral: for instance, given two functions $f : \Omega \times \mathbb{R} \to \overline{\mathbb{R}}$ and $u : \Omega \to \mathbb{R}$ such that $x \mapsto f(x,u(x))$ is integrable over Ω, we write its integral as $\int_\Omega f(x,u)\,\mathrm{d}x$ omitting the variable x. We denote with $N \in \mathcal{N}(\Omega)$ a subset $N \subset \Omega$ such that $|N| = 0$ and we say that $\gamma \in \mathrm{Lip}(\Omega)$ is *transversal* to N, if

$$\mathcal{L}^1(\{t \in [0,1] \,:\, \gamma(t) \in N\}) = 0, \tag{1.1.5}$$

denoting it with $\mathcal{P}(\Omega, N)$. ■

1.2 Metrics in the Euclidean space

Let us begin introducing a key concept in order to achieve the most important results: the class of metrics $\mathcal{M}(\Omega)$.

Definition 1.2.1. *We denote with $\mathcal{M}(\Omega)$ the class of maps $\varphi : \Omega \times \mathbb{R}^n \longrightarrow [0,+\infty)$ that satisfy the following conditions:*

(1) *$\varphi(\cdot, v)$ is measurable for all $v \in \mathbb{R}^n$ and $\varphi(x,\cdot)$ is continuous for a.e. $x \in \Omega$;*

(2) *$\varphi(x, \lambda v) = |\lambda|\varphi(x,v)$ for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$;*

(3) *there exist $c, C > 0$ with $c < C$ such that $c|v| \leq \varphi(x,v) \leq C|v|$.*

We say that $\varphi \in \mathcal{M}(\Omega)$ is *convex* if, for almost all $x \in \Omega$, the function $v \mapsto \varphi(x,v)$ is convex. This means that for almost all $x \in \Omega$ and all $v_1, v_2 \in \mathbb{R}^n$:

$$\varphi(x, v_1 + v_2) \leq \varphi(x, v_1) + \varphi(x, v_2). \tag{1.2.1}$$

Classically, every $\varphi \in \mathcal{M}(\Omega)$ and convex is called *Finsler metric* but, for the reader convenience, we will omit the term Finsler, when it is clear that we refer to this definition. The literature on the subject is wide and an introduction is supplied, for instance, by [10, 22]. In particular, it is shown in [40] that *smooth* Finsler metrics are dense in $\mathcal{M}(\Omega)$. Moreover, since weak regularity assumptions were of physical interest, measurable Finsler metrics have been deeply studied in [41, 42, 46–48]. Now, we recall the notion of lower and upper semicontinuity in the Euclidean space.

Definition 1.2.2. *Given a map $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$, we say that φ is* upper *or* lower *semi-continuous at the point $(x, v) \in \Omega \times \mathbb{R}^n$ if, for every sequence $(x_n)_n$ converging to x in Ω, we have that $\varphi(x, v) \geq \limsup_{n\to\infty} \varphi(x_n, v)$ or $\varphi(x, v) \leq \liminf_{n\to\infty} \varphi(x_n, v)$.*

In this kind of analysis, in order to infer some properties about metrics, an auxiliary and very important object is the *dual metric* (or sometimes called *conjugate* function).

Definition 1.2.3. *Given a metric $\varphi \in \mathcal{M}(\Omega)$ the* dual metric *$\varphi^\star : \Omega \times \mathbb{R}^n \to [0, +\infty)$ is defined by*

$$\varphi^\star(x, v) \coloneqq \sup\left\{ \frac{|\langle v, w\rangle|}{\varphi(x, w)} \;:\; w \in \mathbb{R}^n \setminus \{0\} \right\}. \tag{1.2.2}$$

Theorem 1.2.4. *Let $\varphi \in \mathcal{M}(\Omega)$. Then the following properties hold:*

i) *$\varphi^\star \in \mathcal{M}(\Omega)$ and it is convex;*

ii) *$\varphi^\star(x, \cdot)$ is a Lipschitz function for a.e. $x \in \Omega$;*

iii) *$|\langle w, v\rangle| \leq \varphi^{\star\star}(x, v)\varphi^\star(x, w) \leq \varphi^\star(x, w)\varphi(x, v)$ for every $x \in \Omega$ and $v, w \in \mathbb{R}^n$;*

iv) *if φ is upper (lower) semicontinuous then $\varphi^\star$ is lower (upper) semicontinuous;*

v) *$\varphi^\star(x, v) = \sup_w \big\{\langle w, v\rangle \;:\; \varphi(x, w) \leq 1\big\}$.*

Proof. i) First, we want to prove that $\varphi^\star \in \mathcal{M}(\Omega)$. Let us choose a dense sequence $(w_n)_n$ in $\mathbb{R}^n \setminus \{0\}$. For every $x \in \Omega$ and thus for any $v \in \mathbb{R}^n$ we can write

$$\varphi^\star(x, v) = \sup_{n\in\mathbb{N}} \frac{|\langle v, w_n\rangle|}{\varphi(x, w_n)}.$$

This proves that $\Omega \ni x \mapsto \varphi^\star(x, v)$ is measurable. It is straightforward to show the 1-homogeneity with respect to $v \in \mathbb{R}^n$.
In order to prove the estimate (3) of Definition 1.2.1, let us fix $w \in \mathbb{R}^n$. Then, it holds that

$$\frac{1}{C}\frac{|\langle v, w\rangle|}{|w|} \leq \frac{|\langle v, w\rangle|}{\varphi(x, w)} \leq \frac{1}{c}\frac{|\langle v, w\rangle|}{|w|},$$

and we can take the supremum over all $w \in \mathbb{R}^n \setminus \{0\}$, obtaining the desired estimate. The convexity follows since, taking $x \in \Omega$ and $v_1, v_2 \in \mathbb{R}^n$, we clearly have $|\langle v_1 + v_2, w\rangle| \leq |\langle v_1, w\rangle| + |\langle v_2, w\rangle|$. Dividing both by $\varphi(x, w)$ and passing to the supremum in both the sides, we get the thesis.

Therefore, $\varphi^\star(x, \cdot)$ is a norm, thus in particular it is continuous and hence $\varphi^\star \in \mathcal{M}(\Omega)$.

ii) From the latter we get that $\varphi^\star(x, \cdot)$ is Lipschitz.

iii) The claim comes from the definition of $\varphi^\star$ and the expression of the so-called bidual metric $\varphi^{\star\star}$.

iv) We will prove the first claim since the opposite is very similar. Let $(x_n)_n \subset \Omega$ a sequence of points converging to $x \in \Omega$. We must check that

$$\limsup_{n \to +\infty} \varphi^\star(x_n, v) \leq \varphi^\star(x, v) \quad \text{ for every } v \in \mathbb{R}^n.$$

Without loss of generality, we may assume that the lim sup is in fact a limit. Let us consider a sequence $(w_n)_{n\in\mathbb{N}} \subset \mathbb{R}^n$ such that $\varphi(x_n, w_n) = 1$ and $\varphi^\star(x_n, v) = |\langle v, w_n\rangle|$. Since φ satisfies assumption (3) of Definition 1.2.1, then $(w_n)_n$ is bounded and we may extract a subsequence $(w_{n_k})_k$ converging to some $w \in \mathbb{R}^n$. Since φ is lower semicontinuous, we get that

$$\lim_{n\to+\infty} \varphi^\star(x_n, v) = \lim_{n\to+\infty} |\langle v, w_n\rangle| = |\langle v, w\rangle| \leq \varphi(x, v)$$

which concludes the proof.

v) By the 1-homogeneity of the metric in $\mathcal{M}(\Omega)$, the assertion easily follows. $\square$

Proposition 1.2.5. *Let us consider $\varphi \in \mathcal{M}(\Omega)$. Then φ is convex if and only if $\varphi(x, v) = \varphi^{\star\star}(x, v)$ for every $(x, v) \in \Omega \times \mathbb{R}^n$.*

Proof. $\boxed{\Leftarrow}$ Since φ is 1-homogeneous in the second entry, in order to prove that $\varphi(x, \cdot)$ is convex, it is sufficient to prove that $\varphi(x, v_1 + v_2) \leq \varphi(x, v_1) + \varphi(x, v_2)$ for every $x \in \Omega$ and $v_1, v_2 \in \mathbb{R}^n$. By assumption $\varphi(x, v) = \varphi^{\star\star}(x, v)$ for all $(x, v) \in \Omega \times \mathbb{R}^n$, then for every

$v_1, v_2 \in \mathbb{R}^n$ we can write

$$\begin{aligned}\varphi(x, v_1 + v_2) &= \varphi^{\star\star}(x, v_1 + v_2) \\ &\leq \sup\left\{ \frac{|\langle v_1, w\rangle_x|}{\varphi^\star(x, w)} + \frac{|\langle v_2, w\rangle_x|}{\varphi^\star(x, w)} \ : \ w \in \mathbb{R}^n, \ w \neq 0 \right\} \\ &\leq \varphi^{\star\star}(x, v_1) + \varphi^{\star\star}(x, v_2) \\ &= \varphi(x, v_1) + \varphi(x, v_2).\end{aligned}$$

$\boxed{\Rightarrow}$ Given that $\varphi(x, \cdot)$ is convex and 1-homogeneous, $\varphi(x, \cdot)$ is a norm on $\mathbb{R}^n$ and $\varphi^{\star\star}(x, \cdot)$ represents its bidual norm. Since the space is finite-dimensional, the conclusion follows. □

1.2.1 Class of distances $\mathcal{D}(\Omega)$ and metric derivative

The forthcoming class is nothing but the set of geodesic distances locally equivalent to the Euclidean one.

Definition 1.2.6. *We say that a distance $d : \Omega \times \Omega \to [0, +\infty)$ belongs to $\mathcal{D}(\Omega)$ if it satisfies the following assumptions:*

i) *d is geodesic, as in* (1.1.2)*;*

ii) *there exist $c, C > 0$ with $c < C$ such that*

$$c|x - y| \leq d(x, y) \leq C|x - y| \quad \textit{for every } x, y \in \Omega. \tag{1.2.3}$$

We endow $\mathcal{D}(\Omega)$ with the metric given by the uniform convergence of compact subsets of $\Omega \times \Omega$ and we denote it with $d_n \to d$ in $\Omega \times \Omega$.

Remark 1.2.7. Notice that we may have $\mathcal{D}(\Omega) = \emptyset$ if the domain Ω is disconnected or it has an irregular boundary. However, the bound $c > 0$ in (1.2.3) implies that d is non-degenerate and, when Ω is closed, then $\mathcal{D}(\Omega)$ is a metrizable compact space. (cf. [28, Theorem 3.1]). Clearly, any distance in $\mathcal{D}(\Omega)$ induces on Ω a topology which is equivalent to the Euclidean one.

Now we are ready to introduce a central object for our analysis, which plays the same role of the classical metric derivative (1.1.3).

Definition 1.2.8 (Metric Derivative)**.** *Let $d \in \mathcal{D}(\Omega)$. Then the* metric derivative *is the map $\varphi_d : \Omega \times \mathbb{R}^n \to [0, +\infty)$ defined by*

$$\varphi_d(x, v) := \limsup_{t \to 0^+} \frac{d(x, x+tv)}{t}.$$

Lemma 1.2.9. *Let $d \in \mathcal{D}(\Omega)$. Then the following properties hold:*

(a) *$\varphi_d(\cdot, v)$ is measurable for every $v \in \mathbb{R}^n$ and φ_d is 1-homogeneous;*

(b) *$c|v| \le \varphi_d(x, v) \le C|v|$ for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$;*

(c) *$|\varphi_d(x, v_1) - \varphi_d(x, v_2)| \le C|v - w|$ for every $x \in \Omega$ and $v_1, v_2 \in \mathbb{R}^n$.*

Proof. (a) By definition, $x \mapsto \varphi_d(x, v)$ may be regarded as the limit of lower semicontinuous functions and, as a consequence, a Borel measurable map with respect to x. Moreover, for every fixed $\lambda > 0$ we have that:

$$\varphi_d(x, \lambda v) = \limsup_{t \to 0^+} \frac{d(x, x+t\lambda v)}{t} = \lambda \limsup_{t \to 0^+} \frac{d(x, x+t\lambda v)}{t\lambda} = \lambda \varphi_d(x, v).$$

If $\lambda < 0$, the proof is similar performing a change of variable.

(b) We know that we can pass to the limsup as below

$$\limsup_{t \to 0^+} \frac{c|tv|}{t} \le \limsup_{t \to 0^+} \frac{d(x, x+tv)}{t} \le \limsup_{t \to 0^+} \frac{C|tv|}{t},$$

and we obtain the thesis. Therefore, φ_d belongs to $\mathcal{M}(\Omega)$.

(c) Since $d \in \mathcal{D}(\Omega)$, for every $v_1, v_2 \in \mathbb{R}^n$ it holds that

$$d(x, x+tv_1) \le d(x, x+tv_2) + d(x+tv_2, x+tv_1) \le d(x, x+tv_2) + tC|v_2 - v_1|$$

whence $\varphi_d(x, v_1) \le \varphi_d(x, v_2) + C|v_2 - v_1|$ and interchanging the role of v_2 and v_1 we get $|\varphi_d(x, v_1) - \varphi_d(x, v_2)| \le C|v_2 - v_1|$.

$\square$

In the sequel, for any $d \in \mathcal{D}(\Omega)$ and $a \in \Omega$, we denote with $d_a : \Omega \to [0, \infty)$ the fixed-point distance map $d_a(x) \coloneqq d(a, x)$. Clearly, d_a is Lipschitz with respect to Euclidean distance and then, by Rademacher's Theorem, d_a is differentiable for a.e. $x \in \Omega$.

Proposition 1.2.10. *Let $d \in \mathcal{D}(\Omega)$, and let $a \in \Omega$ be a fixed point. Then it holds that:*

$$\|\varphi_d^\star(x, Dd_a(x)\|_\infty \leq 1, \quad \textit{for every } x \in \Omega.$$

Proof. For almost every $x \in \Omega$ and for all $v \in \mathbb{R}^n$ we have that $d_a(x + tv) - d_a(x) = t\langle Dd_a(x), w\rangle + o(|t|)$, from which we get

$$\langle Dd_a(x), v\rangle + \frac{o(|t|)}{t} = \frac{d(a, x + tv) - d(a, x)}{t} \leq \frac{d(x, x + tv)}{t}.$$

Therefore, passing to the limsup for $t \to 0$ we get that:

$$\langle Dd_a(x), v\rangle \leq \varphi_d(x, v) \;\Rightarrow\; \varphi_d^\star(x, Dd_a(x)) \leq 1.$$

□

Definition 1.2.11. *If $\varphi \in \mathcal{M}(\Omega)$ we denote the* length functional $\mathbb{L}_\varphi$ *by the formula*

$$\mathbb{L}_\varphi(\gamma) \coloneqq \int_0^1 \varphi(\gamma(t), \dot\gamma(t))\mathrm{d}t, \quad \gamma \in \mathrm{Lip}([0, 1], \Omega). \tag{1.2.4}$$

Since the map $t \mapsto (\gamma(t), \dot\gamma(t))$ is Borel measurable on $\Omega \times \mathbb{R}^n$ and φ satisfies (1) in Definition 1.2.1, then $\mathbb{L}_\varphi(\gamma)$ is well-defined.

Remark 1.2.12. By assumption (2) of Definition 1.2.1, $\mathbb{L}_\varphi(\gamma)$ does not depend on the way the Lipschitz curve γ is parametrized, namely, if $\eta = \gamma \circ \rho$, where $\rho : \mathbb{R} \to \mathbb{R}$ is a Lipschitz diffeomorphism, then $\mathbb{L}_\varphi(\gamma) = \mathbb{L}_\varphi(\eta)$. For this reason, it is not restrictive to assume γ to be defined on $[0, 1]$. Further, let us notice that $\mathbb{L}_\varphi$ is lower semicontinuous on $\mathrm{Lip}([0, 1], \bar\Omega)$, where $\bar\Omega$ is the topological closure of Ω. This happens, for instance, when φ is lower semicontinuous on $\Omega \times \mathbb{R}^n$ and φ is convex for every $x \in \bar\Omega$ (cf. [24, Theorem 4.1.1]).

Now we present the *length representation result* and we give the proof in a simpler formulation with respect to [46, Theorem 2.5]. Again, we recall that this is a particular case of equality (1.1.4) and [40, Theorem 1.3], in the setting of metric spaces.

Theorem 1.2.13. *Let $d \in \mathcal{D}(\Omega)$, then for every $\gamma \in \mathrm{Lip}(\Omega)$ it holds that:*

$$L_d(\gamma) = \mathbb{L}_{\varphi_d}(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot{\gamma}(t))\mathrm{d}t. \tag{1.2.5}$$

In particular, $d = d_{\varphi_d}$.

Proof. We adapt the proof contained in [46, Theorem 2.5] in the Euclidean space. Pick a partition of $[0,1]$ and, since γ is Lipschitz, we have that $\mathrm{d}(\gamma(t_i), \gamma(t))$ are differentiable a.e. $t \in [0,1]$ and hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} d(\gamma(t_i), \gamma(t)) &= \lim_{h \to 0} \frac{d(\gamma(t_i), \gamma(t) + \dot{\gamma}(t_i)h + o(h)) - d(\gamma(t_i), \gamma(t))}{h} \\ &\leq \liminf_{h \to 0} \frac{d(\gamma(t), \gamma(t) + \dot{\gamma}(t_i)h)}{h}.\end{aligned}$$

This implies that

$$d(\gamma(t_i), \gamma(t_{i+1})) \leq \int_{t_i}^{t_{i+1}} \liminf_{h \to 0} \frac{d(\gamma(t), \gamma(t) + \dot{\gamma}(t_i)h)}{h} \mathrm{d}t$$

for which

$$L_d(\gamma) \leq \int_0^1 \liminf_{h \to 0} \frac{d(\gamma(t), \gamma(t+h))}{h} \mathrm{d}t. \tag{1.2.6}$$

Since the function $L_d(\gamma(t)) = L_d(\gamma|_{[0,t]})$ is increasing, applying a corollary of Fatou Lemma, in the differentiability points of $L_d(\gamma(t))$ and $\gamma(t)$ we have

$$\begin{aligned}L_d(\gamma) \geq \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} L_d(\gamma(t))\mathrm{d}t &= \int_0^1 \lim_{h \to 0} \frac{L_d(\gamma(t+h)) - L_d(\gamma(t))}{h} \, \mathrm{d}t \\ &\geq \int_0^1 \limsup_{h \to 0} \frac{d(\gamma(t), \gamma(t+h))}{h} \mathrm{d}t.\end{aligned}$$

Now by (1.2.6) we have,

$$\int_0^1 \limsup_{h \to 0} \frac{d(\gamma(t), \gamma(t+h))}{h} \mathrm{d}t \leq L_d(\gamma) \leq \int_0^1 \liminf_{h \to 0} \frac{d(\gamma(t), \gamma(t+h))}{h} \mathrm{d}t,$$

but this is false and then the assertion follows.

To get the second assertion, first we have to pass to the supremum over all the partitions on the right hand side of (1.2.5). After that we compute the infimum over all $\gamma \in \mathrm{Lip}([0,1], \Omega)$ on the left hand side of (1.2.5). Since d is a geodesic distance we obtain the thesis. $\square$

1.2.2 Metric Derivative's convexity in $\mathbb{R}^n$

The main goal of this section is to prove that, given a distance in $\mathcal{D}(\Omega)$ the induced metric derivative is a convex metric in the sense of Definition 1.2.1. This means that $\varphi_d(x,\cdot)$ is a norm for every fixed $x \in \Omega$ and, in order to show it, we recall two results proved in [94, Lemma 3.1, Lemma 3.2].

Lemma 1.2.14. *Let $d \in \mathcal{D}(\Omega)$ and let $\varphi \in \mathcal{M}(\Omega)$ be upper semicontinuous. Let $N \in \mathcal{N}(\Omega)$ and let us suppose that for every $\gamma \in \mathcal{P}(\Omega, N)$ it holds*

$$d(\gamma(0),\gamma(1)) \leq \int_0^1 \varphi(\gamma(t),\dot\gamma(t))\mathrm{d}t.$$

Then for every fixed $a \in \Omega$, for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$ we have:

$$|\langle Dd_a(x), v\rangle| \leq \liminf_{t\to 0} \frac{d(x,x+tv)}{t} \leq \limsup_{t\to 0} \frac{d(x,x+tv)}{t} \leq \varphi(x,v).$$

Proof. Fix $v \in \mathbb{R}^n$ and for $a \in \Omega$, let E_a^v be the full measure set where d_a is differentiable and let $\gamma(t) = x + tv$ the curve belonging to $\mathcal{P}(\Omega, N)$. If t is small enough, it holds that

$$\varphi(x,v) = \lim_{t\to 0}\frac{1}{t}\int_0^t \varphi(x+sv,v)\mathrm{d}s.$$

By Rademacher's Theorem $|\Omega \setminus E_a^v| = 0$ and, when $x \in E_a^v$, we have that

$$\begin{aligned}|\langle Dd_a(x), v\rangle| &= \lim_{t\to 0}\left|\frac{d(a,x+tv)-d(a,x)}{t}\right| \leq \liminf_{t\to 0}\frac{d(x,x+tv)}{|t|}\\ &\leq \limsup_{t\to 0}\frac{d(x,x+tv)}{|t|} \leq \lim_{t\to 0}\frac{1}{t}\int_0^t \varphi(x+sv,v)\mathrm{d}s = \varphi(x,v).\end{aligned}$$

Pick a countable dense subset $F \subset \mathbb{R}^n$ and set $E(a) = \cap_{y\in F} E_a^y$. Then $|\Omega \setminus E(a)| = 0$ and for every $x \in E(a)$ and all $v \in \mathbb{R}^n$ we obtain the estimate

$$|\langle Dd_a(x), v\rangle| \leq \liminf_{t\to 0} \frac{d(x,x+tv)}{|t|} \leq \limsup_{t\to 0} \frac{d(x,x+tv)}{|t|} \leq \varphi(x,v).$$

By the upper semicontinuity of φ, the above inequalities hold for almost all $x \in \Omega$ and for every $v \in \mathbb{R}^n$. $\square$

Theorem 1.2.15. *Suppose $d \in \mathcal{D}(\Omega)$. Then φ_d is a convex metric and for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$, it holds*

$$\varphi_d(x, v) = \lim_{t\to 0} \frac{d(x, x + tv)}{|t|}.$$

Proof. We follow a slightly modified version of the proof in [42, Theorem 1.11].
Let $G \subset \Omega$ be a countable dense subset in Ω. For each $x \in \Omega$, let E_a the negligible Borel subset of Ω containing the points where d_a is not differentiable. For every $(x, v) \in \Omega \times \mathbb{R}^n$ we set

$$\xi(x, v) = \begin{cases} \sup_{a\in G} |\langle Dd_a(x), v\rangle| & \text{if } \ x \in \Omega \setminus E_a \\ 0 \quad \text{otherwise.} \end{cases}$$

Clearly $\xi \in \mathcal{M}(\Omega)$ it is convex and upper semicontinuous. Now, considering $E := \cup_{a\in G} E_a$, since G is countable we have that $|E| = 0$. Hence for every $\gamma \in \mathcal{P}(\Omega, E)$ we obtain:

$$\begin{aligned} d(\gamma(0), d(\gamma(1))) &= \sup_{a\in G} \Big(d_a(\gamma(1)) - d_a(\gamma(0)) \Big) = \sup_{a\in G} \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} d_a(\gamma(t))\mathrm{d}t \\ &= \sup_{a\in G} \int_0^1 |\langle Dd_a(\gamma(t)), \dot\gamma(t)\rangle| \,\mathrm{d}t \le \int_0^1 \xi(\gamma(t), \dot\gamma(t))\mathrm{d}t. \end{aligned}$$

Therefore we can apply Lemma 1.2.14 to obtain

$$|\langle Dd_a(x), v\rangle| \le \liminf_{t\to 0} \frac{d(x, x + tv)}{|t|} \le \limsup_{t\to 0} \frac{d(x, x + tv)}{|t|} \le \xi(x, v),$$

for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$. The claim then easily follows by taking the supremum over $a \in G$ on the left-hand side term. $\square$

1.3 A quick overview on Intrinsic Distances

Now, in order to develop the analytic study of the class of intrinsic distances, we introduced the notion given by De Cecco and Palmieri in [47, Definition 1.4]. In particular, considering only the infimum over Lipschitz curves connecting two points of Ω, the quantity we get is only a pseudodistance. In order to obtain a distance we need further to compute a supremum over all curves transversal to some negligible set.

Definition 1.3.1. *Given any $\varphi \in \mathcal{M}(\Omega)$, we define the induced* intrinsic distance *as*

$$d_\varphi(x,y) \coloneqq \sup_N \inf_\gamma \int_0^1 \varphi(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \qquad \text{for every } x,y \in \Omega, \tag{1.3.1}$$

where the supremum is taken over all subsets $N \subset \Omega$ such that $|N| = 0$ and the infimum is taken over all curves belonging to $\mathcal{P}(\Omega, N)$ and joining x and y.

The quantity $d_\varphi(x,y)$ is well-defined and (1.3.1) is invariant with respect to modifications of the metric φ on negligible subsets of $\bar\Omega$.

Remark 1.3.2. It is well know that $L_{d_\varphi}(\gamma) \le \mathbb{L}_\varphi(\gamma)$ for any $\gamma \in \mathrm{Lip}(\Omega)$ (see for instance [42, Theorem 1.18]). This yields to the inequality $\varphi_{d_\varphi}(x,v) \le \varphi(x,v)$ for every $(x,v) \in \Omega \times \mathbb{R}^n$, but this can be strict when φ is not continuous, see e.g. [46, Example 5.1]. Moreover, if $\varphi \in \mathcal{M}(\Omega)$ and when $c > 0$ in (1.2.3), in [42, Proposition 2.7] the authors showed that actually L_{d_φ} can be characterized as the *relaxed functional* of $\mathbb{L}_\varphi$ on $\mathrm{Lip}([0,1],\bar\Omega)$. In other words it holds that

$$L_{d_\varphi}(\gamma) = \inf\Big\{\liminf_{n\to+\infty} \mathbb{L}_\varphi(\gamma_n) \ : \ (\gamma_n)_n \subset \mathrm{Lip}(\bar\Omega) \quad \gamma_n \to \gamma \ \text{ in } \ \mathrm{Lip}(\bar\Omega)\Big\}.$$

Lemma 1.3.3. *If $\varphi \in \mathcal{M}(\Omega)$ is a convex metric, then $d_\varphi \in \mathcal{D}(\Omega)$.*

Proof. Thanks to [46, Theorem 3.7] we know that d_φ is geodesic, therefore we need to prove that it is a distance. First of all, $d_\varphi(x,y) \ge 0$ for every $x,y \in \Omega$ since the integral of $\varphi(\gamma(\cdot),\dot\gamma(\cdot))$ is non-negative. In order to prove the symmetry, let us consider $\gamma \in \mathrm{Lip}([0,1],\Omega)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Set $\xi : [0,1] \to \Omega$ as $\xi(t) = \gamma(1-t)$, hence this is a horizontal curve in $[0,1]$. By the 1-homogeneity of $\varphi(x,\cdot)$, we get that

$$\begin{aligned}\int_0^1 \varphi(\xi(t),\dot\xi(t))\,\mathrm{d}t &= \int_0^1 \varphi(\gamma(1-t),-\dot\gamma(1-t))\,\mathrm{d}t = \int_0^1 \varphi(\gamma(s),-\dot\gamma(s))\,\mathrm{d}s \\ &= \int_0^1 \varphi(\gamma(s),\dot\gamma(s))\,\mathrm{d}s.\end{aligned}$$

So now, passing to the infimum over $\gamma \in \mathrm{Lip}([0,1],\Omega)$ we get that $d_\varphi(x,y) = d_\varphi(y,x)$.

To prove the triangle inequality, let $x, y, z \in \Omega$ and $\gamma_1, \gamma_2 \in \mathrm{Lip}([0,1], \Omega)$ be such that $\gamma_1(0) = x, \gamma_1(1) = y = \gamma_2(0)$, and $\gamma_2(1) = z$. Let us define the following curve:

$$\eta : [0,1] \to \Omega, \quad \eta(t) := \begin{cases} \gamma_1(2t) & \text{if } t \in [0, \frac{1}{2}]; \\ \gamma_2(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Then we obtain that

$$\begin{aligned} d_\varphi(x,z) \le \int_0^1 \varphi(\eta(t), \dot\eta(t))\,\mathrm{d}t &= \int_0^{\frac{1}{2}} \varphi(\gamma_1(2t), 2\dot\gamma_1(2t))\,\mathrm{d}t + \int_{\frac{1}{2}}^1 \varphi(\gamma_2(2t-1), 2\dot\gamma_2(2t-1))\,\mathrm{d}t \\ &= \int_0^1 \varphi(\gamma_1(s), \dot\gamma_1(s))\,\mathrm{d}s + \int_0^1 \varphi(\gamma_2(s), \dot\gamma_2(s))\,\mathrm{d}s, \end{aligned}$$

where in both integrals we applied a change of variable and the 1-homogeneity of φ. For $i = 1, 2$, passing firstly to the infimum respectively over all $\gamma_i \in \mathcal{P}(\Omega, N_i)$, and to the supremum with respect to N_i such that $|N_i| = 0$, we get the statement. We are left to prove property (3) of Definition 1.2.4. Let $x, y \in \Omega$ and let $\gamma : [0,1] \to \Omega$ be a Lipschitz curve joining x and y. Pick a null set N maximizing (2.5.2) then, by (i) of Theorem 1.2.4, we get that

$$\int_0^1 \varphi(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \le C \int_0^1 |\dot\gamma(t)|\,\mathrm{d}t.$$

Thus, passing to the infimum in the right-hand side we obtain the conclusion and the converse inequality can be achieved by arguing in a similar way. $\square$

The last theorem, proved in [21, Theorem 3.1], asserts that it is possible to drop the condition of supremum in Definition 1.3.1. The idea is contained in the following lemma which states that any Lipschitz curve can be approximated with a transversal one which connects the same end-points (see [21, Lemma 3.2]).

Lemma 1.3.4. *Let $\gamma \in \mathrm{Lip}(\Omega)$ joining $x, y \in \bar\Omega$ and let $N \in \mathcal{N}(\Omega)$. Then for every $\varepsilon > 0$ there exists $\gamma_\varepsilon \in \mathcal{P}(\Omega, N)$ joining x and y, and such that*

$$\sup_{t\in[0,1]} |\gamma(t) - \gamma_\varepsilon(t)| + \sup_{t\in[0,1]} |\dot\gamma(t) - \dot\gamma_\varepsilon(t)| < \varepsilon.$$

Theorem 1.3.5. *If $\varphi \in \mathcal{M}(\Omega)$, then there exists $V \in \mathcal{N}(\Omega)$ such that*

$$d_\varphi(x, y) = \inf_{\gamma \in \mathcal{P}(\Omega, V)} \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t)) \mathrm{d}t.$$

The existence of the negligible set V is guaranteed by [31, Proposition 3.5]. Moreover, Theorem 1.3.5 tells also that, if one sets the metric

$$\widetilde{\varphi}(x, v) \coloneqq \varphi(x, v)\chi_{\bar{\Omega}\setminus V}(x) + C|v|\chi_V(x),$$

one gets $d_\varphi = d_{\widetilde{\varphi}}$. This implies that the set of intrinsic distances is a proper subset of $\mathcal{D}(\Omega)$.

1.3.1 Relation between φ and φ_{d_φ}

In [94, Theorem 4.3] we can find a simple property that links the norm of the metric derivative and the bound between φ and φ_{d_φ}.

Remark 1.3.6. Let $d \in \mathcal{D}(\Omega)$ and $\varphi \in \mathcal{M}(\Omega)$, then we have that

$$\varphi_d(x, v) \le \varphi(x, v) \iff \varphi_d^\star(x, v) \ge \varphi^\star(x, v) \quad \text{for a.e. } x \in \Omega \text{ and for every } v \in \mathbb{R}^n.$$

Moreover, these are equivalent to $\|\varphi^\star(x, Dd_a(x))\|_\infty \le 1$, for every fixed $a \in \Omega$.

Given a metric $\varphi \in \mathcal{M}(\Omega)$, one could ask when the metric derivative induced by the intrinsic distance d_φ coincides with the original metric. The following example shows that it is possible to construct a distance d_φ in $\mathcal{D}(\Omega)$ such that φ_{d_φ} does not come from a scalar product. This is due to the possible lack of regularity of metrics in $\mathcal{M}(\Omega)$.

Example 1.3.7. Let $E \coloneqq \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\}$ and define

$$\varphi(x, v) \coloneqq a(x)|v|, \quad \text{with } a(x) = \chi_E(x) + \beta\chi_{\mathbb{R}^2\setminus E}(x).$$

If $C > 0$ is such that

$$C\sqrt{|x_1|^2 + |x_2|^2} \ge |x_1| + |x_2| \quad \text{for every } x \in \mathbb{R}^2,$$

the intrinsic distance is $d_\varphi(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Hence, we obtain that $\varphi_{d_\varphi}(x, v) = |v_1| + |v_2|$, so that $\varphi_{d_\varphi}(x, v) \neq \varphi(x, v)$ everywhere.

Even if φ_{d_φ} need not be equal to φ, some relations between them were studied and now we present some results in this direction. The first one, and the relative proof, are contained in [21, Proposition 2.9].

Proposition 1.3.8. *For every $\varphi \in \mathcal{M}(\Omega)$, it holds that*

$$\varphi_{d_\varphi}(x, v) \leq \varphi(x, v) \quad \textit{for a.e. } x \in \Omega \ \textit{ and for every } \ v \in \mathbb{R}^n. \tag{1.3.2}$$

Moreover, if φ is upper semicontinuous, (1.3.2) *holds for every $x \in \Omega$ and $v \in \mathbb{R}^n$.*

Proof. Let us fix $v \in \mathbb{S}^{n-1}$ and, for every $a \in \Omega$, let us set $\gamma_a(s) := a + sv$. For $h > 0$ small enough we have

$$\frac{1}{h}\int_t^{t+h} \varphi(\gamma_a(s), v)\mathrm{d}s = \frac{1}{h}\int_0^1 \varphi(\gamma_a(t + hr), hv)\mathrm{d}r \geq \frac{d(\gamma_a(t), \gamma_a(t) + hv)}{h},$$

where t is a Lebesgue point for the map $s \mapsto \varphi(\gamma_a(s), v)$. Hence, by taking the limsup as $h \to 0^+$, we get that $\varphi_d \leq \varphi$ for all the points of γ_a. Since we choose arbitrarily the point $a \in \Omega$ and the Lebesgue point $t \in \mathbb{R}$, by Fubini's Theorem we get that $\varphi_d(x, v) \leq \varphi(x, v)$ for a.e. $x \in \Omega$. Taking a dense sequence $(v_n)_{n\in\mathbb{N}} \in \mathbb{S}^{n-1}$, repeating the argument above, by density $\varphi_{d_\varphi}(x, v) \leq \varphi(x, v)$ for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$ and hence the thesis.
Assuming that φ is upper semicontinuous, fixing $(x, v) \in \Omega \times \mathbb{S}^{n-1}$ there exists $r > 0$ such that $B_r(x) \subset \Omega$ and $\varphi(y, v) < \varphi(x, v) + \varepsilon$ for every $y \in B_r(x)$. For t small enough, $\gamma_t(s) := x + s(tv)$ stays in $B_r(x)$ and then we have

$$d(x, x + tv) \leq \int_0^1 \varphi(x + s(tv), tv)\mathrm{d}s \leq \int_0^1 (\varphi(x, tv) + \varepsilon t)\mathrm{d}s = t(\varphi(x, v) + \varepsilon),$$

and hence

$$\frac{d(x, x + tv)}{t} \leq \varphi(x, v) + \varepsilon.$$

By taking the limsup, since $\varepsilon > 0$, $x \in \Omega$ and $v \in \mathbb{S}^{n-1}$ are arbitrary, we obtain the claim. $\square$

On the other hand, when we consider a lower semicontinuous metric, the relation (1.3.2) changes. We present here the result contained in [21, Proposition 2.9 (ii)]) because it will be important in order to extend the claim in Chapter 2.

Theorem 1.3.9. *Let $d \in \mathcal{D}(\Omega)$. Then for every convex and lower semicontinuous metric $\varphi \in \mathcal{M}(\Omega)$, we have that:*

$$\varphi_{d_\varphi}(x,v) \geq \varphi(x,v) \quad \text{for every } (x,v) \in \Omega \times \mathbb{R}^n.$$

Proof. If $(x,v) \in \Omega \times \mathbb{S}^{n-1}$, by lower semicontinuity, for every $\varepsilon > 0$ there exists $r = r(\varepsilon, x) > 0$ such that $B_r(x) \subset \Omega$ and $\varphi(y,v) > \varphi(x,v) - \varepsilon$ for every $y \in B_r(x)$. Since φ is convex, by (ii) of Theorem 1.2.3 it is also Lipschitz and by possibly choosing a smaller r, the previous inequality holds in $B_r(x) \times B_r(v)$. Since $\mathbb{S}^{n-1}$ is compact, the same holds for every $(y,v) \in B_r(x) \times \mathbb{S}^{n-1}$ as well. Choosing a d-minimizing sequence of paths $(\gamma_n)_n \subset \mathrm{Lip}(\Omega)$, connecting x and $x + tv$, we have that $\gamma_n([0,t]) \subset B_r(x)$ for t small enough. If n is big enough we get

$$\mathbb{L}_\varphi(\gamma_n) \geq \int_0^1 (\varphi(x, \dot\gamma_n(s)) - \varepsilon|\dot\gamma_n(s)|)\mathrm{d}s \geq t\Big(\varphi(x,v) - 2\frac{C}{c}\varepsilon\Big),$$

where in the last estimate we applied Jensen's inequality applied to $\varphi(x,\cdot)$ in combination with $c\int_0^1 |\dot\gamma_n(s)\mathrm{d}s \leq \mathbb{L}_\varphi(\gamma_n) \leq 2d(x, x+tv) \leq 2Ct$, when n is large enough. Letting n goes to $+\infty$ in the above inequality we obtain that

$$\frac{d(x,x+tv)}{t} \geq \varphi(x,v) - 2\frac{C}{c}\varepsilon. \tag{1.3.3}$$

By taking the liminf of (1.3.3) as $t \to 0^+$ and since $\varepsilon > 0$, $x \in \Omega$ and $v \in \mathbb{S}^{n-1}$ were arbitrary, we get

$$\varphi_{d_\varphi}(x,v) \geq \liminf_{t\to 0^+} \frac{d(x,x+tv)}{t} \geq \varphi(x,v) \quad \text{for every } (x,v) \in \Omega \times \mathbb{S}^{n-1},$$

and the claim follows by 1-homogeneity in v. $\square$

1.4 Some results involving δ_φ

Whenever φ is a convex metric, it is possible to state a relation between Lipschitz functions $f : \Omega \to \mathbb{R}$ satisfying the bound $\|\varphi(x, Df(x))\|_\infty \leq 1$ and the following family of distances (cf. [45, 47]).

Definition 1.4.1. *If $\varphi \in \mathcal{M}(\Omega)$ is a convex metric, for every $x, y \in \Omega$ we define:*

$$\delta_\varphi(x,y) \coloneqq \sup\left\{|f(x)-f(y)| \;:\; f \in \mathrm{Lip}(\Omega),\ \|\varphi(x, Df(x))\|_\infty \leq 1\right\}. \tag{1.4.1}$$

Recall that Rademacher's Theorem grants that $Df(x)$ exists at almost every $x \in \Omega$ and thus the above definition makes sense. From now on, we will say that any Lipschitz function satisfying the conditions in (1.4.1) is a *competitor* for δ_φ. For instance, thanks to Proposition 1.2.10 it is easy to verify that the fixed-point distance d_a is a competitor for δ_{φ_d}. This is a direct consequence of Remark 1.3.6.

Lemma 1.4.2. *For every convex $\varphi \in \mathcal{M}(\Omega)$, we have that $\delta_\varphi : \Omega \times \Omega \to [0, +\infty)$ is a distance.*

Proof. Clearly, we have that $\delta_\varphi(x,y) \geq 0$ for every $x, y \in \Omega$ and $\delta_\varphi(x,y) > 0$ if $x \neq y$. The symmetry comes from the fact that $|f(x)-f(y)| = |f(y)-f(x)|$. Also, δ_φ satisfies the triangle inequality since for every $x, y, z \in \Omega$ we have $\delta_\varphi(x,y)+\delta_\varphi(y,z) \geq |f(x)-f(y)|+|f(y)-f(z)| \geq |f(x)-f(z)|$. Passing to the supremum on the right-hand side for every $f \in \mathrm{Lip}(\Omega)$ such that $\|\varphi(x, Df(x))\|_\infty \leq 1$, we get that $\delta_\varphi(x,y) + \delta_\varphi(y,z) \geq \delta_\varphi(x,z)$. $\square$

Albeit we already show the general inequality result in Proposition 1.3.8, we recall the inequality involving the metric derivative with respect to δ_φ. Indeed, in [65, Proposition 1.6] it is shown that $\varphi_{\delta_\varphi}(x,v) \leq \varphi(x,v)$ for a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^n$. The following example (see [65, Example 1.8]) proves that there exist a geodesic distance contained in $\mathcal{D}(\Omega)$ which is not intrinsic, in the sense of Definition 1.3.1.

Example 1.4.3. Let $\Omega = (-1,1) \times (-1,1)$ and consider the subset $S = (-1,1) \times \{0\}$. If we define the following metric as

$$\varphi(x,v) = \begin{cases} \beta|v| & \text{if } x \in \Omega \setminus S \\ \alpha|v| & \text{if } x \in S, \end{cases}$$

with $0 < \alpha < \beta$, then the distance d_φ is clearly different from $\beta|x-y|$. In particular, for many pairs of points (x,y) near S we have that $d(x,y) < \beta|x-y|$. However the metric derivative φ_{d_φ} coincides with $\beta|v|$ in $\Omega \setminus S$ and thus $\delta_{\varphi_{d_\varphi}}(x,y) = \beta|x-y|$. This means that the class of intrinsic distances is strictly contained in the set of distances $\mathcal{D}(\Omega)$. Another similar example of this phenomenon can be find in [21, Corollary 3.4].

Now, the forthcoming crucial theorem, proved in [47, Theorem 2.10], assures us that δ_φ coincides with the intrinsic distance, which is geodesic by Lemma 1.3.3. An alternative proof of the fact that δ_φ is a geodesic distance is given in [65, Theorem 3.9].

