Quantum Systems and their Classical Limit

A C^*-Algebraic Approach

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Christiaan Jozef Farielda van de Ven

born on March 17, 1991
in Venlo (The Netherlands)

Supervisor
Prof. dr. Valter Moretti

Scientific Committee
Prof. dr. Michele Correggi, referee
Prof. dr. Claudio Dappiaggi, referee

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Publications and preprints

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1. C. J. F. van de Ven, G. C. Groenenboom, R. Reuvers, N. P. Landsman,
Quantum spin systems versus Schrödinger operators: A case study in spontaneous symmetry breaking.
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`https://scipost.org/SciPostPhys.8.2.022`

2. K. Landsman, V.Moretti, C.J.F. van de Ven,
Strict deformation quantization of the state space of $M_k(\mathbb{C})$ *with applications to the Curie-Weiss model.*
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3. V. Moretti, C.J.F. van de Ven,
Bulk-boundary asymptotic equivalence of two strict deformation quantizations.
In *Letters. Math. Phys.* Vol. 110, Iss. 11 (2020).
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5. S. Murro, C.J.F. van de Ven,
Injective tensor products in strict deformation quantization.
arXiv: 2010.03469 (2021).
Accepted for publication in Mathematical Physics, Analysis and Geometry.

6. Valter Moretti, C.J.F. van de Ven,
The classical limit of Schrödinger operators in the framework of Berezin quantization and spontaneous symmetry breaking as emergent phenomenon.
arXiv: 2103.11914 (2021).
Accepted for publication in International Journal of Geometric Methods in Modern Physics.

7. C.J.F. van de Ven,
The classical limit and spontaneous symmetry breaking in algebraic quantum theory.
arXiv: 2109.05653v2 (2021), under submission.

The following above mentioned publications have evolved from my doctoral dissertation: $2, 3, 4, 5, 6, 7$. **Publication** 1 **is an outcome of my M.Sc. thesis.**

Preface

This thesis is the result of the author's PhD research, supervised by professor Valter Moretti. This research was financially supported by means of a fellowship awarded by the Istituto Nazionale di Alta Matematica "Francesco Severi" as part of the European project INdAM-DP-Cofund-2015 "INdAM Doctoral Programme in Mathematics and/or Application Cofunded by Marie Skłodowska-Curie Actions" under grant number: 800 713485.

First and foremost, I want to express my gratitude to Valter Moretti for his support during this PhD, for giving me the freedom to choose own topics in the field of mathematical physics and for encouraging me to collaborate with prominent researchers in Italy and abroad. The results in this thesis would not have been possible without his helpful suggestions, insightful conversations and useful comments. Moreover, I am undoubtedly very grateful to INdAM for giving me the opportunity to do a PhD in Italy. I would like to thank Mrs. Esposito for her help in solving bureaucratic issues. In addition, I thank my other co-authors Klaas Landsman and Simone Murro I have collaborated with during this PhD. Furthermore, I am grateful to Nicolò Drago for sharing his expertise in the field of symplectic geometry, all my other colleagues at the mathematics department, and I thank my friends for their companionship over the last years.

As a mathematical physicist, it is an honor and privilege to do this work, and I can say that I feel very lucky and satisfied to have been in the position to do this PhD in Trento. In particular, the freedom of choice of research topics that came with this fellowship allowed me to combine several of my research interests: mathematics and physics. I had a great time the last three years in Trento, during which I learned a great deal about mathematical physics, but I also developed myself as a person. Emigrating abroad as a foreigner was certainly not easy for me at first. Soon I found my way thanks in part to the support of my family and a close Italian group of friends of which I particularly would like to thank Luciano, Francesco and Giuseppe.

Let me conclude with some words of gratitude to my family and my family in-law for their care in my development and supporting my decisions. I thank my twin brother Mathijs for his many phone calls and his companionship at distance. Finally, I want to express eternal gratitude to my wife Bilge and my mother Francine for stimulating me to follow my dream to find a PhD position in Trento, and for always being there to help and give positive advice.

Contents

1. Introduction

Inspired by the approach of semi-classical analysis to study the link between quantum and classical mechanics, I decided to immerse myself into the theory of deformation quantization, i.e. a mathematically precise way to describe the transition from quantum to classical theories based on *C^*-algebras*. The classical theories typically arise as "higher-level" theories H which are limiting cases of "lower-level" theories L (viz. quantum theories). The higher-level theories are well defined and understood by itself (typically predating L) and have properties that cannot be described by L. Examples of such properties are spontaneous symmetry breaking, Bose-Einstein condensation and phase transitions [57].

Motivated by physics let us give a few examples of such pairs (H, L). In the first example we consider a theory H describing classical mechanics of a particle on the real line with phase space $\mathbb{R}^2 = \{(p, q)\}$ and ensuing C^*-algebra of observables given by $\mathfrak{A}_0 = C_0(\mathbb{R}^2)$, i.e. the continuous (complex-valued) functions on $\mathbb{R}^2$ that vanish at infinity, under pointwise operations and supremum norm. Then, L is quantum mechanics with pertinent C^*-algebra $\mathfrak{A}_\hbar$ $(\hbar > 0)$ taken to be the compact operators $\mathfrak{B}_\infty(L^2(\mathbb{R}))$ on the Hilbert space $L^2(\mathbb{R})$ for each non-zero $\hbar$. Another example, originating in the field of spin systems, concerns the case in which H describes classical thermodynamics on the C^*-algebra $C(B^3)$, with $B^3 \subset \mathbb{R}^3$ the closed unit 3-ball, and L is given by the N-fold tensor product of the matrix algebra $M_2(\mathbb{C})$ with itself, used to describe statistical mechanics of finite quantum spin systems.

In these two examples the algebra on which theory H is described is commutative. This however does not always have to be the case. Indeed, if we consider the relation between statistical mechanics of finite quantum and infinite quantum spin systems, often characterized by a procedure called the thermodynamic limit, then H is statistical mechanics of an infinite quantum spin system, typically given by the (highly non-commutative) quasi-local algebra being the infinite (projective) tensor product of $M_k(\mathbb{C})$ with itself, and L is the N-fold (projective) tensor product of $M_k(\mathbb{C})$ with itself. In view of the *classical limit* that is central to this thesis, however, we focus on pairs (H, L) such that H is always given by a classical theory whose structure is encoded by a commutative C^*-algebra.

As already mentioned the pairs (H, L) have in common that the limiting theory H has features that at first sight cannot be explained by the lower-level theory L, because apparently L lacks a property inducing those features in the limit to H. This principle is called *asymptotic emergence*, whose precise concept was first introduced in [9], and reformulated in terms of C^*-algebras in [57]. In this algebraic framework the limiting relationship between theories H and L is specified by a *continuous bundle of C^*-algebras*, i.e. a mathematical framework providing a bridge between the two (a priori) different theories. A major advantage of this structure is that classical and quantum theory are both described by a single C^*-bundle allowing one to study such emergent features in a complete algebraic way. Additionally, since this C^*-algebraic framework is encapsulated by relatively well manageable algebraic relations, it often simplifies the analysis in the pertinent semi-classical limit.

In this thesis we particularly focus on the natural phenomenon of spontaneous symmetry breaking (SSB). The theory of spontaneous symmetry breaking is a topic of great interest in mathematical physics. It is responsible for many physical phenomena, like phase transitions in condensed-matter systems, superconductivity of metals and it is the origin of particle masses in the standard model, described by the Higgs mechanism [19, 20, 86].

In view of the above discussion, spontaneous symmetry breaking is considered as an emergent phenomenon feature of H: it does not occur in the underlying quantum theory L [57]. In this thesis we make this precise for several pairs (H, L), i.e. we provide a C^*-algebraic formalism showing that SSB occurs when passing from the quantum realm to the classical world by switching off a so-called *semi-classical* parameter. In the context of Schrödinger operators describing a quantum theory this process is achieved by the limit in Planck's constant $\hbar \to 0$ yielding classical mechanics on the phase space $\mathbb{R}^{2n}$ (cf. Chapter 9). In the context of quantum spin systems on a finite lattice this mechanism means that the number of particles N is sent to infinity using a suitable set of physical observables (see Chapter 3 and Chapter 8), which in turn produces a classical theory on the Poisson manifold $S(M_k(\mathbb{C}))$, i.e. the algebraic phase space of the $k \times k$-matrices. Moreover, we will see that for $k = 2$ this limit also relates to a classical theory on S^2, the unit 2-sphere embedded in $\mathbb{R}^3$.

1.1 Different limits

A common way to investigate physical properties in infinite quantum systems exists under the name *thermodynamic limit*, where typically the limit of the number of particles N at fixed density N/V is sent to infinity and where V the volume of the system is sent to infinity, as well. In such works the system constructed in the limit $N = \infty$ is quantum statistical mechanics in infinite volume, whose existence (followed by the establishment of e.g. phase transitions and spontaneous symmetry breaking in infinite quantum systems) was a major achievement of mathematical physics [57]. This highly non-commutative limit is obtained by considering so-called *quasi-local observables* and goes beyond the scope of this thesis. Instead, particularly when dealing with quantum spin systems described by an N-particle Hamiltonian, our goal in taking the limit $N \to \infty$ is quite different, in that the limiting system will be classical and therefore defined on a *commutative* C^*-algebra. To this end, we look at *macroscopic* (also called global or quasi-symmetric) observables rather than quasi-local observables. These generate a commutative C^*-algebra of observables of an infinite quantum system, describing classical thermodynamics as a limit of quantum statistical mechanics [55, 58, 99]. We will see in Chapter 3 and Chapter 8 that macroscopic observables are indeed the correct ones to study the classical limit of quantum spin systems since these precisely relate to deformation quantization (and thus to a classical theory).

1.2 Classical limit

The theory of quantum mechanics provides an accurate description of systems containing tiny particles, but in principle it can be applied to any physical system. As known from centuries of experience, classical physics, in turn, is a theory that deals with large and familiar objects. One may therefore expect that if quantum mechanics is applied to such objects, it reproduces classical results. Roughly speaking, this is what we call the *classical limit*, and refers to a way connecting quantum with classical theories[1], rather than with infinite quantum systems. Let us illustrate this with two examples [56, 79]. Consider first the following Schrödinger operator for a particle with mass m and a potential V in one dimension:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V. \tag{1.2.1}$$

If we apply this equation to a particle with large mass m, then according to the above discussion, the Schrödinger equation should yield classical mechanics. To make clear what this means, we have to introduce a typical energy scale ϵ (like $\sup_x |V(x)|$) and a typical length scale λ, such as $\lambda = \epsilon/|\sup_x \nabla V(x)|$, provided these quantities are finite.[2] Consequently, we perform a scale separation by considering the Hamiltonian as $\tilde{H} = H/\epsilon$ which we write it in terms of the dimensionless variable $\tilde{x} = x/\lambda$:

$$\tilde{H} = -\tilde{\hbar}^2\frac{d^2}{d\tilde{x}^2} + \tilde{V}(\tilde{x}), \tag{1.2.2}$$

[1] From a C^*-algebraic point of view central to this thesis, the classical limit refers to a rigorous and correct way connecting non-commutative C^*-algebras $\mathfrak{A}_\hbar$ (describing quantum theories) with commutative C^*-algebras $\mathfrak{A}_0$ (encoding classical theories) by means of convergence of algebraic states with respect to so-called quantization maps (see in particular Section 1.3).

[2] Strictly speaking, $\hbar$ is a dimensionful constant. In order to study the semi-classical limit of a given quantum theory one has to form a dimensionless combination of $\hbar$ and other parameters, which in turn re-enters the theory as if it were a dimensionless version of $\hbar$ that can indeed be varied.

where we have introduced the dimensionless quantity $\tilde{\hbar} = \hbar/\lambda\sqrt{2m\epsilon}$ and $\tilde{V}(\tilde{x}) = V(\lambda\tilde{x})$. Now, $\tilde{\hbar}$ is dimensionless, and one might study the regime where it is small [44]. As a result, large mass effectively means small $\hbar$.

Another example is Planck's radiation formula:

$$\frac{E_\nu}{N_\nu} = \frac{h\nu}{e^{h\nu/k_BT} - 1}, \tag{1.2.3}$$

with temperature T as variable. As observed by Einstein and Planck, in the limit of the dimensionless quantity $h\nu/k_BT \to 0$ this formula converges to the classical equipartition law $E_\nu/N_\nu = k_BT$.

In both examples, the classical limit corresponds to the limit where a certain dimensionless quantity becomes small. In the first example, this can be physically interpreted by means of a large mass, whilst in the second example this is due to a high temperature.

We would like to point out to the reader that these kind of scale separations can be realized in several contexts, e.g. micro to macro, strong or weak coupling, small to large systems, etc. It is realized by a limit $\tilde{\hbar} \to 0$ of a parameter $\tilde{\hbar}$ giving the ratio between the corresponding characteristic length scales. The classical limit then corresponds to the limit where such a parameter becomes small.

In view of the above discussion this thesis places special emphasis on quantum spin systems in the limit $N \to \infty$, where N could be the principal quantum number labeling orbits in atomic physics, the number of particles or lattice sites corresponding to a quantum spin system, or the spin quantum number of a single quantum spin system. The idea is exactly the same: the limit $N \to \infty$ should reproduce a classical theory in the sense that the limiting theory is described by (usually) classical thermodynamics [57]. As already indicated in Section 1.1 this strongly depends on the choice of observables (viz. Chapters 3,8). Since a classical theory is encoded by a commutative set of observables, the correct observables to consider are macroscopic averages (i.e. macroscopic observables) as they *asymptotically commute* in the pertinent limit $N \to \infty$.

With slightly abuse of notation we will often refer to the classical limit as $\hbar \to 0$, keeping in mind that the meaning of this limit can always be realized by a genuine physical limit that is well understood. More details about the relationship between classical and quantum theory and the importance to the philosophy of physics can be found in [56].

1.3 Quantization theory

Quantization refers to the passage from a classical to a corresponding quantum theory. This notion goes back to the time that the correct formalism of quantum mechanics was beginning to be discovered. There is in principle no general recipe working in all cases, and different so-called quantization schemes may lead to inequivalent results with respect to other quantization methods. This is certainly unsatisfactory and depending on the precise purpose, each method has its pros and cons.

For example, in geometric quantization (GQ) one aims to obtain a quantum mechanical system given a classical mechanical system whose procedure basically consists of the following three steps: prequantization of the classical system, a polarization method, and finally a metaplectic correction in order to obtain a nonzero quantum Hilbert space [8]. This quantization scheme focuses on the space of states and therefore on the Schrödinger picture. A major advantage of GQ is that this technique is very efficient for controlling the physics of the quantum system.

Formal deformation quantization (FDQ) instead is based on the construction of the quantum theory via a so-called $\star$- product defined in terms of a formal parameter (typically Planck's constant $\hbar$). FDQ is useful for example for the construction of quantum states in terms of classical ones.

The aforementioned quantization procedures are used to obtain quantum mechanics from classical methods. Even though such approaches often give accurate results, they also have their drawbacks: the quantum theory is pre-existing compared with its classical limit and not vice versa. Therefore, one should be able to address the classical limit without the need of imposing a given structure of the quantum model, i.e. that it is obtained as a suitable quantization of a classical one. It is precisely the latter point of view on which this thesis is based. Indeed, we see quantization as the study of the possible correspondence between a given classical theory, defined by a Poisson algebra or a Poisson manifold possibly equipped with a (classical) Hamiltonian, and a given quantum theory, mathematically expressed as a certain algebra of observables or a pure state space, and perhaps a time evolution and (quantum) Hamiltonian. For this purpose it is not at all necessary that the quantum theory be formulated in terms of classical structures. On the basis of this understanding quantization and the classical limit can be seen as two sides of the same coin.

A mathematically correct approach that encompasses this perspective exists under the name *deformation quantization*. Although, several notions of a deformation quantization exist in literature [57, 55], in this work we shall mainly focus on the concept of (strict) deformation quantization[3] developed in the 1970s (Berezin [12]; Bayen et al. [10]), where non-commutative algebras characteristic of quantum mechanics arise as deformations of commutative Poisson algebras characterizing classical theories. In Rieffel's [80, 83] approach to deformation quantization, further developed by Landsman [55], the deformed algebras are C^*-algebras, and hence the apparatus of operator algebras and non-commutative geometry (Connes, [29]) becomes available. In short, a deformation quantization focuses on the algebras of observables of a physical system (classical and quantum), and hence on the Heisenberg picture.

Remark 1.3.1. We would like to point out that even though this concept seems to be a correct formalism to study the classical limit, only a few pairs of a classical and a quantum C^*-algebra are known to connect in this rigorous manner [81, 82]. The search for and examination of such pairs is an important question in modern mathematical physics. ■

To be more precise, the idea of a deformation quantization is to consider a classical theory, whose observables are described by sufficiently regular functions over a space of phases X (assumed to be locally compact and Hausdorff), as the zero-limit of a sequence of usually non-commutative or quantum theories[4] labeled by a semi-classical parameter $\hbar$, whose observables are represented by self-adjoint operators on a corresponding sequence of Hilbert spaces, or more generally, seen as elements of a sequence of $*$-algebras. Obviously, only a selection of sequences of observables, parametrized by $\hbar \geq 0$, makes physical sense in order to establish this limit. These are sequences with a suitable continuity property reformulated in terms of $*$-algebras (more precisely, C^*-algebras as explained below).[5] [6] Technical problems often arising in the setting of Hilbert spaces are typically avoided in this way. In this *algebraic approach*, as already mentioned, the quantum observables are given by self-adjoint elements in a family of abstract $*$-algebras $\{\mathfrak{A}_\hbar\}_\hbar$ of formal operators $a \in \mathfrak{A}_\hbar$ and the algebraic states are linear complex-valued functionals on such algebra $\omega_\hbar : \mathfrak{A}_\hbar \to \mathbb{C}$ with the physical meaning of $\omega_\hbar(a)$ as the expectation values of the observable $a = a^*$ in the state $\omega_\hbar$.

Another benefit arising from the use of the algebraic approach is that, differently from the Hilbert space formulation, the algebraic approach is suitable even for classical theories. This is because the set of (sufficiently regular) functions f on the space of phases X representing observables (also extending the functions to complex valuated maps) has a natural structure of *commutative* $*$-algebra $\mathfrak{A}_0$. The algebraic states are there nothing but probability measures over the space of phases: $\omega_0(f) = \int_X f d\mu_\omega$.[7] To avoid technical problems with topologies,

[3] This also exists under the name C^*-algebraic deformation quantization.

[4] For the purpose of this thesis we focus on the physical systems proper of quantum and quantum statistical mechanics although quantization theory can be profitably exploited to study quantum field theory and quantum gravity as well [57].

[5] Concretely, such sequences are nothing else than a subclass of the continuous cross-sections of a continuous bundle of $*$- (or C^*-) algebras (cf. Definition 2.1.1).

[6] We stress that such an assumption excludes various physical models when one can prove continuity (in extremely weak sense) at the best for the expectations on some states of certain specific observables, rather than on a general class of sequences. Nonetheless, one can always try to weaken the assumptions on the classical limit and the set of observables to capture also these models.

[7] In general, the algebra $\mathfrak{A}_0$ can be any commutative $*$-algebra. This thesis is however based on quantization of a Poisson manifold X, which naturally corresponds to the C^*-algebra $\mathfrak{A}_0 = C_0(X)$, as indicated in [57, Chapter 7]. In this setting, strictly speaking, the classical observables should be elements of a dense $*$-Poisson subalgebra of $C_0(X)$ (which itself is not a Poisson algebra) in order to define a Poisson bracket and therefore a classical theory.

the family of algebras $\{\mathfrak{A}_\hbar\}_{\hbar\geq 0}$ is chosen to be made of more tamed C^*-algebras rather than $*$-algebras. It is important to stress that this more abstract viewpoint actually encompasses the Hilbert space formulation. It is because the celebrated GNS reconstruction theorem (see e.g. [57, 64]) permits to recast the abstract algebraic perspective to a standard Hilbert space framework. As already mentioned, the sequence of C^*-algebras $\{\mathfrak{A}_\hbar\}_{\hbar\geq 0}$, where $\mathfrak{A}_0$ is the algebra of classical observables, is formally encoded by the structure of a *C^*-bundle* whose precise details will be discussed in Chapter 2.

A machinery of utmost relevance in this framework is the notion of *quantization map*. Its design can be traced back to Dirac's foundational ideas on quantum theory and, from a modern point of view, it consists of a map $Q_\hbar : \mathfrak{A}_0 \ni f \mapsto Q_\hbar(f) \in \mathfrak{A}_\hbar$ which associates classical observables to quantum ones. Obviously the quantization map is requested to satisfy a number of conditions of various nature. For instance, one of them regards the interplay of the quantum commutator and the Poisson bracket referred to the Poisson structure of the space of phases X. After Dirac, one expected that $\frac{1}{\hbar}[Q_\hbar(f), Q_\hbar(g)]$ tends to $Q_\hbar(\{f, g\})$ as $\hbar \to 0^+$. This condition stated within a suitable topological formulation is nowadays known as the *Dirac-Groenewold-Rieffel condition*. As is known, the set of all naturally expected requirements on $Q_\hbar$ is contradictory as proved in the various versions of the *Groenewold-van Hove theorem* [57]. These no-go theorems gave rise to the birth of a number of different types of quantization maps whose distinct nature depends on the specific choice of a subset of mutually compatible requirements.

The most popular quantization map is the one attributed to Weyl, denoted by $Q_\hbar^W$, whereas one of the most effective quantization maps, is the so-called *Berezin quantization map* [12], indicated by $Q_\hbar^B$, which is reviewed in Section 2.1.1 and plays a crucial role in the study of Schrödinger operators reviewed in Chapter 9. It is worth mentioning that in the special case when dealing with a compact Kähler manifold, the theory of *Toeplitz quantization* has also proved its great importance [17, 88].

Using the aforementioned concepts, in this thesis we attempt to bring forward two important topics in this area of mathematical physics:

(1) existence of the *classical limit* of a sequence of $\hbar$-indexed eigenvectors $\{\psi_\hbar\}_{\hbar>0}$ corresponding to a quantum Hamiltonian $H_\hbar$;

(2) occurrence of spontaneous symmetry breaking (SSB) as emergent phenomenon arising in the classical limit $\hbar \to 0$.

In this setting we remind the reader that $\hbar$ has several interpretations depending on physical system one considers (e.g. Schrödinger operators for which $\hbar$ occurs as Planck's constant, or quantum spin systems where $\hbar$ plays the role of $1/N$, with N denotes the number particles, etc.), and letting $\hbar \to 0$ (provided this limit is taken correctly) should be understood as a way to generate a classical theory.

It turns out that the concept of the classical limit provides a rigorous meaning of the convergence of quantum algebraic states $\omega_\hbar$ to classical algebraic ω_0 (i.e. probability measures) on the commutative algebra on $\mathfrak{A}_0$, when $\hbar \to 0^+$. Given a sequence of quantization maps $Q_\hbar : \mathfrak{A}_0 \ni f \mapsto Q_\hbar(f) \in \mathfrak{A}_\hbar$, we then say that a sequence of states $\omega_\hbar : \mathfrak{A}_\hbar \to \mathbb{C}$ is said to be have a **classical limit** if the following limit exists and defines a state ω_0 on $\mathfrak{A}_0$ (or a substructure of it),

$$\lim_{\hbar\to 0} \omega_\hbar(Q_\hbar(f)) = \omega_0(f), \ \ (f \in \mathfrak{A}_0). \tag{1.3.4}$$

Regarding (1) above, the issue is to study whether or not the sequence of algebraic quantum (vector) states $\omega_\hbar(Q_\hbar(f)) := \langle\psi_\hbar, Q_\hbar(f)\psi_\hbar\rangle$ tends to some classical algebraic state $\omega_0(f)$ for any classical observable $f \in \mathfrak{A}_0$, when $\hbar \to 0^+$. This issue has been presented from a technical perspective in Section 6.2. We show that this C^*-algebraic approach is perfectly suitable and offers a complete interpretation of the classical limit of quantum systems, even though eigenvectors of such operators in general do not admit a limit in the pertinent Hilbert space.

The found results about the classical limit distinguish between the case where a symmetry group G acts on the physical system at classical and quantum level, or there is not such a group. This distinction plays a central role in developing issue (2), introduced above. In case that a symmetry group G exists and the considered states are *ground states* of a given Hamiltonian (quantum or classical), spontaneous symmetry breaking occurs if there are no *extreme* (roughly speaking *pure*) ground states that are invariant under the action of G. It is interesting to study if the phenomenon of spontaneous symmetry breaking arises as an emergent phenomenon, i.e., it only occurs in the limit $\hbar \to 0^+$ (or in the case of spin systems, $N \to \infty$). To this end, in Chapters 8, 9 we prove that this is the case for a large class of Hamiltonians, that includes, in particular, Schrödinger operators with a potential trap defined in terms of the *double well system* or the *Mexican hat system* (where typically a topological compact group is used) and mean-field quantum spin systems (whose symmetry is implemented by a finite cyclic group).

1.4 Contributions

In this section we shortly explain our main contributions starting with the results obtained in Chapter 2, Chapter 3, etc.

Contributions Chapter 2

In this chapter we introduce the definitions of a C^*-bundle (cf. Definition 2.1.1) and a deformation quantization (cf. Definition 2.1.3), followed by recalling some general results on Weyl and Berezin quantization maps on $\mathbb{R}^{2n}$. We finally prove

- Proposition 2.1.8;
- Proposition 2.1.13 .

In Proposition 2.1.8 several cases are investigated for which the Weyl quantization map uniquely determines a true operator $Q_\hbar^W(f)$ which in general is an unbounded and densely defined operator on $L^2(\mathbb{R}^n, dx)$. This happens when the arguments of the Weyl quantization map are actually functions of various spaces rather than distributions. We furthermore investigate the cases for which the quantized functions $f : \mathbb{R}^{2n} \to \mathbb{C}$ are constant in one of the variables and polynomially bounded in the other, and show that the ensuing quantization map $Q_\hbar^W(f)$ defines a bounded operator on $L^2(\mathbb{R}, dx)$, defined by spectral calculus.

Proposition 2.1.13 yields a result on the asymptotically norm-equivalence between the Berezin quantization map, denoted by $Q_\hbar^B(f)$, and the multiplication operator corresponding to multiplication with the function $f \in C_0(\mathbb{R}^n)$ interpreted as a function on $\mathbb{R}^{2n}$ constant in the variable p. If the function is constant in the variable q then a similar statement holds, in that the operator $Q_\hbar^B(f)$ is asymptotically norm-equivalent to the operator $\check{f}_\hbar\, *$, i.e. the convolution with the inverse Fourier transform. Special emphasis is given to the function $p \mapsto e^{-tp^2}$, whose Berezin quantization is norm-equivalent to the operator $e^{t\hbar^2\Delta}$, where Δ denotes the Laplacian on $L^2(\mathbb{R}^n)$. This proposition is essential for the proof of Lemma 9.3.2 used to prove Proposition 9.3.1 (see Chapter 9).

These results are part of the paper accepted for publication in [66].

Contributions Chapter 3

The existence of a deformation quantization of the algebraic state space $S(M_k(\mathbb{C}))$ of the matrix algebra $M_k(\mathbb{C})$ is proved. To this end, in Section 3.1 we investigate the smooth structures and algebraic properties of $S(M_k(\mathbb{C}))$ allowing us to show that $S(M_k(\mathbb{C}))$ is a Poisson manifold with stratified boundary (see i.e. Definition 3.1.3 for the definition of the Poisson bracket). Subsequently, Section 3.3.1 (particularly Lemma 3.3.2) concerns the construction of a suitable dense Poisson subalgebra of $C(S(M_k(\mathbb{C})))$ on which quantization maps will be defined. Finally, a deformation quantization is constructed in Section 3.3 (cf. Theorem 3.3.4). As explained in Chapter 4 a particular case of physical interest holds for $k = 2$, where $S(M_2(\mathbb{C})) \cong B^3 \subset \mathbb{R}^3$ the closed unit three-ball in $\mathbb{R}^3$.

These results have been published in [58].

Contributions Chapter 4

The deformation quantization of the state space $S(M_2(\mathbb{C})) \cong B^3$ (cf. Chapter 3) is related to the deformation quantization of the two-sphere $S^2 \subset \mathbb{R}^3$ (cf. Section 2.2.2). This yields a natural physical interpretation for quantum spin systems typically arising as quantized functions on the 2-sphere in $\mathbb{R}^3$, also called the Bloch sphere from physics. The relevant results are proved by means of

- Proposition 4.1.1;
- Theorem 4.2.1;
- Proposition 4.3.1.

Proposition 4.1.1 proves that, on the domain $P_N(S^2) \subset C(S^2)$, the quantization maps defined by (2.2.40) are in bijection with $\mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$, the algebra of bounded operators on the symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes^N \mathbb{C}^2$. The set $P_N(S^2)$ is defines as follows. We first consider the complex vector space of polynomials in the variables $x, y, z \in \mathbb{R}^3$ of degree $\leq N$ where $N \geq 1$, and then let $P_N(S^2)$ be the vector space made of the restrictions to S^2 of those polynomials.

This proposition serves as a useful tool in the proof of our main theorem, Theorem 4.2.1. This theorem states a result on the relation between quantization maps (3.3.56)–(3.3.57) and the maps (2.2.40). More precisely, it shows that in a certain manner both quantization maps are asymptotically norm-equivalent.

Subsequently, Proposition 4.3.1 deals with an application to the quantum Curie-Weiss spin Hamiltonian, initially defined on the Hilbert space $\bigotimes^N \mathbb{C}^2$. The proposition states a result on the relation between the quantum spin Curie-Weiss Hamiltonian restricted to the symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes^N \mathbb{C}^2$ and a quantized function on the unit 2-sphere $S^2 \subset \mathbb{R}^3$, also called *classical symbol*.

The found results have been published in [65].

Contributions Chapter 5

The theory of a continuous bundle of C^*-algebras (cf. Definition 2.1.1) and a (strict) deformation quantization (cf. Definition 2.1.3) is applied to a certain tensor product of C^*-algebras. This is the injective tensor product. Exploiting the properties of this tensor product a natural framework is provided to study products of KMS states and the correspondence between quantum and classical Hamiltonians in spin systems and Schrödinger operators for non-interacting many particle systems. The main contributions are

- Theorem 5.2.3;
- Theorem 5.3.3.

Theorem 5.2.3 provides criteria for the existence of a deformation quantization of the algebraic tensor product $\tilde{\mathfrak{A}}_0 \otimes \tilde{\mathfrak{B}}_0$, where $\tilde{\mathfrak{A}}_0$ and $\tilde{\mathfrak{B}}_0$ are assumed to admit a deformation quantization in the sense of Definition 2.1.3.

