

BI-SKEW BRACES AND REGULAR SUBGROUPS OF THE HOLOMORPH

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ABSTRACT. L. Childs has defined a skew brace (G, \cdot, \circ) to be a bi-skew brace if (G, \circ, \cdot) is also a skew brace, and has given applications of this concept to the equivalent theory of Hopf-Galois structures.

The goal of this paper is to deal with bi-skew braces (G, \cdot, \circ) from the yet equivalent point of view of regular subgroups of the holomorph of (G, \cdot) . In particular, we find that certain groups studied by T. Kohl, F. Dalla Volta and the author, and C. Tsang all yield examples of bi-skew braces.

Building on a construction of Childs, we also give various methods for exhibiting further examples of bi-skew braces.

1. INTRODUCTION

Let $G = (G, \cdot)$ be a group. A (right) *skew brace* with *additive group* (G, \cdot) is a triple (G, \cdot, \circ) , where \circ is an operation on G such that (G, \circ) is also a group, and the following axiom holds

$$(xy) \circ z = (x \circ z) \cdot z^{-1} \cdot (y \circ z), \quad \text{for } x, y, z \in G. \quad (1.1)$$

The group (G, \circ) is called the *circle group* of the skew brace.

According to L. Childs [Chi19], such a skew brace is called a *bi-skew brace* if (G, \circ, \cdot) is also a skew brace. Given an arbitrary group (G, \cdot) , the trivial skew brace (G, \cdot, \cdot) is clearly a bi-skew brace. If G is non-abelian, setting $x \circ y = y \cdot x$ we obtain another (trivial) example (G, \cdot, \circ) of a bi-skew brace.

It is well known that the specification of a skew brace with additive group (G, \cdot) corresponds to the choice of a regular subgroup of the holomorph $\text{Hol}(G)$ of (G, \cdot) ; we recall the details of this correspondence in Section 2. The two trivial examples of bi-skew braces we have mentioned in the previous paragraph correspond to the images of the right resp. left regular representation of (G, \cdot) .

The goal of this paper is to discuss the concept of a bi-skew brace from the point of view of regular subgroups, and of the associated

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gamma functions, as defined and used by F. Dalla Volta, E. Campedel, I. Del Corso and the author in [CDVS06, CDV17, CDV18, Car18, CCDC20]. From this discussion, which we carry out in Section 3, it will follow among others that the multiple holomorphs studied by T. Kohl [Koh15], F. Dalla Volta and the author [CDV17, CDV18], the author [Car18], and C. Tsang [Tsa20] all yield examples of bi-skew braces.

Finally, building on a construction of Childs, in Section 4 we exhibit various methods for constructing more examples of bi-skew braces, and review from the point of view of bi-skew braces results of J. C. Ault and J. F. Watters [AW73] about the occurrence of nilpotent groups of nilpotence class 2 as circle groups of skew braces.

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2. PRELIMINARIES

We reprise the following from [CCDC20]. Let $G = (G, \cdot)$ be a group. There is a one-to-one correspondence between binary operations \circ on G , and maps $\gamma : G \rightarrow G^G$, where G^G is the set of maps from G to G , given by

$$x^{\gamma(y)} = (x \circ y) \cdot y^{-1}, \quad \text{and} \quad x \circ y = x^{\gamma(y)}y, \quad \text{for } x, y \in G.$$

In Table 1 we exhibit a correspondence between certain properties of the operation \circ , and certain properties of the corresponding map γ .

| Property of \circ | Property of γ |
|-------------------------|---|
| Axiom (1.1) holds | $\gamma(g) \in \text{End}(G, \cdot)$, for $g \in G$ |
| \circ is associative | $\gamma(x^{\gamma(y)}y) = \gamma(x)\gamma(y)$ holds, for $x, y \in G$ |
| \circ admits inverses | $\gamma(g)$ is bijective, for $g \in G$ |

Table 1: \circ and γ

The properties on the first line are equivalent. The properties on the second line are equivalent, under the assumption that the properties on the first line hold. On the third line, the property on the right implies the property on the left, while to prove the left-to-right implication one need to assume the property on the right in the second line. The fact that (G, \circ) has the same identity as (G, \cdot) follows from the properties in the first line.

Taken together, the properties in the left column state that (G, \cdot, \circ) is a (right) skew brace. A function γ satisfying the properties in the

right column is called a gamma function, or a GF for short. We will actually need the following

Definition 2.1 ([CCDC20, Definition 2.1]). Let G be a group, $A \leq G$, and $\gamma : A \rightarrow \text{Aut}(G)$ a function.

γ is said to satisfy the *gamma functional equation* (or *GFE* for short) if

$$\gamma(g^{\gamma(h)}h) = \gamma(g)\gamma(h), \quad \text{for all } g, h \in A.$$

γ is said to be a *relative gamma function* (or *RGF* for short) on A if it satisfies the gamma functional equation, and A is $\gamma(A)$ -invariant.

If $A = G$, a relative gamma function is simply called a *gamma function* (or *GF* for short) on G .

In Section 3 we will be using gamma functions only; relative gamma functions will be used for two examples in Section 4.