Theorem 1.4.4. *Let $\varphi \in \mathcal{M}(\Omega)$ be a convex and lower semicontinuous metric. Then we have that*

$$\delta_\varphi(x,y) = d_{\varphi^\star}(x,y) \quad \text{for every} \quad x,y \in \Omega. \tag{1.4.2}$$

Let us observe that the equality (1.4.2) can be stated in the dual way. Indeed, assuming $\varphi \in \mathcal{M}(\Omega)$ upper semicontinuous, then $\varphi^\star$ will be lower semicontinuous by Theorem 1.2.4 (iv). Therefore, by Theorem 1.4.4, we obtain that $\delta_{\varphi^\star}(x,y) = d_{\varphi^{\star\star}}(x,y) = d_\varphi(x,y)$ for every $x,y \in \Omega$. This is the reason why, in the literature, one can find the previous claims stated in the two different perspectives.

1.4.1 Equality among metric and Lipschitz pointwise constant

In this section we will state two results regarding the intrinsic analysis of metrics $\varphi \in \mathcal{M}(\Omega)$. Our purpose is to generalize them in the next Chapter, in the context of Carnot groups. First, we show a basic result contained in [66, Lemma 2.1].

Lemma 1.4.5. *Let $d \in \mathcal{D}(\Omega)$. Then for any $f \in \mathrm{Lip}_d(\Omega)$ it holds that*

$$\|Df\|_{L^\infty(\Omega)} = \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{d(x,y)}.$$

The previous lemma inspired the forthcoming result contained in [70, Theorem 5.2]. It states that the Lipschitz pointwise constant of a Lipschitz function f, with respect to the distance δ_φ, coincides with the metric evaluated in the direction of Df. It can be regarded as an improved version of [65, Proposition 2.4] from L^∞-norm to pointwise equality.

Theorem 1.4.6. *Let $\varphi \in \mathcal{M}(\Omega)$ be convex and upper semicontinuous on Ω. Then for every function $f \in \mathrm{Lip}_{\delta_\varphi}(\Omega)$ we have that*

$$\mathrm{Lip}_{\delta_\varphi} f(x) = \varphi(x, Df(x)) \quad \textit{for a.e. } x \in \Omega.$$

Proof. $\boxed{\leq}$ Let f be an arbitrary Lipschitz function with respect to δ_φ. Since both sides are positively 1-homogeneous with respect to f, we only need to show that if $\varphi(x, \nabla f(x)) \leq 1$ then $\mathrm{Lip}_{\delta_\varphi} f(x) \leq 1$. By the definition of δ_φ, if $\varphi(x, \nabla f(x)) \leq 1$, then $|f(x) - f(y)| \leq \delta_\varphi(x, y)$ for all $x, y \in \Omega$, which implies that

$$\sup_{x \in \Omega} \mathrm{Lip}_{\delta_\varphi} f(x) \leq 1$$

and we obtain the first inequality.

$\boxed{\geq}$ Thanks to the previous argument, we only need to show that for a.e. $x \in \Omega$, if $\mathrm{Lip}_{\delta_\varphi} f(x) \leq 1$, then $\varphi(x, \nabla f(x)) \leq 1$. Since φ is upper semicontinuous, we can apply Theorem 1.4.4 and we obtain that $\delta_\varphi(y, z) = d_{\varphi^\star}(y, z)$, for every $y, z \in \Omega$. Then $\mathrm{Lip}_{\delta_\varphi} f(x) = \mathrm{Lip}_{d_{\varphi^\star}} f(x)$ for a.e. $x \in \Omega$. Then, fixing such a point x and for each $v \in \mathbb{R}^n$, we can infer that

$$\begin{aligned}
\langle \nabla f(x), v \rangle &= \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} \\
&\leq \limsup_{t \to 0} \frac{d_{\varphi^\star}(x + tv, x)}{t} \limsup_{t \to 0} \frac{f(x + tv) - f(x)}{d_{\varphi^\star}(x + tv, x)} \\
&\leq \varphi_{d_{\varphi^\star}}(x, v) \, \mathrm{Lip}_{d_{\varphi^\star}} u(x) \leq \varphi^\star(x, v).
\end{aligned}$$

Then, since φ is convex we get that $\varphi(x, \nabla u(x)) = \varphi^{\star\star}(x, \nabla u(x)) \leq 1$ and hence the thesis. $\square$

Remark 1.4.7. We finally emphasize that the distance δ_φ has been analyzed in several cases. For instance, when $\varphi(x, v) = \left(\sum_{i,j} a_{ij} v_i v_j\right)^{\frac{1}{2}}$ with (a_{ij}) elliptic matrix in Ω, or when φ is a Riemannian metric on a Lipschitz manifold (see the next section). In this case, it has a relevant role in the study of the heat flow associated to Dirichlet forms on smooth and Lipschitz manifold (cf. [88]). More recently, in the context of diffusion problem, the authors of [74] establish when the intrinsic differential and the local intrinsic structures coincide.

1.5 Lipschitz manifolds and Finsler Structures

Most of the previous results and main ideas originally were stated in the general context of Riemannian and Lipschitz manifolds, by De Cecco and Palmieri (see [45, 46]).
Indeed, setting the length of a regular curve, they constructed an intrinsic geodesic distance (1.3.1) on a smooth Riemannian manifold (M, g). On the other hand, given a geodesic distance d on M, they recovered the metric g by differentiation, since the directional derivative along a vector coincides with the induced Riemannian norm evaluated on the same vector in the corresponding point. Indeed

Remark 1.5.1. Let (U, Φ) be a chart at $z \in \mathrm{M}$ where $x = \Phi(z)$ and let v be a vector of $\Phi(U) \subset \mathbb{R}^n$. Then, Definition 1.2.8 can be reformulated in the following:

$$\varphi_d(x, v) \coloneqq \lim_{t \to 0} \frac{d(\Phi^{-1}(x), \Phi^{-1}(x + tv))}{t} = \sqrt{\sum_{i,j=1}^{n} g_{ij}(z) v_i v_j}. \tag{1.5.1}$$

Equality (1.5.1) still holds whenever g_{ij} are measurable but it is not longer true when we consider only Borel measurable metrics. This phenomenon arises when we deal with the Lipschitz manifolds because, differently from the case of smooth manifold equipped of an irregular metric, one could also treat the case of singularities carried by the manifold.

Definition 1.5.2. *Let* M *be a connected oriented and locally compact manifold of dimension* n *and let* $\Lambda \subset \mathbb{N}$. *A LIP atlas* $\mathcal{A}$ *on* M *is a family of charts* $\big(U_\alpha, \Phi_\alpha\big)_{\alpha \in \Lambda}$ *where* $(U_\alpha)_{\alpha \in \Lambda}$ *is an open cover of* M *and* $\Phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ *is a diffeomorphism such that, for every* $\alpha, \beta \in \Lambda$ *the following map defines a diffeomorphism*

$$\Phi_{\alpha\beta} \coloneqq \Phi_\beta \circ \Phi_\alpha^{-1} : \Phi_\alpha(U_\alpha \cap U_\beta) \to \Phi_\beta(U_\alpha \cap U_\beta).$$

We say that M *is a* Lipschitz manifold *if it is equipped of an equivalence class of LIP atlases.*

Clearly, $\Phi_{\alpha\beta}$ is a Lipschitz map and therefore, by Rademacher's theorem it is differentiable a.e. $x \in \mathrm{M}$.

Example 1.5.3. Any smooth, in particular differentiable and any piecewise-linear manifold is a Lipschitz manifold.

Now we can introduce a reasonable generalization of the metric stated in Definition 1.2.1.

Definition 1.5.4. *A* Finsler structure F *on* M *is a collection of maps* $F_\alpha : V_\alpha \times \mathbb{R}^n \to [0, +\infty)$ *such that* $F_\alpha \in \mathcal{M}(V_\alpha)$ *for every* $\alpha \in \Lambda$ *and, for every* $\alpha, \beta \in \Lambda$, *the* compatibility condition *holds:*

$$F_\alpha(x, v) = F_\beta(\Phi_{\alpha\beta}(x), D\Phi_{\alpha\beta}(x)[v]) \quad \textit{for a.e. } x \in V_\alpha, \quad \forall\, v \in \mathbb{R}^n, \tag{1.5.2}$$

where $\Phi_\alpha^{-1}(x) \in U_\alpha \cap U_\beta$.

In other words, a Finsler structure is the collection $F = (F_\alpha)_{\alpha\in\Lambda}$ defined on every open set of any chart. This behaves well under the changes of chart and, by property (3) of Definition 1.2.1, for every $\alpha \in \Lambda$ there exist two positive constants c_α and C_α such that

$$c_\alpha |v| \le F_\alpha(x, v) \le C_\alpha |v| \quad \text{a.e. } x \in V_\alpha \text{ and for every } \ v \in \mathbb{R}^n. \tag{1.5.3}$$

Remark 1.5.5. We want to point out that the Definition 1.5.4 is not standard in the literature. Indeed, a Finsler structure, on a smooth manifold M, is given by a function $F : T\mathrm{M} \to \mathbb{R}$ that is smooth on the complement of the zero section of TM and such that the restriction of F to any tangent space $T_p\mathrm{M}$ is a symmetric norm. Moreover, this norm is not necessarily induced by a Riemannian scalar product product.

We denote with $\mathcal{AC}(\mathrm{M})$ the set of absolutely continuous curves $\gamma : [0, 1] \to \mathrm{M}$ joining two points $x, y \in \mathrm{M}$ such that $\Phi_\alpha \circ \gamma \in \mathcal{AC}(V_\alpha)$ is also absolutely continuous, for every $\alpha \in \Lambda$. Finally, $\mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N)$ will denote the set of curves $\gamma \in \mathcal{AC}(\mathrm{M})$ transversal to $N \in \mathcal{N}(\mathrm{M})$, in the sense of (1.1.5).

Definition 1.5.6. *Let* $\mathcal{A}$ *be a LIP atlas of* M *and let* $N \in \mathcal{N}(\mathrm{M})$. *We define the* length *with respect to a Finsler structure* F *by*

$$L_{\mathcal{A}}(\gamma, F) := \begin{cases} \int_0^1 F(\gamma(t), \dot\gamma(t))\mathrm{d}t & \textit{if } \gamma \in \mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N), \\ +\infty & \textit{otherwise}; \end{cases} \tag{1.5.4}$$

where the index $\mathcal{A}$ means that the integral and N depend on the chosen atlas.

Some facts have to be mentioned. Firstly, if we fix a null set $N \subset \mathrm{M}$, by [47, Lemma 2.2] we always have that $\mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N) \neq \emptyset$. Moreover, despite the value of the length depends on the chosen atlas, with a suitable choice of the negligible set $N = N(\mathcal{A})$ it is possible to give a meaning to (1.5.4), as we are going to show. Secondly, even if F_α is continuous on $V_\alpha \times \mathbb{R}^n$, this does not ensure that the changes of charts behave well, because $\Phi_{\alpha\beta}$ are Lipschitz but $D\Phi_{\alpha\beta}$ are only measurables maps.

Remark 1.5.7. Thanks to a lemma contained in [47], we can avoid the problem above. Indeed, if $F = (F_\alpha)$ is a Finsler structure on a Lipschitz manifold M, then there exists a negligible set N_1 such that, for every $\gamma \in \mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N_1)$, the map

$$t \mapsto F_\alpha\big(\Phi_\alpha(\gamma(t)), \frac{\mathrm{d}}{\mathrm{d}t}\Phi_\alpha(\gamma(t))\big)$$

is Borel measurable. Actually, let N_2 be the set where the compatibility condition (1.5.2) and (1.5.3) are not satisfied. By Rademacher's Theorem, $|N_1 \cup N_2| = 0$ and picking a curve $\gamma \in \mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N_1 \cup N_2)$, then we can find a partition $0 = t_0 < t_1 < \dots t_k = 1$ such that $\gamma([t_{h-1}, t_h]) \subset U_h$ for $h = 1, \dots, k$. Then we can set

$$L_{\mathcal{A}}(\gamma, F) = \sum_{h=1}^{k} \int_{t_{h-1}}^{t_h} F_h\Big(\Phi_h(\gamma(t), \frac{\mathrm{d}}{\mathrm{d}t}\Phi_h(\gamma(t)))\Big)\mathrm{d}t := \int_0^1 F(\gamma(t), \dot\gamma(t))\mathrm{d}t.$$

1.5.1 Metric Derivative on Lipschitz Manifolds

In this section our aim is to transpose the Definition 1.2.8 in the context of Lipschitz manifolds. First of all, we set a class of distances on an open cover of M.

Definition 1.5.8. *Let* M *be a Lipschitz manifold. A geodesic distance* $d : \mathrm{M} \times \mathrm{M} \to [0, +\infty)$ *belongs to* $\mathcal{D}(\mathrm{M})$ *if, for all* $\alpha \in \Lambda$*, there exist* $c_\alpha, C_\alpha > 0$ *such that:*

$$c_\alpha|x - y| \leq d(\Phi_\alpha^{-1}(x), \Phi_\alpha^{-1}(y)) \leq C_\alpha|x - y| \quad \textit{for every } x, y \in V_\alpha \subset \mathbb{R}^n.$$

In particular, we set $\sigma_\alpha(x, y) := d(\Phi_\alpha^{-1}(x), \Phi_\alpha^{-1}(y))$ *for every* $x, y \in V_\alpha \subset \mathbb{R}^n$*, which belongs to* $\mathcal{D}(V_\alpha)$*, for every* $\alpha \in \Lambda$*.*

Definition 1.5.9. *Let* $d \in \mathcal{D}(\mathrm{M})$*. Then we define the* upper *and* lower *metric derivative* φ_d^α, $\underline{\varphi}_d^\alpha : V_\alpha \times \mathbb{R}^n \to [0, +\infty)$ *as:*

$$\varphi_d^\alpha(x, v) \coloneqq \limsup_{t \to 0^+} \frac{\sigma_\alpha(x, x + tv)}{t}, \quad \underline{\varphi}_d^\alpha(x, v) \coloneqq \liminf_{t \to 0^+} \frac{\sigma_\alpha(x, x + tv)}{t},$$

for every $\alpha \in \Lambda$*,* $x \in V_\alpha$ *and* $v \in \mathbb{R}^n$*.*

Clearly φ_d^α and $\underline{\varphi}_d^\alpha$ are functions that depend on the charts. For the sake of brevity, we omit the index α when there is no risk of ambiguity. The fact that σ_α belongs to $\mathcal{D}(V_\alpha)$ allows us to simplify computations in the next results.

Lemma 1.5.10. *If* $d \in \mathcal{D}(\mathrm{M})$*, then* $\varphi_d = (\varphi_d^\alpha)_{\alpha \in \Lambda}$ *is a Finsler structure on* M*.*

Proof. The first assertion comes from Lemma 1.2.9. It actually shows that the upper derivative φ_d^α and the lower derivative $\underline{\varphi}_d^\alpha$ define a collection of metrics which makes $\varphi = (\varphi_d^\alpha)_{\alpha \in \Lambda}$ a Finsler structure.
It remains to prove that φ_d satisfies the compatibility condition. If $\Phi_\alpha^{-1}(x) \in U_\alpha \cap U_\beta$ then, for $t > 0$ small enough, we have that $\Phi_\alpha^{-1}(x + tv) \in U_\alpha \cap U_\beta$. Since $\Phi_{\alpha\beta}$ is differentiable for a.e. $x \in V_\alpha$, we get that

$$\begin{aligned}\sigma_\alpha(x, x + tv) &= d(\Phi_\alpha^{-1}(x), \Phi_\alpha^{-1}(x + tv)) = \sigma_\beta(\Phi_{\alpha\beta}(x), \Phi_{\alpha\beta}(x + tv)) \\ &= \sigma_\beta(\Phi_{\alpha\beta}(x), \Phi_{\alpha\beta}(x) + tD\Phi_{\alpha\beta}(x)[v] + o(t)).\end{aligned}$$

Then the compatibility condition holds, i.e. $\varphi_d^\alpha(x, v) = \varphi_d^\beta(\Phi_{\alpha\beta}(x), D\Phi_{\alpha\beta}(x)[v])$. □

Therefore, given a distance $d \in \mathcal{D}(\mathrm{M})$ one obtains that M inherits a Finsler structure given by φ_d. Moreover, by [46, Corollary 4.2] and Theorem 1.2.15, $\varphi_d(x, \cdot)$ is convex for all $x \in \Omega$ and this lead to show the counterpart of Theorem 1.2.13, where now the length is independent on the chosen atlas.

Corollary 1.5.11. *Let* $d \in \mathcal{D}(\mathrm{M})$ *and let* $\varphi_d = (\varphi_d^\alpha)$ *be the metric derivative defined on the chart* (U_α, Φ_α)*, for every* $\alpha \in \Lambda$*. Then for every* $\gamma \in \mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N)$ *we have that:*

$$L_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot\gamma(t))\mathrm{d}t = \int_0^1 \underline{\varphi}_d(\gamma(t), \dot\gamma(t))\mathrm{d}t. \tag{1.5.5}$$

Moreover, $\varphi_d(x,v) = \underline{\varphi}_d(x,v)$ *for a.e.* $x \in \mathrm{M}$ *and for every* $v \in \mathbb{R}^n$ *and hence*

$$\varphi_d(x,v) = \lim_{t\to 0} \frac{\sigma(x, x+tv)}{t} \quad \textit{for a.e. } x \in \mathrm{M}, \quad \textit{and for every } v \in \mathbb{R}^n.$$

Proof. Thanks to Lemma 1.5.10 it is sufficient to prove the thesis in a single chart $(U_\alpha, \varphi_d^\alpha)$. By Theorem 1.2.13 and since in every chart (U_α, Φ_α) every curve γ belongs to $\mathcal{P}(V_\alpha, N)$ where $N = \{(x,v) \in V_\alpha \times \mathbb{R}^n : \varphi_d(x,v) > \underline{\varphi}_d(x,v)\}$ is a negligible set, we get the assertion. The second claim comes from [46, Corollary 2.7]. $\square$

In particular, (1.5.5) is equivalent to assert that $L_d(\gamma) = \mathbb{L}_{\varphi_d}(\gamma)$ and this means that the metric space (M, d) becomes a length space.

Corollary 1.5.12. *Let* $d \in \mathcal{D}(\mathrm{M})$ *and let* $\varphi_d = (\varphi_d^\alpha)$ *be the metric derivative on the chart* (U_α, Φ_α)*, for every* $\alpha \in \Lambda$*. Then we have that*

$$d(x,y) = \inf\Big\{ \int_0^1 \varphi_d(\gamma(t), \dot\gamma(t))\mathrm{d}t \ : \ \gamma \in \mathcal{AC}(\mathrm{M}) \Big\} \quad \textit{for every } \ x, y \in \mathrm{M}.$$

Moreover, the metric derivative is symmetric with respect to the second entry, namely, $\varphi_d(x,v) = \varphi_d(x,-v)$ *for a.e.* $x \in \mathrm{M}$ *and for every* $v \in \mathbb{R}^n$*.*

Proof. The first assertion is a direct consequence of Corollary 1.5.11 while the second one is proved in [46, Corollary 2.7]. $\square$

Similarly to Definition 1.3.1, starting from the length functional 1.5.4 now we give the concept of intrinsic distance in the context of Lipschitz manifold.

Proposition 1.5.13. *Let* F *be a Finsler structure* F *on a Lipschitz manifold* M*. For every* $x, y \in \mathrm{M}$*, let us set*

$$d_F(x,y) := \sup_{N \in \mathcal{N}(\mathrm{M})} \inf \left\{ \int_0^1 F(\gamma(t), \dot\gamma(t))\mathrm{d}t \ : \ \gamma \in \mathcal{P}_{\mathcal{AC}}(\mathrm{M}, N) \right\}. \tag{1.5.6}$$

Then, d_F *is a geodesic distance on* M *and it is independent on the atlas* $\mathcal{A}$*. Moreover, the topology induced by* d_F *coincides with the topology of the manifold.*

Proof. The proof of the previous facts can be found in [47, Theorem 2.10, Theorem 2.8]. □

Similarly to Proposition 1.3.8, if F is an upper semicontinuous Finsler structure on a Lispchitz manifold M, then $\varphi_{d_F}(x,v) \leq F(x,v)$ for every $x \in \mathrm{M}$ and $v \in \mathbb{R}^n$. In the same hypothesis, the transversality condition and the supremum on N can be left out in the right hand side of (1.5.6).

Corollary 1.5.14. *Let* M *be a smooth manifold equipped with a* continuous *Finsler structure* F*, namely for every* $v \in \mathbb{R}^n$, $F(\cdot, v)$ *is a norm. Then we obtain that:*

$$\varphi_{d_F}(x,v) = F(x,v) \quad \text{for a.e. } x \in \mathrm{M} \text{ and } v \in \mathbb{R}^n.$$

Proof. See [47, Theorem 7.2]. □

The next result is proved in [47, Theorem 4.7], where the authors consider the distance in Definition 1.4.1 and the constraint is given by the dual metric, properly defined on $\mathrm{M} \times \mathbb{R}^n$.

Theorem 1.5.15. *Let* F *be a Finsler structure on a Lipschitz manifold* M*, then*

$$\delta_{F^\star}(x,y) = d_{F^{\star\star}}(x,y) \quad \text{for every } \ x,y \in \mathrm{M}.$$

Inequality ($\leq$) is the counterpart of Theorem 1.4.4 where we substitute the Finsler structure with the dual $F^\star$, while inequality ($\geq$) is based on the following classical approximation result for intrinsic distances.

Lemma 1.5.16. *Let* F *and* $(F_n)_{n\in\mathbb{N}}$ *be a sequence of Finsler structures on a Lipschitz manifold* M *such that, for every* $n \in \mathbb{N}$ *it holds that* $F_n(x,v) \geq F(x,v)$ *and*

$$\lim_{n\to+\infty} F_n(x,v) = F(x,v) \quad \text{for a.e. } x \in \mathrm{M}, \ \text{for every } v \in \mathbb{R}^n.$$

Then we have that: $\lim_{n\to+\infty} d_{F_n}(x,y) = d_F(x,y)$ *for every* $x,y \in \mathrm{M}$.

To conclude, we can deduce some basic consequences from the previous facts. Thanks to Theorem 1.2.4, point iii), we know that $F^{\star\star}(x,v) \leq F(x,v)$, for a.e. $x \in \mathrm{M}$ and for every

$v \in \mathbb{R}^n$. By the monotonicity of the intrinsic distance we have that $d_{F^{\star\star}}(x, y) \leq d_F(x, y)$ and hence, applying Theorem 1.5.15, we infer that $\delta_{F^\star}(x, y) \leq d_F(x, y)$ for every $x, y \in \mathrm{M}$. Moreover, if the Finsler structure F is convex, by Proposition 1.2.5, we know that $F(x, v) = F^{\star\star}(x, v)$ for a.e. $x \in \mathrm{M}$ and for every $v \in \mathbb{R}^n$ and thus $d_{F^{\star\star}}(x, y) = d_F(x, y)$ for every $x, y \in \mathrm{M}$. Therefore, thanks to Theorem 1.5.15, we obtain that $d_F(x, y) = \delta_{F^\star}(x, y)$ and this still holds when F is continuous but not necessarily convex (see [23]).

Chapter 2

Sub-Finsler metrics in Carnot groups

In this chapter, our aim is to introduce the main concepts concerning the sub-Riemannian geometry. The reader can find more detailed information in references [52, 69, 76, 84, 85]. Then, we will apply all the techniques developed in the first chapter in order to generalize the main results in Carnot groups, a special class of Carnot–Carathéodory spaces associated with a system of bracket-generating vector fields.

2.1 Carnot Groups and Sub-Riemannian structures

Let us recall that a Lie group $(\mathbb{G}, \cdot)$ is a smooth manifold endowed also with a group structure, such that the multiplication and inversion are smooth. Given a point $x \in \mathbb{G}$, we denote by $\tau_x \colon \mathbb{G} \to \mathbb{G}$ the *left translation* by x, which is given by

$$\tau_x z \coloneqq x \cdot z \quad \text{for every } z \in \mathbb{G},$$

where $\cdot$ is the group law in $\mathbb{G}$. Moreover, it holds that the map τ_x is a smooth diffeomorphism, thus we can consider its differential $\mathrm{d}_y \tau_x : T_y\mathbb{G} \to T_{x \cdot y}\mathbb{G}$ at any point $y \in \mathbb{G}$. We will say that a vector field X, section of the tangent bundle $T\mathbb{G}$, is left invariant if

$$(Xf)(\tau_z(x)) = X(f \circ \tau_z)(x) \quad \text{for all } x, z \in \mathbb{G} \ \text{ and } f \in C^\infty(\mathbb{G}).$$

The Lie algebra $\mathfrak{g}$ associated to a Lie group $\mathbb{G}$ can be characterized as the set of left invariant vector fields. Indeed it is a vector space, it is closed under the Lie bracket $[\cdot,\cdot]$ defined on smooth functions by

$$[X,Y](f) = X(Y(f)) - Y(X(f)),$$

and it is canonically isomorphic to the tangent space of $\mathbb{G}$, at the origin, via the identification of X and $X(e)$, see e.g. [95].

Definition 2.1.1. *A connected and simply connected Lie group $\mathbb{G}$ is said to be a Carnot group of step k if its Lie algebra $\mathfrak{g}$ admits a step k stratification, namely, there exist linear subspaces $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$ of $\mathfrak{g}$ such that*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, \quad [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, \quad \mathfrak{g}_k \neq \{0\}, \quad [\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}, \tag{2.1.1}$$

where $[\mathfrak{g}_1, \mathfrak{g}_i]$ is the subspace of $\mathfrak{g}$ generated by $[X,Y]$ with $X \in \mathfrak{g}_1$ and $Y \in \mathfrak{g}_i$.

Let $n := \dim \mathfrak{g} = m_1 + \ldots + m_k$ be the topological dimension of $\mathbb{G}$, where $m_j := \dim \mathfrak{g}_j$. We will denote with $m := m_1 < n$ the dimension of the so called *first stratum* of the stratification, which generates the whole Lie algebra $\mathfrak{g}$. Choose a basis $e_1, \ldots, e_n$ of $\mathfrak{g}$ adapted to the stratification, namely

$$e_{h_{j-1}+1}, \ldots, e_{h_j} \quad \text{is a basis of } \mathfrak{g}_j \quad \text{for each} \quad j = 1, \ldots, k.$$

Let $X_1, \ldots, X_n$ be the family of left invariant vector fields such that, at the identity e of $\mathbb{G}$, they satisfy $X_i(e) = e_i$ for every $i = 1, \ldots, n$. By (2.1.1), we will refer to $X_1, \ldots, X_m$ as generating horizontal vector fields. Indeed, they satisfy the *Hörmander condition* (see [72]), that is each vector field X_j is smooth and

$$\dim(\mathcal{L}(X_1, \ldots, X_m))(x) = n \quad \text{for all } x \in \mathbb{G},$$

where $\mathcal{L}(X_1, \ldots, X_m)$ is the linear span of the vector fields $X_1, \ldots, X_m$ and their commutators of any order. For this reason, they play a crucial role in the theory, as the following can suggests.

Definition 2.1.2. *The subbundle of the tangent bundle $T\mathbb{G}$ that is spanned by $X_1, \dots, X_m$ is called the* horizontal bundle $H\mathbb{G}$ *and the fibers are given by*

$$H_x\mathbb{G} = \operatorname{span}\{X_1(x), \dots, X_m(x)\} \qquad \text{for every } x \in \mathbb{G}.$$

Furthermore, in [3, Lemma 7.48]. it is stated that the tangent bundle $T\mathbb{G}$ of a Lie group $\mathbb{G}$ is always trivializable. We then have an isomorphism between $T\mathbb{G}$ and $\mathbb{G} \times T_e\mathbb{G}$, acting in the following way:

$$T\mathbb{G} \ni (x, v) \mapsto (x, \mathrm{d}_x\tau_{x^{-1}}[v]) \in \mathbb{G} \times T_e\mathbb{G}.$$

A sub-Riemannian structure can be defined on $\mathbb{G}$ in the following way. Consider a scalar product $\langle \cdot, \cdot \rangle_e$ on $\mathfrak{g}_1 = H_e\mathbb{G}$ that makes $\{X_1, \dots, X_m\}$ an orthonormal basis. Moreover, by left translating the horizontal fiber in the identity, we obtain that $H_x\mathbb{G} = \mathrm{d}_e\tau_x(\mathfrak{g}_1)$.

Definition 2.1.3. *A* sub-Riemannian structure *on a Carnot group $\mathbb{G}$ is given by the scalar product $\langle \cdot, \cdot \rangle_x$ on $H_x\mathbb{G}$ defined as*

$$\langle v, w \rangle_x := \left\langle \mathrm{d}_x\tau_{x^{-1}}[v], \mathrm{d}_x\tau_{x^{-1}}[w] \right\rangle_e \quad \text{for every } v, w \in H_x\mathbb{G}.$$

Moreover, we denote by $\|\cdot\|_x$ the norm induced by $\langle \cdot, \cdot \rangle_x$, namely $\|\cdot\|_x := \sqrt{\langle v, v \rangle_x}$ for every $v \in H_x\mathbb{G}$.

Since $\mathbb{G}$ is finite dimensional, every choice of the norm $\|\cdot\|_x$ would not change the biLipschitz equivalence class of the sub-Riemannian structure. This is the reason why we may assume that the norm $\|\cdot\|_x$ is coming from a scalar product (see [52]).

2.1.1 Exponential map

We want to recall the definition of the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ in the context of Carnot group. Given any vector $v \in \mathfrak{g} = T_e\mathbb{G}$ and denoting by $\gamma\colon [0,1] \to \mathbb{G}$ the (unique) smooth curve satisfying the ODE

$$\begin{cases} \dot\gamma(t) = \mathrm{d}_e\tau_{\gamma(t)}[v] \quad \text{for every } t \in [0,1], \\ \gamma(0) = e, \end{cases} \tag{2.1.2}$$

we define $\exp(v) = e^v \coloneqq \gamma(1)$, where $\mathrm{d}_e\tau_{\gamma(t)}[v]$ is a left-invariant vector field. It holds that exp is smooth and $(d\exp)_0 = \mathrm{id}_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{g}$ is the identity map. Hence exp is a local diffeomorphism of a small neighborhood of 0 in $\mathfrak{g}$ onto a neighborhood of e in $\mathbb{G}$. Then, any $p \in \mathbb{G}$ can be written in a unique way as

$$p = \exp(p_1X_1 + \cdots + p_nX_n) = e^{p_1X_1+\cdots+p_nX_n}, \qquad \text{where } v = \sum_{i=1}^{n} p_iX_i.$$

Proposition 2.1.4. *Let X be an element of the Lie algebra $\mathfrak{g}$ of a Lie group $\mathbb{G}$. Then the curve $\gamma(t) \coloneqq y \cdot \exp(tX)$ is the flow of X starting at y and we have that:*

- $\exp(s+t)X = \exp(sX) \cdot \exp(tX),$ *for $s, t \in \mathbb{R}$;*
- $\exp(-X) = (\exp(X))^{-1}$.

The underlying manifold of a Carnot group can always be chosen to be $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Indeed, we can identify p with the n-tuple $(p_1, \ldots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$ where the group operation $\cdot$ satisfies (see [85, Section 7] and [15])

$$x \cdot y = \exp\left(\exp^{-1}(x) \star \exp^{-1}(y)\right) \quad \text{for every } x, y \in \mathbb{G},$$

where $\star$ denotes the group operation determined by the Campbell–Baker–Hausdorff formula, see e.g. [16, 79]. In other words, the latter links Lie groups to Lie algebras, by expressing the inverse of exponential, i.e., the logarithm $\log(e^Xe^Y)$ of the product of two Lie group elements as a Lie algebra element.

Definition 2.1.5. *If $y = (y_1, \ldots, y_n) \in \mathbb{G}$ and $x \in \mathbb{G}$, then we set the* projection map *as:*

$$\pi_x : \mathbb{G} \to H_x\mathbb{G} \quad as \quad \pi_x(y) = \sum_{j=1}^{m} y_jX_j(x).$$

The map $y \mapsto \pi_x(y)$ is a smooth section of $H_x\mathbb{G}$ and it is linear in y. Finally, if $v \in \mathfrak{g}_1$, by exponential coordinates it holds that $v = \sum_{i=1}^{m} v_iX_i$ and then

$$\pi_x(e^v) = \pi_x(v_1, \ldots, v_m, 0\ldots, 0) = \sum_{i=1}^{m} v_iX_i(x) = \mathrm{d}_e\tau_x[v]. \quad \text{for every } x \in \mathbb{G}. \tag{2.1.3}$$

2.1.2 Dilations and Carnot–Carathéodory distance

Definition 2.1.6 (Dilations). *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ be a step k Lie algebra of a Carnot group $\mathbb{G}$. For any $\lambda > 0$, we denote by $\delta^\star_\lambda : \mathfrak{g} \to \mathfrak{g}$ the unique linear map such that*

$$\delta^\star_\lambda X = \lambda^i X, \qquad \forall X \in \mathfrak{g}_i.$$

The maps $\delta^\star_\lambda$ are Lie algebra automorphisms, namely, $\delta^*_\lambda([X,Y]) = [\delta^*_\lambda X, \delta^*_\lambda Y]$ for all $X, Y \in \mathfrak{g}$ and they satisfy $\delta^*_\lambda \circ \delta^*_\eta = \delta^*_{\lambda\eta}$ for all $\lambda, \eta > 0$ (see [76, Lemma 6.1.17]). Moreover, for every $\lambda > 0$, the map δ^*_λ naturally induces an automorphism $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ on the Carnot group, by the identity

$$\delta_\lambda(x) = (\exp \circ \delta^\star_\lambda \circ \log)(x). \tag{2.1.4}$$

In the future we will both call δ^*_λ and δ_λ *dilations of factor* λ.

Lemma 2.1.7. *For all $\lambda, \eta > 0$ and for every $v \in \mathfrak{g}_1$, the following properties hold:*

(i) $\delta_{\lambda+\eta} \exp(v) = \delta_\lambda \exp(v) \cdot \delta_\eta \exp(v)$*;*

(ii) $\pi_x(\delta_\lambda \exp(v)) = \lambda \pi_x(\exp(v))$ *for all $x \in \Omega$;*

(iii) $\delta_\lambda \exp(v) = \exp(\lambda v)$*.*

Remark 2.1.8. According to [90], we can extend the dilations also to negative parameters $\lambda < 0$, denoting $\delta^\star_{|\lambda|}(X) = \delta^\star_\lambda(-X) = |\lambda|^i(-X)$ for $X \in \mathfrak{g}_i$. Note that we will exploit this fact only on the fibers of the horizontal bundle. In the same way, it is possible to extend the dilations on the entire $T_x\mathbb{G}$, for every $x \in \mathbb{G}$. In particular, on $H_x\mathbb{G}$ we have that $\delta^*_\lambda(v) = \lambda v$ for every $\lambda > 0$ and $x \in \mathbb{G}$.

Now we are ready to present the following crucial definition.

Definition 2.1.9. *An absolutely continuous curve $\gamma \colon [a,b] \to \mathbb{G}$ is said to be* horizontal *if there exists a vector of measurable functions $h = (h_1(t), \ldots h_m(t)) : [a,b] \to \mathbb{R}^m$ called the vector of canonical coordinates, such that*

- $\dot{\gamma}(t) = \sum_{i=1}^{m} h_i(t) X_i(\gamma(t))$ *for a.e.* $t \in [a, b]$*;*

- $|h| \in L^\infty(a, b)$.

The length *of such a curve is given by* $L_{\mathbb{G}}(\gamma) \coloneqq \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)}\,\mathrm{d}t$.

In other words, the allowed curves are constrained to have their velocities in a lower dimensional subspace of the tangent space of the manifold. However, the Chow-Rashevskii Theorem, proved by L.W. Chow in [33] and independently by P.K. Rashevskii in [91] guarantees that any pair of points in a Carnot group can be connected by a horizontal curve (see [16, Theorem 19.1.3] and [69] for a exhaustive discussion of this fact). Therefore, the following definition is well-posed.

Definition 2.1.10. *For every* $x, y \in \mathbb{G}$*, the* Carnot–Carathéodory distance *is defined by*

$$d_{cc}(x, y) \coloneqq \inf\left\{L_{\mathbb{G}}(\gamma) \colon \gamma \text{ is a horizontal curve joining } x \text{ and } y\right\}.$$

We remark that, since the generating vector fields are bracket-generating, the Carnot–Carathéodory distance is finite, and it is homogeneous with respect to dilations and left translations. More precisely, for every $\lambda > 0$ and for every $x, y, z \in \mathbb{G}$ one has

$$d_{cc}(\delta_\lambda x, \delta_\lambda y) = \lambda d_{cc}(x, y), \quad d_{cc}(\tau_x y, \tau_x z) = d_{cc}(y, z).$$

The reader can be find a proof of the previous equality respectively in [76, Proposition 6.2.13] and in [85, Proposition 1.7.3]. Moreover $L_{\mathbb{G}}(\delta_\lambda(\gamma)) = \lambda L_{\mathbb{G}}(\gamma)$ and this immediately implies that $\tau_x(B_r(y)) = B_r(\tau_x y)$ and $\delta_\lambda B_r(y) = B_{\lambda r}(\delta_\lambda y)$, where

$$B_r(x) = \left\{y \in \mathbb{G} : d_{cc}(y, x) < r\right\}$$

is the open ball centered at $x \in \mathbb{G}$ with radius $r > 0$. The following crucial estimate is proved in [87, Proposition 1.1].

Theorem 2.1.11. *Let* $\mathbb{G}$ *be a Carnot group of step* k *and let* $K \subset \mathbb{G}$ *be a compact set. Then there exists* $C_K = C(K) > 1$ *such that*

$$C_K^{-1}|x - y| \leq d_{cc}(x, y) \leq C_K |x - y|^{\frac{1}{k}}, \qquad \forall x, y \in K. \tag{2.1.5}$$

The following lemma shows the biLipschitz equivalence between the Carnot–Carathéodory distance and the norm induced by the scalar product.

Lemma 2.1.12. *There exists a constant $c \geq 1$ such that*

$$\frac{1}{c}\left\|v\right\|_e \leq d_{cc}(e, \exp v) \leq c\left\|v\right\|_e \quad \textit{for every } v \in \mathfrak{g}_1. \tag{2.1.6}$$

Proof. Denote by S the unit sphere of $(H_e\mathbb{G}, \left\|\cdot\right\|_e)$, namely $S := \{v \in H_e\mathbb{G} : \left\|v\right\|_e = 1\}$. Define the function $\eta : S \to [0, +\infty)$ as $\eta(v) := d_{cc}(e, \exp v)$ for every $v \in S$. By Theorem 2.1.11, η is continuous on the compact set S. Then we can find $c \geq 1$ such that $1/c \leq \eta(v) \leq c$ holds for every $v \in S$. We can thus conclude by 1-homogeneity: since $d_{cc}(e, \exp(\lambda v)) = \lambda d_{cc}(e, \exp v)$ for every $\lambda > 0$ and $v \in S$, we deduce that $\eta(v/\|v\|_e) = d_{cc}(e, \exp v)/\|v\|_e$ for every $v \in H_e\mathbb{G} \setminus \{0\}$ and thus

$$\frac{1}{c} \leq \frac{d_{cc}(e, \exp v)}{\|v\|_e} \leq c \quad \text{for every } v \in H_e\mathbb{G} \setminus \{0\},$$

which yields (2.1.6). □

Finally, we introduce the simplest and most important example of Carnot group, i.e. the Heisenberg group, one of the most privileged object of this study (see [32]).

Example 2.1.13. The Heisenberg group $\mathbb{H}^1$ is the connected and simply connected Lie group given by the underlying manifold $\mathbb{R}^3$ with the non commutative group law

$$(x, y, z) \cdot (x', y', z') = \big(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')\big).$$

The unit element is $e = (0, 0, 0)$, and the inverse of the point (x, y, z) is $(-x, -y, -z)$ while the dilations are given by $\delta_\lambda(x, y, z) = (\lambda x, \lambda y, \lambda^2 z)$. The differentiable structure on $\mathbb{H}^1$ is determined by the left invariant vector fields

$$X = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + \frac{1}{2}x\frac{\partial}{\partial z}, \quad Z = [X, Y] = \frac{\partial}{\partial z},$$

which are the push-forward of the orthormal basis through the differential of the left translation. The Lie algebra of the Heisenberg group is the stratified algebra $\mathfrak{h} = \mathbb{R}^3 = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ where $\mathfrak{h}_1 = \text{span}\{X, Y\}$ and $\mathfrak{h}_2 = \text{span}\{Z\}$.

2.1.3 Pansu theorem and Lipschitz functions

We recall some basic definitions regarding differentiability in Carnot groups.

Definition 2.1.14. *A map* $L : \mathbb{G} \to \mathbb{R}$ *is called* homogeneous homomorphism *if*

$$L(x \cdot y) = L(x) + L(y) \quad \textit{and} \quad L(\delta_\lambda(x)) = \lambda L(x) \quad \textit{for every } x, y \in \mathbb{G} \quad \textit{and} \quad \lambda > 0.$$

Now we are ready to introduce the following fundamental notion of intrinsic differentiability, due to Pansu [89].

Definition 2.1.15. *Let* $\Omega \subset \mathbb{G}$ *be an open subset. A map* $f : \Omega \to \mathbb{R}$ *is* Pansu differentiable *at* $x \in \Omega$ *if there exists a homogeneous homomorphism* $L_x : \mathbb{G} \to \mathbb{R}$*, called* Pansu differential *such that*

$$\lim_{y \to x} \frac{f(x) - f(y) - L_x[y^{-1} \cdot x]}{d_{cc}(y, x)} = 0.$$

We notice that, in Carnot groups it only makes sense to consider derivatives in the horizontal directions, since the composition of a Lipschitz function with a non-horizontal curve may not be Lipschitz regular, and consequently may fail to be differentiable.

Remark 2.1.16. Notice that, if $f : \Omega \to \mathbb{R}$ is Pansu differentiable at $x \in \Omega$, then $X_j f(x)$ exists for any $j = 1, \dots, m$, and for any $v \in \mathbb{G}$ we have

$$d_{\mathbb{G}} f(x)[v] = \langle \nabla_{\mathbb{G}} f(x), \pi_x(v) \rangle_x,$$

where the horizontal gradient $\nabla_{\mathbb{G}} f(x)$ is defined as

$$\nabla_{\mathbb{G}} f(x) \coloneqq \sum_{i=1}^{m} X_i f(x) X_i(x). \tag{2.1.7}$$

Moreover, a function $f : \mathbb{G} \to \mathbb{R}$ belongs to $C^1(\mathbb{G})$ if $X_j f : \mathbb{G} \to \mathbb{R}$ exists and is continuous for all $j = 1, \dots, m$.

We stress that the notion of the horizontal gradient only depends on the choice of the horizontal frame $(X_1, \dots, X_m)$ and therefore it is uniquely determined by the sub-Riemannian structure. Now, the notion of Pansu differentiability is motivated by the following result due to Pansu [89] (see also [80] for a similar result in a more general setting).