Theorem 5.3.3 in turn states criteria proving that the product of two KMS_β states is again a KMS_β state.

The found results in this chapter have been submitted for publication in [69].

Contributions Chapter 6

In this chapter the semi-classical properties of Berezin quantization maps are investigated, and moreover our main result (cf. Theorem 6.2.5) concerning the classical limit for eigenvectors of Berezin quantization maps is proved. To this end, our assumption is the existence of a coherent pure state quantization $(\mathcal{H}_\hbar, \Psi_\hbar^\sigma, \mu_\hbar)_{\hbar \in I, \sigma \in S}$ of a symplectic manifold (S, ω_S) (cf. Definition 2.2.2 and Definition 2.2.5). We furthermore assume that the associated Berezin quantization maps satisfy the von Neumann and Rieffel condition (viz. (2.1.2)–(2.1.1)) for elements in the C^*-algebra $C_0(S)$, as typically happens in the additional case of a deformation quantization of S. Our main contributions are listed below.

- Theorem 6.1.2;
- Proposition 6.2.2;
- Proposition 6.2.3;
- Theorem 6.2.4;
- Theorem 6.2.5;
- Proposition 6.2.7.

Theorem 6.1.2 is a result on the semi-classical behavior of the spectrum of Berezin quantization maps. It shows that in the semi-classical limit the spectrum is related to the range of the function that is quantized.

Proposition 6.2.2 yields a result on *equivariance* of Berezin quantization maps. More precisely, it states that an action of a (topological) group G acting by symplectomorphisms on the manifold (S, ω_S) ensures that the quantization maps $Q_\hbar^B$ are equivariant under a suitable unitary representation of G in $\mathcal{H}_\hbar$. This serves as a preparatory result for Theorem 6.2.5.

Proposition 6.2.3 provides criteria concerning the localization of eigenvectors of Berezin quantization maps. This is again a preparatory result for Theorem 6.2.5.

Theorem 6.2.4 yields conditions for the existence of the classical limit of a sequence of eigenvectors of Berezin quantization maps in the specific case when the quantized functions on S are not related by a symmetry.

Finally, this result has been extended to the case where a symmetry implemented by a group G is present (cf. Theorem 6.2.5). The classical limit in turn is defined in terms of Haar integrals.

In Proposition 6.2.7 these integrals are recast in terms of integrals with respect to G-invariant probability measures μ and ν on S with suitable supports.

The found results in this chapter have been accepted for publication in [66].

Contributions Chapter 7

In this chapter the notions of a dynamical symmetry group, ground states and spontaneous symmetry breaking (SSB) are introduced. Our main contributions are

- Proposition 7.2.2;
- Proposition 7.3.1;
- Proposition 7.3.2.

In Proposition 7.2.2 we prove that, when dealing with the algebra of compact operators, the one-parameter subgroup of C^*-algebra automorphisms induced by a self-adjoint generator is strongly continuous (also if the generator of U is unbounded).

Subsequently, we introduce the concepts of an algebraic ground state and spontaneous symmetry breaking. These notions particularly apply to the commutative case, when $\mathfrak{A} := C_0(S)$ endowed with the C^*-norm $||\cdot||_\infty$, referred to a symplectic manifold S and the associated Poisson structure $(C^\infty(S), \{\cdot,\cdot\})$.

Consequently, a characterization of ground states of a commutative C^*-dynamical system has been given (cf. Proposition 7.3.1).

Finally, Proposition 7.3.2 yields an important result stating that under some mild conditions no SSB (or weak symmetry breaking) occurs for any finite $\hbar > 0$. This result explains the fact that in quantum mechanics there is only one G-invariant "ground state".

The found results in this chapter have been accepted for publication in [66].

Contributions Chapter 8

In this chapter we apply our previous findings to mean-field quantum spin systems. We first introduce the concept of the classical symbol and show the correspondence between mean-field quantum spin Hamiltonians and quantization of these symbols, where the unit three-ball $B^3 \subset \mathbb{R}^3$ and its boundary, i.e. the 2-sphere S^2, play a crucial role. In what follows we analyze mean-field quantum spin Hamiltonians in the limit when N, the number of spin particles, is sent to infinity.

In Section 8.2.1 we relate the Heisenberg dynamics induced by these Hamiltonians to the classical dynamics generated by their corresponding symbols on the manifold $S(M_k(\mathbb{C}))$ (cf. Theorem 8.2.8, Proposition 8.2.10).

Consequently, completely analogous to Theorem 6.1.2, we show in Proposition 8.2.11 that the spectrum of a mean-field quantum spin Hamiltonian converges to the range of the corresponding principal symbol and apply Proposition 6.2.8 to the specific case when considering the manifold S^2 (cf. Lemma 8.2.12).

Proposition 8.2.15 provides an alternative proof of Theorem 6.2.5 in the case for mean-field quantum spin systems obeying a discrete symmetry (e.g. $G = \mathbb{Z}_2$).

Finally, in Section 8.3 we apply our findings to the Curie-Weiss model. We hereto prove the existence of the classical limit of a sequence of ground state eigenvectors corresponding the quantum Curie-Weiss model (cf. Theorem 8.3.2). Additionally, an alternative "proof" based on strong numerical evidence is provided. This stands on the following preparatory results stated in the form of a proposition, an assumption and two lemmata,

- Proposition 8.3.4;
- Assumption 8.3.7;
- Lemma 8.3.9;
- Lemma 8.3.11,

To conclude, in Section 8.3.2 the concept of symmetry breaking has been discussed in the context of the Curie-Weiss model showing that weak symmetry breaking occurs in the classical limit as $N \to \infty$.

The results have been published in [58, 98].

Contributions Chapter 9

This chapter starts by repeating the main findings of Chapter 6 specialized to the symplectic manifold $(\mathbb{R}^{2n}, \sum_{k=1}^{n} dp_k \wedge dq^k)$. In Section 9.2 the interplay between Schrödinger operators and Berezin quantization maps is discussed. The main contributions of this section are

- Proposition 9.2.2;
- Proposition 9.2.5;
- Proposition 9.2.7;
- Corollary 9.2.8.

In Section 9.3 the existence of the classical limit of a sequence of eigenvectors of minimal eigenvalues corresponding to Schrödinger operators $H_\hbar$ is proved. Similar as in Section 6.2, we first prove a localization result (cf. Proposition 9.3.1) of such sequences, followed by two main theorems (cf. Theorem. 9.3.3 and Theorem 9.3.4) where again distinction is made between the presence of a symmetry or not.

Subsequently, we prove a result (cf. Proposition 9.3.6) stating that under certain conditions on the potential the ensuing Schrödinger operator $H_\hbar$ commutes with a unitary representation implementing a given symmetry. Finally, it is shown in Corollary 9.3.7 that the classical limit of a sequence of eigenvectors corresponding to $H_\hbar$ coincides with the classical limit of a sequence of eigenvectors of Berezin quantization maps quantizing a suitable function.

In Section 9.4 we prove a result on the classical limit of Gibbs states associated to Schrödinger operators (cf. Proposition 9.4.1).

The last section of this chapter relates the previous findings to symmetry breaking in the context of Schrödinger operators.

The found results in this chapter have been submitted for publication in [66, 99].

1.5 Discussion

The aim of this thesis was to provide a mathematically correct framework describing the transition from quantum to classical theories (viz. classical limit) where in particular special emphasis has been given to the emergent phenomenon of spontaneous symmetry breaking. The C^*-algebraic approach based on the theory of deformation quantization allowed us to give a precise meaning to this transition and to reach rigorous results in a great variety of cases of physical interest, like quantum spin systems and Schrödinger operators.

We preferred to use this C^*-algebraic framework rather than (the perhaps more common approaches as) microlocal analysis and pseudo-differential calculus, for the following reasons.

- The rather technical tools and techniques used in microlocal analysis and pseudo-differential calculus (e.g. [5, 43, 107, 93]) are often based on estimates which can be very well controlled in terms of a semi-classical parameter and therefore from a mathematical point of view such approaches are definitely important. Moreover, these approaches allow to study certain physically relevant questions in great detail. However, the purpose of this thesis is not to give estimates or bounds on for examples norms of eigenvectors, quasimodes or eigenenergies, but rather to provide a natural convenient framework that encodes the physics of both quantum and classical theories allowing a precise meaning of the classical limit whose concept is also perfectly suitable for studying emergent phenomena arising in the classical limit of underlying quantum theories. In this thesis it has been shown that the concept of a continuous bundle of C^*-algebras equipped with a deformation quantization is the natural and correct way to formalize emergent phenomena, since it encodes *both* quantum and classical theory by means of relatively simple algebraic relations inherent to the theory of C^*-algebras. All that is needed are the continuity properties of the C^*-bundle specified by their continuous cross-sections, in turn defined in terms of quantization maps. We point out to the reader that, even though these algebraic methods are relatively "smooth" compared with the usual techniques used in microlocal analysis and pseudo-differential calculus, the algebraic relations inherent to the theory of C^*-algebras often lack enough structure to address complex physical questions: think e.g. of interacting particle systems trapped in a potential.

- In addition to the technical approaches used in microlocal analysis and pseudo-differential calculus where one indeed may obtain very detailed results concerning semi-classical tunneling and SSB (e.g. [6, 92, 46, 47]), this relatively simple algebraic framework in turn enabled us to formalize spontaneous symmetry breaking as a natural emergent phenomenon occurring in the classical limit. It is therefore a promising way to address and study generally emergent phenomena occurring in Nature, like phase transitions, Bose-Einstein condensation, etc.

We provided a theory concerning deformation quantization of several Poisson and symplectic manifolds. Subsequently, by exploiting C^*-algebraic techniques we discussed the classical limit of Berezin quantization maps, mean-field quantum spin systems and Schrödinger operators on a fixed finite dimensional symplectic manifold. Moreover, we presented a complete theory for understanding SSB in all these systems.

For mean-field quantum spin systems, the classical limit corresponds to the growing number of (spin) particles or lattice sites. It is however not yet known if our results and approaches can be extended to general quantum spin systems, where probably infinite dimensional symplectic manifolds (which allow for phase transitions as well) would come into play. Therefore, the interactions between the classical limit and general quantum spin systems need more mathematical investigation and the framework of deformation quantization, which has already shown its importance, might be an outcome.

For Schrödinger operators the classical limit corresponds to the limit in semi-classical parameter $\hbar \to 0$ appearing in front of the Laplacian. Rigorous results have been obtained for a wide class of Schrödinger operators and new light has been shed on the relation with their classical counterparts, i.e. certain smooth functions on $\mathbb{R}^{2n}$. In this context, however, interacting particle systems in the limit/regime of large number of particles $n \to \infty$ have not been investigated and this is still an open problem. Since these systems are of great important in mathematical physics, it is therefore of utmost importance to investigate whether our tools can be generalized in order to detect and prove rigorous results on for example Bose-Einstein-condensation and the existence of phase transitions in the limit $n \to \infty$.

1.6 Reading this thesis

The body of this thesis is organized as follows. In part I an abstract C^*-algebraic framework is introduced. To this end Chapter 2 contains an introduction in the theory of (strict) deformation quantization, assuming basic knowledge of C^*-algebras. Chapter 3 contains results on quantization theory of the Poisson manifold $S(M_k\mathbb{C})$), which provides an excellent way of studying mean-field quantum spin systems in their classical limit. We show in Chapter 4 that for the specific case when $k = 2$ this manifold is related to quantization of the 2-sphere $S^2 \subset \mathbb{R}^3$. In Chapter 5 we provide a natural framework for quantization theory in many-body quantum systems exploiting properties of the injective tensor product, and Chapter 6 contains a detailed study of semi-classical properties of Berezin quantization maps. Finally, in Chapter 7 we emphasize how this theory can be applied to understand spontaneous symmetry breaking from a algebraic point of view.

Part II focuses on various applications of the theory, starting with mean-field theories in Chapter 8. Here, the correspondence between mean-field quantum spin systems and polynomials (so-called classical symbols) on the unit 3-ball B^3 and its smooth boundary S^2 in the limit $N \to \infty$ is studied, exploiting the tools and techniques of the first part. In Chapter 9 a similar analysis is carried out for Schrödinger operators and Schwartz functions on $\mathbb{R}^{2n}$ where in this case Planck's constant $\hbar$ is sent to zero. In these two chapters special emphasis is given to symmetry breaking seen as emergent phenomenon in the pertinent limit.

In Chapter 10 a short overview with perspectives and further research is presented. Finally, in the appendix basic and useful facts, auxiliary lemmas and numerical analysis concerning quantum spin systems have been presented.

1.6.1 Notations and conventions

The natural numbers $\mathbb{N}$ are assumed to start from 1, i.e. $\mathbb{N} = 1, 2, 3....,$.

If X is a Hausdorff locally compact space, $C_0(X)$ indicates the space of functions $f : X \to \mathbb{C}$ such that, for every $\epsilon > 0$ there is a compact K_ϵ such that $|f(x)| < \epsilon$ if $x \notin K_\epsilon$. $C_c(X)$ and $C_c^\infty(X)$ respectively denote the subspace of continuous compactly supported functions $f : X \to \mathbb{C}$ and the analog for smooth functions when X is a smooth manifold. The space $L^\infty(X) \equiv L^\infty(X, \Sigma, \mu)$ denotes the Banach space (under pointwise operations) of equivalence classes of functions $f : X \to \mathbb{C}$ in the norm $||f||_\infty^{ess} := \inf\{t \in [0, \infty] \mid \mu(\{x \in X \mid |f(x)| > t\}) = 0\}$.

The *Lebesgue measure* on $\mathbb{R}^n$ will be always denoted by dx (or dp, da, etc). In the phase space $\mathbb{R}^{2n}$ we often use the normalized Liouville measure $\mu_\hbar := \frac{d^n p d^n q}{(2\pi\hbar)^n}$, whilst in case of S^2 we also use the measure $\mu_{1/N} = \frac{N+1}{4\pi} \sin\theta d\theta d\phi$. The definition of *Borel measure* and *regular* Borel measure, the statements of corresponding *Riesz' theorems* are adopted from [84].

If Z is a complex Banach space, then $\mathfrak{B}(Z)$ denotes the unital Banach algebra of the bounded operators $A : Z \to Z$ with respect to the standard operator norm. If $\mathcal{H}$ is a complex Hilbert space $\mathfrak{B}_1(\mathcal{H})$, $\mathfrak{B}_2(\mathcal{H})$, $\mathfrak{B}_\infty(\mathcal{H})$ respectively denote the two-sided $*$-ideals of trace class, Hilbert-Schmidt, and compact operators in $\mathfrak{B}(\mathcal{H})$. * is used to denote the adjoint A^* of an operator $A : D(A) \to \mathcal{H}$ with $D(A)$ dense in $\mathcal{H}$, and also the abstract adjoint a^* of an element a in a $*$-algebra $\mathfrak{A}$.

If $\mathfrak{A}$ is a C^*-algebra without unit, an *algebraic state* is a positive ($\omega(a^*a) \geq 0$), continuous, and norm-1 linear functional $\omega : \mathfrak{A} \to \mathbb{C}$. (As is known, these requirements are equivalent to positivity and normalization $\omega(\mathbb{1}) = 1$ if $\mathfrak{A}$ admits unit $\mathbb{1}$ for a linear functional $\omega : \mathfrak{A} \to \mathbb{C}$).

The *Fourier transform* [75] and its inverse are respectively defined as follows, for f, g in the *Schwartz space* $\mathcal{S}(\mathbb{R}^n)$,

$$\mathcal{F}_\hbar(f)(p) := \int_{\mathbb{R}^n} e^{-ip\cdot x/\hbar} f(x) \frac{dx}{(2\pi\hbar)^{n/2}} \; ; \qquad \mathcal{F}_\hbar^{-1}(g)(x) := \int_{\mathbb{R}^n} e^{ip\cdot x/\hbar} \hat{g}(p) \frac{dp}{(2\pi\hbar)^{n/2}} \; ,$$

and we also use the notation $\hat{f}_\hbar(p) := \mathcal{F}_\hbar(f)(p)$, $\check{g}_\hbar(x) := \mathcal{F}_\hbar^{-1}(g)(x)$. We shall also take advantage of the natural continuous linear extensions of $\mathcal{F}_\hbar$ and its inverse (1) to the space $\mathcal{S}'(\mathbb{R}^n)$ of *Schwartz distributions* and (2) to $L^2(\mathbb{R}^n, dx)$. The latter is known as the *Fourier-Plancherel transformation* which will be denoted by $F_\hbar$ and is a unitary operator on $L^2(\mathbb{R}^n, dx)$. With our conventions, the *convolution theorem* in the Schwartz space reads $\mathcal{F}_\hbar(f * g) = (2\pi\hbar)^{n/2}\mathcal{F}(f)_\hbar$, where the *convolution* of f and g is defined as $(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)dy$ as usual. The analogous statement is valid for the said pair of extensions of $\mathcal{F}_\hbar$. When $\hbar = 1$, we simply omit the index writing for instance $\mathcal{F}$, $\hat{g}$, and F.

I Algebraic methods

This part is developed to study the algebraic properties of quantization maps on Poisson and symplectic manifolds. In Chapter 2 a detailed overview regarding quantization procedures and their properties are presented. In Chapter 3 the existence of a deformation quantization of the algebraic state space of the $k \times k$ matrices (which is canonically a Poisson manifold with stratified boundary) is proved. Additionally, the interplay of different quantization maps is compared (Chapter 4) and quantization theory on tensor products is studied (Chapter 5). Moreover, in Chapter 6 a detailed study on Berezin quantization maps on a symplectic manifold S is carried out and several results in the semi-classical regime exploiting C^*-algebraic techniques are proved. Finally, a complete theory concerning the classical limit is developed. We hereto show under which conditions to Husimi density function converges to a probability measure on S. This in turn yields a notion of SSB as emergent phenomenon. To this end an extensive analysis is provided in Chapter 7 where also the relation with quantization theory is outlined.

2. Quantization procedures

Quantization theory aims to describe the process of transition from a classical theory to a newer understanding of a quantum theory. This can be done in several ways of rigour. In particular, several (inequivalent) quantization procedures are known and each has its pros and cons.

The aim of quantization theory presented in this thesis is to relate Poisson algebras or Poisson manifolds (encoding a classical theory) to non-commutative C^* -algebras which are used to describe quantum theories. This C^*-algebraic approach is a natural way to formalize emergent phenomena, the details of which are explained in Chapter 7. In this chapter we will introduce the necessary definitions and recall and summarize the main results useful for what is coming in the next chapters.

2.1 Strict deformation quantization

In this section we concentrate on the construction of a strict deformation quantization. In order to do so the following technical definition of *C^*-bundle* has to be introduced [57, Definition C 121]. This structure is a quite general arena where a quantization procedure based on C^*-algebras is performed (cf. Definition 2.1.3).

Definition 2.1.1. *Let I be a locally compact Hausdorff space. A* **C^*-bundle**[1] *over I consists of a complex C^*-algebra $\mathfrak{A}$, a collection of C^*-algebras $\{\mathfrak{A}_\hbar\}_{\hbar\in I}$, and surjective homomorphisms $\pi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar$ for each $\hbar \in I$, such that*

(i) $\|a\| = \sup_{\hbar\in I} \|\pi_\hbar(a)\|_\hbar$, where $\|\cdot\|$ (resp. $\|\cdot\|_\hbar$) denotes the C^*-norm of $\mathfrak{A}$ (resp. $\mathfrak{A}_\hbar$);

(ii) there exist an action $\rho : C_0(I) \times \mathfrak{A} \to \mathfrak{A}$ satisfying $\pi_\hbar(\rho(f, a)) = f(\hbar)\pi_\hbar(a)$ for any $\hbar \in I$ and $f \in C_0(I)$.

A **section** *of the bundle is an element $\{a_\hbar\}_{\hbar\in I}$ of $\Pi_{\hbar\in I}\mathfrak{A}_\hbar$ for which there exists an $a \in \mathfrak{A}$ such that $a_\hbar = \pi_\hbar(a)$ for each $\hbar \in I$. A* **C^*-bundle** *$\mathcal{A}$ is said to be* **continuous***, and its sections are called* **continuous sections***, if it satisfies*

(iii) for $a \in \mathfrak{A}$, the norm function $I \ni \hbar \mapsto \|\pi_\hbar(a)\|_\hbar$ is in $C_0(I)$.

A (continuous) bundle of C^*-algebras over I is also indicated by $(\mathfrak{A}, \{\mathfrak{A}_\hbar, \phi_\hbar\}_{\hbar\in I})$ or by the triple $\mathcal{A} := (I, \mathfrak{A}, \pi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar)$. The C^*-algebra $\mathfrak{A}$ is also called the C^*-algebra of the (continuous) bundle. ■

Remark 2.1.2. Since the $\pi_\hbar$ are homomorphisms of C^*-algebras, the $*$-algebra operations in $\mathfrak{A}$ are induced by the corresponding pointwise operations of the sections $I \ni \hbar \mapsto \pi_\hbar(a)$. It is therefore clear that $\mathfrak{A}$ can be identified with the space of sections of the bundle, seen as a C^*-algebra under pointwise scalar multiplication, addition, adjointing, and operator multiplication, by means of $\{\pi_\hbar(a)\}_{\hbar\in I} \leftrightarrow a$. Furthermore, condition (ii) reinforces the linearity preservation condition permitting coefficients continuously depend on $\hbar$.

We finally stress that in the case the C^*-algebras $\mathfrak{A}$ and/or $A_\hbar$ are non-unital, one can always unitize the algebra(s) and extend $\pi_\hbar$ to the (unique) unital homomorphism of these unitized C^*-algebras [68]. ■

Having defined a continuous bundle of C^*-algebras let us introduce the notion of *deformation quantization* of classical structures, a Poisson manifold in particular, according to [57, Definition 71]. The above *C^*-bundle* is therefore specialized to the case where $\mathfrak{A}_0$ is a commutative C^*-algebra representing the classical structure achieved in the *classical limit* $\hbar \to 0^+$ from corresponding quantum structures defined in the quantum fibers $\mathfrak{A}_\hbar$ with $\hbar > 0$. The *quantization maps* act along the opposite direction, associating to a classical observable $f \in \mathfrak{A}_0$ (or a substructure of it) a quantum observable $Q_\hbar(f) \in \mathfrak{A}_\hbar$.

[1]Called *continuous field of C^*-algebras* in [55].

Definition 2.1.3. *A* **deformation quantization**[2] *of a Poisson manifold* $(X, \{\cdot,\cdot\})$ *consists of:*

(1) A continuous C^**-bundle* $\mathcal{A} = (I, \mathfrak{A}, \pi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar)$*, where* I *is a subset of* $\mathbb{R}$ *containing* 0 *as accumulation point and* $\mathfrak{A}_0 = C_0(X)$ *equipped with norms* $||\cdot||_\hbar$*;*

(2) a dense $*$*-subalgebra* $\tilde{\mathfrak{A}}_0$ *of* $C_0(X)$ *closed under the action Poisson brackets (so that* $(\tilde{\mathfrak{A}}_0, \{\cdot,\cdot\})$ *is a complex Poisson algebra);*

(3) a collection of **quantization maps** $\{Q_\hbar\}_{\hbar\in I}$*, namely linear maps* $Q_\hbar : \tilde{\mathfrak{A}}_0 \to \mathfrak{A}_\hbar$ *(possibly defined on* $\mathfrak{A}_0$ *itself and next restricted to* $\tilde{\mathfrak{A}}_0$*) such that:*

(i) Q_0 *is the inclusion map* $\tilde{\mathfrak{A}}_0 \hookrightarrow \mathfrak{A}_0$ *(and* $Q_\hbar(\mathbb{1}_{\mathfrak{A}_0}) = \mathbb{1}_{\mathfrak{A}_\hbar}$ *if* $\mathfrak{A}_0$*, and* $\mathfrak{A}_\hbar$ *are unital for all* $\hbar \in I$*);*

(ii) $Q_\hbar(\overline{f}) = Q_\hbar(f)^*$*, where* $\overline{f}(x) := \overline{f(x)}$*;*

(iii) for each $f \in \tilde{\mathfrak{A}}_0$*, the assignments* $0 \mapsto f$*,* $\hbar \mapsto Q_\hbar(f)$ *when* $\hbar \in I \setminus \{0\}$*, define a continuous section of* $(I, \mathfrak{A}, \pi_\hbar)$*, meaning that there exists an element* $a^f \in \mathfrak{A}$ *such that* $\pi_\hbar(a^f) = Q_\hbar(f)$ *for each* $\hbar \in I$*.*

(iv) each pair $f, g \in \tilde{\mathfrak{A}}_0$ *satisfies the* **Dirac-Groenewold-Rieffel condition***:*

$$\lim_{\hbar\to 0} \left|\left| \frac{i}{\hbar}[Q_\hbar(f), Q_\hbar(g)] - Q_\hbar(\{f,g\}) \right|\right|_\hbar = 0.$$

If $Q_\hbar(\tilde{\mathfrak{A}}_0)$ *is dense in* $\mathfrak{A}_\hbar$ *for every* $\hbar \in I$*, then the deformation quantization is called* **strict***. (If* $Q_\hbar$ *is defined on the whole* $C_0(X)$*, all conditions except (iv) are assumed to be valid on* $C_0(X)$*.)* ■

Elements of I are interpreted as possible values of Planck's constant $\hbar$ and $\mathfrak{A}_\hbar$ is the quantum algebra of observables of the theory at the given value of $\hbar \neq 0$. For real-valued f, the operator $Q_\hbar(f)$ is the *quantum observable* associated to the classical observable f. This is possible because of condition (ii) in Definition 2.1.3.

It immediately follows from the definition of a continuous bundle of C^*-algebras that for any $f \in \tilde{\mathfrak{A}}_0$ the continuity property, called the **Rieffel condition**

$$\lim_{\hbar\to 0} \|Q_\hbar(f)\|_\hbar = \|f\|_\infty \ , \tag{2.1.1}$$

holds. Also the so-called **von Neumann condition**

$$\lim_{\hbar\to 0} \|Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)\|_\hbar = 0 \tag{2.1.2}$$

is valid. Indeed, the section $I \ni \hbar \mapsto Q_\hbar(f)Q_\hbar(g) - Q_\hbar(fg)$ is a continuous section because, it is constructed with the pointwise operations of the C^*-algebra $\mathfrak{A}$ and $(I, \mathfrak{A}, \pi_\hbar)$ is a continuous C^*-bundle, finally $Q_0(f)Q_0(g) - Q_0(fg) = fg - fg = 0$, hence (iii) in Definition 2.1.1 implies (2.1.2).

Remark 2.1.4. Suppose we have a strict deformation quantization according to the previous definition. If we also require the quantization maps $Q_\hbar$ to be injective for each $\hbar$ and that $Q_\hbar(\tilde{\mathfrak{A}}_0)$ is a $*$-subalgebra of $\mathfrak{A}_\hbar$ (for each $\hbar \in I$), then a (non-commutative, associative) **quantization deformation product** $\star_\hbar$ turns out to be implicitly defined [55] in $\tilde{\mathfrak{A}}_0$ from[3] $Q_\hbar(f \star_\hbar g) = Q_\hbar(f)Q_\hbar(g)$ whose anti-symmetric part is related to the Poisson structure. ■

Remark 2.1.5. In view of Definition 2.1.3 the Poisson $*$-algebra $\tilde{\mathfrak{A}}_0$ is assumed to be dense in $C_0(X)$, for some Poisson manifold X. This definition can be easily generalized to arbitrary Poisson $*$-algebras densely contained in the self-adjoint part of a commutative C^*-algebra $\mathfrak{A}_0$ [55, Definition II 1.1.1]. ■

[2] Named *continuous quantization* of a Poisson manifold in [55, Definition II 1.2.5].

[3] The reader should pay attention to the fact that, in [55], these hypotheses are included in the definition of *strict deformation quantization* as stated in [55, II Definition 1.1.2]. Furthermore our notion of *deformation quantization* adopted from [57] is named *continuous quantization* in [55].

To define a deformation quantization it is not necessary starting from a continuous C^*-bundle, but it is sufficient to assign quantization maps satisfying some conditions. That is due to the following result adapted from [55, Theorem II 1.2.4].

Theorem 2.1.6 ([55, II. Thm. 1.2.4]). *Let X be a smooth manifold (possibly Poisson), $\{\mathfrak{A}_\hbar\}_{\hbar\in I}$ a collection of C^*-algebras where $\mathfrak{A}_0 := C_0(X)$ and $I \subset \mathbb{R}$, with $0 \in I$ as accumulation point. Consider a collection of maps $Q_\hbar : \tilde{\mathfrak{A}}_0 \to \mathfrak{A}_\hbar$, $\hbar \in I$, $\tilde{\mathfrak{A}}_0 \subset \mathfrak{A}_0$ being a dense $*$-subalgebra (possibly $\mathfrak{A}_0$ itself or a Poisson algebra), satisfying*

- *(a) (i) and (ii) in (3) Definition 2.1.3 ((iv) possibly);*
- *(b) $I \ni \hbar \mapsto ||Q_\hbar(f)||_\hbar$ is in $C_0(I)$ for every $f \in \tilde{\mathfrak{A}}_0$;*
- *(c) the Rieffel condition (viz.* (2.1.1)*);*
- *(d) the von Neumann condition (viz.* (2.1.2)*);*
- *(e) $\overline{Q_\hbar(\tilde{\mathfrak{A}}_0)} = \mathfrak{A}_\hbar$ for every $\hbar \in I$;*
- *(f) I is discrete, or the C^*-algebras $\mathfrak{A}_\hbar$ are identical for $\hbar \in I \setminus \{0\}$ and $I \ni \hbar \mapsto Q_\hbar(f)$ is continuous for every $f \in \tilde{\mathfrak{A}}_0$.*

Then the following facts are true.

- *(1) There exists a unique continuous C^*-bundle $(I, \mathfrak{A}, \pi_\hbar)$ such that every $\{Q_\hbar(f)\}_{\hbar\in I}$, $f \in \tilde{\mathfrak{A}}_0$ is a continuous section. In this case Definition 2.1.3 is valid (up to (iv) in (3) possibly).*
- *(2) If a family of maps $Q'_\hbar : \tilde{\mathfrak{A}}_0 \to \mathfrak{A}_\hbar$, $\hbar \in I$ satisfies the hypotheses (a)-(f) and*

$$||Q'_\hbar(f) - Q_\hbar(f)||_\hbar \to 0 \quad \text{for } \hbar \to 0 \text{ and every } f \in \tilde{\mathfrak{A}}_0,$$

then the maps $Q'_\hbar$ determine the same C^*-bundle $(I, \mathfrak{A}, \pi_\hbar)$ as the maps $Q_\hbar$.[4]

■

A natural question one can ask is to investigate whether or not it is possible to extend the quantization maps to all of $\mathfrak{A}_0$. The precise statement is given in the next lemma.