Remark 2.2. Note that, referring to Table 1, a GF γ on G is a homomorphism of groups $\gamma : (G, \circ) \rightarrow \text{Aut}(G, \cdot)$. In particular $\ker(\gamma) = \{g \in G : \gamma(g) = 1\}$ is a normal subgroup of (G, \circ) , but in general only a subgroup of (G, \cdot) .

Recall that the right regular representation is the homomorphism

$$\begin{aligned} \rho : G &\rightarrow S(G) \\ g &\mapsto (x \mapsto xg), \end{aligned}$$

where $S(G)$ is the group of permutations on the set G . The (permutational) holomorph of a group G is the normaliser in $S(G)$ of the image of the right regular representation ρ of G ,

$$\text{Hol}(G) = N_{S(G)}(\rho(G)) = \text{Aut}(G)\rho(G),$$

where the latter is a semidirect product.

Let N be a regular subgroup of $\text{Hol}(G)$, that is, N is transitive, and the one-point stabilisers are trivial. The map $N \rightarrow G$ which takes $n \in N$ to the image 1^n of $1 \in G$ under n is thus a bijection. Let $\nu : G \rightarrow N$ be its inverse. Thus $\nu(g)$ is the unique element of N such that $1^{\nu(g)} = g$, for all $g \in G$. Since $N \leq \text{Hol}(G)$, we can write $\nu(g) = \gamma(g)\rho(g)$, for a suitable function $\gamma : G \rightarrow \text{Aut}(G)$. It is immediate to see that γ is a gamma function, and that conversely every gamma function γ determines a regular subgroup N of $\text{Hol}(G)$ as $N = \{\gamma(g)\rho(g) : g \in G\}$. It follows that the data of Table 1 are equivalent to specifying a regular subgroup N of the holomorph $\text{Hol}(G)$, as the set of maps

$$N = \{x \mapsto x \circ y : y \in G\} = \{\gamma(y)\rho(y) : y \in G\},$$

as for $x, y \in G$ one has $x^{\gamma(y)\rho(y)} = x^{\gamma(y)}y = x \circ y$. For such an N , the map

$$\begin{aligned} \nu : (G, \circ) &\rightarrow N \\ y &\mapsto (x \mapsto x \circ y) \end{aligned}$$

is an isomorphism of groups.

For the details, see the discussions in [CDV18, CCDC20].

Remark 2.3. *In the rest of the paper, we will freely use the fact that, given a group (G, \cdot) , any of the following data uniquely determines one of the others*

- (1) an operation \circ on G such that (G, \cdot, \circ) is a skew brace,
- (2) a regular subgroup $N \leq \text{Hol}(G)$, and
- (3) a gamma function $\gamma : G \rightarrow \text{Aut}(G)$.

We also recall from [CCDC20, Proposition 2.21] that, given a regular subgroup $N \leq \text{Hol}(G)$ with associated GF γ , the GF associated to the regular subgroup $N^{\text{inv}} \leq \text{Hol}(G)$ is

$$\bar{\gamma}(y) = \gamma(y^{-1})\rho(y^{-1}).$$

Here $\text{inv} \in S(G)$ is the inverse map $x \mapsto x^{-1}$ on G , which normalises $\text{Hol}(G)$, and thus acts by conjugacy on the set of regular subgroups of $\text{Hol}(G)$. This construction is equivalent to the opposite skew brace construction of [KT20].

3. REGULAR SUBGROUPS OF THE HOLOMORPH, AND GAMMA FUNCTIONS

3.1. Regular subgroups. A skew brace (G, \cdot, \circ) is said to be a bi-skew brace if (G, \circ, \cdot) is also a skew brace [Chi19].

We reinterpret this definition in terms of regular subgroups and gamma functions. Let

- (1) γ be the GF associated to (G, \cdot, \circ) , that is

$$x \circ y = x^{\gamma(y)}y, \text{ for } x, y \in G,$$

and

- (2) γ' the GF associated to (G, \circ, \cdot) , that is

$$xy = x^{\gamma'(y)} \circ y, \text{ for } x, y \in G.$$

Therefore, for $x, y \in G$, we have

$$xy = x^{\gamma'(y)} \circ y = x^{\gamma'(y)\gamma(y)}y,$$

so that for $y \in G$ we have

$$\gamma'(y) = \gamma(y)^{-1}. \tag{3.1}$$

This implies, as per Remark 2.2, that $\ker(\gamma) = \ker(\gamma')$ is normal in both (G, \cdot) and (G, \circ) .

We now state the following theorem, in which we use the convention of Remark 2.3, and write

$$\begin{aligned} \iota : G &\rightarrow \text{Aut}(G) \\ y &\mapsto (x \mapsto y^{-1}xy) \end{aligned}$$

for the map taking $y \in G$ to the inner automorphism of G it induces.