Theorem 2.1.17. *Let $\Omega \subset \mathbb{G}$ be an open subset. Then for every Lipschitz function $f : \Omega \to \mathbb{R}$ we have that f is Pansu differentiable at $\mathcal{L}^n$-a.e. $x \in \Omega$.*

On the other hand, if we consider $x \in \mathbb{G}$ and $\bar{v} \in \mathfrak{g}_1$, the map $t \mapsto x \cdot \delta_t \exp(\bar{v})$ is Lipschitz and hence, if $f : \mathbb{G} \to \mathbb{R}$ is Lipschitz, then the composition $t \mapsto f(x \cdot \delta_t \exp(\bar{v}))$ is a Lipschitz mapping from $\mathbb{R}$ to itself. By Theorem 2.1.17, it is Pansu differentiable. Furthermore, the next result allows us to represent the Pansu differential with respect to vectors in the first stratum of the Lie algebra $\mathfrak{g}$.

Lemma 2.1.18. *Let $f : \mathbb{G} \to \mathbb{R}$ be a Lipschitz function. Then we have that*

$$\langle \nabla_{\mathbb{G}} f(x), v \rangle = \lim_{t \to 0} \frac{f(x \cdot \delta_t e^{\bar{v}}) - f(x)}{t} \quad \text{for every } (x, v) \in H\mathbb{G}, \tag{2.1.8}$$

where $\bar{v} = \mathrm{d}_x \tau_{x^{-1}}[v]$.

Proof. We know that, if $X_1, \dots, X_m$ is a generating family of vector fields and $f : \mathbb{G} \to \mathbb{R}$ is Lipschitz, the (linear) action of X_j on f is given by the equation

$$X_j f(x) = \lim_{t \to 0} \frac{f(x \cdot e^{tX_j}) - f(x)}{t} = \frac{\mathrm{d}}{\mathrm{d}t} f(x \cdot e^{tX_j})_{|t=0}, \tag{2.1.9}$$

for every $x \in \mathbb{G}$ and $j = 1, \dots, m$. Now, let us take $v \in H_x\mathbb{G}$ and hence we can write $v = \sum_{i=1}^m v_i X_i(x)$ where v_i is constant for all $i = 1, \dots, m$. Since, by definition, the vector fields are left invariant, we have that

$$\bar{v} = \mathrm{d}_x \tau_{x^{-1}}[v] = \mathrm{d}_x \tau_{x^{-1}} \left[\sum_{i=1}^m v_i X_i(x) \right] = \sum_{i=1}^m v_i \mathrm{d}_x \tau_{x^{-1}}[X_i(x)] = \sum_{i=1}^m v_i X_i.$$

Then, by Remark 2.1.16 and (2.1.9), we get

$$\begin{aligned}
\langle \nabla_{\mathbb{G}} f(x), v \rangle_x &= \sum_{i=1}^m X_i f(x) v_i \langle X_i(x), X_i(x) \rangle_x \\
&= \Big(\sum_{i=1}^m v_i X_i \Big) f(x) := \lim_{t \to 0} \frac{f(x \cdot e^{\sum_{i=1}^m v_i (tX_i)}) - f(x)}{t} \\
&= \lim_{t \to 0} \frac{f(x \cdot \delta_t e^{\sum_{i=1}^m v_i X_i}) - f(x)}{t} = \lim_{t \to 0} \frac{f(x \cdot \delta_t e^{\bar{v}}) - f(x)}{t}.
\end{aligned}$$

□

In particular, if not otherwise stated, for every $v \in T_x\mathbb{G}$ and $x \in \mathbb{G}$ we will denote with $\bar{v} \coloneqq \mathrm{d}_x\tau_{x^{-1}}[v]$ the representative vector of v in the Lie algebra $\mathfrak{g}$.

2.2 Sub-Finsler Metrics and Duality

Now we want to present a central notion of the present chapter: the sub-Finsler metric. Differently from the sub-Riemannian structure, a priori this does not arise from a scalar product, even if it is defined only on on a subspace of the Lie algebra. Nevertheless, it has to satisfy a sort of biLipschitz equivalence with respect to the norm induced by the sub-Riemannian structure.

Definition 2.2.1. *For $\alpha \geq 1$, we define $\mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ as the family of maps $\varphi : H\mathbb{G} \to [0, +\infty)$, that we will call metrics on $H\mathbb{G}$, verifying the following properties:*

(1) $\varphi : H\mathbb{G} \to \mathbb{R}$ *is Borel measurable, where $H\mathbb{G}$ is endowed with the product σ-algebra;*

(2) $\varphi(x, \delta^*_\lambda v) = |\lambda|\varphi(x, v)$ *for every $(x, v) \in H\mathbb{G}$ and $\lambda \in \mathbb{R}$;*

(3) $\frac{1}{\alpha}\|v\|_x \leq \varphi(x, v) \leq \alpha\|v\|_x$ *for every $(x, v) \in H\mathbb{G}$.*

Moreover, we will say that $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ is a sub-Finsler convex metric *if*

$$\varphi(x, v_1 + v_2) \leq \varphi(x, v_1) + \varphi(x, v_2) \tag{2.2.1}$$

for every $x \in \mathbb{G}$ and $v_1, v_2 \in H_x\mathbb{G}$ (or equivalently if $\varphi(x, \cdot)$ is a norm for every $x \in \mathbb{G}$).

According to the preliminaries, conditions (2) and (3) are well-defined with respect to the exponential and the dilation map (cf. Remark 2.1.8). Moreover, let us remark that condition (1) is equivalent to the Borel measurability with respect to the product space $\mathbb{G} \times \mathfrak{g}_1$.

Definition 2.2.2 (Dual Metric)**.** *Let us take $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$. We define the* dual metric *$\varphi^\star : H\mathbb{G} \to [0, +\infty)$ of φ as*

$$\varphi^\star(x, v) \coloneqq \sup\left\{\frac{|\langle v, w\rangle_x|}{\varphi(x, w)} \ : \ w \in H_x\mathbb{G},\ w \neq 0\right\}. \tag{2.2.2}$$

Compared to the classical Definition 1.2.3 and in order to adapt it on the sub-Riemannian structure, we have some asymmetry. The reason is that, for every fixed point $x \in \mathbb{G}$, the domain of the dual metric is the m-dimensional vector space $H_x\mathbb{G}$. Notice that this notion, known as *Fenchel transform*, was already developed in the Heisenberg group (see [30]). Now, as in Section 1.2, we prove that the dual metric enjoys some useful properties.

Proposition 2.2.3. *For any $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$, it holds that $\varphi^\star$ is a sub-Finsler convex metric, and in particular*

$$\frac{1}{\alpha}\|v\|_x \leq \varphi^\star(x,v) \leq \alpha\|v\|_x \quad \textit{for every } (x,v) \in H\mathbb{G}. \tag{2.2.3}$$

Proof. It is straightforward to prove property (2) since for every $v, w \in H_x\mathbb{G}$ and $\lambda \in \mathbb{R}$ we have that $\langle \delta^*_\lambda v, w\rangle_x = \lambda\langle v, w\rangle_x$. Passing to the supremum over all $w \in H_x\mathbb{G} \setminus \{0\}$, we obtain that $\varphi^\star(x, \delta^*_\lambda v) = |\lambda|\varphi^\star(x,v)$. The convexity on the horizontal bundle is a consequence of property (i) of Theorem (1.2.4). Moreover, accordingly to property (3) of Definition 2.2.1, taking $w \in H_x\mathbb{G} \setminus \{0\}$ it holds that

$$\frac{1}{\alpha}\frac{|\langle v, w\rangle_x|}{\|w\|_x} \leq \frac{|\langle v, w\rangle_x|}{\varphi(x,w)} \leq \alpha\frac{|\langle v, w\rangle_x|}{\|w\|_x}.$$

By taking the supremum over all $w \in H_x\mathbb{G} \setminus \{0\}$, we obtain (2.2.3). Therefore, $\varphi^\star(x,\cdot)$ is a norm, thus in particular it is continuous. Finally, chosen a dense sequence $(w_n)_n$ in $\mathfrak{g}_1 \setminus \{0\}$, we have that $(\mathrm{d}_e\tau_x[w_n])_n$ is dense in $H_x\mathbb{G}$ for every $x \in \mathbb{G}$, thus for any $v \in \mathfrak{g}_1$ we can write

$$\varphi^\star(x, \mathrm{d}_e\tau_x[v]) = \sup_{n\in\mathbb{N}} \frac{|\langle \mathrm{d}_e\tau_x[v], \mathrm{d}_e\tau_x[w_n]\rangle_x|}{\varphi(x, \mathrm{d}_e\tau_x[w_n])} = \sup_{n\in\mathbb{N}} \frac{|\langle v, w_n\rangle_e|}{\varphi(x, \mathrm{d}_e\tau_x[w_n])}$$

for every $x \in \mathbb{G}$, which shows that $\mathbb{G} \ni x \mapsto \varphi^\star(x, \mathrm{d}_e\tau_x[v])$ is measurable and accordingly property (1) of Definition 2.2.1 is satisfied. All in all, $\varphi^\star$ is a sub-Finsler convex metric. $\square$

Remark 2.2.4. In every Carnot groups $\mathbb{G}$, when we move points, the fibers of the horizontal bundle cannot be identify, even if they are isomorphic. Then we need to slightly modify Definition 1.2.2 in the following way. We will say that $\varphi : H\mathbb{G} \to \mathbb{R}$ is *lower* (or *upper*) *semicontinuous* at $(x,v) \in H\mathbb{G}$ if, for every sequence $(x_n, v_n) \in H\mathbb{G}$ converging to (x,v), in the sense that $d_{cc}(x_n,x) + \left\| \mathrm{d}_{x_n}\tau_{x_n^{-1}}[v_n] - \mathrm{d}_x\tau_{x^{-1}}[v] \right\|_e \to 0$, we have that

$$\varphi(x,v) \leq \liminf_{n\to\infty} \varphi(x_n,v_n) \quad \text{ or } \quad \varphi(x,v) \geq \limsup_{n\to\infty} \varphi(x_n,v_n).$$

Lemma 2.2.5. *Let $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then the following hold:*

(a) *If φ is lower semicontinuous, then $\varphi^\star$ is upper semicontinuous.*

b) *If φ is upper semicontinuous, then $\varphi^\star$ is lower semicontinuous.*

In particular, if φ is continuous, then $\varphi^\star$ is continuous.

Proof. To prove (a) suppose φ is lower semicontinuous. Fix $(x,v) \in H\mathbb{G}$ and $(x_n,v_n) \in H\mathbb{G}$ such that $(x_n,v_n) \to (x,v)$, in the sense of Remark 2.2.4. Possibly passing to a not relabeled subsequence, we can assume that $\limsup_n \varphi^\star(x_n,v_n)$ is actually a limit. Given any $n \in \mathbb{N}$, there exists $w_n \in H_{x_n}\mathbb{G}$ such that $\varphi(x_n,w_n) = 1$ and $\varphi^\star(x_n,v_n) = |\langle v_n, w_n\rangle_{x_n}|$. By compactness, there exists $w \in H_x\mathbb{G}$ such that (up to a not relabeled subsequence) $(x_n,w_n) \to (x,w)$. Being φ lower semicontinuous, we deduce that

$$\varphi(x,w) \leq \liminf_{n\to\infty} \varphi(x_n,w_n) \leq 1.$$

Therefore, we conclude that

$$\varphi^\star(x,v) \geq \frac{|\langle v,w\rangle_x|}{\varphi(x,w)} \geq \lim_{n\to\infty} |\langle v_n,w_n\rangle_{x_n}| = \limsup_{n\to\infty} \varphi^\star(x_n,v_n),$$

which proves that $\varphi^\star$ is upper semicontinuous.
Item (b) can be proved noticing that if φ is upper semicontinuous, then $\varphi^\star$ is lower semicontinuous as it can be expressed as a supremum of lower semicontinuous functions. □

Finally, we can characterize sub-Finsler convex metrics $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ in terms of the bidual metric $\varphi^{\star\star}$, exactly as in Proposition 1.2.5, substituting the real space $\mathbb{R}^n$ with the m-dimensional vector space $H_x\mathbb{G}$.

2.3 Metric Derivative in Carnot groups

Inspired by [94] and Definition 1.2.6, now we introduce the following class of distances.

Definition 2.3.1. *Let $\mathbb{G}$ be a Carnot group and let $\Omega \subset \mathbb{G}$ be an open set. If $\alpha \geq 1$, we introduce the family $\mathcal{D}_{cc}(\Omega)$ of all geodesic distances $d : \Omega \times \Omega \to [0, +\infty)$ verifying*

$$\frac{1}{\alpha} d_{cc}(x, y) \leq d(x, y) \leq \alpha d_{cc}(x, y) \qquad \forall x, y \in \Omega. \tag{2.3.1}$$

Remark 2.3.2. The set $\mathcal{D}_{cc}(\Omega)$ depends on α and we omit such dependence for the sake of brevity. Clearly $\mathcal{D}_{cc}(\Omega) \neq \emptyset$ for every $\Omega \subset \mathbb{G}$ open, connected and provided that the boundary is regular enough. We will endow $\mathcal{D}_{cc}(\Omega)$ with the topology of the uniform convergence on compact subsets of $\Omega \times \Omega$ and we will see in the proof of Theorem 4.4.1 that $\mathcal{D}_{cc}(\Omega)$ is compact with respect to such topology.

Now, given a geodesic distance $d \in \mathcal{D}_{cc}(\mathbb{G})$, it is natural to consider the associated metric given by differentiation. This is inspired by the ones proposed in [90, 94] but we necessarily have to define it on the horizontal bundle $H\mathbb{G}$.

Definition 2.3.3. *Given $d \in \mathcal{D}_{cc}(\mathbb{G})$, we define the* metric derivative $\varphi_d : H\mathbb{G} \to [0, +\infty)$ *as the map*

$$\varphi_d(x, v) := \limsup_{t \to 0} \frac{d(x, x \cdot \delta_t \exp \mathrm{d}_x \tau_{x^{-1}}[v])}{|t|} \quad \textit{for every } (x, v) \in H\mathbb{G}.$$

Let us notice that we translate the vector $v \in H_x\mathbb{G}$ to e via the differential of the left traslation, because the exponential map is defined on the first stratum $\mathfrak{g}_1 = H_e\mathbb{G}$. The next lemma tells us that the metric derivative is actually a metric.

Lemma 2.3.4. *For every $d \in \mathcal{D}_{cc}(\mathbb{G})$ we have that $\varphi_d \in \mathcal{M}^{c\alpha}_{cc}(\mathbb{G})$, for some $c \geq 1$ independent of d.*

Proof. In order to prove (1) of Definition 2.2.1, let just observe that

$$\varphi_d(x, v) = \lim_{n \to \infty} \sup_{\substack{t \in \mathbb{Q}: \\ |t| < 1/n}} \frac{d(x, x \cdot \delta_t \exp \mathrm{d}_x \tau_{x^{-1}}[v])}{|t|} \quad \text{for every } (x, v) \in H\mathbb{G}.$$

Let us verify (2). Pick $x \in \mathbb{G}$, $v \in H_x\mathbb{G}$ and $t, \lambda \in \mathbb{R}$. Since the differential of the left translation is a diffeomorphism and, by property (iii) of Lemma 2.1.7, we have that $\delta_t \exp\left(\mathrm{d}_x\tau_{x^{-1}}[\delta^\star_\lambda(v)]\right) = \delta_t\delta_\lambda \exp\left(\mathrm{d}_x\tau_{x^{-1}}[v]\right)$. Therefore

$$\varphi_d(x, \delta^*_\lambda v) = \limsup_{t\to 0} \frac{d(x, x\cdot \delta_t\delta_\lambda e^{\mathrm{d}_x\tau_{x^{-1}}[v]})}{|t|} = |\lambda| \limsup_{t\to 0} \frac{d(x, x\cdot \delta_{t\lambda} e^{\mathrm{d}_x\tau_{x^{-1}}[v]}))}{|t\lambda|} = |\lambda|\varphi(x, v).$$

In order to show (3), fix $x \in \mathbb{G}$ and $v \in H_x\mathbb{G}$. Since $d \in \mathcal{D}_{cc}(\mathbb{G})$ we can write

$$\varphi_d(x, v) \le \alpha \limsup_{t\to 0} \frac{d_{cc}(x, x\cdot \delta_t e^{\mathrm{d}_x\tau_{x^{-1}}[v]})}{|t|} = \alpha\, d_{cc}(e, \exp \mathrm{d}_x\tau_{x^{-1}}[v]) \le c\,\alpha\|\mathrm{d}_x\tau_{x^{-1}}[v]\|_e$$

where in the last inequality we applied Lemma 2.1.12. The estimate from below can be proved similarly. Finally, using the left invariance of the norm, for every $(x, v) \in H\mathbb{G}$, we get that

$$\frac{1}{c\alpha}\|v\|_x \le \varphi_d(x, v) \le c\,\alpha\|v\|_x$$

and the conclusion follows. □

Notation. We denote with $\mathcal{H}([0, 1], \Omega)$ the set of horizontal curves and, for every Lebesgue null set $N \subset \Omega$, we set $\mathcal{P}(\Omega, N)$ the class of all horizontal curves such that (1.1.5) holds. Clearly $\mathcal{P}(\Omega, N) \neq \emptyset$ and we denote with $H\Omega := \{(x, v) \in H\mathbb{G} : x \in \Omega\}$ the restriction of the horizontal bundle $H\mathbb{G}$ to Ω. Finally, for any $d \in \mathcal{D}_{cc}(\Omega)$ and $a \in \Omega$, we denote by $d_a(x) := d(a, x)$ the fixed-point distance map that is a Lipschitz function and then, by Theorem 2.1.17, is Pansu differentiable for a.e. $x \in \Omega$. ■

The following result asserts that Lipschitz curves and horizontal ones essentially coincide when the L^∞-norm of the canonical coordinates is finite.

Proposition 2.3.5. *A curve $\gamma : [a, b] \to \mathbb{G}$ is Lipschitz, with constant L, if and only if it is horizontal and $\|h\|_{L^\infty(a,b)} \le L$.*

Proof. See [85, Proposition 1.3.3] for the precise statement and see [3, Proposition 3.50] for a general proof. □

2.3.1 Length Representation result

Now, given any $\varphi \in \mathcal{M}^{\alpha}_{cc}(\Omega)$, we define the *length functional* $\mathbb{L}_\varphi$ through the formula

$$\mathbb{L}_\varphi(\gamma) \coloneqq \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t \quad \gamma \in \mathcal{H}\big([0,1], \Omega\big). \tag{2.3.2}$$

The latter is well defined since $t \mapsto (\gamma(t), \dot{\gamma}(t))$ is Borel measurable on $H_{\gamma(t)}\mathbb{G}$ and φ satisfies assumption (1) of Definition 2.2.1. Furthermore, let us observe that, thanks to property (2) of Definition 2.2.1, $\mathbb{L}_\varphi(\gamma)$ does not depend on the chosen parametrization for γ. For this reason, it is not restrictive to assume γ to be defined on the closed unit interval and parametrized with constant velocity.

Now, since Carnot groups are naturally endowed with Carnot–Carathéodory distances, this make them interesting examples of geodesic metric spaces $(\mathbb{G}, d_{cc})$. In particular, the metric derivative can be explicitly computed (see [85, Theorem 1.3.5]).

Lemma 2.3.6. *Let $\gamma : [0,1] \to \mathbb{G}$ be a Lipschitz curve and let $h \in L^\infty(0,1)^m$ be its vector of canonical coordinates. Then*

$$|\dot{\gamma}(t)|_{d_{cc}} = \lim_{s \to 0} \frac{d_{cc}(\gamma(t+s), \gamma(t))}{|s|} = |h(t)| \quad \textit{for a.e. } t \in [0,1]$$
$$\textit{and} \quad \lim_{s \to 0} \delta_{\frac{1}{s}} \big(\gamma(t)^{-1} \cdot \gamma(t+s)\big) = (h_1(t), \dots, h_m(t), 0, \dots, 0) \quad \textit{for a.e.} \quad t \in [0,1].$$

The second claim is proved in [85, Lemma 2.1.4] and it gives a characterization of horizontal curves in terms of canonical coordinates. Therefore, by Proposition 2.3.5, a Lipschitz curve is horizontal and, with abuse of notation, we set the following quantity:

$$\exp \dot{\gamma}(t) \coloneqq \exp \mathrm{d}_{\gamma(t)} \tau_{\gamma(t)^{-1}}[\dot{\gamma}(t)] = \big(h_1(t), \dots, h_m(t), 0 \dots, 0\big), \qquad \text{for a.e. } t \in [0,1].$$

Thanks to the previous argument and to Lemma 2.3.6, it holds that $\gamma(t+s) = \gamma(t) \cdot (sh_1(t), \dots, sh_m(t), 0, \dots, 0) \cdot o(s) = \gamma(t) \cdot \delta_s e^{\dot{\gamma}(t)} \cdot o(s)$ where $\delta_{\frac{1}{s}} o(s) \to 0$ as $s \to 0$. This allows us to show that the metric derivative and the classical one coincide almost everywhere on the sets of horizontal curves.

Lemma 2.3.7. *If $\gamma : [0,1] \to \mathbb{G}$ is a horizontal curve and $d \in \mathcal{D}_{cc}(\mathbb{G})$, then*

$$|\dot\gamma(t)|_d = \varphi_d(\gamma(t), \dot\gamma(t)) \quad \text{for a.e. } t \in [0,1]. \tag{2.3.3}$$

Proof. Let $t \in [0,1]$ be such that $|\dot\gamma(t)|_d$ exists. By the triangle inequality we have that

$$\begin{aligned}
|\dot\gamma(t)|_d &= \limsup_{h\to 0} \frac{d(\gamma(t), \gamma(t+h))}{h} \\
&\le \limsup_{h\to 0} \frac{d(\gamma(t), \gamma(t)\cdot \delta_h e^{\dot\gamma(t)})}{h} + \limsup_{h\to 0} \frac{d(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t+h))}{h} \\
&= \varphi_d(\gamma(t), \dot\gamma(t)) + \limsup_{h\to 0} \frac{d(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t+h))}{h}.
\end{aligned}$$

We want to show that

$$\limsup_{h\to 0} \frac{d(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t+h))}{h} = 0.$$

Since $\gamma(t+h) = \gamma(t)\cdot\delta_h e^{\dot\gamma(t)} \cdot o(h)$, where $\delta_{\frac{1}{h}} o(h) \to 0$ as $h \to 0$, we get

$$\begin{aligned}
\limsup_{h\to 0} \frac{d(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t)\cdot \delta_h e^{\dot\gamma(t)}\cdot o(h))}{h} &\le \alpha \limsup_{h\to 0} \frac{d_{cc}(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t)\cdot \delta_h e^{\dot\gamma(t)}\cdot o(h))}{h} \\
&= \alpha \limsup_{h\to 0} \frac{d_{cc}(e, o(h))}{h} \\
&\le \alpha \limsup_{h\to 0} |\delta_{\frac{1}{h}} o(h)|^{\frac{1}{k}} = 0.
\end{aligned}$$

This proves that

$$|\dot\gamma(t)|_d \le \varphi_d(\gamma(t), \dot\gamma(t)).$$

In order to get the opposite inequality, we can estimate

$$\begin{aligned}
|\dot\gamma(t)|_d &\ge \limsup_{h\to 0} \frac{d(\gamma(t), \gamma(t)\cdot \delta_h e^{\dot\gamma(t)})}{h} - \limsup_{h\to 0} \frac{d(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t+h))}{h} \\
&= \varphi_d(\gamma(t), \dot\gamma(t)) - \limsup_{h\to 0} \frac{d(\gamma(t)\cdot \delta_h e^{\dot\gamma(t)}, \gamma(t+h))}{h}
\end{aligned}$$

and the conclusion follows as before. $\square$

Theorem 2.3.8. *Let $d \in \mathcal{D}_{cc}(\mathbb{G})$. Then, for every horizontal curve $\gamma : [0,1] \to \mathbb{G}$ we have*

$$L_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot\gamma(t))\, \mathrm{d}t. \tag{2.3.4}$$

Moreover, for every $x, y \in \mathbb{G}$ we have that $d = d_{\varphi_d}$, namely,

$$d(x,y) = \inf\left\{\int_0^1 \varphi_d(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \;:\; \gamma \in \mathrm{Lip}([0,1],\mathbb{G}),\;\; \gamma(0)=x,\; \gamma(1)=y\right\}.$$

Proof. Pick a curve $\gamma : [0,1] \to \mathbb{G}$ in $\mathrm{Lip}(\mathbb{G})$, then, by Proposition 2.3.5 we have that $\gamma \in \mathcal{H}([0,1],\mathbb{G})$. Arguing as in [6], it is easy to verify that $\varphi_d(\gamma(t),\dot\gamma(t)) = |\dot\gamma(t)|_d$, thanks to (1.1.4) and applying Lemma 2.3.7. On the other hand, the second claim holds because d is a geodesic distance. $\square$

2.4 Metric Derivative's convexity in $H_x\mathbb{G}$

The aim of the present section is to prove that if $d \in \mathcal{D}_{cc}(\mathbb{G})$, then φ_d is also a sub-Finsler convex metric. In order to make this, first we have to show some technical results.

Lemma 2.4.1. *Let $\psi\colon \mathbb{G} \to \mathbb{R}$ be a locally bounded, Borel function and $v \in H_x\mathbb{G}\setminus\{0\}$. Then*

$$\psi(x) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi(x\cdot\delta_s e^{\bar v})\,\mathrm{d}s, \qquad \textit{for } \mathcal{L}^n\textit{-a.e. } x \in \mathbb{G}. \tag{2.4.1}$$

Proof. Given any fixed $y \in \mathbb{G}$, we have that $\mathbb{R} \ni t \mapsto \psi(y\cdot\delta_t e^{\bar v}) \in \mathbb{R}$ is a locally bounded and Borel function, thus an application of Lebesgue's differentiation theorem guarantees that for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$

$$\psi(y\cdot\delta_r e^{\bar v}) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi(y\cdot\delta_{r+s}e^{\bar v})\;\mathrm{d}s. \tag{2.4.2}$$

In particular, an application of Fubini's theorem ensures that the set

$$\Gamma \coloneqq \big\{(y,r) \in \mathbb{G}\times\mathbb{R} \;\big|\; (2.4.2) \text{ holds}\big\}$$

has $\mathcal{L}^{n+1}$ zero measure, thus for $\mathcal{L}^1$-a.e. $r \in \mathbb{R}$ we have that (2.4.2) holds for $\mathcal{L}^n$-a.e. $y \in \mathbb{G}$. Fix any such $r \in \mathbb{R}$ and a $\mathcal{L}^n$-negligible set $N \subset \mathbb{G}$ satisfying (2.4.2) for every $y \in \mathbb{G}\setminus N$. Calling $\sigma_z : \mathbb{G} \to \mathbb{G}$ the right-translation map $\sigma_z w \coloneqq w\cdot z$ for every $z, w \in \mathbb{G}$ and defining $N' \coloneqq \sigma_{\delta_r e^{\bar v}}(N)$, we thus have that $\psi(x) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi(x\cdot\delta_s e^{\bar v})\,\mathrm{d}s$ holds for every $x \in \mathbb{G}\setminus N'$. Here, we also used the fact that $\delta_{r+s}e^{\bar v} = \delta_r e^{\bar v}\cdot\delta_s e^{\bar v}$, which is in turn guaranteed by the fact that $\bar{v}$ belongs to the first layer (see [90, Lemma 2.2] or property (ii) of Lemma 2.1.7). Therefore, in order to prove (2.4.1) it is only left to check that N' is $\mathcal{L}^n$-negligible. This can be achieved by exploiting the right-invariance of the measure $\mathcal{L}^n$ (see e.g. [85, Proposition 1.7.7]), namely the fact that $\mathcal{L}^n(E \cdot z) = \mathcal{L}^n(E)$ holds whenever $E \subset \mathbb{G}$ is a Borel set and $z \in \mathbb{G}$. Indeed, this implies that $(\sigma_{\delta_r e^{\bar{v}}})_\# \mathcal{L}^n = \mathcal{L}^n$, because for any Borel set $E \subset \mathbb{G}$ it holds that

$$(\sigma_{\delta_r e^{\bar{v}}})_\# \mathcal{L}^n(E) = \mathcal{L}^n(\sigma^{-1}_{\delta_r e^{\bar{v}}}(E)) = \mathcal{L}^n(\sigma_{\delta_{-r} e^{\bar{v}}}(E)) = \mathcal{L}^n(E \cdot \delta_{-r} e^{\bar{v}}) = \mathcal{L}^n(E).$$

In particular, we conclude that $\mathcal{L}^n(N') = (\sigma_{\delta_r e^{\bar{v}}})_\# \mathcal{L}^n(N') = \mathcal{L}^n(N) = 0$, as required. □

Lemma 2.4.2. *Let $d \in \mathcal{D}_{cc}(\Omega)$, $\varphi \in \mathcal{M}^\alpha_{cc}(\Omega)$, and $N \subset \Omega$ be such that $|N| = 0$. Suppose that for every $\gamma \in \mathcal{P}(\Omega, N)$ we have that*

$$d(\gamma(0), \gamma(1)) \leq \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t))\, \mathrm{d}t.$$

Then for every fixed $a \in \Omega$ and a.e. $(x, v) \in H\Omega$ we have:

$$|\langle \nabla_{\mathbb{G}} d_a(x), v \rangle_x| \leq \liminf_{t \to 0} \frac{d(x, x \cdot \delta_t e^{\bar{v}})}{t} \leq \limsup_{t \to 0} \frac{d(x, x \cdot \delta_t e^{\bar{v}})}{t} \leq \varphi(x, v).$$

Proof. Let N be as in the hypothesis and $v \in H_x\mathbb{G}$. For $a \in \Omega$, let $E(a, v)$ be the set of all $x \in \Omega$ for which d_a is Pansu differentiable for a.e. $x \in \Omega$ and the map $[0, 1] \ni t \mapsto x \cdot \delta_t e^{\bar{v}}$ belongs to $\mathcal{P}(\Omega, N)$, with t small enough. Moreover, thanks to Lemma 2.4.1, we can assume that

$$\lim_{t \searrow 0} \frac{1}{t} \int_0^t \varphi(x \cdot \delta_s e^{\bar{v}}, v)\, \mathrm{d}s = \varphi(x, v).$$

By Pansu–Rademacher Theorem $|\Omega \setminus E(a, v)| = 0$ and, by (2.1.8) if $x \in E(a, v)$, we have that

$$\lim_{t \to 0} \frac{d_a(x) - d_a(x \cdot \delta_t e^{\bar{v}}) - \langle \nabla_{\mathbb{G}} d_a(x), \pi_x(\delta_t e^{-\bar{v}}) \rangle_x}{|t|} = 0.$$

Hence, by the reverse triangle inequality we can assert that

$$\begin{aligned} |\langle \nabla_{\mathbb{G}} d_a(x), v \rangle_x| &\leq \left| \liminf_{t \to 0} \frac{d_a(x \cdot \delta_t e^{\bar{v}}) - d_a(x)}{t} \right| \leq \liminf_{t \to 0} \frac{d(x, x \cdot \delta_t e^{\bar{v}})}{t} \\ &\leq \limsup_{t \to 0} \frac{d(x, x \cdot \delta_t e^{\bar{v}})}{t} \leq \lim_{t \searrow 0} \frac{1}{t} \int_0^t \varphi(x \cdot \delta_s e^{\bar{v}}, v)\, \mathrm{d}s = \varphi(x, v). \end{aligned}$$

Pick a countable dense subset $F \subset H_x\mathbb{G}$ and put $E(a) = \cap_{y\in F} E(a,y)$. Then $|\Omega \setminus E(a)| = 0$ and for all $x \in E(a)$ and all $v \in H_x\mathbb{G}$ we obtain the same estimate above. $\square$

We observe that the previous Lemma could be proved under more general conditions. Indeed, the distance needs only to be geodesic in its domain.

Lemma 2.4.3. *Let $\varphi \in \mathcal{M}^{\alpha}_{cc}(\Omega)$ be a sub-Finsler convex metric, let $d \in \mathcal{D}_{cc}(\Omega)$ and $\Theta \subset \Omega$ be a countable dense set of Ω. If*

$$\|\varphi(x, \nabla_{\mathbb{G}} d_a(x))\|_{\infty} \leq 1 \quad \forall a \in \Theta,$$

then there exists $N \in \mathcal{N}(\Omega)$ and for every $\gamma \in \mathcal{P}(\Omega, N)$

$$d(\gamma(0), \gamma(1)) \leq \int_0^1 \varphi^{\star}(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t.$$

Proof. The proof is the same as in [94, Lemma 3.2]. $\square$

Theorem 2.4.4. *Let $d \in \mathcal{D}_{cc}(\Omega)$. Then φ_d is a sub-Finsler convex metric. In particular, for almost all $x \in \Omega$ and all $v \in H_x\mathbb{G}$*

$$\varphi_d(x, v) = \lim_{t\to 0} \frac{d(x, x\cdot \delta_t e^{\bar{v}})}{|t|}. \tag{2.4.3}$$

Proof. Take a countable dense subset Θ of Ω and, for each $a \in \Theta$, we consider Σ_a a negligible Borel subset of Ω which contains all points where d_a is not Pansu-differentiable. For every $(x, v) \in H\Omega$ we define

$$\xi(x, v) := \begin{cases} \sup_{a\in\Theta} \left|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\right| & \text{if } x \in \Omega \setminus \bigcup_{a\in\Theta} \Sigma_a; \\ 0 \quad \text{otherwise.} \end{cases}$$

For $\varepsilon > 0$ we define $\xi_\varepsilon : H\Omega \to [0, +\infty)$ as $\xi_\varepsilon(x, v) := \xi(x, v) + \varepsilon\|v\|_x$, that is a Borel measurable function in $H\Omega$ and it is a sub-Finsler convex metric. Indeed, if we take $v_1, v_2 \in H_x\mathbb{G}$ we can estimate in this way

$$\begin{aligned} \xi_\varepsilon(x, v_1 + v_2) &= \sup_{a\in\Theta} \left|\langle \nabla_{\mathbb{G}} d_a(x), v_1 + v_2\rangle_x\right| + \varepsilon\|v_1 + v_2\|_x \\ &\leq \xi(x, v_1) + \xi(x, v_2) + \varepsilon\|v_1 + v_2\|_x \leq \xi_\varepsilon(x, v_1) + \xi_\varepsilon(x, v_2). \end{aligned}$$

The homogeneity w.r.t. the second variable comes from the equality $\mathrm{d}_e\tau_x[\delta^\star_\lambda \bar{v}] = \lambda \mathrm{d}_e\tau_x[\bar{v}]$ where $\bar{v} = \mathrm{d}_x\tau_{x^{-1}}[v]$. Moreover, if $a \in \Theta$ we get that

$$\big|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\big| \le \xi(x,v) \le \xi_\varepsilon(x,v) \quad \text{for a.e. } x\in\Omega \text{ and } v \in H_x\mathbb{G}.$$

Thus, by definition of dual metric, we have

$$\frac{\big|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\big|}{\xi_\varepsilon(x,v)} \le 1 \quad \Rightarrow \quad \big\|\xi^\star_\varepsilon(x, \nabla_{\mathbb{G}} d_a(x))\big\|_\infty \le 1. \tag{2.4.4}$$

Being Θ countable, by Lemma 2.4.3, there exists a Lebesgue null set $N \subset \Omega$ such that, the horizontal curve $\gamma(t) = x \cdot \delta_t e^{\bar{v}}$ belongs to $\mathcal{P}(\Omega, N)$, and for every small $t > 0$ we can infer that

$$d(x, x\cdot \delta_t e^{\bar{v}}) = d(\gamma(0), \gamma(t)) \le \int_0^t \xi_\varepsilon(\gamma(s), \dot{\gamma}(s))\,\mathrm{d}s.$$

Now, we are in position to apply Lemma 2.4.2 to the metric ξ_ε: for every fixed $a \in \Omega$, a.e. $x \in \Omega$ and all $v \in H_x\mathbb{G}$

$$\big|\langle \nabla_{\mathbb{G}} d_a(x), v\rangle_x\big| \le \liminf_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{t} \le \limsup_{t\to 0}\frac{d(x, x\cdot\delta_t e^{\bar{v}})}{t} \le \xi_\varepsilon(x,v).$$

Taking the least upper bound w.r.t. $a \in \Theta$ and letting $\varepsilon \to 0$, we obtain

$$\xi(x,v) \le \liminf_{t\to 0} \frac{d(x, x\cdot\delta_t e^{\bar{v}})}{|t|} \le \limsup_{t\to 0}\frac{d(x, x\cdot\delta_t e^{\bar{v}})}{|t|} \le \xi(x,v).$$

This proves the convexity of the limit, i.e. of the metric derivative on the horizontal bundle.

$\square$

2.5 Definitions of δ_φ and Intrinsic Distance

The present section is devoted to the generalization of the metric results contained in [70]. To this aim, we set two distances introduced respectively in Definition 1.4.1 and Definition 1.3.1 which involve the structure of the sub-Finsler metric.

Definition 2.5.1. *If $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ is a sub-Finsler convex metric, for every $x, y \in \mathbb{G}$ we define the following quantity:*

$$\delta_\varphi(x,y) := \sup\big\{|f(x) - f(y)| \,\big|\, f\colon \mathbb{G}\to\mathbb{R} \text{ Lipschitz}, \ \|\varphi(x, \nabla_{\mathbb{G}} f(x))\|_\infty \le 1\big\}. \tag{2.5.1}$$

Recall that Pansu's Theorem assures that $\nabla_{\mathbb{G}} f(x)$ exists at almost every $x \in \mathbb{G}$ and thus the above definition makes sense. Indeed, we have that δ_φ is a distance on $\mathbb{G}$ and the proof is similar to Lemma 1.4.2.

Definition 2.5.2. *Given any $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$, we define its induced* intrinsic distance *d_φ as*

$$d_\varphi(x,y) \coloneqq \inf_\gamma \int_0^1 \varphi(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \quad \text{for every } x,y \in \mathbb{G}, \tag{2.5.2}$$

where the infimum is taken over all curves $\gamma \in \mathcal{H}\big([0,1],\mathbb{G}\big)$ joining x and y.

The intrinsic distance (2.5.2) shows some differences with respect to Definition 1.3.1. Indeed, the latter is computed over Lipschitz curves and, mostly, on the entire tangent bundle, avoiding transversal curves on null sets. In the light of Theorem 1.3.5 and the results in Chapter 1, we will give some semicontinuity assumptions on the sub-Finsler metric, in order to avoid computations on null and very bad sets.

Lemma 2.5.3. *If $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ is a sub-Finsler convex metric, then d_φ is a geodesic distance belonging to $\mathcal{D}_{cc}(\mathbb{G})$.*

Proof. Since $(\mathbb{G}, d_\varphi)$ is a complete and a locally compact length space, then d_φ is a geodesic distance, thanks to the general result contained in [22, Theorem 2.5.23].
To prove the last assertion it is enough to slightly modify the proof of Lemma 1.3.3 and we are done. □

A further key notion for our treatment is the notion of Finsler metrics on Carnot groups.

Definition 2.5.4. *Given a Carnot group $\mathbb{G}$, we say that a map $F : T\mathbb{G} \to [0,+\infty)$ is a* Finsler metric *if*

- *F is continuous on $T\mathbb{G}$ and smooth on $T\mathbb{G} \setminus \{0\}$,*
- *the Hessian of F^2 is positive definite at any vector $v \in T_x\mathbb{G} \setminus \{0\}$ for every $x \in \mathbb{G}$.*

Moreover, we denote by d_F the length distance on $\mathbb{G}$ induced by F, as in Definition 2.5.2, where the infimum is taken among all curves $\gamma \in \mathrm{Lip}([0,1],\mathbb{G})$ joining x and y.

Now we want to recall a very important approximation result contained in [77] and due to E. Le Donne, D. Lučić and E. Pasqualetto. They introduce the concept of *generalised metric*, which is a lower semicontinuous function on the horizontal bundle and a norm on the fibers of the latter. The original statements are produced in the setting of sub-Finsler manifold, that allows the distribution $X_1, \dots, X_m$ to be rank-varying. Clearly, since Carnot groups have a stratification so that every layers have constant dimension, then we can simplify their claims without difficulties.

Proposition 2.5.5. *Let $\mathbb{G}$ be a Carnot group and let d be any distance on $\mathbb{G}$ that induces the manifold topology. Let us assume that φ is a lower semicontinuous sub-Finsler convex metric. Then there exists a sequence $(F_n)_n$ of Finsler metrics over $H\mathbb{G}$ such that*

$$F_{n-1}(x,v) < F_n(x,v) < \varphi(x,v) \quad \textit{for every } \ n \in \mathbb{N},\, x \in \mathbb{G} \ \ \textit{and}\, v \in \mathbb{R}^m \setminus \{0\}.$$

Moreover, it holds that $F_n(x,v) \searrow \varphi(x,v)$ for every $x \in \mathbb{G}$ and $v \in \mathbb{R}^m$.

In particular, they show in [77, Theorem 3.11] that it is possible to approximate a lower semicontinuous metric which is defined on the entire tangent bundle $T\mathbb{G}$.

The last result of this subsection states that any Carnot–Carathéodory distance can be monotonically approximated by distances associated to suitable Finsler metrics. More precisely, the theorems are proved in [77, Theorem 5.1, Corollary 5.2] respectively in the setting of sub-Finsler and sub-Riemannian manifolds.