Lemma 2.1.7. *Assume we are given a strict deformation quantization such that the associated quantization maps $Q_\hbar$ satisfy*

$$||Q_\hbar(f)||_\hbar \le C||f||_\infty, (f \in \tilde{\mathfrak{A}}_0,\ \hbar \in I); \tag{2.1.3}$$

for some C that does not depend on $\hbar$. Then the maps $Q_\hbar$ extend to all of $\mathfrak{A}_0$ and in particular Rieffel's and von Neumann's condition hold.

Proof. The extension from $\tilde{\mathfrak{A}}_0$ to $\mathfrak{A}_0$ is the continuous extension: every $f \in \mathfrak{A}_0$ is a limit of a Cauchy sequence of elements (f_k) in the subalgebra $\tilde{\mathfrak{A}}_0$. As a result of (2.1.3) it follows that also $Q_\hbar(f_k)$ is a Cauchy sequence in $\mathfrak{A}_\hbar$, and hence it has a limit herein. Define $Q_\hbar(f) := \lim_{k\to\infty} Q_\hbar(f_k)$. It is not difficult to show that $Q_\hbar : \mathfrak{A}_0 \to \mathfrak{A}_\hbar$ is linear and that (2.1.3) holds for any $f \in \mathfrak{A}_0$.

In order to conclude we only prove Rieffel's condition since von Neumann's condition goes in a similar fashion. Take a sequence (f_k) in $\tilde{\mathfrak{A}}_0$ with $f_k \to f$ in $\mathfrak{A}_0$. Then, by the reverse triangle inequality

$$\begin{aligned}
\Big| ||Q_\hbar(f)|| - ||f||_\infty \Big| &\le \Big| ||Q_\hbar(f)|| - ||Q_\hbar(f_k)|| \Big| + \Big| ||Q_\hbar(f_k)|| - ||f_k||_\infty \Big| + \Big| ||f_k|| - ||f||_\infty \Big| \\
&\le ||Q_\hbar(f - f_k)|| + \Big| ||Q_\hbar(f_k)|| - ||f_k||_\infty \Big| + ||f_k - f||_\infty \\
&\le C||f - f_k||_\infty + \Big| ||Q_\hbar(f_k)|| - ||f_k||_\infty \Big| + ||f_k - f||_\infty .
\end{aligned} \tag{2.1.4}$$

[4] We recall from [55, II. Lemma 1.2.2, II. Prop. 1.2.3] that a continuous C^* bundle $(I, \mathfrak{A}, \pi_\hbar)$ is uniquely determined by its quantization maps $Q_\hbar$ (assuming they exist) in the sense that $\mathfrak{A}$ consists of all $\{a_\hbar\}_{\hbar\in I}$ in $\Pi_{\hbar\in I}\mathfrak{A}_\hbar$ for which the function $\hbar \mapsto ||a_\hbar - Q_\hbar(f)||_\hbar$ is in $C_0(I)$ for each $f \in C_0(X)$.

Given $\epsilon > 0$, take $k \in \mathbb{N}$ such that $||f - f_k||_\infty \leq \frac{\epsilon}{2}\frac{1}{C+1}$ and take $\hbar_0$ such that for all $\hbar < \hbar_0$, $\Big| ||Q_\hbar(f_k)|| - ||f_k||_\infty \Big| < \epsilon/2$ (which is possible due to the fact that Rieffel's condition (2.1.1) applies to each $f_k \in \tilde{\mathfrak{A}}_0$). Then, by construction,

$$\Big| ||Q_\hbar(f)|| - ||f||_\infty \Big| \leq \epsilon, \tag{2.1.5}$$

concluding the proof. □

2.1.1 Weyl and Berezin quantizations maps on $\mathbb{R}^{2n}$

In this section we focus on Weyl and Berezin quantization of the manifold $\mathbb{R}^{2n}$. These maps play a crucial role in the theory of Schrödinger operators as will become clear in Chapter 9. We start with some general results, most of them also stated in e.g. [3, Sect. B.1],[12, 28, 48, 107].

Let us first consider the commutative C^*-algebra $C_0(\mathbb{R}^{2n})$, where $\mathbb{R}^{2n}$ plays the role of the classical *phase space* equipped with the standard symplectic structure. The natural symplectic coordinates of $\mathbb{R}^{2n}$ will be denoted by $(q,p) = (q^1, \dots, q^n, p_1, \dots, p_n)$. The Liouville measure will therefore coincide with the standard $2n$-dimensional Lebesgue measure $dqdp$ and the Poisson bracket will take the form

$$\{f,g\} := \sum_{k=1}^n \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} - \sum_{k=1}^n \frac{\partial g}{\partial p_k}\frac{\partial f}{\partial q^k}.$$

The *strict deformation Weyl quantization* is a subcase of a more general quantization procedure acting on the space of Schwartz distributions $f \in \mathcal{S}'(\mathbb{R}^{2n})$ to which it associates a possibly unbounded operator, or more generally, just a quadratic form corresponding to the *quantum-Fourier antitransform of* f:

$$Q_\hbar^W(f) := \int_{\mathbb{R}^{2n}} e^{i\overline{a\cdot X + b\cdot P}} \widehat{f}(a,b)\frac{dadb}{(2\pi)^n} = \int_{\mathbb{R}^{2n}} e^{ia\cdot X} e^{ib\cdot P} e^{-\frac{i\hbar}{2}a\cdot b}\widehat{f}(a,b)\frac{dadb}{(2\pi)^n}, \tag{2.1.6}$$

where $\widehat{f} \in \mathcal{S}'(\mathbb{R}^{2n})$ is the Fourier transform of f and $\overline{a\cdot X + b\cdot P}$ denotes the closure of the (essentially self-adjoint) operator $a\cdot X + b\cdot P$. The $\hbar$ appearing in $Q_\hbar^W$ takes place both in the exponents $e^{\pm i\hbar a\cdot b/2}$ and in the definition of P_k, that is the unique self-adjoint extension of $-i\hbar\frac{\partial}{\partial x^k}$ acting on $\mathcal{S}(\mathbb{R}^n)$. As said, $Q_\hbar^W(f)$ is only defined in the sense of *quadratic forms* in general: the **Weyl quantization map**, in this sense, is defined as

$$\langle \psi, Q_\hbar^W(f)\phi\rangle := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \langle e^{-ia\cdot X}\psi, e^{ib\cdot P}\phi\rangle e^{-\frac{i\hbar}{2}a\cdot b}\widehat{f}(a,b)\, dadb\,, \quad \hbar > 0; \tag{2.1.7}$$

for all $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$. By construction $\mathbb{R}^{2n} \ni (a,b) \mapsto \langle e^{-ia\cdot X}\psi, e^{ib\cdot P}\phi\rangle e^{-\frac{i\hbar}{2}a\cdot b}$ is an element of $\mathcal{S}(\mathbb{R}^{2n})$ as it is, up to a phase, the Fourier transform of a function in that space, so that the definition is well posed. There are cases where the quadratic form uniquely determines a true operator $Q_\hbar^W(f)$ which is in general *unbounded and densely defined.* This happens when the distributions f are actually functions of various spaces. Further restricting the space of functions finally yields an everywhere defined and bounded operator as is proper for the C^*-algebraic approach. Let us examine some of those cases.

Proposition 2.1.8. *With the given definition of the quadratic form* $\langle \psi, Q_\hbar^W(f)\phi\rangle$ *(2.1.7), where* $\hbar > 0$, $f \in \mathcal{S}(\mathbb{R}^{2n})$ *and* $\psi, \phi \in \mathcal{S}(\mathbb{R}^n)$*, the following facts are valid.*

(1) If $f \in \mathcal{S}(\mathbb{R}^{2n})$*, then there is a unique* $Q_\hbar^W(f) \in \mathfrak{B}(L^2(\mathbb{R}^n, dx))$ *whose quadratic form on* $\mathcal{S}(\mathbb{R}^n)$ *is (2.1.7) and*

$$||Q_\hbar^W(f)|| \leq \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} |\widehat{f}(a,b)| dadb\,. \tag{2.1.8}$$

where the right-hand side does not depend on $\hbar$*. With the considered hypotheses (2.1.7) is valid for generic* $\psi, \phi \in L^2(\mathbb{R}^n, dx)$ *and the everywhere defined operator* $Q_\hbar^W(f)$*.*

(2) If $f : \mathbb{R}^{2n} \to \mathbb{C}$ is a Borel function which is constant in the variable p and polynomially bounded in the variable x, then there is a unique operator $Q^W_\hbar(f)$ on $\mathcal{S}(\mathbb{R}^n)$ whose quadratic form is (2.1.7), that is $Q^W_\hbar(f) = f(X)|_{\mathcal{S}(\mathbb{R}^n)}$, where $f(X)$ is defined by spectral calculus. In particular,

$$||Q^W_\hbar(f)|| = ||f(X)|_{\mathcal{S}(\mathbb{R}^n)}|| \leq ||f||_\infty \leq +\infty,$$

and the bound does not depend on $\hbar$.

(3) If $f : \mathbb{R}^{2n} \to \mathbb{C}$ is a Borel function which is constant in the variable x and polynomially bounded in the variable p, then there is a unique operator $Q^W_\hbar(f)$ on $\mathcal{S}(\mathbb{R}^n)$ whose quadratic form is (2.1.7), that is $Q^W_\hbar(f) = f(P)|_{\mathcal{S}(\mathbb{R}^n)}$, where $f(P)$ is defined by spectral calculus. In particular,

$$||Q^W_\hbar(f)|| = ||f(P)|_{\mathcal{S}(\mathbb{R}^n)}|| \leq ||f||_\infty \leq +\infty,$$

where the bound does not depend on $\hbar$.

(4) If $f(q,p) = f_1(q)f_2(p)$ with $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$, then the operator $Q^W_\hbar(f_1f_2) \in \mathfrak{B}(L^2(\mathbb{R}^n, dx))$ that exists due to the case (1), is completely determined by its quadratic form:

$$\langle \psi, Q^W_\hbar(f_1f_2))\phi\rangle = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \widehat{f_2}(b) \left\langle \psi, e^{ib\cdot P} f_1\left(X + \frac{\hbar}{2}bI\right)\phi\right\rangle db \tag{2.1.9}$$

for $\psi, \phi \in L^2(\mathbb{R}^n, dx)$.

Proof. (1) If $f \in \mathcal{S}(\mathbb{R}^{2n})$ then $\widehat{f} \in \mathcal{S}(\mathbb{R}^{2n})$ as well and

$$|\langle \psi, Q^W_\hbar(f)\phi\rangle| \leq ||\psi||\,||\phi||\,\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} |\widehat{f}(a,b)|dadb,$$

so that $Q^W_\hbar(f)$ exists due to the Riesz lemma and is bounded and thus it can be extended to the whole Hilbert space. The estimate (2.1.8) holds trivially. Finally observe that in (2.1.7) $\widehat{f} \in \mathcal{S}(\mathbb{R}^{2n})$ so that the right-hand side is defined for generic vectors $\psi, \phi \in L^2(\mathbb{R}^2, dx)$ since the map $a, b \mapsto \langle e^{-ia\cdot X}\psi, e^{ib\cdot P}\phi\rangle e^{-\frac{i\hbar}{2}a\cdot b}$ is bounded. A direct application of the Lebesgue dominated convergence theorem also using the fact that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n, dx)$ proves that it is valid for generic vectors ψ, ϕ and where $Q_\hbar(f)$ is the unique bounded linear extension of the operator initially defined on $\mathcal{S}(\mathbb{R}^n)$.
(2) $f \in \mathcal{S}'(\mathbb{R}^{2n})$ so that $\langle \psi, Q^W_\hbar(f)\phi\rangle$ equals

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \langle e^{-ia\cdot X}\psi, e^{ib\cdot P}\phi\rangle e^{-\frac{i\hbar}{2}a\cdot b}(2\pi)^{n/2}\delta(b)\widehat{f}(a)dadb = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^{2n}} \langle e^{-ia\cdot X}\psi, \phi\rangle \widehat{f}(a)da\,.$$

We can now exploit the spectral decomposition [64] of $e^{-ia\cdot X}$, $\langle e^{-ia\cdot X}\psi, \phi\rangle = \int_{\mathbb{R}} e^{ia\cdot\lambda}d\mu_{\psi,\phi}(\lambda)$, noticing that this function of a belongs to $\mathcal{S}(\mathbb{R}^n)$(it is the Fourier transform of a function of that space). From the definition of Fourier transform of distributions and the definition of Fourier transform of (finite) complex measures, we conclude that

$$\langle \psi, Q^W_\hbar(f)\phi\rangle = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \widehat{f}(a)\int_{\mathbb{R}} e^{ia\cdot\lambda}d\mu_{\psi,\phi}(\lambda)da = \int_{\mathbb{R}} f(\lambda)d\mu_{\psi,\phi}(\lambda)\,.$$

Hence $\langle \psi, Q^W_\hbar(f)\phi\rangle = \langle \psi, f(X)\phi\rangle$ and the final bound in the thesis is valid.
(3) The proof is strictly analogous to that of (2).
(4) With a procedure analogous to the one of the previous cases using in particular the last expression in (2.1.6), we easily prove that the operator $Q^W_\hbar(f_1f_2)$, which is bounded and everywhere defined in view of the case (1), is completely determined by its quadratic form (2.1.9). □

When restricting to $\mathcal{S}(\mathbb{R}^{2n})$, Q^W defines a deformation quantization map according to Definition 2.1.3 and re-adapting [55, Theorem 2.6.1] to our definitions.[5] Remarkably, it turns

[5]This follows from Theorem 2.1.6 using [55, Theorem II 2.6.1] combined with the fact that the map $\hbar \mapsto$

in particular out that $Q_\hbar^W(\mathcal{S}(\mathbb{R}^{2n})) \subset \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$.

Theorem 2.1.9 ([55, II. Cor. 2.5.4, II. Thm. 2.6.1])**.** *The family of maps* $Q_\hbar^W : \mathcal{S}(\mathbb{R}^{2n}) \to \mathfrak{B}(L^2(\mathbb{R}^n, dx))$, $\quad \hbar > 0$, *constructed as in (1) of Proposition 2.1.8 together with* $Q_0^W := id_{\mathcal{S}(\mathbb{R}^{2n})}$ *defines a deformation quantization of the Poisson manifold* $(\mathbb{R}^{2n}, \{\cdot,\cdot\})$ *according to definition 2.1.3, where* $I := [0, +\infty)$ *and*

$$Q_\hbar^W(\mathcal{S}(\mathbb{R}^{2n})) \subset \mathfrak{A}_\hbar = \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$$

for all $\hbar > 0$*. This deformation quantization enjoys the following properties*

(1) it is strict, i.e., $\overline{Q_\hbar^W(\mathcal{S}(\mathbb{R}^{2n}))} = \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$ *for all* $\hbar \in I \setminus \{0\}$*,*

(2) the maps $Q_\hbar^W$ *are injective for all* $\hbar \in I$*,*

(3) $Q_\hbar^W(\mathcal{S}(\mathbb{R}^{2n}))$ *is a* $*$*-subalgebra of* $\mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$ *for all* $\hbar \in I \setminus \{0\}$ *so that* $\star_\hbar$ *can be defined.*

■

The **Berezin quantization map** on $\mathbb{R}^{2n}$, as a sesquilinear form, is defined as in [55] (we refer to Section 2.2 for a detailed presentation on completely general symplectic manifolds in place of $\mathbb{R}^{2n}$ within the coherent and pure state quantization approach)

$$\langle \psi, Q_\hbar^B(f)\phi\rangle := \int_{\mathbb{R}^{2n}} f(q,p)\langle \psi, \Psi_\hbar^{(q,p)}\rangle\langle \Psi_\hbar^{(q,p)}, \phi\rangle \frac{dqdp}{(2\pi\hbar)^n}, \quad \hbar > 0; \tag{2.1.10}$$

where $\psi, \phi \in L^2(\mathbb{R}^n, dx)$, $f \in L^\infty\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right)$. Above, for any given $(q,p) \in \mathbb{R}^{2n}$,

$$\Psi_\hbar^{(q,p)}(x) := \frac{e^{-\frac{i}{2}p\cdot q/\hbar} e^{ip\cdot x/\hbar} e^{-(x-q)^2/(2\hbar)}}{(\pi\hbar)^{n/4}}\,, \quad x \in \mathbb{R}^n\,, \hbar > 0; \tag{2.1.11}$$

is a *unit* vector in $L^2(\mathbb{R}^n, dx)$ also called a **Schrödinger coherent state** (again, we refer to Section 2.2 for a general construction). It turns out that [55] the integral (2.1.10) satisfies, $\langle \psi, Q_\hbar^B(1)\psi\rangle = 1$ for $\psi \in L^2(\mathbb{R}^n, dx)$ with $||\psi|| = 1$, hence $|\langle \psi, Q_\hbar^B(f)\psi\rangle| \leq ||f||_\infty ||\psi||^2$. By a polarization argument, $|\langle \psi, Q_\hbar^B(f)\phi\rangle| \leq ||f||_\infty ||\psi||\,||\phi||\,, \quad \psi, \phi \in L^2(\mathbb{R}^n, dx)$. Therefore, by the Riesz lemma, $Q_\hbar^B(f) \in \mathfrak{B}(L^2(\mathbb{R}, dx))$ and

$$||Q_\hbar^B(f)|| \leq ||f||_\infty\,, \quad f \in L^\infty\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right)\,. \tag{2.1.12}$$

Let us summarize the general properties of $Q_\hbar^B$ (Theorems II 1.3.3 and II 1.3.5 in [55] specialized to $\mathbb{R}^{2n}$) relevant for our work. Positivity condition (2) is one of the most important improvements which differentiates Berezin quantization from Weyl quantization.

Theorem 2.1.10. *The linear map*

$$Q_\hbar^B : L^\infty\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right) \to \mathfrak{B}(L^2(\mathbb{R}^n, dx)), \quad \hbar > 0;$$

defined as

$$Q_\hbar^B(f) := \int_{\mathbb{R}^{2n}} f(q,p)|\Psi_\hbar^{(q,p)}\rangle\langle\Psi_\hbar^{(q,p)}| \frac{dqdp}{(2\pi\hbar)^n}, \tag{2.1.13}$$

in the sense of (2.1.10) and satisfying (2.1.12) enjoys the following properties.

(1) $Q_\hbar^B(1) = I_{L^2(\mathbb{R}^n, dx)}$*, where* $1 \in L^\infty\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right)$ *is the constant 1 map;*

$Q_\hbar^W(f)$ is continuous for all $f \in \mathcal{S}(\mathbb{R}^{2n})$ (as follows from the proof in [55, Lemma II 2.6.2]) and the continuity property of the norm, in that the map $\hbar \mapsto ||Q_\hbar^W(f)||$ is in $C_0(I)$ for all $f \in \mathcal{S}(\mathbb{R}^{2n})$, as in turn follows from the proof in [55, Thm II 2.6.5].

(2) $f \geq 0$ *a.e. implies* $Q^B_\hbar(f) \geq 0$ *for* $f \in L^\infty\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right)$;

(3) $Q^B_\hbar(\overline{f}) = Q^B_\hbar(f)^*$ *for* $f \in L^\infty\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right)$;

(4) $Q^B_\hbar(C_0(\mathbb{R}^{2n})) = \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$, $\quad Q^B_\hbar(L^1 \cap L^\infty) = \mathfrak{B}_1(L^2(\mathbb{R}^n, dx))$;

(5) $Tr(Q^B_\hbar(f)) = \int_{\mathbb{R}^{2n}} \frac{dqdp}{(2\pi\hbar)^n} f(q,p)$, *if* $f \in L^1(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}) \cap L^\infty(L^2(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}))$.

■

Finally, the Berezin map defines a deformation quantization according to the following pair of theorems (readaptation of Theorem II 2.41 and Proposition II 2.6.3 in [55], item (5) – (2.117) in [55] – is obtained by simply comparing (2.1.11) and (2.1.7) by writing down the action of $e^{\hbar\Delta_{2n}/4}$ in terms of Gaussian convolution).[6]

Theorem 2.1.11 ([55, II. Thm. 2.4.1])**.** *The family of maps* $Q^B_\hbar : C^\infty_c(\mathbb{R}^{2n}) \to \mathfrak{B}(L^2(\mathbb{R}^n, dx))$, $\hbar > 0$, *defined in theorem 2.1.10 together with* $Q^B_0 := id_{C^\infty_c(\mathbb{R}^{2n})}$ *defines a deformation quantization of the Poisson manifold* $(\mathbb{R}^{2n}, \{\cdot,\cdot\})$ *according to definition 2.1.3, where* $\tilde{\mathfrak{A}}_0 := C^\infty_c(\mathbb{R}^{2n})$, $I := [0, +\infty)$, *and*

$$Q^B_\hbar(C^\infty_c(\mathbb{R}^{2n})) \subset \mathfrak{A}_\hbar := \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx)) \quad \text{for all } \hbar > 0.$$

This deformation quantization enjoys the following properties,

(1) it is strict, i.e., $\overline{Q^B_\hbar(C^\infty_c(\mathbb{R}^{2n}))} = \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$ *for all* $\hbar > 0$.

(2) the maps $Q^B_\hbar$ *are injective for all* $\hbar \in I$.

■

A weaker version of the result above is obtained when working directly on $\mathfrak{A}_0 := C_0(\mathbb{R}^{2n})$ since $Q^B_\hbar$ is defined thereon.

Theorem 2.1.12 ([55, II. Thm. 2.4.1, II. Prop 2.6.3])**.** *The family of maps* $Q^B_\hbar : C_0(\mathbb{R}^{2n}) \to \mathfrak{B}(L^2(\mathbb{R}^n, dx))$, $\hbar > 0$, *defined in theorem 2.1.10 together with* $Q^B_0 := id_{C_0(\mathbb{R}^{2n})}$ *gives rise to a deformation quantization of the Poisson manifold* $(\mathbb{R}^{2n}, \{\cdot,\cdot\})$, *except for the Dirac-Groenewold-Rieffel condition, according to definition 2.1.3, where* $I := [0, +\infty)$ *and*

$$Q^B_\hbar(C_0(\mathbb{R}^{2n})) \subset \mathfrak{A}_\hbar = \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx)), \quad \hbar > 0,$$

for all $\hbar > 0$. *This deformation quantization enjoys the following properties*

(1) it is strict and more strongly $Q^B_\hbar(C_0(\mathbb{R}^{2n})) = \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$ *for all* $\hbar > 0$;

(2) it is **asymptotically equivalent** *to* $Q^W_\hbar$ *on* $\mathcal{S}(\mathbb{R}^{2n})$, *i.e., the map*

$$I \ni \hbar \to ||Q^B_\hbar(f) - Q^W_\hbar(f)||_\hbar$$

is continuous if $f \in \mathcal{S}(\mathbb{R}^{2n})$ *and* $||Q^B_\hbar(f) - Q^W_\hbar(f)||_\hbar \to 0$ *for* $\hbar \to 0^+$.

(3) If $f \in \mathcal{S}(\mathbb{R}^{2n})$,

$$Q^B_\hbar(f) = Q^W_\hbar\left(e^{\hbar\Delta_{2n}/4} f\right), \quad \hbar > 0; \tag{2.1.14}$$

where the exponential denotes the one-parameter semigroup generated by the self-adjoint extension on $L^2\left(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n}\right)$ *of* $\Delta_{2n} := \sum_{k=1}^n \frac{\partial^2}{\partial q^{k2}} + \sum_{k=1}^n \frac{\partial^2}{\partial p_k^2}$, *initially defined on* $C^\infty_c(\mathbb{R}^{2n})$.

■

[6] Similar as in the case of Weyl quantization, this follows from Theorem 2.1.6 using [55, Theorem II 2.4.1], the fact that the map $\hbar \mapsto Q^B_\hbar(f)$ is continuous for all $f \in C_c(\mathbb{R}^{2n})$, and that the map $\hbar \mapsto ||Q^B_\hbar(f)||$ is in $C_0(I)$ for all $f \in C_c(\mathbb{R}^{2n})$ (as follows from the proof of [55, Thm II 2.6.5].)

As a final technical result we prove the following which plays a central role in Chapter 9.

Proposition 2.1.13. *Referring to the definition (2.1.13) of $Q^B_\hbar$, the following facts are true.*

(a) Consider $f \in C_0(\mathbb{R}^n)$. If interpreting f as a function in $L^\infty(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n})$ constant in the variable p, then

$$||Q^B_\hbar(f) - f|| \to 0 \quad \text{for } \hbar \to 0^+ \;, \tag{2.1.15}$$

where both operators are defined in $L^2(\mathbb{R}^n, dx)$ and $(f\psi)(x) := f(x)\psi(x)$ if $\psi \in L^2(\mathbb{R}^n, dx)$ and $x \in \mathbb{R}^n$.

(b) Consider $f \in \mathcal{S}(\mathbb{R}^n)$. If interpreting f as a function in $L^\infty(\mathbb{R}^{2n}, \frac{dqdp}{(2\pi\hbar)^n})$ constant in the variable q, then it holds

$$||Q^B_\hbar(f) - \check{f}_\hbar * || \to 0 \quad \text{for } \hbar \to 0^+ \;, \tag{2.1.16}$$

where both operators are defined in $L^2(\mathbb{R}^n, dx)$, where $\check{f}_\hbar := \mathcal{F}^{-1}_\hbar(f)$ and $(\check{f}_\hbar *)(\psi) := \check{f}_\hbar * \psi$ if $\psi \in L^2(\mathbb{R}^n, dx)$. In particular, for every chosen $t > 0$,

$$||Q^B_\hbar(e^{-tp^2}) - e^{t\hbar^2\Delta}|| \to 0 \quad \text{for } \hbar \to 0^+ \;.$$

Proof. (a) Let $f \in C_0(\mathbb{R}^n)$ be a function of the variable q. If $\phi \in \mathcal{S}(\mathbb{R}^n)$, so that all integration can be interchanged and using

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\frac{p(x-x')}{\hbar}} g(x')dx'dp = g(x') \;, \quad (g \in \mathcal{S}(\mathbb{R}^n))\;;$$

we have

$$(Q^B_\hbar(f)\phi)(x) = \frac{1}{2^n(\pi\hbar)^{3n/2}}\int_{\mathbb{R}^{2n}} f(q)e^{-\frac{(x-q)^2}{2\hbar}}\int_{\mathbb{R}^n} e^{i\frac{p(x-x')}{\hbar}}e^{-\frac{(x'-q)^2}{2\hbar}}\phi(x')dx'dqdp$$

$$= \frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n} f(q)e^{-\frac{(x-q)^2}{2\hbar}}e^{-\frac{(x-q)^2}{2\hbar}}\phi(x)dq = \frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n} f(q)e^{-\frac{(x-q)^2}{\hbar}}\phi(x)dq \;.$$

In summary, $(Q^B_\hbar(f)\phi)(x) = \left(\frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n} f(q)e^{-\frac{(x-q)^2}{\hbar}}dq\right)\phi(x)$. An easy density argument of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n, dx)$ extends the above result to the general case of $\phi \in L^2(\mathbb{R}^n, dx)$. To conclude the proof it is sufficient to prove that, if $f \in C_0(\mathbb{R}^n)$, then

$$\frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n} f(q)e^{-\frac{(x-q)^2}{\hbar}}dq \to f(x) \quad \text{uniformly in } x \in \mathbb{R}^n \text{ if } \hbar \to 0^+, \tag{2.1.17}$$

since, considering the functions a multiplicative operators,

$$\left|\left|\frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n} f(q)e^{-\frac{(\cdot-q)^2}{\hbar}}dq - f\right|\right|_{\mathfrak{B}(L^2(\mathbb{R}^n,dx))} = \left|\left|\frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n} f(q)e^{-\frac{(\cdot-q)^2}{\hbar}}dq - f\right|\right|_\infty \;.$$

To prove (2.1.17), observe that since $\int_{\mathbb{R}^n} e^{-\frac{(x-q)^2}{\hbar}}dq = (\pi\hbar)^{n/2}$, (2.1.17) can be re-written

$$I_\hbar(x) := \frac{1}{(\pi\hbar)^{n/2}}\int_{\mathbb{R}^n}(f(q) - f(x))e^{-\frac{(x-q)^2}{\hbar}}dq \to 0 \quad \text{uniformly in } x \in \mathbb{R}^n \text{ if } \hbar \to 0^+. \tag{2.1.18}$$

Let us prove that it is valid. We start from the decomposition

$$I_\hbar(x) = \frac{1}{(\pi\hbar)^{n/2}}\int_{|x-q|<\delta}(f(q) - f(x))e^{-\frac{(x-q)^2}{\hbar}}dq + \frac{1}{(\pi\hbar)^{n/2}}\int_{|x-q|\geq\delta}(f(q) - f(x))e^{-\frac{(x-q)^2}{\hbar}}dq.$$

The crucial observation is that $f \in C_0(\mathbb{R}^n)$ is necessarily uniformly continuous and thus, for every $\epsilon > 0$, there is $\delta > 0$ such that $|f(q) - f(x)| < \epsilon/2$ if $|x - q| < \delta$. Hence,

$$I_\hbar(x) \leq \epsilon/2 + \frac{1}{(\pi\hbar)^{n/2}} \int_{|x-q|\geq\delta} (f(q) - f(x)) e^{-\frac{(x-q)^2}{\hbar}} dq \leq \epsilon/2 + 2||f||_\infty \frac{\hbar^{n/2}}{(\pi)^{n/2}} \int_{|y|\geq\delta/\hbar} e^{-y^2} dy \leq \epsilon/2 + C\hbar^{n/2},$$

where $y = (q - x)/\sqrt{\hbar}$. We can finally choose $H_\epsilon > 0$ such that $C\hbar^{n/2} < \epsilon/2$ if $0 < \hbar < H_\epsilon$, concluding the proof for $f \in C_0(\mathbb{R}^n)$.
(b) Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a function of the variable p. If $\phi \in \mathcal{S}(\mathbb{R}^n)$ so that all integrals can be interchanged, we have

$$(Q_\hbar^B(f)\phi)(x) = \frac{1}{2^n(\pi\hbar)^{3n/2}} \int_{\mathbb{R}^{2n}} f(p) e^{-\frac{(x-q)^2}{2\hbar}} \int_{\mathbb{R}^n} e^{i\frac{p(x-x')}{\hbar}} e^{-\frac{(x'-q)^2}{2\hbar}} \phi(x') dx' dq dp$$

$$= \frac{1}{2^{n/2}(\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \check{f}(x-x') e^{-\frac{(x-q)^2}{2\hbar}} e^{-\frac{(x'-q)^2}{2\hbar}} \phi(x') dx' dq = \frac{1}{2^{n/2}(\pi\hbar)^n} \int_{\mathbb{R}^n} \check{f}(x-x') G(x-x') \phi(x') dx', \tag{2.1.19}$$

where $G(x) := \int_{\mathbb{R}^n} e^{-\frac{(x-z)^2}{2\hbar}} e^{-\frac{z^2}{2\hbar}} dz$. Observe that, the convolution thorem in $\mathcal{S}(\mathbb{R}^n)$ implies that $\widehat{G}(p) = (2\pi\hbar)^{n/2}\hat{g}(p)^2$ where $\hat{g}$ is the Fourier transform of $g(z) := e^{-\frac{z^2}{2\hbar}}$ $\hat{g}(p) = e^{-\frac{p^2}{2\hbar}}$. Using again the convolution theorem in (2.1.19), we conclude that

$$(\widehat{Q_\hbar^B(f)\phi})(p) = \frac{(2\pi\hbar)^{n/2}}{2^{n/2}(\pi\hbar)^n} (f * \hat{g}^2)(p)\hat{\phi}(p) = \left(\frac{1}{(\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{-(p-b)^2/\hbar} f(b) db \right) \hat{\phi}(p) .$$

As $\hat{\phi} \in \mathcal{S}(\mathbb{R}^n)$ (since that space is invariant under Fourier transform), this identity can be extended to every $\hat{\phi} \in L^2(\mathbb{R}^n, dp)$ by means of an elementary density argument. Hence $Q_\hbar^B(f)$ acts as a multiplicative operator on the Fourier-Plancherel transforms $\hat{\phi}$ of the wave functions $\phi \in L^2(\mathbb{R}^n, dx)$. In other words, if $F_\hbar$ denotes the Fourier-Plancherel unitary operator,

$$\left((F_\hbar Q_\hbar^B(f) F_\hbar^{-1}) F_\hbar \phi \right)(p) = \left(\frac{1}{(\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{-(p-b)^2/\hbar} f(b) db \right) (F_\hbar \phi)(p) .$$

Looking at the right-hand side, since $f \in \mathcal{S}(\mathbb{R}^n) \subset C_0(\mathbb{R}^n)$, exploiting the same argument as in (a) $||F_\hbar Q_\hbar^B(f) F_\hbar^{-1} - f|| \to 0$ for $\hbar \to 0$. That is equivalent to $||Q_\hbar^B(f) - F_\hbar^{-1} f F_\hbar|| \to 0$ for $\hbar \to 0$. This is the thesis into an equivalent form. We finally observe that, if $f(p) = e^{-tp^2}$ for a given $t > 0$, then $F_\hbar^{-1} f F_\hbar = e^{t\hbar^2\Delta}$, as is well known. concluding the proof. □

2.2 Coherent pure state quantization

Having introduced quantization theory from the point of view of observables, we now look at quantization from the dual perspective of (pure) states, particularly focusing on $\mathcal{P}(\mathfrak{B}_\infty(\mathcal{H}))$, i.e. the pure state space of the algebra of compact operators on a Hilbert space. This concept may be studied in his own right, even in the absence of a deformation quantization. We will do so in the special case for a symplectic manifold S of dimension $2n < \infty$. To this end we successively present the concept of a pure state quantization of a symplectic manifold, the notion the Berezin quantization map associated to a pure state quantization, and the concept of coherent states. We start with a definition [55, I. Def. 2.5.1].