Theorem 3.1. *Let $G = (G, \cdot)$ be a group.*

The following data are equivalent:

- (1) *A bi-skew brace (G, \cdot, \circ) ;*
- (2) *a regular subgroup N of $\text{Hol}(G)$ which is normalised by $\rho(G)$;*
- (3) *a GF $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies,*

$$\gamma(xy) = \gamma(y)\gamma(x), \text{ for } x, y \in G,$$

that is, γ is an anti-homomorphism;

- (4) *a GF $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies*

$$\gamma(x^{\gamma(y)}) = \gamma(x)^{\gamma(y)}, \text{ for } x, y \in G;$$

- (5) *a function $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies, for $x, y \in G$,*

$$\begin{cases} \gamma(xy) = \gamma(y)\gamma(x), \\ \gamma(x^{\gamma(y)}) = \gamma(x)^{\gamma(y)}; \end{cases}$$

- (6) *a GF $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies*

$$\gamma([G, \bar{\gamma}(G)]) = 1;$$

- (7) *a GF $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies*

$$\gamma(x^{-1}y^{-1}x^{\gamma(y)}y) = 1, \text{ for } x, y \in G;$$

Definition 3.2. We will refer to a GF on G which satisfies the conditions of Theorem 3.1 as a *bi-GF*.

Proof of Theorem 3.1. To say that (G, \cdot, \circ) is a skew brace is equivalent to saying that γ maps G to $\text{Aut}(G)$, and that γ satisfies the GFE

$$\gamma(x^{\gamma(y)}y) = \gamma(x)\gamma(y), \quad \text{for } x, y \in G.$$

If (G, \circ, \cdot) is also a skew brace, then the values of γ' , and by (3.1) also those of γ , are in $\text{Aut}(G, \circ)$. Therefore we have, for $x, y, z \in G$,

$$(x \circ y)^{\gamma(z)} = x^{\gamma(z)} \circ y^{\gamma(z)} = x^{\gamma(z)\gamma(y^{\gamma(z)})}y^{\gamma(z)}$$

and also

$$(x \circ y)^{\gamma(z)} = (x^{\gamma(y)}y)^{\gamma(z)} = x^{\gamma(y)\gamma(z)}y^{\gamma(z)}.$$

It thus follows that for $y, z \in G$ we have

$$\gamma(y^{\gamma(z)}) = \gamma(z)^{-1}\gamma(y)\gamma(z) = \gamma(y)^{\gamma(z)}, \quad (3.2)$$

where in the last term we have used the notation $u^v = v^{-1}uv$ for the conjugate in a group. We have shown that (1) implies (4).

(4) implies that $\gamma : G \rightarrow \text{Aut}(G)$ is an anti-homomorphism, that is, (3), as

$$\gamma(xy) = \gamma(x^{\gamma(y)^{-1}})\gamma(y) = \gamma(x)^{\gamma(y)^{-1}}\gamma(y) = \gamma(y)\gamma(x). \quad (3.3)$$

Conversely, (3) implies (4), so that the two conditions are equivalent. In fact, assuming (3) we have

$$\gamma(x^{\gamma(y)^{-1}}) = \gamma(xy)\gamma(y)^{-1} = \gamma(y)\gamma(x)\gamma(y)^{-1} = \gamma(x)^{\gamma(y)^{-1}}.$$

(Therefore conditions (3.2) and (3.3) are a weaker form of the conditions of Theorem 5.2(2) of [CDV18], see Subsection 3.3 below.)

As (3) and (4) are equivalent, each of them implies (5). Conversely, if (5) holds, we have, for $x, y \in G$,

$$\gamma(x^{\gamma(y)}y) = \gamma(y)\gamma(x^{\gamma(y)}) = \gamma(y)\gamma(x)^{\gamma(y)} = \gamma(x)\gamma(y),$$

that is, γ is a GF, so that (3) and (4) hold.

If γ satisfies (4), and thus also (3), then the map $\gamma' : y \mapsto (x \mapsto x^{\gamma(y)^{-1}})$ is a GF on (G, \circ) , whose associated operation is “ \cdot ”. In fact we have, for $x, y, z \in G$,

$$\begin{aligned} (x \circ y)^{\gamma(z)^{-1}} &= (x^{\gamma(y)}y)^{\gamma(z)^{-1}} \\ &= x^{\gamma(y)\gamma(z)^{-1}}y^{\gamma(z)^{-1}} \\ &= (x^{\gamma(z)^{-1}})^{\gamma(y^{\gamma(z)^{-1}})}y^{\gamma(z)^{-1}} \\ &= x^{\gamma(z)^{-1}} \circ y^{\gamma(z)^{-1}}, \end{aligned}$$

so that $\gamma(z)^{-1} \in \text{Aut}(G, \circ)$. Also,

$$x^{\gamma(y)^{-1}} \circ y = x^{\gamma(y)^{-1}\gamma(y)}y = x \cdot y,$$

and finally

$$\begin{aligned} \gamma(x^{\gamma(y)^{-1}} \circ y)^{-1} &= \gamma(xy)^{-1} \\ &= (\gamma(y)\gamma(x))^{-1} \\ &= \gamma(x)^{-1}\gamma(y)^{-1}, \end{aligned}$$

that is, γ' is a GF on (G, \circ) . This shows that either of (4) or (3) implies (1).

Let

$$N = \{\gamma(x)\rho(x) : x \in G\}$$

be the regular subgroup of $\text{Hol}(G)$ associated to the skew brace (G, \cdot, \circ) and the GF γ . For $x, y \in G$ we have

$$\begin{aligned} [\rho(x), \gamma(y)\rho(y)] &= \rho(x)^{-1}\rho(y)^{-1}\gamma(y)^{-1}\rho(x)\gamma(y)\rho(y) \\ &= \rho(x^{-1}y^{-1}x^{\gamma(y)}y). \end{aligned}$$

Thus N is normalised by $\rho(G)$ if and only if

$$\gamma(x^{-1}y^{-1}x^{\gamma(y)}y) = 1.$$

Therefore (2) and (7) are equivalent.