Theorem 2.5.6. *Let $\mathbb{G}$ be a Carnot group. Then there exists a sequence $(F_n)_n$ of Finsler metrics on $\mathbb{G}$ such that $d_{F_n}(x,y) \searrow d_{cc}(x,y)$ holds for every $x, y \in \mathbb{G}$.*

2.6 Inequality results on $\mathbb{G}$

Before proving one of the main theorems, we recall some basic terminology. Given two Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, and denoting with $\mathrm{L}(\mathbb{B}_1, \mathbb{B}_2)$ the space of all linear and continuous operators $T\colon \mathbb{B}_1 \to \mathbb{B}_2$, it holds that $\mathrm{L}(\mathbb{B}_1, \mathbb{B}_2)$ is a Banach space if endowed with the usual pointwise operations and the operator norm, namely

$$\|T\|_{\mathrm{L}(\mathbb{B}_1,\mathbb{B}_2)} \coloneqq \sup_{v \in \mathbb{B}_1 \setminus \{0\}} \frac{\|T(v)\|_{\mathbb{B}_2}}{\|v\|_{\mathbb{B}_1}} \quad \text{for every } T \in \mathrm{L}(\mathbb{B}_1, \mathbb{B}_2).$$

Remark 2.6.1. Given a smooth map $\varphi\colon M \to N$ between two smooth manifolds M, N and a point $x \in M$, we denote by $\mathrm{d}_x\varphi\colon T_xM \to T_{\varphi(x)}N$ the differential of φ at x. We recall that if $\gamma\colon [0,1] \to M$ is an absolutely continuous curve in M, then $\sigma \coloneqq \varphi \circ \gamma$ is an absolutely continuous curve in N and it holds that

$$\dot{\sigma}(t) = \mathrm{d}_{\gamma(t)}\varphi[\dot{\gamma}(t)] \quad \text{for a.e. } t \in [0,1]. \tag{2.6.1}$$

We also point out that

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta_t e^v = \mathrm{d}_e\tau_{\delta_t e^v}[v] \quad \text{for every } v \in H_e\mathbb{G} \text{ and } t \in (0,1). \tag{2.6.2}$$

Indeed, calling γ the unique curve satisfying (2.1.2) and defining $\gamma^t(s) \coloneqq \gamma(ts)$ for all $t \in (0,1)$ and $s \in [0,1]$, we may compute

$$\frac{\mathrm{d}}{\mathrm{d}s}\gamma^t(s) = t\dot{\gamma}(ts) = t\,\mathrm{d}_e\tau_{\gamma(ts)}[v] = \mathrm{d}_e\tau_{\gamma^t(s)}[tv] \quad \text{for every } s \in (0,1).$$

The latter shows that γ^t fulfills the ODE defining tv, so that property (iii) of Lemma 2.1.7 yields $\gamma(t) = \gamma^t(1) = e^{tv} = \delta_t e^v$ for every $t \in (0,1)$ and accordingly the identity claimed in (2.6.2) is proved.

As we already said, if ψ is a Finsler or sub-Finsler metric, the metric derivative φ_{δ_ψ}, could be very different from ψ (cf. [65, Example 1.5]). Our purpose is to show that, given a sub-Finsler convex metric ψ, the metric derivative with respect to d_ψ is bounded above by ψ almost everywhere. Moreover, we will show that the equality holds, for instance, when ψ is lower semicontinuous. Finally, let us mention that all the results we will present are contained in the work [54, Section 5].

Theorem 2.6.2. *Let $\psi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then the following properties are verified:*

i) *It holds that*

$$\textit{for a.e. } x \in \mathbb{G}, \quad \varphi_{d_\psi}(x,v) \leq \psi(x,v) \quad \textit{ for every } v \in H_x\mathbb{G}.$$

ii) *If ψ is upper semicontinuous, then*

$$\varphi_{d_\psi}(x,v) \leq \psi(x,v) \quad \textit{ for every } (x,v) \in H\mathbb{G}.$$

iii) *If ψ is lower semicontinuous, then*

$$\varphi_{d_\psi}(x,v) \geq \psi(x,v) \quad \textit{ for every } (x,v) \in H\mathbb{G}.$$

In particular, for a.e. $x \in \mathbb{G}$ it holds that $\varphi_{d_\psi}(x,v) = \psi(x,v)$ for every $v \in H_x\mathbb{G}$.

Proof.

$\boxed{i)}$ Given $x \in \mathbb{G}$, $v \in H_e\mathbb{G}$, and $t > 0$, we define the curve $\gamma = \gamma_{x,v,t} \colon [0,1] \to \mathbb{G}$ as

$$\gamma(s) \coloneqq x \cdot \delta_{ts} e^v \quad \text{ for every } s \in [0,1].$$

Notice that γ is horizontal and joins x to $x \cdot \delta_t e^v$. We can compute

$$\dot\gamma(s) = \frac{\mathrm{d}}{\mathrm{d}s}\tau_x\big(\delta_{ts}e^v\big) \overset{(2.6.1)}{=} \mathrm{d}_{\delta_{ts}e^v}\tau_x\left[\frac{\mathrm{d}}{\mathrm{d}s}\delta_{ts}e^v\right] \overset{(2.6.2)}{=} \mathrm{d}_e\tau_{x\cdot\delta_{ts}e^v}[tv] \quad \text{ for every } s \in (0,1).$$

Therefore, we may estimate

$$\begin{aligned} d_\psi(x, x\cdot\delta_t e^v) &\leq \int_0^1 \psi(\gamma(s),\dot\gamma(s))\,\mathrm{d}s = t\int_0^1 \psi\big(x\cdot\delta_{ts}e^v, \mathrm{d}_e\tau_{x\cdot\delta_{ts}e^v}[v]\big)\,\mathrm{d}s \\ &= \int_0^t \psi\big(x\cdot\delta_s e^v, \mathrm{d}_e\tau_{x\cdot\delta_s e^v}[v]\big)\,\mathrm{d}s. \end{aligned} \tag{2.6.3}$$

The next argument closely follows along the lines of Lemma 2.4.1. Fix a dense sequence $(v_i)_i$ in the unit sphere of $H_e\mathbb{G}$ (w.r.t. the norm $\|\cdot\|_e$). Define $v_i(x) \coloneqq \mathrm{d}_e\tau_x[v_i]$ for every $i \in \mathbb{N}$ and $x \in \mathbb{G}$, so that $(v_i(x))_i$ is a dense sequence in the unit sphere of $H_x\mathbb{G}$ (w.r.t. the norm $\|\cdot\|_x$). By using Lebesgue's differentiation theorem and Fubini's theorem, we see that the set Γ_i of all couples $(y,r) \in \mathbb{G}\times\mathbb{R}$ such that

$$\psi\big(y\cdot\delta_r e^{v_i}, \mathrm{d}_e\tau_{y\cdot\delta_r e^{v_i}}[v_i]\big) = \lim_{t\searrow 0}\frac{1}{t}\int_0^t \psi\big(y\cdot\delta_{r+s}e^{v_i}, \mathrm{d}_e\tau_{y\cdot\delta_{r+s}e^{v_i}}[v_i]\big)\,\mathrm{d}s \tag{2.6.4}$$

has zero $\mathcal{L}^{n+1}$-measure. By using Fubini's theorem again, we can find $r \in \mathbb{R}$ such that for any $i \in \mathbb{N}$ there exists a $\mathcal{L}^n$-full set $N_i \subset \mathbb{G}$ such that (2.6.4) holds for every point $y \in \mathbb{G} \setminus N_i$. Let us consider the set $N \coloneqq \bigcup_{i\in\mathbb{N}} \sigma_{\delta_r e^{v_i}}(N_i)$, where $\sigma_z \colon \mathbb{G} \to \mathbb{G}$ stands for the right-translation map $\sigma_z w \coloneqq w \cdot z$. The right-invariance of $\mathcal{L}^n$ grants that N is $\mathcal{L}^n$-negligible. Given that

$$\psi(x, v_i(x)) = \lim_{t\searrow 0} \frac{1}{t} \int_0^t \psi\big(x \cdot \delta_s e^{v_i}, \mathrm{d}_e\tau_{x\cdot\delta_s e^{v_i}}[v_i]\big)\, \mathrm{d}s \quad \text{for every } i \in \mathbb{N} \text{ and } x \in \mathbb{G} \setminus N, \tag{2.6.5}$$

we can conclude that

$$\begin{aligned} \varphi_{d_\psi}(x, v_i(x)) &= \lim_{t\searrow 0} \frac{d_\psi(x, x \cdot \delta_t e^{v_i})}{t} \overset{(2.6.3)}{\leq} \lim_{t\searrow 0} \frac{1}{t} \int_0^t \psi\big(x \cdot \delta_s e^{v_i}, \mathrm{d}_e\tau_{x\cdot\delta_s e^{v_i}}[v_i]\big)\, \mathrm{d}s \\ &\overset{(2.6.5)}{=} \psi(x, v_i(x)) \quad \text{for every } i \in \mathbb{N} \text{ and } x \in \mathbb{G} \setminus N. \end{aligned}$$

Since $\psi(x, \cdot)$ is continuous and positively 1-homogeneous, and $(v_i(x))_i$ is dense in the unit $\|\cdot\|_x$-sphere of $H_x\mathbb{G}$, we deduce that $\varphi_{d_\psi}(x, w) \leq \psi(x, w)$ for every $x \in \mathbb{G} \setminus N$ and $w \in H_x\mathbb{G}$.

$\boxed{ii)}$ Suppose ψ is upper semicontinuous. Let $(x, v) \in H\mathbb{G}$ be fixed. Given any $\varepsilon > 0$, we can thus find $t_\varepsilon > 0$ such that, setting $\bar{v} \coloneqq \mathrm{d}_x\tau_{x^{-1}}[v]$ for brevity, it holds that

$$\psi\big(x \cdot \delta_t e^{\bar{v}}, \mathrm{d}_e\tau_{x\cdot\delta_t e^{\bar{v}}}[\bar{v}]\big) \leq \psi(x, v) + \varepsilon \quad \text{for every } t \in (0, t_\varepsilon). \tag{2.6.6}$$

In particular, we may estimate

$$\varphi_{d_\psi}(x, v) = \lim_{t\searrow 0} \frac{d_\psi(x, x \cdot \delta_t e^{\bar{v}})}{t} \overset{(2.6.3)}{\leq} \lim_{t\searrow 0} \frac{1}{t} \int_0^t \psi\big(x \cdot \delta_s e^{\bar{v}}, \mathrm{d}_e\tau_{x\cdot\delta_s e^{\bar{v}}}[\bar{v}]\big)\, \mathrm{d}s \overset{(2.6.6)}{\leq} \psi(x, v) + \varepsilon.$$

Thanks to the arbitrariness of ε, we can conclude that $\varphi_{d_\psi}(x, v) \leq \psi(x, v)$, as desired.

$\boxed{iii)}$ Suppose ψ is lower semicontinuous. First of all, let us extend $\|\cdot\|_e$ to a Hilbert norm (still denoted by $\|\cdot\|_e$) on the whole $T_e\mathbb{G} = \mathfrak{g}$, then by left-invariance we obtain a Hilbert norm $\|\cdot\|_x$ on each tangent space $T_x\mathbb{G}$. Throughout the rest of the proof, we assume that $T_x\mathbb{G}$ is considered with respect to such norm $\|\cdot\|_x$. Moreover, choose any norm $\mathsf{n} : \mathfrak{g} \to [0, +\infty)$ on the Lie algebra which extends $\psi(e, \cdot)$, so that $\mathsf{n} \leq \lambda\|\cdot\|_e$ for some $\lambda > 0$.

Without loss of generality, up to replacing ψ with the translated metric ψ_x, defined as $\psi_x(y, v) \coloneqq \psi\big(x \cdot y, \mathrm{d}_y\tau_x[v]\big)$ for every $(y, v) \in H\mathbb{G}$, it is sufficient to prove the statement only for $x = e$. Then let $v \in H_e\mathbb{G}$ be fixed. For any $t > 0$ we have that the horizontal curve $[0,1] \ni s \mapsto \delta_{st}e^v \in \mathbb{G}$ is a competitor for $d_\psi(e, \delta_t e^v)$, thus we may estimate

$$\begin{aligned} d_\psi(e, \delta_t e^v) &\leq \int_0^1 \psi(\delta_{st}e^v, t\, \mathrm{d}_e\tau_{\delta_{st}e^v}[v])\, \mathrm{d}s = t \int_0^1 \psi(\delta_{st}e^v, \mathrm{d}_e\tau_{\delta_{st}e^v}[v])\, \mathrm{d}s \\ &\leq \alpha t \int_0^1 \|\mathrm{d}_e\tau_{\delta_{st}e^v}[v]\|_{\delta_{st}e^v}\, \mathrm{d}s = \alpha t\|v\|_e, \end{aligned}$$

where the last equality comes from the left invariance of the norm. This means that, in order to compute $d_\psi(e, \delta_t e^v)$, it is sufficient to consider those horizontal curves $\gamma_s \colon [0,1] \to \mathbb{G}$ joining e to $\delta_t e^v$ and satisfying $\int_0^1 \|\dot\gamma_s\|_{\gamma_s}\, \mathrm{d}s \leq \alpha \int_0^1 \psi(\gamma_s, \dot\gamma_s)\, \mathrm{d}s \leq \alpha^2 t\|v\|_e$. We can also assume without loss of generality that any such curve γ is parametrized by constant speed with respect to the metric $\|\cdot\|_x$. All in all, we have shown that

$$d_\psi(e, \delta_t e^v) = \inf_{\gamma \in \mathcal{C}_t} \int_0^1 \psi(\gamma_s, \dot\gamma_s)\, \mathrm{d}s \quad \text{for every } t > 0, \tag{2.6.7}$$

where the family $\mathcal{C}_t$ of curves is defined as

$$\mathcal{C}_t \coloneqq \left\{ \gamma \colon [0,1] \to \mathbb{G} \text{ horizontal} \;\middle|\; \gamma_0 = e,\, \gamma_1 = \delta_t e^v,\, \|\dot\gamma_s\|_{\gamma_s} \equiv \int_0^1 \|\dot\gamma_s\|_{\gamma_s}\, \mathrm{d}s \leq \alpha^2 t\|v\|_e \right\}.$$

Now fix any $\varepsilon > 0$. Since the map $\exp^{-1} \colon \mathbb{G} \to \mathfrak{g}$ is a diffeomorphism, we can consider its differential $\mathrm{d}_x\exp^{-1} \colon T_x\mathbb{G} \to T_{\exp^{-1}(x)}\mathfrak{g} \cong \mathfrak{g}$ at any point $x \in \mathbb{G}$. Let us observe that $\exp^{-1}$ is smooth, and $\mathrm{d}_e\exp^{-1} = \mathrm{d}_e\tau_{e^{-1}} = \mathrm{id}_{\mathfrak{g}}$.

Since ψ is lower semicontinuous and by the previous argument, we can find $r > 0$ such that

$$\psi(x, v) \geq \psi(e, \mathrm{d}_x\tau_{x^{-1}}[v]) - \varepsilon \quad \text{for every } x \in B(e,r) \text{ and } v \in H_x\mathbb{G},\ \|v\|_x \leq 1, \tag{2.6.8a}$$

$$\big\|\mathrm{d}_x\exp^{-1} - \mathrm{d}_x\tau_{x^{-1}}\big\|_{\mathrm{L}(T_x\mathbb{G},\mathfrak{g})} \leq \varepsilon \quad \text{for every } x \in B(e,r), \tag{2.6.8b}$$

where $B(e,r) \equiv B_{d_{cc}}(e,r)$. In particular, given any $t > 0$ with $\alpha^2 t\|v\|_e < r$ and $\gamma \in \mathcal{C}_t$, we have that $d_{cc}(e, \gamma_s) \leq s\alpha^2 t\|v\|_e < r$ for every $s \in [0,1]$ and $\|\dot\gamma_s\|_{\gamma_s} \leq \alpha^2 t\|v\|_e$ for a.e. $s \in [0,1]$, thus accordingly (2.6.8a) and (2.6.8b) yield

$$\psi(\gamma_s, \dot\gamma_s) \geq \psi(e, \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]) - \alpha^2 t\|v\|_e\varepsilon \quad \text{for a.e. } s \in [0,1], \tag{2.6.9a}$$

$$\big\|\mathrm{d}_{\gamma_s}\exp^{-1} - \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}\big\|_{\mathrm{L}(T_{\gamma_s}\mathbb{G},\mathfrak{g})} \leq \varepsilon \quad \text{for a.e. } s \in [0,1], \tag{2.6.9b}$$

respectively. Therefore, for any $t > 0$ with $\alpha^2 t\|v\|_e < r$ and $\gamma \in \mathcal{C}_t$, we may estimate

$$
\begin{aligned}
&\left|\psi\left(e, \int_0^1 \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]\,\mathrm{d}s\right) - \mathsf{n}\left(\int_0^1 \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\right)\right| \\
\leq\ & \mathsf{n}\left(\int_0^1 \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]\,\mathrm{d}s - \int_0^1 \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\right) \leq \lambda\left\|\int_0^1 \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s] - \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\right\|_e \\
\leq\ & \lambda\int_0^1 \left\|(\mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}} - \mathrm{d}_{\gamma_s}\exp^{-1})[\dot\gamma_s]\right\|_e \mathrm{d}s \leq \lambda\int_0^1 \left\|\mathrm{d}_{\gamma_s}\exp^{-1} - \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}\right\|_{\mathrm{L}(T_{\gamma_s}\mathbb{G},\mathfrak{g})}\|\dot\gamma_s\|_{\gamma_s}\,\mathrm{d}s \\
\overset{(2.6.9\mathrm{b})}{\leq}\ & \lambda\varepsilon\int_0^1 \|\dot\gamma_s\|_{\gamma_s}\,\mathrm{d}s \leq \lambda\varepsilon\alpha^2 t\|v\|_e,
\end{aligned}
$$

whence it follows that

$$
\begin{aligned}
\int_0^1 \psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s \overset{(2.6.9\mathrm{a})}{\geq}\ & \int_0^1 \psi(e, \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s])\,\mathrm{d}s - \alpha^2 t\|v\|_e\varepsilon \\
\geq\ & \psi\left(e, \int_0^1 \mathrm{d}_{\gamma_s}\tau_{\gamma_s^{-1}}[\dot\gamma_s]\,\mathrm{d}s\right) - \alpha^2 t\|v\|_e\varepsilon \\
\geq\ & \mathsf{n}\left(\int_0^1 \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s\right) - (\lambda+1)\alpha^2 t\|v\|_e\varepsilon.
\end{aligned}
\tag{2.6.10}
$$

where in the second inequality we applied Jensen's inequality to $\psi(e,\cdot)$. Now consider the curve σ in the Hilbert space $(\mathfrak{g},\mathsf{n})$, which is given by $\sigma_s := \exp^{-1}(\gamma_s)$ for every $s \in [0,1]$. It holds that σ is absolutely continuous and satisfies $\dot\sigma_s = \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]$ for a.e. $s \in [0,1]$, thus

$$
\begin{aligned}
tv = tv - 0_{\mathfrak{g}} &= \exp^{-1}(\delta_t e^v) - \exp^{-1}(e) = \exp^{-1}(\gamma_1) - \exp^{-1}(\gamma_0) = \sigma_1 - \sigma_0 \\
&= \int_0^1 \dot\sigma_s\,\mathrm{d}s = \int_0^1 \mathrm{d}_{\gamma_s}\exp^{-1}[\dot\gamma_s]\,\mathrm{d}s.
\end{aligned}
\tag{2.6.11}
$$

By combining (2.6.10) and (2.6.11), we obtain for any $t > 0$ with $\alpha^2 t\|v\|_e < r$ and $\gamma \in \mathcal{C}_t$ that

$$
\int_0^1 \psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s \geq \mathsf{n}(tv) - (\lambda+1)\alpha^2 t\|v\|_e\varepsilon = \big[\psi(e,v) - (\lambda+1)\alpha^2\|v\|_e\,\varepsilon\big]t. \tag{2.6.12}
$$

We are now in a position to conclude the proof of the statement: given $t > 0$ with $\alpha^2 t\|v\|_e < r$, one has that

$$
\frac{d_\psi(e,\delta_t e^v)}{t} \overset{(2.6.7)}{=} \inf_{\gamma\in\mathcal{C}_t}\frac{1}{t}\int_0^1 \psi(\gamma_s,\dot\gamma_s)\,\mathrm{d}s \overset{(2.6.12)}{\geq} \psi(e,v) - (\lambda+1)\alpha^2\|v\|_e\,\varepsilon. \tag{2.6.13}
$$

By letting $t \searrow 0$, we thus deduce that

$$\varphi_{d_\psi}(e,v) = \limsup_{t \searrow 0} \frac{d_\psi(e, \delta_t e^v)}{t} \overset{(2.6.13)}{\geq} \psi(e,v) - (\lambda+1)\alpha^2 \|v\|_e \varepsilon. \tag{2.6.14}$$

Finally, by letting $\varepsilon \searrow 0$ in (2.6.14) we conclude that $\varphi_{d_\psi}(e,v) \geq \psi(e,v)$, as desired. $\square$

Corollary 2.6.3. *If ψ is a continuous sub-Finsler convex metric, then*

$$\varphi_{d_\psi}(x,v) = \psi(x,v) \quad \text{for every } (x,v) \in H\mathbb{G}.$$

Proof. It is an immediate consequence of assertions ii) and iii) of Theorem 2.6.2. $\square$

The crucial observation below states that δ_φ coincides with the intrinsic distance $d_{\varphi^\star}$ when we assume that the sub-Finsler metric is lower semicontinuous. This will allow us to show the same result when φ is upper semicontinuous, thanks to an approximation argument given in Theorem 2.5.6.

Theorem 2.6.4. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then it holds that $\delta_\varphi(x,y) \leq d_{\varphi^\star}(x,y)$. Moreover, if φ is lower semicontinuous, then*

$$\delta_\varphi(x,y) = d_{\varphi^\star}(x,y) \quad \text{for every } x,y \in \mathbb{G}.$$

Proof. Let $x, y \in \mathbb{G}$ be fixed. To prove the first part of the statement, pick any Lipschitz function f with $\|\varphi(z, \nabla_{\mathbb{G}} f(z))\|_\infty \leq 1$ and any horizontal curve $\gamma : [0,1] \to \mathbb{G}$ joining x and y such that

$$\mathcal{H}^1\big(\gamma \cap \{z \in \mathbb{G} \,:\, \varphi(z, \nabla_{\mathbb{G}} f(z)) > 1\}\big) = 0.$$

These are competitors for $\delta_\varphi(x,y)$ and $d_{\varphi^\star}(x,y)$, respectively. Then we can estimate

$$\begin{aligned}
\big|f(x) - f(y)\big| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))\,\mathrm{d}t \right| = \left| \int_0^1 \langle \nabla_{\mathbb{G}} f(\gamma(t)), \dot\gamma(t) \rangle_{\gamma(t)}\,\mathrm{d}t \right| \\
&\leq \int_0^1 \big|\langle \nabla_{\mathbb{G}} f(\gamma(t)), \dot\gamma(t) \rangle_{\gamma(t)}\big|\,\mathrm{d}t \leq \int_0^1 \varphi(\gamma(t), \nabla_{\mathbb{G}} f(\gamma(t)))\, \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \\
&\leq \big\|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\big\|_\infty \int_0^1 \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq \int_0^1 \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t,
\end{aligned}$$

whence it follows that $\delta_\varphi(x,y) \le d_{\varphi^\star}(x,y)$.

Now suppose φ is lower semicontinuous. Define the function $f : \mathbb{G} \to \mathbb{R}$ as $f(\cdot) \coloneqq d_{\varphi^\star}(x,\cdot)$ and since $d_{\varphi^\star}(x,y) \le \alpha^{-1} d_{cc}(x,y)$ everywhere, we have that f is Lipschitz. Fix any point $z \in \mathbb{G}$ such that $\nabla_{\mathbb{G}} f(z)$ exists and a horizontal vector $v \in H_z\mathbb{G}$. Pick a horizontal curve $\gamma\colon [0,\varepsilon] \to \mathbb{G}$ of class C^1 such that $\gamma(0) = z$ and $\dot\gamma(0) = v$. Thanks to the continuity of $t \mapsto (\gamma(t), \dot\gamma(t))$ and the upper semicontinuity of $\varphi^\star$, granted by Lemma 2.2.5, we obtain that $\limsup_{t\searrow 0} \fint_0^t \varphi^\star(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \le \varphi^\star(\gamma(0),\dot\gamma(0)) = \varphi^\star(z,v)$, whence, by (2.1.3), it follows that

$$\begin{aligned}\langle \nabla_{\mathbb{G}} f(z), v\rangle_z &= \lim_{t\searrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} \le \limsup_{t\searrow 0} \frac{f(\gamma(t)) - f(\gamma(0))}{d_{\varphi^\star}(\gamma(t),\gamma(0))} \limsup_{t\searrow 0} \frac{d_{\varphi^\star}(\gamma(t),\gamma(0))}{t} \\ &\le \limsup_{t\searrow 0} \frac{\left|d_{\varphi^\star}(x,\gamma(t)) - d_{\varphi^\star}(x,\gamma(0))\right|}{d_{\varphi^\star}(\gamma(t),\gamma(0))} \limsup_{t\searrow 0} \fint_0^t \varphi^\star(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \le \varphi^\star(z,v).\end{aligned}$$

By arbitrariness of $v \in H_z\mathbb{G}$, we deduce that $\varphi(z, \nabla_{\mathbb{G}} f(z)) \le 1$. Therefore, f is a competitor for $\delta_\varphi(x,y)$. This implies that $\delta_\varphi(x,y) \ge |f(x) - f(y)| = d_{\varphi^\star}(x,y)$. $\square$

In particular, the last part of the proof shows that the supremum appearing in the definition of $\delta_\varphi(x,y)$ is actually a maximum.

The upper semicontinuity of the sub-Finsler metric φ is crucial for our proof, because it allows us to approximate the dual metric $\varphi^\star$ through a family of continuous Finsler metrics as in Proposition 2.5.5. This leads to the approximation of induced intrinsic distances, in the sense of Theorem 2.5.6.

Corollary 2.6.5. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Suppose φ is upper semicontinuous. Then, for every $x, y \in \mathbb{G}$ it holds that $\delta_\varphi(x,y) = d_{\varphi^\star}(x,y)$.*

Proof. Lemma 2.2.5 ensures that $\varphi^\star$ is lower semicontinuous. We set $\widetilde{\varphi}^\star\colon T\mathbb{G} \to [0,+\infty)$ as

$$\widetilde{\varphi}^\star(x,v) \coloneqq \begin{cases} \varphi^\star(x,v), & \text{if } (x,v) \in H\mathbb{G}, \\ +\infty, & \text{if } (x,v) \in T\mathbb{G} \setminus H\mathbb{G}. \end{cases}$$

Observe that $H\mathbb{G}$ is closed in $T\mathbb{G}$ and thus $\widetilde{\varphi}^\star$ is lower semicontinuous. Thanks to [77, Theorem 3.11], there exists a sequence $F_n : T\mathbb{G} \to [0,+\infty)$ of Finsler metrics on $\mathbb{G}$ such that

$F_n(x,v) \nearrow \tilde{\varphi}^\star(x,v)$ for every $(x,v) \in T\mathbb{G}$. Setting

$$\varphi_n \colon H\mathbb{G} \to [0+\infty) \quad \text{as } \varphi_n \coloneqq (F_n|_{H\mathbb{G}})^\star,$$

we obtain that $\varphi_n \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ and $\varphi^\star_n(x,v) \nearrow \varphi^\star(x,v)$ for every $(x,v) \in H\mathbb{G}$. Therefore $\varphi_n(x,v) \searrow \varphi(x,v)$ for every $(x,v) \in H\mathbb{G}$. In particular, the inequality $\varphi_n \geq \varphi$ holds for all $n \in \mathbb{N}$. This implies that any competitor f for δ_{φ_n} is a competitor for δ_φ, so that accordingly

$$\delta_{\varphi_n}(x,y) \leq \delta_\varphi(x,y), \qquad \text{for every } n \in \mathbb{N} \text{ and } x,y \in \mathbb{G}. \tag{2.6.15}$$

Moreover, since the infimum in the definition of d_{F_n} is computed with respect to all Lipschitz curves, while the infimum in the definition of $d_{\varphi^\star_n}$ is just over horizontal curves, for every $x,y \in \mathbb{G}$ we get that

$$d_{F_n}(x,y) \leq d_{\varphi^\star_n}(x,y) \leq d_{\varphi^\star}(x,y) \quad \text{ for every } n \in \mathbb{N}. \tag{2.6.16}$$

From the convergence of F_n to $\widetilde{\varphi}^\star$ we deduce that $d_{F_n}(x,y) \to d_{\varphi^\star}(x,y)$ for every $x,y \in \mathbb{G}$ (cf. the proof of [77, Theorem 5.1]), and thus

$$d_{\varphi^\star}(x,y) = \lim_{n\to\infty} d_{\varphi^\star_n}(x,y) \quad \text{ for every } x,y \in \mathbb{G}. \tag{2.6.17}$$

Finally, since φ_n is lower semicontinuous (actually, continuous) by Lemma 2.2.5, we know from the second part of Theorem 2.6.4 that

$$\delta_{\varphi_n}(x,y) = d_{\varphi^\star_n}(x,y) \quad \text{for every } n \in \mathbb{N}. \tag{2.6.18}$$

All in all, we obtain that

$$d_{\varphi^\star}(x,y) \overset{(2.6.17)}{=} \lim_{n\to\infty} d_{\varphi^\star_n}(x,y) \overset{(2.6.18)}{=} \lim_{n\to\infty} \delta_{\varphi_n}(x,y) \overset{(2.6.15)}{\leq} \delta_\varphi(x,y) \quad \text{ for every } x,y \in \mathbb{G}.$$

Since the converse inequality $d_{\varphi^\star} \geq \delta_\varphi$ is granted by the first part of Theorem 2.6.4, we conclude that $\delta_\varphi = d_{\varphi^\star}$, as required. $\square$

2.7 Main results and consequences on $\mathbb{G}$

Now we are ready to present the generalization of Theorem 1.4.6 in Carnot groups.

Theorem 2.7.1. *Let $\varphi \in \mathcal{M}^{\alpha}_{cc}(\mathbb{G})$ be an upper semicontinuous sub-Finsler convex metric. Then for any locally Lipschitz function $f : \mathbb{G} \to \mathbb{R}$ we have that*

$$\varphi(x, \nabla_{\mathbb{G}} f(x)) = \mathrm{Lip}_{\delta_\varphi} f(x) \quad \textit{for a.e. } x \in \mathbb{G}.$$

Proof.

$\boxed{\leq}$ Since both sides are positively 1-homogeneous with respect to f, we only need to show that, if $\mathrm{Lip}_{\delta_\varphi} f(x) = 1$, then $\varphi(x, \nabla_{\mathbb{G}} f(x)) \leq 1$ for a.e. $x \in \mathbb{G}$. By Corollary 2.6.5, $\mathrm{Lip}_{\delta_\varphi} f(x) = \mathrm{Lip}_{d_{\varphi^\star}} f(x)$, hence if we fix $(x, v) \in H\mathbb{G}$, thanks to the equalities (2.1.3) and (2.1.8), we obtain:

$$\begin{aligned}\langle \nabla_{\mathbb{G}} f(x), v\rangle_x = \lim_{t\to 0} \frac{f(x\cdot \delta_t e^{\bar v}) - f(x)}{t} &\leq \limsup_{t\to 0} \frac{d_{\varphi^\star}(x, x\cdot \delta_t e^{\bar v})}{t} \cdot \limsup_{t\to 0} \frac{|f(x\cdot \delta_t e^{\bar v}) - f(x)|}{d_{\varphi^\star}(x, x\cdot\delta_t e^{\bar v})}\\ &\leq \varphi_{d_{\varphi^\star}}(x, v)\,\mathrm{Lip}_{d_{\varphi^\star}} f(x) \leq \varphi^\star(x, v),\end{aligned}$$

where in the last inequality we used item i) of Theorem 2.6.2. By arbitrariness of $v \in H_x\mathbb{G}$ and the fact that

$$\varphi(x, \nabla_{\mathbb{G}} f(x)) = \varphi^{\star\star}(x, \nabla_{\mathbb{G}} f(x)) \leq 1,$$

we get the conclusion.

$\boxed{\geq}$ Thanks to a convolution argument, we can find a sequence $(f_n)_n \subset C^1(\mathbb{G})$ such that $f_n \to f$ uniformly on compact sets and $\nabla_{\mathbb{G}} f_n \to \nabla_{\mathbb{G}} f$ in the almost everywhere sense. Recall that any C^1-function is locally Lipschitz. Fix any $x \in \mathbb{G}$ such that $\nabla_{\mathbb{G}} f_n(x)$ exists for all $n \in \mathbb{N}$ and $\nabla_{\mathbb{G}} f_n(x) \to \nabla_{\mathbb{G}} f(x)$ as $n \to \infty$. Now let $\varepsilon > 0$ be fixed. Then we can choose $r' > 0$ and $\bar n \in \mathbb{N}$ so that

$$\sup_{B(x,2r')} |f_{\bar n} - f| \leq \varepsilon \quad \text{and} \quad \varphi(x, \nabla_{\mathbb{G}} f_{\bar n}(x) - \nabla_{\mathbb{G}} f(x)) \leq \varepsilon,$$

where the ball is with respect to the distance $d^\star_\varphi$. Calling $g \coloneqq f_{\bar n}$ and being $z \mapsto \nabla_{\mathbb{G}} g(z)$ continuous, we deduce that $z \mapsto \varphi(z, \nabla_{\mathbb{G}} g(z))$ is upper semicontinuous, thus there exists

$r < r'$ such that

$$\varphi(y, \nabla_{\mathbb{G}} g(y)) \leq \varphi(x, \nabla_{\mathbb{G}} g(x)) + \varepsilon \quad \text{for every } y \in B_{2r}(x).$$

Fix any point $y \in B_r(x)$ and consider a horizontal curve $\gamma : [0,1] \to \mathbb{G}$ such that $\gamma(0) = x$, $\gamma(1) = y$ with $\gamma([0,1]) \subset B_{2r}(x)$. We can estimate in this way:

$$\begin{aligned}
|f(x) - f(y)| \leq |g(x) - g(y)| + 2\varepsilon &\leq \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g(\gamma(t))\,\mathrm{d}t + 2\varepsilon \\
&\leq \int_0^1 \varphi(\gamma(t), \nabla_{\mathbb{G}} g(\gamma(t)))\varphi^\star(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t + 2\varepsilon \\
&\leq \Big(\varphi(x, \nabla_{\mathbb{G}} g(x)) + \varepsilon\Big) \int_0^1 \varphi^\star(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t + 2\varepsilon \\
&\leq \Big(\varphi(x, \nabla_{\mathbb{G}} f(x)) + 2\varepsilon\Big) \int_0^1 \varphi^\star(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t + 2\varepsilon.
\end{aligned}$$

By taking the infimum over all $\gamma \in \mathcal{H}\big([0,1], B_{2r}(x)\big)$, we obtain that

$$|f(x) - f(y)| \leq \Big(\varphi(x, \nabla_{\mathbb{G}} f(x)) + 2\varepsilon\Big) d_{\varphi^\star}(x,y) + 2\varepsilon,$$

whence by letting $\varepsilon \to 0$ we obtain that

$$\frac{|f(x) - f(y)|}{d_{\varphi^\star}(x,y)} \leq \varphi(x, \nabla_{\mathbb{G}} f(x)).$$

Finally, by letting $y \to x$ we conclude that

$$\operatorname{Lip}_{\delta_\varphi} f(x) = \operatorname{Lip}_{d_{\varphi^\star}} f(x) \leq \varphi(x, \nabla_{\mathbb{G}} f(x)),$$

as required. □

To conclude, in Proposition 2.7.3 we prove that in the definition (2.5.1) of the distance δ_φ it is sufficient to consider smooth functions. Before passing to the proof of this claim, we prove the following technical result.

Lemma 2.7.2. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Then it holds that*

$$\operatorname{LIP}_{d_{\varphi^\star}}(f) = \operatorname*{ess\,sup}_{x \in \mathbb{G}} \operatorname{Lip}_{d_{\varphi^\star}} f(x) \quad \text{for every } f \in \operatorname{LIP}_{d_{\varphi^\star}}(\mathbb{G}), \tag{2.7.1}$$

where $\operatorname{LIP}_{d^\star_\varphi}(f) \in [0, +\infty)$ is the (global) Lipschitz constant of $f \in \operatorname{LIP}_{d^\star_\varphi}(\mathbb{G})$.

Proof. The inequality ($\geq$) is trivial. To prove the converse inequality, we argue by contradiction: suppose there exist $x, y \in \mathbb{G}$ with $x \neq y$, a negligible Borel set $N \subseteq \mathbb{G}$ and $\delta > 0$ such that

$$\frac{\big|f(x) - f(y)\big|}{d_{\varphi^\star}(x,y)} \geq \sup_{z \in \mathbb{G} \setminus N} \mathrm{Lip}_{d_{\varphi^\star}} f(z) + \delta.$$

Given any $\varepsilon > 0$, we can find $\gamma \in \mathcal{H}([0,1], \mathbb{G})$ such that $\gamma(0) = x$, $\gamma(1) = y$, and

$$\int_0^1 \varphi^*(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq d_{\varphi^\star}(x,y) + \varepsilon.$$

Since $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is Lipschitz, hence Pansu-differentiable almost everywhere, we deduce that

$$\begin{aligned}
\big|f(x) - f(y)\big| &\leq \int_0^1 \big|(f \circ \gamma)'(t)\big|\,\mathrm{d}t \leq \int_0^1 \varphi(\gamma(t), \nabla_{\mathbb{G}} f(\gamma(t)))\,\varphi^*(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \\
&= \int_0^1 \mathrm{Lip}_{d_{\varphi^\star}} f(\gamma(t))\,\varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \leq \sup_{z \in \mathbb{G} \setminus N} \mathrm{Lip}_{d_{\varphi^\star}} f(z) \int_0^1 \varphi^\star(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \\
&\leq \left[\frac{\big|f(x) - f(y)\big|}{d_{\varphi^\star}(x,y)} - \delta\right]\big(d_{\varphi^\star}(x,y) + \varepsilon\big).
\end{aligned}$$

By letting $\varepsilon \searrow 0$ in the above estimate, we get $0 \leq -\delta\, d_{\varphi^\star}(x,y)$, which leads to a contradiction. Therefore, also the inequality ($\leq$) in (2.7.1) is proved, whence the statement follows. $\square$

Proposition 2.7.3. *Let $\varphi \in \mathcal{M}^\alpha_{cc}(\mathbb{G})$ be a sub-Finsler convex metric. Suppose φ is upper semicontinuous. Then for any $x, y \in \mathbb{G}$ it holds that*

$$\delta_\varphi(x,y) = \sup\left\{\big|f(x) - f(y)\big| \;\Big|\; f \in C^\infty(\mathbb{G}),\ \big\|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot)\big\|_\infty \leq 1\right\}. \tag{2.7.2}$$

Proof. Denote by $\tilde\delta_\varphi(x,y)$ the quantity in the right-hand side of (2.7.2). Since any competitor for $\tilde\delta_\varphi(x,y)$ is a competitor for $\delta_\varphi(x,y)$, we have that $\delta_\varphi(x,y) \geq \tilde\delta_\varphi(x,y)$. To prove the converse inequality, fix any Lipschitz function $f\colon \mathbb{G} \to \mathbb{R}$ such that $\big\|\varphi(\cdot, \nabla_{\mathbb{G}} f(\cdot))\big\|_\infty \leq 1$. Corollary 2.6.5 and Theorem 2.7.1 grant that $\operatorname{ess\,sup} \mathrm{Lip}_{d_{\varphi^\star}} f \leq 1$, thus Lemma 2.7.2 yields $\mathrm{LIP}_{d_{\varphi^\star}}(f) \leq 1$. Given that $d_{\varphi^\star}$ is an increasing, pointwise limit of Finsler distances by [77, Theorem 3.11], we are in a position to apply Theorem 2.7.4. Thus we obtain a sequence $(f_n)_n \subseteq C^\infty(\mathbb{G}) \cap \mathrm{LIP}_{d_{\varphi^\star}}(\mathbb{G})$ such that $\mathrm{LIP}_{d_{\varphi^\star}}(f_n) \leq 1$ for all $n \in \mathbb{N}$ and $f_n \to f$ uniformly on compact sets. Corollary 2.6.5 and Theorem 2.7.1 imply that $\left\|\varphi(\cdot, \nabla_{\mathbb{G}} f_n(\cdot))\right\|_\infty = \sup \mathrm{Lip}_{d_{\varphi^\star}} f_n \leq 1$, thus f_n is a competitor for $\tilde{\delta}_\varphi(x,y)$. Then we conclude that $\left|f(x) - f(y)\right| = \lim_n \left|f_n(x) - f_n(y)\right| \leq \tilde{\delta}_\varphi(x,y)$, whence it follows that $\delta_\varphi(x,y) \leq \tilde{\delta}_\varphi(x,y)$ by arbitrariness of f. $\square$

2.7.1 Smooth approximation of Lipschitz functions on generalized sub-Finsler manifolds

The purpose is to prove an approximation result for real-valued Lipschitz functions defined on some very weak kind of sub-Finsler manifold. More precisely, we consider a distance d on a smooth manifold that can be obtained as the monotone increasing limit of Finsler distances; this notion covers the case of possibly rank-varying sub-Finsler manifolds, thanks to [77, Theorem 3.11]. In this framework, we prove (see Theorem 2.7.4 below) that any Lipschitz function can be approximated (uniformly on compact sets) by smooth functions having the same Lipschitz constant. This generalizes previous results that were known on sub-Riemannian manifolds, see e.g. [71] and the references therein.

In a metric space (X, d), if $f \in \mathrm{LIP}_d(X)$, we denote by $\mathrm{LIP}_d(f) \in [0, +\infty)$ the (global) Lipschitz constant. Moreover, given a smooth manifold M equipped of a Finsler metric F, we denote by d_F the usual length distance on M induced by F, as in Definition 2.5.2.