Definition 2.2.1 (I. Def. 2.5.1). *The* **projective space** $\mathbb{P}\mathcal{H}$ *of a Hilbert space* $\mathcal{H}$ *is the space of one-dimensional complex linear subspaces of* $\mathcal{H}$*. Equivalently,* $\mathbb{P}\mathcal{H}$ *is the quotient* $\mathbb{S}\mathcal{H}/U(1)$ *of the unit sphere*

$$\mathbb{S}\mathcal{H} := \{\Psi \in \mathcal{H} \mid \langle \Psi, \Psi \rangle = 1\} \tag{2.2.20}$$

by the action of $U(1) \cong \mathbb{T}$*, given* $z : \Psi \mapsto z\Psi$ *where* $|z| = 1$*.* ■

The identification of vector states in $\mathcal{H}$, i.e. one-dimensional projections on $\mathcal{H}$, and points of $\mathbb{PH}$ directly follows from this realization. As a result, $\mathbb{P}\mathcal{H} \cong \mathcal{P}(\mathfrak{B}_\infty(\mathcal{H}))$ (via $\omega(\cdot) = Tr(\rho\cdot)$ with ρ a density matrix[7]), and $\mathbb{P}\mathcal{H} \subset \mathcal{P}(\mathfrak{B}(\mathcal{H}))$ when $\mathcal{H}$ is infinite-dimensional.

The space $\mathbb{P}\mathcal{H}$ can be topologized by restricting the usual (norm) Hilbert space topology on $\mathcal{H}$ to $\mathbb{S}\mathcal{H}$, and taking the quotient to $\mathbb{P}\mathcal{H} \cong \mathbb{S}\mathcal{H}/U(1)$. We will denote the image of $\Psi \in \mathbb{S}\mathcal{H}$ in $\mathbb{P}\mathcal{H}$ under the canonical projection $\tau : \mathbb{S}\mathcal{H} \to \mathbb{P}\mathcal{H}$ by ψ. Conversely, given $\psi \in \mathbb{P}\mathcal{H}$ such a $\Psi \in \mathbb{S}\mathcal{H}$ will stand for an arbitrary preimage of ψ under the map τ.

A Hilbert space $\mathcal{H}$ is in particular a symplectic manifold, with symplectic form $\omega(\Phi, \Omega) = 2Im\langle\Phi, \Omega\rangle$ (here the inner product $\langle\ ,\ \rangle$ on $\mathcal{H}$ is linear in the second entry). This form is invariant under the standard action $\exp(i\alpha) : \Psi \mapsto \exp(i\alpha)\Psi$ of $U(1)$ on $\mathcal{H}$, so that the quotient $\mathcal{H}^*/U(1)$ is a Poisson manifold (here $H^* = H \setminus \{0\}$). The symplectic leaves of $H^*/U(1)$ are the spaces $S_r = \mathcal{H}_r/U(1)$, where $\mathcal{H}_r = \{\Psi \in \mathcal{H} \mid \langle\Psi, \Psi\rangle = r^2\}$. It follows from the above that the projective space $\mathbb{P}\mathcal{H}$ may be identified with S_1, and is therefore a symplectic manifold, with symplectic form $\omega_{\mathbb{P}\mathcal{H}}$.

In addition, $\mathbb{P}\mathcal{H}$ is equipped with a transition probability $p : \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to [0, 1]$, given by $p(\psi, \phi) = |\langle\Psi, \Phi\rangle|^2$; here Ψ and Φ are arbitrary lifts of ψ and ϕ to unit vectors in $\mathcal{H}$. Equipped with these transition probabilities and with the Poisson bracket defined by $\omega_{\mathbb{P}\mathcal{H}}$, the manifold $\mathbb{P}\mathcal{H}$ can be seen as the pure state space of a quantum system. Furthermore, it can be shown that $\mathbb{P}\mathcal{H}$ can be equipped with the structure of a real manifold.

The pure state space of a classical system is a symplectic manifold (S, ω_S), supporting the Liouville measure μ_L on S. This measure μ_L is defined by

$$\mu_L(f) = \frac{1}{n!}\int_S f\omega_S^n, \tag{2.2.21}$$

where $2n$ is the dimension of the symplectic manifold S. Such a classical pure state space may be seen as carrying the "classical" transition probability p_0, defined by $p_0(\rho, \sigma) = \delta_{\rho,\sigma}$. This yields the following definition [55, II. Def. 1.3.3].

Definition 2.2.2 (II. Def. 1.3.3)**.** *Let $I \subset \mathbb{R}$ as in Definition 2.1.3 and put $I_0 = I\backslash\{0\}$. A* **pure state quantization** *of a symplectic manifold (S, ω_S) consists of a collection of Hilbert spaces $\{\mathcal{H}_\hbar\}_{\hbar \in I_0}$ and a collection of smooth injections $\{q_\hbar : S \to \mathbb{P}\mathcal{H}_\hbar\}_{\hbar \in I_0}$ for which the following requirements are satisfied.*

(1) There exists a positive function $c : I_0 \to \mathbb{R} \setminus \{0\}$ such that for all $\hbar \in I_0$ and all $\psi \in \mathbb{P}\mathcal{H}_\hbar$ one has

$$c(\hbar)\int d\mu_L(\sigma)p(q_\hbar(\sigma), \psi) = 1. \tag{2.2.22}$$

(2) For all fixed $f \in C_c(S)$ and $\rho \in S$ the function

$$\hbar \to \int_S d\mu_L(\sigma)p(q_\hbar(\rho), q_\hbar(\sigma))f(\sigma); \tag{2.2.23}$$

is continuous on I_0 and satisfies

$$\lim_{\hbar\to 0} c(\hbar)\int_S d\mu_L(\sigma)p(q_\hbar(\rho), q_\hbar(\sigma))f(\sigma) = f(\rho); \tag{2.2.24}$$

(3) For each $\hbar \in I_0$ the map $q_\hbar$ is a symplectomorphism, that is,

$$q_\hbar^*\omega_{\mathbb{P}\mathcal{H}_\hbar} = \omega_S, \tag{2.2.25}$$

where $\omega_{\mathbb{P}\mathcal{H}_\hbar}$ denotes the canonical symplectic form on $\mathbb{P}\mathcal{H}_\hbar$.

■

[7] In the case of the algebra of compact operators the density matrix associated to a pure state is a one-dimensional projection on $\mathcal{H}$. This relies on the fact that pure states on $\mathfrak{B}_\infty(\mathcal{H})$ coincide with normal pure states on $\mathfrak{B}(\mathcal{H})$ which in turn can be identified with one-dimensional projections on $\mathcal{H}$.

By Urysohn's lemma, it is not difficult to show that

$$\lim_{\hbar\to 0} p(q_\hbar(\rho), q_\hbar(\sigma)) = \delta_{\rho,\sigma} \tag{2.2.26}$$

In quantizing pure states the quantum-mechanical transition probabilities should therefore converge to the classical ones for $\hbar \to 0$.

Furthermore, it can be shown that the volume $\mathrm{vol}_\hbar(S)$ of S with respect to the measure

$$\mu_\hbar := c(\hbar)\mu_L \tag{2.2.27}$$

is found to be $\mathrm{vol}_\hbar(S) = \dim(\mathcal{H}_\hbar)$. It follows that S is compact if and only if $\mathcal{H}_\hbar$ is finite dimensional, and that only certain discrete values of $\hbar$ are allowed in that case. Given a pure state quantization of a symplectic manifold (S, ω_S), depending on the situation, we shall use both the Liouville measure μ_L as well as the family of measures $\mu_\hbar$ defined by (2.2.27).

A pure state quantization naturally leads to the quantization of observables [55, II. Def. 1.3.3].

Definition 2.2.3 (II. Def. 1.3.4)**.** *Let $\{\mathcal{H}_\hbar, q_\hbar\}_{\hbar\in I_0}$ be a pure state quantization of a symplectic manifold S. The* **Berezin quantization** *of a function $f \in L^\infty(S)$ is the family of operators $\{Q_\hbar^B(f)\}_{\hbar\in I_0}$, where $Q_\hbar^B(f) \in \mathfrak{B}(\mathcal{H}_\hbar)$ is defined by polarizing*

$$\psi(Q_\hbar^B(f)) := \int d\mu_\hbar(\sigma) p(q_\hbar(\sigma), \psi) f(\sigma). \tag{2.2.28}$$

Here $\psi \in \mathbb{P}\mathcal{H}$, and the integral converges because of Definition 2.2.2 (1). In case that $f \in L^1(S) \cap L^\infty(S)$, the operator $Q_\hbar(f)$ may be written as:

$$Q_\hbar^B(f) = \int d\mu_\hbar(\sigma) f(\sigma)[q_\hbar(\sigma)], \tag{2.2.29}$$

where $[q_\hbar(\sigma)]$ is the projection onto the one-dimensional subspace in $\mathcal{H}_\hbar$ whose image in $\mathbb{P}\mathcal{H}_\hbar$ is $q_\hbar(\sigma)$. ■

Remark 2.2.4. As a result of [55, II. Thm. 1.3.5] and [55, II. Cor. 1.4.5], Theorem 2.1.10 (except perhaps for (4)) also applies to the Berezin quantization maps $Q_\hbar^B$ associated to a pure state quantization of a general symplectic manifold S. In particular, (always assuming $f \in L^\infty(S)$) the operators $Q_\hbar^B(f)$, $(f \in C_0(S))$ are compact on $\mathcal{H}_\hbar$, and any compact operator on $\mathcal{H}_\hbar$ can be approximated (w.r.t. the operator norm) by a sequence of quantization maps $\{Q_\hbar^B(f_n)\}_n$, where $f_n \in C_0(S)$. ■

To move on our discussion we now make a further assumption on a given pure state quantization. Fortunately, this is satisfied in many cases of physical interests.

Definition 2.2.5. *A pure state quantization $\{\mathcal{H}_\hbar, q_\hbar, \mu_\hbar\}_{\hbar\in I_0}$ of S is said to be* **coherent** *if each $q_\hbar(\sigma) \in \mathbb{P}\mathcal{H}_\hbar$ can be lifted to a unit vector $\Psi_\hbar^\sigma \in \mathcal{H}_\hbar$ and the ensuing map $\sigma \mapsto \Psi_\hbar^\sigma$ from S to $\mathcal{H}_\hbar$ is continuous. The unit vectors $\Psi_\hbar^\sigma$ coming from a coherent pure state quantization are called* **coherent states***.* ■

2.2.1 Coherent pure state quantization of $\mathbb{R}^{2n}$

We return to quantization of $\mathbb{R}^{2n}$. We have seen that $\mathbb{R}^{2n}$ admits a strict deformation quantization. It turns out that this manifold admits a coherent pure state quantization as well. We hereto recal a result in [55, II. Prop 2.3.1].

Theorem 2.2.6 (II. Prop. 2.3.1)**.** *Let $I = [0, \infty)$ and $\mathcal{H}_\hbar = L^2(\mathbb{R}^n, dx)$ for each $\hbar > 0$. Denote by μ_L the Liouville measure on $\mathbb{R}^{2n}$ which coincides with the standard $2n$-dimensional Lebesgue measure d^npd^nq. For any $(p, q) \in \mathbb{R}^{2n}$ define a unit vector $\Psi_\hbar^{(q,p)} \in \mathcal{H}_\hbar$ by (2.1.11). Denote the* *projection of* $\Psi_\hbar^{(q,p)} \in \mathbb{S}\mathcal{H}$ *to* $\mathbb{P}\mathcal{H}$ *by* $\psi_\hbar^{(q,p)}$*. Then the choices,*

$$q_\hbar(p,q) = \psi_\hbar^{(q,p)}; \tag{2.2.30}$$

$$c(\hbar) = \frac{1}{(2\pi\hbar)^n}, \tag{2.2.31}$$

so that $d\mu_\hbar(p,q) = \frac{d^npd^nq}{(2\pi\hbar)^n}$ *yield a coherent pure state quantization of* $\mathbb{R}^{2n}$*. The Berezin quantization map* $Q_\hbar^B$ *is precisely given by* (2.1.13). ■

Remark 2.2.7. The fact that the manifold $\mathbb{R}^{2n}$ admits a strict deformation quantization as well as a coherent pure state quantization in general does not hold: there exists strict deformation quantizations for which the corresponding quantizations maps are not defined in terms of coherent states. We will see an explicit example in Chapter 3. ■

2.2.2 Strict deformation quantization and coherent pure state quantization of S^2

In this section we prove the existence of a strict deformation quantization of the two-sphere $S^2 \subset \mathbb{R}^3$ with ensuing Poisson bracket given by

$$\{f,g\}(\mathbf{x}) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a}\frac{\partial g}{\partial x_b}, \quad \mathbf{x} \in S^2, \tag{2.2.32}$$

where $f,g \in C^\infty(S^2)$. To this end we first consider the symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$ defined by

$$\mathrm{Sym}^N(\mathbb{C}^2) := T_N(\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2), \tag{2.2.33}$$

where $T_N : \bigotimes_{n=1}^N \mathbb{C}^2 \to \bigotimes_{n=1}^N \mathbb{C}^2$ denotes the symmetrizer and is given by unique linear extension of the following map acting on elementary tensors

$$T_N(v_1 \otimes \cdots \otimes v_N) = \frac{1}{N!}\sum_{\sigma \in \mathcal{P}(N)} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(N)}, \tag{2.2.34}$$

with $\mathcal{P}(N)$ the permutation group consisting of $N!$ elements. It is not difficult to see that T_N is a projection and $\dim(\mathrm{Sym}^N(\mathbb{C}^2)) = N+1$. Indicating the algebra of bounded operators on $\mathrm{Sym}^N(\mathbb{C}^2)$ by $\mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$, it is known [57, Theorem 8.1] that

$$\mathfrak{A}'_0 := C(S^2); \tag{2.2.35}$$

$$\mathfrak{A}'_{1/N} := \mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2)) \cong M_{N+1}(\mathbb{C}), \tag{2.2.36}$$

are the fibers of a continuous bundle of C^*-algebras over base space

$$I = \{1/N \mid N \in \mathbb{N}\} \cup \{0\} \equiv (1/\mathbb{N}) \cup \{0\}, \tag{2.2.37}$$

with the topology inherited from $[0,1]$. That is, we put $\hbar = 1/N$, where $N \in \mathbb{N}$ is interpreted a the number of sites of the model; the interest is the limit $N \to \infty$. The continuous cross-sections are given by all sequences $(a_{1/N})_{N\in\mathbb{N}} \in \Pi_{n\in\mathbb{N}}\mathfrak{A}'_{1/N}$ for which $a_0 \in C(S^2)$ and $a_{1/N} \in \mathfrak{A}'_{1/N}$ and such that the sequence $(a_{1/N})_{N\in\mathbb{N}}$ is asymptotically equivalent to $(Q'_{1/N}(a_0))_{N\in\mathbb{N}}$, in the sense that

$$\lim_{N\to\infty} ||a_{1/N} - Q'_{1/N}(a_0)||_N = 0. \tag{2.2.38}$$

Here, the symbol $Q'_{1/N}$ denotes the quantization maps

$$Q'_{1/N} : \tilde{\mathfrak{A}}'_0 \to \mathfrak{A}'_{1/N}, \tag{2.2.39}$$

where $\tilde{\mathfrak{A}}'_0 \subset C^\infty(S^2) \subset \mathfrak{A}'_0$ is the dense Poisson subalgebra made of polynomials in three real variables restricted to the sphere S^2. The maps $Q'_{1/N}$ are defined by[8] the integral computed in weak sense

$$Q'_{1/N}(p) := \frac{N+1}{4\pi} \int_{S^2} p(\Omega) |\Psi_N^\Omega\rangle\langle\Psi_N^\Omega| d\Omega \,, \tag{2.2.40}$$

where p denotes an arbitrary polynomial restricted to S^2, $d\Omega$ indicates the unique $SO(3)$-invariant Haar measure on S^2 with $\int_{S^2} d\Omega = 4\pi$, and $|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega| \in \mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$ are so-called N-coherent spin states defined below. In particular, if 1 is the constant function $1(\Omega) = 1$, $(\Omega \in S^2)$, and 1_N is the identity on $\mathfrak{A}'_{1/N} = \mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$, the previous definition implies

$$Q'_{1/N}(1) = 1_N \,. \tag{2.2.41}$$

Indeed, it can be shown that the quantization maps (2.2.40) - (2.2.41) satisfy the axioms of Definition 2.1.3, which implies the existence of a deformation quantization of S^2.[9] Moreover, as a result of Proposition 4.1.1 they map surjectively onto the algebra $\mathfrak{A}'_{1/N}$ and therefore the deformation quantization is strict as well.

In order to define the N-coherent spin states we shall use the bra-ket notation. Let $|\uparrow\rangle, |\downarrow\rangle$ be the eigenvectors of σ_3 in $\mathbb{C}^2$, so that $\sigma_3|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_3|\downarrow\rangle = -|\downarrow\rangle$, and where $\Omega \in S^2$, with polar angles $\theta_\Omega \in (0,\pi)$, $\phi_\Omega \in (-\pi,\pi)$, we then define the unit vector

$$|\Omega\rangle_1 = \cos\frac{\theta_\Omega}{2}|\uparrow\rangle + e^{i\phi_\Omega}\sin\frac{\theta_\Omega}{2}|\downarrow\rangle. \tag{2.2.42}$$

If $N \in \mathbb{N}$, the associated N**-coherent spin state** $\Psi_N^\Omega \in \mathrm{Sym}^N(\mathbb{C}^2)$, equipped with the usual scalar product $\langle\cdot,\cdot\rangle_N$ inherited from $(\mathbb{C}^2)^N$, is defined as follows [71]:

$$|\Psi_N^\Omega\rangle := \underbrace{|\Omega\rangle_1 \otimes \cdots \otimes |\Omega\rangle_1}_{N\ times} \,. \tag{2.2.43}$$

Depending on the situation this vector is also written in ordinary notation, i.e. Ψ_N^Ω. We also use the notation $|\Omega_{\theta,\phi}\rangle$ to emphasize the dependence on the angles θ,ϕ. We refer to Appendix A.1 for a more detailed description of coherent spin states.

Remark 2.2.8. For any polynomial p in three real variables restricted to S^2 the quantization maps satisfy

$$||Q'_{1/N}(p)|| \leq ||p||_\infty, \ (N \in \mathbb{N}). \tag{2.2.44}$$

As a result of Lemma 2.1.7 the quantization maps can be extended to all of $C(S^2)$ such that (except for the Dirac-Groenewold-Rieffel condition) the von Neumann and Rieffel condition hold. ■

These observations furthermore yield a general result proving the existence of a coherent pure state quantization of the spheres S_j^2, where $j \in \mathbb{N}_0/2$ denotes the radius. To see this we first consider the highest weight representation U_j of $SU(2)$ onto the algebra of bounded operators on the vector space $\mathcal{V}_j$ of dimension $2j+1$,

$$U_j : SU(2) \to \mathfrak{B}(\mathcal{V}_j). \tag{2.2.45}$$

Since jN is still a highest weight for any $N \in \mathbb{N}$, for $\hbar = 1/N$, $(N \in \mathbb{N})$ we can therefore consider

$$U_{j/\hbar} : SU(2) \to \mathfrak{B}(\mathcal{V}_{j/\hbar}), \tag{2.2.46}$$

[8] Equivalent definitions of these quantization maps are used in literature, see e.g. [57, 71].

[9] We remark that S^2 is a special case of a regular integral coadjoint orbit in the dual of the Lie algebra associated to $SU(2)$, which can be identified with $\mathbb{R}^3$. In fact, this theory can be generalized to arbitrary compact connected Lie groups [55].

which is the carrier space of the highest weight representation $U_{j/\hbar}$ of dimension $\dim(\mathcal{V}_{j/\hbar}) = 2Nj+1$. It can be shown that the spheres $S_j^2 \subset \mathbb{R}^3$ (where $j \in \mathbb{N}/2$ denotes the radius) admit a coherent pure state quantization with Hilbert spaces $\{\mathcal{V}_{j/\hbar}\}_{1/\hbar}$ (see [54, Thm. 1] for a general discussion in the context of integral coadjoint orbits). For our purpose we fix $j = 1/2$ yielding the carrier space $\mathcal{V}_{N/2}$ of dimension $N+1$.[10] The precise result is stated in the following proposition providing a coherent pure state quantization of $S^2_{1/2} \subset \mathbb{R}^3$, the two-sphere of radius $1/2$ in $\mathbb{R}^3$.

Proposition 2.2.9. *Let $I = (1/\mathbb{N})\cup\{0\}$ (with topology inherited from $[0,1]$), $\hbar = 1/N$, ($N \in I$) and fix $j = 1/2$. Define $\mathcal{H}_\hbar = \mathcal{V}_{1/2\hbar}$, i.e. the carrier space of the irreducible representation $U_{1/2\hbar}$ with highest weight $1/2\hbar = N/2$. Denote by μ_L the Liouville measure on $S^2_{1/2}$ which coincides with the spherical measure $\sin\theta d\theta d\phi$ ($\theta \in (0,\pi)$ $\phi \in (0,2\pi)$). For any $\Omega := (\theta,\phi) \in S^2_{1/2}$ define a unit vector $\Psi_N^\Omega \in \mathcal{H}_\hbar$ by (2.2.43). Denote the projection of $\Psi_N^\Omega \in \mathbb{S}\mathcal{H}_\hbar$ to $\mathbb{P}\mathcal{H}_\hbar$ by ψ_N^Ω. Then the choices,*

$$q_\hbar(\Omega) = \psi_N^\Omega; \tag{2.2.47}$$
$$c(\hbar) = (N+1)/4\pi; \tag{2.2.48}$$

so that $\mu_{1/N}(\theta,\phi) = \frac{N+1}{4\pi} sin\theta d\theta d\phi$ yield a coherent pure state quantization of $S^2_{1/2}$ on I. ■

Remark 2.2.10. As spheres of arbitrary radius are diffeomorphic it follows that the specific choice of the radius of the sphere is irrelevant for the ensuing Berezin quantization maps: any sphere of fixed radius $j \in \mathbb{N}/2$ provides the same image under Berezin quantization $Q^B_{1/N} : C(S_j^2) \to \mathfrak{B}(\mathcal{V}_{N/2})$ defined by (2.2.40). ■

As a result of Remark 2.2.10, Remark 2.2.8 and [54] the Berezin quantization map $Q^B_{1/N}$ associated to this coherent pure state quantization is precisely given by extension of the map (2.2.40) to all of $C(S_1^2)$, where we have identified $\mathcal{V}_{N/2} \cong \mathrm{Sym}^N(\mathbb{C}^2)$, i.e. the vector space corresponding to the fibers (2.2.35)–(2.2.36). In the context of the symplectic manifold $S^2 \equiv S^2_1$ we use both notations $Q'_{1/N}$ and $Q^B_{1/N}$ to indicate the Berezin quantization map defined by (2.2.40), extended to all of $C(S^2)$.

[10] We stress that, as opposed to the pure state quantization of $\mathbb{R}^{2n}$ (cf. Subsection 2.2.1) the dimension of the Hilbert space now *does* depend on the semi-classical parameter.

3. Deformation quantization of the algebraic state space of $M_k(\mathbb{C})$

In this chapter we prove the existence of a deformation quantization of the algebraic state space $S(M_k(\mathbb{C}))$ of the matrix algebra $M_k(\mathbb{C})$ according to Definition 2.1.3. To this end, in Section 3.1 we investigate the structure and algebraic properties of $S(M_k(\mathbb{C}))$ and see how Poisson geometry comes into play. We particularly show that $S(M_k(\mathbb{C}))$ can be equipped with a suitable Poisson bracket, transforming it into a Poisson manifold (with stratified boundary). Consequently, in Section 3.2 we introduce the notion of quasi-symmetric sequences. These sequences play a key role in Section 3.3 where we finally prove the existence of a deformation quantization of $S(M_k(\mathbb{C}))$.

3.1 Structures on $\mathfrak{A} = M_k(\mathbb{C})$

Unless stated otherwise, $\mathfrak{A} = M_k(\mathbb{C})$ is the unital C^*-algebra of $k \times k$ complex matrices equipped with the natural C^*-norm, whose unit element is denoted by I_k and whose $*$-operation is the standard hermitian conjugation. Furthermore, $M_k^h(\mathbb{C})$ is the real linear subspace of $M_k(\mathbb{C})$ containing all hermitian $k \times k$ matrices.

3.1.1 The state space of $M_k(\mathbb{C})$ as a set

The state space $S(\mathfrak{A})$ of a general unital C^*-algebra $\mathfrak{A}$ with unit $I_{\mathfrak{A}}$ is defined as the set of linear functionals $\omega : \mathfrak{A} \to \mathbb{C}$ that satisfy $\omega(I_{\mathfrak{A}}) = 1$ and $\omega(a^*a) \geq 0$ for any $a \in \mathfrak{A}$. It follows that $S(\mathfrak{A}) \subset \mathfrak{A}^*$ (the Banach dual of $\mathfrak{A}$), but $S(\mathfrak{A})$ will always be equipped with the topology of pointwise convergence, i.e., the weak*-topology (rather than the norm-topology inherited from $\mathfrak{A}^*$; for finite-dimensional B this difference does not matter, though). In this topology $S(\mathfrak{A})$ is a compact convex set. For $\mathfrak{A} = M_k(\mathbb{C})$, regarded as $\mathfrak{A} = \mathfrak{B}(\mathbb{C}^k)$, the algebra of (automatically) bounded linear operators on the Hilbert space $\mathbb{C}^k$, all states are normal and hence bijectively correspond with *density matrices* (i.e. positive matrices ρ with unit trace) via

$$\omega_\rho(a) = \mathrm{tr}(\rho a) \quad \text{for every } a \in \mathfrak{A}. \tag{3.1.1}$$

The set of density matrices on $\mathbb{C}^k$ is denoted by $\mathcal{D}_k$. These form a convex set in their own right, and hence $\mathcal{D}_k \cong S(M_k(\mathbb{C}))$ via (3.1.1) as an affine bijection (i.e. isomorphism) of convex sets. We also give $\mathcal{D}_k$ the unique topology making this bijection a homeomorphism and in practice we often identify $\mathcal{D}_k$ and $S(M_k(\mathbb{C}))$.

We proceed by introducing some useful coordinate systems on $\mathcal{D}_k$ [11, 21].

Definition 3.1.1. A **parametrization** $(\mathcal{Q}_k, F_k)$ of $\mathcal{D}_k$ consists of:

(a) a parameter set $\mathcal{Q}_k \subset \mathbb{R}^m$, where m depends on k, i.e., $m = m(k)$;

(b) a bijective map $F_k : \mathcal{Q}_k \to \mathcal{D}_k$.

The parametrization is said to be **affine** if it is (the restriction to $\mathcal{Q}_k$ of) an affine map with respect to the natural real linear space structures of $\mathbb{R}^m$ and $M_k^h(\mathbb{C})$. ∎

Remark 3.1.2. The inverse map $F_k^{-1} : \mathcal{D}_k \to \mathcal{Q}_k \subset \mathbb{R}^{m(k)}$ defines a (global) coordinate system on $\mathcal{D}_k$, though in a somewhat extended sense compared to the standard definition for smooth manifolds (with boundary) when $k > 2$. This is because, as we shall see shortly, $\mathcal{Q}_k$ has a more complicated structure than an open set possibly bounded by an embedded submanifold of $\mathbb{R}^{m(k)}$. ∎

Clearly, the case $k = 1$ is trivial, because $D_1 = \{1\}$. Therefore, in what follows we assume $k \geq 2$. We start with the simplest and simultaneously physically most relevant case $k = 2$.

