SLICE FUETER-REGULAR FUNCTIONS

RICCARDO GHILONI

ABSTRACT. Slice Fueter-regular functions, originally called slice Dirac-regular functions, are generalized holomorphic functions defined over the octonion algebra $\mathbb{O}$, recently introduced by M. Jin, G. Ren and I. Sabadini. A function $f : \Omega_D \subset \mathbb{O} \to \mathbb{O}$ is called (quaternionic) slice Fueter-regular if, given any quaternionic subalgebra $\mathbb{H}_\mathbb{I}$ of $\mathbb{O}$ generated by a pair $\mathbb{I} = (I, J)$ of orthogonal imaginary units I and J ($\mathbb{H}_\mathbb{I}$ is a 'quaternionic slice' of $\mathbb{O}$), the restriction of f to $\mathbb{H}_\mathbb{I}$ belongs to the kernel of the corresponding Cauchy-Riemann-Fueter operator $\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1} + J\frac{\partial}{\partial x_2} + (IJ)\frac{\partial}{\partial x_3}$.

The goal of this paper is to show that slice Fueter-regular functions are standard (complex) slice functions, whose stem functions satisfy a Vekua system having exactly the same form of the one characterizing axially monogenic functions of degree zero. The mentioned standard sliceness of slice Fueter-regular functions is able to reveal their 'holomorphic nature': slice Fueter-regular functions have Cauchy integral formulas, Taylor and Laurent series expansions, and a version of Maximum Modulus Principle, and each of these properties is global in the sense that it is true on genuine 8-dimesional domains of $\mathbb{O}$. Slice Fueter-regular functions are real analytic. Furthermore, we introduce the global concepts of spherical Dirac operator Γ and of slice Fueter operator $\overline{\vartheta}_F$ over octonions, which allow to characterize the slice Fueter-regular functions as the $\mathscr{C}^2$-functions in the kernel of $\overline{\vartheta}_F$ satisfying a second order differential system associated with Γ.

1. INTRODUCTION

In [13] M. Jin, G. Ren and I. Sabadini introduced and studied the class of slice Dirac-regular functions. Slice Dirac-regularity is a new concept of 'holomorphy over octonions', which merges the classical quaternionic Fueter-regularity and the more recent slice regularity ideas.

In the present paper we use the term *slice Fueter-regular* function to indicate a slice Dirac-regular function in the sense of [13]. See final Subsection 4.2 for the reason of this choice.

Our aim is to prove that slice Fueter-regular functions are standard slice functions satisfying a Vekua-type system, whose form is identical to the one characterizing axially monogenic functions of degree zero. The mentioned standard sliceness has some basic consequences. Slice Fueter-regular functions $f : \Omega \subset \mathbb{O} \to \mathbb{O}$ satisfy Cauchy-type integral formulas of *global nature*, i.e. valid on the whole domain Ω of f. At each point y of Ω, f admits 'Fueter polynomial' series expansions of *global nature*, i.e. converging in standard 8-dimensional Euclidean neighborhoods of y. In particular, slice Fueter-regular functions are real analytic. Furthermore, thanks to a new result which allows to pointwise 'annihilate' the camshaft effect occuring in the slice product between any given slice function and certain pairs of octonionic constants, we are able to show that slice Fueter-regular functions satisfy a version of Maximum Modulus Principle.

In the 1934 paper [14], R. Fueter introduced a notion of 'holomorphic function over quaternions', making use of the generalized Cauchy-Riemann operator $\overline{\mathcal{D}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$.

Riccardo Ghiloni
ORCID: 0000-0003-4189-2597
riccardo.ghiloni@unitn.it

Dipartimento di Matematica, Via Sommarive 14, Università di Trento, 38123 Povo-Trento, Italy.

Given a non-empty open subset Ω of the quaternion algebra $\mathbb{H}$, a function $f : \Omega \to \mathbb{H}$ is nowadays called *(left) Fueter-regular* if it is of class $\mathscr{C}^1$ in the usual real sense, and

$$\overline{\mathcal{D}}f(x) = \frac{\partial f}{\partial x_0}(x) + i\frac{\partial f}{\partial x_1}(x) + j\frac{\partial f}{\partial x_2}(x) + k\frac{\partial f}{\partial x_3}(x) = 0 \quad \text{for each } x \in \Omega,$$

where (x_0, x_1, x_2, x_3) are the real coordinates of x, i.e. $x = x_0 + x_1 i + x_2 j + x_3 k$. The operator $\overline{\mathcal{D}}$ is called Cauchy-Riemann-Fueter operator, even though it has been originally introduced by G. C. Moisil in [38]. The theory of Fueter regularity is now well-established, including its generalizations over Clifford algebras (see e.g. [3, 10, 35, 50]). It shares several fundamental analytic results with the theory of holomorphic functions of a complex variable; for instance, it has its own Cauchy integral formula, Taylor and Laurent series expansions, and Maximum Modulus Principle. However, the algebraic nature is lost in the sense that classical polynomials with quaternionic coefficients on the right are almost always not Fueter-regular. For instance, given the monomial xa with $a \in \mathbb{H}$, $\overline{\mathcal{D}}(xa) = -2a$ so xa is Fueter-regular if and only if $a = 0$.

In 2006 G. Gentili and D. C. Struppa [17, 18] introduced a new notion of regularity for quaternionic functions $f : \Omega \subset \mathbb{H} \to \mathbb{H}$, the one of *slice regular* function, starting from an idea of G. C. Cullen [9]. These functions have remarkable properties. Besides Cauchy-type integral formulas, Taylor- and Laurent-type series expansions, and Maximum and Minimum Modulus Principles, they include all classical quaternionic polynomials $\sum_{h=0}^{d} x^h a_h$ and their multiplicative structure. Furthermore, the theory of slice regular functions has applications to functional calculus and mathematical foundation of quantum mechanics (see e.g. [7, 21, 22] and their references), classification of orthogonal complex structures in $\mathbb{R}^4$ (see e.g. [1, 2, 15]), and operator semigroup theory (see e.g. [4, 33, 34]). Slice regularity was extended to Clifford algebras and octonions in [6, 19, 20]. Paper [25] contains an unified approach valid over all real alternative *-algebras, based on the notion of *stem* function. The latter notion allows to define the concept of *slice* function, which extends the one of slice regular function to the set-theoretic, not necessarily holomorphic, setting. It is important to remember that the germ of the idea of slice function was already contained in [9, 14, 43] for associative algebras and in [11, 45] for non-associative algebras.

1.1. **Structure of the paper.** In the remaining part of present introductory section we briefly recall basic concepts and results from the theories of slice functions and slice Fueter-regular functions over the octonion algebra $\mathbb{O}$. We conclude the section fixing some notations we will use throughout the next two sections of the paper.

Section 2 is devoted to the presentation of our results.

First, we compare the mentioned theories in Subsection 2.1. In Subsection 2.2 we introduce and study the concepts of spherical Dirac operator Γ and of slice Fueter operator $\overline{\vartheta}_F$ over $\mathbb{O}$, which allow to describe slice Fueter-regular functions as the $\mathscr{C}^2$-functions in the kernel of $\overline{\vartheta}_F$ satisfying a second order differential system associated with Γ.

In Sections 5 and 6 of [13], the authors prove that, given any slice Fueter-regular function f and fixed any pair $\mathbb{I} = (I, J)$ of orthogonal imaginary units I and J of $\mathbb{O}$, the restriction of f to the subalgebra $\mathbb{H}_{\mathbb{I}}$ of $\mathbb{O}$ generated by I and J satisfies a version of Cauchy integral formula, and it admits a Taylor or Laurent series expansion via Fueter-type polynomials or rational functions. All these results, including Theorems 5.4, 6.4 and 6.5, are *not global* in the sense that they give information on f only over $\mathbb{H}_{\mathbb{I}}$, see Remarks 6.2 and 6.3 of [13]. In Subsections 2.3, 2.4 and 2.5 below, we present global versions of the mentioned results. Subsection 2.3 contains also a global slice version of Borel-Pompeiu formula. In Subsection 2.6, we present a version of Maximum Modulus Principle for slice Fueter-regular functions. As we said, this Principle is based on a new result, Lemma 48, which is able to pointwise 'annihilate' the camshaft effect occurring in our non-associative setting.

The proofs of our results are postponed to Section 3.

Section 4 contains some final comments concerning possible generalizations of the concept of slice Fueter-regular function.

In the present paper we formulate eight open problems, see *Questions* 7, 9, 21, 23, 34, 38, 51 and 52. We conjecture that *Questions* 7, 9, 21 and 34 have substantially affirmative, possibly demanding, solutions.

1.2. **Slice functions over** $\mathbb{O}$.

Over octonions the stem function approach to slice regular functions is as follows. By the Cayley-Dickson process, the division algebra of octonions can be defined as $\mathbb{O} := \mathbb{H} + \ell\mathbb{H}$ with $(a + \ell b)(c + \ell d) := (ac - d\bar{b}) + \ell(\bar{a}d + cb)$ and $\overline{a + \ell b} := \bar{a} - \ell b$ for each $a, b, c, d \in \mathbb{H}$, where $\bar{a}$ denotes the usual quaternionic conjugation of a. As a consequence, $(1, i, j, k, \ell, \ell i, \ell j, \ell k)$ can be identified with the canonical real vector basis of $\mathbb{O}$, and $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ with the real subalgebras of $\mathbb{O}$ generated by $\{1\}$, $\{i\}$ and $\{i, j\}$, respectively. We assume that $\mathbb{O}$ is equipped with the Euclidean topology of $\mathbb{R}^8$, and $|x|$ is the usual Euclidean norm of x in $\mathbb{R}^8$.

Let $\mathbb{S} = \{I \in \mathbb{O} : I^2 = -1\}$ be the 6-sphere of imaginary units of $\mathbb{O}$. For each $I \in \mathbb{S}$, we indicate $\mathbb{C}_I$ the real subalgebra of $\mathbb{O}$ generated by I, which coincides with its real vector subspace $\mathrm{Span}(1, I)$. The octonion algebra $\mathbb{O}$ decomposes into 'complex slices' as follows:

$$\mathbb{O} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I, \quad \text{where } \mathbb{C}_I \cap \mathbb{C}_J = \mathbb{R} \text{ if } I \neq \pm J.$$

Let D be a subset of $\mathbb{C} = \mathbb{R}^2$, invariant under the complex conjugation $(\alpha, \beta) \mapsto (\alpha, -\beta)$. Define the circularization Ω_D of D in $\mathbb{O}$ by setting

$$\Omega_D := \{\alpha + \beta I \in \mathbb{O} : (\alpha, \beta) \in D, I \in \mathbb{S}\}.$$

Note that, if D is open in $\mathbb{C}$, then Ω_D is open in $\mathbb{O}$. A subset of $\mathbb{O}$ is said to be circular if it is equal to some Ω_D. Suppose that $D \neq \emptyset$. A function $F = (F_1, F_2) : D \to \mathbb{O}^2$ is said to be a *stem function* if F is conjugation-intrinsic in the sense that $F(\bar{z}) = \overline{F}(z)$ for each $z \in D$, where $\overline{F}(z) := (F_1(z), -F_2(z))$. Equivalently, F is a stem function if and only if

$$F_1(\alpha, -\beta) = F_1(\alpha, \beta) \text{ and } F_2(\alpha, -\beta) = -F_2(\alpha, \beta) \quad \text{for each } (\alpha, \beta) \in D. \tag{1}$$

We denote $\mathrm{Stem}(D, \mathbb{O}^2)$ the set of all stem functions from D to $\mathbb{O}^2$. A function $f : \Omega_D \to \mathbb{O}$ is called *(left) slice function* if there exists a stem function $F = (F_1, F_2) : D \to \mathbb{O}^2$ such that, if $x = \alpha + \beta I \in \Omega_D$ for some $I \in \mathbb{S}$ and $z = (\alpha, \beta) \in D$, then

$$f(\alpha + \beta I) = F_1(z) + IF_2(z).$$

Note that, thanks to (1), the above definition of f is well-posed. Indeed, if $x \in \mathbb{R}$, then $\beta = 0$ and I can be choosen arbitrarily in $\mathbb{S}$; anyway, $F_2(z) = 0$ so $f(x) = F_1(z)$. If $x \notin \mathbb{R}$, then there exist, and are unique, $(\alpha, \beta) \in D$ and $I \in \mathbb{S}$ such that $\beta > 0$ and $x = \alpha + \beta I = \alpha + (-\beta)(-I)$; anyway, $F_1(\alpha, \beta) + IF_2(\alpha, \beta) = F_1(\alpha, -\beta) + (-I)F_2(\alpha, -\beta)$. We say that f is induced by F and we write $f = \mathcal{I}(F)$. The slice function f is induced by a unique stem function F. Indeed, given any $J \in \mathbb{S}$, $F_1(\alpha, \beta) = \frac{1}{2}(f(\alpha + \beta J) + f(\alpha - \beta J))$ and $F_2(\alpha, \beta) = -\frac{1}{2}J(f(\alpha + \beta J) - f(\alpha - \beta J))$. We denote $\mathcal{S}(\Omega_D, \mathbb{O})$ the set of all slice functions from Ω_D to $\mathbb{O}$. As we just know, the map $\mathcal{I} : \mathrm{Stem}(D, \mathbb{O}^2) \to \mathcal{S}(\Omega_D, \mathbb{O})$ is a bijection. For details concerning the relations between the differential regularity of stem functions and the one of the induced slice functions, we refer the reader to [25, Proposition 7].

In general the pointwise product of two slice functions is not a slice function. Let us recall the notion of slice product. Let $f = \mathcal{I}(F), g = \mathcal{I}(G) \in \mathcal{S}(\Omega_D, \mathbb{O})$ with $F = (F_1, F_2)$ and $G = (G_1, G_2)$. Equip $\mathbb{O}^2$ with the multiplication induced by the tensor product $\mathbb{O} \otimes_\mathbb{R} \mathbb{C}$: $(x, y)(x', y') := (xx' - yy', xy' + yx')$. Now the pointwise product FG is again a stem function, so we can define the *slice product* of f and g as $f \cdot g := \mathcal{I}(FG)$. If f is slice preserving, i.e. F_1 and F_2 are real-valued, then $f \cdot g$ is equal to the pointwise product fg. For instance, for each $n, m \in \mathbb{N}$ and $a \in \mathbb{O}$, the pointwise-defined monomial function $\mathbb{O} \to \mathbb{O}$, $x \mapsto x^n \bar{x}^m a = \bar{x}^m x^n a$ is a slice function. The same is true for finite sums of such monomial functions. If f and g are $\mathbb{C}_I$-preserving for some $I \in \mathbb{S}$, i.e. $f(\Omega_D \cap \mathbb{C}_I) \subset \mathbb{C}_I$ and $g(\Omega_D \cap \mathbb{C}_I) \subset \mathbb{C}_I$, then $f \cdot g = fg$

on $\Omega_D \cap \mathbb{C}_I$. Furthermore, by Artin's theorem [46, Theorem 3.1], if f is $\mathbb{C}_I$-preserving and g is constantly equal to an octonion c, then $f \cdot g = fc$ on $\Omega_D \cap \mathbb{C}_I$. The slice product makes $\mathcal{S}(\Omega_D, \mathbb{O})$ a non-associative real alternative algebra, whose center contains all slice preserving functions.

Let $F^c : D \to \mathbb{O}^2$ be the stem function $F^c(z) := (\overline{F_1(z)}, \overline{F_2(z)})$ and let $f^c := \mathcal{I}(F^c)$. The slice preserving function $N(f) : \Omega_D \to \mathbb{O}$ defined by $N(f) := f \cdot f^c$ is called *normal function* of f. This is a very important function. For instance, $N(f \cdot g) = N(f)N(g)$, and the zero set $V(N(f))$ of $N(f)$ is the smallest circular subset of $\mathbb{O}$ containing the zero set $V(f)$ of f. Furthermore, if $V(N(f)) \neq \Omega_D$ then the function $f^{-\bullet} : \Omega_D \setminus V(N(f)) \to \mathbb{O}$ defined by $f^{-\bullet} := (N(f))^{-1}f^c$ is the unique slice function such that $f \cdot f^{-\bullet} = f^{-\bullet} \cdot f = 1$ on $\Omega_D \setminus V(N(f))$. The function $f^{-\bullet}$ is called *slice reciprocal*, or *slice multiplicative inverse*, of f. If f is slice preserving then $f^{-\bullet}(x) = (f(x))^{-1}$ for each $x \in \Omega_D \setminus V(N(f))$. If f is $\mathbb{C}_I$-preserving then $f^{-\bullet}(x) = (f(x))^{-1}$ for each $x \in (\Omega_D \setminus V(N(f))) \cap \mathbb{C}_I$.

Given $\xi \in \mathbb{O}$, the real polynomial $\Delta_\xi(x) = x^2 - 2x\mathrm{Re}(\xi) + |\xi|^2$ is equal to the normal function $N(\xi - x)$ of $\xi - x$, so the zero set of Δ_ξ coincides with the set $\mathbb{S}_\xi := \{\mathrm{Re}(\xi) + I|\mathrm{Im}(\xi)| \in \mathbb{O} : I \in \mathbb{S}\}$, where $\mathrm{Re}(\xi) := \frac{1}{2}(\xi + \bar{\xi}) \in \mathbb{R}$ and $\mathrm{Im}(\xi) := \frac{1}{2}(\xi - \bar{\xi})$. It turns out that $\mathbb{S}_\xi$ is equal to the conjugation class of ξ in $\mathbb{O}$, i.e. $\mathbb{S}_\xi = \{x\xi x^{-1} \in \mathbb{O} : x \in \mathbb{O} \setminus \{0\}\}$. For further details on slice product, normal function and slice reciprocal, see [25, 30, 32].

Suppose that D is open in $\mathbb{C}$ and consider a stem function $F = (F_1, F_2) : D \to \mathbb{O}^2$ of class $\mathscr{C}^1$ in the usual real sense. Define the functions $\partial F/\partial \bar{z} : D \to \mathbb{O}^2$ and $F_2' : D \setminus \mathbb{R} \to \mathbb{O}$ by setting

$$\frac{\partial F}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial F_1}{\partial \alpha} - \frac{\partial F_2}{\partial \beta}, \frac{\partial F_1}{\partial \beta} + \frac{\partial F_2}{\partial \alpha}\right) \quad \text{and} \quad F_2'(\alpha, \beta) := \frac{F_2(\alpha, \beta)}{\beta}.$$

It is immediate to verify that $\frac{\partial F}{\partial \bar{z}}$ and $(F_2', 0)$ are stem functions. Moreover, if $D \cap \mathbb{R} \neq \emptyset$ then F_2' extends to a continuous function G on the whole D such that $G(\alpha, 0) = \frac{\partial F_2}{\partial \beta}(\alpha, 0)$ for each $(\alpha, 0) \in D \cap \mathbb{R}$. We define the slice functions $\frac{\partial f}{\partial \bar{x}} : \Omega_D \to \mathbb{O}$ and $f_s' : \Omega_D \setminus \mathbb{R} \to \mathbb{O}$ by $\frac{\partial f}{\partial \bar{x}} := \mathcal{I}\left(\frac{\partial F}{\partial \bar{z}}\right)$ and $f_s' := \mathcal{I}((F_2', 0))$. *For short we also use the symbol $\overline{\partial} f$ to denote $\frac{\partial f}{\partial \bar{x}}$.* The slice function f_s' is called *spherical derivative* of f. One can also define the *spherical value* of f as the slice function $f_s^\circ : \Omega_D \to \mathbb{O}$ induced by the stem function $(F_1, 0)$. Note that $f(x) = f_s^\circ(x) + \mathrm{Im}(x)f_s'(x)$ for each $x \in \Omega_D \setminus \mathbb{R}$. For further information on f_s° and f_s', we refer the reader to [30, 32].

The slice function $f = \mathcal{I}(F) : \Omega_D \to \mathbb{O}$ is said to be *slice regular* if $\frac{\partial F}{\partial \bar{z}} = 0$ on D or, equivalently, if $\overline{\partial} f = 0$ on Ω_D. It is easy to see that the slice function $f = \mathcal{I}(F) : \Omega_D \to \mathbb{O}$ is slice regular if and only if, for some (and hence for all) $I \in \mathbb{S}$, the restriction of f to $\Omega_D \cap \mathbb{C}_I$ is holomorphic in the following sense: $\left(\frac{\partial}{\partial \alpha} + I\frac{\partial}{\partial \beta}\right)f(\alpha + I\beta) = 0$. Polynomial functions $\sum_{h=0}^d x^h a_h :$ $\mathbb{O} \to \mathbb{O}$ with coefficients a_h in $\mathbb{O}$ are examples of slice regular functions. For further details on the general stem function approach to the theory of slice regularity and on the particular case of octonions, we refer the reader to [23–32].

1.3. **Slice Fueter-regular functions.** As we said, in [13] M. Jin, G. Ren and I. Sabadini introduced and studied slice Dirac-regular functions over $\mathbb{O}$, we call slice Fueter-regular functions. Their idea is to mime the construction described above by decomposing the octonion algebra $\mathbb{O}$ in 'quaternionic slices' and by using the operator $\overline{\mathcal{D}}$ instead of $\frac{\partial}{\partial \bar{z}}$. Professor G. Ren presented this idea at the 12th ISAAC Congress held in Aveiro on summer 2019.

We will use notations slightly different from the ones of [13].

Let $\mathcal{N} := \{(I, J) \in \mathbb{S} \times \mathbb{S} : I \perp J\}$ and, for each $\mathbb{I} = (I, J) \in \mathcal{N}$, let $\mathbb{H}_\mathbb{I}$ be the subalgebra of $\mathbb{O}$ generated by $\{I, J\}$. Note that $\mathbb{H}_\mathbb{I}$ coincides with the real vector subspace of $\mathbb{O}$ generated by the orthonormal set $\{1, I, J, IJ\}$, and it is isomorphic to $\mathbb{H}$. The octonion algebra $\mathbb{O}$ decomposes into 'quaternionic slices':

$$\mathbb{O} = \bigcup_{\mathbb{I} \in \mathcal{N}} \mathbb{H}_\mathbb{I}.$$

Identify $O(3)$ with the group of 4×4 orthogonal real matrices A fixing the first vector of the canonical vector basis of $\mathbb{R}^4$, i.e. $A1 = 1$. Note that each matrix A in $O(3)$ defines a linear trasformation both in $\mathbb{R}^4$ and in $\mathbb{O}^4$, via the usual matrix-column multiplication $v \mapsto Av$. Here a vector v in $\mathbb{R}^4$, or in $\mathbb{O}^4$, is considered as a column; anyway, we often write v as a row.

Let E be a subset of $\mathbb{R}^4$. Suppose that it is $O(3)$-invariant, i.e. $Av \in E$ for each $v \in E$ and $A \in O(3)$. Define the circularization Θ_E of E in $\mathbb{O}$ by setting

$$\Theta_E := \{x_0 + x_1 I + x_2 J + x_3 IJ \in \mathbb{O} \; : \; (x_0, x_1, x_2, x_3) \in E, (I, J) \in \mathcal{N}\}.$$

If E is open in $\mathbb{R}^4$, then Θ_E is open in $\mathbb{O}$. Suppose that $E \neq \emptyset$. A function $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ is said to be a $O(3)$-*stem function* if $\mathcal{F}$ is $O(3)$-intrinsic in the sense that

$$\mathcal{F}(Av) = A\mathcal{F}(v) \quad \text{for each } v \in E \text{ and } A \in O(3). \tag{2}$$

We denote $\mathrm{Stem}_{O(3)}(E, \mathbb{O}^4)$ the set of all $O(3)$-stem functions from E to $\mathbb{O}^4$. We say that a function $f : \Theta_E \to \mathbb{O}$ is a *(left) $O(3)$-slice function* if there exists a $O(3)$-stem function $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ such that, if $x = x_0 + x_1 I + x_2 J + x_3 IJ$ for some $v = (x_0, x_1, x_2, x_3) \in E$ and $(I, J) \in \mathcal{N}$, then

$$f(x) = \mathcal{F}_0(v) + I\mathcal{F}_1(v) + J\mathcal{F}_2(v) + (IJ)\mathcal{F}_3(v).$$

It is proven in [13, Proposition 3.8] that, thanks to (2), the preceding definition is well-posed. We say that f is induced by $\mathcal{F}$ and we write $f = \mathcal{I}_{O(3)}(\mathcal{F})$. The $O(3)$-slice function f is induced by a unique $O(3)$-stem function, see the proof of Theorem 3.10 of [13]. We denote $\mathcal{S}_{O(3)}(\Theta_E, \mathbb{O})$ the set of all $O(3)$-slice functions from Θ_E to $\mathbb{O}$. The map $\mathcal{I}_{O(3)} : \mathrm{Stem}_{O(3)}(E, \mathbb{O}^4) \to \mathcal{S}_{O(3)}(\Theta_E, \mathbb{O})$ is a bijection.