Theorem 2.7.4. *Let M be a smooth manifold. Let d be a distance on M having the following property: there exists a sequence $(F_i)_i$ of Finsler metrics on M such that*

$$d_{F_i}(x,y) \nearrow d(x,y) \quad \textit{for every } x, y \in M.$$

Then for any $f \in \mathrm{LIP}_d(M)$ there exists a sequence $(f_n)_n \subseteq C^\infty(M) \cap \mathrm{LIP}_d(M)$ such that

$$\sup_{n\in\mathbb{N}} \mathrm{LIP}_d(f_n) \leq \mathrm{LIP}_d(f), \qquad f_n \to f \ \textit{ uniformly on compact sets.}$$

Proof. Denote $L \coloneqq \mathrm{LIP}_d(f)$ and $d_i \coloneqq d_{F_i}$ for every $i \in \mathbb{N}$. Choose any countable, dense subset $(x_j)_j$ of (M,d). Given any $n \in \mathbb{N}$, we define the function $h_n \in \mathrm{LIP}_d(M)$ as

$$h_n(x) \coloneqq \big(-L\,d(x,x_1)+f(x_1)\big)\vee\cdots\vee\big(-L\,d(x,x_n)+f(x_n)\big)-\frac{1}{n} \quad \text{for every } x \in M.$$

Observe that $\mathrm{LIP}_d(h_n) \leq L$ and that $h_n(x) < h_{n+1}(x) < f(x)$ for every $n \in \mathbb{N}$ and $x \in M$. We claim that $h_n(x) \nearrow f(x)$ for all $x \in M$. In order to prove it, fix any $x \in M$ and $\varepsilon > 0$. Pick some $\bar{n} \in \mathbb{N}$ such that $1/\bar{n} < \varepsilon$ and $d(x,x_{\bar{n}}) < \varepsilon$. Then for every $n \geq \bar{n}$ it holds that

$$h_n(x) \geq -L\,d(x,x_{\bar{n}})+f(x_{\bar{n}})-\frac{1}{n} \geq -L\,\varepsilon+\big(f(x)-L\,d(x,x_{\bar{n}})\big)-\frac{1}{\bar{n}} \geq f(x)-(2L+1)\varepsilon,$$

thus proving the claim. Fix an increasing sequence $(K_n)_n$ of compact sets in M satisfying the following property: given any compact set $K \subseteq M$, there exists $n \in \mathbb{N}$ such that $K \subseteq K_n$. In particular, one has that $\bigcup_n K_n = M$. Notice that $h_n + \frac{1}{n(n+1)} \leq h_{n+1}$ on K_n for all $n \in \mathbb{N}$. Since $d_{F_i} \nearrow d$, there exists $i_n \in \mathbb{N}$ such that the function $g_n \colon M \to \mathbb{R}$, given by

$$g_n(x) \coloneqq \big(-L\,d_{i_n}(x,x_1)+f(x_1)\big)\vee\cdots\vee\big(-L\,d_{i_n}(x,x_n)+f(x_n)\big)-\frac{1}{n} \quad \text{for every } x \in M,$$

satisfies $h_n < g_n < h_n + \frac{1}{n(n+1)}$ on K_n. Note that $g_n \in \mathrm{LIP}_{d_{i_n}}(M)$ and $\mathrm{LIP}_{d_{i_n}}(g_n) = L$. Thanks to a mollification argument, it is possible to build a function $f_n \in C^\infty(M) \cap \mathrm{LIP}_{d_{i_n}}(M)$ such that $\mathrm{LIP}_{d_{i_n}}(f_n) \leq L$ and $g_n < f_n < g_{n+1}$ on K_n. Therefore, for any $n \in \mathbb{N}$ and $x \in K_n$ it holds that the sequence $\big(f_j(x)\big)_{j \geq n}$ is strictly increasing and converging to $f(x)$. This grants that $f_j \to f$ uniformly on K_n for any given $n \in \mathbb{N}$. Hence, our specific choice of $(K_n)_n$ implies that $f_n \to f$ uniformly on compact sets. Finally, the inequality $d_{i_n} \leq d$ yields $f_n \in \mathrm{LIP}_d(M)$ and $\mathrm{LIP}_d(f_n) \leq \mathrm{LIP}_{d_{i_n}}(f_n) \leq L$ for all $n \in \mathbb{N}$, whence the statement follows. $\square$

Chapter 3

Integral Representation of three classes of functionals

3.1 Background and Preliminaries

The second part of this Ph.D. dissertation is devoted to the analysis of general Carathéodory integrand f, with the aim to obtain an integral representation for integral functionals which are not translation-invariant. For this purpose, we need to introduce a further generalization of the ambient space which is given by a general family of linear vector fields with Lipschitz regularity. This class embraces many relevant families studied in literature, among the others, the already developed Carnot groups and Carnot–Carathéodory spaces.

Notation. We let $1 \leq p < +\infty$ and $m, n \in \mathbb{N} \setminus \{0\}$ with $m \leq n$, we denote by Ω an open and bounded subset of $\mathbb{R}^n$ and by $\mathcal{A}$ the family of all open subsets of Ω. Given two open sets A and B, we write $A \Subset B$ whenever $\overline{A} \subseteq B$. We set $\mathcal{A}_0$ to be the subfamily of $\mathcal{A}$ of all the open subsets A of Ω such that $A \Subset \Omega$ and by $\mathcal{B}$ the family of all Borel subsets of Ω. Given $x \in \mathbb{R}^n, r > 0$ we let $B_r(x) := \{y \in \mathbb{R}^n \,:\, |x - y| < r\}$ and given an integrable function $f : B_r(x) \to \mathbb{R}$ we denote its integral average by $⨍_{B_r(x)} f \,\mathrm{d}x := \frac{1}{|B_r(x)|} \int_{B_r(x)} f \,\mathrm{d}x$. Finally, for

$x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ we set the linear affine function

$$\varphi_{x,u,\xi}(y) \coloneqq u + \langle \xi, y - x \rangle. \tag{3.1.1}$$

■

3.1.1 Basic Definitions and X-Sobolev Spaces

We will identify a first order differential operator $X \coloneqq \sum_{i=1}^{n} c_i \frac{\partial}{\partial x_i}$ with the map $X(x) \coloneqq (c_1(x), \ldots, c_n(x)) : \Omega \to \mathbb{R}^n$ and let $m \leq n$.

Definition 3.1.1. *We say that* $\boldsymbol{X} \coloneqq (X_1, \ldots, X_m)$ *is a* family of Lipschitz vector fields *on* Ω *if for any* $j = 1, \ldots, m$ *and for any* $i = 1, \ldots, n$ *there exists a function* $c_{j,i} \in \mathrm{Lip}(\Omega)$ *such that* $X_j(x) = (c_{j,1}(x), \ldots, c_{j,n}(x))$*. We denote by* $C(x)$ *the* $m \times n$ *matrix defined as*

$$C(x) \coloneqq [c_{j,i}(x)]_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}.$$

We say that $\boldsymbol{X}$ *satisfies the* linear independence condition (LIC) *on* Ω *if the set*

$$N_X \coloneqq \{x \in \Omega \,:\, X_1(x), \ldots, X_m(x) \text{ are linearly dependent } \}$$

is such that $|N_X| = 0$*. In this case we set* $\Omega_X \coloneqq \Omega \setminus N_X$*.*

As we already said, (LIC) is a very general request and many relevant families of vector fields embraces this condition as, for instance, the Euclidean space, the Heisenberg group and the Grushin space. Furthermore, (LIC) is a weaker assumption with respect to the request that the X-gradient induces a Carnot–Carathéodory distance in Ω (see for instance [16]).

Definition 3.1.2. *Let* $m \leq n$*,* $u \in L^1_{\mathrm{loc}}(\Omega)$ *and* $v \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$*, and let* $\boldsymbol{X}$ *be a family of Lipschitz vector fields. We say that* v *is the* X-gradient *of* u *if for any* $\varphi \in C^\infty_c(\Omega, \mathbb{R}^m)$ *it holds that*

$$-\int_\Omega u \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (c_{j,i}\varphi_j)\, \mathrm{d}x = \int_\Omega \varphi \cdot v \mathrm{d}x.$$

Whenever it exists, the X-gradient is shown to be unique a.e. and we set $Xu := v$. Moreover, if $p \in [1, +\infty]$ we define the vector spaces

$$W^{1,p}_X(\Omega) := \{u \in L^p(\Omega) \,:\, Xu \in L^p(\Omega)\}$$

and

$$W^{1,p}_{X,\mathrm{loc}}(\Omega) := \{u \in L^p_{\mathrm{loc}}(\Omega) \,:\, u|_{A'} \in W^{1,p}_X(A'), \quad \forall\, A' \in \mathcal{A}_0\}.$$

We refer to them as X-Sobolev spaces*, and to their elements as* X-Sobolev functions*.*

Remark 3.1.3. If $u \in L^1_{\mathrm{loc}}(\Omega)$, the previous definition is equivalent to set the distributional *X-gradient* as

$$\langle Xu, \varphi\rangle := -\int_\Omega u\,\mathrm{div}(\varphi(x)\cdot C(x))\mathrm{d}x \qquad \text{for any } \varphi \in C^\infty_c(\Omega, \mathbb{R}^m).$$

In the particular case in which $\mathbf{X}$ is the family of generating horizontal vector fields of a Carnot group, then the X-gradient reduces to the classical *horizontal gradient* in (2.1.7).

The next proposition can be found in [59].

Proposition 3.1.4. *Let $p \in [1, +\infty]$. Then the vector space $W^{1,p}_X(\Omega)$, endowed with the norm*

$$\|u\|_{W^{1,p}_X(\Omega)} := \|u\|_{L^p(\Omega)} + \|Xu\|_{L^p(\Omega,\mathbb{R}^m)},$$

is a Banach space. Moreover, if $1 < p < +\infty$ it is a reflexive Banach space.

The following simple proposition tells us that X-Sobolev spaces are actually a generalization of the classical ones. The first reason is because each Sobolev function is in particular an X-Sobolev one, whatever $\mathbf{X}$ is the standard family of vector fields $(\partial_1, \ldots, \partial_n)$ which gives rise to the classical Sobolev spaces. The second is that the X-gradient can be computed starting from the Euclidean gradient in a very simple way, whenever the function is regular enough.

Proposition 3.1.5. *The following facts hold:*

(i) if $n = m$ *and* $c_{j,i}(x) = \delta_{j,i}$ *for every* $i, j = 1, \ldots, n$*, then* $W^{1,p}(\Omega) = W^{1,p}_X(\Omega)$*;*

(ii) $W^{1,p}(\Omega) \subseteq W^{1,p}_X(\Omega)$*, the inclusion is continuous and*

$$Xu(x) = C(x)Du(x) \quad \textit{for every } \ u \in W^{1,p}(\Omega) \ \textit{ and a.e. } x \in \Omega.$$

Let us notice that, being Ω bounded, we have that

$$W^{1,\infty}(\Omega) \subseteq W^{1,p}(\Omega) \subseteq W^{1,p}_X(\Omega)$$

for any family $\mathbf{X}$ of Lipschitz vector fields. The following proposition tells us that the weak convergence in $W^{1,p}_X$ is weaker than the weak*- convergence in $W^{1,\infty}$.

Proposition 3.1.6. *Let* $\boldsymbol{X}$ *be a family of Lipschitz vector fields. Then, for any sequence* $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ *and any* $u \in W^{1,\infty}(\Omega)$*, it follows that*

$$u_h \rightharpoonup^* u \ \textit{in}\ W^{1,\infty}(\Omega) \quad \Longrightarrow \quad u_h \rightharpoonup u \ \textit{in}\ W^{1,p}_X(\Omega).$$

Proof. Follows easily from [20, Theorem 3.10]. □

Remark 3.1.7. In the classical Sobolev spaces, a typical strategy to get an integral representation of the form

$$F(u, A) = \int_A f_e(x, u(x), Du(x))\mathrm{d}x$$

for linear or affine functions, is to exploit classical differentiation theorems for measures. Then one can combine some semicontinuity properties of the functional together with approximation results by means of piecewise affine functions (see for instance [53, Chapter X, Proposition 2.9]), in order to extend the integral representation to all Sobolev functions. However, one of the main difficulties is that an analogue of [53, Chapter X, Proposition 2.9]) does not hold in our setting. Indeed, if we call X-affine a C^∞ function such that Xu is constant, then there are choices of $\mathbf{X}$ for which not all X-Sobolev functions can be approximated in $W^{1,p}_X$ by piecewiese X-affine functions [81, Section 2.3].

Hence, we have to adopt a different strategy but we present some useful Meyers-Serrin type results that are still true in this non Euclidean framework and that allow us to approximate X-Sobolev functions with smooth ones. Finally, similarly to the Euclidean case, a Meyers–Serrin approximation result holds (cf. [61, Theorem 1.2]).

Theorem 3.1.8. *Let Ω be an open subset of $\mathbb{R}^n$. For any $u \in W^{1,p}_X(\Omega)$ there exists a sequence $u_\epsilon \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ such that*

$$u_\epsilon \to u \text{ in } W^{1,p}_X(\Omega) \text{ as } \epsilon \to 0.$$

In other words, we have that

$$\overline{W^{1,p}_X(\Omega) \cap C^\infty(\Omega)} = W^{1,p}_X(\Omega),$$

where the closure is with respect to the metric topology of $(W^{1,p}_X(\Omega), \|\cdot\|_{W^{1,p}_X(\Omega)})$.

Proposition 3.1.9. *Let $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ and $A' \Subset \Omega$, then there exists $v \in W^{1,p}_X(\Omega)$ which coincides with u on A'.*

Proof. Let φ be a smooth cut-off function between A' and Ω. It is straightforward to verify that the function $v(x) := \varphi(x)u(x)$ satisfies the desired requirements. $\square$

Proposition 3.1.9, together with Theorem 3.1.8, allows to prove the following result.

Proposition 3.1.10. *Let $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ and let $A' \Subset \Omega$. Then there exists a sequence $(u_\epsilon)_\epsilon \subseteq W^{1,p}_X(\Omega)$ such that*

$$u_\epsilon|_{A'} \in W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_\epsilon|_{A'} \longrightarrow u|_{A'} \text{ in } W^{1,p}_X(A').$$

Proof. Let us fix $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$ and $A' \in \mathcal{A}_0$. By Proposition 3.1.9 we can find a function $\tilde{u} \in W^{1,p}_X(\Omega)$ such that $u|_{A'} = \tilde{u}|_{A'}$, and by Theorem 3.1.8 there exists a sequence $(u_\epsilon)_\epsilon \subseteq W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ converging to $\tilde{u}$ in $W^{1,p}_X(\Omega)$. It is easy to see that $(u_{\varepsilon|A'})_\epsilon \subseteq W^{1,p}_X(A') \cap C^\infty(A')$; moreover, since $u|_{A'} = \tilde{u}|_{A'}$, we conclude that $u_\epsilon|_{A'} \longrightarrow u|_{A'}$ in $W^{1,p}_X(A')$. $\square$

When dealing with integral representation in classical Sobolev spaces one might exploit the following Lusin-type result (cf. [29, Theorem 13]):

Proposition 3.1.11. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, $1 \leq p \leq +\infty$ and $u \in W^{1,p}(\Omega)$. Then, for any $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{A}$ and $v \in C^1(\overline{\Omega})$ such that $|A_\epsilon| \leq \epsilon$ and $u|_{\Omega\setminus A_\epsilon} = v|_{\Omega\setminus A_\epsilon}$.*

Under reasonable assumptions (cf. [26, Lemma 2.7]) this result allows to extend an integral representation result from $C^1(\overline{\Omega}) \times \mathcal{A}$ to $W^{1,p}(\Omega) \times \mathcal{A}$. The following counterexample shows that an analogue of Proposition 3.1.11 does not hold in a general X-Sobolev space. In the following example we speak about *approximate differentiability* and *approximate partial derivatives* according to [57, Section 3.1.2].

Example 3.1.12. Let us take $\Omega = (0,1) \times (0,1)$ and $\mathbf{X} = X_1 = \frac{\partial}{\partial x}$ which clearly satisfies the (LIC). Let us consider a function $w : (0,1) \to \mathbb{R}$ which is bounded, continuous but which is not approximately differentiable for a.e. $x \in (0,1)$ (see for instance [92, p. 297]), and define the function $u : \Omega \to \mathbb{R}$ as

$$u(x,y) := w(y).$$

We have that $u \in L^\infty(\Omega)$ and it is constant with respect to x. Thus, for any $\varphi \in C_c^\infty(\Omega)$, we have that

$$-\int_\Omega u\frac{\partial\varphi}{\partial x}\mathrm{d}x = -\int_0^1 w(y)\mathrm{d}y \int_0^1 \frac{\partial\varphi}{\partial x}\mathrm{d}x = 0,$$

and hence $Xu = 0$. Then $u \in W_X^{1,\infty}(\Omega)$ and, in particular, we have that $u \in W_X^{1,p}(\Omega)$ for any $p \in [1,+\infty]$. If it was the case that u satisfies the desired property, then we would have that, for a.e. (x,y) in Ω, u is approximately differentiable at (x,y) (see [78, Theorem 1]). Thus, according to [92, Theorem 12.2] and to the fact that u is constant w.r.t. x, we would have that for any $x \in (0,1)$ and for a.e. $y \in (0,1)$, the function $z \mapsto u(x,z) = w(z)$ is approximately differentiable at y, but this claim is in contradiction with our choice of w.

We conclude this section with a Leibniz-type property for the X-gradient, which is a direct consequence of the previous result and which will be very useful in the sequel.

Proposition 3.1.13. *For any $u, v \in W^{1,p}_X(\Omega)$, it holds that*

$$X(uv) = (Xu)v + u(Xv) \qquad a.e.\ on\ \Omega.$$

Proof. Assume first that $u, v \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$. Then it follows that

$$\begin{aligned} X(uv) &= D(uv) \cdot C^T = [(Du)v + u(Dv)] \cdot C^T \\ &= Du \cdot C^T v + uDv \cdot C^T = (Xu)v + u(Xv) \end{aligned} \tag{3.1.2}$$

everywhere on Ω. Let now $A' \Subset \Omega$, $u \in W^{1,p}_X(\Omega)$ and $v \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$. From Theorem 3.1.8 we know in particular that there exists a sequence $(u_h)_h \subseteq W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$ converging to u in the strong topology of $W^{1,p}_X(A')$, and clearly $v \in C^\infty(\overline{A'})$. It is easy to see that the sequence $(vu_h)_h$ belongs to $W^{1,p}_X(A') \cap C^\infty(\overline{A'})$ and converges to uv in the strong topology of $W^{1,p}_X(A')$. This fact, together with (3.1.2) and recalling that $\sup_{A'} |Xv| < +\infty$ since $\sup_{A'} |Dv| < +\infty$, yields that

$$\begin{aligned} &\|X(uv) - (Xu)v - u(Xv)\|_{L^p(A',\mathbb{R}^m)} \\ &\le \|X(uv) - X(u_h v)\|_{L^p(A',\mathbb{R}^m)} + \|(Xv)u_h + vX(u_h) - (Xu)v - u(Xv)\|_{L^p(A',\mathbb{R}^m)}, \end{aligned}$$

and so, passing to the limit as $h \to \infty$, we conclude that

$$X(uv) = (Xu)v + u(Xv) \qquad \text{a.e. on } A'.$$

Since Ω can be approximated by a countable family of open sets $A' \Subset \Omega$, we conclude that

$$X(uv) = (Xu)v + u(Xv) \qquad \text{a.e. on } \Omega$$

for any $u \in W^{1,p}_X(\Omega)$ and $v \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$. Repeating once more the same procedure, the thesis follows. □

3.1.2 Algebraic Properties of X

Here we present some algebraic properties of the coefficient matrix $C : \Omega \to \mathbb{R}^{m\times n}$. The following results have been achieved in [81, Section 3.2].

Definition 3.1.14. *Let* $\boldsymbol{X}$ *be a family of Lipschitz vector fields. For any* $x \in \Omega$ *we define the linear map*

$$L_x : \mathbb{R}^n \to \mathbb{R}^m \quad \textit{by} \quad L_x(v) := C(x)v \ \ \textit{if } v \in \mathbb{R}^n$$

and

$$N_x := \ker(L_x), \quad V_x := \{C(x)^T z \ : \ z \in \mathbb{R}^m\}.$$

By easy arguments from linear algebra, we know that $\mathbb{R}^n = N_x \oplus V_x$*, and then, for any* $x \in \Omega$ *and* $\xi \in \mathbb{R}^n$*, there is a unique choice of* $\xi_{N_x} \in N_x$ *and* $\xi_{V_x} \in V_x$ *such that*

$$\xi = \xi_{N_x} + \xi_{V_x}.$$

Finally we define $\Pi_x : \mathbb{R}^n \to V_x \subset \mathbb{R}^n$ *as the* projection $\Pi_x(\xi) := \xi_{V_x}$.

These definitions make sense for a generic family of Lipschitz vector fields, but the following two propositions list some very useful invertibility and continuity properties that are typical of those families of vector fields satisfying the (LIC).

Proposition 3.1.15. *Let* $\boldsymbol{X}$ *be a family of Lipschitz vector fields satisfying the (LIC) on* Ω*. Then the following facts hold:*

(i) $\dim V_x = m$ *for each* $x \in \Omega_X$ *and* $L_x(V_x) = \mathbb{R}^m$*, in particular* $L_x : V_x \to \mathbb{R}^m$ *is an isomorphism.*

(ii) For every $x \in \Omega$*, let us set* $B(x) := C(x)\,C^T(x)$*. Then, for each* $x \in \Omega_X$*,* $B(x)$ *is a symmetric invertible matrix of order* m*. Moreover the map* $B^{-1}\colon \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$*, defined as*

$$B^{-1}(x)(z) := B(x)^{-1} z \quad \textit{if } z \in \mathbb{R}^m\,,$$

is continuous.

(iii) For each $x \in \Omega_X$*, the projection* Π_x *can be represented as*

$$\Pi_x(\xi) = \xi_{V_x} = C(x)^T B(x)^{-1} C(x)\,\xi, \quad \forall\, \xi \in \mathbb{R}^n\,.$$

It is easy to verify that $N_X = \{x \in \Omega : \det B(x) = 0\}$. Therefore N_X is closed in Ω.

Proposition 3.1.16. *Let $\boldsymbol{X}$ be a family of Lipschitz vector fields satisfying the (LIC) on Ω. Then $L_x : V_x \to \mathbb{R}^m$ is invertible and the map $L^{-1} : \Omega_X \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ defined by*

$$L^{-1}(x) := L_x^{-1} \text{ if } x \in \Omega_X$$

belongs to $\mathbf{C}^0(\Omega_X, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$.

3.1.3 Local Functionals

We conclude this section by giving some definitions about increasing set functions, for which we refer to [38, Chapter 14]. From now on we assume that $\mathbf{X}$ is a family of Lipschitz vector fields satisfying the (LIC) on Ω.

Definition 3.1.17. *Let $\alpha : \mathcal{A} \to [0, +\infty]$ be a function. We say that α is*

(i) increasing *if it holds that $\alpha(A) \le \alpha(B)$ for any $A, B \in \mathcal{A}$ s.t. $A \subseteq B$;*

(ii) inner regular *if it is increasing and $\alpha(A) = \sup\{\alpha(A') : A' \Subset A\}$ for any $A \in \mathcal{A}$;*

(iii) subadditive *if it is increasing and, for any $A, B, C \in \mathcal{A}$ with $A \subseteq B \cup C$,*

$$\alpha(A) \le \alpha(B) + \alpha(C);$$

(iv) superadditive *if it is increasing and, for any $A, B, C \in \mathcal{A}$ with $A \cap B = \emptyset$ and $A \cup B \subseteq C$,*

$$\alpha(C) \ge \alpha(A) + \alpha(B);$$

(v) a measure *if it is increasing and the restriction to $\mathcal{A}$ of a non-negative Borel measure.*

Before introducing the main assumptions for local functionals, we have to mention some facts. In the integral representation theorems we will handle with functionals defined on $W^{1,p}_{X,\mathrm{loc}}(\Omega)$ while, in the Γ-convergence analysis, we will use mostly the functionals defined on

$L^p(\Omega)$ or $W^{1,p}_X(\Omega)$. As a consequence, even if in the following notations we consider functionals defined on $L^p(\Omega)$, the only assumptions that changes when F is defined on $W^{1,p}_X(\Omega)$ or $W^{1,p}_{X,\mathrm{loc}}(\Omega)$ are the assumptions (a) and (b).

Definition 3.1.18. *Let $1 \leq p < +\infty$ and let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded subset. Let $\boldsymbol{X}$ be a family of Lipschitz vector fields and let us consider the functional*

$$F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty].$$

We say that F is:

(a) *a* measure *if $F(u, \cdot)$ is a measure for any $u \in L^p(\Omega)$;*

(b) local *if, for any $A \in \mathcal{A}$ and $u, v \in L^p(\Omega)$, then*

$$u|_A = v|_A \implies F(u, A) = F(v, A).$$

(c) convex *if, for any $A' \in \mathcal{A}_0$, the function $F(\cdot, A') : W^{1,p}_X(\Omega) \longrightarrow [0, +\infty]$ is convex;*

(d) p-bounded *if there exist $a \in L^1_{\mathrm{loc}}(\Omega)$ and $b, c > 0$ such that, for any $A \in \mathcal{A}_0$ and for any $u \in W^{1,p}_X(\Omega)$, it holds that*

$$F(u, A) \leq \int_A a(x) + b|Xu|^p + c|u|^p \mathrm{d}x;$$

(e) L^p-lower semicontinuous *(resp.* $W^{1,p}_X$-lower semicontinuous*) if $F(\cdot, A)$ is L^p-lower semicontinuous (resp. $W^{1,p}_X$-lower semicontinuous) for any $A \in \mathcal{A}$, i.e. for any $A \in \mathcal{A}_0$, $(u_h)_h \subseteq L^p(\Omega)$ and $u \in L^p(\Omega)$ it holds that*

$$u_h \to u \text{ in } L^p(\Omega) \quad \implies \quad F(u, A') \leq \liminf_{h \to +\infty} F(u_h, A'),$$

and the same holds in the $W^{1,p}_X$-case;

(f) weakly sequentially lower semicontinuous *if for any $A' \in \mathcal{A}_0$, $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ and $u \in W^{1,p}_X(\Omega)$ it holds that*

$$u_h \rightharpoonup u \text{ in } W^{1,p}_X(\Omega) \quad \implies \quad F(u, A') \leq \liminf_{h \to +\infty} F(u_h, A');$$

(g) weakly*- sequentially lower semicontinuous *if, for any* $A' \in \mathcal{A}_0$, $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ *and* $u \in W^{1,\infty}(\Omega)$ *it holds that*

$$u_h \rightharpoonup^* u \text{ in } W^{1,\infty}(\Omega) \quad \Longrightarrow \quad F(u, A') \leq \liminf_{h \to +\infty} F(u_h, A').$$

In the following proposition we prove that the notion of lower semicontinuity is actually equivalent to a more useful condition.

Proposition 3.1.19. *Let* $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *be a measure and a local functional. Then the following conditions are equivalent:*

(a) F *is lower semicontinuous;*

(b) $\forall A' \in \mathcal{A}_0$, $F_{A'} : (\{u|_{A'} : u \in W^{1,p}_X(\Omega)\}, \|\cdot\|_{W^{1,p}_X(A')}) \to [0, +\infty]$ *defined as*

$$F_{A'}(u|_{A'}) := F(u, A') \quad \text{is lower semicontinuous.}$$

Proof. $(b) \implies (a)$. It is straightforward.

$(a) \implies (b)$. Fix an open set $A' \in \mathcal{A}_0$ and take $(u_h)_h, u$ in $W^{1,p}_X(\Omega)$ such that $\|u_h|_{A'} - u|_{A'}\|_{W^{1,p}(A')} \to 0$. Now, for any $k \in \mathbb{N}$, take an open set A_k such that $A_k \Subset A_{k+1} \Subset A'$ and $\bigcup_{k=0}^{+\infty} A_k = A'$, and a smooth cut-off function φ_k between A_k and A'. For any $h, k \in \mathbb{N}$, define the functions $v^k := \varphi_k u$ and $v^k_h := \varphi_k u_h$. We have that, for any $h, k \in \mathbb{N}$, v^k_h, v^k belong to $W^{1,p}_X(\Omega)$, $v^k_h|_{A_k} = u_h|_{A_k}$, $v^k|_{A_k} = u|_{A_k}$ and moreover $\lim_{h\to\infty} \|v^k_h - v^k\|_{W^{1,p}_X(\Omega)} = 0$ for any $k \in \mathbb{N}$. Applying assumptions (i) and (ii) we get

$$\begin{aligned} F(u, A') &= \lim_{k\to\infty} F(u, A_k) = \lim_{k\to\infty} F(v^k, A_k) \\ &\leq \lim_{k\to\infty} \liminf_{h\to\infty} F(v^k_h, A_k) = \lim_{k\to\infty} \liminf_{h\to\infty} F(u_h, A_k) \\ &\leq \lim_{k\to\infty} \liminf_{h\to\infty} F(u_h, A') = \liminf_{h\to\infty} F(u_h, A'). \end{aligned}$$

□

3.1.4 From Euclidean to non Euclidean Lagrangian

In order to characterize the class of convex local functionals defined on $W_X^{1,p}$, our aim is to exploit [25, Lemma 4.1] to get an integral representation of the form

$$F(u, A) = \int_A f_e(x, u, Du)\mathrm{d}x \quad \forall A \in \mathcal{A},\, \forall u \in W^{1,p}(\Omega). \tag{3.1.3}$$

For this purpose, we need the forthcoming Propositions 3.1.20 and 3.1.21 which guarantee the existence of a non Euclidean Lagrangian f such that

$$\int_A f(x, u, Xu)\mathrm{d}x = \int_A f_e(x, u, Du)\mathrm{d}x \quad \forall A \in \mathcal{A},\, \forall u \in C^\infty(A). \tag{3.1.4}$$

The following propositions, which are almost totally inspired by [81, Theorem 3.5] and [81, Lemma 3.13], allow us to pass from an Euclidean to a non Euclidean integral representation.

Proposition 3.1.20. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a Carathéodory function and let us define $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ as*

$$f(x, u, \eta) := \begin{cases} f_e(x, u, L^{-1}(x)(\eta)) & \text{if } (x, u, \eta) \in \Omega_X \times \mathbb{R} \times \mathbb{R}^m \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.5}$$

Then the following facts hold:

(i) *f is a Carathéodory function;*

(ii) *if $f_e(x, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$, then $f(x, \cdot, \cdot)$ is convex for a.e. $x \in \Omega$;*

(iii) *if $f_e(x, u, \cdot)$ is convex for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$, then $f(x, u, \cdot)$ is convex for a.e. $x \in \Omega$ and for any $u \in \mathbb{R}$;*

(iv) *If we assume that*

$$f_e(x, u, \xi) = f_e(x, u, \Pi_x(\xi)) \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.1.6}$$

then it follows that

$$\int_A f_e(x, u, Du)\,\mathrm{d}x = \int_A f(x, u, Xu)\,\mathrm{d}x \quad \forall A \in \mathcal{A},\, \forall u \in C^\infty(A). \tag{3.1.7}$$

Proof. (i) First we want to show that, for any $(u,\eta) \in \mathbb{R} \times \mathbb{R}^m$, the function $x \mapsto f(x,u,\eta)$ is measurable. Let us fix then $(u,\eta) \in \mathbb{R} \times \mathbb{R}^m$, define the function $\Phi : \Omega_X \to \mathbb{R} \times \mathbb{R}^n$ as $\Phi(x) \coloneqq (u, L^{-1}(x)(\eta))$ and extend it to be zero on $\Omega \setminus \Omega_X$. By Proposition 3.1.16, $\Phi|_{\Omega_X}$ is continuous, and so in particular Φ is measurable. Noticing that

$$f(x,u,\eta) = f_e(x, \Phi(x)) \quad \forall x \in \Omega_X,$$

being f_e a Carathéodory function and recalling [36, Proposition 3.7] we conclude that $x \mapsto f(x,u,\eta)$ is measurable. Let us define now the function $\Psi : \Omega_X \times \mathbb{R} \times \mathbb{R}^m \to \Omega_X \times \mathbb{R} \times \mathbb{R}^n$ as $\Psi(x,u,\eta) \coloneqq (x,u,L^{-1}(x)(\eta))$. Since on Ω_X we have that $f = f_e \circ \Psi$, then, for any fixed $x \in \Omega_X$ such that $f_e(x,\cdot,\cdot)$ is continuous, $f(x,\cdot,\cdot)$ is the composition of a continuous function and a linear function, and so it is continuous.
(ii) If $x \in \Omega_X$ is such that $f_e(x,\cdot,\cdot)$ is convex, then $f = f_e \circ \Psi$ is the composition of a convex function and a linear function, and so it is convex.
(iii) Follows as (ii).
(iv) Assume that (3.1.6) holds. Let us fix $A \in \mathcal{A}$ and $u \in C^\infty(A)$. From the regularity of u we have that $Xu(x) = C(x)Du(x)$. By Proposition 3.1.15 we get

$$\begin{aligned} L_x(\Pi_x(Du)) &= L_x(C(x)^T B(x)^{-1} C(x) Du) = C(x)C(x)^T B(x)^{-1} C(x) Du \\ &= B(x)B(x)^{-1}C(x)Du = C(x)Du = L_x(Du), \end{aligned}$$

and

$$\begin{aligned} f(x,u,Xu) &= f(x,u,C(x)Du) = f(x,u,L_x(Du)) = f(x,u,L_x(\Pi_x(Du))) \\ &= f_e(x,u,L_x^{-1}(L_x(\Pi_x(Du)))) = f_e(x,u,\Pi_x(Du)) = f_e(x,u,Du). \end{aligned}$$

Now (3.1.7) follows by integrating over A.

□

In the following result we provide some sufficient conditions to guarantee (3.1.6).

Proposition 3.1.21. *Let $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,+\infty]$ be a Carathéodory function such that*

(i) *$f_e(x,u,\cdot)$ is convex for a.e $x \in \Omega$, for any $u \in \mathbb{R}$;*

(ii) there exist $a \in L^1_{\mathrm{loc}}(\Omega)$ *and* $b, c > 0$ *such that*

$$f_e(x, u, \xi) \le a(x) + b|C(x)\xi|^p + c|u|^p \tag{3.1.8}$$

for a.e. $x \in \Omega$*, for any* $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$*.*

Then f_e *satisfies* (3.1.6)*.*

Proof. Follows with some trivial modifications as in [81, Lemma 3.13]. □

3.2 Integral Representation of convex functionals

Let us now state and prove the main theorem of this section. Thanks to the previous results, it is possible to extend the integral representation to the whole $W^{1,p}_{X,\mathrm{loc}}(\Omega)$.

Theorem 3.2.1. *Let* $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *be such that:*

(i) F *is a measure;*

(ii) F *is local;*

(iii) F *is convex;*

(iv) F *is p-bounded.*

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ *such that*

$$(u, \xi) \mapsto f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega, \tag{3.2.1}$$

$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall\ (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{3.2.2}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)\mathrm{d}x \quad \forall u \in W^{1,p}_{X,\mathrm{loc}}(\Omega),\ \forall A \in \mathcal{A}. \tag{3.2.3}$$

Moreover, if $f_1, f_2 : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ *are two Carathéodory functions satisfying* (3.2.1)*,* (3.2.2) *and* (3.2.3)*, then there exists* $\tilde{\Omega} \subseteq \Omega$ *such that* $|\tilde{\Omega}| = |\Omega|$ *and*

$$f_1(x, u, \xi) = f_2(x, u, \xi) \quad \forall x \in \tilde{\Omega},\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m. \tag{3.2.4}$$

Proof. **First Step.** Let

$$C := \max\{\sup\{|c_{j,i}(x)| \,:\, x \in \Omega\} \,:\, i = 1, \ldots, n,\ j = 1, \ldots, m\}.$$

Then from our assumptions on $\mathbf{X}$ it follows that $0 < C < +\infty$. Let $\tilde{b} := C^p b$. Using (iv) and recalling that for all $u \in W^{1,p}(\Omega)$ we have that $Xu(x) = C(x)Du(x)$ it follows that

$$F(u, A') \leq \int_{A'} a(x) + c|u|^p + \tilde{b}|Du|^p \mathrm{d}x \quad \forall A' \in \mathcal{A}_0,\ \forall u \in W^{1,p}(\Omega). \tag{3.2.5}$$

Thus we can apply [25, Lemma 4.1] to get a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ such that

$$F(u, A) = \int_A f_e(x, u, Du) \mathrm{d}x \quad \forall A \in \mathcal{A},\ \forall u \in W^{1,p}_{\mathrm{loc}}(\Omega), \tag{3.2.6}$$

$$f_e(x, u, \xi) \leq a(x) + \tilde{b}|\xi|^p + c|u|^p \quad \text{ for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{3.2.7}$$

and

$$f_e(x, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^n \to [0, \infty] \text{ is convex for a.e. } x \in \Omega.$$

Second Step. We want to prove that f_e satisfies (3.1.6). By Proposition 3.1.21 and the convexity of f_e we only need to prove (3.1.8). Let us take then $\Omega' \subseteq \Omega$ such that $|\Omega'| = |\Omega|$ and

$$(u, \xi) \mapsto f_e(x, u, \xi) \text{ is convex and finite } \forall x \in \Omega', \tag{3.2.8}$$

and fix $x \in \Omega'$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$. By (3.2.6), for any $R > 0$ small enough to ensure that $B_R(x) \Subset \Omega$, we have that

$$F(\varphi_{x,u,\xi}, B_R(x)) = \int_{B_R(x)} f_e(y, u + \langle \xi, y - x \rangle, \xi)\, \mathrm{d}y$$

and from (iv) we have that

$$F(\varphi_{x,u,\xi}, B_R(x)) \leq \int_{B_R(x)} a(y) + c|u + \langle \xi, y - x \rangle|^p + b|C(y)\xi|^p \, \mathrm{d}y,$$

where $\varphi_{x,u,\xi}$ is as in (3.1.1). Combining these two facts and dividing by $|B_R(x)|$ we obtain that

$$\fint_{B_R(x)} f_e(y, u + \langle \xi, x - y \rangle, \xi) \mathrm{d}y \leq \fint_{B_R(x)} a(y) + c|u + \langle \xi, y - x \rangle|^p + b|C(y)\xi|^p \mathrm{d}y. \tag{3.2.9}$$

Since the right integrand is in $L^1_{\mathrm{loc}}(\Omega)$, and (3.2.9) holds indeed for all $A' \in \mathcal{A}_0$, the left one is in $L^1_{\mathrm{loc}}(\Omega)$ as well. Therefore, thanks to Lebesgue Theorem we can find $\Omega_{u,\xi} \subseteq \Omega'$ such that $|\Omega_{u,\xi}| = |\Omega|$ and

$$f_e(x,u,\xi) \le a(x) + c|u|^p + b|C(x)\xi|^p \quad \forall x \in \Omega_{u,\xi}.$$

Setting $\tilde{\Omega} := \bigcap_{(u,\xi)\in\mathbb{Q}\times\mathbb{Q}^n} \Omega_{u,\xi}$, it holds that $|\tilde{\Omega}| = |\Omega|$ and

$$f_e(x,u,\xi) \le a(x) + c|u|^p + b|C(x)\xi|^p \quad \forall x \in \tilde{\Omega},\, \forall (u,\xi) \in \mathbb{Q}\times\mathbb{Q}^n.$$

Since the map $(u,\xi) \mapsto f_e(x,u,\xi)$ is continuous for any $x \in \tilde{\Omega}$ and $\mathbb{Q}\times\mathbb{Q}^n$ is dense in $\mathbb{R}\times\mathbb{R}^n$ then (3.1.8) holds and the conclusion follows.

Third Step. Thanks to the previous step we can apply (iv) of Proposition 3.1.20. Hence we get

$$\int_A f_e(x,u,Du)\mathrm{d}x = \int_A f(x,u,Xu)\mathrm{d}x \quad \forall A \in \mathcal{A},\, u \in C^\infty(A), \tag{3.2.10}$$

where $f : \Omega\times\mathbb{R}\times\mathbb{R}^m \to [0,+\infty]$ is the function defined in (3.1.5). First of all we can assume that f is finite up to modifying it on a set of measure zero. Moreover, thanks to the convexity of f_e and (ii) of Proposition 3.1.20 we have that f satisfies (3.2.1). Now we want to prove that f satisfies (3.2.2). Let us fix $x \in \Omega$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$: by (iv), (3.2.6) and (3.2.10) we have that

$$\begin{aligned}\int_{B_R(x)} f(y,\varphi_{x,u,\xi},X\varphi_{x,u,\xi})\,\mathrm{d}y &\le \int_{B_R(x)} a(y) + c|\varphi_{x,u,\xi}|^p + b|X\varphi_{x,u,\xi}|^p\mathrm{d}y\\ &= \int_{B_R(x)} a(y) + c|u + \langle\xi, y-x\rangle|^p + b|C(y)\xi|^p\,\mathrm{d}y,\end{aligned}$$

and so, dividing by $|B_R(x)|$, we get that

$$\fint_{B_R(x)} f(y, u+\langle\xi,y-x\rangle, C(y)\xi)\mathrm{d}y \le \fint_{B_R(x)} a(y) + c|u+\langle\xi,y-x\rangle|^p + b|C(y)\xi|^p\mathrm{d}y.$$

Arguing as in the second step we can conclude that

$$f(x,u,C(x)\xi) \le a(x) + b|C(x)\xi|^p + c|u|^p \quad \text{for a.e. } x\in\Omega,\, \forall (u,\xi)\in\mathbb{R}\times\mathbb{R}^n.$$

Finally, recalling that for $x \in \Omega_X$ the map $L_x : V_x \to \mathbb{R}^m$ is surjective, (3.2.2) follows.

Fourth Step. Let us show that (3.2.3) holds. Let us fix $u \in W^{1,p}_X(\Omega)$ and $A' \in \mathcal{A}_0$, and consider the two functionals

$$F_{A'}, G_{A'} : (\{v|_{A'} \ : \ v \in W^{1,p}_X(\Omega)\}, \|\cdot\|_{W^{1,p}_X(A')}) \longrightarrow [0, +\infty]$$

defined as $F_{A'}(v|_{A'}) \coloneqq F(v, A')$ and $G_{A'}(v|_{A'}) \coloneqq \int_{A'} f(x, v, Xv)\mathrm{d}x$ respectively. Thanks to (*iii*), (*iv*), (3.2.1) and (3.2.2), they are convex and bounded on bounded sets on $\{v|_{A'} \ : \ v \in W^{1,p}_X(\Omega)\}$. Hence, they are continuous (cf. [53, Lemma 2.1]). Moreover, from Proposition 3.1.10 we can find a sequence $(u_\epsilon)_\epsilon \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_\epsilon|_{A'})_\epsilon \subseteq W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_\epsilon|_{A'} \longrightarrow u|_{A'} \text{ in } W^{1,p}_X(A').$$

From (3.2.6) and (3.2.10) we get that

$$\begin{aligned} F(u, A') &= \lim_{\epsilon \to 0} F(u_\epsilon, A') = \lim_{\epsilon \to 0} \int_{A'} f_e(x, u_\epsilon, Du_\epsilon)\mathrm{d}x \\ &= \lim_{\epsilon \to 0} \int_{A'} f(x, u_\epsilon, Xu_\epsilon)\mathrm{d}x = \int_{A'} f(x, u, Xu)\mathrm{d}x, \end{aligned}$$

and so we assert that

$$F(u, A') = \int_A f(x, u, Xu)\mathrm{d}x \quad \forall u \in W^{1,p}_X(\Omega),\ \forall A' \in \mathcal{A}_0. \tag{3.2.11}$$

Let us take now $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$, $A \in \mathcal{A}$ and $A' \Subset A$, and, thanks to Proposition 3.1.9, take a function $v \in W^{1,p}_X(\Omega)$ such that $u|_{A'} = v|_{A'}$. Thus, from hypothesis (*ii*) and from (3.2.11), we have that

$$F(u, A') = F(v, A') = \int_{A'} f(x, v, Xv)\mathrm{d}x = \int_{A'} f(x, u, Xu)\mathrm{d}x. \tag{3.2.12}$$

Since by hypothesis the function $B \mapsto F(u, B)$ is inner regular (cf. [38, Theorem 14.23]), and noticing that the function $B \mapsto \int_B f(x, u, Xu)\mathrm{d}x$ is inner regular, thanks to (3.2.12) we have that

$$\begin{aligned} F(u, A) &= \sup\{F(u, A') \ : \ A' \Subset A\} \\ &= \sup\left\{\int_{A'} f(x, u, Xu)\mathrm{d}x \ : \ A' \Subset A\right\} = \int_A f(x, u, Xu)\mathrm{d}x, \end{aligned}$$

and so we can conclude that (3.2.3) holds.