(6) is a restatement of (7), as

$$x^{-1}y^{-1}x^{\gamma(y)}y = x^{-1}x^{\gamma(y)\iota(y)} = [x, \bar{\gamma}(y^{-1})].$$

Assuming (3) and (4), we get (7), as for $x, y \in G$ we have

$$\gamma(x^{-1}y^{-1}x^{\gamma(y)}y) = \gamma(y)\gamma(x)^{\gamma(y)}\gamma(y)^{-1}\gamma(x)^{-1} = 1$$

We conclude the proof by showing that (7) implies (3). Property (7) can be rewritten as

$$\gamma((yx)^{-1}(x^{\gamma(y)}y)) = 1$$

for all $x, y \in G$, that is, there is $k \in \ker(\gamma)$ such that

$$x^{\gamma(y)}y = yxk,$$

so that we have, for all $x, y \in G$,

$$\gamma(x)\gamma(y) = \gamma(x^{\gamma(y)}y) = \gamma((yx)k) = \gamma((yx)^{\gamma(k)^{-1}})\gamma(k) = \gamma(yx),$$

that is, (3) holds. \square

The equivalence of (3), (4) and (5) can be stated as the following

Lemma 3.3. *Let G be a group, and $\gamma : G \rightarrow \text{Aut}(G)$ a function.*

Any two of the following conditions imply the third.

- (1) γ is a GF on G ,
- (2) γ is an anti-homomorphism, and
- (3) $\gamma(x^{\gamma(y)}) = \gamma(x)^{\gamma(y)}$, for $x, y \in G$.

3.2. Cubes. In [CDVS06, FCC12] it is proved that if $(G, +, 0)$ is an abelian group, then the abelian regular subgroups $N \leq \text{Hol}(G)$ can be described via the structures of commutative, radical rings $(G, +, *)$. In [CDVS06], the condition that N normalises $\rho(G)$ was shown to be equivalent in this context to

$$G * G * G = \{0\} \tag{3.4}$$

We now show that Theorem 3.1(7) translates as expected to (3.4) when G and N are abelian. Recalling from [CDVS06, FCC12] that we have

$$x^{\gamma(y)} = x + x * y,$$

the additive version of Theorem 3.1(7) reads

$$\begin{aligned} 0 &= x^{\gamma(y^{\gamma(z)}-y)} - x \\ &= x^{\gamma(y*z)} - x \\ &= x * y * z \end{aligned}$$

for $x, y, z \in G$, that is, (3.4) holds.

3.3. A stronger condition. In [CDV17, CDV18, Car18] the regular subgroups N of $\text{Hol}(G)$ which are normal in $\text{Hol}(G) = \text{Aut}(G)\rho(G)$ were studied. Clearly this is a stronger condition than Theorem 3.1(2). We also note that regular subgroups $N \trianglelefteq \text{Hol}(G)$ correspond to the GF's γ on G which satisfy

$$\gamma(x^\beta) = \gamma(x)^\beta \text{ for } x \in G, \beta \in \text{Aut}(G),$$

which is again a stronger condition than Theorem 3.1(4).

We obtain the following.

Theorem 3.4. *Let $G = (G, \cdot)$ be a group.*

The following data are equivalent:

- (1) *A skew brace (G, \cdot, \circ) , such that $\text{Aut}(G, \cdot) \leq \text{Aut}(G, \circ)$;*
- (2) *a skew brace (G, \cdot, \circ) which is unique of its isomorphism type among skew braces with additive group (G, \cdot) ;*
- (3) *a regular subgroup $N \trianglelefteq \text{Hol}(G)$;*
- (4) *a regular subgroup $N \leq \text{Hol}(G)$ which is normalised by $\text{Aut}(G)$;*
- (5) *a GF $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies*

$$\gamma(x^\beta) = \gamma(x)^\beta \text{ for } x \in G, \beta \in \text{Aut}(G);$$

- (6) *a function $\gamma : G \rightarrow \text{Aut}(G)$ which satisfies*

$$\begin{cases} \gamma(xy) = \gamma(y)\gamma(x), \text{ for } x, y \in G, \text{ and} \\ \gamma(x^\beta) = \gamma(x)^\beta \text{ for } x \in G, \beta \in \text{Aut}(G). \end{cases}$$

The skew braces appearing in the statement are all bi-skew braces.

In a number of papers, the regular subgroups which are normal in the holomorph of a group in certain classes of groups have been described, thereby providing examples of non-trivial bi-skew braces satisfying the conditions of the Theorem. In Table 2 we list these papers, briefly describing the corresponding classes of groups for the convenience of the reader.

| | |
|---------|--|
| [Koh15] | Dihedral and quaternionic groups |
| [CDV17] | Finitely generated abelian groups |
| [CDV18] | Finite, perfect, centreless groups |
| [Car18] | Finite p -groups of nilpotence class two |
| [Tsa20] | Finite groups of squarefree orders, and finite p -groups of nilpotence class $< p$ |

Table 2: Examples

Proof of Theorem 3.4. (1), (3), (5) and (6) are shown to be equivalent in [CDV18].