Suppose that E is open in $\mathbb{R}^4$. Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ be a $O(3)$-stem function of class $\mathscr{C}^1$ and let $f = \mathcal{I}_{O(3)}(\mathcal{F}) : \Theta_E \to \mathbb{O}$ be the $O(3)$-slice function induced by $\mathcal{F}$. According to [13, Definition 4.1], we say that f is a *(left $O(3)$-)slice Fueter-regular function* if $\mathcal{F}$ satisfies the following system:

$$\begin{cases} \dfrac{\partial \mathcal{F}_0}{\partial x_0} - \dfrac{\partial \mathcal{F}_1}{\partial x_1} - \dfrac{\partial \mathcal{F}_2}{\partial x_2} - \dfrac{\partial \mathcal{F}_3}{\partial x_3} = 0 \\[2mm] \dfrac{\partial \mathcal{F}_0}{\partial x_1} + \dfrac{\partial \mathcal{F}_1}{\partial x_0} - \dfrac{\partial \mathcal{F}_2}{\partial x_3} + \dfrac{\partial \mathcal{F}_3}{\partial x_2} = 0 \\[2mm] \dfrac{\partial \mathcal{F}_0}{\partial x_2} + \dfrac{\partial \mathcal{F}_1}{\partial x_3} + \dfrac{\partial \mathcal{F}_2}{\partial x_0} - \dfrac{\partial \mathcal{F}_3}{\partial x_1} = 0 \\[2mm] \dfrac{\partial \mathcal{F}_0}{\partial x_3} - \dfrac{\partial \mathcal{F}_1}{\partial x_2} + \dfrac{\partial \mathcal{F}_2}{\partial x_1} + \dfrac{\partial \mathcal{F}_3}{\partial x_0} = 0 \,. \end{cases} \tag{3}$$

For each $\mathbb{I} = (I, J) \in \mathcal{N}$, we define the set $\Theta_{\mathbb{I}} := \Theta_E \cap \mathbb{H}_{\mathbb{I}}$, and the functions $f_{\mathbb{I}} : E \to \mathbb{O}$ and $\overline{\mathcal{D}}_{\mathbb{I}} f : \Theta_{\mathbb{I}} \to \mathbb{O}$ by setting

$$f_{\mathbb{I}}(v) := f(x_0 + x_1 I + x_2 J + x_3 IJ)$$

and

$$\overline{\mathcal{D}}_{\mathbb{I}} f(x) := \frac{\partial f_{\mathbb{I}}}{\partial x_0}(v) + I\frac{\partial f_{\mathbb{I}}}{\partial x_1}(v) + J\frac{\partial f_{\mathbb{I}}}{\partial x_2}(v) + (IJ)\frac{\partial f_{\mathbb{I}}}{\partial x_3}(v)$$

for each $v = (x_0, x_1, x_2, x_3) \in E$, where $x = x_0 + x_1 I + x_2 J + x_3 IJ$. Note that $\overline{\mathcal{D}}_{\mathbb{I}}$ is a '$\mathbb{H}_{\mathbb{I}}$-slice' version of the Cauchy-Riemann-Fueter operator. In [13, Proposition 4.2], it is proven that, given any $O(3)$-slice function $f = \mathcal{I}_{O(3)}(\mathcal{F})$ with $\mathcal{F}$ of class $\mathscr{C}^1$, it holds:

$$f \text{ is slice Fueter-regular if and only if } \overline{\mathcal{D}}_{\mathbb{I}} f = 0 \text{ on } \Theta_{\mathbb{I}} \textit{ for all } \mathbb{I} \in \mathcal{N}. \tag{4}$$

Actually, one can prove some more:

$$f \text{ is slice Fueter-regular if and only if } \overline{\mathcal{D}}_{\mathbb{I}} f = 0 \text{ on } \Theta_{\mathbb{I}} \textit{ for at least one } \mathbb{I} \in \mathcal{N}. \tag{5}$$

In Section 3 below, we will give the proof of (5), including also a new proof of (4). We have decided to add such a new proof, because the original proof of (4), see [13, Proposition 4.2],

contains an implicit computation of $\overline{\mathcal{D}_{\mathbb{I}}}f$, which is not easy to make explicit in the present (non-associative) octonionic setting. See equality (53) in Remark 39 below.

1.4. **Notations.** Identify $\mathbb{C} = \mathbb{R}^2$ with the real vector subspace $\mathbb{R}^2 \times \{(0,0)\}$ of $\mathbb{R}^4$.

Throughout the remaining part of the paper, E denotes a non-empty $O(3)$-invariant open subset of $\mathbb{R}^4$, and D the non-empty open subset of $\mathbb{R}^2$ defined by $D := E \cap \mathbb{R}^2$. Note that D is a subset of $\mathbb{R}^2$ invariant under the complex conjugation. In what follows, we identify the real coordinates (α, β) and $(x_0, x_1, 0, 0)$ of points in D, used in above Subsections 1.2 and 1.3, respectively; namely, we set $(\alpha, \beta) = (x_0, x_1)$.

2. The results

2.1. **Comparison of the theories.** Our first observation is that

$$\Theta_E = \Omega_D. \tag{6}$$

This equality is very easy to prove. Indeed, given any $x = x_0 + x_1 I + x_2 J + x_3 IJ \in \Theta_E$ with $v = (x_0, x_1, x_2, x_3) \in E$ and $(I, J) \in \mathcal{N}$, if $x_1 = x_2 = x_3 = 0$ then $x = x_0 \in E \cap \mathbb{R} = D \cap \mathbb{R} \subset \Omega_D$. Otherwise, $r := (x_1^2 + x_2^2 + x_3^2)^{1/2} > 0$ and there exists $A \in O(3)$ such that $Av = (x_0, r, 0, 0)$; hence $(x_0, r, 0, 0) \in E \cap \mathbb{R}^2 = D$. Since $H := \frac{1}{r}(x_1 I + x_2 J + x_3 IJ)$ belongs to $\mathbb{S}$ and $x = x_0 + rH$, it follows that $x \in \Omega_D$. This shows that $\Theta_E \subset \Omega_D$. Given any $x = x_0 + x_1 I \in \Omega_D$ with $(x_0, x_1) \in D$ and $I \in \mathbb{S}$, the point $(x_0, x_1, 0, 0)$ belongs to E so, choosing any $J \in \mathbb{S}$ with $(I, J) \in \mathcal{N}$, we have $x = x_0 + x_1 I + 0J + 0IJ \in \Theta_E$. This proves the inverse inclusion $\Omega_D \subset \Theta_E$ and hence the desired equality $\Theta_E = \Omega_D$.

From now on, we use Ω_D to denote Θ_E also.

If Ω_D is connected and $\Omega_D \cap \mathbb{R} \neq \emptyset$, then we say that Ω_D is a *slice domain*. If Ω_D is connected and $\Omega_D \cap \mathbb{R} = \emptyset$, then Ω_D is called *product domain*. Note that, in the first case, D is connected and $D \cap \mathbb{R} \neq \emptyset$; in the second, D has two connected components D^+ and D^- switched by complex conjugation, and Ω_D is homeomorphic to the topological product $\mathbb{S} \times D^+$. In general Ω_D decomposes into the disjoint union of its connected components, which are slice domains or product domains. In this way, from now on, we assume that

$$\Omega_D \text{ is either a slice domain or a product domain.} \tag{7}$$

Our first main result establishes the equivalence between $O(3)$-sliceness and sliceness.

Theorem 1. *A function $f : \Omega_D \to \mathbb{O}$ is $O(3)$-slice if and only if it is slice, namely $\mathcal{S}_{O(3)}(\Omega_D, \mathbb{O}) = \mathcal{S}(\Omega_D, \mathbb{O})$. More precisely, the following holds. Let $\Phi : \mathrm{Stem}_{O(3)}(E, \mathbb{O}^4) \to \mathrm{Stem}(D, \mathbb{O}^2)$ be the map sending each $O(3)$-stem function $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ into the stem function $(F_1, F_2) : D \to \mathbb{O}^2$ defined by*

$$F_1(x_0, x_1) := \mathcal{F}_0(x_0, x_1, 0, 0) \quad and \quad F_2(x_0, x_1) := \mathcal{F}_1(x_0, x_1, 0, 0). \tag{8}$$

Then it holds:

(1) *Φ is bijective and $\mathcal{I}_{O(3)} = \mathcal{I} \circ \Phi$. Equivalently, the following is a commutative diagram of bijective maps:*

$$
\begin{array}{ccc}
\mathrm{Stem}_{O(3)}(E, \mathbb{O}^4) & \xrightarrow{\ \Phi\ } & \mathrm{Stem}(D, \mathbb{O}^2) \\
{\scriptstyle \mathcal{I}_{O(3)}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{I}} \\
\mathcal{S}_{O(3)}(\Theta_E, \mathbb{O}) & =\!=\!=\!=\!= & \mathcal{S}(\Omega_D, \mathbb{O})
\end{array}
\tag{9}
$$

(2) *If $F = (F_1, F_2) : D \to \mathbb{O}^2$ is a stem function and $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = \Phi^{-1}(F)$, then given any $v = (x_0, x_1, x_2, x_3) \in E \setminus \mathbb{R}$ it holds*

$$\begin{cases} \mathcal{F}_0(v) = F_1(x_0, r), \\[2mm] \mathcal{F}_h(v) = \dfrac{x_h}{r} F_2(x_0, r) \quad \text{for each } h \in \{1, 2, 3\}, \end{cases} \tag{10}$$

where $r := (x_1^2 + x_2^2 + x_3^2)^{1/2}$. In particular, we have:

$$F \text{ is real analytic if and only if } \mathcal{F} = \Phi^{-1}(F) \text{ is.} \tag{11}$$

A relevant consequence is as follows.

Theorem 2. *All slice Fueter-regular functions are real analytic. Equivalently, the stem function inducing any slice Fueter-regular function is real analytic.*

The concept of slice Fueter-regularity can be explicitly described in 'purely slice' terms; this is the aim of our third main result.

Theorem 3. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function and let $F = (F_1, F_2) : D \to \mathbb{O}^2$ be the stem function inducing f. The following assertions are equivalent.*

(1) *f is slice Fueter-regular, i.e. $\mathcal{F} = \Phi^{-1}(F)$ is of class $\mathscr{C}^1$ and satisfies (3).*
(2) *F_1 is of class $\mathscr{C}^2$, F_2 is of class $\mathscr{C}^3$ and they satisfy the following Vekua system on $D \setminus \mathbb{R}$:*

$$\begin{cases} \dfrac{\partial F_1}{\partial x_0}(x_0, x_1) - \dfrac{\partial F_2}{\partial x_1}(x_0, x_1) = \dfrac{2F_2(x_0, x_1)}{x_1} \\[3mm] \dfrac{\partial F_1}{\partial x_1}(x_0, x_1) + \dfrac{\partial F_2}{\partial x_0}(x_0, x_1) = 0 \,. \end{cases} \tag{12}$$

(2a) *F_1 is of class $\mathscr{C}^2$, F_2 is of class $\mathscr{C}^3$ and $\overline{\partial} f = f'_s$ on $\Omega_D \setminus \mathbb{R}$.*
(2b) *F_1 is of class $\mathscr{C}^2$ and there exists a function $G_2 : D \to \mathbb{O}$ of class $\mathscr{C}^3$ such that $F_2(x_0, x_1) = x_1 G_2(x_0, x_1)$ and*

$$\begin{cases} \dfrac{\partial F_1}{\partial x_0}(x_0, x_1) - x_1 \dfrac{\partial G_2}{\partial x_1}(x_0, x_1) = 3G_2(x_0, x_1) \\[3mm] \dfrac{\partial F_1}{\partial x_1}(x_0, x_1) + x_1 \dfrac{\partial G_2}{\partial x_0}(x_0, x_1) = 0. \end{cases} \tag{13}$$

on D.

(2c) *If D^* denotes the set $\{(x_0, x_1) \in \mathbb{R}^2 : x_1 \geq 0, (x_0, \sqrt{x_1}) \in D\}$, then there exist an open neighborhood D^{**} of D^* in $\mathbb{R}^2$ and two functions $H_1, H_2 : D^{**} \to \mathbb{O}$ such that H_1 is of class $\mathscr{C}^2$, H_2 is of class $\mathscr{C}^3$, and $F_1(x_0, x_1) = H_1(x_0, x_1^2)$, $F_2(x_0, x_1) = x_1 H_2(x_0, x_1^2)$ and*

$$\begin{cases} \dfrac{\partial H_1}{\partial x_0}(x_0, x_1) - 2x_1 \dfrac{\partial H_2}{\partial x_1}(x_0, x_1) = 3H_2(x_0, x_1) \\[3mm] 2\dfrac{\partial H_1}{\partial x_1}(x_0, x_1) + \dfrac{\partial H_2}{\partial x_0}(x_0, x_1) = 0. \end{cases} \tag{14}$$

on the open subset D^* of $\mathbb{R}^2$.

Furthermore, if the preceding assertions are verified, then F, $\mathcal{F}$ and f are real analytic.

The reader notes that, thanks to (10), above system (13) coincides exactly with (4.5) of [13]. Furthermore, if Ω_D is a product domain, then in the preceding statement one can replace 'of class $\mathscr{C}^2$' and 'of class $\mathscr{C}^3$' with 'of class $\mathscr{C}^1$'.

In point (2) of the preceding statement, we referred to (12) as a Vekua system. For the theory of Vekua systems, see [36, Section 6.2] and references mentioned therein. Actually, Vekua systems of form (12) characterize axially monogenic functions of degree zero. For the theory of axially monogenic functions of degree zero and also of positive degree, we refer the

reader to [8, 37, 47, 48] and [10, Chapter II, Section 2.4]. For recent developments on this topic, see [5, 12, 39] and their references.

Two consequences of Theorem 3 are as follows:

Corollary 4. *Only the constant functions are both slice Fueter-regular and slice regular.*

Corollary 5. *The slice product $f \cdot g$ of two slice Fueter-regular functions $f, g : \Omega_D \to \mathbb{O}$ is again a slice Fueter-regular function if and only if either f is constant or g is.*

Remark 6. Let $F = (F_1, F_2) : D \to \mathbb{R}^2 \subset \mathbb{O}^2$ be a stem function such that F_1 is of class $\mathscr{C}^2$, F_2 is of class $\mathscr{C}^3$ and they satisfy differential system (12). Theorem 3 ensures that F is real analytic. As a matter of fact, we have that the theory of slice Fueter-regularity allows to prove standard regularity results for $\mathscr{C}^3$-solutions of a 2×2 differential system as (12). ∎

Question 7. Is the argument based on the theory of slice Fueter-regular functions, described in above Remark, an example of a general strategy to prove regularity results for solutions of certain real differential systems, via the embedding of $\mathbb{R}$ into higher dimensional real algebras, as $\mathbb{O}$?

Remark 8. As we have just said, the stem function $F = (F_1, F_2) : D \to \mathbb{O}^2$ inducing a slice Fueter-regular function is real analytic. In particular, both F_1 and F_2 are of class $\mathscr{C}^2$. In this way, we can differentiate both the equations of (12) with respect to $\frac{\partial}{\partial x_0}$ and $\frac{\partial}{\partial x_1}$. We deduce:

$$\Delta F_1 = -\frac{2}{x_1} \frac{\partial F_1}{\partial x_1} \quad \text{and} \quad \Delta F_2 = \frac{2}{x_1^2} F_2 - \frac{2}{x_1} \frac{\partial F_2}{\partial x_1} \quad \text{on } D \setminus \mathbb{R}.$$

For instance, if we differentiate the first equation of (12) by $\frac{\partial}{\partial x_0}$ and the second by $\frac{\partial}{\partial x_1}$, we obtain: $\frac{\partial^2 F_1}{\partial x_0^2} - \frac{\partial^2 F_2}{\partial x_0 \partial x_1} = \frac{2}{x_1} \frac{\partial F_2}{\partial x_0}$ and $\frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial^2 F_2}{\partial x_0 \partial x_1} = 0$. Adding these two equations and using the second equation of (12), we have that $\Delta F_1 = -\frac{2}{x_1} \frac{\partial F_1}{\partial x_1}$. ∎

We recall that, if $f = \mathcal{I}(F)$ is slice regular, then F is harmonic, i.e. $\Delta F_1 = \Delta F_2 = 0$. Making use of the harmonicity of F_2, A. Perotti proved in [42, Section 6] that all quaternionic slice regular functions are biharmonic.

Question 9. Have slice Fueter-regular functions some harmonicity properties, similar to the biharmonicity of quaternionic slice regular functions?

Let us present some examples of slice Fueter-regular functions.

Examples 10. (1) Let $(F_1, F_2) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{O}^2$ be a stem function of class $\mathscr{C}^1$ and let $f : \mathbb{O} \setminus \mathbb{R} \to \mathbb{O}$ be the slice function induced by (F_1, F_2). Define $\mathbb{R}_+ := \{t \in \mathbb{R} : t > 0\}$. Suppose that there exist four functions $\gamma, \delta, \eta, \xi : \mathbb{R}_+ \to \mathbb{O}$ of class $\mathscr{C}^1$ such that

$$F_1(x_0, x_1) = \gamma(x_0) + \delta(x_1), \tag{15}$$

$$F_2(x_0, x_1) = x_1 \eta(x_0) + x_1 \xi(x_1) \tag{16}$$

if $x_1 > 0$. Thanks to equations (12) and some elementary computations, one easily deduces that $f = \mathcal{I}((F_1, F_2))$, with (F_1, F_2) of the forms (15) and (16), is slice Fueter-regular if and only if there exist octonionic constants a, b, c, d such that

$$F_1(x_0, x_1) = (3x_0^2 - x_1^2)a + 3x_0 b + c, \tag{17}$$

$$F_2(x_0, x_1) = x_1 \left(2x_0 a + b + \frac{d}{x_1^3}\right) \tag{18}$$

if $x_1 > 0$. The slice Fueter-regular function $f = \mathcal{I}((F_1, F_2)) : \mathbb{O} \setminus \mathbb{R} \to \mathbb{O}$ induced by the stem function defined by (17) and (18) has the following form:

$$f(x) = (3\mathrm{Re}(x)^2 + 2\mathrm{Re}(x)\mathrm{Im}(x) + \mathrm{Im}(x)^2)a + 3\mathrm{Re}(x)b + \mathrm{Im}(x)b + c + \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|^3} d. \tag{19}$$

Let us prove the latter equation. Let $x \in \mathbb{O} \setminus \mathbb{R}$. Define $x_0 := \mathrm{Re}(x)$, $x_1 := |\mathrm{Im}(x)|$ and $I := \frac{\mathrm{Im}(x)}{x_1}$. Note that $x_1^2 = -\mathrm{Im}(x)^2$. It holds:

$$f(x) = F_1(x_0, x_1) + IF_2(x_0, x_1) = (3x_0^2 - x_1^2)a + 3x_0b + c + I\left(x_1\left(2x_0a + b + \frac{d}{x_1^3}\right)\right) =$$

$$= (3\mathrm{Re}(x)^2 + \mathrm{Im}(x)^2)a + 3\mathrm{Re}(x)b + c + \mathrm{Im}(x)\left(2\mathrm{Re}(x)a + b + \frac{d}{|\mathrm{Im}(x)|^3}\right) =$$

$$= (3\mathrm{Re}(x)^2 + 2\mathrm{Re}(x)\mathrm{Im}(x) + \mathrm{Im}(x)^2)a + 3\mathrm{Re}(x)b + \mathrm{Im}(x)b + c + \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|^3}\,d.$$

This proves (19). Note that, if $d = 0$, then the function f extends to a slice Fueter-regular function on the whole $\mathbb{O}$. Such extensions include the example given in [13, Example 4.4].

Equations (17) and (18) imply that a slice Fueter-regular function $f : \mathbb{O} \setminus \mathbb{R} \to \mathbb{O}$ depends only on $\mathrm{Im}(x)$, i.e. it is of the form $f(x) = \varphi(\mathrm{Im}(x))$ for some $\varphi : \mathbb{O} \setminus \mathbb{R} \to \mathbb{O}$ of class $\mathscr{C}^1$ if and only if $f(x) = c + \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|^3}\,d$ for some $c, d \in \mathbb{O}$.

(2) If in the preceding example we replace (16) with the following

$$F_2(x_0, x_1) = \eta(x_0) + x_1\xi(x_1) \quad \text{for } x_1 > 0, \tag{20}$$

then $f = \mathcal{I}((F_1, F_2))$, with (F_1, F_2) of the forms (15) and (20), is slice Fueter-regular if and only if there exist $b, c, d \in \mathbb{O}$ such that

$$F_1(x_0, x_1) = 3x_0b + c, \tag{21}$$

$$F_2(x_0, x_1) = x_1b + \frac{d}{x_1^2} \tag{22}$$

if $x_1 > 0$. Note that equations (21) and (22) are particular cases of (17) and (18). However, (21) and (22) prove that a slice Fueter-regular function $f : \mathbb{O} \setminus \mathbb{R} \to \mathbb{O}$ depends only on $\mathrm{Re}(x)$ if and only if it is constant.

(3) Slice Fueter powers associated to a slice Fueter-regular function that we will introduce in Definitions 24 and 30 below are examples of homogeneous slice Fueter-regular functions, see Corollaries 26 and 32 below. ∎

For the comparison between Fueter-regular functions and slice regular functions on Clifford algebras $\mathbb{R}_n = C\ell(0, n)$ for $n \geq 2$, we refer the reader to papers [40–42].

2.2. **Slice Fueter operator.** Given $p, q, h \in \mathbb{N}$ with $p > 0$ and $q > 0$, and a non-empty open subset O of $\mathbb{R}^p$, we denote $\mathscr{C}^h(O, \mathbb{R}^q)$ the set of all functions from O to $\mathbb{R}^q$ of class $\mathscr{C}^h$.

Let us rename the real vector basis $(1, i, j, k, \ell, \ell i, \ell j, \ell k)$ of $\mathbb{O}$ by $(1, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$, i.e.

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7) := (i, j, k, \ell, \ell i, \ell j, \ell k).$$

For each $m, n \in \{1, \ldots, 7\}$ with $m < n$, we denote $L_{mn} : \mathscr{C}^1(\mathbb{O}, \mathbb{O}) \to \mathscr{C}^0(\mathbb{O}, \mathbb{O})$ the usual spherical tangential operator defined by

$$L_{mn} := x_m \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_m}.$$

With regard to the preceding definition of L_{mn}, we implicitly mean that, given any non-empty open subset O of $\mathbb{O}$ and any $f \in \mathscr{C}^1(O, \mathbb{O})$, $L_{mn}(f)$ denotes the function in $\mathscr{C}^0(O, \mathbb{O})$ given by $L_{mn}(f)(x) = x_m \frac{\partial f}{\partial x_n}(x) - x_n \frac{\partial f}{\partial x_m}(x)$ for each $x \in O$. Note that, if O' is a non-empty open subset of O, then $L_{mn}(f) = 0$ on O' if and only if $L_{mn}(f|_{O'}) = 0$, i.e. $L_{mn}(f|_{O'})$ is constantly equal to zero on O'. The same clarification is valid for all differential operators we will consider below.

Given any $f \in \mathscr{C}^1(\Omega_D, \mathbb{O})$ and any $x \in \Omega_D \setminus \mathbb{R}$, the differential at x of the restriction $f|_{\mathbb{S}_x}$ of f to $\mathbb{S}_x$ vanishes if and only if $\nabla f(x) \in \mathrm{Span}(1, x)$, which is in turn equivalent to say that $L_{mn}(f)(x) = 0$ for each $n, m \in \{1, \ldots, 7\}$ with $m < n$. Since $\mathbb{S}_x$ is connected, it follows that $f|_{\mathbb{S}_x}$ is constant if and only if

$$L_{mn}(f) = 0 \text{ on } \mathbb{S}_x \text{ for each } n, m \in \{1, \ldots, 7\} \text{ with } m < n. \tag{23}$$

The spherical Dirac operator is a well-known differential operator used in Clifford analysis, see [3, Section 8.7] and [10, Chapter II, Section 0.1.4, p. 139]. See also [44, 51] and [42, Sections 3,4,6] for a recent development.

In the next definition, we introduce the concept of spherical Dirac operator over octonions: the defining formula is essentially the usual one, but now we have to choose the 'right' position of the parentheses. To the best of our knowledge, the concept we are going to introduce is new.

Definition 11. We define the *(octonionic) spherical Dirac operator* $\Gamma : \mathscr{C}^1(\mathbb{O}, \mathbb{O}) \to \mathscr{C}^0(\mathbb{O}, \mathbb{O})$ by setting

$$\Gamma := -\sum_{m<n} e_m(e_n L_{mn}),$$

where the symbol '$\sum_{m<n}$' denotes '$\sum_{m,n\in\{1,\dots,7\},m<n}$'. ∎

Note that, if $f \in \mathscr{C}^1(\Omega_D, \mathbb{O})$, then

$$\Gamma(f)(x) = -\sum_{m<n} e_m\left(e_n\left(x_m\frac{\partial f}{\partial x_n}(x) - x_n\frac{\partial f}{\partial x_m}(x)\right)\right)$$

for each $x = (x_0,\dots,x_7) \in \Omega_D$.