Fifth Step. Let us show the uniqueness of the Lagrangian. Fix then $x \in \Omega$, $u \in \mathbb{Q}$ and $\xi \in \mathbb{Q}^n$: since (3.2.3) holds both for f_1 and f_2, for any $R > 0$ small enough we have that

$$\fint_{B_R(x)} f_1(y, u + \langle \xi, y - x\rangle, C(y)\xi)\mathrm{d}y = \fint_{B_R(x)} f_2(y, u + \langle \xi, y - x\rangle, C(y)\xi)\mathrm{d}y.$$

Since both integrand functions satisfy (3.2.2), then they are both in $L^1_{\mathrm{loc}}(\Omega)$. Again, thanks to Lebesgue theorem, there exists $\Omega_{u,\xi} \subseteq \Omega$ such that $|\Omega_{u,\xi}| = |\Omega|$ and

$$f_1(x, u, C(x)\xi) = f_2(x, u, C(x)\xi) \quad \forall x \in \Omega_{u,\xi}.$$

If we set

$$\tilde{\Omega} := \bigcap_{(u,\xi)\in\mathbb{Q}\times\mathbb{Q}^n} \Omega_{u,\xi} \cap \{x \in \Omega \,:\, \text{(3.2.1) and (3.2.2) hold for } f_1 \text{ and } f_2\} \cap \Omega_X,$$

clearly we have $|\tilde{\Omega}| = |\Omega|$ and it holds that

$$f_1(x, u, C(x)\xi) = f_2(x, u, C(x)\xi) \quad \forall x \in \tilde{\Omega}, \forall (u,\xi) \in \mathbb{Q} \times \mathbb{Q}^n. \tag{3.2.13}$$

Since $(u, \xi) \mapsto f_1(x, u, \xi)$ and $(u, \xi) \mapsto f_2(x, u, \xi)$ are continuous for any $x \in \tilde{\Omega}$, and recalling again that for any $x \in \Omega_X$ L_x is surjective, then (3.2.4) follows. □

The following theorem tells us that all the hypotheses of Theorem 3.2.1 are also necessary.

Theorem 3.2.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ be a Carathéodory function such that*

$$(u, \xi) \mapsto f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega, \tag{3.2.14}$$

$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega, \forall\, (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{3.2.15}$$

for some $b, c > 0$ and $a \in L^1_{\mathrm{loc}}(\Omega)$. Let $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ be the functional defined by

$$F(u, A) := \int_A f(x, u(x), Xu(x))\mathrm{d}x \quad \forall u \in W^{1,p}_{X,\mathrm{loc}}(\Omega), \forall A \in \mathcal{A},$$

then F satisfies hypotheses $(i) - (iv)$ of Theorem 3.2.1.

Proof. Let us fix $u \in W^{1,p}_{X,\mathrm{loc}}(\Omega)$: our aim is to prove that $\alpha(A) := F(u, A)$ is a measure. Notice that, being $f \geq 0$, α is increasing, and of course $\alpha(\emptyset) = 0$. Then, according to [38, Theorem 14.23], it suffices to show that α is subadditive, superadditive and inner regular. The first two properties are trivial, so let us focus on the third one. Let us fix $A \in \mathcal{A}$ and define the sequence of sets $(A_h)_h$ as $A_h := \{x \in A : \mathrm{dist}(x, \partial A) > \frac{1}{h}\}$. We have that $(A_h)_h \subseteq \mathcal{A}_0$, $A_h \Subset A_{h+1} \Subset A$ and $\bigcup_{h\in\mathbb{N}_+} A_h = A$. Thus by the Monotone Convergence Theorem we conclude that

$$\int_A f(x, u, Xu)\mathrm{d}x = \int_A \lim_{h\to+\infty} \chi_{A_h} f(x, u, Xu)\mathrm{d}x = \lim_{h\to+\infty} \int_{A_h} f(x, u, Xu)\mathrm{d}x,$$

and so α is a measure. Property (ii) is straightforward, noticing that the X-gradients of two a.e. equal functions coincide a.e. Finally, (iii) and (iv) follow from (3.2.14) and (3.2.15). □

3.2.1 Strong and Weak condition (ω)

Definition 3.2.3. *We say that* $\omega : \Omega \times [0, +\infty) \longrightarrow [0, +\infty)$ *is a* locally integrable modulus of continuity *if and only if*

$$r \mapsto \omega(x, r) \textit{ is increasing, continuous and } \omega(x, 0) = 0 \quad \textit{for a.e. } x \in \Omega$$

and

$$x \mapsto \omega(x, r) \in L^1_{\mathrm{loc}}(\Omega) \quad \forall r \geq 0.$$

Definition 3.2.4. *Let us consider a functional* $F : \mathcal{F}\times\mathcal{A} \longrightarrow [0, +\infty]$*, where* $\mathcal{F}$ *is a functional space such that* $C^1(\overline{\Omega}) \subseteq \mathcal{F}$*. We say that:*

(i) F *satisfies the* strong condition (ω) *if there exists a sequence* $(\omega_k)_k$ *of locally integrable moduli of continuity such that*

$$|F(v, A') - F(u, A')| \leq \int_{A'} \omega_k(x, r)\,\mathrm{d}x \tag{3.2.16}$$

for any $k \in \mathbb{N}$*,* $A' \in \mathcal{A}_0$*,* $r \in [0, \infty)$*,* $u, v \in C^1(\overline{\Omega})$ *such that*

$$|u(x)|, |v(x)|, |Du(x)|, |Dv(x)| \leq k$$

$$|u(x) - v(x)|, |Du(x) - Dv(x)| \leq r;$$

for all $x \in A'$.

(ii) F satisfies the weak condition (ω) *if there exists a sequence $(\omega_k)_k$ of locally integrable moduli of continuity such that*

$$|F(u+s,A') - F(u,A')| \leq \int_{A'} \omega_k(x,|s|)\mathrm{d}x$$

for any $k \in \mathbb{N}$, $A' \in \mathcal{A}_0$, $s \in \mathbb{R}$, $u \in C^1(\overline{\Omega})$ such that

$$|u(x)|, |u(x)+s|, |s| \leq k \quad \forall x \in A'.$$

Remark 3.2.5. The classical strong and weak condition (ω) were introduced in [26] in order to guarantee the continuity of the Lagrangian f when proving an integral representation result. In particular, the strong condition (ω) guarantees that $f(x,\cdot,\cdot)$ is continuous, while the weak condition (ω) implies the continuity of $f(x,\cdot,\xi)$. Though, the strong condition (ω) implies the weak condition (ω), it is difficult to verify the strong condition (ω), whereas the weak condition (ω) is easier. On the other hand, if we require only the weak condition (ω), we have to add an extra hypothesis in order to get the equivalence, i.e. the weak*-sequential lower semicontinuity of the functional. Indeed, it is well known that (see [2]) for an integral functional of the form

$$F(u,A) := \int_A f_e(x,u(x),Du(x))\mathrm{d}x,$$

the weak*- lower semicontinuity is equivalent to the convexity in the third entry of f_e.

3.3 Integral Representation of w*-sequentially lower semicontinuous functionals

Now, our aim is to characterize a class of local functionals defined on $W^{1,p}_X$ for which we do not require neither translations-invariance nor convexity, but which are weakly*- sequentially lower semicontinuous in $W^{1,\infty}$. We exploit the ideas contained in [55] adopting the same strategy employed in the previous section and using [26, Theorem 1.10] to get an Euclidean integral representation of the form (3.1.3). In the same way, Propositions 3.1.20 and 3.1.21 guarantee the existence of a non Euclidean Lagrangian f such that (3.1.4) holds. We start by proving the usually known *Carathéodory continuity theorem* adapted to the case of $W^{1,p}_X$.

Theorem 3.3.1. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ be a Carathéodory function such that there exist $a \in L^1_{\text{loc}}(\Omega)$ and $b, c > 0$ such that*

$$f(x, u, \xi) \leq a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Then it holds that, for any $A' \in \mathcal{A}_0$, the functional $F : W^{1,p}_X(A') \longrightarrow [0, +\infty)$ defined as

$$F(u) := \int_{A'} f(x, u, Xu) \mathrm{d}x$$

is continuous with respect to the strong topology of $W^{1,p}_X(A')$.

Proof. **First Step.** Let us prove that F is lower semicontinuous. Fix $u \in W^{1,p}_X(A')$ and let $(u_h)_h \subseteq W^{1,p}_X(A')$ be a sequence converging to u and such that

$$\exists \lim_{h \to +\infty} F(u_h) < +\infty.$$

Up to a subsequence we can assume that $(u_h(x))_h$ converges to $u(x)$ and $(Xu_h(x))_h$ converges to $Xu(x)$ for a.e. $x \in A'$. Being f a Carathéodory function, it follows that $\lim_{h\to\infty} f(x, u_h(x), Xu_h(x)) = f(x, u(x), Xu(x))$ for a.e. $x \in \Omega$. Thanks to Fatou's Lemma we conclude that

$$\begin{aligned} F(u) = \int_{A'} f(x, u, Xu) \mathrm{d}x &= \int_{A'} \liminf_{h \to +\infty} f(x, u_h, Xu_h) \mathrm{d}x \\ &\leq \liminf_{h \to +\infty} \int_{A'} f(x, u_h, Xu_h) \mathrm{d}x = \lim_{h \to +\infty} F(u_h). \end{aligned}$$

Second Step. Let us show that F is upper semicontinuous. Let us fix $u \in W^{1,p}_X(A')$ and let us take a sequence $(u_h)_h \subseteq W^{1,p}_X(A')$ converging to u and such that

$$\exists \lim_{h \to +\infty} F(u_h) > -\infty.$$

Up to a subsequence, we can assume that $(u_h(x))_h$ converges to $u(x)$ and $(Xu_h(x))_h$ converges to $Xu(x)$ for almost every $x \in A'$. Let us define the sequence of functions

$$g_h(x) \coloneqq -f(x, u_h, Xu_h) + C(|Xu_h|^p + |u_h|^p)$$

where $C \coloneqq \max\{b, c\} > 0$. Using the p-boundness of f we get

$$g_h(x) \geq -a(x) \text{ for a.e. } x \in A',$$

and so, since the right side belongs to $L^1(A')$, we can apply Fatou's Lemma and get that

$$\begin{aligned}
\int_{A'} -f(x,u,Xu)\mathrm{d}x + \|u\|_{W^{1,p}_X(A')} &= \int_{A'} \liminf_{h\to+\infty} g_h(x,u,Xu)\mathrm{d}x \\
&= \int_{A'} \liminf_{h\to+\infty}(-f(x,u_h,Xu_h) + C(|Xu_h|^p + |u_h|^p))\mathrm{d}x \\
&\leq \liminf_{h\to+\infty} \int_{A'} -f(x,u_h,Xu_h) + C(|Xu_h|^p + |u_h|^p))\mathrm{d}x \\
&= \lim_{h\to+\infty} \int_{A'} -f(x,u_h,Xu_h) + C \lim_{h\to+\infty} \|u_h\|_{W^{1,p}_X(A')} \\
&= \lim_{h\to+\infty} \int_{A'} -f(x,u_h,Xu_h) + \|u\|_{W^{1,p}_X(A')}.
\end{aligned}$$

□

We are ready to state the main result of this section.

Theorem 3.3.2. *Let* $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *be such that:*

(i) *F is a measure;*

(ii) *F is local;*

(iii) *F satisfies the weak condition (ω);*

(iv) *F is p-bounded;*

(v) *F is weakly*- sequentially lower semicontinuous;*

(vi) *F is lower semicontinuous.*

Then there exists a unique Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0, +\infty)$ *such that*

$$\xi \mapsto f(x, u, \xi) \text{ is convex for a.e. } x \in \Omega, \, \forall u \in \mathbb{R}, \tag{3.3.1}$$

$$f(x, u, \xi) \leq a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega, \, \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{3.3.2}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu) \mathrm{d}x \quad \forall u \in W^{1,p}_{X,\mathrm{loc}}(\Omega), \, \forall A \in \mathcal{A}. \tag{3.3.3}$$

Remark 3.3.3. If we substitute hypotheses (v) and (vi) with

(v$'$) F is weakly sequentially lower semicontinuous,

then the conclusions of Theorem 3.3.2 still hold. Indeed, thanks to Proposition 3.1.6 the latter is stronger than both (v) and (vi), even if not equivalent in general.

Proof. **First Step.** Arguing as in the first step of the proof of Theorem 3.2.1, the restriction of F to $W^{1,p}_{\mathrm{loc}}(\Omega) \times \mathcal{A}$ satisfies all the hypotheses of [26, Theorem 1.10]. Thus there exist $\tilde{b} > 0$ and a Carathéodory function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \longrightarrow [0, +\infty]$ such that

$$F(u, A) = \int_A f_e(x, u, Du) \mathrm{d}x \quad \forall A \in \mathcal{A}, \, \forall u \in W^{1,p}_{\mathrm{loc}}(\Omega), \tag{3.3.4}$$

$$f_e(x, u, \xi) \leq a(x) + \tilde{b}|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega, \, \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n \tag{3.3.5}$$

and

$$f_e(x, u, \cdot) : \mathbb{R}^n \to [0, \infty] \text{ is convex} \quad \text{for a.e. } x \in \Omega, \, \forall u \in \mathbb{R}. \tag{3.3.6}$$

Now, arguing as in the second step of the proof of Theorem 3.2.1, from (3.3.5) and (3.3.6) and recalling Propositions 3.1.20 and 3.1.21, we obtain that

$$\int_A f_e(x, u, Du) \mathrm{d}x = \int_A f(x, u, Xu) \mathrm{d}x \quad \forall A \in \mathcal{A}, \, u \in C^\infty(A), \tag{3.3.7}$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ is the Carathéodory function defined in (3.1.5). Up to modifying f on a set of measure zero, we can assume that it is finite. Moreover, arguing as in the third step of the proof of Theorem 3.2.1, f satisfies (3.3.1) and (3.3.2).

Second Step. Here we prove that (3.3.3) holds. Let us start by fixing $u \in W^{1,p}_X(\Omega)$ and $A' \in \mathcal{A}_0$. Thanks to Proposition 3.1.10 we can find a sequence $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_h|_{A'})_h \subseteq W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_h|_{A'} \longrightarrow u|_{A'} \text{ in } W^{1,p}_X(A').$$

From this, (vi), (3.3.4), (3.3.7), Theorem 3.3.1 and Proposition 3.1.19 it follows that

$$\begin{aligned} F(u,A') &\leq \liminf_{h\to+\infty} F(u_h,A') = \liminf_{h\to+\infty} \int_{A'} f_e(x,u_h,Du_h)\mathrm{d}x \\ &= \lim_{h\to+\infty} \int_{A'} f(x,u_h,Xu_h)\mathrm{d}x = \int_{A'} f(x,u,Xu)\mathrm{d}x, \end{aligned}$$

and hence we obtain that

$$F(u,A') \leq \int_{A'} f(x,u,Xu)\mathrm{d}x \quad \forall A' \in \mathcal{A}_0,\ \forall u \in W^{1,p}_X(\Omega). \tag{3.3.8}$$

To prove the converse inequality, fix $u_0 \in W^{1,p}_X(\Omega)$ and set $H : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ as $H(u,A) := F(u+u_0,A)$. It is straightforward to check that H satisfies all the hypotheses of the theorem. Hence there exist a Carathéodory function $h : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty)$, $a_H \in L^1_{\mathrm{loc}}(\Omega)$ and $b_H, c_H > 0$ such that

$$h(x,u,\xi) \leq a_H(x) + b_H|\xi|^p + c_H|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Moreover, it holds that

$$H(u,A) = \int_A h(x,u,Xu)\mathrm{d}x \quad \forall A \in \mathcal{A},\ \forall u \in C^\infty(A) \tag{3.3.9}$$

and

$$H(u,A') \leq \int_{A'} h(x,u,Xu)\mathrm{d}x \quad \forall A' \in \mathcal{A}_0,\ \forall u \in W^{1,p}_X(\Omega). \tag{3.3.10}$$

Fix then $A' \in \mathcal{A}_0$. Arguing as before we can find a sequence $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ such that

$$(u_h|_{A'})_h \subseteq W^{1,p}_X(A') \cap C^\infty(A') \text{ and } u_h|_{A'} \to u_0|_{A'} \text{ in } W^{1,p}_X(A').$$

Thus, thanks to Theorem 3.3.1, and the following chain of inequalities we get that

$$\begin{aligned}\int_{A'} h(x,0,0) &\overset{(3.3.9)}{=} H(0,A') = F(u_0,A') \overset{(3.3.8)}{\leq} \int_{A'} f(x,u_0,Xu_0)\mathrm{d}x \\ &= \lim_{h\to+\infty}\int_{A'} f(x,u_h,Xu_h)\mathrm{d}x = \lim_{h\to+\infty} F(u_h,A') = \lim_{h\to+\infty} H(u_h-u_0,A') \\ &\overset{(3.3.10)}{\leq} \lim_{h\to+\infty}\int_{A'} h(x,u_h-u_0,Xu_h-Xu_0)\mathrm{d}x = \int_{A'} h(x,0,0)\mathrm{d}x,\end{aligned}$$

and all inequalities are indeed equalities. Being u_0 arbitrarily chosen, we conclude that

$$F(u,A') = \int_{A'} f(x,u,Xu)\mathrm{d}x \quad \forall u \in W^{1,p}_X(\Omega),\ \forall A' \in \mathcal{A}_0. \tag{3.3.11}$$

The rest of the proof follows as in the proof of Theorem 3.2.1. $\square$

The following theorem shows that the hypotheses of Theorem 3.3.2 are also necessary.

Theorem 3.3.4. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \longrightarrow [0,+\infty)$ be a Carathéodory function such that*

$$\xi \mapsto f(x,u,\xi) \text{ is convex for a.e. } x \in \Omega,\ \forall u \in \mathbb{R}, \tag{3.3.12}$$

$$f(x,u,\xi) \leq a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u,\xi) \in \mathbb{R}\times\mathbb{R}^m \tag{3.3.13}$$

for $b,c>0$ and $a \in L^1_{\mathrm{loc}}(\Omega)$, and define the functional $F : W^{1,p}_{X,\mathrm{loc}}(\Omega)\times\mathcal{A} \longrightarrow [0,+\infty]$ as

$$F(u,A) := \int_A f(x,u,Xu)\mathrm{d}x \quad \forall u \in W^{1,p}_{X,\mathrm{loc}}(\Omega),\ \forall A \in \mathcal{A}.$$

Then F satisfies hypotheses $(i)-(vi)$ of Theorem 3.3.2.

Proof. (i) Follows as in the proof of Theorem 3.2.2, while (ii) is trivial. In order to prove (iii) let us show that F satisfies the strong property (ω). This suffices, according to [26]. Since f is Carathéodory, then the set $\Omega' := \{x \in \Omega \,:\, (u,\xi) \mapsto f(x,u,\xi) \text{ is continuous}\}$ satisfies $|\Omega'| = |\Omega|$. For any $k \in \mathbb{N}$ and $\epsilon > 0$ set $E^k_\epsilon \subseteq \mathbb{R}\times\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^m$ as

$$E^k_\epsilon := \{(u,v,\xi,\eta) \,:\, |u|,|v|,|\xi|,|\eta| \leq k, |u-v|,|\xi-\eta| \leq \epsilon\}$$

and the function

$$\omega_k(x,\epsilon) := \begin{cases} \sup\{|f(x,u,\xi)-f(x,v,\eta)| \,:\, (u,v,\xi,\eta) \in E^k_\epsilon\} & \text{if } x \in \Omega', \\ 0 & \text{otherwise.}\end{cases}$$

We show that, for any k, ω_k is a locally integrable modulus of continuity. Let us fix then $\epsilon \geq 0$: since $(u,\xi) \mapsto f(x,u,\xi)$ is continuous for almost every $x \in \Omega$, then the supremum in the definition of ω_k can be taken over a countable subset of E^k_ϵ. Since for any (u,v,ξ,η) the function $x \mapsto |f(x,u,\xi) - f(x,v,\eta)|$ is measurable, then $\omega_k(\cdot,\epsilon)$ is measurable. We are left to show that it belongs to $L^1_{\mathrm{loc}}(\Omega)$. Observe that by (3.3.13) it follows that, for any $(u,v,\xi,\eta) \in E^k_\epsilon$,

$$\begin{aligned}|f(x,u,\xi) - f(x,v,\eta)| &\leq 2|a(x)| + b|\xi|^p + b|\eta|^p + c|u|^p + c|v|^p \\ &\leq 2|a(x)| + 4k(b+c).\end{aligned}$$

Since the right side does not depend on $(u,v,\xi,\eta) \in E^k_\epsilon$, we conclude that

$$\omega_k(x,\epsilon) \leq 2|a(x)| + 4k(b+c).$$

Hence $\omega_k(\cdot,\epsilon) \in L^1_{\mathrm{loc}}(\Omega)$. Fix now $x \in \Omega'$. Since $E^k_\epsilon \subseteq E^k_\delta$ for any $\epsilon \leq \delta$, then $\omega_k(x,\cdot)$ is increasing, and $\omega_k(x,0) = 0$. Finally its continuity follows from the continuity of $f(\cdot,u,\xi)$. Then $(\omega_k)_k$ is a sequence of locally integrable moduli of continuity. Let us recall that, if we define $C \coloneqq \max\{\sup\{|c_{j,i}(x)| \ : \ x \in \Omega\} \quad i = 1,\dots,n,\ j = 1,\dots,m\}$, it holds that $0 < C < +\infty$. Let us define now, for any $k \in \mathbb{N}$, the function

$$\tilde{\omega}_k(x,\epsilon) \coloneqq \omega_{(\lfloor C \rfloor + 1)k}(x, C\epsilon) \quad \forall x \in \Omega,\ \forall \epsilon \geq 0.$$

Of course we have that $(\tilde{\omega}_k)_k$ is still a sequence of locally integrable moduli of continuity: we show that such a sequence satisfies (3.2.16). Take $A' \in \mathcal{A}_0$, $k \in \mathbb{N}$, $\epsilon \geq 0$, $u, v \in C^1(\overline{\Omega})$ such that

$$|u(x)|, |v(x)|, |Du(x)|, |Dv(x)| \leq k,\ |u(x) - v(x)|, |Du(x) - Dv(x)| \leq \epsilon \quad \forall x \in A'.$$

Then it follows that

$$|Xu(x)| = |C(x)Du(x)| \leq C|Du(x)| \leq Ck \leq (\lfloor C \rfloor + 1)k,$$

$$|Xv(x)| = |C(x)Dv(x)| \leq C|Dv(x)| \leq Ck \leq (\lfloor C \rfloor + 1)k$$

and

$$|Xu(x) - Xv(x)| = |C(x)(Du(x) - Dv(x))| \leq C|Du(x) - Dv(x)| \leq C\epsilon.$$

Thus we conclude that

$$|F(u, A') - F(v, A')| \leq \int_{A'} |f(x, u, Xu) - f(x, v, Xv)| \mathrm{d}x \leq \int_{A'} \tilde{\omega}_k(x, \epsilon) \mathrm{d}x,$$

and so also (iii) is proved. (iv) follows easily from (3.3.13), while (vi) is a direct consequence of Theorem 3.3.1. Let us now define $H : W^{1,\infty}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ as the restriction to $W^{1,\infty}(\Omega) \times \mathcal{A}$ of F. Then, since for every $u \in W^{1,\infty}(\Omega)$ it holds that $Xu(x) = C(x)Du(x)$, if we define $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty)$ as

$$f_e(x, u, \xi) \coloneqq f(x, u, C(x)\xi)$$

we can easily notice that f_e is a Carathéodory function, convex in the third argument and such that

$$H(u, A) = \int_A f_e(x, u, Du) \mathrm{d}x.$$

Applying [2, Theorem 2.1], condition (v) holds for H and hence for F. □

3.3.1 X-convexity

Let us mention that, in Proposition 3.1.21, the convexity of f is a crucial assumption in order to guarantee the identity (3.1.6). Indeed, the latter is not verified when we drop that condition, as showed in [55, Example 5.1].

Example 3.3.5. Let us take $\Omega = B_1(0) \subseteq \mathbb{R}^2$, $m = 1$ and

$$X_1 \coloneqq x\frac{\partial}{\partial y}.$$

Then X_1 is a Lipschitz vector field satisfying the (LIC) on Ω, with $N_X \coloneqq \{(x, y) \in \Omega : x = 0\}$. Clearly, for all $(x, y) \in \Omega_X$ we have

$$C((x,y))^T \cdot B^{-1}((x,y)) \cdot C((x,y)) = \begin{bmatrix} 0 \\ x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{x^2} \end{bmatrix} \cdot \begin{bmatrix} 0 & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

thus by Proposition 3.1.15 it follows that

$$\Pi_{(x,y)}(\xi_1,\xi_2) = (0,\xi_2) \quad \forall(\xi_1,\xi_2) \in \mathbb{R}^2,\ \forall(x,y) \in \Omega_X. \tag{3.3.14}$$

Let us define the map $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^2 \longrightarrow [0,+\infty)$ as

$$f_e((x,y),u,(\xi_1,\xi_2)) := \begin{cases} 1-\xi_1^2-\xi_2^2 & \text{if } \xi_1^2+\xi_2^2 \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, f_e is a bounded Carathéodory function not convex in the third entry. Moreover, for any $(x,y) \in \Omega_X$ and $(\xi_1,\xi_2) \in \mathbb{R}^2$ with $\xi_1^2+\xi_2^2 \le 1$, thanks to (3.3.14) it holds that

$$f_e((x,y),u,\Pi_{(x,y)}(\xi_1,\xi_2)) = 1-\xi_2^2.$$

We conclude that (3.1.6) does not hold.

On the other hand, it is easy to see that there are cases when Proposition 3.1.21 still holds even if the Lagrangian is not convex in the third argument, as the following example shows.

Example 3.3.6. Let us take $n, m, \mathbf{X}$ and Ω as in the previous example, and define the function $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to [0,+\infty)$ as

$$f_e((x,y),u,(\xi_1,\xi_2)) := \begin{cases} 1-\xi_2^2 & \text{if } |\xi_2| \le 1 \\ 0 & \text{otherwise} \end{cases}.$$

Then f_e is again a bounded Carathéodory function which is not convex in the third entry. Anyway we can easily see that f_e satisfies (3.1.6).

At this point we may ask if there is a way to weaken the convexity of f_e in the third entry which is still able to guarantee the validity of (3.1.6). In the previous example we see that, despite f_e is not globally convex in the third entry, however it is convex along the direction given by N_x (see Definition 3.1.14). This leads us to the following crucial new notion.

Definition 3.3.7. *Let* $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ *be a Carathéodory function. We say that* f_e *is* X-convex *if, for a.e.* $x \in \Omega$ *and for any* $u \in \mathbb{R}$*,* $t \in (0,1)$ *and* $\xi_1, \xi_2 \in \mathbb{R}^n$ *such that* $\xi_2 - \xi_1 \in N_x$*, it holds that*

$$f_e(x, u, t\xi_1 + (1-t)\xi_2) \leq t f_e(x, u, \xi_1) + (1-t) f_e(x, u, \xi_2).$$

The next proposition tells us that X-convexity is the proper requirement that we have to assume on the Euclidean Lagrangian.

Proposition 3.3.8. *Let* $f_e : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, +\infty]$ *be a Carathéodory function such that there exist* $a \in L^1_{\mathrm{loc}}(\Omega)$ *and* $b, c > 0$ *such that*

$$f_e(x, u, \xi) \leq a(x) + b|C(x)\xi|^p + c|u|^p \text{ for a.e. } x \in \Omega, \ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n. \tag{3.3.15}$$

Then the following facts are equivalent:

(i) f_e *is* X*-convex;*

(ii) for a.e. $x \in \Omega$ *and for any* $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$*, the function* $g : N_x \to [0, +\infty]$ *defined as* $g(\eta) \coloneqq f_e(x, u, \xi + \eta)$ *is constant;*

(iii) $f_e(x, u, \xi) = f_e(x, u, \Pi_x(\xi))$ *for a.e.* $x \in \Omega$*,* $\forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^n$*;*

Proof. $(ii) \Leftrightarrow (iii)$ Fix $x \in \Omega$ such that (ii) holds. For any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$, we have that

$$f_e(x, u, \xi) = f_e(x, u, \xi_{N_x} + \Pi_x(\xi)) = f_e(x, u, \Pi_x(\xi)).$$

Conversely, take $x \in \Omega$ such that (iii) holds. For any $(u, \xi) \in \mathbb{R} \times \mathbb{R}^n$ and $\eta \in N_x$, it holds that

$$f_e(x, u, \xi + \eta) = f_e(x, u, \Pi_x(\xi + \eta)) = f_e(x, u, \Pi_x(\xi)) = f_e(x, u, \xi).$$

$(i) \Leftrightarrow (ii)$ The right implication is trivial. Conversely, assume (i) and fix $x \in \Omega$ such that (i) holds and $a(x) < +\infty$. Thanks to (3.3.15) we have that, for any fixed $u \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $\eta \in N_x$,

$$\begin{aligned} g(\eta) = f_e(x, u, \xi + \eta) &\leq a(x) + b|C(x)\xi + C(x)\eta|^p + c|u|^p \\ &= a(x) + b|C(x)\xi|^p + c|u|^p < +\infty. \end{aligned}$$

Since the right side does not depend on η, then g is bounded on N_x. Since by assumption it is also convex on N_x, then g is constant. $\square$

In order to guarantee the X-convexity of the Euclidean Lagrangian we exploit the zig-zag argument employed in [26, Lemma 2.11].

Lemma 3.3.9. *Let* $F : W^{1,p}_{\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *be such that*

(i) For every $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$*, the map* $A \mapsto F(u, A)$ *is a measure;*

(ii) For every $u, v \in W^{1,p}_{\mathrm{loc}}(\Omega), \forall A' \in \mathcal{A}_0$*,* $u|_{A'} = v|_{A'} \implies F(u, A') = F(v, A')$*;*

(iii) F *satisfies the weak condition* (ω)*;*

(iv) For any $A' \in \mathcal{A}_0$ *and* $(u_h)_h \subseteq W^{1,p}(\Omega), u \in W^{1,p}(\Omega)$ *such that* $\lim_{h\to\infty} \|u_h - u\|_{W^{1,p}_X(\Omega)} = 0$*, then* $F(u, A') \leq \liminf_{h\to\infty} F(u_h, A')$*;*

Then, if for any $x \in \Omega$*,* $u \in \mathbb{R}$ *and* $\xi \in \mathbb{R}^n$ *we define*

$$f_e(x, u, \xi) \coloneqq \limsup_{R\to 0} \frac{F(\varphi_{x,u,\xi}, B_R(x))}{|B_R(x)|} \tag{3.3.16}$$

it holds that f_e *is* X*-convex.*

Proof. A slight modification of [26, Lemma 2.10] ensures the existence of a sequence $(\omega_k)_k$ of locally integrable moduli of continuity and a set $\Omega' \subseteq \Omega$ such that $|\Omega'| = |\Omega|$ and all the points in Ω' are Lebesgue points of $x \mapsto \omega_k(x, r)$ for any $k \in \mathbb{N}$ and for any $r \geq 0$. Moreover

$$|f_e(x, u, \xi) - f_e(x, v, \xi)| \leq \omega_k(x, |u - v|) \tag{3.3.17}$$

for any $x \in \Omega'$, $k \in \mathbb{N}$, $u, v \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$ such that

$$|\xi|, |u|, |v| \leq k.$$

Pick $x \in \Omega'$, $z \in \mathbb{R}$, $t \in (0, 1)$, $\xi_1 \neq \xi_2$ in $\mathbb{R}^n$ such that $\xi_2 - \xi_1 \in N_x$, and set $\xi \coloneqq t\xi_1 + (1-t)\xi_2$. We want to prove that

$$f_e(x, z, \xi) \leq t f_e(x, z, \xi_1) + (1 - t) f_e(x, z, \xi_2). \tag{3.3.18}$$

Let us define

$$\xi_0 := \frac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|},$$

and, for any $h \in \mathbb{N}$, $k \in \mathbb{Z}$ and $i = 1, 2$, let us set

$$\Omega^1_{h,k} := \left\{ y \in \Omega \,:\, \frac{k-1}{h} \le (\xi_0, y) < \frac{k-1+t}{h} \right\};$$

$$\Omega^2_{h,k} := \left\{ y \in \Omega \,:\, \frac{k-1+t}{h} \le (\xi_0, y) < \frac{k}{h} \right\};$$

$$\Omega^i_h := \bigcup_{k \in \mathbb{Z}} \Omega^i_{h,k};$$

$$u(y) := z + (\xi, y - x) \quad \forall y \in \Omega;$$

$$v_h(y) := \begin{cases} (1-t)\frac{k-1}{h}|\xi_2 - \xi_1| + z + \langle \xi_1, y - x \rangle & \text{if } y \in \Omega^1_{h,k} \\ -t\frac{k}{h}|\xi_2 - \xi_1| + z + \langle \xi_2, y - x \rangle & \text{if } y \in \Omega^2_{h,k} \end{cases}.$$

Arguing as in the proof of [25, Lemma 2.11] we have that $v_h \to u$ uniformly on Ω. Hence, in particular, $v_h \to u$ strongly in $L^p(\Omega)$. Moreover, since $\xi_2 - \xi_1$ belongs to N_x and ξ is a convex combination of ξ_1 and ξ_2, then both $\xi - \xi_1$ and $\xi - \xi_2$ belong to N_x. Thus for $i = 1, 2$ and for any $y \in \Omega^i_{h,k}$ we have that

$$|Xu(y) - Xv_h(y)| = |C(x)\xi - C(x)\xi_i| = |C(x)(\xi - \xi_i)| = 0.$$

Therefore v_h converges to u strongly in $W^{1,p}_X(\Omega)$. Take now $k \in \mathbb{N}^+$ such that, for any $y \in \Omega$ and for any $h \in \mathbb{N}^+$,

$$|\xi_1|, |\xi_2|, |u_1(y)|, |u_2(y)|, |v_h(y)| \le k.$$

Then, thanks to (3.3.17) and thanks to (see [26, Lemma 2.4])

$$F(u, A) = \int_A f_e(x, u, Du)\mathrm{d}x \quad \forall u \text{ affine on } \Omega, \forall A \in \mathcal{A};$$

we can argue as in [25, Lemma 2.11] and we set $B^i_{h,R}(x) := B_R(x) \cap \Omega^i_h$ for $i = 1, 2$ and for any $R > 0$ such that $B_R(x) \Subset \Omega$. Then, it holds that

$$F(v_h, B_R(x)) \le \int_{B^1_{h,R}(x)} f_e(y, u_1, Du_1)\mathrm{d}y + \int_{B^2_{h,R}(x)} f_e(y, u_2, Du_2)\mathrm{d}y + \int_\Omega w_k(y, aR + \frac{b}{h}),$$

with $a := |\xi_2 - \xi_1|$ and $b := at(1-t)$. Since v_h converges to u strongly in $W^{1,p}_X(\Omega)$ and thanks to hypothesis (iv) it is easy to see that

$$F(u, B_R(x)) \leq tF(u_1, B_R(x)) + (1-t)F(u_2, B_R(x)) + \int_\Omega w_k(y, \epsilon),$$

where this inequality holds for any $\epsilon > 0$ and for any $R \in (0, \frac{\epsilon}{a}]$. Dividing both sides by $|B_R(x)|$, passing to the limsup and recalling that x is a Lebesgue point of $y \mapsto w_k(y, \epsilon)$, we have that

$$f_e(x, z, \xi) \leq tf_e(x, z, \xi_1) + (1-t)f_e(x, z, \xi_2) + w_k(x, \epsilon).$$

Letting ϵ go to zero, the thesis is proved. □

3.4 Integral Representation of Non-convex functionals

To resume, we developed all the tools to exploit [26, Theorem 1.8]. This allows us to characterize the class of local functionals for which we do not require neither translations-invariance nor convexity, and for which we want to weaken the assumption of weak*- sequential lower semicontinuity in Theorem 3.3.2. Thanks to the previous results, we are now ready to show and prove the main result of this section.

Theorem 3.4.1. *Let $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ be such that:*

(i) *F is a measure;*

(ii) *F is local;*

(iii) *F satisfies the strong condition (ω);*

(iv) *F is p-bounded;*

(v) *F is lower semicontinuous.*

Then there exists a unique Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ such that

$$f(x, u, \xi) \leq a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\ \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m \tag{3.4.1}$$

and the following representation formula holds:

$$F(u, A) = \int_A f(x, u, Xu)\mathrm{d}x \quad \forall u \in W^{1,p}_{X,\mathrm{loc}}(\Omega),\, \forall A \in \mathcal{A}. \tag{3.4.2}$$

Proof. Let us consider the restriction of F to $W^{1,p}_{\mathrm{loc}}(\Omega) \times \mathcal{A}$. Arguing as in the first step of the proof of Theorem 3.2.1, it is easy to see that it satisfies all the hypotheses of [26, Theorem 1.8]. Thus, if f_e is defined as in (3.3.16), it is a Carathéodory function and moreover there exists $\tilde{b} > 0$ such that

$$F(u, A) = \int_A f_e(x, u, Du)\mathrm{d}x \quad \forall A \in \mathcal{A},\, \forall u \in W^{1,p}_{\mathrm{loc}}(\Omega)$$

and

$$f_e(x, u, \xi) \le a(x) + \tilde{b}|\xi|^p + c|u|^p \qquad \text{for a.e. } x \in \Omega,\, \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m.$$

Moreover, thanks to Lemma 3.3.9, f_e is X-convex. Therefore, recalling the thesis of Proposition 3.3.8 and (iv) of Proposition 3.1.20, we obtain that

$$\int_A f_e(x, u, Du)\mathrm{d}x = \int_A f(x, u, Xu)\mathrm{d}x \quad \forall A \in \mathcal{A},\, u \in C^\infty(A),$$

where $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty]$ is the function defined in (3.1.5). Such an f can be supposed to be finite up to modifying it on a set of measure zero. Arguing as in the third step of the proof of Theorem 3.2.1, (3.4.1) holds, while (3.4.2) follows exactly as in the last step of the proof or Theorem 3.3.2. Finally, uniqueness follows as usual. □

Corollary 3.4.2. *Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, +\infty)$ be a Carathéodory function such that*

$$f(x, u, \xi) \le a(x) + b|\xi|^p + c|u|^p \quad \text{for a.e. } x \in \Omega,\, \forall (u, \xi) \in \mathbb{R} \times \mathbb{R}^m,$$

for $b, c > 0$ and $a \in L^1_{\mathrm{loc}}(\Omega)$. Setting the functional $F : W^{1,p}_{X,\mathrm{loc}}(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ by

$$F(u, A) := \int_A f(x, u, Xu)\mathrm{d}x \quad \forall u \in W^{1,p}_{X,\mathrm{loc}}(\Omega),\, \forall A \in \mathcal{A},$$

then F satisfies hypotheses $(i) - (v)$ of Theorem 3.4.1.

Proof. Arguing exactly as in Theorem 3.3.4 we obtain the thesis. □

Chapter 4

Γ-compactness of Integral Functionals in L^p and $W^{1,p}_X$

The aim of the present chapter is to generalize the Γ-compactness result presented in [81] to the previous classes of integral functionals which are not assumed to be translations-invariant. Let us mention that the main results are contained in the work [56].

4.1 Basic Notions of Γ-convergence

First of all, we collect some basic notions and results concerning Γ-convergence's theory, which are contained in the fundamental monograph [38] and to which we will refer through this section. We also recommend monograph [17] as exhaustive account on this topic, containing also interesting applications of Γ-convergence.

Definition 4.1.1. *If (X, τ) is a first-countable topological space and $(F_h)_h$ is a sequence of functions defined on (X, τ) with values in $\overline{\mathbb{R}}$, we define the* Γ-lower limit *and* Γ-upper limit *respectively as*

$$\Gamma - \liminf_{h\to\infty} F_h(u) := \inf \left\{ \liminf_{h\to\infty} F_h(u_h) \,:\, u_h \xrightarrow{\tau} u \right\}$$

and

$$\Gamma - \limsup_{h\to\infty} F_h(u) := \inf\left\{\limsup_{h\to\infty} F_h(u_h) \,:\, u_h \xrightarrow{\tau} u\right\},$$

and we say that $(F_h)_h$ Γ-converges *to* $F : (X,\tau) \longrightarrow \overline{\mathbb{R}}$ *if it holds that*

$$\Gamma - \liminf_{h\to\infty} F_h(u) = \Gamma - \limsup_{h\to\infty} F_h(u) \quad \textit{for any } u \in X.$$

In this case we say that F *is the* $\Gamma-$*limit of* $(F_h)_h$ *and we write* $F = \Gamma - \lim_{h\to\infty} F_h$.

The next results give an idea of some basic properties involving Γ-limits.

Theorem 4.1.2. *Let* F_h *and* F *be functionals from space* (X,τ) *to* $[-\infty,+\infty]$, $h \in \mathbb{N}$.

- [38, Proposition 6.1] *If* $(F_h)_h$ $\Gamma(\tau)$*-converges to* F*, then each of its subsequences still* $\Gamma(\tau)$*- converges to* F.

- [38, Proposition 6.3] *Let* τ_1 *and* τ_2 *be two topologies on* X *such that* τ_1 *is weaker than* τ_2. *If* $(F_h)_h$ $\Gamma(\tau_1)$*-converges to* F_1 *and* $(F_h)_h$ $\Gamma(\tau_2)$*-converges to* F_2*, then* $F_1 \le F_2$.