3.1.2 Smooth structure of the state space of $M_k(\mathbb{C})$

We investigate the properties of the state space of $M_k(\mathbb{C})$, starting with $k = 2$. The *Pauli matrices* $\sigma_1, \sigma_2, \sigma_3$ together with the identity I_2 form a complex basis of the complex vector space $M_2(\mathbb{C})$, and a real basis of $M_2^h(\mathbb{C})$, i.e.

$$a = \tfrac{1}{2}(x_0 I + x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3), \tag{3.1.2}$$

for any $a \in M_2^h(\mathbb{C})$, where $x_j \in \mathbb{R}$ $(j = 0, 1, 2, 3)$. Then a is a density matrix, i.e. $a \in \mathcal{D}_2 \subset M_2^h(\mathbb{C})$, iff $x_0 = 1$ and $\mathbf{x} = (x_1, x_2, x_3)$ lies in the parameter set

$$\mathcal{Q}_2 = \{\mathbf{x} \in \mathbb{R}^3 \mid |\mathbf{x}| \le 1\} = B^3, \tag{3.1.3}$$

the closed unit ball in $\mathbb{R}^3$, ensuring that a is indeed positive as follows by a simple computation. The corresponding map $F_2 : \mathcal{Q}_2 \to \mathcal{D}_2$ is given by

$$F_2(\mathbf{x}) = \tfrac{1}{2}I_2 + \tfrac{1}{2}\sum_{j=1}^{3} x_j\sigma_j. \tag{3.1.4}$$

By construction, this map is onto $\mathcal{D}_2$, and it is affine. An elementary argument based on the identity $\mathrm{tr}(\sigma_k\sigma_l) = 2\delta_{kl}$ shows that F_2 is also injective. Hence $(\mathcal{Q}_2, F_2)$ is an affine parametrization of $\mathcal{D}_2$ with $m = 3 = k^2 - 1$, for $k = 2$.

The key to generalize this construction to $k > 2$ lies in the fact that the anti-hermitian traceless matrices $(i\sigma_1, i\sigma_2, i\sigma_3)$ form a basis of the Lie algebra $\mathfrak{su}(2)$ of the Lie group $SU(2)$; adding iI_2 gives a basis $(iI_2, i\sigma_1, i\sigma_2, i\sigma_3)$ of the Lie algebra $\mathfrak{u}(2)$ of the Lie group $U(2)$.

Similarly, for $k \ge 2$ every $\rho \in \mathcal{D}_k$ is hermitian and hence it can be written as

$$\rho = \frac{1}{k}I_k + \sum_{j=1}^{k^2-1} x_j b_j, \tag{3.1.5}$$

where $x_j \in \mathbb{R}$ and $b_j = iT_j$, for some basis $(T_j)_{j=1,\dots,k^2-1}$ of the Lie algebra $\mathfrak{su}(k)$ of $SU(k)$, consisting of all traceless anti-hermitian $k \times k$ complex matrices, so that

$$b_j^* = b_j, \quad \mathrm{tr}(b_j) = 0, \quad (j = 1, \dots, k^2 - 1). \tag{3.1.6}$$

Since the T_j are a basis of $\mathfrak{su}(k)$ as a vector space, as usual we also have

$$[T_r, T_s] = \sum_{l=1}^{k^2-1} C_{rs}^l T_l; \qquad [b_r, b_s] = i\sum_{l=1}^{k^2-1} C_{rs}^l b_l, \tag{3.1.7}$$

for some real constants C_{rs}^l antisymmetric in the lower indices and satisfying the Jacobi identity. The second part of (3.1.6) guarantees $\mathrm{tr}(\rho) = 1$ in (3.1.5), but to turn ρ into a density matrix the real numbers $x_1, \dots, x_{k^2-1}$ must also be constrained in order that $\rho \ge 0$. As for $k = 2$, this defines a set $\mathcal{Q}_k \subset \mathbb{R}^{k^2-1}$ which we use to construct an affine parametrization of $\mathcal{D}_k$ based on (3.1.5). For the moment we assume that $\mathcal{Q}_k$ has been defined that way, so that the map F_k (3.1.9) below is surjective. Compactness of the Lie group $SU(k)$ implies that the matrices (b_j), which so far merely satisfy (3.1.6), can be chosen so as to also satisfy[1]

$$\mathrm{tr}(b_i b_j) = \delta_{ij}. \tag{3.1.8}$$

From (3.1.8) and the same argument as for $k = 2$, it follows that the surjective map

$$F_k : \mathcal{Q}_k \ni (x_1, \dots x_{k^2-1}) \mapsto \frac{1}{k}I_k + \sum_{j=1}^{k^2-1} x_j b_j \in \mathcal{D}_k, \tag{3.1.9}$$

[1]With this choice of the normalization, for $k = 2$, we find $b_j = 2^{-1/2}\sigma_j$ and also the coordinates x_j in (3.1.9) below correspond to $2^{-1/2}x_j$ in (3.1.4).

is also injective. Indeed, multiplying both sides of (3.1.5) with b_i, taking the trace, and using (3.1.8) and the second identity in (3.1.6), the inverse of F_k reads

$$F_k^{-1}(\rho) = (\mathrm{tr}(\rho b_1), \dots, \mathrm{tr}(\rho b_{k^2-1})), \quad \rho \in \mathcal{D}_k. \tag{3.1.10}$$

In terms of the state $\omega \in S(M_k(\mathbb{C}))$ related to the density matrix ρ, this gives an explicit coordinate system $\omega \mapsto (x_1(\omega), \dots, x_{k^2-1}(\omega))$ of the former, given by

$$x_j(\omega) = \omega(b_j) = \mathrm{tr}(\rho b_j) \ \ (j = 1, \dots, k^2 - 1). \tag{3.1.11}$$

To find $\mathcal{Q}_k$ more explicitly, we note that the eigenvalues of $\rho \in \mathcal{D}_k$ are the roots $\lambda \in \mathbb{R}$ of the characteristic polynomial $\det(\lambda I_k - \rho)$, which has a unique representation

$$\det(\lambda I_k - \rho) = \sum_{j=1}^{k} (-1)^j a_j \lambda^{k-j}, \quad a_0 = 1. \tag{3.1.12}$$

Here the coefficients a_j are uniquely determined by the choice of the generators b_j and are polynomials in the parameters $\mathbf{x} = (x_1, \dots, x_{k^2-1})$, and hence they define continuous functions $a_j = a_j(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^{k^2-1}$. If $\lambda_1, \dots, \lambda_k$ denote the roots of $\det(\lambda I_k - \rho)$, we obviously have

$$\sum_{j=1}^{k} (-1)^j a_j \lambda^{k-j} = \prod_{j=1}^{k} (\lambda - \lambda_j). \tag{3.1.13}$$

From this, the characterization of the non-negativity of the eigenvalues follows:

$$\lambda_j \geq 0 \ \ (j = 1, ..., k) \ \text{ if and only if } \ a_j \geq 0 \ \ (j = 1, ..., k). \tag{3.1.14}$$

By definition, $\mathcal{Q}_k$ is then the following subset in $\mathbb{R}^{k^2-1}$:

$$\mathcal{Q}_k = \{\mathbf{x} \in \mathbb{R}^{k^2-1} \mid a_j(\mathbf{x}) \geq 0, \ j = 1, ..., k\}. \tag{3.1.15}$$

Being the intersection of closed sets (note that the maps a_j are continuous), $\mathcal{Q}_k$ is closed. Also note that $\mathcal{Q}_k$ has non-empty interior, because the set

$$\{\mathbf{x} \in \mathbb{R}^{k^2-1} \mid a_j(\mathbf{x}) > 0, \ j = 1, ..., k\} \subset \mathcal{Q}_k \tag{3.1.16}$$

is open as a finite intersection of open sets, and is not empty since it contains the density matrix $\rho = I_k$, whose coordinates are $\mathbf{x}(I) = (0, 0, \dots, 0)$, so that $a_j(0, 0, \dots, 0) = \frac{1}{k^j}\binom{k}{j} > 0$ for all j. We now also show that $\mathcal{Q}_k$ is bounded in $\mathbb{R}^{k^2-1}$. Since $\rho \in \mathcal{D}_k$ is hermitian, $\rho \geq 0$, and $\mathrm{tr}(\rho) = 1$, we have

$$\mathrm{tr}(\rho^2) \leq \mathrm{tr}(\rho) = 1, \tag{3.1.17}$$

as can be seen e.g. by diagonalizing ρ. Representing ρ as in (3.1.5) and taking advantage of (3.1.8) and the second identity in (3.1.6), the condition $\mathrm{tr}(\rho^2) \leq 1$ can be rephrased in a way that makes boundedness of $\mathcal{Q}_k$ obvious, viz.[2]

$$\frac{1}{k} + \sum_{j=1}^{k^2-1} |x_j|^2 \leq 1 \quad \text{if } \mathbf{x} \in \mathcal{Q}_k. \tag{3.1.18}$$

Therefore, with $\mathcal{Q}_k$ defined as in (3.1.15) and F_k defined in (3.1.9), the pair $(\mathcal{Q}_k, F_k)$ is an affine parametrization for density matrices $\rho \in \mathcal{D}_k$ with $m = k^2 - 1$, and (3.1.11) defines a global coordinate system over $\mathcal{D}_k \equiv S(M_k(\mathbb{C}))$ in the sense of Remark 3.1.2. Coming from an affine map, this coordinate system preserves the convex structure of $S(\mathfrak{A})$, so that $\mathcal{Q}_k$ is a compact

[2]The pure states are exactly those points in $\mathcal{Q}_k$ that saturate this inequality, since their density matrices satisfy $tr\rho^2 = tr\rho = 1$. The pure states form $\partial_e \mathcal{Q}_k \cong \mathbb{CP}^{k-1}$ with canonical (Fubini–Study) Poisson structure, cf. §3.1.3 below. Strict deformation quantization of complex projective spaces is well known, for example as a special case of the constructions in [17, 99] or [57, §8.1].

convex subset of $\mathbb{R}^{k^2-1}$ with non-empty interior. To conclude this section, few remarks about the differentiable structure of $\mathcal{Q}_k$ are in order. We have seen that $\mathcal{Q}_2 \cong B^3$ is a 3-dimensional manifold with boundary $\partial B^3 \cong S^2$ (the two-sphere), where the topological boundary also coincides with the extreme boundary $\partial_e \mathcal{Q}_2$ as defined in convexity theory (which defines the pure states). However, this simple picture is misleading, since for $k > 2$ the set $\mathcal{Q}_k$ is no longer a (smooth) manifold with boundary [42], as the boundary is not a manifold but a stratified space [72]. Indeed, for $k > 2$, we have the following situation:

(1) Under the isomorphism $\mathcal{Q}_k \cong \mathcal{D}_k$ the interior $\text{int}(\mathcal{Q}_k)$ of $\mathcal{Q}_k$ corresponds to the rank-k density matrices and is a connected k^2-1 dimensional smooth manifold. Points in the interior precisely correspond to *faithful states* on $M_k(\mathbb{C})$.[3]

(2) The topological boundary $\partial\mathcal{Q}_k$ now differs from the extreme boundary $\partial_e\mathcal{Q}_k$:

- $\partial\mathcal{Q}_k$ is the disjoint union of $k-1$ smooth embedded submanifolds $\mathcal{Q}_k^{(l)}$ of $\mathbb{R}^{k^2-1}$, where $l = 1, \dots, k-1$, and $\mathcal{Q}_k^{(l)}$ contains all points corresponding to density matrices with rank l (rank $l = k$ corresponding to the interior).
- $\partial_e\mathcal{Q}_k = \mathcal{Q}_k^{(1)} \subset \partial\mathcal{Q}_k \subset \mathcal{Q}_k$ corresponds to the pure state space on $M_k(\mathbb{C})$.

(3) Every point of $\partial\mathcal{Q}_k$ is a limit point of $\text{int}(\mathcal{Q}_k)$ and clearly $\mathcal{Q}_k = \text{int}(\mathcal{Q}_k) \cup \partial\mathcal{Q}_k$.

Finally, all properties of $\mathcal{Q}_k$ we established are independent of the choice of the basis $\{ib_j\}_{j=1,\dots,k^2-1}$ used to define $(\mathcal{Q}_k, F_k)$, as one easily proves: each different choice of basis just defines a different global coordinate system compatible with the linear structure, the topology, and the differentiable structures involved. In that sense, these properties are *intrinsic*, and eventually come from $\mathcal{D}_k \cong S(M_k(\mathbb{C}))$.

3.1.3 Poisson structure of state space of $M_k(\mathbb{C})$

We now show that the state space X_k, so far realized in three different ways as

$$X_k = S(M_k(\mathbb{C})) \cong \mathcal{D}_k \cong \mathcal{Q}_k, \tag{3.1.19}$$

carries a canonical Poisson structure [16, 37, 57].[4] If the X_k were a manifold, this structure would be defined as a Poisson bracket on $C^\infty(X_k)$, but we have just seen that X_k is not even a manifold with boundary. We circumvent this problem by recalling

$$\mathcal{Q}_k \subset \mathbb{R}^{k^2-1}, \tag{3.1.20}$$

with $\dim(\text{int}(\mathcal{Q}_k)) = k^2-1$, as shown in the previous section, and hence we simply define $f \in C^\infty(\mathcal{Q}_k)$ iff f is the restriction of some $\tilde{f} \in C^\infty(\mathbb{R}^{k^2-1})$.

We also recall that if $\mathfrak{g}$ is any (finite-dimensional) Lie algebra, then the dual space $\mathfrak{g}^*$ has a canonical Poisson structure coming from the Lie bracket on $\mathfrak{g}$ [63]. The Poisson bracket is completely defined by its value on linear functions on $\mathfrak{g}^*$; each $X \in \mathfrak{g}$ defines such a function $\hat{X}$ through $\hat{X}(\theta) = \theta(X)$, where $\theta \in \mathfrak{g}^*$, and

$$\{\hat{X}, \hat{Y}\} = \widehat{[X,Y]}. \tag{3.1.21}$$

If $(T_1, \dots, T_n)$ is a basis of $\mathfrak{g}$ ($n = \dim(\mathfrak{g})$) with structure constants C_{ab}^c given by

$$[T_a, T_b] = \sum_c C_{ab}^c T_c, \tag{3.1.22}$$

[3] A state ω on a C^*algebra B is called *faithful* if $\omega(a^*a) = 0$ implies $a = 0$, for $a \in B$.

[4] A Poisson bracket $\{\cdot,\cdot\}$ on a commutative algebra A is a Lie bracket satisfying the Leibniz rule $\{a, bc\} = \{a,b\}c + \{a,c\}b$, or: for each $a \in A$ the (linear) map $\delta_a : A \to A$ defined by $\delta_a(b) = \{a,b\}$ is a derivation, i.e. $\delta_a(bc) = \delta_a(b)c + \delta_a(c)b$. We take $A = C^\infty(X_k)$ with pointwise mutliplication.

then one has an identification $\mathfrak{g}^* \cong \mathbb{R}^n$ in that $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ corresponds to $\theta = \sum_a x_a \omega^a$, where (ω^a) is the dual basis to (T_a) (i.e., $\omega^a(T_b) = \delta^a_b$), so that

$$\{f, g\}(\mathbf{x}) = \sum_{a,b,c=1}^{n} C^c_{ab} x_c \frac{\partial f(\mathbf{x})}{\partial x_a} \frac{\partial g(\mathbf{x})}{\partial x_b}. \tag{3.1.23}$$

In particular, the coordinate functions $f(\mathbf{x}) = x_a$ reproduce the Lie bracket, i.e.,

$$\{x_a, x_b\} = \sum_{c=1}^{k^2-1} C^c_{ab} x_c. \tag{3.1.24}$$

Applying this to $\mathfrak{g} = \mathfrak{su}(k)$, so $n = k^2 - 1$, see (3.1.7), then gives a Poisson structure on $\mathbb{R}^{k^2-1}$ and hence, by restriction, on $\mathcal{Q}_k$:

Definition 3.1.3. The **Poisson bracket** of $f, g \in C^\infty(\mathcal{Q}_k)$ is given by

$$\{f, g\} = \{\tilde{f}, \tilde{g}\}_{|\mathcal{Q}_k}, \tag{3.1.25}$$

where $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{R}^{k^2-1})$ are arbitrary extensions of f and g respectively, cf. (3.1.20), and the Poisson bracket $\{\tilde{f}, \tilde{g}\}$ on $C^\infty(\mathbb{R}^{k^2-1})$ is defined by (3.1.23) for $\mathfrak{g} = \mathfrak{su}(k)$. ■

This definition is meaningful because of the following facts:

1. The bracket $\{f, g\}$ does not depend on the choice of the extensions $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{R}^{k^2-1})$, because every point of $\partial\mathcal{Q}_k$ is a limit point of the interior of $\mathcal{Q}_k$.
2. The function $\{f, g\}$ trivially lies in $C^\infty(\mathcal{Q}_k)$, which by definition means that it has a smooth extension to $\mathbb{R}^{k^2-1}$, since $\{\tilde{f}, \tilde{g}\}$ is such an extension.
3. The bracket does not depend on the choice of the basis $\{T_j\}_{j=1,\ldots,k^2-1}$ of $\mathfrak{su}(k)$ (with $b_j = iT_j$), since a linear change of basis induces a change in the structure constants C^c_{ab} in (3.1.22) *and* a linear change of the coordinates in $\mathbb{R}^{k^2-1}$ coming from identifying $\mathfrak{su}(k) \cong \mathbb{R}^{k^2-1}$, cancel out in (3.1.23) and hence in (3.1.25).

The last point can also be seen from the more intrinsic form the bracket takes in terms of the other two entries in (3.1.19). First, for the density matrices $\mathcal{D}_k$ we have

$$\mathcal{D}_k \subset M^h_k(\mathbb{C})_1 \cong M^h_k(\mathbb{C})_0 = i\mathfrak{su}(k) \cong i\mathfrak{su}(k)^*, \tag{3.1.26}$$

where $M^h_k(\mathbb{C})_t$ is the space of hermitian $k \times k$ matrices ρ with trace t. The first inclusion is given by (3.1.5), the subsequent isomorphism is given by $(I_k/k) + b \mapsto b$, where $b \in M^h_k(\mathbb{C})_0$, and the last isomorphism $\mathfrak{su}(k) \cong \mathfrak{su}(k)^*$ comes from (minus) the (negative definitie) Cartan–Killing inner product on $\mathfrak{su}(k)$, which is given by

$$-B(X, Y) = -2k\,\mathrm{tr}(XY) = 2k\,\mathrm{tr}(X^*Y) = 2k\langle X, Y\rangle_{HS}, \tag{3.1.27}$$

where the right-hand side is the Hilbert–Schmidt inner product on $M_k(\mathbb{C})$. If we now equip $\mathcal{D}_k$ with a differentiable structure through the last isomorphism in (3.1.19), as detailed in the previous section, and define $f \in C^\infty(\mathcal{D}_k)$ iff f is the restriction of some $\tilde{f} \in C^\infty(M^h_k(\mathbb{C})_1)$, or, by (3.1.26), iff it is the restriction of some $\tilde{f} \in C^\infty(i\mathfrak{su}(k))$, and transfer the Poisson structure on $\mathfrak{su}(k)^*$ to $\mathfrak{su}(k)$ through (3.1.26), then we clearly obtain an intrinsic Poisson structure on $\mathcal{D}_k$, essentially also given by (3.1.21).

Finally, perhaps as the mother of all of the above, for any unital finite-dimensional C^*-algebra $\mathfrak{A}$ (and with due modifications, even for infinite-dimensional ones), the state space $S(\mathfrak{A})$ has a natural structure as a Poisson manifold (with stratified boundary, as above). The Poisson bracket is most easily written down through the isomorphism $B^h \cong C_a(S(\mathfrak{A}))$ of real Banach spaces, where $\mathfrak{A}^h$ is the set of hermitian (= self-adjoint) elements of $\mathfrak{A}$ and for any compact convex set K, $C_a(K)$ is the space of continuous real-valued affine functions on K, equipped with the supremum-norm. This isomorphism is given by $b \mapsto \hat{b}$, where $b \in \mathfrak{A}^h$ and

$\hat{b} \in C_a(S(\mathfrak{A}))$ is given by $\hat{b}(\omega) = \omega(b)$, and, as in (3.1.21), the Poisson bracket is fully defined by

$$\{\hat{a}, \hat{b}\} = i\widehat{[a,b]}. \tag{3.1.28}$$

The relationship with the previous constructions may be inferred from the inclusion

$$S(M_k(\mathbb{C})) \subset M_k^h(\mathbb{C})_1^* \cong (M_k^h(\mathbb{C})_0)^* = (i\mathfrak{su}(k))^*, \tag{3.1.29}$$

where $M_k^h(\mathbb{C})_1^*$ is the set of linear functionals $\varphi : M_k(\mathbb{C}) \to \mathbb{C}$ that are hermitian ($\varphi(a^*) = \overline{\varphi(a)}$) and normalized ($\varphi(I_k) = 1$); the remainder is obvious from (3.1.26).

3.2 Intermezzo: quasi-symmetric sequences

In this section we introduce the notion of symmetric and quasi-symmetric sequences. These sequences play a key rol in the definition of the quantization maps, as explained in the next section.

Let us start recalling some facts from [57, Sec. 8.2]. For any unital C^*-algebra $\mathfrak{A}$, the following fibers may be turned into a continuous bundle of C^*-algebras (with the C^*-algebra of the continuous bundle denoted by $\mathfrak{A}^{(c)}$),[5] and base space $I = \{0\} \cup 1/\mathbb{N} \subset [0,1]$ (with relative topology, so that $(1/N) \to 0$ as $N \to \infty$):

$$\mathfrak{A}_0^{(c)} = C(S(\mathfrak{A})); \tag{3.2.30}$$

$$\mathfrak{A}_{1/N}^{(c)} = \mathfrak{A}^{\otimes N}. \tag{3.2.31}$$

As before, $S(\mathfrak{A})$ is the (algebraic) state space of $\mathfrak{A}$ equipped with the weak*-topology (in which it is a compact convex set) and $\mathfrak{A}^{\otimes N}$ is the N^{th} *projective* (also called maximal) tensor power of $\mathfrak{A}$ (often called $\mathfrak{A}^N$ in what follows).[6] As in the case of vector bundles, the continuity structure of a bundle of C^*-algebras may be defined (indirectly) by specifying what the continuous cross-sections are. To do so for the fibers ($\mathfrak{A}_0^{(c)}$, $\mathfrak{A}_{1/N}^{(c)}$), we need the *symmetrization operator* $S_N : \mathfrak{A}^N \to \mathfrak{A}^N$, defined as the unique linear continuous extension of the following map on elementary tensors:

$$S_N(a_1 \otimes \cdots \otimes a_N) = \frac{1}{N!} \sum_{\sigma \in \mathcal{P}(N)} a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(N)}. \tag{3.2.32}$$

Furthermore, for $N \geq M$ we need to generalize the definition of S_N to give a bounded operator $S_{M,N} : \mathfrak{A}^M \to \mathfrak{A}^N$, defined by linear and continuous extension of

$$S_{M,N}(b) = S_N(b \otimes \underbrace{I \otimes \cdots \otimes I}_{N-M \text{times}}), \quad b \in \mathfrak{A}^M. \tag{3.2.33}$$

Clearly, $S_{N,N} = S_N$. We write cross-sections a of ($\mathfrak{A}_0^{(c)}$, $\mathfrak{A}_{1/N}^{(c)}$) as sequences $(a_0, a_{1/N})_{N\in\mathbb{N}}$, where $a(0) = a_0$ etc. Following [74], the part of the cross-section $(a_{1/N})_{N\in\mathbb{N}}$ away from zero (i.e. with a_0 omitted) is called **symmetric** if there exist $M \in \mathbb{N}$ and $a_{1/M} \in \mathfrak{A}^{\otimes M}$ such that

$$a_{1/N} = S_{M,N}(a_{1/M}) \text{ for all } N \geq M, \tag{3.2.34}$$

and **quasi-symmetric** if $a_{1/N} = S_N(a_{1/N})$ if $N \in \mathbb{N}$, and for every $\epsilon > 0$, there is a symmetric sequence $(b_{1/N})_{N\in\mathbb{N}}$ as well as $M \in \mathbb{N}$ (both depending on ϵ) such that

$$\|a_{1/N} - b_{1/N}\|_{1/N} < \epsilon \text{ for all } N > M. \tag{3.2.35}$$

[5]The superscript c occurring in $\mathfrak{A}^{(c)}$ indicates that this algebra corresponds to a commutative C^*-algebra of observables of infinite quantum systems (describing classical thermodynamics as a limit of quantum statistical mechanics). For details we refer to Chapter 8.

[6]Although this choice is irrelevant for our main application $\mathfrak{A} = M_k(\mathbb{C})$, for general C^*-algebras $\mathfrak{A}$ one should equip $\mathfrak{A}^N$ with this maximal C^*-norm $\| \ \|_N$.

The continuous cross-sections of the bundle $(\mathfrak{A}_0^{(c)}, \mathfrak{A}_{1/N}^{(c)})$ then, are the sequences $(a_0, a_{1/N})_{N\in\mathbb{N}}$ for which the part $(a_{1/N})_{N\in\mathbb{N}}$ away from zero is quasi-symmetric and

$$a_0(\omega) = \lim_{N\to\infty} \omega^N(a_{1/N}), \tag{3.2.36}$$

where $\omega \in S(\mathfrak{A})$, and $\omega^N = \underbrace{\omega \otimes \cdots \otimes \omega}_{N \text{ times}} \in S(\mathfrak{A}^{\otimes N})$, is the unique (norm) continuous linear extension of the following map that is defined on elementary tensors:

$$\omega^N(b_1 \otimes \cdots \otimes b_N) = \omega(b_1)\cdots\omega(b_N). \tag{3.2.37}$$

The limit in (3.2.36) exists and defines a function in $C(S(B))$ provided that $(a_{1/N})_{N\in\mathbb{N}}$ is quasi-symmetric, and by [57, Theorem 8.4], this choice of continuous cross-sections uniquely defines (or identifies) a continuous bundle of C^*-algebras over I with fibers $(\mathfrak{A}_0, \mathfrak{A}_{1/N})$.

The set of (quasi-) symmetric sequences admits an additional structure. Let us recall some of its properties. For a more detailed discussion we refer to [74].

We indicate by $\mathfrak{Y}$ the set of symmetric sequences and by $\tilde{\mathfrak{Y}}$ the set of quasi-symmetric sequences. It turns out that $\tilde{\mathfrak{Y}}$ forms a subalgebra of $\mathfrak{A}^\infty \equiv \bigotimes_{N\geq 1}\mathfrak{A}$, where $\mathfrak{A}^\infty$ is the C^* inductive limit of the sequences $\mathfrak{A}^{\otimes N}$ with natural injections, N-wise addition and N-wise multiplication. In addition, $\tilde{\mathfrak{Y}}$ carries a seminorm, defined by

$$||x|| = \limsup_{N\to\infty} ||x_{1/N}||_N, \tag{3.2.38}$$

where $x := (x_{1/N})_{N\in\mathbb{N}} \in \tilde{\mathfrak{Y}}$. Using the fact that the sequence $(x_{1/N})_{N\in\mathbb{N}}$ is norm-decreasing, and bounded, it follows that the the limit of the sequence $||x_{1/N}||_N$ exists and therefore, one can write

$$||x|| := \lim_{N\to\infty} ||x_{1/N}||_N. \tag{3.2.39}$$

Since pointwise operations do not preserve symmetric sequences, the set $\mathfrak{Y}$ (equiped with these operations) is not a subalgebra of $\mathfrak{A}^\infty$. There exists however a commutative product $\star$ on $\mathfrak{Y}$ turning $(\mathfrak{Y}, \star)$ into a commutative subalgebra of $\mathfrak{A}^\infty$. This product is defined in terms of the symmetrized tensor product $S_{M,N}$ in the following way. For $x = x_1 \otimes \cdots \otimes x_N \in \mathfrak{A}^N$ and $y = y_1 \otimes \cdots \otimes y_M \in \mathfrak{A}^M$ let

$$x \star_{N,M} y = S_{M+N}(x \otimes y) \tag{3.2.40}$$

Clearly, this extends by linearity and norm continuity to a bilinear map $\star_{N,M} : \mathfrak{A}^N \times \mathfrak{A}^M \to \mathfrak{A}^N \otimes \mathfrak{A}^M$. Moreover, this product is associative and a sequence is symmetric precisely iff for all K,

$$x_{N+K} = x_N \star_{N,K} \underbrace{I \otimes \cdots \otimes I}_{K\text{times}}. \tag{3.2.41}$$

The product $\star : Y \times Y \to Y$ is now defined by

$$(x \star y)_{N+M} = x_N \star_{N,M} y_M \tag{3.2.42}$$

for all N, M such that both x_N and y_M are defined. It can be shown that $\star$ does not depend on the choice of the representation with $r = N + M$ and that indeed $x \star y \in \mathfrak{Y}$. It is easy to verify that with these operations $(\mathfrak{Y}, \star)$ becomes a (semi)-normed unital $*$-algebra with semi-norm given by (3.2.39).

We will now see that the algebras $\mathfrak{Y}$ and $\tilde{\mathfrak{Y}}$ are related to a classical theory, as predicted by (3.2.36). To this end we consider the map

$$j_N : \mathfrak{A}^{\otimes N} \to C(S(\mathfrak{A}));$$
$$j_N(x_{1/N})(\omega) := \omega^N(x_{1/N}), \ (\omega \in S(B)). \tag{3.2.43}$$

It can be shown that for each $x := (x_{1/N})_N \in \tilde{\mathfrak{Y}}$ the norm limit $j(x) := \lim_{N\to\infty} j_N(x_{1/N})$ exists in $C(S(\mathfrak{A}))$.[7] In fact, the map

$$j : \tilde{\mathfrak{Y}} \to C(S(\mathfrak{A}))$$
$$x \mapsto j(x) \tag{3.2.44}$$

is an isometric $*$-homomorphism from $\tilde{\mathfrak{Y}}$ onto $C(S(\mathfrak{A}))$ for the $\star$-product [74].[8] In addition, j restricted to the algebra of symmetric sequences $(\mathfrak{Y}, \star)$ is an isometric $*$ homomorphism of $\mathfrak{Y}$ onto a *dense* subalgebra of $C(S(\mathfrak{A}))$. This particularly shows the compatibility with (3.2.36). Indeed, any quasi-symmetric sequence $(x_{1/N})_N \in \tilde{\mathfrak{Y}}$ with limit $j(x)$ is nothing else than the function $x_0 \in C(S(\mathfrak{A}))$ defined by (3.2.36).