Recall that, given a slice function $f : \Omega_D \to \mathbb{O}$ induced by the stem function (F_1, F_2), then the spherical derivative $f'_s : \Omega_D \setminus \mathbb{R} \to \mathbb{O}$ of f and the spherical value $f^\circ_s : \Omega_D \to \mathbb{O}$ of f are the slice functions induced by the stem functions $(x_0, x_1) \mapsto (x_1^{-1}F_2(x_0, x_1), 0)$ and $(x_0, x_1) \mapsto (F_1(x_0, x_1), 0)$, respectively.

Making use of the first Moufang identity (see [46, (3.4), p. 28]), we obtain:

Lemma 12. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function of class $\mathscr{C}^1$. Then it holds:*

$$f'_s = \tfrac{1}{6}\mathrm{Im}^{-1}\Gamma(f) \qquad on \ \Omega_D \setminus \mathbb{R}, \tag{24}$$

where '$\mathrm{Im}^{-1}\Gamma(f)$' denotes the function $\Omega_D \setminus \mathbb{R} \to \mathbb{O}$, $x \mapsto (\mathrm{Im}(x))^{-1}(\Gamma(f)(x))$.

It is worth noting that, given f as in the statement of the preceding lemma, $f^\circ_s = f - \tfrac{1}{6}\Gamma(f)$ on Ω_D. In the case $f(x) = \mathrm{Im}(x)$, (24) reduces to $\Gamma(\mathrm{Im}(x)) = 6\mathrm{Im}(x)$ on $\mathbb{O}$.

As a consequence of Lemma 12, we obtain the following *differential* sliceness criterion; for other sliceness criterions, see [27, Lemma 3.2] and Lemma 44 below.

Lemma 13 (Differential sliceness criterion)**.** *Let $f : \Omega_D \to \mathbb{O}$ be a function of class $\mathscr{C}^2$. The following assertions are equivalent.*

(1) *f is a slice function.*
(2) *$f|_{\Omega_D \setminus \mathbb{R}} \in \bigcap_{m<n} \ker\left(L_{mn}(\mathrm{Im}^{-1}\Gamma)\right) \cap \bigcap_{m<n} \ker\left(L_{mn}(\mathrm{Id} - \tfrac{1}{6}\Gamma)\right)$ or, equivalently*

$$\begin{cases} L_{mn}(\mathrm{Im}^{-1}\Gamma(f)) = 0 & \text{for each } m, n \in \{1,\dots,7\} \text{ with } m < n, \\ L_{mn}(f - \tfrac{1}{6}\Gamma(f)) = 0 & \text{for each } m, n \in \{1,\dots,7\} \text{ with } m < n \end{cases} \tag{25}$$

on $\Omega_D \setminus \mathbb{R}$.

Let us introduce the concept of slice Fueter operator.

Definition 14. We define the *slice Fueter operator* $\overline{\vartheta}_F : \mathscr{C}^1(\mathbb{O}\setminus\mathbb{R}, \mathbb{O}) \to \mathscr{C}^0(\mathbb{O}\setminus\mathbb{R}, \mathbb{O})$ as follows:

$$\overline{\vartheta}_F := \frac{\partial}{\partial x_0} - \mathrm{Im}^{-1}\mathrm{E} - \tfrac{1}{6}\mathrm{Im}^{-1}\Gamma, \tag{26}$$

where E is the Euler operator $\sum_{h=1}^7 x_h\frac{\partial}{\partial x_h}$. ∎

Note that $\frac{\partial}{\partial x_0} - \mathrm{Im}^{-1}\mathrm{E}$ is the operator $\overline{\vartheta}$ introduced in [27], so we have:

$$\overline{\vartheta}_F = \overline{\vartheta} - \tfrac{1}{6}\mathrm{Im}^{-1}\Gamma. \tag{27}$$

Given any $f \in \mathscr{C}^1(\Omega_D, \mathbb{O})$, we have:

$$\overline{\vartheta}_F(f) = \frac{\partial f}{\partial x_0}(x) - \mathrm{Im}(x)^{-1}\left(\sum_{h=1}^{7} x_h \frac{\partial f}{\partial x_h}(x)\right) - \frac{1}{6}\mathrm{Im}(x)^{-1}(\Gamma(f)(x)).$$

In Theorem 2.2(ii) of [27], it is proven that, if f is slice, then $\overline{\partial}f = \overline{\vartheta}(f)$ on $\Omega_D \setminus \mathbb{R}$. Combining the latter equality with equivalence (1) $\Longleftrightarrow$ (2a) of Theorem 3 and with Lemma 13, we obtain:

Theorem 15. *Let $f : \Omega_D \to \mathbb{O}$ be a function of class $\mathscr{C}^3$. The following assertions are equivalent:*

    (1) *f is a slice Fueter-regular function.*
    (2) *f is a slice function and $\overline{\vartheta}_F(f) = 0$ on $\Omega_D \setminus \mathbb{R}$.*
    (3) *f satisfies (25) and $\overline{\vartheta}_F(f) = 0$ on $\Omega_D \setminus \mathbb{R}$.*

The reader observes that above equivalence (1) $\Longleftrightarrow$ (3) allows to describe the slice Fueter-regular functions f as the $\mathscr{C}^2$-solutions of the second order differential system given by (25) and $\overline{\vartheta}_F(f) = 0$, without any request concerning the sliceness of f.

2.3. **Slice Borel-Pompeiu and Cauchy integral formulas.** *Fix $I \in \mathbb{S}$. Let $\pi_I : \mathbb{O} \to \mathbb{C}_I$ be the orthogonal projection of $\mathbb{O}$ onto $\mathbb{C}_I$. For each $\xi \in \mathbb{O}$, we define $\xi_I := \pi_I(\xi)$, $\xi_I^\perp := \xi - \xi_I$, the slice function $g_{I,\xi} : \mathbb{O} \to \mathbb{O}$ by*

$$g_{I,\xi}(x) := (\xi_I - x) \cdot (\overline{\xi_I} - \overline{x}) + |\xi_I^\perp|^2 = |x|^2 - x\overline{\xi_I} - \overline{x}\xi_I + |\xi|^2, \tag{28}$$

the subset $\mathbb{S}_{I,\xi}$ of $\mathbb{O}$ by

$$\mathbb{S}_{I,\xi} := \begin{cases} \mathbb{S}_\xi & \text{if } \xi \in \mathbb{C}_I \\ \emptyset & \text{if } \xi \notin \mathbb{C}_I \end{cases}$$

and the closed subset Γ_I of $\mathbb{O}^2$ by

$$\Gamma_I := \{(x, \xi) \in \mathbb{O}^2 : \xi \in \mathbb{C}_I, x \in \mathbb{S}_\xi\} = \bigcup_{\xi \in \mathbb{O}} (\mathbb{S}_{I,\xi} \times \{\xi\}).$$

Note that $(\mathbb{O}^2 \setminus \Gamma_I) \cap (\mathbb{O} \times \{\xi\}) = (\mathbb{O} \setminus \mathbb{S}_{I,\xi}) \times \{\xi\}$ for each $\xi \in \mathbb{O}$.

Lemma 16. *Given any $\xi \in \mathbb{O}$, the following assertions hold.*

    (1) *$N(g_{I,\xi})(x) = |\Delta_{\xi_I}(x)|^2 + 2|\xi_I^\perp|^2(|x|^2 - 2\mathrm{Re}(x)\mathrm{Re}(\xi_I) + |\xi_I|^2) + |\xi_I^\perp|^4 \geq 0$ for each $x \in \mathbb{O}$.*
    (2) *The zero set of $N(g_{I,\xi})$ is equal to $\mathbb{S}_{I,\xi}$.*

Fix $J \in \mathbb{S}$ with $I \perp J$. Define $\mathbb{I} := (I, J) \in \mathcal{N}$ and $\Omega_{\mathbb{I}} := \Omega_D \cap \mathbb{H}_{\mathbb{I}}$. Note that $\Omega_{\mathbb{I}} = \Theta_{\mathbb{I}}$. Assume that D is bounded and its boundary ∂D in $\mathbb{R}^2$ is of class $\mathscr{C}^1$. It follows that $\Omega_{\mathbb{I}}$ is bounded and its boundary $\partial \Omega_{\mathbb{I}}$ in $\mathbb{H}_{\mathbb{I}}$ is of class $\mathscr{C}^1$ as well. We denote $\mathbf{n}_{\mathbb{I}} : \partial \Omega_{\mathbb{I}} \to \mathbb{H}_{\mathbb{I}} \subset \mathbb{O}$ the outer unit normal vector field of $\partial \Omega_{\mathbb{I}}$ in $\mathbb{H}_{\mathbb{I}}$. We indicate $\mathrm{ds}_{\mathbb{I}}(\xi)$ the surface element of $\partial \Omega_{\mathbb{I}}$ and $\mathrm{dv}_{\mathbb{I}}(\xi)$ the volume element of $\Omega_{\mathbb{I}} \subset \mathbb{H}_{\mathbb{I}}$.

Fix a slice function $f : \Omega_D \to \mathbb{O}$ of class $\mathscr{C}^1$. Thanks to Lemma 16, given any $\xi \in \partial \Omega_{\mathbb{I}}$, we can define the slice function $S_{f,\mathbb{I},\xi} : \mathbb{O} \setminus \mathbb{S}_{I,\xi} \to \mathbb{O}$ by setting

$$S_{f,\mathbb{I},\xi}(x) := \frac{1}{2\pi^2}(g_{I,\xi}(x)^{-\bullet})^{\bullet 2} \cdot ((\overline{\xi} - \overline{x})(\mathbf{n}_{\mathbb{I}}(\xi)f(\xi)))$$

or, equivalently,

$$S_{f,\mathbb{I},\xi}(x) := \frac{1}{2\pi^2}(N(g_{I,\xi})(x))^{-2} \cdot (g_{I,\xi}^c(x))^{\bullet 2} \cdot ((\overline{\xi} - \overline{x})(\mathbf{n}_{\mathbb{I}}(\xi)f(\xi))), \tag{29}$$

where $(g_{I,\xi}(x)^{-\bullet})^{\bullet 2}$ and $(g_{I,\xi}^c(x))^{\bullet 2}$ denote the slice squares of $g_{I,\xi}^{-\bullet}$ and $g_{I,\xi}^c$ respectively, i.e. $g_{I,\xi}^{-\bullet} \cdot g_{I,\xi}^{-\bullet}$ and $g_{I,\xi}^c \cdot g_{I,\xi}^c$ (see above Section 1.2 for the definitions of $g_{I,\xi}^c$ and $g_{I,\xi}^{-\bullet}$). Note that $(N(g_{I,\xi}))^{-2}$ is a slice preserving function, so it belongs to the center of $\mathcal{S}(\mathbb{O} \setminus \mathbb{S}_{I,\xi}, \mathbb{O})$. Hence it is not necessary to parenthesize the three-terms slice product in definition (29). Furthermore,

for each octonionic constant a, the pointwise product function $x \mapsto \overline{\xi}a - \overline{x}a = (\overline{\xi} - \overline{x})a$ coincides with the slice product $x \mapsto (\overline{\xi} - \overline{x}) \cdot a$. As a consequence, the function $x \mapsto (\overline{\xi} - \overline{x})(\mathtt{n}_{\mathbb{I}}(\xi)f(\xi))$ is slice and hence also the function $x \mapsto S_{f,\mathbb{I},\xi}(x)$ is.

By Lemma 16, given any $\xi \in \Omega_{\mathbb{I}}$, we can also define the slice function $V_{f,\mathbb{I},\xi} : \mathbb{O} \setminus \mathbb{S}_{I,\xi} \to \mathbb{O}$ by setting

$$V_{f,\mathbb{I},\xi}(x) := \frac{1}{2\pi^2}(g_{I,\xi}(x)^{-\bullet})^{\bullet 2} \cdot \left((\overline{\xi} - \overline{x})(\overline{\mathcal{D}}_{\mathbb{I}}f(\xi))\right)$$

or, equivalently,

$$V_{f,\mathbb{I},\xi}(x) := \frac{1}{2\pi^2}(N(g_{I,\xi})(x))^{-2} \cdot (g_{I,\xi}^{c}(x))^{\bullet 2} \cdot \left((\overline{\xi} - \overline{x})(\overline{\mathcal{D}}_{\mathbb{I}}f(\xi))\right). \tag{30}$$

We define

$$\partial^{*}\Omega_{\mathbb{I}} := (\mathbb{O}^2 \setminus \Gamma_I) \cap (\mathbb{O} \times \partial\Omega_{\mathbb{I}}) = \bigcup_{\xi \in \partial\Omega_{\mathbb{I}}} \left((\mathbb{O} \setminus \mathbb{S}_{I,\xi}) \times \{\xi\}\right) \tag{31}$$

and

$$\Omega_{\mathbb{I}}^{*} := (\mathbb{O}^2 \setminus \Gamma_I) \cap (\mathbb{O} \times \Omega_{\mathbb{I}}) = \bigcup_{\xi \in \Omega_{\mathbb{I}}} \left((\mathbb{O} \setminus \mathbb{S}_{I,\xi}) \times \{\xi\}\right). \tag{32}$$

Definition 17. We define the *slice Cauchy surface $(f,\mathbb{I})$-integrand* $S_{f,\mathbb{I}} : \partial^{*}\Omega_{\mathbb{I}} \to \mathbb{O}$ and the *slice Cauchy volume $(f,\mathbb{I})$-integrand* $V_{f,\mathbb{I}} : \Omega_{\mathbb{I}}^{*} \to \mathbb{O}$ by

$$S_{f,\mathbb{I}}(x,\xi) := S_{f,\mathbb{I},\xi}(x) \tag{33}$$

and

$$V_{f,\mathbb{I}}(x,\xi) := V_{f,\mathbb{I},\xi}(x). \ \blacksquare \tag{34}$$

Next result deals with pointwise-product formulas for slice Cauchy integrands $S_{f,\mathbb{I}}$ and $V_{f,\mathbb{I}}$.

Lemma 18. *Let $Q_I : \mathbb{O}^3 \to \mathbb{O}$, $M_I : \mathbb{O}^3 \to \mathbb{O}$ and $N_I : \mathbb{O}^2 \to \mathbb{O}$ be the functions defined by*

$$Q_I(x,\xi,a) := (|x|^2 + |\xi|^2)^2 a - 2(|x|^2 + |\xi|^2)(x(\xi_I a) + \overline{x}(\overline{\xi_I}a)) +$$
$$+ x^2(\xi_I^2 a) + \overline{x}^2(\overline{\xi_I}^2 a) + 2|x|^2|\xi_I|^2 a,$$
$$M_I(x,\xi,a) := Q_I(x,\xi,\overline{\xi}a) - \overline{x}Q_I(x,\xi,a),$$
$$N_I(x,\xi) := |\Delta_{\xi_I}(x)|^2 + 2|\xi_I^{\perp}|^2\left(|x|^2 - 2\mathrm{Re}(x)\mathrm{Re}(\xi_I) + |\xi_I|^2\right) + |\xi_I^{\perp}|^4.$$

Then the zero set of N_I coincides with Γ_I, and it holds:

$$S_{f,\mathbb{I}}(x,\xi) = \frac{M_I(x,\xi,\mathtt{n}_{\mathbb{I}}(\xi)f(\xi))}{2\pi^2 N_I(x,\xi)^2} \quad \text{for each } (x,\xi) \in \partial^{*}\Omega_{\mathbb{I}} \tag{35}$$

and

$$V_{f,\mathbb{I}}(x,\xi) = \frac{M_I(x,\xi,\overline{\mathcal{D}}_{\mathbb{I}}f(\xi))}{2\pi^2 N_I(x,\xi)^2} \quad \text{for each } (x,\xi) \in \Omega_{\mathbb{I}}^{*}. \tag{36}$$

Note that, for each fixed ξ, the functions $x \mapsto S_{f,\mathbb{I}}(x,\xi)$ and $x \mapsto V_{f,\mathbb{I}}(x,\xi)$ turn out to be slice, and rational with respect to the eight real components of $x = (x_0, \ldots, x_7) \in \mathbb{O}$.

Remark 19. Note that, due to the non-associativity of $\mathbb{O}$, the slice Cauchy integrands $S_{f,\mathbb{I}}$ and $V_{f,\mathbb{I}}$ cannot be written as the pointwise product of a slice function in x (the 'potential' octonionic slice Cauchy kernel) and the functions $\xi \mapsto \mathtt{n}_{\mathbb{I}}(\xi)f(\xi)$ and $\xi \mapsto \overline{\mathcal{D}}_{\mathbb{I}}f(\xi)$, respectively. In this pointwise-product sense, we can say that an octonionic slice Cauchy kernel, independent from f, does not exist. This is a typical issue of the non-associative case, see [25, 29]. $\blacksquare$

Given any subset S of $\mathbb{O}$, we denote $\mathrm{cl}(S)$ the Euclidean closure of S in $\mathbb{O} = \mathbb{R}^8$.

Our slice Borel-Pompeiu and Cauchy integral formulas read as follows.

Theorem 20. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function of class $\mathscr{C}^1$, and let $\mathbb{I} \in \mathcal{N}$. Suppose that D is bounded with boundary of class $\mathscr{C}^1$, and f admits a continuous extension on $\mathrm{cl}(\Omega_D)$. Then the following integral formula for f holds:*

$$f(x) = \int_{\partial\Omega_{\mathbb{I}}} S_{f,\mathbb{I}}(x,\xi)\,\mathrm{ds}_{\mathbb{I}}(\xi) - \int_{\Omega_{\mathbb{I}}} V_{f,\mathbb{I}}(x,\xi)\,\mathrm{dv}_{\mathbb{I}}(\xi) \quad \text{for each } x \in \Omega_D. \tag{37}$$

In particular, if f is slice Fueter-regular then

$$f(x) = \int_{\partial\Omega_{\mathbb{I}}} S_{f,\mathbb{I}}(x,\xi)\,\mathrm{ds}_{\mathbb{I}}(\xi) \quad \text{for each } x \in \Omega_D \tag{38}$$

and

$$0 = \int_{\partial\Omega_{\mathbb{I}}} S_{f,\mathbb{I}}(x,\xi)\,\mathrm{ds}_{\mathbb{I}}(\xi) \quad \text{for each } x \in \mathbb{O} \setminus \mathrm{cl}(\Omega_D). \tag{39}$$

Question 21. Let f and D be as in the statement of the preceding theorem, and let $I \in \mathbb{S}$. Define: $D_I := \Omega_D \cap \mathbb{C}_I$, ∂D_I as the boundary of D_I in $\mathbb{C}_I$, $\partial^* D_I := (\mathbb{O}^2 \setminus \Gamma_I) \cap (\mathbb{O} \times \partial D_I)$ and $D_I^* := (\mathbb{O}^2 \setminus \Gamma_I) \cap (\mathbb{O} \times D_I)$. Do there exist two real analytic functions $S_{f,I} : \partial^* D_I \to \mathbb{O}$ and $V_{f,I} : D_I^* \to \mathbb{O}$ such that: each ξ-restriction $x \mapsto S_{f,I}(x,\xi)$ is a slice Fueter-regular function, each ξ-restriction $x \mapsto V_{f,I}(x,\xi)$ is a slice function, $V_{f,I} = 0$ if f is slice Fueter-regular, and the following 'curve-plane' Borel-Pompeiu formula holds:

$$f(x) = \int_{\partial D_I} S_{f,I}(x,\xi)\,\mathrm{d}\xi - \int_{D_I} V_{f,I}(x,\xi)\left(\tfrac{1}{2}I\mathrm{d}\xi \wedge \mathrm{d}\overline{\xi}\right) \quad \text{for each } x \in \Omega_D?$$

2.4. **Slice Taylor expansion.** Consider $\mathbb{C}_i = \{x_0 + ix_1 \in \mathbb{H} \ : \ x_0, x_1 \in \mathbb{R}\} \subset \mathbb{H}$ and write $\mathbf{x} = x_0 + ix_1$. Denote $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3 : \mathbb{C}_i \to \mathbb{H}$ the quaternionic Fueter variables restricted to $\mathbb{C}_i$, i.e.

$$\mathbf{z}_1(\mathbf{x}) := -\mathbf{x}i = x_1 - ix_0,$$
$$\mathbf{z}_2(\mathbf{x}) := -\mathrm{Re}(\mathbf{x})j = -x_0 j,$$
$$\mathbf{z}_3(\mathbf{x}) := -\mathrm{Re}(\mathbf{x})k = -x_0 k.$$

Note that

$$\mathbf{z}_2\mathbf{z}_1 = \overline{\mathbf{z}_1}\mathbf{z}_2, \qquad \mathbf{z}_3\mathbf{z}_1 = \overline{\mathbf{z}_1}\mathbf{z}_3, \qquad \mathbf{z}_3\mathbf{z}_2 = -\mathbf{z}_2\mathbf{z}_3. \tag{40}$$

Let $\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{N}^3$, let $|\kappa| := \kappa_1 + \kappa_2 + \kappa_3$ and let $\kappa! := \kappa_1!\kappa_2!\kappa_3!$. Suppose that $\kappa \neq (0,0,0)$ and hence $|\kappa| \geq 1$. Consider the sequence of indices $k_1, \ldots, k_{|\kappa|}$ such that the first κ_1 indices are equal to 1, the next κ_2 indices are equal to 2 and the last κ_3 indices are equal to 3. Let $\Sigma(\kappa)$ be the group of permutations of $\{1, 2, \ldots, |\kappa|\}$. For each $\sigma \in \Sigma(\kappa)$, we denote $\mathbf{z}_{\kappa,\sigma} : \mathbb{C}_i \to \mathbb{H}$ the polynomial function given by setting

$$\mathbf{z}_{\kappa,\sigma} := \mathbf{z}_{k_{\sigma(1)}}\mathbf{z}_{k_{\sigma(2)}}\cdots\mathbf{z}_{k_{\sigma(|\kappa|)}}.$$

Let Q_8 be the quaternion group $\{1, i, j, k, -1, -i, -j, -k\}$. Thanks to (40), there exist, and are unique, $a(\kappa, \sigma), b(\kappa, \sigma) \in \mathbb{N}$ and $L_{\kappa,\sigma} \in Q_8$ such that $\kappa_1 = a(\kappa, \sigma) + b(\kappa, \sigma)$ and

$$\mathbf{z}_{\kappa,\sigma}(\mathbf{x}) = \mathrm{Re}(\mathbf{x})^{\kappa_2+\kappa_3}\mathbf{x}^{a(\kappa,\sigma)}\overline{\mathbf{x}}^{b(\kappa,\sigma)}L_{\kappa,\sigma} = x_0^{\kappa_2+\kappa_3}(x_0 + ix_1)^{a(\kappa,\sigma)}(x_0 - ix_1)^{b(\kappa,\sigma)}L_{\kappa,\sigma}.$$

We define the polynomial function $\mathcal{P}_\kappa : \mathbb{C}_i \to \mathbb{H}$ by $\mathcal{P}_\kappa := \frac{1}{|\kappa|!}\sum_{\sigma\in\Sigma(\kappa)}\mathbf{z}_{\kappa,\sigma}$ or, equivalently,

$$\mathcal{P}_\kappa(x_0 + ix_1) := x_0^{\kappa_2+\kappa_3}\sum_{\sigma\in\Sigma(\kappa)}(x_0 + ix_1)^{a(\kappa,\sigma)}(x_0 - ix_1)^{b(\kappa,\sigma)}\frac{L_{\kappa,\sigma}}{|\kappa|!}. \tag{41}$$

For convention, we set $\mathcal{P}_{(0,0,0)} := 1$. Note that, if $P_\kappa : \mathbb{H} \to \mathbb{H}$ is the standard κ^{th} Fueter polynomial in $\mathbb{H}$ (see [35, Definition 6.1]), then $\mathcal{P}_\kappa$ is the restriction of P_κ on $\mathbb{C}_i$, i.e. we have:

$$\mathcal{P}_\kappa = P_\kappa|_{\mathbb{C}_i}. \tag{42}$$

The next result follows immediately from definition (41).