Since N is regular, we have

$$\text{Hol}(G) = \text{Aut}(G)\rho(G) = \text{Aut}(G)N, \quad (3.5)$$

as $\text{Aut}(G)$ is the stabiliser of 1 in $\text{Hol}(G)$, so that (3) is equivalent to (4).

As to (2), the isomorphism classes of skew braces (G, \cdot, \circ) correspond to the conjugacy classes of regular subgroups N of $\text{Hol}(G)$, or equivalently, according to (3.5), the orbits of these regular subgroups under the action of $\text{Aut}(G)$. \square

We now recall a result of Miller [Mil08].

Theorem 3.5. *The group*

$$T(G) = N_{S(G)}(\text{Hol}(G)) / \text{Hol}(G)$$

acts regularly on the set

$$\mathcal{H}(G) = \left\{ N \leq S(G) : N \text{ is regular, } N \cong G \text{ and } N_{S(G)}(N) = \text{Hol}(G) \right\}.$$

In particular, the map

$$\begin{aligned} \mathcal{T} : T(G) &\rightarrow \mathcal{H}(G) \\ \text{Hol}(G)\tau &\mapsto \rho(G)^\tau. \end{aligned}$$

is well defined and bijective.

Note that $\rho(G)^\tau = \tau^{-1}\rho(G)\tau$ denotes the conjugate of the subgroup $\rho(G)$ of $S(G)$ by the element $\tau \in S(G)$.

The inversion map $\text{inv} : x \mapsto x^{-1}$ normalises $\text{Hol}(G)$, as it centralises $\text{Aut}(G)$ and satisfies $\rho(G)^{\text{inv}} = \lambda(G)$. If G is non-abelian, then $\text{inv} \notin \text{Aut}(G)$, so that $\text{Hol}(G)\text{inv}$ is an element of $T(G)$ different from the identity.

Note that a regular subgroup $N \in \mathcal{H}(G)$ satisfies $N \trianglelefteq \text{Hol}(G)$, and thus yields a bi-skew brace. We obtain

Corollary 3.6. *Let $G = (G, \cdot)$ be a group.*

- (1) \mathcal{T} yields an injective map from $T(G)$ to the set of isomorphism classes of bi-skew braces of Theorem 3.4.
- (2) In particular,
 - (a) if G is non-abelian, and $|T(G)| > 2$, or
 - (b) if G is abelian and $T(G) \neq \{1\}$,
 then there are non-trivial examples of bi-skew braces with additive group G .

However,

- (1) there are non-abelian groups (G, \cdot) which are the additive group of a bi-skew brace, such that the latter corresponds to a regular subgroup N which is not in the image $\mathcal{H}(G)$ of \mathcal{T} , and
- (2) there are abelian groups (G, \cdot) with $T(G) = \{1\}$, for which there are non-trivial bi-skew braces with additive group G .

We postpone the proof of the last two statements to Subsection 4.5.

Note that the bi-skew braces (G, \cdot, \circ) in the image of \mathcal{T} all have $(G, \cdot) \cong (G, \circ)$. The referee has asked whether the converse holds. In Example 3.8, we show that this is not the case.

We first note the following Lemma, whose proof is immediate, and which will also be used in Subsection 4.4.

Lemma 3.7. *Let G be a group, and γ a GF on G . Any two of the following conditions imply the third.*

- (1) $\gamma : (G, \cdot) \rightarrow \text{Aut}(G, \cdot)$ is a homomorphism,
- (2) $\gamma(G)$ is abelian,

(3) $\gamma : (G, \cdot) \rightarrow \text{Aut}(G, \cdot)$ is an anti-homomorphism.

The third condition states, according to Theorem 3.1, that γ is a bi-GF.

Example 3.8. Let $p > 2$ and q be primes such that $q \mid p - 1$. Let G be the non-trivial semidirect product of a cyclic group of order p^2 by a cyclic group of order q .

In [CCDC20, Proposition 4.5] it is shown that there are regular subgroups $N \leq \text{Hol}(G)$ such that $(G, \circ) \cong N \cong (G, \cdot)$, and $N_{\text{Hol}(G)}(N)$ has index p^2 in $\text{Hol}(G)$; in particular, $N \not\trianglelefteq \text{Hol}(G)$, so that N is not in the image of \mathcal{T} . We will show that the GF associated to these N satisfy Theorem 3.1(3).

In [CCDC20, 4.4.1], the GF γ associated to these regular subgroups N are described. These are the morphisms $\gamma : (G, \cdot) \rightarrow \text{Aut}(G, \cdot)$ which have the Sylow p -subgroup of G as their kernel, and have thus a cyclic image of order q . According to Lemma 3.7, condition (3) of Theorem 3.1 holds, so that γ is a bi-GF.

4. SOME CONSTRUCTIONS

We now record some constructions for bi-skew braces, which in particular cover, and generalise, the examples of [CDV18, Car18, Tsa20].

4.1. A result of Childs. We begin with two generalisations of the following result of Childs, which we reformulate in our terminology.