3.3 Deformation quantization of $S(M_k(\mathbb{C}))$

In this section we state and prove the existence of a deformation quantization of the state space $X_k = S(M_k(\mathbb{C}))$.[9] To this end we apply the ideas of Section 3.2. Indeed, as a result of [57, Thm. 8.4] the choice of $\mathfrak{A} = M_k(\mathbb{C})$, i.e.,

$$\mathfrak{A}_0^{(c)} = C(S(M_k(\mathbb{C})) \equiv C(X_k); \tag{3.3.45}$$
$$\mathfrak{A}_{1/N}^{(c)} = M_k(\mathbb{C})^{\otimes N} \cong M_{k^N}(\mathbb{C}) \tag{3.3.46}$$

are the fibers of a continuous bundle of C^*-algebras over the base space $I = \{0\} \cup 1/\mathbb{N}$ whose continuous cross-sections are given by all quasi-symmetric sequences. As before, $\mathfrak{A}^{(c)}$ denotes the C^*-algebra of the continuous bundle. In view of Remark 2.1.2 the maps $\pi_{1/N}$ correspond to the evaluation map, i.e. $\pi_{1/N}(x) = x_{1/N}, \ (x \in \mathfrak{A}^{(c)})$. From now on we omit the superscript c in $\mathfrak{A}_0^{(c)}$ and $\mathfrak{A}_{1/N}^{(c)}$ and $\mathfrak{A}^{(c)}$.

As already mentioned $X_k = S(M_k(\mathbb{C}))$ is canonically a compact Poisson manifold, so that one may start looking for suitable Poisson subalgebras $\tilde{\mathfrak{A}}_0 \subset C^\infty(X_k)$ on which, in accordance with Definition 2.1.3, quantization maps

$$Q_{1/N} : \tilde{\mathfrak{A}}_0 \to M_k(\mathbb{C})^{\otimes N} \tag{3.3.47}$$

may be constructed. This can indeed be done. First, we show that in suitable coordinates $\tilde{\mathfrak{A}}_0$ essentially consists of polynomial functions on X_k (see section 3.3.1). Second, the construction of the maps $Q_{1/N} : \tilde{\mathfrak{A}}_0 \to M_k(\mathbb{C})^N$ is given in section 3.3.2, see especially (3.3.56). Although the space we quantize is X_k, we will (often without comment) use both identifications $X_k \cong \mathcal{D}_k$ and $X_k \cong \mathcal{Q}_k$ explained in the previous chapter, the latter equipped with the Poisson structure of Definition 3.1.3.

3.3.1 Choice of the Poisson subalgebra $\tilde{\mathfrak{A}}_0$

As before, we choose a basis $\{b_1, ...b_{k^2-1}\}$ of $i\mathfrak{su}(k)$ satisfying (3.1.6) and (3.1.7), where $\mathfrak{su}(k)$ is a *real* vector space. Using *complex* coefficients, the hermitian matrices $(I_k, b_1, ..., b_{k^2-1})$ then

[7] Note that $j_N(x_{1/N})(\omega)$ is just the same as equation (3.2.37).

[8] Observe that $||x|| = ||j(x)||_\infty = 0$ implies that $j(x) = 0$, but not necessarily that $x = 0$, as the norm on $\tilde{\mathfrak{Y}}$ is a seminorm.

[9] Some ideas in the proof were inspired by techniques in [16, 37, 74], as rewritten in terms of continuous bundles of C^*-algebras in [57, Ch. 8]. The relationship between the deformation quantization of X_k (constructed below) and of its extreme boundary $\mathbb{CP}^{k-1}$ (cf. footnote 2 and [17, 57]) is in general unclear. Even though the fiber algebras $\mathfrak{A}_{1/N}$ are different, namely $\mathfrak{A}_{1/N} = M_2(\mathbb{C})^N$ for $X_2 \cong B^3$ and $\mathfrak{A}'_{1/N} = M_{N+1}(\mathbb{C})$ for $\mathbb{CP}^1 \cong S^2$, the case $k = 2$ is fully understood and is the topic of Chapter 4.

form a basis of the complex vector space $M_k(\mathbb{C})$. We introduce a subspace of $\bigoplus_{M=0}^{\infty}\mathfrak{A}^M$ making use of the symmetrized tensor product

$$a_1 \otimes_s \cdots \otimes_s a_N = S_N(a_1 \otimes \cdots \otimes a_N), \tag{3.3.48}$$

where S_N is defined in (3.2.32) and we adopt the Einstein summation convention. We define $Z \subset \bigoplus_{M=0}^{\infty}\mathfrak{A}^M$ as the subspace consisting of all elements of the form

$$z = c_0 I_k \oplus c_1^{j_1} b_{j_1} \oplus c_2^{j_1 j_2} b_{j_1} \otimes_s b_{j_2} \oplus ... \oplus c_M^{j_1 \cdots j_M} b_{j_1} \otimes_s \cdots \otimes_s b_{j_M}, \quad (M = 0, 1, \ldots), \tag{3.3.49}$$

where the coefficients $c_L^{j_1 \cdots j_L} \in \mathbb{C}$ are symmetric, $j_i \in \{1, ..., k^2 - 1\}$, and $i = 1, ..., N$.

Remark 3.3.1. (1) The matrices I_k and all of the $b_{j_1} \otimes_s \cdots \otimes_s b_{j_N}$, where $j_i \in \{1, ..., k^2 - 1\}$ and $i = 1, ..., N$, are linearly independent and form a basis of Z.

(2) Z does not depend on the initial choice of the basis $\{b_1, ..., b_{k^2-1}\}$ of $i\mathfrak{su}(k)$. ■

We now introduce an important auxiliary linear map $\chi : Z \to C(S(\mathfrak{A}))$, through which we will construct $\tilde{\mathfrak{A}}_0$. By linearity, χ is completely defined if, for $\omega \in S(B)$,

$$\chi(I_k)(\omega) = 1, \quad \text{i.e.} \quad \chi(I_k) = 1_{S(B)}; \tag{3.3.50}$$
$$\chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_N})(\omega) = \omega^N(b_{j_1} \otimes_s \cdots \otimes_s b_{j_N}) = \omega(b_{j_1}) \cdots \omega(b_{j_N}). \tag{3.3.51}$$

By definition of weak $*$-topology we have $\chi(z) \in C(X_k)$, since $z \in Z$ is a finite sum.

Lemma 3.3.2. *The map $\chi : Z \to C(S(\mathfrak{A}))$, is injective, so that in particular all functionals $1_{S(\mathfrak{A})}$ and $\chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_N})$ are linearly independent.*

Proof. Since χ is linear, the claim is equivalent to the implication $\chi(z) = 0 \implies z = 0$, where $z \in Z$ has the generic form

$$z = c_0 I_k \oplus c_1^{j_1} b_{j_1} \oplus c_2^{j_1 j_2} b_{j_1} \otimes_s b_{j_2} \oplus ... \oplus c_M^{j_1 \cdots j_M} b_{j_1} \otimes_s \cdots \otimes_s b_{j_M}, \tag{3.3.52}$$

The requirement $\chi(z) = 0$ means $\chi(z)(\omega) = 0$, for all $\omega \in C(S(\mathfrak{A}))$. Thinking of the states ω as density matrices of $\mathcal{D}_k$ represented by the affine parametrization $(\mathcal{Q}_k, F_k)$ defined in (3.1.9), the map $S(B) \ni \omega \mapsto \chi(z)(\omega)$ is clearly the restriction of a polynomial in $k^2 - 1$ variables $(x_1, \ldots, x_{k^2-1}) \in \mathbb{R}$, which determine ω through F_k when restricted to $\mathcal{Q}_k$, that is,

$$\chi(z)(\omega) = c_0 + c_1^{j_1} x_{j_1} + c_2^{j_1 j_2} x_{j_1} x_{j_2} + \cdots + c_M^{j_1 \cdots j_M} x_{j_1} \cdots x_{j_M}, \tag{3.3.53}$$

where we have taken (3.1.11) into account. Since the interior of $\mathcal{Q}_k$ is not empty (and open by definition) and the polynomial therefore vanishes on some open nonempty set, it vanishes everywhere, hence all coefficients $c_N^{j_1 \cdots j_N}$ are zero. From (3.3.52), we have proved that, for $z \in Z$ the condition $\chi(z) = 0$ implies that $z = 0$, as wanted. □

We can now define our Poisson subalgebra as

$$\tilde{\mathfrak{A}}_0 = \chi(Z). \tag{3.3.54}$$

Then $\tilde{\mathfrak{A}}_0$ is a $\| \cdot \|_\infty$ dense subspace of $C(S(\mathfrak{A}))$ by injectivity of χ and the Stone–Weierstrass theorem (if necessary using the identification $S(B) \cong \mathcal{Q}_k$, or directly in its C^*-algebraic version). Indeed, it follows from (3.1.11) and (3.3.50) - (3.3.51) that (using the Einstein summation convention) generic elements of $\tilde{\mathfrak{A}}_0$ take the form

$$\begin{aligned}&\chi(c_0 I_k \oplus c_1^{j_1} b_{j_1} \oplus c_2^{j_1 j_2} b_{j_1} \otimes_s b_{j_2} \oplus \cdots \oplus c_M^{j_1 \cdots j_M} b_{j_1} \otimes_s \cdots \otimes_s b_{j_M})(\omega) \\ &= c_0 + c_1^{j_1} x_{j_1} + c_2^{j_1 j_2} x_{j_1} x_{j_2} + \cdots + c_M^{j_1 \cdots j_M} x_{j_1} \cdots x_{j_M}.\end{aligned} \tag{3.3.55}$$

Since (under $X_k \cong \mathcal{Q}_k$) elements of $\tilde{\mathfrak{A}}_0$ are polynomials, we also have $\tilde{\mathfrak{A}}_0 \subset C^\infty(X_k)$, and using Definition 3.1.3, it is also clear that $\tilde{\mathfrak{A}}_0$ is a Poisson subalgebra of $C^\infty(X_k)$.

3.3.2 Quantization maps

Finally, we are in the position to define the desired quantization maps $Q_{1/N} : \tilde{\mathfrak{A}}_0 \to M_k(\mathbb{C})^{\otimes N}$. it suffices to define $Q_{1/N}$ by linear extension of its values on the basis vectors $\chi(I_k)$ and $\chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L})$ of $\tilde{\mathfrak{A}}_0$ $(L \in \mathbb{N})$, (3.3.50) - (3.3.51). On those, we define

$$Q_{1/N}(\chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L})) = \begin{cases} S_{L,N}(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}), & \text{if } N \geq L \\ 0, & \text{if } N < L, \end{cases} \tag{3.3.56}$$

$$Q_{1/N}(\chi(I_k)) = \underbrace{I_k \otimes \cdots \otimes I_k}_{N\ times}. \tag{3.3.57}$$

Remark 3.3.3. Suppose that $z \in Z$ takes the form (3.3.49) with not all coefficients $c_M^{j_1 \cdots j_M}$ vanishing. Then there exists $z_1 \in \mathfrak{A}^M$, such that

$$Q_{1/N}(\chi(z)) = S_{M,N}(z_1) \quad \text{if } N \geq M. \tag{3.3.58}$$

To construct z_1 from z, it is sufficient to replace every summand

$$c_L^{j_1 \cdots j_L} b_{j_1} \otimes_s \cdots \otimes_s b_{j_L} \in \mathfrak{A}^L \tag{3.3.59}$$

in (3.3.49) by a corresponding term

$$c_L^{j_1 \cdots j_L} b_{j_1} \otimes_s \cdots \otimes_s b_{j_L} \otimes_s I_k \otimes_s \cdots \otimes_s I_k \in \mathfrak{A}^M, \tag{3.3.60}$$

where the factor I_k occurs $M - L$ times, so that

$$\begin{aligned} z_1 = (c_0 \underbrace{I_k \otimes_s \cdots \otimes_s I_k}_{M\ times}) &\oplus (c_1^{j_1} b_{j_1} \otimes_s \underbrace{I_k \otimes_s \cdots \otimes_s I_k}_{M-1\ times}) \\ &\oplus \cdots \oplus (c_M^{j_1 \cdots j_M} b_{j_1} \otimes_s \cdots \otimes_s b_{j_M}), \end{aligned} \tag{3.3.61}$$

With z as in (3.3.49), where not all $C_M^{j_1 \cdots j_M}$ vanish, and $z_1 \in \mathfrak{A}^M$ as in (3.3.61), it immediately follows from the definition of $Q_{1/N}$ that (3.3.58) holds. ■

This yields the following overarching theorem.

Theorem 3.3.4. *Let $S(M_k(\mathbb{C}))$ be the state space of $M_k(\mathbb{C})$. The following data give a deformation quantization of $S(M_k(\mathbb{C}))$ in the sense of Definition 2.1.3:*

1. *The continuous bundle of C^*-algebras over the base space $I = \{0\} \cup 1/\mathbb{N}$ with fibers* (3.3.45) - (3.3.46)*, and continuity structure as explained after these equations (see Section 3.2);*
2. *The (canonical) Poisson structure on $S(M_k(\mathbb{C}))$ defined in section 3.1.3:*
3. *The dense Poisson subalgebra $\tilde{\mathfrak{A}}_0 \subset C^\infty(S(M_k(\mathbb{C}))) \subset \mathfrak{A}_0$ defined by (3.3.54);*
4. *The maps $Q_{1/N} : \tilde{\mathfrak{A}}_0 \to M_k(\mathbb{C})^{\otimes N}$ defined by linear extension of* (3.3.56) - (3.3.57)*.*

Before proving the theorem we stress that the above constructed quantization maps neither define a Berezin quantization (cf. Definition 2.2.3), nor coherent states are involved in their definition.

Proof. For each $a_0 \in \tilde{\mathfrak{A}}_0$, the following map is a continuous section of the bundle:

$$0 \to a_0 \tag{3.3.62}$$

$$1/N \to Q_{1/N}(a_0) \quad (N > 0). \tag{3.3.63}$$

This is true because continuous sections are given by (quasi) symmetric sequences and the sequence of $Q_{1/N}(a_0)$ defined in (3.3.56) - (3.3.57) is even symmetric due to (3.3.58). The only nontrivial part of the proof is the Dirac-Groenewold-Rieffel condition

$$\lim_{N\to\infty} \big|\big| iN[Q_{1/N}(f), Q_{1/N}(g)] - Q_{1/N}(\{f, g\}) \big|\big|_N = 0, \tag{3.3.64}$$

where $h, g \in \tilde{\mathfrak{A}}_0$. Since both terms in the norm in (3.3.64) are bilinear in f and g, and the case where f or g equals $1_{S(M_k(\mathbb{C}))}$ is trivially satisfied (since $Q_{1/N}(1_{S(M_k(\mathbb{C}))})$ is the unit operator in $\mathfrak{A}^N$), it is is sufficient to prove this for basis elements of $\tilde{\mathfrak{A}}_0$:

$$f = \chi(b_{i_1} \otimes_s \cdots \otimes_s b_{i_M}), \quad g = \chi(b_{j_1} \otimes_s \cdots \otimes_s b_{j_L}). \tag{3.3.65}$$

For these functions, we have by definition

$$f(x_1, \ldots, x_{k^2-1}) = x_{i_1} \cdots x_{i_M}; \qquad g(x_1, \ldots, x_{k^2-1}) = x_{j_1} \cdots x_{j_L}. \tag{3.3.66}$$

As a consequence of (3.1.23) and (3.1.24), we obtain

$$\begin{aligned}\{f, g\} = \Bigg(& \sum_l C^l_{i_1 j_1} x_l x_{i_2} \cdots x_{i_M} x_{j_2} \cdots x_{j_L} \\ & + \sum_l C^l_{i_1 j_2} x_l x_{i_2} \cdots x_{i_M} x_{j_1} x_{j_3} \cdots x_{j_L} \\ & + \cdots + \sum_l C^l_{i_M j_L} x_l x_{i_1} x_{j_1} \cdots x_{i_{M-1}} x_{j_1} x_{j_3} \cdots x_{j_{L-1}} \Bigg),\end{aligned}$$

where all possible Poisson brackets $\{x_{i_l}, x_{j_m}\} = \sum_l C^l_{i_l j_m} x_l$ are considered for $l = 1, \ldots, M$, $m = 1, \ldots, L$. From this expression we compute $Q_{1/N}(\{f, g\})$ in (3.3.64):

$$\begin{aligned}Q_{1/N}(\{f, g\}) = S_{M+L-1,N} \Bigg(& \sum_l C^l_{i_1 j_1} b_l \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_2} \otimes \cdots \otimes b_{j_L} \\ & + \sum_l C^l_{i_1 j_2} b_l \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_L} \\ & + \cdots + \\ & + \sum_l C^l_{i_M j_L} b_l \otimes b_{i_1} \otimes \cdots \otimes b_{i_{M-1}} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_{L-1}} \Bigg).\end{aligned} \tag{3.3.67}$$

Let us pause to analyze the remaining term in the norm in (3.3.64), more precisely,

$$[Q_{1/N}(f), Q_{1/N}(g)] = [S_{M,N}(f^{-1}(f)), S_{L,N}(f^{-1}(g))]. \tag{3.3.68}$$

Lemma 3.3.5. *Consider elements $a_1 \otimes \cdots \otimes a_N$ and $b_1 \otimes \cdots \otimes b_N$ of $\mathfrak{A}^N$. Then*

$$\begin{aligned}& [S_N(a_1 \otimes \cdots \otimes a_N), S_N(a'_1 \otimes \cdots \otimes a'_N)] \\ & = \frac{1}{N!} \sum_{\pi \in \mathcal{P}(N)} \left(S_N \left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)} \right) - S_N \left(a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N \right) \right).\end{aligned} \tag{3.3.69}$$

Proof. See Appendix B.1. ☐

Let us to evaluate the commutator

$$[Q_{1/N}(f), Q_{1/N}(g)] = [S_{M,N}(f^{-1}(f)), S_{L,N}(f^{-1}(g))] \tag{3.3.70}$$

in the concrete case from where f and g are given by (3.3.65). Then the relevant sequences in $\mathfrak{A}^N$ are

$$a_1 \otimes \cdots \otimes a_N = b_{i_1} \otimes \cdots \otimes b_{i_M} \otimes I_k \otimes \cdots \otimes I_k; \tag{3.3.71}$$

$$a'_1 \otimes \cdots \otimes a'_N = b_{j_1} \otimes \cdots \otimes b_{j_L} \otimes I_k \otimes \cdots \otimes I_k, \tag{3.3.72}$$

since, from (3.3.65) and the definition of $S_{P,N}$, i.e.,

$$S_{M,N}(f^{-1}(f)) = S_N(a_1 \otimes \cdots \otimes a_N); \tag{3.3.73}$$
$$S_{L,N}(f^{-1}(g)) = S_N(a'_1 \otimes \cdots \otimes a'_N). \tag{3.3.74}$$

Keeping (3.3.71) and (3.3.72), for $L \leq M$ fixed and large N there are three types of permutations $\pi \in \mathcal{P}(N)$ classified by the following distinct properties of the elements

$$a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)} \qquad \text{or} \qquad a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N)$$

in the right-hand side of (3.3.69):

I. For every factor $a_l a'_{\pi(l)}$ (resp. $a'_{\pi(l)} a_l$), either $a_l = I_k$ or $a'_{\pi(l)} = I_k$ (or both);

II. There is exactly one factor $a_l a'_{\pi(l)}$ (resp. $a'_{\pi(l)} a_l$) with both $a_l \neq I_k$ and $a'_{\pi(l)} \neq I_k$;

III. There is more than one factor $a_l a'_{\pi(l)}$ (resp. $a'_{\pi(l)} a_l$) with both $a_l \neq I_k$ and $a'_{\pi(l)} \neq I_k$.

We accordingly decompose $\mathcal{P}(N)$ into three pairwise disjoint parts as

$$\mathcal{P}(N) = \mathcal{P}(N)_I \cup \mathcal{P}(N)_{II} \cup \mathcal{P}(N)_{III}. \tag{3.3.75}$$

This decomposition induces a corresponding decomposition of $[Q_{1/N}(f), Q_{1/N}(g)]$ arising from the right-hand side of (3.3.69), taking (3.3.74) into account, where a sum over $\pi \in \mathcal{P}(N)$ shows up. We symbolically write this decomposition as

$$\begin{aligned}[Q_{1/N}(f), Q_{1/N}(g)] = {} & [Q_{1/N}(f), Q_{1/N}(g)]_I \\ & + [Q_{1/N}(f), Q_{1/N}(g)]_{II} \\ & + [Q_{1/N}(f), Q_{1/N}(g)]_{III}.\end{aligned} \tag{3.3.76}$$

It should be clear that

$$\sum_{\pi \in \mathcal{P}(N)_I} \left(S_N\left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right) - S_N\left(a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N\right)\right) = 0, \tag{3.3.77}$$

so that $[Q_{1/N}(f), Q_{1/N}(g)]_I = 0$. The term $[Q_{1/N}(f), Q_{1/N}(g)]_{II}$ is proportional to

$$\begin{aligned}&\sum_{\pi \in \mathcal{P}(N)_{II}} \left(S_N\left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right) - S_N\left(a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N\right)\right) \\ = & \sum_{\pi \in \mathcal{P}(N)_{II}} S_N\left([a_1, a'_{\pi(1)}] \otimes a_2 a'_{\pi(2)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right) \\ + & \sum_{\pi \in \mathcal{P}(N)_{II}} S_N\left(a_1 a'_{\pi(1)} \otimes [a_2, a'_{\pi(2)}] \otimes \cdots \otimes a_N a'_{\pi(N)}\right) + \cdots \\ + & \sum_{\pi \in \mathcal{P}(N)_{II}} S_N\left(a_1 a'_{\pi(1)} \otimes a_2 a'_{\pi(2)} \otimes \cdots \otimes [a_N, a'_{\pi(N)}]\right),\end{aligned} \tag{3.3.78}$$

where, for each fixed $\pi \in \mathcal{P}(N)_{II}$, there is exactly one pair $a_l, a'_{\pi(l)}$ with both $a_l \neq I_k$ and $a'_{\pi(l)} \neq I_k$ (so that at most the commutator $[a_l, a'_{\pi(l)}]$ does not vanish and the overall sum above contains at most one non-vanishing summand depending on π).

Let us focus on the first summand in the right-hand side of (3.3.78) and consider the generic summand therein for some $\pi \in \mathcal{P}(N)_{II}$, namely

$$S_N\left([a_1, a'_{\pi(1)}] \otimes a_2 a'_{\pi(2)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right), \tag{3.3.79}$$

where we assume, to avoid a trivial case, that $a_1 \neq I_k$ and $a'_{\pi(1)} \neq I_k$. Recall that

$$a_1 \otimes \cdots \otimes a_N = b_{i_1} \otimes \cdots \otimes b_{i_M} \otimes I_k \otimes \cdots \otimes I_k; \tag{3.3.80}$$
$$a'_1 \otimes \cdots \otimes a'_N = b_{j_1} \otimes \cdots \otimes b_{j_L} \otimes I_k \otimes \cdots \otimes I_k, \tag{3.3.81}$$

where we assume $M \geq L$. Since, in every pair $a_j, a'_{\pi(j)}$ with $j > 2$ at least one of the elements must coincide with I_k, the following identity must hold:

$$\begin{aligned} & S_N\left([a_1, a'_{\pi(1)}] \otimes a_2 a'_{\pi(2)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right) = \\ & S_N\left([b_{i_1}, b_{j_{\pi(1)}}] \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_{\pi(2)}} \otimes \cdots \otimes b_{j_{\pi(L)}} \otimes I_k \otimes \cdots \otimes I_k\right). \end{aligned} \tag{3.3.82}$$

The number of all permutations π of type II and with fixed value $\pi(1)$ can be easily evaluated as (see Appendix B.1 for a more general formula)

$$C_N = \frac{(N-L)!(N-M)!}{(N-L-M+1)!}. \tag{3.3.83}$$

Each of these permutations makes the same contribution (3.3.82) to the first summand in the right-hand side of (3.3.78), because changing π in this way just amounts to keeping the factor $[b_{i_1}, b_{j_{\pi(1)}}]$ and permuting the remaining factors in the argument of S_N in the right-hand side of (3.3.82). This cannot change the final value in view of the very presence of the symmetrizer S_N. An identical argument applies to the remaining terms in the right-hand side of (3.3.78). Summing up, we can now write

$$\begin{aligned} & C_N^{-1} \sum_{\pi \in \mathcal{P}(N)_{II}} \left(S_N\left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right) - S_N\left(a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N\right)\right) \\ & = S_N\left([b_{i_1}, b_{j_1}] \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_2} \otimes \cdots \otimes b_{j_L} \otimes I_k \otimes \cdots \otimes I_k\right) \\ & + S_N\left([b_{i_1}, b_{j_2}] \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_L} \otimes I_k \otimes \cdots \otimes I_k\right) \\ & + \cdots + \\ & + S_N\left(b_{i_1} \otimes \cdots \otimes b_{i_{M-1}} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_{L-1}} \otimes [b_{i_M}, b_{j_L}] \otimes I_k \otimes \cdots \otimes I_k\right), \end{aligned}$$

where all possible commutators $[b_{i_l}, b_{j_m}]$ are considered for $l = 1, \ldots, M$ and $m = 1, \ldots, L$. We finally have that the term $iN[Q_{1/N}(f), Q_{1/N}(g)]_{II}$ equals

$$\begin{aligned} & iN[Q_{1/N}(f), Q_{1/N}(g)]_{II} = \\ & i\frac{N}{N!} \sum_{\pi \in \mathcal{P}(N)_{II}} \left(S_N\left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right) - S_N\left(a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N\right)\right) \\ = & \frac{-C_N}{(N-1)!} S_N\left(\sum_l C^l_{i_1 j_1} b_l \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_2} \otimes \cdots \otimes b_{j_L} \otimes I_k \otimes \cdots \otimes I_k\right. \\ & + \sum_l C^l_{i_1 j_2} b_l \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_L} \otimes I_k \otimes \cdots \otimes I_k \\ & + \cdots + \\ & \left. + \sum_l C^l_{i_M j_L} b_l \otimes b_{i_1} \otimes \cdots \otimes b_{i_{M-1}} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_{L-1}} \otimes I_k \otimes \cdots \otimes I_k\right), \end{aligned}$$

which can be rearranged to

$$\begin{aligned}
&iN[Q_{1/N}(f), Q_{1/N}(g)]_{II} = \\
&\frac{-C_N}{(N-1)!} S_{M+L-1,N} \left(\sum_l C^l_{i_1 j_1} b_l \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_2} \otimes \cdots \otimes b_{j_L} \right. \\
&+ \sum_l C^l_{i_1 j_2} b_l \otimes b_{i_2} \otimes \cdots \otimes b_{i_M} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_L} \\
&+ \cdots + \\
&\left. + \sum_l C^l_{i_M j_L} b_l \otimes b_{i_1} \otimes \cdots \otimes b_{i_{M-1}} \otimes b_{j_1} \otimes b_{j_3} \otimes \cdots \otimes b_{j_{L-1}} \right), \qquad (3.3.84)
\end{aligned}$$

where we have freely rearranged the order of some factors (which we may do since this order is irrelevant in view of the presence of the symmetizator S_N).
We now observe that the last expression for $iN[Q_{1/N}(f), Q_{1/N}(g)]_{II}$ is identical to the expression of $Q_{1/N}(\{f, g\})$ found in (3.3.67), up to the factor $\frac{C_N}{(N-1)!}$. However, a direct computation using Stirling's formula proves that, for fixed M, L,

$$\frac{C_N}{(N-1)!} \to 1 \quad \text{for } N \to \infty. \qquad (3.3.85)$$

To conclude the proof of (3.3.64), exploiting (3.3.85) and the triangle inequality for the norm in (3.3.64), it is therefore sufficient to prove that

$$\left\| \left[\frac{C_N}{(N-1)!} - 1 \right] Q_{1/N}(\{f, g\}) \right\|_N \to 0; \qquad (3.3.86)$$

$$\| iN[Q_{1/N}(f), Q_{1/N}(g)]_{III} \|_N \to 0 \qquad (3.3.87)$$

are both valid as $N \to \infty$. The former is true as a consequence of (3.3.67) and the following property of the maps $S_{M,L}$ for $M \leq L$, which is easy to prove:

$$\| S_{M,L}(a_1 \otimes \cdots \otimes a_M) \|_M \leq \max\{ \|a_j\|^M \mid j = 1, \ldots, M \}. \qquad (3.3.88)$$

This implies that $\|Q_{1/N}(\{f, g\})\|$ is a bounded function of N (for f and g given as above), so that (3.3.85) implies (3.3.86). Regarding the latter, we observe that the conjunction of (3.3.74), the property

$$N[Q_{1/N}(f), Q_{1/N}(g)] = N[S_{M,N}(f^{-1}(f)), S_{L,N}(f^{-1}(g))], \qquad (3.3.89)$$

and Lemma 3.3.5 imply

$$\| iN[Q_{1/N}(f), Q_{1/N}(g)]_{III} \|_N \leq \frac{2C}{(N-1)!} \# \mathcal{P}(N)_{III} \qquad (3.3.90)$$

for the constant

$$C = \max \left\{ \|b_{i_m}\|^M \|b_{j_l}\|^L \;\middle|\; m = 1, \ldots, M,\, l = 1, \ldots, L \right\}. \qquad (3.3.91)$$

Referring to the discussion just before (3.3.75), one can prove (see Appendix B.1) that the number $\#\mathcal{P}(N)_K$ of elements $\pi \in \mathcal{P}(N)$ for which the string

$$a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)},$$

or, equivalently,

$$a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N,$$

in the right-hand side of (3.3.69) includes *exactly* K factors $a_l a'_{\pi(l)}$ (resp. $a'_{\pi(l)} a_l$) with both

$a_l \neq I_k$ and $a'_{\pi(l)} \neq I_k$ is equal to

$$\#\mathcal{P}(N)_K = \frac{L!M!(N-L)!(N-M)!}{K!(L-K)!(M-K)!(N-L-M+K)!}, \tag{3.3.92}$$

where we assumed $0 \leq K \leq L \leq M$ and N large. Hence

$$\frac{\#\mathcal{P}(N)_{III}}{(N-1)!} = \frac{1}{(N-1)!}\sum_{K=2}^{L} \frac{L!M!(N-L)!(N-M)!}{K!(N-M-L+K)!(L-K)!(M-K)!}. \tag{3.3.93}$$

As a consequence, for some constant $A > 0$ depending on L, M, we have

$$\begin{aligned} \frac{\#\mathcal{P}(N)_{III}}{(N-1)!} &\leq \frac{A(N-L)!(N-M)!}{(N-1)!(N-M-L+2)!} \\ &= \frac{AC_N}{(N-1)!}\frac{1}{(N-M-L+2)}, \end{aligned} \tag{3.3.94}$$

where we used (3.3.83). Taking advantage of (3.3.85), we obtain

$$\frac{\#\mathcal{P}(N)_{III}}{(N-1)!} \to 0 \quad \text{for } N \to \infty. \tag{3.3.95}$$

This result implies that (3.3.87) holds because of (3.3.90), which concludes the proof. □

Remark 3.3.6. Observe that we can rearrange (3.3.92) as

$$\#\mathcal{P}(N)_K = (N-L)!L!\binom{M}{K}\binom{N-M}{L-K}. \tag{3.3.96}$$

As a consequence, exploiting the well-known *Chu-Vandermonde identity*, we find

$$\begin{aligned} \sum_{K=0}^{L} \#\mathcal{P}(N)_K &= (N-L)!L!\sum_{K=0}^{L}\binom{M}{K}\binom{N-M}{L-K} \\ &= (N-L)!L!\binom{N}{L} = N!, \end{aligned} \tag{3.3.97}$$

that is,

$$\sum_{K=0}^{L} \#\mathcal{P}(N)_K = \#\mathcal{P}(N), \tag{3.3.98}$$

as it must be. ■

We finally mention that, according to Definition 2.1.3 the deformation quantization constructed in Theorem 3.3.4 is not strict as the image $Q_{1/N}(\tilde{\mathfrak{A}}_0)$ is clearly not dense in $M_k(\mathbb{C})^{\otimes N}$. There exist however several definitions of a (strict) deformation quantization in literature [55, 57, 58]. For example, according to the definition followed in [58, 57] the deformation quantization constructed in Theorem 3.3.4 would be a strict deformation quantization.