Lemma 22. *Let $\kappa \in \mathbb{N}^3$ and let $\mathrm{k} := |\kappa|$. Then there exists, and is unique, a sequence $\{\ell_{\kappa,h}\}_{h=0}^{\mathrm{k}}$ of $(\mathrm{k}+1)$ octonions such that $\mathcal{P}_\kappa(x_0 + ix_1) = \sum_{h=0}^{\mathrm{k}} x_0^{\mathrm{k}-h}(ix_1)^h \ell_{\kappa,h}$ or, equivalently,*

$$\mathcal{P}_\kappa(\mathbf{x}) = \sum_{h=0}^{\mathrm{k}} \mathrm{Re}(\mathbf{x})^{\mathrm{k}-h}\mathrm{Im}(\mathbf{x})^h \ell_{\kappa,h}.$$

Note that, if $\lfloor r \rfloor$ denotes the integer part of $r \in \mathbb{R}$, then we can also write:

$$\mathcal{P}_\kappa(\mathbf{x}) = \sum_{h=0}^{\lfloor \frac{\mathrm{k}}{2} \rfloor} \mathrm{Re}(\mathbf{x})^{\mathrm{k}-2h}\mathrm{Im}(\mathbf{x})^{2h}\ell_{\kappa,2h} + \sum_{h=0}^{\lfloor \frac{\mathrm{k}-1}{2} \rfloor} \mathrm{Re}(\mathbf{x})^{\mathrm{k}-2h-1}\mathrm{Im}(\mathbf{x})^{2h+1}\ell_{\kappa,2h+1}. \tag{43}$$

For instance, the $\mathcal{P}_\kappa$'s with $|\kappa| \leq 3$ are $\mathcal{P}_{(0,0,0)} = 1$ and the following:

- $\mathcal{P}_{(0,0,1)} = \mathbf{z}_3 = -\mathrm{Re}(\mathbf{x})k$, $\mathcal{P}_{(0,1,0)} = \mathbf{z}_2 = -\mathrm{Re}(\mathbf{x})j$, $\mathcal{P}_{(1,0,0)} = \mathbf{z}_1 = -\mathbf{x}i$;
- $\mathcal{P}_{(0,0,2)} = \mathbf{z}_3^2 = -\mathrm{Re}(\mathbf{x})^2$, $\mathcal{P}_{(0,1,1)} = 0$, $\mathcal{P}_{(0,2,0)} = \mathbf{z}_2^2 = -\mathrm{Re}(\mathbf{x})^2 = \mathcal{P}_{(0,0,2)}$, $\mathcal{P}_{(1,0,1)} = \mathrm{Re}(\mathbf{z}_1)\mathbf{z}_3 = -\mathrm{Re}(\mathbf{x})\mathrm{Im}(\mathbf{x})j$, $\mathcal{P}_{(1,1,0)} = \mathrm{Re}(\mathbf{z}_1)\mathbf{z}_2 = \mathrm{Re}(\mathbf{x})\mathrm{Im}(\mathbf{x})k$, $\mathcal{P}_{(2,0,0)} = \mathbf{z}_1^2 = -\mathbf{x}^2$;
- $\mathcal{P}_{(0,0,3)} = \mathbf{z}_3^3 = \mathrm{Re}(\mathbf{x})^3k$, $\mathcal{P}_{(0,1,2)} = \frac{1}{3}\mathbf{z}_2\mathbf{z}_3^2 = \frac{1}{3}\mathrm{Re}(\mathbf{x})^3j$, $\mathcal{P}_{(0,2,1)} = \frac{1}{3}\mathbf{z}_2^2\mathbf{z}_3 = \frac{1}{3}\mathrm{Re}(\mathbf{x})^3k$, $\mathcal{P}_{(0,3,0)} = \mathbf{z}_2^3 = \mathrm{Re}(\mathbf{x})^3j$, $\mathcal{P}_{(1,0,2)} = \frac{1}{3}(2\mathbf{z}_1+\overline{\mathbf{z}_1})\mathbf{z}_3^2 = \frac{1}{3}\mathrm{Re}(\mathbf{x})^2(\mathrm{Re}(x)+3\mathrm{Im}(x))i$, $\mathcal{P}_{(1,1,1)} = 0$, $\mathcal{P}_{(1,2,0)} = \frac{1}{3}(2\mathbf{z}_1 + \overline{\mathbf{z}_1})\mathbf{z}_2^2 = \frac{1}{3}\mathrm{Re}(\mathbf{x})^2(\mathrm{Re}(x) + 3\mathrm{Im}(x))i = \mathcal{P}_{(1,0,2)}$, $\mathcal{P}_{(2,0,1)} = \frac{1}{3}(\mathbf{z}_1^2 + \overline{\mathbf{z}_1}^2 + |\mathbf{z}_1|^2)\mathbf{z}_3 = \frac{1}{3}\mathrm{Re}(\mathbf{x})(\mathrm{Re}(\mathbf{x})^2 + 3\mathrm{Im}(\mathbf{x})^2)k$, $\mathcal{P}_{(2,1,0)} = \frac{1}{3}(\mathbf{z}_1^2 + \overline{\mathbf{z}_1}^2 + |\mathbf{z}_1|^2)\mathbf{z}_2 = \frac{1}{3}\mathrm{Re}(\mathbf{x})(\mathrm{Re}(\mathbf{x})^2 + 3\mathrm{Im}(\mathbf{x})^2)j$, $\mathcal{P}_{(3,0,0)} = \mathbf{z}_1^3 = \mathbf{x}^3i$.

Note that the $\mathcal{P}_\kappa$'s are not $\mathbb{R}$-linearly independent; for instance, $\mathcal{P}_{(0,1,1)} = \mathcal{P}_{(1,1,1)} = 0$, $\mathcal{P}_{(0,2,0)} = \mathcal{P}_{(0,0,2)}$ and $\mathcal{P}_{(1,2,0)} = \mathcal{P}_{(1,0,2)}$. It would be important to solve the following problem:

Question 23. Let k be any natural number, $\mathbb{N}_{\mathrm{k}}^3 := \{\kappa \in \mathbb{N}^3 : |\kappa| = \mathrm{k}\}$ and let P_{k} be the real vector subspace of $\mathscr{C}^0(\mathbb{C}_i, \mathbb{O})$ generated by the set $\{\mathcal{P}_\kappa\}_{\kappa \in \mathbb{N}_{\mathrm{k}}^3}$. Find a subset B_{k} of $\mathbb{N}_{\mathrm{k}}^3$ and, for each $\lambda \in \mathbb{N}_{\mathrm{k}}^3 \setminus B_{\mathrm{k}}$, real coefficients $\{r_{\lambda,\kappa}\}_{\kappa \in B_{\mathrm{k}}}$ such that: $\{\mathcal{P}_\kappa\}_{\kappa \in B_{\mathrm{k}}}$ is a real vector basis of P_{k} and, for each $\lambda \in \mathbb{N}_{\mathrm{k}}^3 \setminus B_{\mathrm{k}}$, $\mathcal{P}_\lambda = \sum_{\kappa \in B_{\mathrm{k}}} r_{\lambda,\kappa}\mathcal{P}_\kappa$.

Thanks to the preceding list, the coefficients $\ell_{\kappa,h}$ for $|\kappa| \leq 2$ are as follows:

- $\ell_{(0,0,0),0} = 1$;
- $\ell_{(0,0,1),0} = -k$, $\ell_{(0,0,1),1} = 0$; $\ell_{(0,1,0),0} = -j$, $\ell_{(0,1,0),1} = 0$; $\ell_{(1,0,0),0} = -i$, $\ell_{(1,0,0),1} = -i$;
- $\ell_{\kappa,0} = -1$, $\ell_{\kappa,1} = \ell_{\kappa,2} = 0$ if $\kappa \in \{(0,0,2),(0,2,0)\}$; $\ell_{(0,1,1),0} = \ell_{(0,1,1),1} = \ell_{(0,1,1),2} = 0$; $\ell_{(1,0,1),0} = 0$, $\ell_{(1,0,1),1} = -j$, $\ell_{(1,0,1),2} = 0$; $\ell_{(1,1,0),0} = 0$, $\ell_{(1,1,0),1} = k$, $\ell_{(1,1,0),2} = 0$; $\ell_{(2,0,0),0} = -1$, $\ell_{(2,0,0),1} = -2$, $\ell_{(2,0,0),2} = -1$.

Fix $\mathbb{I} = (I, J) \in \mathcal{N}$. We indicate $\psi_\mathbb{I} : \mathbb{H} \to \mathbb{O}$ the real algebra embedding such that $\psi_\mathbb{I}(1) = 1$, $\psi_\mathbb{I}(i) = I$, $\psi_\mathbb{I}(j) = J$ and hence $\psi_\mathbb{I}(k) = IJ$. Given $\kappa \in \mathbb{N}^3$ and $h \in \{0, \ldots, |\kappa|\}$, we define $\ell_{\mathbb{I},\kappa,h} \in \mathbb{H}_\mathbb{I}$ by setting

$$\ell_{\mathbb{I},\kappa,h} := \psi_\mathbb{I}(\ell_{\kappa,h}). \tag{44}$$

For each $a, b \in \mathbb{O}$, we denote $\langle I; a, b \rangle$ the following octonion:

$$\langle I; a, b \rangle := -I((Ia)b). \tag{45}$$

Furthermore, if $\mathrm{k} := |\kappa|$, we define the function $\mathcal{P}_{\mathbb{I},\kappa} : \mathbb{O}^2 \to \mathbb{O}$ as follows:

$$\mathcal{P}_{\mathbb{I},\kappa}(x, c) := \sum_{h=0}^{\lfloor \frac{\mathrm{k}}{2} \rfloor} \mathrm{Re}(x)^{\mathrm{k}-2h}\mathrm{Im}(x)^{2h}(\ell_{\mathbb{I},\kappa,2h}c) + \sum_{h=0}^{\lfloor \frac{\mathrm{k}-1}{2} \rfloor} \mathrm{Re}(x)^{\mathrm{k}-2h-1}\mathrm{Im}(x)^{2h+1}\langle I; \ell_{\mathbb{I},\kappa,2h+1}, c \rangle. \tag{46}$$

Note that, for each $c \in \mathbb{O}$, the c-restriction $x \mapsto \mathcal{P}_{\mathbb{I},\kappa}(x, c)$ is a variant of the slice product between $x \mapsto \mathcal{P}_\kappa(x)$ and the constant c.

Let $f : \Omega_D \to \mathbb{O}$ be a slice function of class $\mathscr{C}^{|\kappa|}$. Recall that $f_{\mathbb{I}} : E \to \mathbb{O}$ is the function defined by $f_{\mathbb{I}}(v) := f(x_0 + x_1 I + x_2 J + x_3 IJ)$ if $v = (x_0, x_1, x_2, x_3) \in E$. Since $f_{\mathbb{I}}$ is of class $\mathscr{C}^{|\kappa|}$, we can define $\partial_{\mathbb{I},\kappa} f : \Omega_{\mathbb{I}} \to \mathbb{O}$ by setting

$$\partial_{\mathbb{I},\kappa} f(x) := \frac{\partial^{|\kappa|} f_{\mathbb{I}}}{\partial x_1^{\kappa_1} \partial x_2^{\kappa_2} \partial x_3^{\kappa_3}}(\psi_{\mathbb{I}}^{-1}(x)).$$

Let us introduce the slice Fueter counterpart of Taylor powers at the origin, associated to f.

Definition 24. Let $\mathrm{k} \in \mathbb{N}$. For each sequence $\mathcal{C} = \{c_\kappa\}_{\kappa \in \mathbb{N}^3}$ in $\mathbb{O}$, we define the *slice Fueter* $(\mathbb{I}, \mathrm{k})$-*power associated to* $\mathcal{C}$ as the slice function $\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{C}} : \mathbb{O} \to \mathbb{O}$ given by

$$\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{C}}(x) := \sum_{\kappa \in \mathbb{N}^3, |\kappa| = \mathrm{k}} \mathcal{P}_{\mathbb{I},\kappa}(x, c_\kappa).$$

In the case $c_\kappa = \frac{\partial_{\mathbb{I},\kappa} f(0)}{\kappa!}$ for each $\kappa \in \mathbb{N}^3$, we call $\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{C}}$ the *slice Fueter* $(\mathbb{I}, \mathrm{k})$-*power associated to f and centered at* 0, and we write $\mathcal{P}_{\mathbb{I},\mathrm{k};f,0}$ in place of $\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{C}}$. In other words, we set:

$$\mathcal{P}_{\mathbb{I},\mathrm{k};f,0}(x) := \sum_{\kappa \in \mathbb{N}^3, |\kappa| = \mathrm{k}} \frac{1}{\kappa!} \mathcal{P}_{\mathbb{I},\kappa}(x, \partial_{\mathbb{I},\kappa} f(0)) = \sum_{\kappa \in \mathbb{N}^3, |\kappa| = \mathrm{k}} \mathcal{P}_{\mathbb{I},\kappa}\left(x, \frac{\partial_{\mathbb{I},\kappa} f(0)}{\kappa!}\right) =$$

$$= \sum_{\kappa \in \mathbb{N}^3, |\kappa| = \mathrm{k}} \left(\sum_{h=0}^{\lfloor \frac{\mathrm{k}}{2} \rfloor} \mathrm{Re}(x)^{\mathrm{k}-2h} \mathrm{Im}(x)^{2h} \left(\ell_{\mathbb{I},\kappa,2h}\left(\frac{\partial_{\mathbb{I},\kappa} f(0)}{\kappa!}\right)\right) + \right.$$

$$\left. + \sum_{h=0}^{\lfloor \frac{\mathrm{k}-1}{2} \rfloor} \mathrm{Re}(x)^{\mathrm{k}-2h-1} \mathrm{Im}(x)^{2h+1} \left\langle I; \ell_{\mathbb{I},\kappa,2h+1}, \frac{\partial_{\mathbb{I},\kappa} f(0)}{\kappa!}\right\rangle \right). \blacksquare$$

Fix $r > 0$. We denote $B(r)$ the open ball of $\mathbb{O}$ centered at 0 with radius r.

Theorem 25. *Each slice Fueter-regular function $f : B(r) \to \mathbb{O}$ can be expanded as follows:*

$$f(x) = \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};f,0}(x) \quad \text{for each } x \in B(r),$$

where the series converges uniformly on each compact subset of $B(r)$.

Corollary 26. *Let $f : B(r) \to \mathbb{O}$ be a slice Fueter-regular function. Then, for each $\mathrm{k} \in \mathbb{N}$, the slice Fueter $(\mathbb{I}, \mathrm{k})$-power $\mathcal{P}_{\mathbb{I},\mathrm{k};f,0} : \mathbb{O} \to \mathbb{O}$ is slice Fueter-regular.*

Example 27. Let $a, b \in \mathbb{O}$ and let $f : \mathbb{O} \to \mathbb{O}$ be the slice Fueter-regular function we considered in (19) with $c = d = 0$, i.e.

$$f(x) = 3\mathrm{Re}(x)b + \mathrm{Im}(x)b + 3\mathrm{Re}(x)^2 a + 2\mathrm{Re}(x)\mathrm{Im}(x)a + \mathrm{Im}(x)^2 a.$$

Choose $\mathbb{I} = (i, j)$ in $\mathcal{N}$. By homogeneity's reasons, it is evident that $\mathcal{P}_{(\mathbb{I},0;f,0)} = 0$, $\mathcal{P}_{\mathbb{I},1;f,0}(x) = 3\mathrm{Re}(x)b + \mathrm{Im}(x)b$, $\mathcal{P}_{\mathbb{I},2;f,0}(x) = 3\mathrm{Re}(x)^2 a + 2\mathrm{Re}(x)\mathrm{Im}(x)a + \mathrm{Im}(x)^2 a$; moreover, by Corollary 26, these functions are slice Fueter-regular.

We would like to verify the latter assertions by direct computations. First, we note that

$$f_{\mathbb{I}}(x_0 + ix_1 + jx_2 + kx_3) = 3x_0 b + (3x_0^2 - x_1^2 - x_2^2 - x_3^2)a + (ix_1 + jx_2 + kx_3)(2x_0 a + b).$$

Hence

- $\frac{\partial_{\mathbb{I},(0,0,0)} f(0)}{(0,0,0)!} = f(0) = 0,$

- $\frac{\partial_{\mathbb{I},(0,0,1)} f(0)}{(0,0,1)!} = kb, \ \frac{\partial_{\mathbb{I},(0,1,0)} f(0)}{(0,1,0)!} = jb, \ \frac{\partial_{\mathbb{I},(1,0,0)} f(0)}{(1,0,0)!} = ib,$

- $\frac{\partial_{\mathbb{I},(0,0,2)} f(0)}{(0,0,2)!} = \frac{\partial_{\mathbb{I},(0,2,0)} f(0)}{(0,2,0)!} = \frac{\partial_{\mathbb{I},(2,0,0)} f(0)}{(2,0,0)!} = -a, \ \frac{\partial_{\mathbb{I},(0,1,1)} f(0)}{(0,1,1)!} = \frac{\partial_{\mathbb{I},(1,0,1)} f(0)}{(1,0,1)!} = \frac{\partial_{\mathbb{I},(1,1,0)} f(0)}{(1,1,0)!} = 0$

and, by Artin's theorem [46, Theorem 3.1],

- $\mathcal{P}_{\mathbb{I},(0,0,0)}\big(x, \frac{\partial_{\mathbb{I},(0,0,0)}f(0)}{(0,0,0)!}\big) = 0,$

- $\mathcal{P}_{\mathbb{I},(0,0,1)}\big(x, \frac{\partial_{\mathbb{I},(0,0,1)}f(0)}{(0,0,1)!}\big) = \mathrm{Re}(x)(-k)(kb) = \mathrm{Re}(x)b,$

  $\mathcal{P}_{\mathbb{I},(0,1,0)}\big(x, \frac{\partial_{\mathbb{I},(0,1,0)}f(0)}{(0,1,0)!}\big) = \mathrm{Re}(x)(-j)(jb) = \mathrm{Re}(x)b,$

  $\mathcal{P}_{\mathbb{I},(1,0,0)}\big(x, \frac{\partial_{\mathbb{I},(1,0,0)}f(0)}{(1,0,0)!}\big) = \mathrm{Re}(x)(-i)(ib) + \mathrm{Im}(x)\langle i; -i, ib\rangle = \mathrm{Re}(x)b + \mathrm{Im}(x)(-i((i(-i))(ib))) = \mathrm{Re}(x)b + \mathrm{Im}(x)b,$

- $\mathcal{P}_{\mathbb{I},(0,0,2)}\big(x, \frac{\partial_{\mathbb{I},(0,0,2)}f(0)}{(0,0,2)!}\big) = \mathcal{P}_{\mathbb{I},(0,2,0)}\big(x, \frac{\partial_{\mathbb{I},(0,2,0)}f(0)}{(0,2,0)!}\big) = \mathrm{Re}(x)^2(-1)(-a) = \mathrm{Re}(x)^2 a,$

  $\mathcal{P}_{\mathbb{I},(2,0,0)}\big(x, \frac{\partial_{\mathbb{I},(2,0,0)}f(0)}{(2,0,0)!}\big) = \mathrm{Re}(x)^2(-1)(-a) + \mathrm{Im}(x)^2(-1)(-a) + \mathrm{Re}(x)\mathrm{Im}(x)\langle i, -2; -a\rangle = \mathrm{Re}(x)^2 a + \mathrm{Im}(x)^2 a + 2\mathrm{Re}(x)\mathrm{Im}(x)a,$

  $\mathcal{P}_{\mathbb{I},(0,1,1)}\big(x, \frac{\partial_{\mathbb{I},(0,1,1)}f(0)}{(0,1,1)!}\big) = \mathcal{P}_{\mathbb{I},(1,0,1)}\big(x, \frac{\partial_{\mathbb{I},(1,0,1)}f(0)}{(1,0,1)!}\big) = \mathcal{P}_{\mathbb{I},(1,1,0)}\big(x, \frac{\partial_{\mathbb{I},(1,1,0)}f(0)}{(1,1,0)!}\big) = 0.$

As a consequence, we have: $\mathcal{P}_{\mathbb{I},0;f,0}(x) = 0$, $\mathcal{P}_{\mathbb{I},1;f,0}(x) = \sum_{\kappa\in\mathbb{N}^3, |\kappa|=1} \mathcal{P}_{\mathbb{I},\kappa}\big(x, \frac{\partial_{\mathbb{I},\kappa}f(0)}{\kappa!}\big) = 3\mathrm{Re}(x)b + \mathrm{Im}(x)b$ and $\mathcal{P}_{\mathbb{I},2;f,0}(x) = \sum_{\kappa\in\mathbb{N}^3, |\kappa|=2} \mathcal{P}_{\mathbb{I},\kappa}\big(x, \frac{\partial_{\mathbb{I},\kappa}f(0)}{\kappa!}\big) = 3\mathrm{Re}(x)^2 a + 2\mathrm{Re}(x)\mathrm{Im}(x)a + \mathrm{Im}(x)^2 a$, as claimed.

Finally, if (F_1, F_2) and (G_1, G_2) are the stem functions inducing $\mathcal{P}_{\mathbb{I},1;f,0}$ and $\mathcal{P}_{\mathbb{I},2;f,0}$ respectively, then we have: $F_1(x_0, x_1) = 3x_0 b$, $F_2(x_0, x_1) = x_1 b$, $G_1(x_0, x_1) = 3x_0^2 a - x_1^2 a$ and $G_2(x_0, x_1) = 2x_0 x_1 a$. Consequently, $\frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_1} = 2b = 2\frac{F_2}{x_1}$, $\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_0} = 0$, and $\frac{\partial G_1}{\partial x_0} - \frac{\partial G_2}{\partial x_1} = 4x_0 a = 2\frac{G_2}{x_1}$, $\frac{\partial G_1}{\partial x_1} + \frac{\partial G_2}{\partial x_0} = 0$. In other words, $\mathcal{P}_{\mathbb{I},1;f,0}$ and $\mathcal{P}_{\mathbb{I},2;f,0}$ are slice Fueter-regular, as claimed. ∎

Remark 28. Let $y \in \mathbb{R}$, let $B_y(r) := y + B(r)$, let $f : B_y(r) \to \mathbb{O}$ be a slice Fueter-regular function and let $\mathcal{C}_y = \{c_\kappa\}_{\kappa\in\mathbb{N}^3}$ be the sequence given by $c_\kappa := \frac{\partial_{\mathbb{I},\kappa}f(y)}{\kappa!}$. We define the *slice Fueter* $(\mathbb{I}, \mathrm{k})$-*power associated to f and centered at y* as the slice function $\mathcal{P}_{\mathbb{I},\mathrm{k};f,y} : \mathbb{O} \to \mathbb{O}$ defined by $\mathcal{P}_{\mathbb{I},\mathrm{k};f,y}(x) := \mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{C}_y}(x - y)$. In this situation, Theorem 25 and Corollary 26 can be restated in the same way by replacing $B(r)$ and $\mathcal{P}_{\mathbb{I},\mathrm{k};f,0}$ with $B_y(r)$ and $\mathcal{P}_{\mathbb{I},\mathrm{k};f,y}$, respectively. ∎

2.5. **Slice Laurent expansion.** Let $\mathcal{E} : \mathbb{H} \setminus \{0\} \to \mathbb{H}$ be the rational function defined by

$$\mathcal{E}(x) := \frac{1}{2\pi^2}\frac{\overline{x}}{|x|^4}.$$

Given any $\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \mathbb{N}^3$, we denote $Q_\kappa : \mathbb{H} \setminus \{0\} \to \mathbb{H}$ the ∂_κ-derivative $\partial_\kappa\mathcal{E}$ of $\mathcal{E}$, i.e.

$$Q_\kappa(x) := \partial_\kappa\mathcal{E}(x) = \frac{\partial^{|\kappa|}\mathcal{E}}{\partial x_1^{\kappa_1}\partial x_2^{\kappa_2}\partial x_3^{\kappa_3}}(x). \tag{47}$$

Next result is an immediate consequence of Proposition 7.27 of [35].

Lemma 29. *Let* $\mathrm{x} \in \mathbb{C}_i \setminus \{0\} \subset \mathbb{H}$, *let* $\kappa \in \mathbb{N}^3$ *and let* $\mathrm{k} := |\kappa|$. *Then there exists, and is unique, a sequence* $\{m_{\kappa,h}\}_{h=0}^{\mathrm{k}+1}$ *of* $(\mathrm{k} + 2)$ *octonions such that*

$$Q_\kappa(\mathrm{x}) = \frac{1}{|\mathrm{x}|^{4+2\mathrm{k}}}\sum_{h=0}^{\mathrm{k}+1} \mathrm{Re}(\mathrm{x})^{\mathrm{k}+1-h}\mathrm{Im}(\mathrm{x})^h m_{\kappa,h}$$

or, equivalently,

$$Q_\kappa(\mathrm{x}) = \frac{1}{|\mathrm{x}|^{4+2\mathrm{k}}}\sum_{h=0}^{\lfloor\frac{\mathrm{k}+1}{2}\rfloor} \mathrm{Re}(\mathrm{x})^{\mathrm{k}+1-2h}\mathrm{Im}(\mathrm{x})^{2h} m_{\kappa,2h} + \frac{1}{|\mathrm{x}|^{4+2\mathrm{k}}}\sum_{h=0}^{\lfloor\frac{\mathrm{k}}{2}\rfloor} \mathrm{Re}(\mathrm{x})^{\mathrm{k}-2h}\mathrm{Im}(\mathrm{x})^{2h+1} m_{\kappa,2h+1}.$$

Above *Question* 23 can be repeated replacing the $\mathcal{P}_\kappa$'s with the $\mathcal{Q}_\kappa$'s.