Proposition 4.1 ([Chi19, Proposition 7.1]). *Let the group G be the semidirect product of $K \trianglelefteq G$ by $H \leq G$.*

Then the map

$$\begin{aligned} \gamma : G &\rightarrow \text{Aut}(G) \\ hk &\mapsto \iota(h^{-1}) \end{aligned}$$

for $h \in H$ and $k \in K$, is a bi-GF on G .

We recall a result from [CCDC20, Proposition 2.14].

Proposition 4.2. *Let G be a group, and $H, K \leq G$ such that $G = HK$.*

Let $\gamma' : H \rightarrow \text{Aut}(G)$ be a RGF such that

- (1) $\gamma'(H \cap K) \equiv 1$,
- (2) K is invariant under $\{\gamma'(h)\iota(h) : a \in H\}$.

Then the map

$$\gamma(hk) = \gamma'(h), \text{ for } h \in H, k \in K,$$

is a well defined GF on G , and $\ker(\gamma) = \ker(\gamma')K$.

4.2. Allowing central intersections. Our first result is a straightforward generalisation of Proposition 4.1, in which we allow H and K to intersect centrally.

Proposition 4.3. *Let (G, \cdot) be a group. Let $H, K \leq G$ such that*

- (1) $K \trianglelefteq G$,
- (2) $HK = G$, and
- (3) $H \cap K \leq Z(G)$.

Then the map $\gamma : G \rightarrow \text{Aut}(G)$ given by, for $h \in H$ and $k \in K$,

$$\gamma(hk) = \iota(h^{-1}) \tag{4.1}$$

is well defined, is a GF on G , and defines a bi-skew brace (G, \cdot, \circ) .

Moreover, the following are equivalent:

- (1) $\bar{\gamma}$ is also a bi-GF, and
- (2) $H \trianglelefteq G$.

The bi-skew braces constructed in this way comprise in particular the bi-skew braces arising from the groups of [CDV18]. Examples in finite p -groups, like the following one, are easy to construct.

Example 4.4. Let $p > 2$ be a prime, and let K be an elementary abelian group of order p^n , for $p > n \geq 2$, with generators a_0, \dots, a_{n-1} . There is a unique automorphism β of K such that $a_i \mapsto a_i a_{i+1}$ for $i = 0, \dots, n-1$ (where we take $a_n = 1$), and β has order p . The group presented as

$$G = \langle K, b : b^p = a_{n-1}, b^{-1}ab = a^\beta \text{ for } a \in K \rangle$$

satisfies the hypotheses of Proposition 4.3, with $H = \langle b \rangle$, and does not split as a semidirect product over K , as K has index p in G , and all elements outside K have order p^2 . It is easy to see that (G, \circ) is abelian here.

Proof of Proposition 4.3. γ is clearly well defined because the intersection of H and K lies in the centre.

Now γ restricted to H is a RGF. Proposition 4.2 shows that the extension (4.1) is a GF on G , which is readily seen to satisfy Theorem 3.1(3), as for $j_1, h_2 \in H$ and $k_1, k_2 \in K$ we have

$$\begin{aligned} \gamma(h_1 k_1 h_2 k_2) &= \gamma(h_1 h_2 k_1^{h_2} k_2) \\ &= \iota(h_1 h_2)^{-1} \\ &= \iota(h_2)^{-1} \iota(h_1)^{-1} \\ &= \gamma(h_2 k_2) \gamma(h_1 k_1). \end{aligned}$$

Therefore γ is a bi-GF.

If $\bar{\gamma}$ is a bi-GF, then $H = \ker(\bar{\gamma}) \trianglelefteq G$. Conversely, if $H \trianglelefteq G$, then we can exchange the roles of H and K . \square

4.3. Pairs of compatible automorphisms of semidirect products. With the second extension of Childs's result we are back to semidirect products, where we allow further GF's on H .

We start by recalling a fact from [Cur08, Theorem 1] about automorphisms of semidirect products. Let G be a semidirect product of $K \trianglelefteq G$ by $H \leq G$. An automorphism d of H can be extended to an automorphism of G that leaves K invariant, acting on K as $a \in \text{Aut}(K)$, if and only if the relation

$$\iota(h)^a = \iota(h^d), \text{ for } h \in H \quad (4.2)$$

holds, where

$$\begin{aligned} \iota : H &\mapsto \text{Aut}(K) \\ h &\mapsto (k \mapsto k^h). \end{aligned}$$

Note that d only determines a up to an element of $C_{\text{Aut}(K)}(\iota(H))$.

We can thus consider the subset \mathcal{P} of $\text{Aut}(G)$ given by all automorphisms of G which leave H and K invariant, and act as $d \in \text{Aut}(H)$ on H , and as $a \in \text{Aut}(K)$ on K , where a and d are related by (4.2); we write such an automorphism as da .

Now \mathcal{P} is clearly a subgroup of $\text{Aut}(G)$, as if $d_1a_1, d_2a_2 \in \mathcal{P}$, then for their product $d_1a_1d_2a_2 = d_1d_2a_1a_2$ (4.2) yields, for $h \in H$,

$$\iota(h)^{a_1a_2} = \iota(h^{d_1})^{a_2} = \iota(h^{d_1d_2}),$$

so that $d_1d_2a_1a_2 \in \mathcal{P}$.