4. Bulk-boundary asymptotic equivalence

In this chapter we relate the deformation quantization of the state space of $M_k(\mathbb{C})$ (cf. Theorem 3.3.4 in Chapter 3) to the deformation quantization of the two-sphere $S^2 \subset \mathbb{R}^3$ (cf. Section 2.2.2), for the special case when $k = 2$. In this setting special emphasis is given to one-dimensional quantum spin systems defined on $M_2(\mathbb{C})^{\otimes N}$, where each site of such a spin chain is exactly described by the algebra of (2×2)-matrices. This matrix algebra is related to the – also called – *Bloch sphere* S^2 which acts as a classical phase space and may correspond to a single spin system of total spin $J = N/2$, but it can also be used to describe a collection of N two-level atoms [11] corresponding to a spin chain of N sites, which is for example the case for the quantum Curie-Weiss model [58]. Inspired by that model, which admits a classical limit[1] on S^2 (i.e. the smooth boundary of $X_2 = S(M_2(\mathbb{C})) \cong B^3$), one can ask if the quantization maps $Q_{1/N}$ (viz. (3.3.56)–(3.3.57)) quantizing X_2 could in general be related to another well-known strict deformation quantization of S^2 whose details are explained in what follows.[2]

From the mathematical side, we observe that $k = 2$ is the unique case for which X_k admits a smooth boundary, as said, $X_2 = B^3$ and $\partial X_2 = S^2$. Furthermore S^2 is a Poisson submanifold of B^3, when the latter is equipped with the Poisson structure (4.0.2) specialized to $k = 2$, so that $C^a_{bc} = \epsilon_{abc}$. This is because S^2 (and also B^3) is invariant under the flow of the Hamilton vector fields of $\mathbb{R}^{k^2-1}$ constructed out of the Poisson bracket (4.0.2). For $k = 2$, we precisely have

$$\{f, g\}^{(B^3)}|_{S^2} = \{f|_{S^2}, g|_{S^2}\}^{(S^2)} \quad \text{if } f, g \in \tilde{\mathfrak{A}}_0, \tag{4.0.1}$$

with obvious notation and the right-hand side is defined by (2.2.32). In particular,

$$\{f, g\}^{(B^3)}(\mathbf{x}) = \sum_{a,b,c=1}^{3} \epsilon_{abc} x_c \frac{\partial f}{\partial x_a} \frac{\partial g}{\partial x_b}, \quad \mathbf{x} \in B^3. \tag{4.0.2}$$

In a sense, we promote at quantum level the illustrated interplay of the two symplectic (or Poisson) structures of $X_2 = B^3$ and $\partial B^3 = S^2$. As a matter of fact, we consider the quantization elements $Q_{1/N}(f) \in \mathfrak{B}((\mathbb{C}^2)^{\otimes N})$ under the maps (3.3.56)-(3.3.57) referred to the Poisson structure of B^3. Next we restrict the operators $Q_{1/N}(f)$ to a suitable common invariant subspace of $(\mathbb{C}^2)^{\otimes N}$. It turns out that, for large N, these restricted operators correspond to the image of *another* quantization map acting on $C(\partial B^3)$ and referring to the symplectic structure of ∂B^3.

The said invariant subspace[3] is $\mathrm{Sym}^N(\mathbb{C}^2) \subset (\mathbb{C}^2)^{\otimes N}$, for which the corresponding algebras $\mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$ exactly correspond to the fibers (for $N \neq 0$) of a different continuous bundle of C^*-algebras given by (2.2.35) - (2.2.36). In Chapter 2 we have seen that these fibers together with quantization maps $Q'_{1/N}$ defined by (2.2.40) - (2.2.41) give rise to a strict deformation quantization of S^2 according to Definition 2.1.3. It is precisely the maps $Q'_{1/N}$ that relate to the operators $Q_{1/N}(f)$ when restricted to the symmetric subspace.

These ideas are made precise by means of Theorem 4.2.1, the main result of this chapter. It provides an asymptotic relation connecting the bulk (i.e. B^3) and the boundary (i.e. $\partial B^3 = S^2$) quantization maps. We stress that the validity of the *Dirac-Groenewold-Rieffel condition* (cf.

[1] This means that $\langle \Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)} \rangle$ admits a limit (if $N \to \infty$) as a probability measure on S^2 for any function $f \in \tilde{\mathfrak{A}}'_0$, where $\Psi_N^{(0)}$ denotes the ground state eigenvector of the quantum CW Hamiltonian (see Chapters 6 and 8 for details).

[2] Of course, one can always try to restrict $\tilde{\mathfrak{A}}_0$ to $\tilde{\mathfrak{A}}'_0$ but in that case the same manifolds are quantized which is not of particular new interest.

[3] This space is clearly invariant under the maps (3.3.56) - (3.3.57).

Definition 2.1.3) for both maps is possible just thank to (4.0.1). The chapter is organized as follows. In Section 4.1 we state and prove our main theorem (Theorem 4.2.1) establishing a connection between the strict deformation quantization of X_2 and the one of S^2 defined above. In Section 4.3 we apply our theorem to the Curie-Weiss model which links the corresponding quantum Hamiltonian to its classical counter part on the sphere.

4.1 Interplay of bulk quantization map $Q_{1/N}$ and boundary quantization map $Q'_{1/N}$

In order to arrive at the main theorem of this Chapter we first introduce some vector spaces. We let P_N to be the complex vector space of polynomials in the variables $x, y, z \in \mathbb{R}^3$ of degree $\leq N$ where $N \geq 1$, and let $P_N(S^2)$ be the vector space made of the restrictions to S^2 of those polynomials.

4.1.1 Harmonic polynomials

As a result of Remark 2.2.8, Definition 2.2.40 can actually be stated replacing the polynomial p by a generic $f \in C(S^2)$, though its meaning as a quantization map (Dirac-Groenewold-Rieffel condition) is valid for the domain of the polynomials restricted to S^2 as indicated in Section 2.2.2. The map associating $f \in C(S^2)$ with

$$Q'_{1/N}(f) : \mathrm{Sym}^N(\mathbb{C}^2) \to \mathrm{Sym}^N(\mathbb{C}^2); \tag{4.1.3}$$

$$Q'_{1/N}(f) := \frac{N+1}{4\pi} \int_{S^2} f(\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega| d\Omega, \tag{4.1.4}$$

is well-defined and *it is surjective on* $\mathfrak{B}(Sym^N(\mathbb{C}^2))$ since, for every $A : \mathrm{Sym}^N(\mathbb{C}^2) \to \mathrm{Sym}^N(\mathbb{C}^2)$, there exists a function $p \in P_N(S^2)$ such that $A = Q'_{1/N}(p)$. Indeed, that the function

$$p(\Omega) := tr(A\Delta_N^{(1)}(\Omega)) \,, \tag{4.1.5}$$

where $\Omega \in S^2$ and $\Omega \mapsto \Delta_N^{(1)} \in \mathrm{Sym}^N(\mathbb{C}^2)$ is defined by Definition (2.6) in [53], defines a polynomial on the sphere, i.e.

$$tr(A\Delta_N^{(1)}) \in P_N(S^2). \tag{4.1.6}$$

In particular, we realize that the *linear* map (4.1.4) *cannot be injective on the domain* $C(S^2)$ since this space is infinite dimensional whereas the co-domain is finite dimensional. Nevertheless, if restricting the domain to $P_N(S^2)$, the said map turns out to be bijective.

Proposition 4.1.1. *The map*

$$P_N(S^2) \ni p \mapsto Q'_{1/N}(p) := \frac{N+1}{4\pi} \int_{S^2} p(\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega| d\Omega \in \mathfrak{B}(Sym^N(\mathbb{C}^2)) \tag{4.1.7}$$

is a bijection for $N > 1$.

Proof. The said map is obviously surjective, as already observed, because, by defining $p(\Omega) := tr(A\Delta_N^{(1)}(\Omega))$ for $A \in \mathfrak{B}(Sym^{(N)}(\mathbb{C}^2))$, we have $A = Q'_{1/N}(p)$. Let us prove injectivity. It is well known [78] that $\dim(P_N(S^2)) = (N+1)^2$ if $N > 1$. On the other hand, $\dim(\mathfrak{B}(Sym^{(N)}(\mathbb{C}^2))) = (N+1)^2$ as one immediately proves. As $\dim(\mathfrak{B}(Sym^{(N)}(\mathbb{C}^2))) = \dim(P_N(S^2)) < +\infty$, surjectivity implies injectivity from elementary results of linear algebra. □

Going back to Weyl, let us recall a few results on the theory of $SO(3)$ representations of polynomials restricted to the unit sphere. The group $SO(3)$ admits a natural representation on $P_N(S^2)$ given by

$$SO(3) \ni \mathbb{R} \mapsto \rho_R \,, \quad (\rho_R p)(\Omega) := p(R^{-1}\Omega) \quad \forall p \in P_N(S^2) \,, \forall \Omega \in S^2. \tag{4.1.8}$$

In turn, the space $P_N(S^2)$ admits a direct decomposition into *invariant and irreducible* subspaces under the action of ρ, viz.

$$P_N(S^2) = \bigoplus_{j=0,1,\dots,N} P_N^{(j)}(S^2).$$

Each subspace $P_N^{(j)}(S^2)$ consists of the restrictions to S^2 of the homogeneous polynomials of order j that are also harmonic functions. $P_N^{(j)}(S^2)$ has dimension $2j+1$.

Example 4.1.2. If $N = 2$

$$P_2(S^2) = P_2^{(0)}(S^2) \oplus P_2^{(1)}(S^2) \oplus P_2^{(2)}(S^2)\,.$$

In the right-hand side, the first subspace is the span of the restriction to S^2 of the constant polynomial $p(x,y,z) := 1$, the second one is the span of the restrictions of the three polynomials $p_j(x,y,z) := x_j$, $j = 1,2,3$ where $x_1 = x, x_2 = y, x_3 = z$, and the third one is the the span of the restrictions to S^2 of five elements suitably chosen[4] of the six polynomials $p_{ij}(x,y,z) := x_i x_j - \frac{1}{3}\delta_{ij}(x^2+y^2+z^2)$ for $i,j \in \{1,2,3\}$. ■

If $\rho_R^{(j)}$ is the restriction of ρ_R to $P_N^{(j)}(S^2)$ and $\{p_m^{(j)}\}_{m=-j,-j+1,\dots,j-1,j}$ is a basis of $P_N^{(j)}(S^2)$, we find

$$\rho_R^{(j)} p_m^{(j)} = \sum_{m'=-j}^{j} D_{mm'}^{(j)}(R^{-1}) p_{m'}^{(j)}\,. \tag{4.1.9}$$

Each class of matrices $\{D^{(j)}(R)\}_{R\in SO(3)}$ defines an irreducible representation of $SO(3)$ in $\mathbb{C}^{2j+1}$. These representations are completely fixed by their dimension i.e., by j, up to equivalence given by similarity transformations, and different j correspond to similarity inequivalent representations. Every irreducible representation of $SO(3)$ is unitarily equivalent to one of the $D^{(j)}$.

4.2 Bulk-boundary asymptotic equivalence

Before arriving at the main theorem, we recall that by construction the space $\tilde{\mathfrak{A}}_0$ is the complex vector space of polynomials in three variables on the closed unit ball B^3 which in particular contains all polynomials of P_M ($M \in \mathbb{N}$) restricted to B^3 [58]. In the proof of the theorem we occasionally use the space $\tilde{\mathfrak{A}}_0$ as well as P_M, where the former is the domain of the quantization maps $Q_{1/N}$, whereas the latter is used to underline the degree of the polynomial in question.

Theorem 4.2.1. *If $p \in \tilde{\mathfrak{A}}_0$, then*

$$\left|\left|Q_{1/N}(p)|_{Sym^N(\mathbb{C}^2)} - Q'_{1/N}(p|_{S^2})\right|\right|_N \to 0 \quad \textit{for } N \to +\infty\,,$$

the (operator) norm being the one on $\mathfrak{B}(Sym^N(\mathbb{C}^2))$.

Remark 4.2.2. We stress that the result does not automatically imply that the cross-sections (3.3.56) - (3.3.57) whose images are restricted to $\mathrm{Sym}^N(\mathbb{C}^2)$ are also continuous cross-sections of the fibers defined in (2.2.35) - (2.2.36), since $f \in \mathfrak{A}_0 = C(B^3)$ does *not* imply that $f \in \mathfrak{A}'_0 = C(S^2)$. ■

Proof. We start the proof by discussing the interplay between the action of $SO(3)$ and the quantization maps $Q_{1/N}$, defined in (3.3.56). We first focus on a homogeneous polynomial of order $M < N$.[5] If $k_1,\dots,k_M$ are taken in $\{1,2,3\}$ and

$$p_{k_1\cdots k_M}(x,y,z) := x_{k_1}\cdots x_{k_M}\,, \tag{4.2.10}$$

[4] The restrictions to S^2 of these six polynomials and the one of the above p form a linearly dependent set.
[5] As we are dealing with a limit in N, we can safely take N such that $M < N$.

the representation (4.1.8) implies that

$$(\rho_R p_{k_1\cdots k_M})(x,y,z) = (R_U^{-1})_{k_1}{}^{j_1}\cdots(R_U^{-1})_{k_M}{}^{j_M} p_{k_1\cdots k_M}(x,y,z)\,. \tag{4.2.11}$$

We stress that when restricting to S^2, every $p_m^{(j)}$ is a linear combination of the restrictions of the polynomials $p_{k_1\cdots k_M}$ so that, by extending (4.2.11) by linearity and working on $p_m^{(j)}$, (4.2.11) must coincide with (4.1.9)

$$\left(\rho_R^{(j)} p_m^{(j)}\right)(x,y,z) = \sum_{m'=-j}^{j} D^{(j)}_{mm'}(R^{-1}) p^{(j)}_{m'}(x,y,z)\,, \quad (x,y,z)\in S^2$$

Since both sides are restrictions of homogeneous polynomials of the same degree j, this identity is valid also removing the constraint $(x,y,z)\in S^2$:

$$\left(\rho_R^{(j)} p_m^{(j)}\right)(x,y,z) = \sum_{m'=-j}^{j} D^{(j)}_{mm'}(R^{-1}) p^{(j)}_{m'}(x,y,z)\,, \quad (x,y,z)\in \mathbb{R}^3\,. \tag{4.2.12}$$

where now the $p_m^{(j)}$ are homogeneous polynomials in P_M whose restrictions are the basis elements of $P_M^{(j)}(S^2)$ with the same name. We underline that for our quantization maps $Q_{1/N}$ we need $p_m^{(j)}$ to be a polynomial in $\tilde{\mathfrak{A}}_0$, rather than in P_M. However, since $\tilde{\mathfrak{A}}_0$ contains all polynomials of P_M restricted to B^3 which has non-empty interior, polynomials of P_M are in one-to-one correspondence with those of $\tilde{\mathfrak{A}}_0$. Therefore, in view of (4.2.12) the same statement holds when we replace $(x,y,z)\in\mathbb{R}^3$ by $(x,y,z)\in B^3$. Now, by definition of the quantization maps $Q_{1/N}$ we know that

$$Q_{1/N}(p_{k_1\cdots k_M}) = S_{N,M}\left(\sigma_{k_1}\otimes\cdots\otimes\sigma_{k_M}\otimes\underbrace{I\otimes\cdots\otimes I}_{N-M\ times}\right)\,.$$

Let us indicate by $R_U \in SO(3)$ the image of $U\in SU(2)$ through the universal covering homomorphism $\Pi : SU(2)\to SO(3)$. This covering homomorphism as is well known satisfies (using the summation convention on repeated indices)

$$U\sigma_j U^* = (R_U^{-1})_j{}^k \sigma_k. \tag{4.2.13}$$

Remembering that $\mathrm{Sym}^N(\mathbb{C}^2)$ is invariant under the tensor representation $\underbrace{U\otimes\cdots\otimes U}_{N\ times}$, we have

$$\begin{aligned}
&\underbrace{U\otimes\cdots\otimes U}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}\, Q_{1/N}(p_{k_1\cdots k_M})|_{\mathrm{Sym}^N(\mathbb{C}^2)}\, \underbrace{U^*\otimes\cdots\otimes U^*}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}\\
&= \left(\underbrace{U\otimes\cdots\otimes U}_{N\ times} Q_{1/N}(p_{k_1\cdots k_M})\underbrace{U^*\otimes\cdots\otimes U^*}_{N\ times}\right)|_{\mathrm{Sym}^N(\mathbb{C}^2)}\\
&= (R_U^{-1})_{k_1}{}^{j_1}\cdots(R_U^{-1})_{k_M}{}^{j_M} Q_{1/N}(p_{j_1\dots j_M})|_{\mathrm{Sym}^N(\mathbb{C}^2)}\,.
\end{aligned}$$

Let us consider linear combinations $p_m^{(j)}$ of polynomials $p_{k_1\cdots k_M}$ whose restriction to S^2 define the basis element, indicated with the same symbol, $p_m^{(j)}\in P_M^{(j)}(S^2)$. Since the map $Q_{1/N}$ is linear, from (4.2.12) we have

$$\begin{aligned}
&\left(\underbrace{U\otimes\cdots\otimes U}_{N\ times} Q_{1/N}(p_m^{(j)})\underbrace{U^*\otimes\cdots\otimes U^*}_{N\ times}\right)|_{\mathrm{Sym}^N(\mathbb{C}^2)}\\
&\qquad = \sum_{m'} D^{(j)}_{mm'}(R^{-1}) Q_{1/N}(p^{(j)}_{m'})|_{\mathrm{Sym}^N(\mathbb{C}^2)}\,.
\end{aligned} \tag{4.2.14}$$

Let us now pass to the other quantization map $Q'_{1/N}$ observing that (4.2.14) and Proposition 4.1.1 entail

$$Q_{1/N}(p_m^{(j)})|_{\mathrm{Sym}^N(\mathbb{C}^2)} = Q'_{1/N}(q_m^{(j)}) = \frac{N+1}{4\pi}\int_{S^2} q_m^{(j)}(\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega|d\Omega \tag{4.2.15}$$

for some $q_m^{(j)} \in P_N(S^2)$ (where $N > M$ in general) is the unknown restriction to S^2 of a polynomial in P_N. Exploiting (4.2.14) and linearity we find

$$\underbrace{U\otimes\cdots\otimes U}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}\, Q_{1/N}(p_m^{(j)})|_{\mathrm{Sym}^N(\mathbb{C}^2)}\, \underbrace{U^*\otimes\cdots\otimes U^*}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$$

$$= \frac{N+1}{4\pi}\int_{S^2}\sum_{m'} D_{m'}^{(j)}(R^{-1})q_{m'}^{(j)}(\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega|d\Omega\,. \tag{4.2.16}$$

Again, from (4.1.4) we have the general relation

$$VA_f^{(N)}V^* = \frac{N+1}{4\pi}\int_{S^2} f(\Omega)V|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega|V^*d\Omega\,.$$

Specializing to $V = \underbrace{U\otimes\cdots\otimes U}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ we obtain (see Lemma B.2.1 in Appendix B.2)

$$V|\Psi_N^\Omega\rangle = e^{i\alpha_{U,\Omega}}|R_U\Psi_N^\Omega\rangle \tag{4.2.17}$$

where the phase is irrelevant as it disappears in view of later computations, hence

$$VA_f^{(N)}V^* = \frac{N+1}{4\pi}\int_{S^2} f(\Omega)|R_U\Psi_N^\Omega\rangle\langle R_U\Psi_N^\Omega|d\Omega$$

$$= \frac{N+1}{4\pi}\int_{S^2} f(R_U^{-1}\Omega)|R_U R_U^{-1}\Psi_N^\Omega\rangle\langle R_U R_U^{-1}\Psi_N^\Omega|dR_U^{-1}\Omega,$$

namely

$$\underbrace{U\otimes\cdots\otimes U}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}\, A\,\underbrace{U^*\otimes\cdots\otimes U^*}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)} = \frac{N+1}{4\pi}\int_{S^2} f(R_U^{-1}\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega|d\Omega \tag{4.2.18}$$

where we took advantage of $d\Omega = dR^{-1}\Omega$ if $R \in SO(3)$. To conclude, if $A = Q_{1/N}(p_m^{(j)})$, identity (4.2.16) yields

$$\int_{S^2}\sum_{m'} D_{mm'}^{(j)}(R^{-1})q_{m'}^{(j)}(\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega|d\Omega = \int_{S^2} q_m^{(j)}(R_U^{-1}\Psi_N^\Omega)|\Psi_N^\Omega\rangle\langle\Psi_N^\Omega|d\Omega\,.$$

Since the map (4.1.4) is bijective on $P_N(S^2)$ it must be

$$q_m^{(j)}(R^{-1}\Psi_N^\Omega) = \sum_{m'} D_{mm'}^{(j)}(R^{-1})q_{m'}^{(j)}(\Omega)\,, \quad \forall\Omega\in S^2\,, \forall R\in SO(3) \tag{4.2.19}$$

Linearity and bijectivity of the map (4.1.7) also implies that, varying $m = -j, -j+1, \ldots, j-1, j$ the functions $q_m^{(j)}$ form a basis of a $2j+1$ dimensional subspace of $P_N(S^2)$. We can expand each of these functions over the basis of functions $p_m^{(j)}$ of $P_N(S^2)$:

$$q_{m'}^{(j')} = \sum_{j=0}^{N}\sum_{m=-j}^{j} C_{m'm}^{(j',j)}p_m^{(j)}\,, \tag{4.2.20}$$

where both sides are now and henceforth evaluated on S^2. Here (4.1.9) and (4.2.19) together imply

$$\sum_{j,m,\kappa} D^{(j)}_{m'\kappa}(R) C^{(j',j)}_{\kappa m} p^{(j)}_m = \sum_{j,m,\ell} C^{(j',j)}_{m'm} D^{(j)}_{m\ell}(R) p^{(j)}_\ell ,$$

that is

$$\sum_{j,\ell,\kappa} D^{(j)}_{m'\kappa}(R) C^{(j',j)}_{\kappa\ell} p^{(j)}_\ell = \sum_{j,m,\ell} C^{(j',j)}_{m'm} D^{(j)}_{m\ell}(R) p^{(j)}_\ell .$$

Since the set of the (restrictions of to the sphere of the) $p^{(j)}$ is a basis,

$$\sum_m D^{(j)}_{m'm}(R) C^{(j',j)}_{m\ell} = \sum_m C^{(j',j)}_{m'm} D^{(j)}_{m\ell}(R) .$$

Since the representation $D^{(j)}$ is irreducible, Schur's lemma implies that there are complex numbers $C^{(j',j)}$ such that

$$C^{(j',j)}_{m\ell} = C^{(j',j)} \delta_{m\ell} .$$

In summary, (4.2.21) reduces to

$$q^{(j')}_m = \sum_{j=0}^{M} C^{(j',j)} p^{(j)}_m|_{S^2} . \tag{4.2.21}$$

However, since the elements in the left-hand side are $2j'+1$ whereas, for every j in the right-hand side we have $2j+1$ elements and the spaces of these representations transform separately, the only possibility is that $C^{(j',j)} = 0$ if $j \neq j'$. In other words,

$$q^{(j,N)}_m = C^{(j)}_N p^{(j)}_m|_{S^2} \quad \text{for every given } j = 0, 1, 2, \ldots, M . \tag{4.2.22}$$

where,

(i) we have terminated j to $M < N$ because the initial polynomial $p^{(j)}_m$ has been chosen in $P^{(j)}_M(S^2)$;

(ii) we have restored the presence of N, since $C^{(j)}_N$ may depend on N.

Let us examine what happens to $C^{(j)}_N$ at large N. First observe that (4.2.22) immediately implies

$$Q_{1/N}(p^{(j)}_m)|_{\mathrm{Sym}^N(\mathbb{C}^2)} = C^{(j)}_N \frac{N+1}{4\pi} \int_{S^2} p^{(j)}_m(\Omega) |\Psi^\Omega_N\rangle\langle\Psi^\Omega_N| d\Omega.$$

Taking the expectation value $\langle\Psi^{\Omega'}_N| \cdot |\Psi^{\Omega'}_N\rangle$, we find

$$p^{(j)}_m(\Omega') = C^{(j)}_N \int_{S^2} p^{(j)}_m(\Omega) \frac{N+1}{4\pi} |\langle\Psi^{\Omega'}_N|\Psi^\Omega_N\rangle|^2 d\Omega . \tag{4.2.23}$$

In Lemma B.2.2 (Appendix B.2) it is proved that $\lim_{N\to+\infty} C^{(j)}_N$ exists and is finite. Hence,

$$p^{(j)}_m(\Omega') = \left(\lim_{N\to+\infty} C^{(j)}_N\right) p^{(j)}_m(\Omega'),$$

where we exploited Proposition 8.3.4 (a) so that [6]

$$\lim_{N\to+\infty} C^{(j)}_N = 1 .$$

This reasoning implies the claim for the considered special polynomials since, for $N \to +\infty$,

$$\left|\left| Q_{1/N}(p^{(j)}_m)|_{\mathrm{Sym}^N(\mathbb{C}^2)} - \frac{N+1}{4\pi} \int_{S^2} p^{(j)}_m|_{S^2}(\Omega) |\Psi^\Omega_N\rangle\langle\Psi^\Omega_N| d\Omega \right|\right|_N$$

[6] This also follows from the fact that the the manifold S^2 admits a coherent pure state quantization according to Definition 2.2.2 (viz. (2.2.24)).

$$= |C_N - 1| \left|\left| \frac{N+1}{4\pi} \int_{S^2} p_m^{(j)}|_{S^2}(\Omega) |\Psi_N^\Omega\rangle\langle\Psi_N^\Omega| d\Omega \right|\right|_N \leq |C_N - 1| ||p_m^{(j)}|_{S^2}||_\infty \to 0 \tag{4.2.24}$$

The found result immediately extends to every polynomial of given degree M which can be written as a linear combination of the $p_m^{(j)}$ viewed as polynomials. To pass to a generic polynomial in $\tilde{\mathfrak{A}}_0$ (say of degree M) we observe that, as a consequence of known results [78], the map

$$\tilde{\mathfrak{A}}_0 \ni p \mapsto p|_{S^2} \in P_M(S^2)$$

has a kernel made of all possible polynomials of the form $q(x, y, z)(x^2 + y^2 + z^2 - 1)$ with $q \in P_{M-2}$. Furthermore, Proposition B.2.3 (Appendix B.2) proves that, for every $q \in P_{M-2}$,

$$||Q_{1/N}(q(x, y, z)(x^2 + y^2 + z^2 - 1))|_{\mathrm{Sym}^N(\mathbb{C}^2)}||_N \to 0 \quad \text{as } N \to +\infty. \tag{4.2.25}$$

So, if $p \in \tilde{\mathfrak{A}}_0$ is a polynomial of degree M, then we can write for a finite number of coefficients $C^{(j,m)}$ and some polynomial $q \in P_{M-2}$,

$$p = \sum_{j,m} C^{(j,m)} p_m^{(j)} + q(x, y, z)(x^2 + y^2 + z^2 - 1), \tag{4.2.26}$$

where the $p_m^{(j)}$ and q are here interpreted as elements of P_M and P_{M-2} respectively, restricted to B^3. Hence,

$$Q_{1/N}(p)|_{\mathrm{Sym}^N(\mathbb{C}^2)} = \sum_{j,m} C^{(j,m)} Q_{1/N}(p_m^{(j)})|_{\mathrm{Sym}^N(\mathbb{C}^2)}$$
$$+ Q_{1/N}(q(x, y, z)(x^2 + y^2 + z^2 - 1))|_{\mathrm{Sym}^N(\mathbb{C}^2)}\,.$$

The former term on the right-hand side tends to $Q'_{1/N}(p|_{S^2})$, the latter vanishes as $N \to +\infty$ proving the thesis. □

4.3 Application to the quantum Curie-Weiss model

We apply the previous theorem to the (quantum) Curie-Weiss model[7], which is an exemplary quantum mean-field spin model.[8] We recall that the scaled **quantum Curie Weiss**[9] defined on a lattice with N sites is

$$h_{1/N}^{CW} : \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N\ times} \to \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N\ times}; \tag{4.3.27}$$

$$h_{1/N}^{CW} = \frac{1}{N} \left(-\frac{J}{2N} \sum_{i,j=1}^N \sigma_3(i)\sigma_3(j) - B \sum_{j=1}^N \sigma_1(j) \right). \tag{4.3.28}$$

Here $\sigma_k(j)$ stands for $I_2 \otimes \cdots \otimes \sigma_k \otimes \cdots \otimes I_2$, where σ_k, the k^{th} spin Pauli matrix ($k = 1, 2, 3$) occupies the j-th slot, and $J, B \in \mathbb{R}$ are given constants defining the strength of the spin-spin coupling and the (transverse) external magnetic field, respectively. Note that

$$h_{1/N}^{CW} \in \mathrm{Sym}(M_2(\mathbb{C})^{\otimes N}), \tag{4.3.29}$$

[7] This model exists in both a classical and a quantum version and is a mean-field approximation to the Ising model. See e.g. [39] for a mathematically rigorous treatment of the classical version, and [27, 49] for the quantum version. For our approach the papers [16, 37, 74] played an important role. See also [2] for a very detailed discussion of the quantum Curie–Weiss model.