Fix $\mathbb{I} = (I, J) \in \mathcal{N}$. For $h \in \{0, \ldots, \mathrm{k} + 1\}$, let $m_{\mathbb{I},\kappa,h}$ be the octonion in $\mathbb{H}_\mathbb{I}$ given by

$$m_{\mathbb{I},\kappa,h} := \psi_{\mathbb{I}}(m_{\kappa,h}). \tag{48}$$

We define the function $\mathcal{Q}_{\mathbb{I},\kappa} : (\mathbb{O} \setminus \{0\}) \times \mathbb{O} \to \mathbb{O}$ by setting

$$\mathcal{Q}_{\mathbb{I},\kappa}(x,c) := \frac{1}{|x|^{4+2\mathrm{k}}} \sum_{h=0}^{\lfloor \frac{\mathrm{k}+1}{2} \rfloor} \mathrm{Re}(x)^{\mathrm{k}+1-2h} \mathrm{Im}(x)^{2h} (m_{\mathbb{I},\kappa,2h} c) + \tag{49}$$

$$+ \frac{1}{|x|^{4+2\mathrm{k}}} \sum_{h=0}^{\lfloor \frac{\mathrm{k}}{2} \rfloor} \mathrm{Re}(x)^{\mathrm{k}-2h} \mathrm{Im}(x)^{2h+1} \langle I; m_{\mathbb{I},\kappa,2h+1}, c \rangle.$$

Definition 30. Let $\mathrm{k} \in \mathbb{N}$. For each sequence $\mathcal{A} = \{a_\kappa\}_{\kappa \in \mathbb{N}^3}$ in $\mathbb{O}$, we define the *slice Fueter* $(\mathbb{I}, \mathrm{k})$-*reciprocal power associated to* $\mathcal{A}$ as the slice function $\mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{A}} : \mathbb{O} \setminus \{0\} \to \mathbb{O}$ given by

$$\mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x) := \sum_{\kappa \in \mathbb{N}^3, |\kappa| = \mathrm{k}} \mathcal{Q}_{\mathbb{I},\kappa}(x, a_\kappa). \quad \blacksquare$$

Fix $r_1, r_2, r \in \mathbb{R} \cup \{+\infty\}$ *with* $0 \le r_1 < r < r_2 \le +\infty$. We denote $B(r_1, r_2)$ the open spherical corona of $\mathbb{O}$ centered at 0 with radii r_1 and r_2, i.e. $B(r_1, r_2) = B(r_2) \setminus (\mathrm{cl}(B(r_1)) \cup \{0\})$. We define $S_{\mathbb{I}}(r) := \{x \in \mathbb{H}_{\mathbb{I}} : |x| = r\}$, $m_{\mathbb{I}} : S_{\mathbb{I}}(r) \to \mathbb{H}_{\mathbb{I}} \subset \mathbb{O}$ as the outer unit normal vector field of $S_{\mathbb{I}}(r)$ in $\mathbb{H}_{\mathbb{I}}$ and $d\sigma_{\mathbb{I}}(\xi)$ as the surface element of $S_{\mathbb{I}}(r)$. Recall that P_κ is the standard κ^{th} Fueter polynomial in $\mathbb{H}$ and $\psi_{\mathbb{I}} : \mathbb{H} \to \mathbb{O}$ is the algebra embedding such that $\psi_{\mathbb{I}}(i) = I$ and $\psi_{\mathbb{I}}(j) = J$. Define the functions $\mathcal{Q}^*_{\mathbb{I},\kappa}, \mathcal{P}^*_{\mathbb{I},\kappa} : S_{\mathbb{I}}(r) \to \mathbb{H}_{\mathbb{I}} \subset \mathbb{O}$ by setting $\mathcal{Q}^*_{\mathbb{I},\kappa}(x) := \psi_{\mathbb{I}}(\mathcal{Q}_\kappa(\psi_{\mathbb{I}}^{-1}(x)))$ and $\mathcal{P}^*_{\mathbb{I},\kappa}(x) := \psi_{\mathbb{I}}(\mathcal{P}_\kappa(\psi_{\mathbb{I}}^{-1}(x)))$ for each $x \in S_{\mathbb{I}}(r)$.

Theorem 31. *Let* $g : B(r_1, r_2) \to \mathbb{O}$ *be a slice Fueter-regular function. Then there exist sequences* $\mathcal{A} = \{a_\kappa\}_{\kappa \in \mathbb{N}^3}$ *and* $\mathcal{B} = \{b_\kappa\}_{\kappa \in \mathbb{N}^3}$ *in* $\mathbb{O}$ *such that* g *can be expanded as follows:*

$$g(x) = \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x) + \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}(x) \quad \text{for each } x \in B(r_1, r_2),$$

where both the series converge uniformly on each compact subset of $B(r_1, r_2)$, *and it holds*

$$a_\kappa = (-1)^{|\kappa|} \int_{S_{\mathbb{I}}(r)} \mathcal{Q}^*_{\mathbb{I},\kappa}(\xi)(m_{\mathbb{I}}(\xi)g(\xi)) \, d\sigma_{\mathbb{I}}(\xi),$$

$$b_\kappa = (-1)^{|\kappa|} \int_{S_{\mathbb{I}}(r)} \mathcal{P}^*_{\mathbb{I},\kappa}(\xi)(m_{\mathbb{I}}(\xi)g(\xi)) \, d\sigma_{\mathbb{I}}(\xi).$$

Corollary 32. *Let* $g : B(r_1, r_2) \to \mathbb{O}$, $\mathcal{A}$ *and* $\mathcal{B}$ *be as in the statement of the preceding theorem. Then, for each* $\mathrm{k} \in \mathbb{N}$, $\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}$ *and* $\mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}$ *are slice Fueter-regular functions on* $B(r_1, r_2)$.

Remark 33. Let $y \in \mathbb{R}$, let $B_y(r_1, r_2) := y + B(r_1, r_2)$ and let $g : B_y(r_1, r_2) \to \mathbb{O}$ be a slice Fueter-regular function. Define the sequences $\mathcal{A} = \{a_\kappa\}_{\kappa \in \mathbb{N}^3}$ and $\mathcal{B} = \{b_\kappa\}_{\kappa \in \mathbb{N}^3}$ in $\mathbb{O}$ by

$$a_\kappa = (-1)^{|\kappa|} \int_{S_{\mathbb{I}}(r)} \mathcal{Q}^*_{\mathbb{I},\kappa}(\xi)(m_{\mathbb{I}}(\xi)g(\xi + y)) \, d\sigma_{\mathbb{I}}(\xi),$$

$$b_\kappa = (-1)^{|\kappa|} \int_{S_{\mathbb{I}}(r)} \mathcal{P}^*_{\mathbb{I},\kappa}(\xi)(m_{\mathbb{I}}(\xi)g(\xi + y)) \, d\sigma_{\mathbb{I}}(\xi).$$

Then the functions $\mathbb{O} \setminus \{y\} \to \mathbb{O}$, $x \mapsto \mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x-y)$ and $x \mapsto \mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}(x-y)$ are slice Fueter-regular, the series $\sum_{\mathrm{k} \in \mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x-y)$ and $\sum_{\kappa \in \mathbb{N}^3} \mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}(x-y)$ converges uniformly on compact subsets of $B_y(r_1, r_2)$, and the sum of the series $\sum_{\mathrm{k} \in \mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x-y) + \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}(x-y)$ is equal to $g(x)$ for each $x \in B_y(r_1, r_2)$. $\quad \blacksquare$

Let $y \in \Omega_D$. In paper [30], generalizing a quaternionic pioneering idea of C. Stoppato [49], it is proven the existence of a family $\{\mathscr{S}_{y,k} : \mathbb{O} \to \mathbb{O}\}_{k \in \mathbb{N}}$ of slice regular polynomial functions with the following property: each slice regular function $f : \Omega_D \to \mathbb{O}$ can be expanded as $f(x) = \sum_{k \in \mathbb{N}} \mathscr{S}_{y,k}(x) \cdot a_k$ on certain open circular neighborhoods U of $\mathbb{S}_y$ in $\mathbb{O}$, where $\{a_k\}_{k \in \mathbb{N}}$ is a sequence in $\mathbb{O}$ uniquely determined by f, and the series converges uniformly on compact subsets of U. In addition, there exists another family $\{\mathscr{S}_{y,k} : \mathbb{O} \setminus \mathbb{S}_y \to \mathbb{O}\}_{k < 0}$ of slice regular rational functions such that, given any slice regular function $g : \Omega_D \setminus \mathbb{S}_y \to \mathbb{O}$, g can be expanded as $g(x) = \sum_{k \in \mathbb{Z}} \mathscr{S}_{y,k}(x) \cdot b_k$ on $U \setminus \mathbb{S}_y$, where the sequence $\{b_k\}_{k \in \mathbb{N}}$ in $\mathbb{O}$ is unique and the

series converges uniformly on compact subsets of $U \setminus \mathbb{S}_y$. It is worth recalling that, if $y \in \mathbb{R}$, then $\mathbb{S}_y = \{y\}$, $\mathscr{S}_{y,k}$ is the usual k^{th}-power centered at y, i.e. $\mathscr{S}_{y,k}(x) = (x - y)^k$, and the neighborhoods U are standard open balls of $\mathbb{O}$ centered at y.

Our Theorems 25 and 31 stated above are slice Fueter-regular versions of the just mentioned results of [30] in the case $y = 0$, or better $y \in \mathbb{R}$, see Remarks 28 and 33.

Let y be any point of Ω_D, and let $f : \Omega_D \to \mathbb{O}$ and $g : \Omega_D \setminus \mathbb{S}_y \to \mathbb{O}$ be slice Fueter-regular functions.

Question 34. Is it possible to generalize Theorems 25 and 31 to the case in which $y \notin \mathbb{R}$? More precisely, given any $y \in \Omega_D \setminus \mathbb{R}$, do there exist slice Fueter-regular functions $\mathcal{P}_{\mathbb{I},\mathrm{k};f,y} : \mathbb{O} \to \mathbb{O}$, $\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}} : \mathbb{O} \to \mathbb{O}$ and $\mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}} : \mathbb{O} \setminus \mathbb{S}_y \to \mathbb{O}$ for each $\mathrm{k} \in \mathbb{N}$ such that the $\mathcal{P}_{\mathbb{I},\mathrm{k};f,y}(x)$'s and the $\mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x)$'s are polynomial functions in the eight real components $(x_0, \ldots, x_7)$ of $x \in \mathbb{O}$ of degree k, the $\mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}(x)$'s are rational functions in $(x_0, \ldots, x_7)$ of degree $-\mathrm{k}$, and these functions satisfy the following two assertions?

Assertion 35. *There exists an open circular neighborhood U of $\mathbb{S}_y$ in $\mathbb{O}$ such that*
$$f(x) = \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};f,y}(x) \quad \textit{for each } x \in U,$$
where the series converges uniformly on each compact subset of U.

Assertion 36. *There exists an open circular neighborhood U of $\mathbb{S}_y$ in $\mathbb{O}$ such that*
$$g(x) = \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};\mathcal{A}}(x) + \sum_{\mathrm{k} \in \mathbb{N}} \mathcal{Q}_{\mathbb{I},\mathrm{k};\mathcal{B}}(x) \quad \textit{for each } x \in U \setminus \mathbb{S}_y,$$
where both the series converge uniformly on each compact subset of $U \setminus \mathbb{S}_y$.

2.6. **Maximum Modulus Principle.** Slice Fueter-regular functions enjoy the following version of Maximum Modulus Principle. Recall that, given $f \in \mathcal{S}(\Omega_D, \mathbb{O})$, $V(f_s')$ denotes the zero set of the spherical derivative f_s' of f, i.e. $V(f_s') = \{x \in \Omega_D \setminus \mathbb{R} : f_s'(x) = 0\}$.

Theorem 37. *Let $f : \Omega_D \to \mathbb{O}$ be a slice Fueter-regular function. Suppose that f satisfies one of the following conditions:*

   (1) *$|f|$ has a maximum at a point of Ω_D.*
   (2) *$|f|$ has a local maximum at a point of $\Omega_D \setminus V(f_s')$.*

Then f is constant.

It would be very interesting to know the answer to the following problem:

Question 38. Does there exist a non-constant slice Fueter-regular function $f : \Omega_D \to \mathbb{O}$ such that $|f|$ has a local maximum at a point of $V(f_s')$?

3. THE PROOFS

Proofs of Section 2.1 and proofs of assertions (4) and (5).

Proof of Theorem 1. Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ be a $O(3)$-stem function, and let $(F_1, F_2) : D \to \mathbb{O}^2$ be the map given by $F_1(x_0, x_1) := \mathcal{F}_0(x_0, x_1, 0, 0)$ and $F_2(x_0, x_1) := \mathcal{F}_1(x_0, x_1, 0, 0)$. Consider the matrices $R_1.R_2, R_3$ in $O(3)$ such that
$$R_1(x_0, x_1, x_2, x_3) = (x_0, -x_1, x_2, x_3),$$
$$R_2(x_0, x_1, x_2, x_3) = (x_0, x_1, -x_2, x_3),$$
$$R_3(x_0, x_1, x_2, x_3) = (x_0, x_1, x_2, -x_3).$$

Since $\mathcal{F}(R_h v) = R_h \mathcal{F}(v)$ for each $v \in \Omega_D$ and $h \in \{1, 2, 3\}$, if we set $v = (x_0, x_1, 0, 0) \in D$, we obtain that (F_1, F_2) is a stem function and $\mathcal{F}_2(x_1, x_2, 0, 0) = \mathcal{F}_3(x_1, x_2, 0, 0) = 0$.

Let x be any point of Ω_D. Write $x = x_0 + x_1 I \in \Omega_D$ for some $x_0, x_1 \in \mathbb{R}$ and $I \in \mathbb{S}$. Define $v := (x_0, x_1, 0, 0) \in E$ and choose $J \in \mathbb{S}$ such that $I \perp J$, so $x = x_0 + x_1 I + 0 J + 0(IJ) \in \Theta_E = \Omega_D$. Since $\mathcal{F}_2(v) = \mathcal{F}_3(v) = 0$, it follows that

$$((\mathcal{I} \circ \Phi)(\mathcal{F}))(x) = (\mathcal{I}(\Phi(\mathcal{F})))(x) = \mathcal{F}_0(v) + I\mathcal{F}_1(v) =$$
$$= \mathcal{F}_0(v) + I\mathcal{F}_1(v) + J\mathcal{F}_2(v) + (IJ)\mathcal{F}_3(v) =$$
$$= \mathcal{I}_{O(3)}(\mathcal{F})(x).$$

In particular, this proves that $\mathcal{S}_{O(3)}(\Omega_D, \mathbb{O}) \subset \mathcal{S}(\Omega_D, \mathbb{O})$.

Consider now $F = (F_1, F_2) \in \mathrm{Stem}(D, \mathbb{O}^2)$ and define the map $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ by equalities (10) if $x \in E \setminus \mathbb{R}$, and $\mathcal{F}(x_0, 0, 0, 0) := (F_1(x_0, 0), 0, 0, 0)$ for each $x_0 \in E \cap \mathbb{R} = D \cap \mathbb{R}$.

We have to show that $\mathcal{F}$ is a $O(3)$-stem function; note that, if this is true, then $\mathcal{I}_{O(3)}(\mathcal{F})$ makes sense and it is equal to $\mathcal{I}(F)$, because $F = \Phi(\mathcal{F})$. To show that $\mathcal{F}$ is a $O(3)$-stem function, it suffices to prove that $\mathcal{F}(Bw) = B\mathcal{F}(w)$ for each $B \in O(3)$ and for each $w = (w_0, w_1, 0, 0) \in E$ with $w_1 > 0$. Indeed, given any $v \in E \setminus \mathbb{R}$, there exist $C \in O(3)$ and $w = (w_0, w_1, 0, 0) \in E$ with $w_1 > 0$ such that $v = Cw$; hence, given any $A \in O(3)$, it holds:

$$\mathcal{F}(Av) = \mathcal{F}(A(Cw)) = \mathcal{F}((AC)w) = (AC)\mathcal{F}(w) = A(C\mathcal{F}(w)) = A\mathcal{F}(Cw) = A\mathcal{F}(v).$$

Consider $w = (w_0, w_1, 0, 0) \in E$ with $w_1 \geq 0$, and let $w' := (w_1, 0, 0) \in \mathbb{R}^3$, so $w = (w_0, w')$. Choose arbitrarily $B \in O(3)$ and denote by B^* the orthogonal 3×3 real matrix such that

$$B = \left(\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{0} & B^* \end{array} \right),$$

where $\mathbf{0}$ denotes both a null 3-entries row and a null 3-entries column. Note that $Bw = (w_0, B^*w')$ and $|B^*w'| = |w'| = w_1$. Indicate $B_1^* \in \mathbb{R}^3$ the first column of B^*.

If $w_1 = 0$, then $w \in \mathbb{R}$, $Bw = (w_0, 0, 0, 0) = w$, and hence $\mathcal{F}(Bw) = \mathcal{F}(w) = (F_1(w_0, 0), 0, 0, 0) = B\mathcal{F}(w)$, as desired.

Now suppose that $w_1 > 0$. For each $v = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4$, we denote $\mathrm{Diag}(v)$ the 4×4 diagonal matrix whose diagonal entries are x_0, x_1, x_2 and x_3. Furthermore, if $v' := (x_1, x_2, x_3)$ and hence $v = (x_0, v') \in E$, we define $\mathrm{Y}(v) \in \mathbb{O}^4$ by setting

$$\mathrm{Y}(v) = \mathrm{Y}(x_0, v') := (F_1(x_0, |v'|), F_2(x_0, |v'|), F_2(x_0, |v'|), F_2(x_0, |v'|)).$$

Thanks to (10), we know that

$$\mathcal{F}(w) = \mathrm{Diag}((1, w'/|w'|))\mathrm{Y}(w) = \mathrm{Diag}((1, 1, 0, 0))\mathrm{Y}(w) =$$
$$= (F_1(w_0, w_1), F_2(w_0, w_1), 0, 0),$$
$$\mathcal{F}(Bw) = \mathcal{F}((w_0, B^*w')) = \mathrm{Diag}((1, B^*w'/|B^*w'|))\mathrm{Y}(w_0, B^*w') =$$
$$= \mathrm{Diag}((1, B^*w'/w_1))\mathrm{Y}(w_0, w') = \mathrm{Diag}((1, B_1^*))\mathrm{Y}(w_0, w') =$$
$$= (F_1(w_0, w_1), B_1^* F_2(w_0, w_1)).$$

It follows that

$$B\mathcal{F}(w) = (F_1(w_0, w_1), B_1^* F_2(w_0, w_1)) = \mathcal{F}(Bw),$$

as desired. In particular, we have: $\mathcal{S}(\Omega_D, \mathbb{O}) \subset \mathcal{S}_{O(3)}(\Omega_D, \mathbb{O})$, so $\mathcal{S}(\Omega_D, \mathbb{O}) = \mathcal{S}_{O(3)}(\Omega_D, \mathbb{O})$.

Finally, we have to show that $F = (F_1, F_2)$ is real analytic if and only if $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ is. If $\mathcal{F}$ is real analytic, then by (8) it is evident that F is real analytic as well. Suppose that F is real analytic. Let $y \in E$. If $y \notin \mathbb{R}$, then $\mathcal{F}$ is real analytic locally at y in E, because the function $E \to \mathbb{R}$, $v = (x_0, x_1, x_2, x_3) \mapsto r(v) = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ is. Let $y \in \mathbb{R}$. By [52], there exist an open neighborhood U of y in D, an open neighborhood V of y in $\mathbb{R}^2$ and real analytic functions $G_1, G_2 : V \to \mathbb{O}$ such that, for each $(x_0, x_1) \in U$, we have: $(x_0, x_1^2) \in V$, $F_1(x_0, x_1) = G_1(x_0, x_1^2)$ and $F_2(x_0, x_1) = x_1 G_2(x_0, x_1^2)$. Then the functions $\mathcal{F}_0(v) = G_1(x_0, r(v)^2)$ and $\mathcal{F}_h(v) = x_h G_2(x_0, r(v)^2)$ for $h \in \{1, 2, 3\}$ are real analytic locally at y in E, because $v \mapsto r(v)^2$ is. $\square$

Proof of Theorem 2. Let $f : \Omega_D \to \mathbb{O}$ be a slice Fueter-regular function. By Theorem 1, f is a slice function. Let (F_1, F_2) be the stem function inducing f. Choose $\mathbb{I} = (I, J) \in \mathcal{N}$. Thanks to Lemma 4.6 of [13] (or equation (6.6) of the same paper), we know that the restriction of f to $\Omega_\mathbb{I} = \Omega_D \cap \mathbb{H}_\mathbb{I}$ is real analytic. In particular, this is true for the restriction of f to $\Omega_D \cap \mathbb{C}_I$. Since $F_1(x_0, x_1) = \frac{1}{2}(f(x_0 + x_1 I) + f(x_0 - x_1 I))$ and $F_2(x_0, x_1) = -\frac{1}{2}I(f(x_0 + x_1 I) - f(x_0 - x_1 I))$ for each $(x_0, x_1) \in D$, it turns out that F_1 and F_2 are real analytic as well. Now Proposition 7(3) of [25] implies that f is real analytic on the whole Ω_D. $\qquad\square$

Proof of Theorem 3. (1) $\implies$ (2). Suppose that $f = \mathcal{I}(F) : \Omega_D \to \mathbb{O}$ is a slice Fueter-regular function. Let $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) : E \to \mathbb{O}^4$ be the $O(3)$-stem function given by $\mathcal{F} = \Phi^{-1}(F)$. By Theorem 2, we know that F is real analytic; in particular, F_1 is of class $\mathscr{C}^2$ and F_2 of class $\mathscr{C}^3$. Moreover, if $v = (x_0, x_1, x_3, x_4) \in E \setminus \mathbb{R}$ and $r = (x_1^2 + x_2^2 + x_3)^{1/2}$, then equations (10) hold. As a consequence, differentiating such equations, we have:

$$\frac{\partial \mathcal{F}_0}{\partial x_0}(v) = \frac{\partial F_1}{\partial x_0}(x_0, r),$$

$$\frac{\partial \mathcal{F}_0}{\partial x_j}(v) = \frac{x_j}{r}\frac{\partial F_1}{\partial x_1}(x_0, r) \quad \text{for each } j \in \{1, 2, 3\},$$

$$\frac{\partial \mathcal{F}_h}{\partial x_0}(v) = \frac{x_h}{r}\frac{\partial F_2}{\partial x_0}(x_0, r) \quad \text{for each } h \in \{1, 2, 3\}, \tag{50}$$

$$\frac{\partial \mathcal{F}_h}{\partial x_j}(v) = \frac{\delta_{hj}r^2 - x_h x_j}{r^3} F_2(x_0, r) + \frac{x_h x_j}{r^2}\frac{\partial F_2}{\partial x_1}(x_0, r) \quad \text{for each } h, j \in \{1, 2, 3\},$$

where $\delta_{hj} := 1$ if $h = j$ and $\delta_{hj} := 0$ if $h \neq j$. Note that $\frac{\partial \mathcal{F}_j}{\partial x_h} = \frac{\partial \mathcal{F}_h}{\partial x_j}$ for each $h, j \in \{1, 2, 3\}$. We have:

$$\frac{\partial \mathcal{F}_0}{\partial x_0}(v) - \frac{\partial \mathcal{F}_1}{\partial x_1}(v) - \frac{\partial \mathcal{F}_2}{\partial x_2}(v) - \frac{\partial \mathcal{F}_3}{\partial x_3}(v) = \frac{\partial F_1}{\partial x_0}(x_0, r) - \frac{\partial F_2}{\partial x_1}(x_0, r) - \frac{2}{r} F_2(x_0, r),$$

$$\frac{\partial \mathcal{F}_0}{\partial x_1}(v) + \frac{\partial \mathcal{F}_1}{\partial x_0}(v) - \frac{\partial \mathcal{F}_2}{\partial x_3}(v) + \frac{\partial \mathcal{F}_3}{\partial x_2}(v) = \frac{x_1}{r}\left(\frac{\partial F_1}{\partial x_1}(x_0, r) + \frac{\partial F_2}{\partial x_0}(x_1, r)\right),$$

$$\frac{\partial \mathcal{F}_0}{\partial x_2}(v) + \frac{\partial \mathcal{F}_1}{\partial x_3}(v) + \frac{\partial \mathcal{F}_2}{\partial x_0}(v) - \frac{\partial \mathcal{F}_3}{\partial x_1}(v) = \frac{x_2}{r}\left(\frac{\partial F_1}{\partial x_1}(x_0, r) + \frac{\partial F_2}{\partial x_0}(x_1, r)\right), \tag{51}$$

$$\frac{\partial \mathcal{F}_0}{\partial x_3}(v) - \frac{\partial \mathcal{F}_1}{\partial x_2}(v) + \frac{\partial \mathcal{F}_2}{\partial x_1}(v) + \frac{\partial \mathcal{F}_3}{\partial x_0}(v) = \frac{x_3}{r}\left(\frac{\partial F_1}{\partial x_1}(x_0, r) + \frac{\partial F_2}{\partial x_0}(x_1, r)\right).$$

It follows at once that systems (3) and (12) are equivalent on $E \setminus \{x_1 x_2 x_3 = 0\}$. By density, F_1 and F_2 are solutions of system (12) on the whole $D \setminus \mathbb{R}$.