We can now state

Proposition 4.5. *Let G be a semidirect product of $K \trianglelefteq G$ by $H \leq G$.*

Let $\gamma' : H \rightarrow \mathcal{P}$ be a RGF that satisfies

$$\gamma'(h_1h_2) = \gamma'(h_2)\gamma'(h_1). \text{ for } h_1, h_2 \in H.$$

In particular, γ' composed with the restriction to H is a bi-GF on H .

Then the map $\gamma : G \rightarrow \text{Aut}(G)$ defined by

$$\gamma(hk) = \gamma'(h), \text{ for } h \in H, k \in K$$

is a bi-GF on G .

Proposition 4.1 of Childs can be regarded as the special case of this, where $\gamma'(h) = \iota(h^{-1})$ maps H onto the subgroup

$$\{x \mapsto h x h^{-1} : h \in H\}$$

of \mathcal{P} .

In [CCDC20] there are examples of the construction of this Proposition, related to the ‘‘lifting’’ of Proposition 4.2.

Example 3.8 above is of this kind, where K is cyclic of order p^2 , H is cyclic of order q , where p and $q \mid p-1$ are primes, and $\gamma'(h) : x \mapsto h^t x h^{-t}$, for $h \in H$, for some $t \neq 1$. Here, however, the component on H of the automorphism in \mathcal{P} is trivial.

Another instance occurs in [CCDC20, 4.2.2], which we describe in the following

Example 4.6. Let $K = \langle k \rangle$ be a cyclic group of order q , and $H = \langle h \rangle$ be a cyclic group of order p^2 , where $p > 2$ and q are primes, with $p \mid q-1$. Let G be the semidirect product of K by H , where H induces on K a group of automorphisms of order p .

Let $\psi \in \text{Aut}(H)$ be the map $x \mapsto x^{1+p}$. Then for each s, t , it is immediate to see that $\iota(h)_{\bar{1}K}^{-s} \psi^t$ is in \mathcal{P} , and that

$$\begin{aligned} \gamma' : H &\rightarrow \text{Aut}(G) \\ h^i &\mapsto \iota(h)_{\bar{1}K}^{-si} \psi^{ti} \end{aligned}$$

is a RGF satisfying the hypothesis of the Proposition. It is shown in [CCDC20, 4.2.2] that when $s \not\equiv 1 \pmod{p}$ we have $(G, \circ) \cong (G, \cdot)$, while for $s \equiv 1 \pmod{p}$, the group (G, \circ) is cyclic.

Proof of Proposition 4.5. Proposition 4.2 yields that γ is a GF on G .

Now for $h_1, h_2 \in H$ and $k_1, k_2 \in K$ we have

$$\begin{aligned} \gamma(h_1 k_1 h_2 k_2) &= \gamma(h_1 h_2 k_1^{h_2} k_2) \\ &= \gamma'(h_1 h_2) \\ &= \gamma'(h_2) \gamma'(h_1) \\ &= \gamma(h_2 k_2) \gamma(h_1 k_1), \end{aligned}$$

so that γ is a bi-GF on G . \square

4.4. Bi-skew braces from bi-homomorphisms into the centre.

We now give a construction that generalises the examples of bi-skew braces coming from [Car18, Theorem 5.5] and [Tsa20].

Proposition 4.7. *Let G be a group, and $K \leq Z(G)$.*

Let

$$\Delta : G/K \times G/K \rightarrow K$$

be a bi-homomorphism, that is, a function that is a homomorphism in each of the two variables.

Then the function $\gamma : G \rightarrow G^G$ given by

$$x^{\gamma(y)} = \Delta(xK, yK)x \tag{4.3}$$

is a bi-GF, which satisfies the conditions of Lemma 3.7.

It is easy to construct examples of this kind in any non-trivial finite p -group G . Given any (non-trivial) subgroup $K \leq Z(G)$, any homomorphism

$$G/G'K \otimes G/G'K \rightarrow K$$

will yield a Δ as in the Theorem; here G' is the derived subgroup of G , and the tensor product is taken over \mathbb{Z} . In the particular case when K

has exponent p , and $K \leq \text{Frat}(G)$, where $\text{Frat}(G)$ denotes the Frattini subgroup of G , we obtain maps Δ from any linear map

$$G/\text{Frat}(G) \otimes G/\text{Frat}(G) \rightarrow K,$$

where $G/\text{Frat}(G)$ and K are regarded as vector spaces over the field with p elements

Proof of Proposition 4.7. For $x, y, z \in G$ we have, since Δ takes values in the centre of G ,

$$\begin{aligned} (xy)^{\gamma(z)} &= \Delta(xyK, zK)xy \\ &= \Delta(xK, zK)\Delta(yK, zK)xy \\ &= \Delta(xK, zK)x\Delta(yK, zK)y \\ &= x^{\gamma(z)}y^{\gamma(z)}, \end{aligned}$$

that is, $\gamma(z) \in \text{End}(G)$ for $z \in G$.