- [38, Theorem 7.8] (Fundamental Theorem of Γ- convergence) Assume that $(F_h)_h$ is equicoercive (on X), that is, for each $t \in \mathbb{R}$ there exists a closed countably compact set $K_t \subset X$ such that

$$\{x \in X \,:\, F_h(x) \le t\} \subset K_t \quad \text{for each } h \in \mathbb{N}.$$

  Let us also assume that $(F_h)_h\,\Gamma(\tau)$-converges to F. Then, F is coercive and

$$\min_{x\in X} F(x) = \lim_{h\to\infty} \inf_{x\in X} F_h(x).$$

Proposition 4.1.3. *Let* F_h *and* F *be functionals from space* (X,τ) *to* $[-\infty,+\infty]$, $h \in \mathbb{N}$, *then the following facts hold.*

· *For any* $u \in X$ *and for any sequence* $(u_h)_h$ *converging to* u *in* X*, it holds that*

$$\Gamma - \liminf_{h\to\infty} F_h(u) \le \liminf_{h\to\infty} F_h(u_h) \quad \textit{and} \quad \Gamma - \limsup_{h\to\infty} F_h(u) \le \limsup_{h\to\infty} F_h(u_h).$$

- *For any $u \in X$ there exist two sequences $(u_h)_h$ and $(v_h)_h$, converging to u in X, which we call* recovery sequences*, such that*

$$\Gamma - \liminf_{h\to\infty} F_h(u) = \liminf_{h\to\infty} F_h(u_h) \quad \textit{and} \quad \Gamma - \limsup_{h\to\infty} F_h(u) = \limsup_{h\to\infty} F_h(v_h).$$

- *For any $u \in X$ and for any sequence $(u_h)_h$ converging to u in X, it holds that*

$$\Gamma - \lim_{h\to\infty} F_h(u) \le \liminf_{h\to\infty} F_h(u_h);$$

- *For any $u \in X$ there exists a sequence $(u_h)_h$ converging to u in X, which we call* recovery sequence*, such that*

$$\Gamma - \lim_{h\to\infty} F_h(u) = \lim_{h\to\infty} F_h(u_h).$$

Beside the notion of Γ-convergence there is a related one, which is more suitable to deal with sequences of local functionals, usually known as $\overline{\Gamma}$-*convergence*.

Definition 4.1.4. *Let $F_h : X \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$ be an increasing sequence of functionals. If we define*

$$F'(\cdot, A) := \Gamma - \liminf_{h\to\infty} F_h(\cdot, A) \quad \textit{and} \quad F''(\cdot, A) := \Gamma - \limsup_{h\to\infty} F_h(\cdot, A)$$

for any $A \in \mathcal{A}$, we say that F_h $\bar{\Gamma}$-converges *to a functional $F : X \times \mathcal{A} \longrightarrow \bar{\mathbb{R}}$ if it holds that*

$$F(\cdot, A) = \sup\{F'(\cdot, A') \,:\, A' \in \mathcal{A},\, A' \Subset A\} = \sup\{F''(\cdot, A') \,:\, A' \in \mathcal{A},\, A' \Subset A\}.$$

In other words, we say that $(F_h)_h$ $\bar{\Gamma}-$converges to F whenever the *inner regular envelopes* of F' and F'' coincide and are equal to F. It is easy to check (cf. [38, Remark 16.3]) that any $\bar{\Gamma}-$limit is increasing, inner regular and lower semicontinuous.
In the sequel, when we will deal with Γ-convergence with respect to the strong topology of L^p or with respect to the strong topology of $W^{1,p}_X$, we will refer respectively to $\Gamma(L^p)-$convergence or $\Gamma(W^{1,p}_X)-$convergence.

4.1.1 Uniform Fundamental Estimate

As one could expect, the notions of $\Gamma-$convergence and $\bar{\Gamma}$-convergence are strongly related. Indeed, let us assume for instance that a sequence of increasing functionals $F_h : L^p(\Omega)\times\mathcal{A} \longrightarrow [0,\infty]$ is such that

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k\to\infty} F_h(\cdot, A) \tag{4.1.1}$$

for any $A \in \mathcal{A}$ and for a suitable measure functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0,\infty]$. Then F is L^p-lower semicontinuous, since it is a Γ-limit (cf. [38, Proposition 6.8]), and also increasing and inner regular, since it is a non-negative measure (cf. [38, Theorem 14.23]). Therefore, thanks to [38, Proposition 16.4], we can conclude that

$$F = \bar{\Gamma}(L^p) - \lim_{h\to\infty} F_h. \tag{4.1.2}$$

The converse implication is usually more delicate because, in general, the $\bar{\Gamma}(L^p)-$limit is not a measure. Indeed, despite the $\bar{\Gamma}-$limit is always increasing, inner regular and, even if superadditivity behaves usually well, there are examples (cf. [38, Example 16.13]) in which F fails to be subadditive. For this reason, when dealing with this issues, it is practise to work within milder classes of local functionals. To this aim, the so-called *uniform fundamental estimates* are introduced. These estimates, although depending in their definition on the chosen topological space, are usually sufficient conditions for the subadditivity of the $\bar{\Gamma}-$limit. To give an instance, we introduce here the standard notion of uniform fundamental estimate (cf. [38, Definition 18.2]) for functional defined on $L^p(\Omega) \times \mathcal{A}$.

Definition 4.1.5. *Let $\mathcal{F}$ be a class of non-negative local functionals defined on $L^p(\Omega) \times \mathcal{A}$. We say that $\mathcal{F}$ satisfies the* uniform fundamental estimate on $L^p(\Omega)$ *if, for any $\varepsilon > 0$ and for any $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ such that for any $u, v \in L^p(\Omega)$ and for any $F \in \mathcal{F}$, there exists a smooth cut-off function φ between A'' and A', such that*

$$\begin{aligned} F\Big(\varphi u + (1-\varphi)v, A' \cup B\Big) &\leq (1+\varepsilon)\Big(F(u, A'') + F(v, B)\Big) + \\ &+ \varepsilon\Big(\|u\|^p_{L^p(S)} + \|v\|^p_{L^p(S)} + 1\Big) + M\|u - v\|_{L^p(S)}, \end{aligned} \tag{4.1.3}$$

where $S = (A'' \setminus A') \cap B$.

The following result, which can be found in [38, Theorem 18.7], tells us that (4.1.2) is sufficient to guarantee (4.1.1), provided that our sequence satisfies the uniform fundamental estimate and that some reasonable boundedness properties hold.

Theorem 4.1.6. *Let* $F_h : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ *be a sequence of functionals for which there exists a functional* $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ *such that* (4.1.2) *holds. Assume in addition that* $(F_h)_h$ *satisfies the uniform fundamental estimate and that there exist constants* $e_1 \geq 1$ *and* $e_2 \geq 0$*, a non-negative increasing functional* $G : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ *and a finite measure* μ *on* Ω *such that*

$$G(u, A) \leq F_h(u, A) \leq e_1 G(u, A) + e_2 \|u\|^p_{L^p(A)} + \mu(A) \tag{4.1.4}$$

for any $u \in L^p(\Omega)$*,* $A \in \mathcal{A}$ *and* $h \in \mathbb{N}$*. Then* (4.1.1) *holds.*

4.2 Γ-compactness in the strong topology of L^p

We are interested to prove a Γ-compactness result for a class of convex integral functionals defined on $L^p(\Omega)$ with respect to the strong topology of L^p. First of all, we introduce a large class of integral functionals for which some important properties are satisfied, for instance the uniform fundamental estimate introduced in Definition 4.1.5.

Definition 4.2.1. *Consider* $1 < p < \infty$*,* $a \in L^1(\Omega)$ *and constants* $0 < c_0 \leq c_1$ *and* $c_2 \geq 0$*. We say that a functional* $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ *belongs to* $\mathfrak{I}_{m,p}(a, c_0, c_1, c_2)$ *if there exists a Carathéodory function* $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ *such that*

$$c_0 |\eta|^p \leq f(x, u, \eta) \leq a(x) + c_1 |\eta|^p + c_2 |u|^p \tag{4.2.1}$$

for any $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$*, for a.e.* $x \in \Omega$*, and it holds that*

$$F(u, A) = \begin{cases} \int_A f(x, u(x), Xu(x))\,\mathrm{d}x & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases}. \tag{4.2.2}$$

In particular, we say that $F \in \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$ *whenever* $F \in \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ *and it holds that*

$$f(x, \cdot, \cdot) \text{ is convex for a.e. } x \in \Omega. \tag{4.2.3}$$

Now, we want to describe some properties of $\Gamma(L^p)$-limits within the class previously defined.

Proposition 4.2.2. *If* $F \in \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$*, then* F *is lower semicontinuous on* $L^p(\Omega)$*.*

Proof. Let us fix $A \in \mathcal{A}$, $u \in L^p(\Omega)$ and a sequence $(u_h)_h$ converging to u w.r.t. the strong topology of $L^p(\Omega)$ and such that

$$\exists \lim_{h\to\infty} F(u_h, A) < +\infty.$$

Since $f \in J_{m,p}(a, c_0, c_1, c_2)$, we have that

$$c_0 \limsup_{h\to\infty} \|Xu\|_p^p \leq \limsup_{h\to\infty} F(u_h, A) < +\infty,$$

hence $(u_h)_h$ is bounded in $W^{1,p}_X(A)$. By reflexivity there exists $v \in W^{1,p}_X(A)$ such that $u_h \rightharpoonup u$ in $W^{1,p}_X(A)$. Since $u_h \rightharpoonup u$ in $L^p(A)$, we conclude that $u|_A = v|_A$, and so in particular $u \in W^{1,p}_X(A)$. Since under our assumptions on f the functional $(u, v) \mapsto \int_A f(x, u, v)\mathrm{d}x$ defined on $L^p(A) \times L^p(A, \mathbb{R}^m)$ satisfies the hypotheses of [24, Theorem 2.3.1], then it is sequentially lower semicontinuous on $L^p(A) \times L^p(A, \mathbb{R}^m)$ with respect to the strong convergence of $L^p(A)$ and the weak convergence of $L^p(A, \mathbb{R}^m)$. Since $u_h \to u$ in L^p and $Xu_h \rightharpoonup Xu$ in $L^p(A, \mathbb{R}^m)$, we conclude that

$$F(u, A) \leq \lim_{h\to\infty} F(u_h, A).$$

$\square$

We recall the following result, which can be found in [81, Lemma 4.15].

Proposition 4.2.3. *Let us define the functional* $\Psi_p : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ *as*

$$\Psi_p(u, A) := \begin{cases} \|Xu\|^p_{L^p(A)} & \text{if } A \in \mathcal{A},\ u \in W^{1,p}_X(A) \\ +\infty & \text{otherwise} \end{cases}.$$

Then Ψ_p is a L^p-lower semicontinuous measure.

Our aim now is to introduce a suitable and bigger class with respect to $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$. Thanks to general results in [38], in Proposition 4.2.5 below we show that $\mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ is closed under Γ-convergence.

Definition 4.2.4. *We say that a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ belongs to $\mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$ if F is a measure and if there exist $d_1 \geq 1$, $d_2, d_3, d_4 \geq 0$, a finite measure μ, independent of F, and a measure $G : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$, which may depend on F, such that*

$$G(u, A) \leq F(u, A) \leq d_1 G(u, A) + d_2 \|u\|_{L^p(A)} + \mu(A) \tag{4.2.4}$$

and

$$G(\varphi u + (1 - \varphi)v, A) \leq d_4(G(u, A) + G(v, A)) + d_3 d_4 (\max |D\varphi|^p) \|u - v\|_{L^p(A)} + \mu(A), \tag{4.2.5}$$

for any $u, v \in L^p(\Omega)$, $A \in \mathcal{A}$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$.

Proposition 4.2.5. *For any sequence $(F_h)_h \subseteq \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ such that*

(a) *F is a measure;*

(b) *F is local;*

(c) *F is L^p-lower semicontinuous;*

(d) *For any $u \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$ it holds that*

$$\int_A c_0 |Xu(x)|^p \mathrm{d}x \leq F(u, A) \leq \int_A a(x) + c_1 |Xu(x)|^p + c_2 |u(x)|^p \mathrm{d}x, \tag{4.2.6}$$

and moreover it holds that

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k \to +\infty} F_{h_k}(\cdot, A) \quad \text{for any } A \in \mathcal{A}. \tag{4.2.7}$$

Proof. First of all, according to [38, Theorem 19.4], we are going to show that $\mathcal{I}_{m,p}(a, c_0, c_1, c_2) \subseteq \mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$. For this purpose, let us define $\mu : \mathbb{B} \to [0, +\infty]$ by

$$\mu(B) \coloneqq \int_B |a(x)| \, \mathrm{d}x.$$

Then μ is a finite measure on Ω. Moreover, thanks to Proposition 4.2.3, the non-negative local functional $G : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ defined as

$$G(u, A) \coloneqq c_0 \Psi_p(u, A) \qquad \text{for any } u \in L^p(\Omega),\ A \in \mathcal{A}$$

is a measure. Let us show (4.2.4). Let us set $d_1 \coloneqq \frac{c_1}{c_0}$ and $d_2 \coloneqq c_2$. If $A \in \mathcal{A}$ and $u \notin W^{1,p}_X(A)$, the estimate is trivial, while if $u \in W^{1,p}_X(A)$, it follows from the definition of $I_{m,p}(a, c_0, c_1, c_2)$. Hence, we are left to show (4.2.5). Fix then $A \in \mathcal{A}$. If either $u \notin W^{1,p}_X(A)$ or $v \notin W^{1,p}_X(A)$ the estimate is trivial. Hence assume that $u, v \in W^{1,p}_X(A)$ and take $\varphi \in C^\infty_c(\Omega)$ such that $0 \le \varphi \le 1$. Then, recalling Proposition 3.1.5, Proposition 3.1.13, the fact that $\eta \mapsto |\eta|^p$ is convex on $\mathbb{R}^m$, and setting

$$C \coloneqq \max\{\|c_{j,i}\|_\infty : j = 1, \dots, m,\ i = 1, \dots, n\}$$

it follows that $0 < C < \infty$ and

$$\begin{aligned}
G(\varphi u + (1-\varphi)v, A) &= c_0 \int_A |X\varphi(u - v) + \varphi Xu + (1-\varphi)Xv|^p \, \mathrm{d}x \\
&= c_0 2^p \int_A \left| \frac{X\varphi(u-v)}{2} + \frac{\varphi Xu + (1-\varphi)Xv}{2} \right|^p \mathrm{d}x \\
&\le c_0 2^{p-1} \int_A |X\varphi(u-v)|^p \, \mathrm{d}x + c_0 2^{p-1} \int_A |\varphi Xu + (1-\varphi)Xv|^p \, \mathrm{d}x \\
&\le c_0 2^{p-1} \int_A |X\varphi(u-v)|^p \, \mathrm{d}x + 2^{p-1}(G(u,A) + G(v,A)) \\
&\le c_0 2^{p-1} (C\sqrt{m})^p (\max |Du|^p) \|u - v\|_{L^p(A)} + 2^{p-1}(G(u,A) + G(v,A)).
\end{aligned}$$

Thus (4.2.5) follows and hence $\mathcal{I}_{m,p}(a, c_0, c_1, c_2) \subseteq \mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$. Therefore, thanks to [38, Theorem 19.5], there exists a subsequence of $(F_h)_h$, still denoted by $(F_h)_h$, and a L^p-lower semicontinuous functional $F \in \mathcal{M}_p(d_1, d_2, d_3, d_4, \mu)$ such that $(F_h)_h$ $\overline{\Gamma}(L^p)$-converges to F. In particular F is a measure. By [38, Proposition 16.4] and [38, Proposition 16.15], F is also local. Furthermore, by Proposition 4.2.3 G is a L^p-lower semicontinuous measure and, since $(F_h)_h$ satisfies the uniform fundamental estimate on $L^p(\Omega)$ according to [38, Theorem 19.4], we can apply Theorem 4.1.6 to conclude that (4.2.7) holds. Finally, we show that F satisfies (4.2.6). Let us fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(\Omega)$, and a sequence $(u_h)_h$ such that

$$F(u, A) = \lim_{h\to+\infty} F_h(u_h, A). \tag{4.2.8}$$

Arguing as above we can assume that $(u_h)_h \subset W^{1,p}_X(A)$. Therefore, thanks to (4.2.8) and Proposition 4.2.3, it follows that

$$c_0 \int_A |Xu|^p \,\mathrm{d}x \le \liminf_{h\to+\infty} \int_A |Xu_h|^p \,\mathrm{d}x \le \liminf_{h\to+\infty} F_h(u_h, A) = F(u, A), \tag{4.2.9}$$

and so the first inequality follows. Finally we have that

$$\begin{aligned} F(u, A) &\le \liminf_{h\to+\infty} F_h(u, A) \le \liminf_{h\to+\infty} \int_A a(x) + c_1|Xu|^p + c_2|u|^p \,\mathrm{d}x \\ &= \int_A a(x) + c_1|Xu|^p + c_2|u|^p \,\mathrm{d}x. \end{aligned}$$

The latter is equivalent to the thesis. □

In order to represent the Γ-limit given by (4.2.7), in an integral form, we need to apply a slight variant of Theorem 3.2.1. Following a remark presented in the introduction of [55], we clearly have to reply the p-boundness (3.2.2) with the new one (4.2.1). Even if the authors did not consider the equivalence between the bound from below of the Lagrangian and the bound from below of the functional, it is clear from their proofs that such an equivalence is trivial, and so we take it for granted.

Theorem 4.2.6. *Let $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ be such that:*

(i) F is a measure;

(ii) F is local;

(iii) F is convex;

(iv) F satisfies (4.2.6).

Then there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ which satisfies (4.2.1) *and* (4.2.3)*, and such that*

$$F(u, A) = \int_A f(x, u(x), Xu(x))\mathrm{d}x \tag{4.2.10}$$

for any $A \in \mathcal{A}$ and for any $u \in W^{1,p}_X(A)$.

Proof. By Theorem 3.2.1, we know that there exists a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ which satisfies (4.2.1) and (4.2.3), and such that

$$F(u, A) = \int_A f(x, u(x), Xu(x))\mathrm{d}x \quad \text{for any} \;\; A \in \mathcal{A},\, u \in W^{1,p}_X(\Omega).$$

Fix now $A \in \mathcal{A}$, $A' \in \mathcal{A}_0$ with $A' \Subset A$ and $u \in L^p(\Omega) \cap W^{1,p}_X(A)$, and let $v := \varphi u$, where φ is a smooth cut-off function between A' and A. Then clearly $v \in W^{1,p}_X(\Omega)$ and $v|_{A'} = u$. Since F is local, it follows that

$$F(u, A') = F(v, A') = \int_{A'} f(x, v(x), Xv(x))\mathrm{d}x = \int_{A'} f(x, u(x), Xu(x))\mathrm{d}x.$$

Since F is a measure, it is in particular inner regular, and so we conclude that (4.2.10) holds. □

As announced, the main result of this section is the Γ-compactness for the class of convex integral functionals.

Theorem 4.2.7. *For any sequence $(F_h)_h \subseteq \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a local functional $F \in \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$ such that*

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k \to +\infty} F_{h_k}(\cdot, A) \qquad \text{for any } A \in \mathcal{A}.$$

Proof. As $F \in \mathcal{K}_{m,p}(a, c_0, c_1, c_2)$, thanks to Proposition 4.2.5 there exists a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ which is a measure, local, satisfies (4.2.6) and such that (4.2.7) holds. Let us show that F is convex on $W^{1,p}_X(\Omega)$. Fix then $A \in \mathcal{A}$ and take $t \in (0, 1)$ and

$u, v \in W_X^{1,p}(\Omega)$. Let $(u_h)_h$ and $(v_h)_h$ be two sequences converging respectively to u and v in $L^p(\Omega)$ and such that

$$F(u, A) = \lim_{h \to +\infty} F_h(u_h, A), \qquad F(v, A) = \lim_{h \to +\infty} F_h(v_h, A). \tag{4.2.11}$$

Since $F(u, A)$ and $F(v, A)$ are finite we can assume that the sequences $(u_h)_h, (v_h)_h$ belong to $W_X^{1,p}(A)$. Therefore, since each $F_h(\cdot, A)$ is convex on $W_X^{1,p}(A)$, recalling (4.2.11) and the fact that $(tu_h + (1-t)v_h)_h$ converges to $tu + (1-t)v$ in $L^p(\Omega)$, it follows that

$$\begin{aligned} F(tu + (1-t)v, A) &\leq \liminf_{h \to +\infty} F_h(tu_h + (1-t)v_h, A) \\ &\leq \liminf_{h \to +\infty} (tF_h(u_h, A) + (1-t)F_h(v_h, A)) \\ &= t \lim_{h \to +\infty} F_h(u_h, A) + (1-t) \lim_{h \to +\infty} F_h(v_h, A) \\ &= tF(u, A) + (1-t)F(v, A). \end{aligned}$$

Therefore we are in position to apply Theorem 4.2.6. Finally, we notice that if $A \in \mathcal{A}$ and $u \in L^p(\Omega) \setminus W_X^{1,p}(A)$, arguing as in (4.2.9) we conclude that $+\infty = c_0 \Psi_p(u, A) \leq F(u, A)$, which implies that

$$\{u \in L^p(\Omega) \,:\, F(u, A) < +\infty\} = W_X^{1,p}(A),$$

and so the thesis follows. □

4.3 Γ-compactness in the strong topology of $W_X^{1,p}$

In this section we show two Γ-compactness results for two family of integral functionals with respect to the strong topology of $W_X^{1,p}$. As aforementioned, working in this framework has the advantages to do not have to assume coercivity on the sequence of Lagrangians and to allow the case $p = 1$. Here we introduce another suitable family of integral functionals.

Definition 4.3.1. *Let $1 \leq p < +\infty$ and let us fix $a \in L^1(\Omega)$ and $c_1, c_2 \geq 0$. We say that a functional $F : W_X^{1,p}(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ belongs to $\mathfrak{U}_{m,p}(a, c_1, c_2)$ if there exists a Carathéodory* *function* $f : \Omega \times \mathbb{R} \times \mathbb{R}^m \to [0, \infty]$ *such that*

$$f(x, u, \eta) \leq a(x) + c_1|\eta|^p + c_2|u|^p \tag{4.3.1}$$

for any $(u, \eta) \in \mathbb{R} \times \mathbb{R}^m$ *and for a.e.* $x \in \Omega$*, and it holds that*

$$F(u, A) = \int_A f(x, u(x), Xu(x))\mathrm{d}x \quad \textit{for any} \;\; A \in \mathcal{A},\, u \in W^{1,p}_X(\Omega).$$

Before presenting the main theorems above we want to present a key notion, strongly inspired by Definition 3.2.3.

Definition 4.3.2. *We say that* $\omega = (\omega_s)_{s \geq 0}$ *is a* family of locally integrable moduli of continuity *if* $\omega_s : \Omega \times [0, +\infty) \longrightarrow [0, +\infty)$ *and*

$$r \mapsto \omega(x, r) \textit{ is increasing, continuous and } \omega(x, 0) = 0 \tag{4.3.2}$$

for a.e. $x \in \Omega$ *and for any* $s \geq 0$*,*

$$s \mapsto \omega_s(x, r) \textit{ is increasing and continuous} \tag{4.3.3}$$

for a.e. $x \in \Omega$ *and for any* $r \geq 0$*, and*

$$x \mapsto \omega_s(x, r) \in L^1_{\mathrm{loc}}(\Omega) \quad \textit{for any } r, s \geq 0.$$

Moreover we say that a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ *satisfies the* strong condition (ωX) with respect to ω *if there exists a family* $\omega = (\omega_s)_{s \geq 0}$ *of locally integrable moduli of continuity such that*

$$|F(v, A') - F(u, A')| \leq \int_{A'} \omega_s(x, r)\, \mathrm{d}x \tag{4.3.4}$$

for any $s \geq 0$*,* $A' \in \mathcal{A}_0$*,* $r \geq 0$*,* $u, v \in W^{1,p}_X(\Omega)$ *such that*

$$|u(x)|, |v(x)|, |Xu(x)|, |Xv(x)| \leq s$$
$$|u(x) - v(x)|, |Xu(x) - Xv(x)| \leq r$$

for a.e. $x \in A'$*.*

This new notion is more flexible, it fits better with the non-Euclidean setting and allows to deal with more general classes of functions. On the other hand, it is quite easy to verify that the previous condition is stronger than Definition 3.2.3, and hence all the integral representation results proved in [26, 55] remain valid. Moreover, differently from [26], we point out that the previous family of moduli of continuity is indexed over a continuous set, and the assumption on the behaviour of $s \mapsto \omega_s(x,r)$ is completely new. Nevertheless we will see below that, when dealing with integral functionals, this original requirement is quite natural.

Proposition 4.3.3. *Let $F \in \mathfrak{U}_{m,p}(a, c_1, c_2)$. Then F satisfies the strong condition (ωX).*

Proof. This proof is based on the proof of [26, Lemma 2.5]. Since f is Carathéodory, then the set $\Omega' := \{x \in \Omega \,:\, (u,\xi) \mapsto f(x,u,\xi) \text{ is continuous}\}$ satisfies $|\Omega'| = |\Omega|$. For any $s, r \geq 0$, set $E_r^s \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$ as

$$E_r^s := \{(u, v, \xi, \eta) \,:\, |u|, |v|, |\xi|, |\eta| \leq s, |u - v|, |\xi - \eta| \leq r\}$$

and the function

$$\omega_s(x,r) := \begin{cases} \sup\{|f(x,u,\xi) - f(x,v,\eta)| \,:\, (u,v,\xi,\eta) \in E_r^s\} & \text{if } x \in \Omega', \\ 0 & \text{otherwise.} \end{cases}$$

We show that $(\omega_s)_{s\geq 0}$ is a family of locally integrable moduli of continuity. Let us fix then $s, r \geq 0$: since $(u,\xi) \mapsto f(x,u,\xi)$ is continuous for almost every $x \in \Omega$, then the supremum in the definition of ω_s can be taken over a countable subset of E_ϵ^k. Since for any (u, v, ξ, η) the function $x \mapsto |f(x,u,\xi) - f(x,v,\eta)|$ is measurable, then $\omega_s(\cdot, r)$ is also measurable. Moreover, thanks to (4.3.1), it follows that, for any $(u, v, \xi, \eta) \in E_\epsilon^k$,

$$\begin{aligned} |f(x,u,\xi) - f(x,v,\eta)| &\leq 2|a(x)| + c_1|\xi|^p + c_1|\eta|^p + c_2|u|^p + c_2|v|^p \\ &\leq 2|a(x)| + 4s(c_1 + c_2). \end{aligned}$$

Since the right hand side does not depend on $(u, v, \xi, \eta) \in E_r^s$, we conclude that

$$\omega_s(x,r) \leq 2|a(x)| + 4s(c_1 + c_2).$$

Hence $\omega_k(\cdot,\epsilon) \in L^1_{\rm loc}(\Omega)$. Fix now $x \in \Omega'$ and $s \geq 0$. Since $E^s_r \subseteq E^s_t$ for any $r \leq t$, then $\omega_s(x,\cdot)$ is increasing, $\omega_k(x,0) = 0$ and the continuity follows from the continuity of $f(\cdot,u,\xi)$. Finally, taking $x \in \Omega'$ and $r \geq 0$ we have again that $E^s_r \subseteq E^t_r$ for any $r \leq t$, hence $s \mapsto \omega_s(x,r)$ is increasing. Once more, from the continuity of $f(\cdot,u,\xi)$ we conclude that $s \mapsto \omega_s(x,r)$ is continuous. Then $(\omega_s)_s$ is a family of locally integrable moduli of continuity. It is straightforward to check that F satisfies the strong condition (ωX) with respect to $(\omega_s)_{s\geq 0}$. □

4.3.1 Uniform Fundamental Estimate on $W^{1,p}_X(\Omega)$

A further key step in order to prove the main results is presenting a suitable notion of uniform fundamental estimate. In order to guarantee a better compatibility with the non-Euclidean setting, we present above a slight modification with respect to the one introduced in [38].

Definition 4.3.4. *Let $\mathcal{F}$ be a class of non-negative local functionals defined on $W^{1,p}_X(\Omega) \times \mathcal{A}$. We say that $\mathcal{F}$ satisfies the* uniform fundamental estimate on $W^{1,p}_X(\Omega)$ *if, for any $\varepsilon > 0$ and for any $A', A'', B \in \mathcal{A}$, with $A' \Subset A''$, there exists a constant $M > 0$ and a finite family $\{\varphi_1,\ldots,\varphi_k\}$ of smooth cut-off functions between A' and A'' such that for any $u,\, v \in W^{1,p}_X(\Omega)$ and for any $F \in \mathcal{F}$, we can choose $\varphi \in \{\varphi_1,\ldots,\varphi_k\}$ such that*

$$F\Big(\varphi u + (1-\varphi)v, A' \cup B\Big) \leq \Big(F(u,A'') + F(v,B)\Big) + \\ + \varepsilon\Big(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\Big) + M\|u-v\|_{L^p(S)},$$

where $S = (A'' \setminus A') \cap B$.

Remark 4.3.5. From one hand, Definition 4.3.4 is stronger than Definition 4.1.5. since the first requires to choose in a finite family of cut-off functions, which depends only on ε, A', A'' and B. This new requirement is crucial to guarantee a uniform estimate for the X-gradients of the test functions. However, in order to avoid the coercivity assumptions on the Lagrangians, we replace some of the L^p norms on the right hand side of (4.1.3) with $W^{1,p}_X$ norms.

The following propositions and the relative proofs are respectively the counterparts of [38, Proposition 19.1] and [38, Proposition 18.3].

Proposition 4.3.6. *$\mathcal{U}_{m,p}(a, c_1, c_2)$ satisfies the uniform fundamental estimate on $W^{1,p}_X(\Omega)$.*

Proof. Let us set $d_1 \coloneqq c_1$, $d_2 \coloneqq c_2$ and $d_4 \coloneqq 2^{p-1}$ and $\sigma(C) \coloneqq \int_C |a(x)| \mathrm{d}x$ for any $C \in \mathcal{B}$. Fix $\varepsilon > 0$, $B \in \mathcal{A}$ and $A', A'' \in \mathcal{A}$ with $A' \Subset A''$. Choose $A \in \mathcal{A}$ with $A' \Subset A \Subset A''$ and $k \in \mathbb{N}$ with

$$\max\left\{\frac{d_1 + d_2 d_4}{k}, \frac{\sigma(A \setminus \overline{A'})}{k}\right\} < \varepsilon.$$

Moreover, choose open sets $A_1, \dots, A_{k+1}$ such that $A' \Subset A_1 \Subset \dots \Subset A_{k+1} \Subset A$, and, for any $i = 1, \dots, k$ take a smooth cut-off function φ_i between A_i and A_{i+1}. Finally, set

$$M \coloneqq \frac{d_1 d_4}{k} \max_{1 \le i \le k} \max_{x \in \Omega} |X\varphi_i(x)|^p.$$

Let $F \in \mathcal{U}_{m,p}(a, c_1, c_2)$ and $u, v \in W^{1,p}_X(\Omega)$. Then, for any $i = 1, \dots, k$, from the choice of φ_i it follows that

$$F(\varphi_i u + (1-\varphi_i)v, A' \cup B) \le F(u, (A' \cup B) \cap \overline{A_i}) + F(v, B \setminus A_{i+1}) + F(\varphi_i u + (1-\varphi_i)v, S_i), \quad (4.3.5)$$

where $S_i \coloneqq B \cap (A_{i+1} \setminus \overline{A_i})$. Setting $I_i \coloneqq F(\varphi_i u + (1-\varphi_i)v, S_i)$, from the bound on the Lagrangian and arguing as in the proof of Proposition 4.2.5, we get that

$$\begin{aligned}
I_i &\le d_1 \int_{S_i} |X(\varphi_i u + (1-\varphi_i)v)|^p \mathrm{d}x + d_2 \int_{S_i} |\varphi_i u + (1-\varphi_i)v|^p \mathrm{d}x + \sigma(S_i) \\
&= d_1 \int_{S_i} |uX\varphi_i + \varphi_i Xu - vX\varphi_i + (1-\varphi_i)Xv)|^p \mathrm{d}x + d_2 \int_{S_i} |u|^p \mathrm{d}x + d_2 \int_{S_i} |v|^p \mathrm{d}x + \sigma(S_i) \\
&= d_1 \int_{S_i} |(\varphi_i Xu + (1-\varphi_i)Xv) + X\varphi_i(u-v)|^p \mathrm{d}x + d_2 \int_{S_i} (|u|^p + |v|^p) \mathrm{d}x + \sigma(S_i) \\
&\le d_1 d_4 \left[\int_{S_i} |\varphi_i Xu + (1-\varphi_i)Xv|^p + \int_{S_i} |X\varphi_i|^p |u-v|^p \mathrm{d}x\right] + d_2 \int_{S_i} (|u|^p + |v|^p) \mathrm{d}x + \sigma(S_i) \\
&\le d_1 d_4 \left[\int_{S_i} |Xu|^p \mathrm{d}x + \int_{S_i} |Xv|^p \mathrm{d}x\right] + kM \int_{S_i} |u-v|^p \mathrm{d}x + d_2 \int_{S_i} (|u|^p + |v|^p) \mathrm{d}x + \sigma(S_i) \\
&\le (d_2 + d_1 d_4) \left(\|u\|^p_{W^{1,p}_X(S_i)} + \|v\|^p_{W^{1,p}_X(S_i)}\right) + kM \|u-v\|^p_{L^p(S_i)} + \sigma(S_i).
\end{aligned}$$

Noticing that σ is a measure and that

$$S_1 \cup \ldots \cup S_k \subseteq (A \setminus \overline{A'}) \cap B \subseteq S,$$

and recalling the choice of k, it follows that

$$\begin{aligned}\min_{1\le i\le k} I_i \le \frac{1}{k}\sum_{i=1}^{k} I_k \le \frac{d_2+d_1d_4}{k}\left(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)}\right) + M\|u-v\|^p_{L^p(S)} + \frac{\sigma(A\setminus\overline{A'})}{k}\\ \le \varepsilon\left(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\right) + M\|u-v\|^p_{L^p(S)}.\end{aligned} \tag{4.3.6}$$

Therefore, if $\varphi_i \in \{\varphi_1, \ldots, \varphi_k\}$ is chosen to realize the minimum, observing that F is a measure, $(A' \cup B) \cap \overline{A_i} \subseteq A''$ and $B \setminus A_{i+1} \subseteq B$, thanks to (4.3.5) and (4.3.6) the thesis follows. $\square$

Proposition 4.3.7. *Let $(F_h)_h \in \mathcal{U}_{m,p}(a, c_1, c_2)$ and let F'' be as in Definition 4.1.4. Then it holds that*

$$F''(u, A' \cup B) \le F''(u, A'') + F''(u, B) \tag{4.3.7}$$

for any $u \in W^{1,p}_X(\Omega)$, $B \in \mathcal{A}$ and $A', A'' \in \mathcal{A}$ with $A' \Subset A''$.

Proof. Choose u, A', A'', B as above and fix $\varepsilon > 0$. Moreover, let $(u_h)_h, (v_h)_h \subseteq W^{1,p}_X(\Omega)$ be two recovery sequences for u with respect to $F''(\cdot, A'')$ and $F''(\cdot, B)$ respectively. From Proposition 4.3.6 we know that $(F_h)_h$ satisfies the uniform fundamental estimate on $W^{1,p}_X(\Omega)$. Therefore there exists $M > 0$ and a finite family $\{\varphi^1, \ldots, \varphi^k\}$ of smooth cut-off functions between A' and A'', depending only on ε, A', A'' and B, and a sequence $(\varphi_h)_h \subseteq \{\varphi^1, \ldots, \varphi^k\}$, such that

$$\begin{aligned}F_h\Big(\varphi_h u_h + (1-\varphi_h)v_h, A' \cup B\Big) \le \Big(F_h(u_h, A'') + F_h(v_h, B)\Big) +\\ + \varepsilon\Big(\|u_h\|^p_{W^{1,p}_X(S)} + \|v_h\|^p_{W^{1,p}_X(S)} + 1\Big) + M\|u_h - v_h\|_{L^p(S)},\end{aligned} \tag{4.3.8}$$

where $S = (A'' \setminus A') \cap B$. Let us define $w_h \coloneqq \varphi_h u_h + (1-\varphi_h)v_h$. Then it follows that

$$\|w_h - u\|_{L^p(\Omega)} = \|\varphi_h(u_h - v_h)\|_{L^p(\Omega)} + \|v_h - u\|_{L^p(\Omega)} \le \|u_h - v_h\|_{L^p(\Omega)} + \|v_h - u\|_{L^p(\Omega)},$$

and moreover

$$\begin{aligned}\|Xw_h - Xu\|_{L^p(\Omega)} &= \|X\varphi_h \cdot u_h + \varphi_h X u_h - X\varphi_h \cdot v_h + (1-\varphi_h)Xv_h - Xu\|_{L^p(\Omega)} \\ &\leq \|X\varphi_h(u_h - v_h)\|_{L^p(\Omega)} + \|\varphi_h(Xu_h - Xv_h)\|_{L^p(\Omega)} + \|Xv_h - Xu\|_{L^p(\Omega)} \\ &\leq \max_{1\leq i\leq k} \||X\varphi^k|^p\|_\infty \cdot \|u_h - v_h\|_{L^p(\Omega)} + \|Xu_h - Xv_h\|_{L^p(\Omega)} + \|Xv_h - Xu\|_{L^p(\Omega)}.\end{aligned}$$

Therefore we conclude that w_h converges to $u \in W^{1,p}_X(\Omega)$. This fact, the choices of u_h and v_h and (4.3.8) allow to conclude that

$$\begin{aligned}F''(u, A' \cup B) &\leq \limsup_{h\to\infty} F''(w_h, A' \cup B) \\ &\leq \limsup_{h\to\infty} F''(u_h, A'') + \limsup_{h\to\infty} F''(v_h, B) \\ &+ \varepsilon\Big(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\Big) \\ &= F''(u, A'') + F''(u, B) + \varepsilon\Big(\|u\|^p_{W^{1,p}_X(S)} + \|v\|^p_{W^{1,p}_X(S)} + 1\Big).\end{aligned}$$

Being ε arbitrary, the thesis follows. $\square$

4.3.2 Main results

To resume, we showed that $\mathcal{U}_{m,p}(a, c_1, c_2)$ satisfies some nice properties but, it is too general to hope to achieve Γ-compactness. Therefore we introduce two subfamilies that are Γ-compact with respect to the strong topology of $W^{1,p}_X(\Omega)$. The first one is given by the sub-class of convex functionals belonging to $\mathcal{U}_{m,p}(a, c_1, c_2)$. In other words

Definition 4.3.8. *We say that $F \in \mathcal{V}_{m,p}(a, c_1, c_2)$ whenever*

$$F \in \mathcal{U}_{m,p}(a, c_1, c_2) \quad \textit{and} \quad f(x, \cdot, \cdot) \textit{ is convex for a.e. } x \in \Omega.$$

In the second subfamily of $\mathcal{U}_{m,p}(a, c_1, c_2)$ we want to drop the convexity assumption and to obtain uniformity in the choice of the family of moduli of continuity. This is due to the fact that, if $(F_h)_h \subseteq \mathcal{U}_{m,p}(a, c_1, c_2)$ and every F_h satisfies the strong condition (ωX), still the family of moduli of continuity strongly depends on h.

Definition 4.3.9. *Let $\omega = (\omega_s)_{s\geq 0}$ be a family as in Definition 4.3.2. We say that a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0,\infty]$ belongs to $\mathcal{W}_{m,p}(a, c_1, c_2, \omega)$ if $F \in \mathcal{U}_{m,p}(a, c_1, c_2)$ and it satisfies the strong condition (ωX) with respect to ω.*

Remark 4.3.10. *Let $(F_h)_h \subseteq \mathcal{U}_{m,p}(a, c_1, c_2)$ be such that there exists $K \in L^1_{\mathrm{loc}}(\Omega)$ such that*

$$|f_h(x, u, \xi) - f_h(x, v, \eta)| \leq |K(x)|(|u - v| + |\xi - \eta|) \tag{4.3.9}$$

for any $u, v \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^m$ and $h \in \mathbb{N}$. If for any $s, r \geq 0$ we define E^s_r as in Proposition 4.3.3 and

$$\tilde{\omega}_s(x, r) := |K(x)| \sup\{(|u - v| + |\xi - \eta|) \,:\, (u, v, \xi, \eta) \in E^s_r\},$$

then it is easy to see that $(F_h)_h$ belongs to $\mathcal{W}_{m,p}(a, c_0, c_1, c_2, \tilde{\omega})$.

As presented in the Introduction, we are now ready to complete the first step of the general scheme, presenting the Γ-compactness result in the strong topology of $W^{1,p}_X(\Omega)$.

Proposition 4.3.11. *For any sequence $(F_h)_h \subseteq \mathcal{U}_{m,p}(a, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0,\infty]$ such that*

(a) *F is a measure;*

(b) *F is local;*

(c) *F is $W^{1,p}_X$−lower semicontinuous;*

(d) *for any $u \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$ it holds that*

$$F(u, A) \leq \int_A a(x) + c_1|Xu(x)|^p + c_2|u(x)|^p \mathrm{d}x \tag{4.3.10}$$

and moreover we have that

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot, A) \tag{4.3.11}$$

for any $A \in \mathcal{A}$.

Proof. Since $(W^{1,p}_X(\Omega), \|\cdot\|_{W^{1,p}_X(\Omega)})$ is a metric space, by [38, Theorem 16.9] we know that, up to a subsequence, $(F_h)_h$ $\bar{\Gamma}(W^{1,p}_X)$-converges to a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow \overline{\mathbb{R}}$. Being F a $\bar{\Gamma}$-limit, we know from [38, Remark 16.3] that F is increasing, inner regular and $W^{1,p}_X-$ lower semicontinuous. Moreover, thanks to [38, Proposition 16.12], we know that F is superadditive. Let us show that F is non-negative. Indeed, fix $A \in \mathcal{A}$ and $u \in W^{1,p}_X(\Omega)$, then we know that

$$F(u, A) = \sup\big\{\inf\{\liminf_{h\to\infty} F_h(u_h, A') \, : \, u_h \to u \text{ in } W^{1,p}_X(\Omega)\} \, : \, A' \in \mathcal{A}, \, A' \Subset A\big\}.$$

Since every $F_h(u_h, A')$ is non-negative, then $F(u, A) \geq 0$. Moreover, in the same way we can see that $F(u, \emptyset) = 0$ for any $u \in W^{1,p}_X(\Omega)$. Now, adapting the proof of [38, Proposition 16.15], we show that F is local. Let us fix $A \in \mathcal{A}$ and $u, v \in W^{1,p}_X(\Omega)$ coinciding a.e. on A. Fix $A' \Subset A$, take a smooth cut-off function φ between A' and A and let $(u_h)_h \subseteq W^{1,p}_X(\Omega)$ be a recovery sequence for u with respect to $F'(\cdot, A')$. We define a new sequence $(v_h)_h$ requiring that

$$v_h \coloneqq \varphi u_h + (1 - \varphi) v.$$

It is clear that

$$\|v_h - v\|_{L^p(\Omega)} = \|\varphi(u_h - v)\|_{L^p(\Omega)} = \|\varphi(u_h - v)\|_{L^p(A)} \leq \|u_h - u\|_{L^p(A)},$$

and moreover

$$\begin{aligned}
\|Xv_h - Xv\|_{L^p(\Omega)} &= \|X\varphi(u_h - v) + \varphi(Xu_h - Xv)\|_{L^p(\Omega)} \\
&\leq \|X\varphi(u_h - v)\|_{L^p(A)} + \|\varphi(Xu_h - Xv)\|_{L^p(A)} \\
&\leq \||X\varphi|^p\|_\infty \|u_h - u\|_{L^p(A)} + \|Xu_h - Xu\|_{L^p(A)}.
\end{aligned}$$

Therefore we have that v_h converges to v in $W^{1,p}_X(\Omega)$. As each F_h is local and $u_h = v_h$ on A', we conclude that

$$F'(v, A') \leq \liminf_{h\to\infty} F_h(v_h, A') = \liminf_{h\to\infty} F_h(u_h, A') = F'(u, A').$$

As the converse inequality can be proved exchanging the roles of u and v, we conclude that $F'(u, A') = F'(v, A')$. Finally, being $A' \Subset A$ arbitrary and recalling the definition of a $\bar{\Gamma}-$limit, we conclude that F is local. Moreover, thanks to Proposition 4.3.7, we can repeat essentially the same steps of the proof of [38, Proposition 18.4] and achieve that F is subadditive. Notice that, thanks to [38, Theorem 14.23] and the previous steps, this suffices to conclude that F is a measure. If we define now $G : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ by

$$G(u, A) := \int_A (a(x) + c_2|u|^p + c_1|Xu|^p)\mathrm{d}x$$

for any $u \in W^{1,p}_X(\Omega)$ and for any $A \in \mathcal{A}$, it is clear that G is a measure and that, thanks to the hypotheses, $F_h \le G$ for any $h \in \mathbb{N}$. Therefore, if $u \in W^{1,p}_X(\Omega)$ and $A \in \mathcal{A}$, it follows that

$$F(u, A) \le \liminf_h F_h(u, A) \le G(u, A).$$

Finally, thanks again to Proposition 4.3.7 and repeating the proof of [38, Theorem 18.7], we conclude that

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{h\to+\infty} F_h(\cdot, A), \tag{4.3.12}$$

for any $A \in \mathcal{A}$. $\square$

We have developed all the tools that we need to state and prove the two main theorems of this Chapter.