(2) $\implies$ (1). Suppose that F_2 is of class $\mathscr{C}^1$, F_2 is of class $\mathscr{C}^3$, and F_1 and F_2 are solutions of system (12) on $D \setminus \mathbb{R}$. By [52] and (10), $\mathcal{F} = \Phi^{-1}(F)$ turns out to be of class $\mathscr{C}^1$. Since systems (3) and (12) are equivalent on $E \setminus \{x_1 x_2 x_3 = 0\}$, by density, $\mathcal{F}$ is a solution of system (3) on the whole E, i.e. f is slice Fueter-regular.

(2) $\iff$ (2a). Since the slice functions $\overline{\partial} f = \mathcal{I}\left(\frac{1}{2}\left(\frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_1}, \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_0}\right)\right)$ and $f'_s = \mathcal{I}\left(\left(\frac{F_2}{x_1}, 0\right)\right)$ are equal if and only if their inducing stem functions $\frac{1}{2}\left(\frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_1}, \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_0}\right)$ and $\left(\frac{F_2}{x_1}, 0\right)$ are, the equivalence between points (2) and (2a) is evident.

(2) $\iff$ (2b). Suppose that (2) holds. Then (1) holds as well, and hence F_1 and F_2 are real analytic. In particular, [52] implies the existence of a real analytic function $G_2 : D \to \mathbb{O}$ such that $F_2(x_0, x_1) = x_1 G_2(x_0, x_1)$ for each $(x_0, x_1) \in D$. Since $\frac{\partial F_2}{\partial x_0} = x_1 \frac{\partial G_2}{\partial x_0}$ and $\frac{\partial F_2}{\partial x_1} = G_2 + x_1 \frac{\partial G_2}{\partial x_1}$, equations (13) are satisfied on $D \setminus \mathbb{R}$, and hence on the whole D by density. This proves implication (2) $\implies$ (2b). The inverse implication is an immediate consequence of the last two equalities; indeed, we have: $\frac{\partial G_2}{\partial x_0} = \frac{1}{x_1}\frac{\partial F_2}{\partial x_0}$ and $\frac{\partial G_2}{\partial x_1} = -\frac{1}{x_1^2}F_2 + \frac{1}{x_1}\frac{\partial F_2}{\partial x_1}$ on $D \setminus \mathbb{R}$.

(2) $\Longleftrightarrow$ (2c). This equivalence can be proven following the same argument used to prove the preceding one. If (2) holds, then F_1 and F_2 are real analytic. Hence, by [52], there exist an open neighborhood D^{**} of D^* in $\mathbb{R}^2$ and real analytic functions $H_1, H_2 : D^{**} \to \mathbb{O}$ such that $F_1(x_0, x_1) = H_1(x_0, x_1^2)$, $F_2(x_0, x_1) = x_1 H_2(x_0, x_1^2)$ for each $(x_0, x_1) \in D$. As a consequence, we have: $\frac{\partial F_1}{\partial x_0}(x_0, x_1) = \frac{\partial H_1}{\partial x_0}(x_0, x_1^2)$, $\frac{\partial F_1}{\partial x_1}(x_0, x_1) = 2x_1 \frac{\partial H_1}{\partial x_1}(x_0, x_1^2)$, $\frac{\partial F_2}{\partial x_0}(x_0, x_1) = x_1 \frac{\partial H_2}{\partial x_0}(x_0, x_1^2)$ and $\frac{\partial F_2}{\partial x_1}(x_0, x_1) = H_2(x_0, x_1^2) + 2x_1^2 \frac{\partial H_2}{\partial x_1}(x_0, x_1^2)$. It follows that H_1 and H_2 satisfy (14) on $D^* \setminus \mathbb{R}$, and by density on the whole D^*. This proves (2c). Finally, if (2c) is satisfied, the latter four equalities imply at once that F_1 and F_2 satisfy (12). $\qquad\square$

Proofs of assertions (4) *and* (5). Let $f : \Omega_D \to \mathbb{O}$ be a $O(3)$-slice function induced by a $O(3)$-stem function $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ of class $\mathscr{C}^1$ and let $(F_1, F_2) := \Phi^{-1}(\mathcal{F})$. Choose $\mathbb{I} = (I, J) \in \mathcal{N}$ and $v = (x_0, x_1, x_2, x_3) \in E$. Define $K := IJ$, $x := x_0 + x_1 I + x_2 J + x_3 K \in \Omega_{\mathbb{I}} \setminus \mathbb{R}$, $r := (x_1^2 + x_2^2 + x_3^2)^{1/2} > 0$, $F_2 := F_2(x_0, r)$, $a := \frac{\partial F_1}{\partial x_0}(x_0, r)$, $b := \frac{\partial F_1}{\partial x_1}(x_0, r)$, $c := \frac{\partial F_2}{\partial x_0}(x_0, r)$, $d := \frac{\partial F_2}{\partial x_1}(x_0, r)$ and $e := d - \frac{F_2(x_0, r)}{r}$.

Thanks to (50), it holds:

$$
\overline{\mathcal{D}}_{\mathbb{I}} f(x) = \left(\frac{\partial}{\partial x_0} + I \frac{\partial}{\partial x_1} + J \frac{\partial}{\partial x_2} + K \frac{\partial}{\partial x_3} \right) (\mathcal{F}_0 + I\mathcal{F}_1 + J\mathcal{F}_2 + K\mathcal{F}_3)(v) =
$$
$$
= \left(\frac{\partial \mathcal{F}_0}{\partial x_0} + I \frac{\partial \mathcal{F}_1}{\partial x_0} + J \frac{\partial \mathcal{F}_2}{\partial x_0} + K \frac{\partial \mathcal{F}_3}{\partial x_0} \right)(v) +
$$
$$
+ I \left(\left(\frac{\partial \mathcal{F}_0}{\partial x_1} + I \frac{\partial \mathcal{F}_1}{\partial x_1} + J \frac{\partial \mathcal{F}_2}{\partial x_1} + K \frac{\partial \mathcal{F}_3}{\partial x_1} \right)(v) \right) +
$$
$$
+ J \left(\left(\frac{\partial \mathcal{F}_0}{\partial x_2} + I \frac{\partial \mathcal{F}_1}{\partial x_2} + J \frac{\partial \mathcal{F}_2}{\partial x_2} + K \frac{\partial \mathcal{F}_3}{\partial x_2} \right)(v) \right) +
$$
$$
+ K \left(\left(\frac{\partial \mathcal{F}_0}{\partial x_3} + I \frac{\partial \mathcal{F}_1}{\partial x_3} + J \frac{\partial \mathcal{F}_2}{\partial x_3} + K \frac{\partial \mathcal{F}_3}{\partial x_3} \right)(v) \right) =
$$
$$
= a + \frac{x_1 I}{r} c + \frac{x_2 J}{r} c + \frac{x_3 K}{r} c +
$$
$$
+ I \left(\frac{x_1}{r} b + I \left(\frac{r^2 - x_1^2}{r^3} F_2 + \frac{x_1^2}{r^2} d \right) + \frac{x_1 x_2}{r^2} Je + \frac{x_1 x_3}{r^2} Ke \right) +
$$
$$
+ J \left(\frac{x_2}{r} b + \frac{x_1 x_2}{r^2} Ie + J \left(\frac{r^2 - x_2^2}{r^3} F_2 + \frac{x_2^2}{r^2} d \right) + \frac{x_2 x_3}{r^2} Ke \right) +
$$
$$
+ K \left(\frac{x_3}{r} b + \frac{x_1 x_3}{r^2} Ie + \frac{x_2 x_3}{r^2} Je + K \left(\frac{r^2 - x_3^2}{r^3} F_2 + \frac{x_3^2}{r^2} d \right) \right) =
$$
$$
= a + \frac{\operatorname{Im}(x)}{r}(c + b) - \left(\sum_{h=1}^{3} \frac{r^2 - x_h^2}{r^3} \right) F_2 - \left(\sum_{h=1}^{3} \frac{x_h^2}{r^2} \right) d +
$$
$$
+ \frac{x_1 x_2}{r^2}(IJ + JI)e + \frac{x_1 x_3}{r^2}(IK + KI)e + \frac{x_2 x_3}{r^2}(JK + KJ)e =
$$
$$
= \left(a - d - \frac{2}{r} F_2 \right) + \frac{\operatorname{Im}(x)}{r}(c + b).
$$

The preceding chain of equalities proves that

$$
\overline{\mathcal{D}}_{\mathbb{I}} f(x) = \left(\frac{\partial F_1}{\partial x_0}(x_0, r) - \frac{\partial F_2}{\partial x_1}(x_0, r) - \frac{2}{r} F_2(x_0, r) \right) + \frac{\operatorname{Im}(x)}{r} \left(\frac{\partial F_1}{\partial x_1}(x_0, r) + \frac{\partial F_2}{\partial x_0}(x_0, r) \right) \quad (52)
$$

for each $x \in \Omega_{\mathbb{I}} \setminus \mathbb{R}$, where $x_0 = \operatorname{Re}(x)$ and $r = |\operatorname{Im}(x)|$.

By Theorem 3, if f is slice Fueter-regular, then $a - d - \frac{2}{r} F_2 = 0$ and $c + b = 0$. This proves that $\overline{\mathcal{D}}_{\mathbb{I}} f = 0$ on $\Omega_{\mathbb{I}}$ for each $\mathbb{I} \in \mathcal{N}$. Conversely, if $\overline{\mathcal{D}}_{\mathbb{I}} f = 0$ on $\Omega_{\mathbb{I}}$ for at least one element $\mathbb{I} = (I, J)$ of $\mathcal{N}$, then (52) ensures that $0 = \overline{\mathcal{D}}_{\mathbb{I}} f(x) = \left(a - d - \frac{2}{r} F_2 \right) + \frac{\operatorname{Im}(x)}{|\operatorname{Im}(x)|}(c + b)$ for each

$x \in \Omega_{\mathbb{I}} \setminus \mathbb{R}$. Since $-x \in \Omega_{\mathbb{I}} \setminus \mathbb{R}$ for each $x \in \Omega_{\mathbb{I}} \setminus \mathbb{R}$, we deduce that $a - d - \frac{2}{r}F_2 = c + b = 0$. Using (51), it follows that (3) holds on $E \setminus \mathbb{R}$, and hence on the whole E by density.

The proof is complete. $\qquad\square$

Remark 39. Let $f : \Omega_D \to \mathbb{O}$ be a $O(3)$-slice function induced by a $O(3)$-stem function $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ of class $\mathscr{C}^1$. Choose $\mathbb{I} = (I, J) \in \mathcal{N}$. Combining equality (52) with equalities (51), we deduce:

$$
\begin{aligned}
\overline{\mathcal{D}}_{\mathbb{I}} f(x) = {} & \left(\tfrac{\partial \mathcal{F}_0}{\partial x_0}(v) - \tfrac{\partial \mathcal{F}_1}{\partial x_1}(v) - \tfrac{\partial \mathcal{F}_2}{\partial x_2}(v) - \tfrac{\partial \mathcal{F}_3}{\partial x_3}(v) \right) + \\
& + I \left(\tfrac{\partial \mathcal{F}_0}{\partial x_1}(v) + \tfrac{\partial \mathcal{F}_1}{\partial x_0}(v) - \tfrac{\partial \mathcal{F}_2}{\partial x_3}(v) + \tfrac{\partial \mathcal{F}_3}{\partial x_2}(v) \right) + \\
& + J \left(\tfrac{\partial \mathcal{F}_0}{\partial x_2}(v) + \tfrac{\partial \mathcal{F}_1}{\partial x_3}(v) + \tfrac{\partial \mathcal{F}_2}{\partial x_0}(v) - \tfrac{\partial \mathcal{F}_3}{\partial x_1}(v) \right) + \\
& + (IJ) \left(\tfrac{\partial \mathcal{F}_0}{\partial x_3}(v) - \tfrac{\partial \mathcal{F}_1}{\partial x_2}(v) + \tfrac{\partial \mathcal{F}_2}{\partial x_1}(v) + \tfrac{\partial \mathcal{F}_3}{\partial x_0}(v) \right)
\end{aligned}
\tag{53}
$$

for each $x = x_0 + x_1 I + x_2 J + x_3 K \in \Omega_{\mathbb{I}}$, where $v := (x_0, x_1, x_2, x_3) \in E$. Note that (53) coincides with equality (4.4) contained in the proof of Proposition 4.2 of [13]. $\blacksquare$

Proof of Corollary 4. If f is slice regular and slice Fueter-regular, then $0 = \frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_1} = \frac{2F_2}{x_1}$ so $F_2 = 0$, $\frac{\partial F_1}{\partial x_0} = 0$ and $\frac{\partial F_1}{\partial x_1} = -\frac{\partial F_2}{\partial x_0} = 0$ on D. By assumption (7), it turns out that $F_1 = c$ on D for some $c \in \mathbb{O}$, and hence $f = c$ on Ω_D. Evidently, each constant function is both slice Fueter-regular and slice regular. $\qquad\square$

Recall that, given any slice function $f = \mathcal{I}((F_1, F_2)) : \Omega_D \to \mathbb{O}$, the spherical value of f is the slice function $f_s^\circ : \Omega_D \to \mathbb{O}$ defined by $f_s^\circ := \mathcal{I}((F_1, 0))$. Let $\mathfrak{I} : \Omega_D \to \mathbb{O}$ be the imaginary part function, i.e. $\mathfrak{I}(x) := \mathrm{Im}(x)$. Note that $\mathcal{I}$ is the slice preserving function induced by the stem function $\mathbb{R}^2 \to \mathbb{O}^2$, $(x_0, x_1) \mapsto (0, x_1)$. It is immediate to verify that $f = f_s^\circ + \mathfrak{I} \cdot f_s' = f_s^\circ + \mathfrak{I} f_s'$.

Proof of Corollary 5. Let $f = \mathcal{I}((F_1, F_2))$ and $g = \mathcal{I}((G_1, G_2))$ be slice Fueter-regular functions. By Theorem 3, we know that $\overline{\partial} f = f_s'$ and $\overline{\partial} g = g_s'$ on $\Omega_D \setminus \mathbb{R}$. By direct computations, we see that $\overline{\partial}(f \cdot g) = (\overline{\partial} f) \cdot g + f \cdot (\overline{\partial} g)$ and $(f \cdot g)_s' = f_s' \cdot g_s^\circ + f_s^\circ \cdot g_s'$. It follows that

$$
\overline{\partial}(f \cdot g) - (f \cdot g)_s' = f_s' \cdot (g - g_s^\circ) + (f - f_s^\circ) \cdot g_s' = f_s' \cdot (\mathfrak{I} \cdot g_s') + (\mathfrak{I} \cdot f_s') \cdot g_s' = 2\,\mathfrak{I}(f_s' g_s').
$$

Hence $\overline{\partial}(f \cdot g) - (f \cdot g)_s' = 0$ if and only if $f_s' g_s' = 0$ on $\Omega_D \setminus \mathbb{R}$. By Theorem 2, f is real analytic. Since $f_s'(x) = \mathrm{Im}(x)^{-1}(f(x) - f(\overline{x}))$, it follows that f_s' is real analytic as well. The same is true for g_s'. By assumption (7), $\Omega_D \setminus \mathbb{R}$ is connected, so the condition ($f_s' g_s' = 0$ on $\Omega_D \setminus \mathbb{R}$) is equivalent to require that either $f_s' = 0$ or $g_s' = 0$ on $\Omega_D \setminus \mathbb{R}$. This is in turn equivalent to the condition (either $F_2 = 0$ or $G_2 = 0$ on D). Note that, if $F_2 = 0$ on D, equations (12) imply that F_1 is constant, and hence f is constant as well. Similarly, if $G_2 = 0$ on D, then g is constant. This completes the proof. $\qquad\square$

Proofs of Section 2.2.

Proof of Lemma 12. Let $m, n \in \{1, \ldots, 7\}$ with $m < n$. Since f_s° and f_s' are constant on each $\mathbb{S}_x$, we have that $L_{mn}(f_s^\circ) = L_{mn}(f_s') = 0$ on the whole $\Omega_D \setminus \mathbb{R}$. Since $f(x) = f_s^\circ(x) + \mathrm{Im}(x) f_s'(x)$ for each $x \in \Omega_D \setminus \mathbb{R}$, it follows that

$$
\begin{aligned}
L_{mn}(f)(x) &= L_{mn}(f_s^\circ)(x) + L_{mn}(\mathrm{Im}(x)) f_s'(x) + \mathrm{Im}(x) L_{mn}(f_s')(x) = \\
&= L_{mn}(\mathrm{Im}(x)) f_s'(x) = (x_m e_n - x_n e_m) f_s'(x).
\end{aligned}
$$

Let $x \in \Omega_D \setminus \mathbb{R}$ and let $a := f_s'(x)$. Thanks to the first Moufang identity [46, (3.4), p. 28], we have: $e_m(e_n(e_m a)) = (e_m e_n e_m)a$. Since $e_m e_n e_m = -e_m e_m e_n = e_n$ for $m < n$, we deduce that

$e_m(e_n(e_m a)) = e_n a$ for $m < n$ and hence, by Artin's theorem,

$$\Gamma(f)(x) = -\sum_{m<n} e_m(e_n L_{mn}(f)(x)) = -\sum_{m<n} e_m(e_n((x_m e_n - x_n e_m)a)) =$$

$$= -\sum_{m<n} e_m(e_n(x_m e_n a - x_n e_m a)) = -\sum_{m<n} e_m(-x_m a - x_n e_n(e_m a)) =$$

$$= -\sum_{m<n} (-x_m e_m a - x_n e_m(e_n(e_m a))) = \sum_{m<n} (x_m e_m a + x_n e_n a) =$$

$$= \left(\sum_{m<n} (x_m e_m + x_n e_n) \right) a = \frac{1}{2} \left(\sum_{m,n=1}^{7} (x_m e_m + x_n e_n) - \sum_{m=1}^{7} (x_m e_m + x_m e_m) \right) =$$

$$= \frac{1}{2} \left(7 \sum_{m=1}^{7} x_m e_m + 7 \sum_{n=1}^{7} x_n e_n - 2 \sum_{m=1}^{7} x_m e_m \right) = 6\operatorname{Im}(x)a,$$

as desired. $\qquad\square$

Lemma 40. *A function $f : \Omega_D \to \mathbb{O}$ is slice if and only if the restriction $f|_{\Omega_D \setminus \mathbb{R}}$ is.*

Proof. We can assume $\Omega_D \cap \mathbb{R} = D \cap \mathbb{R} \neq \emptyset$. If $(F_1, F_2) : D \to \mathbb{O}^2$ is a stem function inducing $f : \Omega_D \to \mathbb{O}$, then the restriction $(F_1, F_2)|_{D \setminus \mathbb{R}}$ is a stem function inducing $f|_{\Omega_D \setminus \mathbb{R}}$. Conversely, if $f : \Omega_D \to \mathbb{O}$ is a function such that $f|_{\Omega_D \setminus \mathbb{R}}$ is a slice function, induced by a certain stem function $(G_1, G_2) : D \setminus \mathbb{R} \to \mathbb{O}^2$, then the function $(F_1, F_2) : D \to \mathbb{O}^2$, defined by $(F_1, F_2) = (G_1, G_2)$ on $D \setminus \mathbb{R}$, $F_1 = f$ on $D \cap \mathbb{R}$ and $F_2 = 0$ on $D \cap \mathbb{R}$, is a stem function such that $\mathcal{I}((F_1, F_2)) = f$. $\quad\square$

Proof of Lemma 13. By Lemma 40, we can assume that $\Omega_D \cap \mathbb{R} = D \cap \mathbb{R} = \emptyset$, i.e. Ω_D is a product domain. If f is slice then Lemma 12 implies that $\frac{1}{6}\operatorname{Im}^{-1}\Gamma(f) = f'_s$, so $f - \frac{1}{6}\Gamma(f) = f^\circ_s$. It follows that the functions $\operatorname{Im}^{-1}\Gamma(f)$ and $f - \frac{1}{6}\Gamma(f)$ are constant on the sphere $\mathbb{S}_x$ for each $x \in \Omega_D \setminus \mathbb{R}$. Hence (25) is verified.

Suppose now that (25) is verified. Let $h \in \mathscr{C}^1(\Omega_D, \mathbb{O})$ and $g \in \mathscr{C}^1(\Omega_D, \mathbb{O})$ be the functions defined by $h := \frac{1}{6}\operatorname{Im}^{-1}\Gamma(f)$ and $g := f - \frac{1}{6}\Gamma(f)$. Note that $f(x) = g(x) + \operatorname{Im}(x)h(x)$ for each $x \in \Omega_D$. Thanks to (25), we know that g and h are constant on each spheres $\mathbb{S}_x$ for $x \in \Omega_D$. Define the stem function $F = (F_1, F_2) : D \to \mathbb{O}$ by setting $F_1(x_0, x_1) := g(x_0 + x_1 i)$ and $F_2(x_0, x_1) := x_1 h(x_0 + x_1 i)$. It remains to show that $\mathcal{I}(F) = f$. Let $x = x_0 + x_1 I \in \Omega_D$ for some $x_0, x_1 \in \mathbb{R}$ and $I \in \mathbb{S}$. Since g and h are constant on $\mathbb{S}_x$, we have:

$$\mathcal{I}(F)(x) = F_1(x_0, x_1) + I F_2(x_0, x_1) =$$
$$= g(x_0 + x_1 i) + I x_1 h(x_0 + x_1 i) =$$
$$= g(x_0 + x_1 I) + x_1 I h(x_0 + x_1 I) =$$
$$= g(x) + \operatorname{Im}(x)h(x) = f(x).$$

The proof is complete. $\qquad\square$

Proof of Theorem 15. First, suppose that f is slice Fueter-regular. By implication (1) $\Longrightarrow$ (2a) of Theorem 3, we know that $\overline{\partial} f - f'_s = 0$ on $\Omega_D \setminus \mathbb{R}$. Theorem 2.2(ii) of [27] and Lemma 12 imply that $\overline{\partial} f = \overline{\vartheta}(f)$ and $f'_s = \frac{1}{6}\operatorname{Im}^{-1}\Gamma(f)$ on $\Omega_D \setminus \mathbb{R}$, respectively. Bearing in mind (27), it follows that $\overline{\vartheta}_F(f) = 0$ on $\Omega_D \setminus \mathbb{R}$. Since f is of class $\mathscr{C}^2$, thanks to implication (1) $\Longrightarrow$ (2) of Lemma 13, we also have that f satisfies (25) on $\Omega_D \setminus \mathbb{R}$. This proves implications (1) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (3) of the present theorem.

Finally, suppose that (3) holds, i.e. f satisfies (25) and $\overline{\vartheta}_F(f) = 0$ on $\Omega_D \setminus \mathbb{R}$. By implication (2) $\Longrightarrow$ (1) of Lemma 13, we know that f is slice. In this way, we can apply again Theorem 2.2(ii) of [27] and Lemma 12. Combining these results with (27) once again, we deduce: $\overline{\partial} f - f'_s = \overline{\vartheta}_F(f) = 0$ on $\Omega_D \setminus \mathbb{R}$. Since f is assumed to be of class $\mathscr{C}^3$, implication (2a) $\Longrightarrow$ (1) of Theorem 3 ensures that (1) holds, i.e. f is slice Fueter-regular. The proof is complete. $\qquad\square$

Proofs of Section 2.3.