Since Δ is a homomorphism in the second variable, we have for $x \in G$

$$\Delta(xK, 1K)\Delta(xK, 1K) = \Delta(xK, 1K),$$

so that $\Delta(xK, 1K) = 1$. It follows that $\gamma(1)$ is the identity map on G . Note next that for $x, y, z \in G$ we have

$$\begin{aligned} z^{\gamma(y)\gamma(x)} &= \Delta(z^{\gamma(y)}K, xK)z^{\gamma(y)} \\ &= \Delta(\Delta(zK, yK)zK, xK)\Delta(zK, yK)z \\ &= \Delta(zK, xK)\Delta(zK, yK)z \\ &= \Delta(zK, xyK)z \\ &= z^{\gamma(xy)}, \end{aligned} \tag{4.4}$$

that is, $\gamma : (G, \cdot) \rightarrow \text{End}(G)$ is an anti-homomorphism. Taking $y = x^{-1}$ in (4.4), we see that $\gamma(x) \in \text{Aut}(G)$ for $x \in G$.

Since Δ takes values in the centre of G , we have

$$\Delta(zK, xK)\Delta(zK, yK) = \Delta(zK, yK)\Delta(zK, xK),$$

in (4.4), which thus also yields $\gamma(y)\gamma(x) = \gamma(x)\gamma(y)$, that is, $\gamma(G)$ is abelian. It follows from Lemma 3.7 that γ is a bi-GF. \square

4.5. Completion of the proof of Corollary 3.6. We deal first with Corollary 3.6(2). In [CDV18, Proposition 7.9] a group $G = (G, \cdot)$ is constructed, which is the central product of two isomorphic, finite, perfect groups H, K , where a centre of order 3 is amalgamated, and it is shown that the construction of Proposition 4.3 yields a group (G, \circ) which is not isomorphic to G . It follows that (G, \cdot, \circ) is a bi-skew brace whose associated regular subgroup $N \cong (G, \circ)$ is not in the image $\mathcal{H}(G)$ of \mathcal{T} .

For this group G [CDV18, Proposition 7.9] yields $|T(G)| = 2$. Now consider, with G as above, the direct product $S = G \times L$, where L is the direct product of $n-1$ pairwise not isomorphic, finite, perfect, centreless

groups, none of them isomorphic to H . Then the results of [CDV18] yield that $|T(S)| = 2^n$. However, the construction of Proposition 4.3, where S is regarded as the semidirect product with amalgamation of KL by H , shows that $S = (S, \cdot)$ is the additive group of a bi-skew brace, (S, \cdot, \circ) , whose associated regular subgroup $N \cong (S, \circ)$ is not isomorphic to S , so that N is not in the image $\mathcal{H}(S)$ of \mathcal{T} .

For a different example, let G be the free group of nilpotence class 2 and exponent 3 on $n \geq 2$ generators. Then [Car18, Theorem 5.2] shows that $|T(G)| = 2$. However, the construction of Proposition 4.7 (which extends to this situation the construction of [Car18, Theorem 5.5]) shows that G has many non-trivial bi-GF's.

For Corollary 3.6(2), an example is provided by the cyclic group G of order 4. It is shown in [CDV17, Proposition 4.1(2)] that $T(G) = 1$, and that the holomorph of G , which is isomorphic to the dihedral group of order 8, contains a non-cyclic normal regular subgroup of order 4.

4.6. Nilpotent groups of class two as circle groups. The construction of Proposition 4.7 is an analogue of the one used in [AW73, Theorem 1] by J. C. Ault and J. F. Watters to show that certain groups of nilpotence class 2 are the circle groups of a (nilpotent) ring, and in turn the circle groups of a brace. (A brace can be defined as a skew brace with abelian additive group, although the concept of a brace predates that of a skew brace [Rum07].) It is still an open problem whether every *infinite* nilpotent group of nilpotence class 2 is the circle group of a brace ([Ced18, Problem 10.5], [Ven19, Problem 32]).

Proposition 4.7 can be used to reformulate the results of Ault and Watters in the context of bi-skew braces. To give an example, let $G = (G, \cdot)$ be a nilpotent group of nilpotence class 2 which is uniquely 2-radicable, that is, each element $g \in G$ has a unique square root $g^{1/2}$. Let $K = Z(G)$. Then

$$\begin{aligned} \Delta : G/K \times G/K &\rightarrow K \\ (xK, yK) &\mapsto [x, y]^{-1/2} \end{aligned}$$

satisfies the hypotheses of Proposition 4.7. (In practice we are appealing to a particular case of the Baer correspondence [Bae38], which is in turn an approximation of the Lazard correspondence and the Baker-Campbell-Hausdorff formulas [Khu98, Ch. 9 and 10].)

For the corresponding γ and \circ we have, for $x, y \in G$,

$$x \circ y = x^{\gamma(y)} y = \Delta(xK, yK) xy = [x, y]^{-1/2} xy,$$

and

$$y \circ x = y^{\gamma(x)} x = \Delta(yK, xK) yx = [y, x]^{-1/2} xy[y, x] = [x, y]^{-1/2} xy,$$

so that (G, \circ) is abelian. (This would also follow from the calculation of [Car18, Lemma 2.4].)

Proposition 4.7 yields that (G, \cdot, \circ) is a bi-skew brace, so that (G, \circ, \cdot) is a brace with the original (G, \cdot) as the circle group.

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