Theorem 4.3.12. *For any sequence $(F_h)_h \subseteq \mathcal{V}_{m,p}(a, c_1, c_2)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F \in \mathcal{V}_{m,p}(a, c_1, c_2)$ such that*

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot, A) \qquad \text{for any } A \in \mathcal{A}.$$

Proof. Since $(F_h)_h \subseteq \mathcal{V}_{m,p}(a, c_1, c_2)$, from Proposition 4.3.11 we know that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ which is a measure, local, satisfies (4.3.10) and such that (4.3.11) holds. Moreover, arguing as in the proof of Theorem 4.2.7, F is convex. Therefore F satisfies all the hypotheses of [55, Theorem 2.3], and so we conclude that $F \in \mathcal{V}_{m,p}(a, c_1, c_2)$. $\square$

In order to prove the counterpart of the Γ-compactness result in $\mathcal{W}_{m,p}(a, c_1, c_2, \omega)$, we wish to apply Theorem 3.4.1 for a suitable functional F. Then, we need to guarantee that the strong condition (ωX) with respect to ω is preserved by the operation of $\Gamma(W^{1,p}_X)$-limit.

Proposition 4.3.13. *If a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ is a measure, it is $W^{1,p}_X$-continuous, it satisfies (4.3.10) for any $u \in W^{1,p}_X(\Omega)$ and for any $B \in \mathcal{B}$ and it satisfies the strong condition (ωX) with respect to ω, then it holds that*

$$|F(v, B') - F(u, B')| \leq \int_{B'} \omega_s(x, r)\mathrm{d}x \tag{4.3.13}$$

for any $s \geq 0$, $B' \in \mathcal{B}_0$, $r \geq 0$, $u, v \in W^{1,p}_X(\Omega)$ such that

$$\begin{aligned} |u(x)|, |v(x)|, |Xu(x)|, |Xv(x)| &\leq s \\ |u(x) - v(x)|, |Xu(x) - Xv(x)| &\leq r \end{aligned} \tag{4.3.14}$$

for a.e. $x \in B'$.

Proof. It is not restrictive to assume that $c_1 = c_2 = 1$. First we show the thesis for regular functions $u, v \in W^{1,p}_X(\Omega) \cap C^\infty(\Omega)$. Let us fix $B' \in \mathcal{B}_0$, and s, r, such that (4.3.14) holds, and let us take $m, M > 0$. Since $F(u, \cdot)$ and $F(v, \cdot)$ are Borel measures, there exists a decreasing sequence of open sets $(A_n)_n \subseteq \mathcal{A}$ such that $B' = \bigcap_{n=1}^\infty A_n$ and moreover

$$F(u, B') = \lim_{n\to\infty} F(u, A_n) \quad \text{and} \quad F(v, B') = \lim_{n\to\infty} F(v, A_n).$$

Furthermore, as $B' \Subset \Omega$, we can assume that $A_n \Subset \Omega$ for each $n \in \mathbb{N}$. Finally, as $u, v \in C^1(\overline{A_0})$ we can assume that

$$\begin{aligned} |u(x)|, |v(x)|, |Xu(x)|, |Xv(x)| &\leq s + \frac{1}{M} \\ |u(x) - v(x)|, |Xu(x) - Xv(x)| &\leq r + \frac{1}{m} \end{aligned}$$

for any $x \in A_n$ and for any $n \geq 0$. We obtain that

$$\begin{aligned} |F(u, B') - F(v, B')| &= \lim_{n\to\infty} |F(u, A_n) - F(v, A_n)| \\ &\leq \lim_{n\to\infty} \int_{A_n} w_{s+\frac{1}{M}}\left(x, r + \frac{1}{m}\right) \mathrm{d}x \\ &= \int_{B'} w_{s+\frac{1}{M}}\left(x, r + \frac{1}{m}\right) \mathrm{d}x. \end{aligned}$$

Therefore, thanks to (4.3.2), (4.3.3) and the Monotone Convergence Theorem we conclude that

$$\begin{aligned} |F(u,B')-F(v,B')| &\le \lim_{m\to\infty}\lim_{M\to\infty}\int_{B'} w_{s+\frac{1}{M}}\left(x,r+\frac{1}{m}\right)\mathrm{d}x \\ &= \lim_{m\to\infty}\int_{B'} w_s\left(x,r+\frac{1}{m}\right)\mathrm{d}x \\ &= \int_{B'} w_s(x,r)\mathrm{d}x. \end{aligned}$$

Let now $B'\in\mathcal{B}_0$, $u,v\in W^{1,p}_X(\Omega)$ and s,r, such that (4.3.14) holds, and fix again $m,M>0$. By Theorem 3.1.8 there are two sequences $(u_h)_h,(v_h)_h\subseteq W^{1,p}_X(\Omega)\cap C^\infty(\Omega)$ converging respectively to u and v in the strong topology of $W^{1,p}_X(\Omega)$. Therefore, thanks to the previous step and the continuity of the functional, we get that

$$|F(u,B')-F(v,B')| = \lim_{h\to\infty}|F(u_h,B')-F(v_h,B')|.$$

Now we want to estimate the right term. For doing this let us define, for any $h\ge 0$,

$$\begin{aligned} A_h &:= \left\{x\in B' \,:\, |u_h(x)|>s+\frac{1}{M}\right\} \quad B_h := \left\{x\in B' \,:\, |v_h(x)|>s+\frac{1}{M}\right\} \\ C_h &:= \left\{x\in B' \,:\, |Xu_h(x)|>s+\frac{1}{M}\right\} \quad D_h := \left\{x\in B' \,:\, |Xv_h(x)|>s+\frac{1}{M}\right\} \\ E_h &:= \left\{x\in B' \,:\, |u_h(x)-v_h(x)|>r+\frac{1}{m}\right\} \\ F_h &:= \left\{x\in B' \,:\, |Xu_h(x)-Xv_h(x)|>r+\frac{1}{m}\right\}, \end{aligned}$$

and let

$$Z_h := A_h\cup B_h\cup C_h\cup D_h\cup E_h\cup F_h. \tag{4.3.15}$$

We claim that

$$\lim_{h\to\infty}|Z_h|=0.$$

Here we only show that $\lim_{h\to\infty}|A_h|=0$, being the other parts of the proof similar. Assume that $x\in A_h$ and assume that (4.3.14) holds in x. Then it follows that

$$|u_h(x)-u(x)|\ge|u_h(x)|-|u(x)|>\frac{1}{M}.$$

and hence

$$x \in \left\{ z \in \Omega \, : \, |u(z) - u_h(z)| > \frac{1}{M} \right\}.$$

Since u_h converges to u in $W^{1,p}_X(\Omega)$, then in particular u_h converges to u in measure, and so the measure of the right set goes to zero as h goes to infinity. We can now estimate in this way:

$$\begin{aligned}
\lim_{h\to\infty} |F(u_h, B') - F(v_h, B')| &\le \liminf_{h\to\infty} |F(u_h, B' \setminus Z_h) - F(v_h, B' \setminus Z_h)| + |F(u_h, Z_h) - F(v_h, Z_h)| \\
&\le \int_{B'} w_{s+\frac{1}{M}} \left(x, r + \frac{1}{m} \right) + \liminf_{h\to\infty} |F(u_h, Z_h)| + |F(v_h, Z_h)| \\
&\le \int_{B'} w_{s+\frac{1}{M}} \left(x, r + \frac{1}{m} \right) \mathrm{d}x + \liminf_{h\to\infty} 2 \int_{Z_h} |a(x)| \mathrm{d}x \\
&+ \liminf_{h\to\infty} \int_{Z_h} |u_h|^p \mathrm{d}x + \int_{Z_h} |v_h|^p \mathrm{d}x + \int_{Z_h} |Xu_h|^p \mathrm{d}x + \int_{Z_h} |Xv_h|^p \mathrm{d}x \\
&\le \int_{B'} w_{s+\frac{1}{M}} \left(x, r + \frac{1}{m} \right) \mathrm{d}x + \liminf_{h\to\infty} 2 \int_{Z_h} |a(x)| \mathrm{d}x \\
&+ \liminf_{h\to\infty} 2^{p-1} \left(\int_{Z_h} |u_h - u|^p \mathrm{d}x + \int_{Z_h} |u|^p \mathrm{d}x + \int_{Z_h} |Xu_h - Xu|^p \mathrm{d}x + \int_{Z_h} |Xu|^p \mathrm{d}x \right) \\
&+ \liminf_{h\to\infty} 2^{p-1} \left(\int_{Z_h} |v_h - v|^p \mathrm{d}x + \int_{Z_h} |v|^p \mathrm{d}x + \int_{Z_h} |Xv_h - Xv|^p \mathrm{d}x + \int_{Z_h} |Xv|^p \mathrm{d}x \right) \\
&\le \int_{B'} w_{s+\frac{1}{M}} \left(x, r + \frac{1}{m} \right) \mathrm{d}x + K \lim_{h\to\infty} \left(\|u - u_h\|_{W^{1,p}_X(\Omega)} + \|v - v_h\|_{W^{1,p}_X(\Omega)} \right) \\
&+ \liminf_{h\to\infty} \int_{B'} \chi_{Z_h} b(x) \mathrm{d}x,
\end{aligned}$$

for a constant $K > 0$ and a suitable function $b \in L^1(B')$. Therefore, thanks to the Dominated Convergence Theorem, we conclude that

$$|F(u, B') - F(v, B')| \le \int_{B'} w_{s+\frac{1}{M}} \left(x, r + \frac{1}{m} \right) \mathrm{d}x.$$

Arguing as in the first step and letting M, m go to infinity, the thesis follows. □

Proposition 4.3.14. *Let $(F_h)_h$ be a sequence in $\mathcal{W}_{m,p}(a, c_1, c_2, \omega)$ and let us assume that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ such that*

$$F(\cdot, A') = \Gamma(W^{1,p}_X) - \lim_{h\to\infty} F_h(\cdot, A') \qquad \textit{for any } A' \in \mathcal{A}_0.$$

Then F satisfies the strong condition (ωX) with respect to ω.

Proof. Let $A' \in \mathcal{A}_0$, $u, v \in W^{1,p}_X(\Omega)$ and $s, r \geq 0$ such that (4.3.14) holds, and fix $m, M > 0$. Let $(u_h)_h$ and $(v_h)_h$ be recovery sequences respectively for u and v. Then it follows that

$$|F(u, A') - F(v, A')| = \lim_{h\to\infty} |F_h(u_h, A') - F_h(v_h, A')|.$$

Notice that, since $F_h \in \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$ then it is a measure, it satisfies the strong condition (ωX) with respect to $(\omega_s)_{s\geq 0}$, and thanks to a slight variant of [55, Theorem 3.1], it is $W^{1,p}_X$-continuous. Moreover, thanks to (4.3.1), it satisfies (4.3.10) for any $u \in W^{1,p}_X(\Omega)$ and for any $B \in \mathcal{B}$. Therefore it satisfies the hypotheses of Proposition 4.3.13. Hence, repeating exactly the same estimates performed in the proof of Proposition 4.3.13, we conclude that

$$\lim_{h\to\infty} |F_h(u_h, A') - F_h(v_h, A')| \leq \int_{A'} \omega_s(x, r)\mathrm{d}x,$$

and so the thesis follows. $\square$

Now, we can state and show the proof of the last main Γ-compactness result.

Theorem 4.3.15. *For any sequence $(F_h)_h \subseteq \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$ there exists a subsequence $(F_{h_k})_k$ and a functional $F \in \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$ such that*

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot, A) \qquad \textit{for any } A \in \mathcal{A}.$$

Proof. Since $(F_h)_h \subseteq \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$, from Proposition 4.3.11 we know that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ which is a measure, local, $W^{1,p}_X$-lower semicontinuous and satisfies (4.3.10), and such that (4.3.11) holds. Moreover, thanks to Proposition 4.3.14, F satisfies the strong condition (ωX) with respect to ω. Therefore F satisfies all the hypotheses of [55, Theorem 4.4], and so we conclude that $F \in \mathcal{W}_{m,p}(a, c_1, c_2, \omega)$. $\square$

4.3.3 Further Remarks and Open Problems

In Section 4.2, we obtained a $\Gamma(L^p)-$compactness result for a class of convex integral functionals defined on $L^p(\Omega)$. Nevertheless, we did not generalize the result when the convexity assumption is dropped. On the other hand, in the last section we took into account the non-convex case, working in a suitable class of integral functionals where the strong condition (ωX) is required uniformly on the class. In the light of these considerations, there are still some questions unsolved. However, we have to introduce the following notion, which is the reasonable counterpart of Definition 4.3.2.

Definition 4.3.16. *If $\omega = (\omega_s)_{s\geq 0}$ is a family of locally integrable moduli of continuity in the sense of Definition 4.3.2. Then, we say that a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ satisfies the* weak condition (ωX) with respect to ω *if*

$$|F(u + r, A') - F(u, A')| \leq \int_{A'} \omega_s(x, |r|)\,\mathrm{d}x \tag{4.3.16}$$

for any $s \geq 0$, $A' \in \mathcal{A}_0$, $r \in \mathbb{R}$, $u \in W^{1,p}_X(\Omega)$ such that

$$|u(x)|, |v(x) + r|, |r| \leq s$$

for a.e. $x \in A'$.

Indeed, if ω is a fixed family of moduli of continuity it is reasonable to ask the following.

Question 1. Let $(F_h)_h \subset \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ be a sequence of functionals satisfying the strong condition (ωX) with respect to ω. Does it exists a subsequence $(F_{h_k})_k$ and a functional $F \in \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$, such that

$$F(\cdot, A) = \Gamma(L^p) - \lim_{k\to+\infty} F_{h_k}(\cdot, A) \qquad \text{for any } A \in \mathcal{A},$$

and F satisfies the strong condition (ωX)?

It is reasonable to demand the same question above when, the sequence of functionals $(F_h)_h \subset \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ is supposed to be weakly*-seq. l.s.c. and satisfying the weak condition (ωX) with respect to ω. Moreover we have

Question 2. Let $(F_h)_h \subset \mathcal{U}_{m,p}(a, c_1, c_2)$ be a sequence of functionals satisfying the weak condition (ωX) with respect to ω and which are weakly*-seq. l.s.c. Does it exists a subsequence

$(F_{h_k})_k$ and a functional $F \in \mathcal{U}_{m,p}(a, c_1, c_2)$, such that

$$F(\cdot, A) = \Gamma(W^{1,p}_X) - \lim_{k\to+\infty} F_{h_k}(\cdot, A) \qquad \text{for any } A \in \mathcal{A},$$

where F is weakly*-seq. l.s.c. and satisfies the weak condition (ωX)?

Unfortunately, in view of Proposition 4.2.5, Proposition 4.3.11 and the integral representation results in [55], the only questions we did not answer are the following.

Open Problem 1. Let $(F_h)_h \subset \mathcal{I}_{m,p}(a, c_0, c_1, c_2)$ be a sequence of functionals that satisfy the weak (resp. strong) condition (ωX). Let $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ be the functional such that

$$F(\cdot, A') = \Gamma(L^p) - \lim_{h\to\infty} F_h(\cdot, A') \qquad \text{for any } A' \in \mathcal{A}_0.$$

Then F satisfies the weak (resp. strong) condition (ωX) with respect to ω.

Open Problem 2. Let $(F_h)_h \subset \mathcal{U}_{m,p}(a, c_1, c_2)$ be a sequence of (possibly not weakly*-seq. l.s.c.) functionals and assume that there exists $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, \infty]$ such that

$$F(\cdot, A') = \Gamma(W^{1,p}_X) - \lim_{h\to\infty} F_h(\cdot, A') \qquad \text{for any } A' \in \mathcal{A}_0.$$

Then F is a weakly*-seq. l.s.c functional.

However we are going to show that the following results are verified.

Proposition 4.3.17. *Let ω be a family of locally integrable moduli of continuity. Let $(F_h)_h$ be a sequence in $\mathcal{U}_{m,p}(a, c_1, c_2)$ and assume that each F_h satisfies the weak condition (ωX) with respect to ω. Assume in addition that there exists a functional $F : W^{1,p}_X(\Omega) \times \mathcal{A} \longrightarrow [0, +\infty]$ such that*

$$F(\cdot, A') = \Gamma(W^{1,p}_X) - \lim_{h\to+\infty} F_h(\cdot, A') \qquad \textit{for any } A' \in \mathcal{A}_0.$$

Then F satisfies the weak condition (ωX) with respect to ω.

Proof. The proof of this result is totally similar to the proofs of Proposition 4.3.13 and Proposition 4.3.14, and so we take it for granted. □

Proposition 4.3.18. *Let $F_h : L^p(\Omega) \times \mathcal{A} \longrightarrow [0,+\infty]$ be a sequence of (not necessary integral) functionals, and assume that there exists a functional $F : L^p(\Omega) \times \mathcal{A} \longrightarrow [0,\infty]$ which is a measure and such that*

$$F(\cdot, A') = \Gamma(L^p) - \lim_{h\to+\infty} F_h(\cdot, A') \qquad \textit{for any } A' \in \mathcal{A}_0.$$

Then F is weakly-seq. lower semicontinuous.*

Proof. Let $A \in \mathcal{A}$, $A' \in \mathcal{A}$ with $A' \Subset A$, $u \in W^{1,\infty}(\Omega)$ and take a sequence $(u_h)_h \subseteq W^{1,\infty}(\Omega)$ which is weakly*-convergent to u. Then, since $A' \Subset A$, it is well known that u_h converges to u strongly in $L^\infty(A')$, and so in particular strongly in $L^p(A')$. Being $F(\cdot, A')$ a $\Gamma(L^p)-$limit, it is L^p-lower semicontinuous. Moreover, being F a measure, it is also increasing. These facts imply that

$$F(u, A') \leq \liminf_{h\to\infty} F(u_h, A') \leq \liminf_{h\to\infty} F(u_h, A).$$

Since F is inner regular and since $A' \Subset A$ is arbitrary, the conclusion follows. $\square$

4.4 A Γ-convergence result on Carnot groups

Strongly inspired by [28, Theorem 3.1], in the last section we want to present a particular Γ-convergence result in a Carnot group of step k. For this purpose, to any distance d in $\mathcal{D}_{cc}(\Omega)$ of Definition 2.3.1, we are going to associate the functionals defined respectively on the class $\mathcal{B}(\Omega)$ of all positive and finite Borel measures μ on $\Omega \times \Omega$ and on $\mathrm{Lip}([0,1],\Omega)$. In other words, we set

$$J_d(\mu) = \int d(x,y)\,\mathrm{d}\mu(x,y), \quad \mu \in \mathcal{B}(\Omega);$$
$$L_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot\gamma(t))\,\mathrm{d}t, \quad \gamma \in \mathrm{Lip}([0,1],\Omega).$$

As already mentioned in Subsection 2.2, we equip $\mathcal{D}_{cc}(\Omega)$ with the topology of the uniform convergence on compact subsets of $\Omega \times \Omega$. Moreover, we endow $\mathcal{B}(\Omega)$ and $\mathrm{Lip}([0,1],\Omega)$ with the topology of weak* convergence and of the uniform convergence, respectively.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{G}$ be an open set in a Carnot group of step k and let $(d_n)_n$ and d belong to $\mathcal{D}_{cc}(\Omega)$. If J_n, L_n and J, L are the functionals associated respectively to d_n and d, defined as before, then the following conditions are equivalent:*

(i) $d_n \to d$ *in* $\mathcal{D}_{cc}(\Omega)$*;*

(ii) $J = \Gamma(\mathcal{B}(\Omega)) - \lim_{n\to\infty} J_n$*;*

(iii) $L = \Gamma(\mathrm{Lip}(\Omega)) - \lim_{n\to\infty} L_n$*;*

Moreover, if Ω is bounded, then (i), (ii) *and* (iii) *are equivalent to the following condition:*

(iv) J_n *continuously converges to J, meaning that $J(\mu) = \lim_n J_n(\mu_n)$ holds whenever the sequence $(\mu_n)_n \subset \mathcal{B}(\Omega)$ weakly* converges to $\mu \in \mathcal{B}(\Omega)$.*

Proof. (i) $\Rightarrow$ (ii). In order to prove the Γ-lim inf inequality, fix $\mu \in \mathcal{B}(\Omega)$ and $(\mu_n)_n \subset \mathcal{B}(\Omega)$ such that μ_n weakly* converges to μ. Fix a sequence $(\eta_k)_k$ of compactly-supported continuous functions $\eta_k : \Omega \times \Omega \to [0,1]$ such that $\eta_k(x) \nearrow 1$ for every $x \in \Omega$. Since $d_n \to d$ in $\mathcal{D}_{cc}(\Omega)$, we deduce that for any $k \in \mathbb{N}$ we have that $\eta_k d_n \to \eta_k d$ uniformly as $n \to \infty$, thus there exists a sequence $(\varepsilon_n^k)_n \subset (0, +\infty)$ such that $\varepsilon_n^k \searrow 0$ as $n \to \infty$ and $|\eta_k d_n - \eta_k d| \le \varepsilon_n^k$ on $\Omega \times \Omega$. Moreover, since μ_n weakly* converges to μ, by using Banach–Steinhaus Theorem we deduce that $\sup_n \mu_n(\Omega \times \Omega) < +\infty$. Therefore, since $\eta_k d$ is continuous and bounded, we get that

$$
\begin{aligned}
&\left| \int \eta_k(x,y) d_n(x,y) \,\mathrm{d}\mu_n(x,y) - \int \eta_k(x,y) d(x,y) \,\mathrm{d}\mu(x,y) \right| \\
\le\, &\varepsilon_n^k \, \mu_n(\Omega \times \Omega) + \left| \int \eta_k(x,y) d(x,y) \,\mathrm{d}\mu_n(x,y) - \int \eta_k(x,y) d(x,y) \,\mathrm{d}\mu(x,y) \right| \to 0 \quad \text{as } n \to \infty,
\end{aligned}
$$

for every $k \in \mathbb{N}$. In particular, for any $k \in \mathbb{N}$ we have that

$$
\int \eta_k(x,y) d(x,y) \,\mathrm{d}\mu(x,y) = \lim_{n\to\infty} \int \eta_k(x,y) d_n(x,y) \,\mathrm{d}\mu_n(x,y) \le \liminf_{n\to\infty} J_n(\mu_n).
$$

By monotone convergence theorem, we conclude that $J(\mu) \le \liminf_n J_n(\mu_n)$, as desired.

Let us pass to the verification of the Γ-lim sup inequality. Fix any $\mu \in \mathcal{B}(\Omega)$. We aim to show that the sequence constantly equal to μ is a recovery sequence, namely $J(\mu) \geq \limsup_n J_n(\mu)$. If $J(\mu) = +\infty$, then there is nothing to prove. Thus suppose that $J(\mu) < +\infty$.

Since $(1/\alpha)d_{cc} \leq d$, we deduce that $d_{cc} \in L^1(\mu)$. By combining this information with the fact that $d_n \leq \alpha d_{cc}$ for all $n \in \mathbb{N}$ and $d_n \to d$ pointwise on $\Omega \times \Omega$, we are in a position to apply the dominated convergence theorem, obtaining that $J(\mu) = \int d(x, y)\, \mathrm{d}\mu(x, y) = \lim_n \int d_n(x, y)\, \mathrm{d}\mu(x, y) = \lim_n J_n(\mu)$.

(i) ⇒ (iii). For every $\gamma \in \mathrm{Lip}(\Omega)$, we have to prove the following two claims:

$$\forall\, \gamma_n \to \gamma \ \text{ in } \ \mathrm{Lip}(\Omega) \ : \ L_d(\gamma) \leq \liminf_{n\to\infty} L_{d_n}(\gamma_n), \tag{4.4.1}$$

$$\exists\, \gamma_n \to \gamma \ \text{ in } \ \mathrm{Lip}(\Omega) \ : \ L_d(\gamma) \geq \limsup_{n\to\infty} L_{d_n}(\gamma_n). \tag{4.4.2}$$

We begin proving (4.4.1). Let $\gamma_n \to \gamma$ in $\mathrm{Lip}(\Omega)$. By definition of $L_d(\gamma)$, for any $\delta \geq 0$ we can find a partition of $[0, 1]$, indexed over a finite set I_δ, such that

$$L_d(\gamma) \leq \delta + \sum_{i \in I_\delta} d(\gamma(t_i), \gamma(t_{i+1})). \tag{4.4.3}$$

Since $\{\gamma_n\}_{n\in\mathbb{N}}$ converges uniformly on $[0, 1]$, we may assume that

$$\exists\ \bar{n} \in \mathbb{N} \ : \ (\gamma_n(s), \gamma_n(t)) \in K \subset \Omega \times \Omega, \ \ \forall s, t \in [0, 1], \ \ \forall n \geq \bar{n},$$

where K is compact. Then, for every $i \in I_\delta$,

$$\begin{aligned}
&|d_n(\gamma_n(t_i), \gamma_n(t_{i+1})) - d(\gamma(t_i), \gamma(t_{i+1}))| \\
&\leq |d_n(\gamma_n(t_i), \gamma_n(t_{i+1})) - d(\gamma_n(t_i), \gamma_n(t_{i+1}))| + |d(\gamma_n(t_i), \gamma_n(t_{i+1})) - d(\gamma(t_i), \gamma(t_{i+1}))| \\
&\leq \sup_K |d_n - d| + d(\gamma(t_i), \gamma_n(t_i)) + d(\gamma(t_{i+1}), \gamma_n(t_{i+1})) \\
&\leq \sup_K |d_n - d| + \alpha\Big(d_{cc}(\gamma(t_i), \gamma_n(t_i)) + d_{cc}(\gamma(t_{i+1}), \gamma_n(t_{i+1}))\Big) \\
&\leq \sup_K |d_n - d| + \alpha C_{K'}\Big(|\gamma(t_i) - \gamma_n(t_i)|^{\frac{1}{k}} + |\gamma(t_{i+1}) - \gamma_n(t_{i+1})|^{\frac{1}{k}}\Big) \\
&\leq \sup_K |d_n - d| + 2\alpha C_{K'} \sup_{[0,1]} |\gamma - \gamma_n|^{\frac{1}{k}} =: \xi_n,
\end{aligned}$$

where $K' \subset \Omega$ is any compact set such that $K \subset K' \times K'$ and $C_{K'}$ is the constant provided by Theorem 2.1.11. Note that $\xi_n \to 0$. We infer from (4.4.3) and from the definition of L_{d_n} that

$$L_d(\gamma) \leq \delta + \sum_{i \in I_\delta} [d_n(\gamma_n(t_i), \gamma_n(t_{i+1})) + \xi_n] \leq \delta + L_{d_n}(\gamma_n) + \xi_n \operatorname{card}(I_\delta).$$

Passing to the lim inf as $n \to +\infty$, we get

$$L_d(\gamma) \leq \liminf_{n \to \infty} L_{d_n}(\gamma_n) + \delta.$$

This yields (4.4.1) by the arbitrariness of $\delta > 0$.

We prove now (4.4.2). Let $\gamma \in \mathrm{Lip}(\Omega)$, let $K \subset \Omega \times \Omega$ compact be chosen as above, and let $r(n) \to \infty$ be a sequence such that

$$\lim_{n \to \infty} r(n) \sup_K |d_n - d| = 0.$$

For every $n \in \mathbb{N}$, let I_n be the partition of $[0, 1]$ into $r(n)$ intervals of equal length, and denote by $\{t_n^i\}, i = 1, \ldots, r(n) + 1$, the endpoints of such intervals. Let γ_n be a curve whose restriction γ_n^i to the interval $[t_n^i, t_n^{i+1}]$ is defined by

$$\gamma_n^i(t_n^i) = \gamma(t_n^i), \quad \gamma_n^i(t_n^{i+1}) = \gamma(t_n^{i+1}), \quad L_{d_n}(\gamma_n^i) \leq d_n(\gamma(t_n^i), \gamma(t_n^{i+1})) + \frac{1}{2^{r(n)}}. \tag{4.4.4}$$

Claim: The sequence $(\gamma_n)_{n \in \mathbb{N}}$ converges to γ in $\mathrm{Lip}(\Omega)$.

Let us prove the claim. Fix a compact set $K \subset \Omega$ such that $\gamma(t), \gamma_n(t) \in K$ for every $n \in \mathbb{N}$ and $t \in [0, 1]$. Given any $n \in \mathbb{N}$ and $t \in (0, 1]$, we denote by $(t_n^-, t_n^+]$ the interval of I_n containing t. Consider the constant C_K given by Theorem 2.1.11. Then it holds that

$$\frac{1}{C_K} |\gamma_n(t) - \gamma(t)| \leq d_{cc}(\gamma_n(t), \gamma(t)) \leq d_{cc}(\gamma_n(t), \gamma_n(t_n^+)) + d_{cc}(\gamma(t_n^+), \gamma(t)) =: A_n + B_n.$$

Now, by the uniform continuity of γ on $[0, 1]$, the term B_n tends to zero as $n \to +\infty$ uniformly with respect to t. The same holds for A_n, since Theorem 2.1.11 implies that

$$\begin{aligned}\frac{1}{\alpha C_K}|\gamma_n(t)-\gamma_n(t_n^+)| &\le \frac{1}{\alpha}d_{cc}(\gamma_n(t),\gamma_n(t_n^+)) \le d_n(\gamma_n(t),\gamma_n(t_n^+)) \le L_{d_n}(\gamma_n|_{[t,t_n^+]})\\ &\le L_{d_n}(\gamma_n|_{[t_n^-,t_n^+]}) \le \alpha d_{cc}(\gamma_n(t_n^-),\gamma_n(t_n^+)) + \frac{1}{2^{r(n)}}\\ &= \alpha d_{cc}(\gamma(t_n^-),\gamma(t_n^+)) + \frac{1}{2^{r(n)}} \le \alpha\cdot C_K|\gamma(t_n^-)-\gamma(t_n^+)|^{\frac{1}{k}} + \frac{1}{2^{r(n)}},\end{aligned}$$

where we applied (4.4.4) and the fact that $d_n \in \mathcal{D}_{cc}(\Omega)$. Now, by definition of $L_d(\gamma)$ and the construction (4.4.4), similarly to [28, Theorem 3.1], we infer that

$$L_d(\gamma) \ge L_{d_n}(\gamma_n) - \frac{r(n)}{2^{r(n)}} + \sum_{i=1}^{r(n)}[d(\gamma(t_n^i),\gamma(t_n^{i+1})) - d_n(\gamma(t_n^i),\gamma(t_n^{i+1}))].$$

To get the required inequality, it is enough to pass to the lim sup in the above inequality, noticing that, by the choice of the sequence $r(n)$, we have

$$\lim_{n\to+\infty}\sum_{i=1}^{r(n)}[d(\gamma(t_n^i),\gamma(t_n^{i+1})) - d_n(\gamma(t_n^i),\gamma(t_n^{i+1}))] \le \lim_{n\to+\infty} r(n)\sup_K|d_n-d| = 0,$$

and then we get the desired conclusion (4.4.2).

(iii) $\Rightarrow$ (i). This implication follows from the following fact:

Claim: The class $\mathcal{D}_{cc}(\Omega)$ is compact.

As we are going to show, the above claim is obtained as a consequence of the Ascoli–Arzelá Theorem and the implication (i) $\Rightarrow$ (iii) already proved. Let $(d_n)_n \subset \mathcal{D}_{cc}(\Omega)$ be a given sequence. First of all, for any $(x,y) \in \Omega\times\Omega$ we have that $(d_n(x,y))_n$ is a bounded sequence, as granted by the following estimate:

$$d_n(x,y) \le \alpha d_{cc}(x,y) \quad \text{for every } x,y\in\Omega \quad \text{and } n\in\mathbb{N}. \tag{4.4.5}$$

Moreover, we have to prove that the sequence $(d_n)_n \in \mathcal{D}_{cc}(\Omega)$ is equi-continuous, in other words, that for every $x,x',y,y' \in \Omega \subset \mathbb{G}$ it holds

$$\forall\varepsilon>0\ \ \exists\,\delta>0\ :\begin{cases}|x'-x|<\delta\\ |y'-y|<\delta\end{cases} \Rightarrow\ |d_n(x,x')-d_n(y,y')|<\varepsilon,\ \ \forall n\in\mathbb{N}.$$

By using the triangle inequality and Theorem 2.1.11, we obtain that

$$\begin{aligned}|d_n(x,y) - d_n(x',y')| &\leq d_n(x,x') + d_n(y,y') \leq \alpha\big(d_{cc}(x,x') + d_{cc}(y,y')\big)\\ &\leq \alpha C_K\big(|x'-x|^{\frac{1}{k}} + |y'-y|^{\frac{1}{k}}\big).\end{aligned}$$

Choosing $\delta = 2\frac{\epsilon^k}{C_K\beta}$, we obtain

$$|d_n(x,y) - d_n(x',y')| \leq \varepsilon.$$

Hence, we may extract a subsequence converging to some element d in $\mathcal{D}_{cc}(\Omega)$.

To prove that d is geodesic, we use the implication (i) $\Rightarrow$ (iii), which ensures that $L = \Gamma(\mathrm{Lip}(\Omega)) - \lim_{n\to\infty} L_n$. Fix $x, y \in \Omega$. We will prove that we have the Γ-convergence for the modified functionals:

$$\tilde{L}_n(\gamma) \coloneqq \begin{cases} L_n(\gamma), & \text{if } \gamma(0) = x \text{ and } \gamma(1) = y;\\ +\infty, & \text{otherwise;}\end{cases}$$

$$\tilde{L}(\gamma) \coloneqq \begin{cases} L(\gamma), & \text{if } \gamma(0) = x \text{ and } \gamma(1) = y;\\ +\infty, & \text{otherwise.}\end{cases}$$

Arguing as in [28, Theorem 3.1], we can show that $\liminf_{n\to\infty} \tilde{L}_n(\gamma_n) \geq \tilde{L}(\gamma)$ whenever $\gamma_n \to \gamma$ in $\mathrm{Lip}(\Omega)$. To conclude we need to prove that, for every $\gamma \in \mathrm{Lip}(\Omega)$, there exists an approximating sequence $\{\tilde{\gamma}_n\}$ satisfying $\limsup_n \tilde{L}_n(\tilde{\gamma}_n) \leq \tilde{L}(\gamma)$. We can assume without loss of generality that $\tilde{L}(\gamma) = L(\gamma)$. Take a sequence $(\gamma_n)_{n\in\mathbb{N}}$ with $\gamma_n \to \gamma$ in $\mathrm{Lip}(\Omega)$ and $\lim_n L_n(\gamma_n) = L(\gamma)$, whose existence follows from the fact that $L = \Gamma - \lim_{n\to\infty} L_n$. Now we construct the optimal sequence $(\tilde{\gamma}_n)$ modifying (γ_n) as follows:

$$\tilde{\gamma}_n(t) \coloneqq \begin{cases} \text{an almost } d_n\text{-geodesic connecting } x \text{ and } \gamma_n(\frac{1}{n}), & \text{if } t \in [0, \frac{1}{n}];\\ \gamma_n(t), & \text{if } t \in [\frac{1}{n}, 1-\frac{1}{n}];\\ \text{an almost } d_n\text{-geodesic connecting } \gamma_n(1-\frac{1}{n}) \text{ and } y, & \text{if } t \in [1-\frac{1}{n}, \frac{1}{n}].\end{cases}$$

Let us now prove that $\tilde{\gamma}_n$ still converges to γ in $\mathrm{Lip}(\Omega)$. Let $\varepsilon > 0$ be fixed. If $t \in [\frac{1}{n}, 1-\frac{1}{n}]$, then the convergence of γ_n to γ in $\mathrm{Lip}(\Omega)$ yields, for n large enough,

$$d_{cc}(\tilde{\gamma}_n(t), \gamma(t)) = d_{cc}(\gamma_n(t), \gamma(t)) \leq \alpha C_K |\gamma_n(t) - \gamma(t)|^{\frac{1}{k}} \leq \varepsilon.$$

If $t \in [0, \frac{1}{n}]$ $\big($the case $t \in [1-\frac{1}{n}, \frac{1}{n}]$ is similar$\big)$, then

$$d_{cc}(\tilde{\gamma}_n(t), \gamma(t)) \leq d_{cc}(\tilde{\gamma}_n(t), x) + d_{cc}(x, \gamma(t)).$$

For n large enough, the second term at the right side is less than ε because γ is a continuous curve; for the first term, we have that

$$\begin{aligned}
\frac{1}{\alpha} d_{cc}\left(\tilde{\gamma}_n(t), x\right) &\leq d_n(\tilde{\gamma}_n(t), x) \leq d_n(\tilde{\gamma}_n\left(\tfrac{1}{n}\right), x) + \varepsilon_n \leq \alpha d_{cc}(\tilde{\gamma}_n\left(\tfrac{1}{n}\right), x) + \varepsilon_n \\
&= \alpha d_{cc}(\gamma_n\left(\tfrac{1}{n}\right), x) + \varepsilon_n \leq \alpha \left[d_{cc}(\gamma_n\left(\tfrac{1}{n}\right), \gamma\left(\tfrac{1}{n}\right)) + d_{cc}(\gamma\left(\tfrac{1}{n}\right), x)\right] + \varepsilon_n \\
&\leq \alpha \cdot C_K \left[\left|\gamma_n\left(\tfrac{1}{n}\right) - \gamma\left(\tfrac{1}{n}\right)\right| + \left|\gamma\left(\tfrac{1}{n}\right) - x\right|\right]^{\frac{1}{k}} + \varepsilon_n,
\end{aligned} \tag{4.4.6}$$

and the last term tends to zero as $n \to \infty$, since $\gamma_n \to \gamma$ in $\mathrm{Lip}(\Omega)$, and γ is a continuous horizontal curve. Thus we have obtained that $\tilde{\gamma}_n \to \gamma$ in $\mathrm{Lip}(\Omega)$. It remains to show the inequality $\limsup_n \tilde{L}_n(\tilde{\gamma}_n) \leq \tilde{L}(\gamma)$. We have that

$$\tilde{L}_n(\tilde{\gamma}_n) \leq d_n(x, \gamma_n\left(\tfrac{1}{n}\right)) + L_n(\gamma_n) + d_n(\gamma_n\left(1 - \tfrac{1}{n}\right), y) + 2\varepsilon_n. \tag{4.4.7}$$

Notice now that, from (4.4.6), it follows in particular that $\lim_n d_n(x, \gamma_n\left(\frac{1}{n}\right)) = 0$. Similarly, one obtains that $\lim_n d_n(\gamma_n\left(1 - \frac{1}{n}\right), y) = 0$, hence passing to the lim sup as $n \to \infty$ in (4.4.7) gives

$$\limsup_{n\to\infty} \tilde{L}_n(\tilde{\gamma}) \leq \limsup_{n\to\infty} L(\gamma_n) = L(\gamma) = \tilde{L}(\gamma).$$

Thus, by the Γ-convergence of L_n to L, we deduce that

$$\inf_{\gamma}\{\tilde{L}(\gamma)\} = \lim_{n\to\infty} \inf_{\gamma}\{\tilde{L}_n(\gamma)\}. \tag{4.4.8}$$

Since d_n are geodesic distances in $\mathcal{D}_{cc}(\Omega)$, the right-hand side is equal to the limit of $d_n(x, y)$, that is $d(x, y)$. Thus equation (4.4.8) means exactly that d is a geodesic distance, as desired.

Finally, assume in addition that Ω is bounded. On the one hand, (iv) trivially implies (ii). On the other hand, we can prove that (i) implies (iv). To this aim, fix any $\mu \in \mathcal{B}(\Omega)$ and $(\mu_n)_n \subset \mathcal{B}(\Omega)$ such that μ_n weakly* converges to μ. Let $\varepsilon > 0$ be fixed. We have that $\sup_n \mu_n(\Omega \times \Omega) < +\infty$ by Banach–Steinhaus Theorem. Moreover, we have that $\{\mu_n\}_n$ is weakly* relatively compact by assumption, thus Prokhorov's Theorem yields the existence of a compact set $K \subset \Omega \times \Omega$ such that $\mu_n((\Omega \times \Omega) \setminus K) \leq \varepsilon$ for every $n \in \mathbb{N}$. Call D the diameter of Ω with respect to d_{cc}. Since $d : \Omega \to \Omega \to \mathbb{R}$ is bounded and continuous, we deduce that

$$\begin{aligned}\big|J_n(\mu_n) - J(\mu)\big| &\leq \int_K |d_n - d| \,\mathrm{d}\mu_n + \int_{(\Omega\times\Omega)\setminus K} |d_n - d| \,\mathrm{d}\mu_n + \big|J(\mu_n) - J(\mu)\big| \\ &\leq \mu_n(\Omega \times \Omega) \max_K |d_n - d| + 2\beta D\varepsilon + \left| \int d \,\mathrm{d}\mu_n - \int d \,\mathrm{d}\mu \right|,\end{aligned}$$

whence by letting $n \to \infty$ we get $\limsup_n |J_n(\mu_n) - J(\mu)| \leq 2\beta D\varepsilon$. By arbitrariness of ε, we finally conclude that $J(\mu) = \lim_n J_n(\mu_n)$, so that (iv) is proved. $\square$

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