Proof of Lemma 16. Since $\overline{\xi_I} \in \mathbb{S}_{\xi_I}$, we have $\Delta_{\overline{\xi_I}}(x) = \Delta_{\xi_I}(x)$ and hence $\Delta_{\overline{\xi_I}}(\overline{x}) = \overline{\Delta_{\xi_I}(x)}$. Bearing in mind the latter equality, it holds:

$$N(g_{I,\xi})(x) = ((\xi_I - x) \cdot (\overline{\xi_I} - \overline{x}) + |\xi_I^\perp|^2) \cdot ((\overline{\xi_I} - x) \cdot (\xi_I - \overline{x}) + |\xi_I^\perp|^2) =$$
$$= N(\xi_I - x)N(\overline{\xi_I} - \overline{x}) + |\xi_I^\perp|^2(|\xi_I|^2 - \overline{x}\xi_I - x\overline{\xi_I} + |x|^2)+$$
$$+ |\xi_I^\perp|^2(|\xi_I|^2 - \overline{x}\overline{\xi_I} - x\xi_I + |x|^2) + |\xi_I^\perp|^4 =$$
$$= \Delta_{\xi_I}(x)\Delta_{\overline{\xi_I}}(\overline{x}) + 2|\xi_I^\perp|^2(|x|^2 - x\mathrm{Re}(\xi_I) - \overline{x}\mathrm{Re}(\xi_I) + |\xi_I|^2) + |\xi_I^\perp|^4 =$$
$$= |\Delta_{\xi_I}(x)|^2 + 2|\xi_I^\perp|^2(|x|^2 - 2\mathrm{Re}(x)\mathrm{Re}(\xi_I) + |\xi_I|^2) + |\xi_I^\perp|^4.$$

Note that

$$|x|^2 - 2\mathrm{Re}(x)\mathrm{Re}(\xi_I) + |\xi_I|^2 = |\mathrm{Im}(x)|^2 + (\mathrm{Re}(x) - \mathrm{Re}(\xi_I))^2 + |\mathrm{Im}(\xi_I)|^2 \geq 0.$$

Hence $N(g_{I,\xi})(x) \geq 0$ for each $x \in \mathbb{O}$, and $N(g_{I,\xi})(x) = 0$ if and only if $\xi_I^\perp = 0$ (i.e. $\xi \in \mathbb{C}_I$, so $\xi = \xi_I$) and $x \in \mathbb{S}_\xi$. The proof is complete. $\square$

Let us recall the representation formula for octonionic slice functions, see [25, Proposition 6] for a proof.

Lemma 41. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function, and let $I, J \in \mathbb{S}$ such that $I \neq J$. For each $(\alpha, \beta) \in D$ and $K \in \mathbb{S}$, it holds:*

$$f(\alpha + K\beta) = (K - J)((I - J)^{-1}f(\alpha + I\beta)) - (K - I)((I - J)^{-1}f(\alpha + J\beta)).$$

For each $I \in \mathbb{S}$, we define $\mathbb{C}_I^+ := \{\alpha + I\beta \in \mathbb{O} : \alpha, \beta \in \mathbb{R}, \beta \geq 0\}$.

An immediate consequence of Lemma 41 is as follows:

Corollary 42 (Identity principle)**.** *Let $f, g : \Omega_D \to \mathbb{O}$ be two slice functions and let $I, J \in \mathbb{S}$ such that $I \neq J$ and $f = g$ on $\Omega_D \cap (\mathbb{C}_I^+ \cup \mathbb{C}_J^+)$. Then $f = g$ on the whole Ω_D.*

In particular, $f = g$ on the whole Ω_D if f and g coincide on $\Omega_D \cap \mathbb{C}_I$.

The slice Cauchy integrands $S_{f,\mathbb{I}}$ and $V_{f,\mathbb{I}}$ can be characterized as follows.

Lemma 43. *The slice Cauchy surface $(f, \mathbb{I})$-integrand $S_{f,\mathbb{I}}$ is the unique function $\mathcal{S} : \partial^*\Omega_\mathbb{I} \to \mathbb{O}$ satisfying the following condition: for each $\xi \in \partial\Omega_\mathbb{I}$, the restriction function $\mathcal{S}|_\xi : \mathbb{O} \setminus \mathbb{S}_{I,\xi} \to \mathbb{O}$, defined by $\mathcal{S}|_\xi(x) := \mathcal{S}(x, \xi)$, is a slice function such that*

$$\mathcal{S}|_\xi(y) = \frac{1}{2\pi^2} \frac{\overline{\xi - y}}{|\xi - y|^4}(\mathfrak{n}_\mathbb{I}(\xi)f(\xi)) \quad \text{for each } y \in \mathbb{C}_I \setminus \mathbb{S}_{I,\xi}. \tag{54}$$

Similarly, the slice Cauchy volume $(f, \mathbb{I})$-integrand $V_{f,\mathbb{I}}$ is the unique function $\mathcal{V} : \Omega_\mathbb{I}^* \to \mathbb{O}$ satisfying the following condition: for each $\xi \in \Omega_\mathbb{I}$, the restriction function $\mathcal{V}|_\xi : \mathbb{O} \setminus \mathbb{S}_{I,\xi} \to \mathbb{O}$, defined by $\mathcal{V}|_\xi(x) := \mathcal{V}(x, \xi)$, is a slice function such that

$$\mathcal{V}|_\xi(y) = \frac{1}{2\pi^2} \frac{\overline{\xi - y}}{|\xi - y|^4}(\overline{\mathcal{D}}_\mathbb{I} f(\xi)) \quad \text{for each } y \in \mathbb{C}_I \setminus \mathbb{S}_{I,\xi}. \tag{55}$$

Proof. Let us prove the first part of the lemma. The proof of the second is similar. During this proof, we will use 'x' as a variable. Let $\xi \in \partial\Omega_\mathbb{I}$ and let $a := \mathfrak{n}_\mathbb{I}(\xi)f(\xi)$. By Corollary 42, it suffices to show that

$$S_{f,\mathbb{I}}(y, \xi) = \frac{1}{2\pi^2} \frac{\overline{\xi}a - \overline{y}a}{|\xi - y|^4} \quad \text{for each } y \in \mathbb{C}_I \setminus \mathbb{S}_{I,\xi}. \tag{56}$$

Choose any $y \in \mathbb{C}_I \setminus \mathbb{S}_{I,\xi}$. Since the slice functions $x \mapsto \xi_I - x$, $x \mapsto \overline{\xi_I} - \overline{x}$ and hence $g_{I,\xi}$ are $\mathbb{C}_I$-preserving, it follows that

$$g_{I,\xi}(y) = (\xi_I - y)(\overline{\xi_I} - \overline{y}) + |\xi_I^\perp|^2 = |\xi_I - y|^2 + |\xi_I^\perp|^2 = |\xi - y|^2,$$

$$g_{I,\xi}^c(y) = |\xi_I - \overline{y}|^2 + |\xi_I^\perp|^2 = |\xi - \overline{y}|^2 = |\overline{\xi} - y|^2,$$

$$N(g_{I,\xi})(y) = |\xi - y|^2 |\overline{\xi} - y|^2$$

and hence

$$((N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2})(y) = (|\xi - y|^2|\overline{\xi} - y|^2)^{-2}|\overline{\xi} - y|^4 = |\xi - y|^{-4},$$

$$((N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot (\overline{\xi}a))(y) = |\xi - y|^{-4}\overline{\xi}a,$$

$$((N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot a)(y) = |\xi - y|^{-4}a.$$

Note that

$$N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot (\overline{x}a) = N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot (\overline{x} \cdot a) = \overline{x}((N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot a).$$

Summarizing, it holds:

$$((N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot (\overline{\xi}a))(y) = |\xi - y|^{-4}\overline{\xi}a,$$

$$(N(g_{I,\xi})^{-2} \cdot (g_{I,\xi}^c)^{\bullet 2} \cdot (\overline{x}a))(y) = \overline{y}|\xi - y|^{-4}a = |\xi - y|^{-4}\overline{y}a.$$

The latter two equalities imply at once that

$$2\pi^2 S_{f,\mathbb{I}}(y,\xi) = \big((N(g_{I,\xi})(x))^{-2} \cdot (g_{I,\xi}^c(x))^{\bullet 2} \cdot (\overline{\xi}a - \overline{x}a)\big)(y) =$$
$$= |\xi - y|^{-4}\overline{\xi}a - |\xi - y|^{-4}\overline{y}a = |\xi - y|^{-4}(\overline{\xi}a - \overline{y}a),$$

which proves (56). $\qquad\square$

Proof of Lemma 18. As above, we will only prove the first part, the proof of the second being similar. Let $\xi \in \partial\Omega_\mathbb{I}$, let $a := \mathbf{n}_\mathbb{I}(\xi)f(\xi)$ and let $y \in \mathbb{C}_I \setminus \mathbb{S}_{I,\xi}$. Since the function $x \mapsto \frac{M_I(x,\xi,a)}{2\pi^2 N_I(x,\xi)^2}$ is slice, by Corollary 42 and Lemma 43, it suffices to show that $\frac{M_I(y,\xi,a)}{N_I(y,\xi)^2} = \frac{\overline{\xi}a - \overline{y}a}{|\xi - y|^4}$. Bearing in mind Artin's theorem and the fact that y and ξ_I belong to $\mathbb{C}_I$, we obtain:

$$N_I(y,\xi) = |(y - \xi_I)(y - \overline{\xi_I})|^2 + |\xi_I^\perp|^2(2|y|^2 - 2\mathrm{Re}(y)2\mathrm{Re}(\xi_I) + 2|\xi_I|^2) + |\xi_I^\perp|^4 =$$
$$= |y - \xi_I|^2|y - \overline{\xi_I}|^2 + |\xi_I^\perp|^2(|y|^2 - y\overline{\xi_I} - \overline{y}\xi_I + |\xi_I|^2 +$$
$$+ |y|^2 - y\xi_I - \overline{y}\overline{\xi_I} + |\xi_I|^2) + |\xi_I^\perp|^4 =$$
$$= |y - \xi_I|^2|y - \overline{\xi_I}|^2 + |\xi_I^\perp|^2(|y - \xi_I|^2 + |y - \overline{\xi_I}|^2) + |\xi_I^\perp|^4 =$$
$$= (|y - \xi_I|^2 + |\xi_I^\perp|^2)(|y - \overline{\xi_I}|^2 + |\xi_I^\perp|^2) = |y - \xi|^2|y - \overline{\xi}|^2,$$

$$Q_I(y,\xi,a) = Q_I(y,\xi,1)a,$$

$$Q_I(y,\xi,1) = (|y|^2 + |\xi|^2)^2 - 2(|y|^2 + |\xi|^2)(y\xi_I + \overline{y}\overline{\xi_I}) + y^2\xi_I^2 + \overline{y}^2\overline{\xi_I}^2 + 2|y|^2|\xi_I|^2 =$$
$$= (|y|^2 + |\xi|^2 - y\xi_I - \overline{y}\overline{\xi_I})^2 = (|y|^2 + |\xi_I|^2 - y\xi_I - \overline{y}\overline{\xi_I} + |\xi_I^\perp|^2)^2 =$$
$$= (|y - \overline{\xi_I}|^2 + |\xi_I^\perp|^2)^2 = |y - \overline{\xi}|^4.$$

As a consequence, we deduce:

$$M_I(y,\xi,a) = Q_I(y,\xi,\overline{\xi}a) - \overline{y}Q_I(y,\xi,a) = Q_I(y,\xi,1)(\overline{\xi}a) - \overline{y}(Q_I(y,\xi,1)a) = |y - \overline{\xi}|^4(\overline{\xi}a - \overline{y}a)$$

and hence

$$\frac{M_I(y,\xi,a)}{N_I(y,\xi)^2} = \frac{|y - \overline{\xi}|^4(\overline{\xi}a - \overline{y}a)}{|y - \xi|^4|y - \overline{\xi}|^4} = \frac{\overline{\xi}a - \overline{y}a}{|\xi - y|^4},$$

as desired. $\qquad\square$

Next result is a version of Lemma 3.2 of [27].

Lemma 44 (Sliceness criterion). *Let $f : \Omega_D \to \mathbb{O}$ be a function. The following assertions are equivalent.*

(1) *f is a slice function.*
(2) *There exists $J \in \mathbb{S}$ with the following property: for each $(\alpha, \beta) \in D$ and for each $I \in \mathbb{S}$, if $x = \alpha + \beta I$ and $y = \alpha + \beta J$, then*

$$f(x) = \frac{1}{2}(f(y) + f(\overline{y})) - \frac{I}{2}(J(f(y) - f(\overline{y}))). \tag{57}$$

(3) *Equation (57) holds for each $x = \alpha + \beta I$ and $y = \alpha + \beta J$ with $(\alpha, \beta) \in D$ and $I, J \in \mathbb{S}$.*

Proof. If f is a slice function induced by the stem function (F_1, F_2) then, given any $J \in \mathbb{S}$, $F_1(\alpha, \beta) = \frac{1}{2}(f(y) + f(\overline{y}))$ and $F_2(\alpha, \beta) = -\frac{1}{2}J(f(y) - f(\overline{y}))$, so (57) holds for each $I \in \mathbb{S}$. This proves that (1) implies (2) and (3).

Suppose that there exists $J \in \mathbb{S}$ as in (2). Define the function $(F_1, F_2) : D \to \mathbb{O}^2$ by setting $F_1(\alpha, \beta) := \frac{1}{2}(f(y) + f(\overline{y}))$ and $F_2(\alpha, \beta) := -\frac{1}{2}J(f(y) - f(\overline{y}))$. Note that $F_1(\alpha, -\beta) = \frac{1}{2}(f(\overline{y}) + f(y)) = F_1(\alpha, \beta)$ and $F_2(\alpha, -\beta) = -\frac{1}{2}J(f(\overline{y}) - f(y)) = -F_2(\alpha, \beta)$. Hence (F_1, F_2) is a stem function. Thanks to (57), we have that $\mathcal{I}((F_1, F_2)) = f$, so f is a slice function. In particular, f satisfies (3). This proves that (2) implies (1) and (3).

Evidently, (3) implies (2). The proof is complete. $\qquad\square$

Corollary 45. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function of class $\mathscr{C}^1$, and let $\mathsf{S}, \mathsf{V} : \Omega_D \to \mathbb{O}$ be the functions defined by setting*

$$\mathsf{S}(x) := \int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(x, \xi) \, ds_\mathbb{I}(\xi) \quad and \quad \mathsf{V}(x) := \int_{\Omega_\mathbb{I}} V_{f,\mathbb{I}}(x, \xi) \, dv_\mathbb{I}(\xi).$$

Then S and V are slice functions.

Proof. By definitions (29), (33) and (30), (34), for each fixed ξ, the functions $x \mapsto S_{f,\mathbb{I}}(x, \xi)$ and $x \mapsto V_{f,\mathbb{I}}(x, \xi)$ are slice functions, so they satisfy (57) for each given $x = \alpha + \beta I$ and $y = \alpha + \beta J$ with $(\alpha, \beta) \in D$ and $I, J \in \mathbb{S}$. As a consequence, we have:

$$\mathsf{S}(x) = \int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(x, \xi) \, ds_\mathbb{I}(\xi) =$$

$$= \int_{\partial\Omega_\mathbb{I}} \left(\frac{1}{2}(S_{f,\mathbb{I}}(y, \xi) + S_{f,\mathbb{I}}(\overline{y}, \xi)) - \frac{I}{2}(J(S_{f,\mathbb{I}}(y, \xi) - S_{f,\mathbb{I}}(\overline{y}, \xi))) \right) ds_\mathbb{I}(\xi) =$$

$$= \frac{1}{2} \left(\int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(y, \xi) \, ds_\mathbb{I}(\xi) + \int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(\overline{y}, \xi) \, ds_\mathbb{I}(\xi) \right) +$$

$$- \frac{I}{2} \left(J \left(\int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(y, \xi) \, ds_\mathbb{I}(\xi) - \int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(\overline{y}, \xi) \, ds_\mathbb{I}(\xi) \right) \right) =$$

$$= \frac{1}{2}(\mathsf{S}(y) + \mathsf{S}(\overline{y})) - \frac{I}{2}(J(\mathsf{S}(y) - \mathsf{S}(\overline{y}))).$$

Implication (3) $\implies$ (1) in Lemma 44 ensures that S is a slice function. The same argument shows that V is a slice function. $\qquad\square$

Proof of Theorem 20. Theorem 5.1 of [13] asserts that, for each $x \in \Omega_\mathbb{I}$, it holds:

$$f(x) = \int_{\partial\Omega_\mathbb{I}} \frac{1}{2\pi^2} \frac{\overline{\xi - x}}{|\xi - x|^4} (\mathbf{n}_\mathbb{I}(\xi) f(\xi)) \, ds_\mathbb{I}(\xi) - \int_{\Omega_\mathbb{I}} \frac{1}{2\pi^2} \frac{\overline{\xi - x}}{|\xi - x|^4} (\overline{\mathcal{D}}_\mathbb{I} f(\xi)) \, dv_\mathbb{I}(\xi).$$

In particular, thanks to Lemma 43, for each $x \in \Omega_D \cap \mathbb{C}_I \subset \Omega_\mathbb{I}$, we have:

$$f(x) = \int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(x, \xi) \, ds_\mathbb{I}(\xi) - \int_{\Omega_\mathbb{I}} V_{f,\mathbb{I}}(x, \xi) \, dv_\mathbb{I}(\xi).$$

By Corollary 45, the function $\Omega_D \to \mathbb{O}, x \mapsto \int_{\partial\Omega_\mathbb{I}} S_{f,\mathbb{I}}(x, \xi) \, ds_\mathbb{I}(\xi) - \int_{\Omega_\mathbb{I}} V_{f,\mathbb{I}}(x, \xi) \, dv_\mathbb{I}(\xi)$ is slice. Hence Corollary 42 implies that the preceding equation holds for each $x \in \Omega_D$, i.e. (37) holds.

If f is slice Fueter-regular, then (4), (30) and (34) imply that $V_{f,\mathbb{I}} = 0$, so (37) reduces to (38). Moreover, equality (5.12) of Corollary 5.2 of [13] ensures that $\int_{\partial\Omega_{\mathbb{I}}} S_{f,\mathbb{I}}(x,\xi)\,\mathrm{ds}_{\mathbb{I}}(\xi) = 0$ for each $x \in \mathbb{H}_{\mathbb{I}} \setminus \mathrm{cl}(\Omega_D)$. Repeating the preceding argument, we obtain that the same equality holds for each $x \in \mathbb{O} \setminus \mathrm{cl}(\Omega_D)$. This proves (39). $\qquad\square$

Proofs of Sections 2.4 and 2.5. Fix $\mathbb{I} = (I, J) \in \mathcal{N}$. Let $\kappa \in \mathbb{N}^3$. Recall that $P_\kappa : \mathbb{H} \to \mathbb{H} \subset \mathbb{O}$ is the standard κ^{th} Fueter polynomial, and $Q_\kappa : \mathbb{H} \setminus \{0\} \to \mathbb{H} \subset \mathbb{O}$ the function $\partial_\kappa\mathcal{E}$, where $\mathcal{E}(x) = \frac{1}{2\pi^2}\frac{\overline{x}}{|x|^4}$. Furthermore, $\psi_{\mathbb{I}} : \mathbb{H} \to \mathbb{O}$ denotes the map sending $x_0 + x_1 i + x_2 j + x_3 k$ into $x_0 + x_1 I + x_2 J + x_3 IJ$. Note that $\psi_{\mathbb{I}}$ is an embedding of real algebras.

Let us give a characterization of $\mathcal{P}_{\mathbb{I},\kappa} : \mathbb{O}^2 \to \mathbb{O}$ and $\mathcal{Q}_{\mathbb{I},\kappa} : (\mathbb{O} \setminus \{0\}) \times \mathbb{O} \to \mathbb{O}$.

Lemma 46. $\mathcal{P}_{\mathbb{I},\kappa}$ *is the unique function* $\mathcal{P} : \mathbb{O}^2 \to \mathbb{O}$ *satisfying the following condition: for each* $c \in \mathbb{O}$, *the restriction function* $\mathcal{P}|_c : \mathbb{O} \to \mathbb{O}$, *defined by* $\mathcal{P}|_c(x) := \mathcal{P}(x,c)$, *is a slice function such that*
$$\mathcal{P}|_c(x) = \psi_{\mathbb{I}}(P_\kappa(\psi_{\mathbb{I}}^{-1}(x)))c \quad \text{for each } x \in \mathbb{C}_I. \tag{58}$$

Similarly, $\mathcal{Q}_{\mathbb{I},\kappa}$ *is the unique function* $\mathcal{Q} : (\mathbb{O}\setminus\{0\})\times\mathbb{O} \to \mathbb{O}$ *satisfying the following condition: for each* $c \in \mathbb{O}$, *the restriction function* $\mathcal{Q}|_c : \mathbb{O} \setminus \{0\} \to \mathbb{O}$, *defined by* $\mathcal{Q}|_c(x) := \mathcal{Q}(x,c)$, *is a slice function such that*
$$\mathcal{Q}|_c(x) = \psi_{\mathbb{I}}(Q_\kappa(\psi_{\mathbb{I}}^{-1}(x)))c \quad \text{for each } x \in \mathbb{C}_I \setminus \{0\}. \tag{59}$$

Proof. We follow the strategy used in the proof of Lemma 43. We prove only the first part of the present lemma, the proof of the second being similar. Let $x = x_0 + x_1 I \in \mathbb{C}_I$ and let $\mathbf{x} \in \mathbb{C}_i$ with $\psi_{\mathbb{I}}(\mathbf{x}) = x$, i.e $\mathbf{x} = x_0 + x_1 i$. By Corollary 42, it suffices to show that $\mathcal{P}_{\mathbb{I},\kappa}(x,c) = \psi_{\mathbb{I}}(P_\kappa(\mathbf{x}))c$. Bearing in mind (44), (45), (46), and using Artin's theorem, if $\mathrm{k} = |\kappa|$, we have:

$$\mathcal{P}_{\mathbb{I},\kappa}(x,c) = \sum_{h=0}^{\lfloor\frac{\mathrm{k}}{2}\rfloor} x_0^{\mathrm{k}-2h}(x_1 I)^{2h}(\ell_{\mathbb{I},\kappa,2h}c) + \sum_{h=0}^{\lfloor\frac{\mathrm{k}-1}{2}\rfloor} x_0^{\mathrm{k}-2h-1}(x_1 I)^{2h+1}(-I((I\ell_{\mathbb{I},\kappa,2h+1})c)) =$$

$$= \sum_{h=0}^{\lfloor\frac{\mathrm{k}}{2}\rfloor} x_0^{\mathrm{k}-2h}(-x_1^2)^h(\ell_{\mathbb{I},\kappa,2h}c) + \sum_{h=0}^{\lfloor\frac{\mathrm{k}-1}{2}\rfloor} x_0^{\mathrm{k}-2h-1}(-x_1^2)^h(x_1 I)(-I((I\ell_{\mathbb{I},\kappa,2h+1})c)) =$$

$$= \sum_{h=0}^{\lfloor\frac{\mathrm{k}}{2}\rfloor} (x_0^{\mathrm{k}-2h}(-x_1^2)^h\ell_{\mathbb{I},\kappa,2h})c + \sum_{h=0}^{\lfloor\frac{\mathrm{k}-1}{2}\rfloor} x_0^{\mathrm{k}-2h-1}(-x_1^2)^h x_1(I\ell_{\mathbb{I},\kappa,2h+1})c =$$

$$= \sum_{h=0}^{\lfloor\frac{\mathrm{k}}{2}\rfloor} (x_0^{\mathrm{k}-2h}(-x_1^2)^h\ell_{\mathbb{I},\kappa,2h})c + \sum_{h=0}^{\lfloor\frac{\mathrm{k}-1}{2}\rfloor} (x_0^{\mathrm{k}-2h-1}(x_1 I)(-x_1^2)^h\ell_{\mathbb{I},\kappa,2h+1})c =$$

$$= \mathcal{P}_{\mathbb{I},\kappa}(x,1)c = \psi_{\mathbb{I}}(P_\kappa(\mathbf{x}))c.$$

The proof is complete. $\qquad\square$

Proof of Theorem 25. Theorem 6.1 of [13] asserts that, for each $x \in B_{\mathbb{I}}(r) := B(r) \cap \mathbb{H}_{\mathbb{I}}$, it holds:
$$f(x) = \sum_{\mathrm{k}\in\mathbb{N}} \sum_{\kappa\in\mathbb{N}^3,|\kappa|=\mathrm{k}} \psi_{\mathbb{I}}(P_\kappa(\psi_{\mathbb{I}}^{-1}(x)))\frac{\partial_\kappa f(0)}{\kappa!}, \tag{60}$$
where the series converges uniformly on compact subsets of $B_{\mathbb{I}}(r)$. Actually, this series converges absolutely in the sense of [35, Section 9.1.3], and hence it converges totally in the sense of [31, Definition 3.1]. In particular, the same is true replacing $B_{\mathbb{I}}(r)$ with $B_I(r) := B_{\mathbb{I}}(r) \cap \mathbb{C}_I = B(r) \cap \mathbb{C}_I$. Combining the latter fact with Lemma 46 and Definition 24, we deduce:
$$f(x) = \sum_{\mathrm{k}\in\mathbb{N}} \sum_{\kappa\in\mathbb{N}^3,|\kappa|=\mathrm{k}} \mathcal{P}_{\mathbb{I},\kappa}(x, \tfrac{\partial_\kappa f(0)}{\kappa!}) = \sum_{\mathrm{k}\in\mathbb{N}} \mathcal{P}_{\mathbb{I},\mathrm{k};f,0}(x) \quad \text{for each } x \in B_I(r)$$

and the series converges uniformly on compact subsets of $B_I(r)$. Now [31, Theorem 3.4] and above Corollary 42 ensure that $f = \sum_{k \in \mathbb{N}} \mathcal{P}_{\mathbb{I},k;f,0}$ on the whole $B(r)$, where the series $\sum_{k \in \mathbb{N}} \mathcal{P}_{\mathbb{I},k;f,0}$ converges uniformly on compact subsets of $B(r)$. $\qquad\square$

Proof of Corollary 26. Given any $n, m \in \mathbb{N}$ and $a \in \mathbb{O}$, it is easy to verify that $\overline{\partial}(\mathrm{Re}(x)^n) = \frac{n}{2}\mathrm{Re}(x)^{n-1}$, $\overline{\partial}(\mathrm{Im}(x)^m) = -\frac{m}{2}\mathrm{Im}(x)^{m-1}$ and hence $\overline{\partial}(\mathrm{Re}(x)^n\mathrm{Im}(x)^m a) = \frac{n}{2}\mathrm{Re}(x)^{n-1}\mathrm{Im}(x)^m a - \frac{m}{2}\mathrm{Re}(x)^n\mathrm{Im}(x)^{m-1}a$. Furthermore, $(\mathrm{Re}(x)^n\mathrm{Im}(x)^m a)'_s = 0$ if m is even and $(\mathrm{Re}(x)^n\mathrm{Im}(x)^m a)'_s = \mathrm{Re}(x)^n\mathrm{Im}(x)^{m-1}a$ if m is odd. By (46) and Definition 24, it follows that the functions $\overline{\partial}\mathcal{P}_{\mathbb{I},k;f,0} : B(r) \to \mathbb{O}$ and $(\mathcal{P}_{\mathbb{I},k;f,0})'_s : B(r) \to \mathbb{O}$ have the following property: each of their eight real components is either the zero function or a homogeneous polynomial function of degree $k - 1$ in the eight real components of x. Thanks to Theorem 3, we know that $\overline{\partial}f = f'_s$ on $B(r) \setminus \mathbb{R}$. Since the series $\sum_{k \in \mathbb{N}} \mathcal{P}_{\mathbb{I},k;f,0}$ converges uniformly on compact subsets of $B(r)$, we deduce that $\sum_{k \in \mathbb{N}} \overline{\partial}\mathcal{P}_{\mathbb{I},k;f,0} = \sum_{k \in \mathbb{N}}(\mathcal{P}_{\mathbb{I},k;f,0})'_s$ on $B(r) \setminus \mathbb{R}$, where the latter two series converges uniformly on compact subsets of $B(r) \setminus \mathbb{R}$. The above homogeneity property of $\overline{\partial}\mathcal{P}_{\mathbb{I},k;f,0}$ and $(\mathcal{P}_{\mathbb{I},k;f,0})'_s$ implies that $\overline{\partial}\mathcal{P}_{\mathbb{I},k;f,0} = (\mathcal{P}_{\mathbb{I},k;f,0})'_s$ on $B(r) \setminus \mathbb{R}$ for each $k \in \mathbb{N}$. Hence each $\mathcal{P}_{\mathbb{I},k;f,0}$ is a slice Fueter-regular function. $\qquad\square$

Proof of Theorem 31 and Corollary 32. To prove these results, it is suffices to repeat the arguments used in the proofs of Theorem 25 and Corollary 26 respectively, replacing [13, Theorem 6.1] and (46) with [13, Theorem 6.5] and (49), respectively. $\qquad\square$

Proofs of Section 2.6. To prove Theorem 37, we need two preliminary results. First, we reformulate a lemma of [32].

Lemma 47 (Lemma 4.14 of [32]). *Let $f : \Omega_D \to \mathbb{O}$ be a slice function and let c be a constant in $\mathbb{O} \setminus \{0\}$. Then the following holds:*

(1) *For each $x \in \Omega_D$, there exists $y_x \in \mathbb{S}_x$ such that $(f \cdot c)(x) = f(y_x)c$.*
(2) *If f is continuous and $\Omega'_D := \Omega_D \setminus (\mathbb{R} \cup V(f'_s)) \neq \emptyset$, then there exists a homeomorphism $\Phi : \Omega'_D \to \Omega'_D$ such that $\Phi(\mathbb{S}_x) = \mathbb{S}_x$ and $(f \cdot c)(x) = f(\Phi(x))c$ for each $x \in \Omega'_D$. Furthermore,*

$$\Phi(x) = ((x(f'_s(x)c))c^{-1})f'_s(x)^{-1}, \tag{61}$$

$$\Phi^{-1}(x) = ((xf'_s(x))c)(c^{-1}f'_s(x)^{-1}) \tag{62}$$

for each $x \in \Omega'_D$.

Proof. The mentioned lemma of [32] implies at once assertion (1), and it ensures that the map $\Phi : \Omega'_D \to \Omega'_D$, defined by formula (61), has the following property: for each $x \in \Omega'_D$, $\Phi(x)$ is the unique element of $\mathbb{S}_x$ such that $(f \cdot c)(x) = f(\Phi(x))c$. Moreover, if f is continuous, then f'_s and hence Φ are continuous as well. Note that $(f \cdot c)'_s = f'_s c$. Hence $V((f \cdot c)'_s) = V(f'_s)$ and, given any $x \in \Omega'_D$, it holds:

$$((f \cdot c) \cdot c^{-1})(\Phi(x)) = f(\Phi(x)) = ((f \cdot c)(x))c^{-1}.$$

In this way, applying Lemma 4.14 of [32] with $f \cdot c$ in place of f and c^{-1} in place of c, we obtain that Φ is bijective and its inverse has the following form:

$$\Phi^{-1}(x) = ((x(((f \cdot c)'_s(x))c^{-1}))c)((f \cdot c)'_s(x))^{-1}$$
$$= ((xf'_s(x))c)(c^{-1}f'_s(x)^{-1}).$$

In particular, Φ^{-1} is continuous as well. $\qquad\square$

Let $f : \Omega_D \to \mathbb{O}$ be a slice function and let x be a point of Ω_D such that $V(f) \cap \mathbb{S}_x = \{x\}$, i.e. x is the unique zero of f on $\mathbb{S}_x$. Given $c \in \mathbb{O} \setminus \{0\}$, there exists a unique point $x_c \in \mathbb{S}_x$ such that $V(f \cdot c) \cap \mathbb{S}_x = \{x_c\}$. Usually, $x_c \neq x$. This phenomenon, called camshaft effect, is due to the non-associativity of $\mathbb{O}$, see [24, 30, 32] for the general version of such an effect.

The mentioned camshaft effect can be also interpreted as a measure of the difference between the pointwise and the slice products. Let $y \in \Omega_D$ and let $g : \Omega_D \to \mathbb{O}$ be the slice function $g(x) := (f \cdot c)(x) - f(y)c = (f(x) - f(y)) \cdot c$. As we said, there exists a unique $y_c \in \mathbb{S}_y$ such that $g(y_c) = 0$ or, equivalently, $(f \cdot c)(y_c) = f(y)c$. Usually, $y_c \neq y$.

The second preliminary result we need is the following lemma, which pointwise 'annihilates' the camshaft effect in the case of a slice product with a certain pair of constants on the right.

Lemma 48. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function induced by the stem function $(F_1, F_2) : D \to \mathbb{O}^2$, let $p = \alpha + I\beta \in \Omega_D$ with $\alpha, \beta \in \mathbb{R}$ and $I \in \mathbb{S}$, and let $w := \alpha + i\beta \in D$. Define $\gamma := F_1(w)$, $\delta := F_2(w)$, $q := f(p) = \gamma + I\delta$ and the slice functions $g_1, g_2 : \Omega_D \to \mathbb{O}$ by setting*

$$g_1 := (f \cdot \overline{\gamma}) \cdot ((-\gamma(\overline{q}I))I) \quad and \quad g_2 := (f \cdot \overline{\delta}) \cdot (\delta\overline{q}).$$

Then $g_1(p) = |\gamma|^2 |q|^2$ and $g_2(p) = |\delta|^2 |q|^2$.

Proof. Let $b := (-\gamma(\overline{q}I))I$. Thanks to Artin's theorem, we have:

$$b = -(\gamma((\overline{\gamma} - \overline{\delta}I)I))I = -(\gamma(\overline{\gamma}I + \overline{\delta}))I = -(|\gamma|^2 I + \gamma\overline{\delta})I = |\gamma|^2 - (\gamma\overline{\delta})I.$$

Let $(G_{1,1}, G_{1,2}) : D \to \mathbb{O}^2$ be the stem function inducing g_1. Since $G_{1,1}(z) = (F_1(z)\overline{\gamma})b$ and $G_{1,2}(z) = (F_2(z)\overline{\gamma})b$ for each $z \in D$, setting $z = w$, we deduce:

$$g_1(p) = (F_1(w)\overline{\gamma})b + I((F_2(w)\overline{\gamma})b) = (\gamma\overline{\gamma})(|\gamma|^2 - (\gamma\overline{\delta})I) + I((\delta\overline{\gamma})(|\gamma|^2 - (\gamma\overline{\delta})I)) =$$
$$= |\gamma|^4 - |\gamma|^2(\gamma\overline{\delta})I + |\gamma|^2 I(\delta\overline{\gamma}) + |\gamma|^2|\delta|^2 = |\gamma|^2(|\gamma|^2 + I(\delta\overline{\gamma}) - (\gamma\overline{\delta})I + |\delta|^2).$$

Bearing in mind that the trace function $\mathrm{t} : \mathbb{O} \to \mathbb{R}$, given by $\mathrm{t}(x) := x + \overline{x}$, vanishes on associators (see e.g. [30, Lemma 5.6]), we deduce:

$$I(\delta\overline{\gamma}) - (\gamma\overline{\delta})I = \mathrm{t}(I(\delta\overline{\gamma})) = \mathrm{t}((I\delta)\overline{\gamma}) = (I\delta)\overline{\gamma} - \gamma(\overline{\delta}I)$$

and hence

$$g_1(p) = |\gamma|^2(|\gamma|^2 + (I\delta)\overline{\gamma} - \gamma(\overline{\delta}I) + |\delta|^2) = |\gamma|^2|\gamma + I\delta|^2 = |\gamma|^2|q|^2.$$

Denote $(G_{2,1}, G_{2,2}) : D \to \mathbb{O}^2$ the stem function inducing g_2. Since $G_{2,1}(z) = (F_1(z)\overline{\delta})(\delta\overline{q})$ and $G_{2,2}(z) = (F_2(z)\overline{\delta})(\delta\overline{q})$ for each $z \in D$, setting $z = w$, we deduce:

$$g_2(p) = (F_1(w)\overline{\delta})(\delta\overline{q}) + I((F_2(w)\overline{\delta})(\delta\overline{q})) = (\gamma\overline{\delta})(\delta\overline{\gamma} - |\delta|^2 I) + I((\delta\overline{\delta})(\delta\overline{\gamma} - |\delta|^2 I)) =$$
$$= |\gamma|^2|\delta|^2 - |\delta|^2(\gamma\overline{\delta})I + |\delta|^2 I(\delta\overline{\gamma}) + |\delta|^4 = |\delta|^2(|\gamma|^2 + I(\delta\overline{\gamma}) - (\gamma\overline{\delta})I + |\delta|^2) =$$
$$= |\delta|^2(|\gamma|^2 + (I\delta)\overline{\gamma} - \gamma(\overline{\delta}I) + |\delta|^2) = |\delta|^2|\gamma + I\delta|^2 = |\delta|^2|q|^2.$$

The proof is complete. $\qquad\square$

The reader notes that $(-\gamma(\overline{q}I))I = \overline{\langle I; q, \overline{\gamma}\rangle}$, because of above definition (45).

Corollary 49. *Let $f : \Omega_D \to \mathbb{O}$ be a slice function, let $p \in \Omega_D$ and let $q := f(p)$. If $q \neq 0$, then there exist $a, b \in \mathbb{O} \setminus \{0\}$ such that $|b| = |a||q|$ and the slice function $g : \Omega_D \to \mathbb{O}$, defined by $g := (f \cdot a) \cdot b$, has the following property:*

$$g(p) = |a|^2|q|^2 > 0.$$

Furthermore, if $p \notin \mathbb{R} \cup V(f'_s)$, then we can set $a = \overline{f'_s(p)}$ and $b = (f'_s(p))\overline{q}$.

Proof. Let (F_1, F_2) be the stem function inducing f, let $p = \alpha + I\beta$ and let $w := \alpha + i\beta$. Note that $0 \neq q = F_1(w) + IF_2(w)$, so either $F_1(w) \neq 0$ or $F_2(w) \neq 0$. Thanks to the preceding lemma, if $F_1(w) \neq 0$, then it suffices to set $g := (f \cdot a) \cdot b$ with $a := \overline{F_1(w)}$ and $b := -((F_1(w))(\overline{q}I))I \neq 0$. If $F_2(w) \neq 0$, we can set $a := \frac{\overline{F_2(w)}}{\beta} = \overline{f'_s(p)}$ and $b := \frac{(F_2(w))\overline{q}}{\beta} = (f'_s(p))\overline{q} \neq 0$. $\qquad\square$

Proof of Theorem 37. Let p be a point of Ω_D.

Let us prove point (1).

Suppose that $|f|$ has a maximum at p. Let $q := f(p)$. If $q = 0$, then $f = 0$ and the proof is complete. Suppose $q \neq 0$. By Corollary 49, there exist $a, b \in \mathbb{O} \setminus \{0\}$ such that $|b| = |a||q|$ and the slice function $g : \Omega_D \to \mathbb{O}$, defined by $g := (f \cdot a) \cdot b$, has the following property: $g(p) = |a|^2 |q|^2 > 0$. Moreover, if $p \notin \mathbb{R} \cup V(f'_s)$, then we can assume $a = \overline{f'_s(p)}$ and $b = (f'_s(p))\overline{q}$. By Corollary 5, we also know that g is slice Fueter-regular. Note that, if (F_1, F_2) and (G_1, G_2) are the stem functions inducing f and g respectively, then $G_h = (F_h a)b$ for $h \in \{1, 2\}$. In this way, f is constant, i.e. F_1 is constant and $F_2 = 0$, if and only if the same is true for G_1 and G_2, i.e. g is constant. We will show that g is constant completing the proof.

Applying twice Lemma 47(1) to g, we see that, given any $x \in \Omega_D$, there exist $y_x, w_x \in \mathbb{S}_x \subset \Omega_D$ such that
$$g(x) = ((f \cdot a)(y_x))b = (f(w_x)a)b,$$
so
$$|g(x)| = |f(w_x)||a|^2|q| \le |a|^2|q|^2 = g(p) = |g(p)|. \tag{63}$$

Choose $\mathbb{I} = (I, J) \in \mathcal{N}$ in such a way that $p \in \mathbb{C}_I$, and let $L \in \mathbb{S}$ be such that $L \perp \mathbb{H}_\mathbb{I}$. Denote $\hat{g} : \Omega_\mathbb{I} = \Omega_D \cap \mathbb{H}_\mathbb{I} \to \mathbb{O}$ the restriction of g to $\mathbb{H}_\mathbb{I}$. By Lemma 4.6 and Remark 4.7 of [13], there exist $\hat{g}_1, \hat{g}_2 : \Omega_\mathbb{I} \to \mathbb{H}_\mathbb{I}$ such that $\hat{g}_1$ is (standard left) Fueter-regular in $\mathbb{H}_\mathbb{I} = \mathbb{H}$, $\hat{g}_2$ is (standard) right Fueter-regular in $\mathbb{H}_\mathbb{I} = \mathbb{H}$ and $\hat{g} = \hat{g}_1 + \hat{g}_2 L$. Note that $\hat{g}_1(p) \in \mathbb{R}$, $\hat{g}_2(p) = 0$ and $|\hat{g}(p)| = g(p) = \hat{g}_1(p) > 0$. By (63), we deduce that p is a maximum of $|\hat{g}|$. Note also that $\Omega_\mathbb{I}$ is connected, because it is dense in $\Omega_\mathbb{I} \setminus \mathbb{R}$ and $\Omega_\mathbb{I} \setminus \mathbb{R}$ is homeomorphic to $D^+ \times \mathbb{S}^2$, where $D^+ = \{(x_0, x_1) \in D : x_1 > 0\}$ is connected by assumption (7).

Now we can adapt to the present situation the argument used in the proof of Theorem 7.1 of [16]. It holds:
$$|\hat{g}_1(p)|^2 = |\hat{g}(p)|^2 \ge |\hat{g}(x)|^2 = |\hat{g}_1(x)|^2 + |\hat{g}_2(x)|^2 \ge |\hat{g}_1(x)|^2 \tag{64}$$
for each $x \in \Omega_\mathbb{I}$. It follows that the modulus of the Fueter-regular function $\hat{g}_1 : \Omega_\mathbb{I} \to \mathbb{H}_\mathbb{I}$ has a maximum at a point of $\Omega_\mathbb{I}$. On the other hand, Fueter-regular functions enjoy Maximum Modulus Principle (see [35, Theorem 7.32]), so $\hat{g}_1$ is constant on its (connected) domain $\Omega_\mathbb{I}$. As a consequence, if $x \in \Omega_\mathbb{I}$, we have:
$$|\hat{g}_2(x)|^2 = |\hat{g}(x)|^2 - |\hat{g}_1(x)|^2 = |\hat{g}(x)|^2 - |\hat{g}(p)|^2 \le 0,$$
hence $\hat{g}_2(x) = 0$ and $\hat{g}(x) = \hat{g}_1(x) + \hat{g}_2(x)L = \hat{g}_1(p) = \hat{g}(p)$. In particular, $\hat{g}$ (and hence g) is constantly equal to $\hat{g}(p) = g(p)$ on $\mathbb{C}_I \cap \Omega_D$. Thanks to Corollary 42, it follows that $g = g(p)$ on $\Omega_D \setminus \mathbb{R}$, and hence on the whole Ω_D by continuity.

It remains to show point (2). Suppose that $p \notin V(f'_s)$ and $|f|$ has a local maximum at p. If $q = f(p) = 0$, then $f = 0$ locally at p in Ω_D. Since Ω_D is connected and f is real analytic, it follows that $f = 0$ on the whole Ω_D, and we are done.

Let $q \neq 0$. We distinguish two cases.

First, we assume that $p \notin \mathbb{R}$, so $p \in \Omega'_D := \Omega_D \setminus (\mathbb{R} \cup V(f'_s))$. As above we define $a := \overline{f'_s(p)}$, $b := (f'_s(p))\overline{q} = \overline{q}a$ and $g := (f \cdot a) \cdot b$. Let U be a neighborhood of p in Ω_D such that $|f(x)| \le |f(p)|$ for each $x \in U$. Since Ω'_D is open in Ω_D and contains x, we can also assume that $U \subset \Omega'_D$. Apply twice Lemma 47(2) to g. We obtain the existence of a homeomorphism $\Psi : \Omega'_D \to \Omega'_D$ such that, for each $x \in \Omega'_D$, $\Psi(\mathbb{S}_x) = \mathbb{S}_x$ and $g(x) = (f(\Psi(x))a)b$.

Let us prove that $\Psi(p) = p$. Since we have
$$(qa)(\overline{q}a) = |q|^2|a|^2 = g(p) = (f(\Psi(p))a)b = (f(\Psi(p))a)(\overline{q}a),$$
dividing first by $\overline{q}a \neq 0$ and then by $a \neq 0$ on the right, we deduce that $f(p) = q = f(\Psi(p))$. On the other hand, f is slice, so $f|_{\mathbb{S}_p}(x) = c + x(f'_s(p))$ for some $c \in \mathbb{O}$. Since $f'_s(p) \neq 0$, $f|_{\mathbb{S}_p}$ turns out to be injective. This proves that $\Psi(p) = p$, as claimed.

Now the continuity of Ψ implies the existence of a neighborhood V of p in Ω'_D such that $\Psi(V) \subset U$. In this way, if $x \in V$, then we have:

$$|g(x)| = |f(\Psi(x))||a|^2|q| \leq |f(p)||a|^2|q| = |a|^2|q|^2 = |g(p)|.$$

Using $\mathbb{I}$, L and $\hat{g} = \hat{g}_1 + \hat{g}_2 L$ as above, we conclude that g is constantly equal to $g(p)$ on the whole Ω_D, completing the proof.

Finally, we assume that $p \in \mathbb{R} \cap \Omega_D$. Applying twice Lemma 47(1) to g again, we obtain that p is a local maximum for $|g|$. Hence, considering $\mathbb{I}$, L and $\hat{g} = \hat{g}_1 + \hat{g}_2 L$ as above, we deduce that $g = g(p)$ locally at p in Ω_D. Since Ω_D is connected and g is real analytic, g turns out to be constantly equal to $g(p)$ on the whole Ω_D. $\square$

4. Final remarks

We conclude the present paper with some comments concerning possible generalizations of the concept of slice Fueter-regular function we studied above.

Let A be an arbitrary finite dimensional real alternative *-algebra with unity, equipped with its natural smooth manifold structure as a real vector space. Consider again a non-empty open subset D of $\mathbb{C}$ invariant under the complex conjugation. Let $F = (F_1, F_2) : D \to A^2$ be a stem function of class $\mathscr{C}^1$. Denote Ω_D the circularization of D in A and $f = \mathcal{I}(F) : \Omega_D \to A$ the slice function induced by F, i.e.

$$f(x_0 + x_1 I) := F_1(x_0, x_1) + IF_2(x_0, x_1)$$

for each $(x_0, x_1) \in D$ and $I \in \mathbb{S}_A := \{x \in A : x^* = -x, x^2 = -1\}$. Recall that Ω_D is an open subset of the quadratic cone Q_A of A. We refer the reader to [25] for details.

4.1. Slice Vekua-regular functions.
Let $\mathsf{V} = (a, b, c, d) : D \setminus \mathbb{R} \to A^4$ be a function *even-odd-odd-even in x_1*, i.e. $a(x_0, -x_1) = a(x_0, x_1)$, $b(x_0, -x_1) = -b(x_0, x_1)$, $c(x_0, -x_1) = -c(x_0, x_1)$ and $d(x_0, -x_1) = d(x_0, x_1)$.

Consider the Vekua-type system S_V induced by V:

$$\mathsf{S}_\mathsf{V} : \begin{cases} \dfrac{\partial F_1}{\partial x_0} - \dfrac{\partial F_2}{\partial x_1} + aF_1 + bF_2 = 0 \\[2mm] \dfrac{\partial F_1}{\partial x_1} + \dfrac{\partial F_2}{\partial x_0} + cF_1 + dF_2 = 0. \end{cases} \tag{65}$$

Note that the function $\frac{\partial F_1}{\partial x_0} - \frac{\partial F_2}{\partial x_1} + aF_1 + bF_2$ is even in x_1 and the function $\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_0} + cF_1 + dF_2$ is odd in x_1. If V is $\mathbb{R}^4$-valued, then above system S_V is a particular case of systems known as *Bers-Vekua* or *Carleman-Bers-Vekua systems*. The study of the latter systems is the core of the pseudoanalytic function theory, see Sections 6.2 and 6.3 of [36] and their references.

A natural possibility to define the concept of slice Vekua-regular function is as follows:

Definition 50. Let $\mathsf{V} = (a, b, c, d) : D \setminus \mathbb{R} \to A^4$ be any given even-odd-odd-even function in x_1. We say that the slice function $f = \mathcal{I}((F_1, F_2)) : \Omega_D \to A$ is a *slice Vekua-regular function with respect to* V, or a *slice V-Vekua-regular function* for short, if the stem function (F_1, F_2) inducing f satisfies system S_V defined in (65). $\blacksquare$

One can ask whether there exist pairs (A, V) such that the corresponding slice V-Vekua-regular functions f share significant results with quasianalytic functions; or better, such that such functions f have a 'holomorphic nature', satisfying for example versions of Cauchy integral formula, Taylor and Laurent series expansions, and Maximum Modulus Principle.

The results of the present paper show that the answer is yes, at least in the case $A := \mathbb{O}$ and $\mathsf{V}(x_0, x_1) := (0, 2x_1^{-1}, 0, 0)$.

Question 51. Do there exist other relevant pairs (A, V)?

4.2. **Slice Dirac-regular functions.** A Dirac operator can be defined as a first order differential operator which factorizes a Laplacian. A remarkable example of Dirac operator is the Cauchy-Riemann-Fueter operator $\overline{\mathcal{D}} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$ on quaternions. Similar examples can be defined over other real alternative *-algebras as Clifford algebras and octonions.

In this way, one can generalize the concept of slice Fueter-regular function as follows. Let A be a finite dimensional real alternative *-algebra with unity, let B be a real subalgebra of A and let $\mathscr{D}$ be a Dirac operator on B. Given a slice function $f : \Omega_D \to A$ of class $\mathscr{C}^1$, we say that f is a *slice $\mathscr{D}$-Dirac-regular function* if the restriction of f to $B \cap \Omega_D$ belongs to the kernel of $\mathscr{D}$. If the triple $(A, B, \mathscr{D})$ is equal to $(\mathbb{O}, \mathbb{H}, \overline{\mathcal{D}})$, then above assertion (5) implies that the just defined concept of slice $\mathscr{D}$-Dirac-regular function coincides with the one of slice Fueter-regular function.

Also in this case, it is natural to ask the following:

Question 52. Do there exist other relevant triples $(A, B, \mathscr{D})$?

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