[8] The geometric configuration including its dimension is irrelevant, as is typical for mean-field models [100], so that we may as well consider the model in one dimension, i.e. defined on a chain.

[9] Inspired by the ideas by Lieb [60],this means that we scale the Hamiltonian by a global factor $1/N$ in front.

where $\mathrm{Sym}(M_2(\mathbb{C})^{\otimes N})$ is the range of the symmetrizer introduced in Section 3.2 (cf. (3.2.32)). Our interest will lie in the limit $N \to \infty$. As such, we rewrite $h_{1/N}^{CW}$ as

$$\begin{aligned} h_{1/N}^{CW} &= -\frac{J}{2N(N-1)} \sum_{i \neq j,\, i,j=1}^{N} \sigma_3(i)\sigma_3(j) - \frac{B}{N}\sum_{j=1}^{N} \sigma_1(j) + O(1/N). \\ &= Q_{1/N}(h_0^{CW}) + O(1/N), \end{aligned} \tag{4.3.30}$$

where $O(1/N)$ is meant in norm (i.e. the operator norm on each space $M_2(\mathbb{C})^{\otimes N}$), and the **classical Curie–Weiss Hamiltonian** is

$$h_0^{CW} : B^3 \mapsto \mathbb{R}; \tag{4.3.31}$$

$$h_0^{CW}(x,y,z) = -\left(\frac{J}{2}z^2 + Bx\right), \quad \mathbf{x} = (x,y,z) \in B^3, \tag{4.3.32}$$

where $B^3 = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| \leq 1\}$ is the closed unit ball in $\mathbb{R}^3$. Therefore, up to a small error as $N \to \infty$, the quantum Curie–Weiss Hamiltonian (4.3.28) is given by deformation quantization of its classical counterpart (4.3.32).

Using these observations we now show that the quantum Curie-Weiss Hamiltonian restricted to the symmetric space is asymptotically norm-equivalent also to the other quantization map $Q'_{1/N}$ applied to $h_0^{CW}|_{S^2}$.

Proposition 4.3.1. *One has*

$$\left|\left| h_{1/N}^{CW}|_{Sym^N(\mathbb{C}^2)} - Q'_{1/N}(h_0^{CW}|_{S^2}) \right|\right|_N \to 0 \quad \text{for } N \to \infty. \tag{4.3.33}$$

Proof. Using (4.3.30) and Theorem (4.2.1),

$$\begin{aligned} &\left|\left| h_{1/N}^{CW}|_{\mathrm{Sym}^N(\mathbb{C}^2)} - Q'_{1/N}(h_0^{CW}|_{S^2}) \right|\right|_N \leq \\ &\left|\left| Q_{1/N}(h_0^{CW})|_{\mathrm{Sym}^N(\mathbb{C}^2)} + O(\frac{1}{N})|_{\mathrm{Sym}^N(\mathbb{C}^2)} - Q'_{1/N}(h_0^{CW}|_{S^2}) \right|\right|_N \leq \\ &\left|\left| Q_{1/N}(h_0^{CW})|_{\mathrm{Sym}^N(\mathbb{C}^2)} - Q'_{1/N}(h_0^{CW}|_{S^2}) \right|\right|_N \to 0 \ \text{ (as } N \to \infty). \end{aligned} \tag{4.3.34}$$

□

This in particular establishes a link between the (compressed) quantum Curie-Weiss spin Hamiltonian and its classical counterpart on the sphere.

5. Injective tensor products in strict deformation quantization

In this chapter we provide necessary and sufficient criteria for the existence of a strict deformation quantization (according to Definition 2.1.3) of algebraic tensor products of Poisson algebras (cf. Theorem 5.2.3), and secondly we discuss the existence of products of KMS states (cf. Theorem 5.3.3). To this end we first apply the theory of continuous bundle of C^*-algebras (according to Definition 2.1.1) to a certain tensor product of C^*-algebras, namely the **injective tensor product** which plays a crucial role in this chapter. As an application, we discuss the correspondence between quantum and classical Hamiltonians in spin systems and finally provide a relation between the resolvent of Schrödinger operators for non-interacting many particle systems and quantization maps.

5.1 The injective tensor product of continuous bundles of C^*-algebras

In this section, we shall collect basic facts about injective tensor products of continuous bundles of C^*-algebras whose definition has been introduced in Definition 2.1.1.

If $\mathcal{A} := (I, \mathfrak{A}, \pi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar)$ and $\mathcal{B} := (I, \mathfrak{B}, \pi_\hbar : \mathfrak{B} \to \mathfrak{B}_\hbar)$ are continuous bundles of C^*-algebras there exists a natural bundle $\mathcal{A} \otimes \mathcal{B}$ over I with bundle algebras given by the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$. Clearly $\mathcal{A} \otimes \mathcal{B}$ is not a bundle of C^*-algebras since the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$ is only a pre-C^*-algebra. Therefore, a suitable completion of $\mathfrak{A} \otimes \mathfrak{B}$ has to be performed to obtain a C^*-algebra. A natural strategy is to embed $\mathfrak{A} \otimes \mathfrak{B}$ as a $*$-subalgebra of algebra of bounded operators $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$: The norm of an element in $\mathfrak{A} \otimes \mathfrak{B}$ will then be the operator norm of the associated bounded operator. The resulting norm on $\mathfrak{A} \otimes \mathfrak{B}$ is usually dubbed *injective tensor norm* (or *spatial norm* or *minimal C^*-norm*) and we will denote it as $\|\cdot\|_\epsilon$. We summarize the above discussion in the following theorem and we refer to [73] for more details.

Theorem 5.1.1 ([73], Theorem B.9). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras and consider two faithful representations $\pi_\mathfrak{A} : \mathfrak{A} \to \mathfrak{B}(\mathcal{H}_\mathfrak{A})$ and $\pi_\mathfrak{B} : \mathfrak{B} \to \mathfrak{B}(\mathcal{H}_\mathfrak{B})$. Then it holds:*

- *There exists a unique $*$-homomorphism $\pi_\mathfrak{A} \otimes \pi_\mathfrak{B} : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{B}(\mathcal{H}_\mathfrak{A} \otimes \mathcal{H}_\mathfrak{B})$ such that $\pi_\mathfrak{A} \otimes \pi_\mathfrak{B}(a \otimes b) = \pi_\mathfrak{A}(a) \otimes \pi_\mathfrak{B}(b)$;*
- *The C^*-norm $\|\cdot\|_\epsilon$ on $\mathfrak{A} \otimes \mathfrak{B}$ defined by*

$$\|\sum_{i=1}^{k} a_i \otimes b_i\|_\epsilon := \|\sum_{i=i}^{k} \pi_\mathfrak{A}(a_i) \otimes \pi_\mathfrak{B}(b_i)\|_{\mathfrak{B}(\mathcal{H}_\mathfrak{A} \otimes \mathcal{H}_\mathfrak{B})}$$

does not depend on the choice of representations and it is a cross-norm, i.e. for all $a_i \in \mathfrak{A}$ and $b_i \in \mathfrak{B}$ it holds

$$\|a_i \otimes b_i\|_\epsilon = \|a_i\|_A \|b_i\|_B \tag{5.1.1}$$

where $\|\cdot\|_\mathfrak{A}$ and $\|\cdot\|_\mathfrak{B}$ are the C^*-norm of $\mathfrak{A}$ and $\mathfrak{B}$ respectively.

■

This yields the following definition.

Definition 5.1.2. *Given two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$, we call* injective tensor product *of $\mathfrak{A}$ and $\mathfrak{B}$ the completion $\mathfrak{A} \hat{\otimes}_\epsilon \mathfrak{B}$ of $\mathfrak{A} \otimes \mathfrak{B}$ with respect to the injective tensor norm $\|\cdot\|_\epsilon$.* ■

Example 5.1.3. There are some basic examples where the injective tensor product of two C^*-algebras takes a familiar form. When one algebra is commutative, for example, we can identify the injective tensor product with an algebra of complex-valued functions. If X is a locally compact Hausdorff space and $\mathfrak{A}$ is a C^*-algebra, then the ring $C_0(X, \mathfrak{A})$ of continuous functions $f : X \to \mathfrak{A}$ such that $x \mapsto \|f(t)\|$ vanishes at infinity is a C^*-algebra with pointwise operations and the supremum norm:

$$fg(x) = f(x)g(x) \qquad f^*(x) = f(x)^* \qquad \|f\|_\infty = \sup_{x\in X} \|f(x)\| .$$

As shown in [73, Corollary B.17,] if X and Y are locally compact Hausdorff spaces, then there is an isomorphism ψ of $C_0(X)\hat{\otimes}_\epsilon C_0(Y)$ onto $C_0(X \times Y)$ such that $\psi(f \otimes g)(x, y) = f(x)g(y)$ for every $f \in C_0(X)$ and $g \in C_0(Y)$. ■

Replacing the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ with the injective tensor product $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$, we thus obtain a bundle of C^*-algebras but this bundle is only lower-semicontinuous as shown by Kirchberg and Wasserman in [51, Proposition 4.9]. A sufficient criteria for continuity is obtained combining Lemma 2.4 and Lemma 2.5 in [51]. We recall the result for sake of completeness.

Lemma 5.1.4 ([51, Remark 2.6.1])**.** *Let $\mathcal{A} = (I, \mathfrak{A}, \pi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar)$ and $\mathcal{B} = (I, \mathfrak{B}, \sigma_\hbar : \mathfrak{B} \to \mathfrak{B}_\hbar)$ be continuous bundles of C^*-algebras. If for every $\hbar \in I$ the algebras $\mathfrak{A}_\hbar$ and $\mathfrak{B}_\hbar$ are nuclear C^*-algebras, then $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is a continuous bundle of $C*$-algebras.* ■

Remark 5.1.5. Clearly, assuming that $\mathfrak{A}_\hbar$ and $\mathfrak{B}_\hbar$ are nuclear is a sufficient but not a necessary condition. On account of [51, Theorem 4.6] one can even take the bundle to be nuclear. ■

A sufficient and necessary condition however was provided by Archbold in [7].

Theorem 5.1.6 ([7, Theorem 3.3])**.** *Let $\mathcal{A} = (I, \mathfrak{A}, \pi_\hbar : \mathfrak{A} \to \mathfrak{A}_x)$ and $\mathcal{B} = (I, \mathfrak{B}, \sigma_\hbar : \mathfrak{B} \to \mathfrak{B}_\hbar)$ be continuous bundles of C^*-algebras. Then for each $\hbar \in I$, the function $\hbar \mapsto \|(\pi_\hbar \otimes \sigma_\hbar)(c)\|_\hbar$ is continuous for all $c \in \mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$ at $\hbar$ if and only if*

$$\ker(\pi_\hbar \otimes \sigma_\hbar) = \ker(\pi_\hbar)\hat{\otimes}_\epsilon\mathfrak{B} + \mathfrak{A}\hat{\otimes}_\epsilon \ker(\sigma_\hbar) .$$

■

5.2 Products of Poisson algebras

Let $\mathfrak{A}$ and $\mathfrak{B}$ two Poisson $*$-algebras (densely contained in C^*-algebras $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$, respectively) and assume that there exists a (strict) deformation quantization of $\mathfrak{A}$ and $\mathfrak{B}$ respectively.[1]

The aim of this section is to provide a necessary and sufficient criteria for the existence of a (strict) deformation quantization of the algebraic tensor product $\mathfrak{A} \otimes \mathfrak{B}$. We start by showing that $\mathfrak{A} \otimes \mathfrak{B}$ is a dense Poisson $*$-subalgebra of $\bar{\mathfrak{A}}\hat{\otimes}_\epsilon\bar{\mathfrak{B}}$.

Lemma 5.2.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be dense Poisson $*$-subalgebras of C^*-algebras $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ respectively. Then there exists a Poisson structure on $\mathfrak{A} \otimes \mathfrak{B}$ and $\mathfrak{A} \otimes \mathfrak{B}$ is dense in $\bar{\mathfrak{A}}\hat{\otimes}_\epsilon\bar{\mathfrak{B}}$.*

Proof. Let $\mathfrak{A} \otimes \mathfrak{B}$ the algebraic tensor product of $\mathfrak{A}$ and $\mathfrak{B}$. For any $f_1 \otimes f_2 \in \mathfrak{A} \otimes \mathfrak{B}$ and $g_1 \otimes g_2 \in C^\infty(X) \otimes C^\infty(Y)$, the map $\{\cdot, \cdot\}_\otimes$ defined by

$$\{f_1 \otimes f_2, g_1 \otimes g_2\}_\otimes := \{f_1, g_1\}_A \otimes f_2g_2 + f_1g_1 \otimes \{f_2, g_2\}_B , \tag{5.2.2}$$

where $\{\cdot, \cdot\}_\mathfrak{A}$ and $\{\cdot, \cdot\}_\mathfrak{B}$ denotes the Poisson bracket on $\mathfrak{A}$ and $\mathfrak{B}$ respectively, is a Poisson bracket on $\mathfrak{A} \otimes \mathfrak{B}$.

To conclude our proof we need to show that $\mathfrak{A} \otimes \mathfrak{B}$ is dense in $\bar{\mathfrak{A}}\hat{\otimes}_\epsilon\bar{\mathfrak{B}}$. But this follows immediately because $\mathfrak{A} \otimes \mathfrak{B}$ is dense (in the cross norm $\| \cdot \|_\epsilon$) in $\bar{\mathfrak{A}} \otimes \bar{\mathfrak{B}}$ which is dense in $\bar{\mathfrak{A}}\hat{\otimes}_\epsilon\bar{\mathfrak{B}}$. □

[1] Strictly speaking we do not require here that the algebras $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{B}}$ are of the form $C_0(X)$ and $C_0(Y)$ (with X, Y Poisson manifolds. In view of Remark 2.1.5 this is not a problem as one can simply generalize the notion of a (strict) deformation quantization to arbitrary Poisson algebras [55].

Corollary 5.2.2. *Let X and Y be locally compact Poisson manifolds. Then there exists a Poisson structure on the manifold $X \times Y$.*

Proof. Since $C_0^\infty(X)$ (resp. $C_0^\infty(Y)$) is a dense Poisson $*$-subalgebra of $C_0(X)$ (resp. $C_0(Y)$), by Lemma 5.2.1 it follows that $C_0^\infty(X) \otimes C_0^\infty(Y)$ is a Poisson algebras densely contained in $C_0(X)\hat{\otimes}_\epsilon C_0(Y)$. By [73, Corollary B.17] we obtain that $C_0(X)\hat{\otimes}_\epsilon C_0(Y) \simeq C_0(X \times Y)$ and we can define a Poisson bracket on $C_0^\infty(X \times Y)$ by declaring

$$\{f, g\}_{C_0^\infty(X\times Y)} := \{f(\cdot, y), g(\cdot, y)\}_{C_0^\infty(X)} + \{f(x, \cdot), g(x, \cdot)\}_{C_0^\infty(Y)} .$$

This concludes our proof. □

With the next theorem we shall provide a criteria for the existence of a deformation quantization of the algebraic tensor product $\tilde{\mathfrak{A}}_0 \otimes \tilde{\mathfrak{B}}_0$, where $\tilde{\mathfrak{A}}_0$ and $\tilde{\mathfrak{B}}_0$ are assumed to admit a deformation quantization in the sense of Definition 2.1.3.

Theorem 5.2.3. *Let $\tilde{\mathfrak{A}}_0$ and $\tilde{\mathfrak{B}}_0$ be Poisson $*$-algebras densely contained in commutative C^*-algebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$ respectively and assume that $\tilde{\mathfrak{A}}_0$ and $\tilde{\mathfrak{B}}_0$ admit a deformation quantization in the sense of Definition 2.1.3. Denote with $\mathcal{A} = (I, \mathfrak{A}, \pi_\hbar)$ (resp. $\mathcal{B} = (I, \mathfrak{B}, \sigma_\hbar)$) the continuous bundle of C^*-algebras and with $Q_\hbar^{\mathfrak{A}}$ (resp $Q_\hbar^{\mathfrak{B}}$) the quantization map for $\tilde{\mathfrak{A}}_0$ (resp. for $\tilde{\mathfrak{B}}_0$). Then there exists a deformation quantization of $\tilde{\mathfrak{A}}_0 \otimes \tilde{\mathfrak{B}}_0$ over the interval I with a quantization map given by $Q_\hbar := Q_\hbar^{\mathfrak{A}} \otimes Q_\hbar^{\mathfrak{B}}$ if and only if for every $\hbar \in I$*

$$\ker(\pi_\hbar \otimes \sigma_\hbar) = \ker(\pi_\hbar)\hat{\otimes}_\epsilon \mathfrak{B} + \mathfrak{A}\hat{\otimes}_\epsilon \ker(\sigma_\hbar) . \tag{5.2.3}$$

Proof. We begin by showing that condition (5.2.3) is a sufficient criterion. By Lemma 5.2.1, $\tilde{\mathfrak{A}}_0 \otimes \tilde{\mathfrak{B}}_0$ are a dense Poisson $*$-subalgebra of $\mathfrak{A}_0\hat{\otimes}_\epsilon\mathfrak{B}_0$. Furthermore, if condition (5.2.3) is satisfied then by Theorem 5.1.6 the bundle $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is continuous.

Now we check that the quantization map $Q_\hbar := Q_\hbar^{\mathfrak{A}} \otimes Q_\hbar^{\mathfrak{B}}$ satisfies properties (i)-(iv) in Definition 2.1.3. By linearity of $Q_\hbar$ it suffices to check this on elementary tensors.

(i) $Q_0 = Q_0^{\mathfrak{A}} \otimes Q_0^{\mathfrak{B}}$ is the inclusion map and $Q_\hbar(1_{\mathfrak{A}_0\otimes\mathfrak{B}_0}) = 1_{\mathfrak{A}_\hbar} \otimes 1_{\mathfrak{B}_\hbar}$ which is the unit of $\mathfrak{A}_\hbar\hat{\otimes}_\epsilon\mathfrak{B}_\hbar$.

(ii) For every $f \otimes g \in \tilde{\mathfrak{A}}_0 \otimes \tilde{\mathfrak{B}}_0$ we have

$$Q_\hbar((f \otimes g)^*) = Q_\hbar^{\mathfrak{A}}\otimes Q_\hbar^{\mathfrak{B}}(f^*\otimes g^*) = Q_\hbar^{\mathfrak{A}}(f^*)\otimes Q_\hbar^{\mathfrak{B}}(g^*) = Q_\hbar^{\mathfrak{A}}(f)^*\otimes Q_\hbar^{\mathfrak{B}}(g)^* = Q_\hbar^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}(f\otimes g)^* ,$$

where we used the fact that $Q_\hbar^{\mathfrak{A}}$ and $Q_\hbar^{\mathfrak{B}}$ are quantization maps.

(iii) Since $Q_\hbar^{\mathfrak{A}}(f)$ and $Q_\hbar^{\mathfrak{B}}(g)$ are continuous section of $\mathfrak{A}_\hbar$ and $\mathfrak{B}_\hbar$ respectively for any $f \in \tilde{\mathfrak{A}}_0$ and $g \in \tilde{\mathfrak{B}}_0$, then the map

$$\begin{aligned} &0 \mapsto f \otimes g; \\ &\hbar \mapsto Q_\hbar(f \otimes g) = Q_\hbar^{\mathfrak{A}}(f) \otimes Q_\hbar^{\mathfrak{B}}(g), \quad (\hbar \in (I \setminus \{0\})) \end{aligned}$$

is a continuous section of $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ by construction. Indeed, the following function is continuous:

$$\hbar \mapsto \|\pi_\hbar(Q_\hbar^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}(f \otimes g))\|_{\hbar,\epsilon} = \|\pi_\hbar(Q_\hbar^{\mathfrak{A}}(f))\|_\hbar \, \|\pi_\hbar(Q_\hbar^{\mathfrak{B}}(g))\|_\hbar .$$

(iv) Each pair $f_1 \otimes g_1, f_2 \otimes g_2 \in \tilde{\mathfrak{A}}_0 \otimes \tilde{\mathfrak{B}}_0$ one has

$$\begin{aligned} [Q_\hbar(f_1 \otimes g_1), Q_\hbar(f_2 \otimes g_2)] =&[Q_\hbar^{\mathfrak{A}}(f_1) \otimes Q_\hbar^{\mathfrak{B}}(g_1), Q_\hbar^{\mathfrak{A}}(f_2) \otimes Q_\hbar^{\mathfrak{B}}(g_2)] = \\ =&[Q_\hbar^{\mathfrak{A}}(f_1), Q_\hbar^{\mathfrak{A}}(f_2)] \otimes Q_\hbar^{\mathfrak{B}}(g_1)Q_\hbar^{\mathfrak{B}}(g_2) \\ &+ Q_\hbar^{\mathfrak{A}}(f_2)Q_\hbar^{\mathfrak{A}}(f_1) \otimes [Q_\hbar^{\mathfrak{B}}(g_1), Q_\hbar^{\mathfrak{B}}(g_2)] \end{aligned}$$

and

$$\begin{aligned} Q_\hbar(\{f_1 \otimes g_1, f_2 \otimes g_2\}_\otimes) =&Q_\hbar(\{f_1, f_2\}_{\mathfrak{A}} \otimes g_1g_2 + f_1f_2 \otimes \{g_1, g_2\}_{\mathfrak{B}}) = \\ =&Q_\hbar^{\mathfrak{A}}(\{f_1, f_2\}_{\mathfrak{A}}) \otimes Q_\hbar^{\mathfrak{B}}(g_1g_2) + Q_\hbar^{\mathfrak{A}}(f_1f_2) \otimes Q_\hbar^{\mathfrak{B}}(\{g_1, g_2\}_{\mathfrak{B}}) \end{aligned}$$

where we used Equation (5.2.2) and $\{\cdot,\cdot\}_{\mathfrak{A}}$ (resp. $\{\cdot,\cdot\}_{\mathfrak{B}}$) denotes the Poisson bracket on $\tilde{\mathfrak{A}}_0$ (resp. $\tilde{\mathfrak{B}}_0$). It then follows

$$\begin{aligned}\|\frac{i}{\hbar}[Q_\hbar(f_1\otimes g_1),&Q_\hbar(f_2\otimes g_2)]-Q_\hbar(\{f_1\otimes g_1,f_2\otimes g_2\})\|_{\hbar,\epsilon}\\ &\leq \|\frac{i}{\hbar}[Q^{\mathfrak{A}}_\hbar(f_1),Q^{\mathfrak{A}}_\hbar(f_2)]\otimes Q^{\mathfrak{B}}_\hbar(g_1)Q^{\mathfrak{B}}_\hbar(g_2)-Q^{\mathfrak{A}}_\hbar(\{f_1,f_2\}_{\mathfrak{A}})\otimes Q^{\mathfrak{B}}_\hbar(g_1g_2)\|_{\hbar,\epsilon}\\ &+\|\frac{i}{\hbar}Q^{\mathfrak{A}}_\hbar(f_2)Q^{\mathfrak{A}}_\hbar(f_1)\otimes[Q^{\mathfrak{B}}_\hbar(g_1),Q^{\mathfrak{B}}_\hbar(g_2)]-Q^{\mathfrak{A}}_\hbar(f_1f_2)\otimes Q^{\mathfrak{B}}_\hbar(\{g_1,g_2\}_{\mathfrak{B}})\|_{\hbar,\epsilon}\,.\end{aligned}$$

The first term in the above inequality can be estimated as follows:

$$\begin{aligned}\lim_{\hbar\to0}\|\frac{i}{\hbar}[Q^{\mathfrak{A}}_\hbar(f_1),&Q^{\mathfrak{A}}_\hbar(f_2)]\otimes Q^{\mathfrak{B}}_\hbar(g_1)Q^{\mathfrak{B}}_\hbar(g_2)-Q^{\mathfrak{A}}_\hbar(\{f_1,f_2\}_{\mathfrak{A}})\otimes Q^{\mathfrak{B}}_\hbar(g_1g_2)\|_{\hbar,\epsilon}\\ &=\lim_{\hbar\to0}\Big\|\Big(\frac{i}{\hbar}[Q^{\mathfrak{A}}_\hbar(f_1),Q^{\mathfrak{A}}_\hbar(f_2)]-Q^{\mathfrak{A}}_\hbar(\{f_1,f_2\}_{\mathfrak{A}})\Big)\otimes Q^{\mathfrak{B}}_\hbar(g_1)Q^{\mathfrak{B}}_\hbar(g_2)\\ &\qquad -Q^{\mathfrak{A}}_\hbar(\{f_1,f_2\}_{\mathfrak{A}})\otimes\Big(Q^{\mathfrak{B}}_\hbar(g_1g_2)-Q^{\mathfrak{B}}_\hbar(g_1)Q^{\mathfrak{B}}_\hbar(g_2)\Big)\Big\|_{\hbar,\epsilon}\\ &\leq\lim_{\hbar\to0}\|\frac{i}{\hbar}[Q^{\mathfrak{A}}_\hbar(f_1),Q^{\mathfrak{A}}_\hbar(f_2)]-Q^{\mathfrak{A}}_\hbar(\{f_1,f_2\}_{\mathfrak{A}})\|_\hbar\|Q^{\mathfrak{B}}_\hbar(g_1)Q^{\mathfrak{B}}_\hbar(g_2)\|_\hbar\\ &\qquad +\|Q^{\mathfrak{A}}_\hbar(\{f_1,f_2\}_{\mathfrak{A}})\|_\hbar\|Q^{\mathfrak{B}}_\hbar(g_1g_2)-Q^{\mathfrak{B}}_\hbar(g_1)Q^{\mathfrak{B}}_\hbar(g_2)\|_\hbar\to0\end{aligned}$$

where we used Equation (5.1.1) together with

$$\lim_{\hbar\to0}\|Q_\hbar(f)\|_\hbar=\|f\|_0,\qquad\text{and}\qquad\lim_{\hbar\to0}\|Q_\hbar(f)Q_\hbar(g)-Q_\hbar(fg)\|_\hbar=0\,,$$

which follows from the definition of a continuous bundle of C^*-algebras. Using a similar argument we obtain

$$\lim_{\hbar\to0}\|\frac{i}{\hbar}Q^{\mathfrak{A}}_\hbar(f_2)Q^{\mathfrak{A}}_\hbar(f_1)\otimes[Q^{\mathfrak{B}}_\hbar(g_1),Q^{\mathfrak{B}}_\hbar(g_2)]-Q^{\mathfrak{A}}_\hbar(f_1f_2)\otimes Q^{\mathfrak{B}}_\hbar(\{g_1,g_2\}_{\mathfrak{B}})\|_{\hbar,\epsilon}\to0.$$

Since given two C^*-algebras, $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$ is the smallest C^*-algebra containing $\mathfrak{A}\otimes\mathfrak{B}$, it follows that $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is the smallest bundle of C^*-algebras containing $\mathcal{A}\otimes\mathcal{B}$. Therefore if there exists another tensor product $\otimes_C$ which makes $\mathfrak{A}\hat{\otimes}_C\mathfrak{B}$ a C^*-algebra, $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is contained in $\mathcal{A}\hat{\otimes}_C\mathcal{B}$. Since condition 5.2.3 is a sufficient and necessary condition to make $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ continuous (cf. Theorem 5.1.6), we can conclude. ☐

Remark 5.2.4. *It may be clear that whenever the deformation quantization of* $\tilde{\mathfrak{A}}_0$ *and* $\tilde{\mathfrak{B}}_0$ *is strict also the deformation quantization of* $\tilde{\mathfrak{A}}_0\otimes\tilde{\mathfrak{B}}_0$ *is strict.* ■

As explained in Section 5.1, given two continuous bundles of C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ over I, the injective tensor product $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is not continuous in general. However for $I=1/\mathbb{N}\cup\{0\}$, $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is a continuous bundle.

Corollary 5.2.5. *Assume the setup of Theorem 5.2.3. If* $I:=1/\mathbb{N}\cup\{0\}$ *then there always exists a deformation quantization of* $\tilde{\mathfrak{A}}_0\hat{\otimes}_\epsilon\tilde{\mathfrak{B}}_0$ *over* I.

Proof. We just need to check that $\mathcal{A}\hat{\otimes}_\epsilon\mathcal{B}$ is a continuous bundle of C^*-algebras. But this follows from the fact that any function is continuous on $1/\mathbb{N}$ and $\tilde{\mathfrak{A}}_0\hat{\otimes}_\epsilon\tilde{\mathfrak{B}}_0$ is a nuclear C^*-algebra (cf. Lemma 5.1.4). ☐

Corollary 5.2.6. *Let* X *and* Y *be Poisson manifold and assume there exists a deformation quantization of* $C_0(X)$ *and* $C_0(Y)$ *over* $I=1/\mathbb{N}\cup\{0\}$. *Then there exists a deformation quantization of* $X\times Y$ *over* I.

Proof. On account of Corollary 5.2.5, there exists a strict deformation quantization of $C_0(X)\hat{\otimes}_\epsilon C_0(Y)$ which is isomorphic to $C_0(X\times Y)$ by [73, Corollary B.17]. To conclude our proof is enough to endow $C_0(X\times Y)$ with the Poisson structure given by Corollary 5.2.2. ☐

5.3 Products of KMS states

The aim of this section is to show that given two KMS$_\beta$ states $\omega_{\mathfrak{A}}$ and $\omega_{\mathfrak{B}}$ for two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ respectively, there exists a KMS$_\beta$-state $\omega_{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}$ for $\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$. For sake of completeness let us recall the definitions of a C^*-dynamical system and a KMS$_\beta$ state.

Definition 5.3.1. *A C^*- dynamical system $(\mathfrak{A},\alpha)$ is a C^*-algebra $\mathfrak{A}$ equipped with a dynamical evolution, i.e., a one-parameter group of C^*-algebra automorphisms $\alpha := \{\alpha_t\}_{t\in\mathbb{R}}$ that is* strongly continuous *on $\mathfrak{A}$: the map $\mathbb{R}\ni t\mapsto \alpha_t(a)\in\mathfrak{A}$ is continuous for every $a\in\mathfrak{A}$.* ■

Definition 5.3.2. *Consider the C^*-dynamical system given by a C^*-algebra $\mathfrak{A}$ and a strongly continuous representation φ_t of $\mathbb{R}$ in the automorphism group of $\mathfrak{A}$. A linear functional $\omega : \mathfrak{A}\to\mathbb{C}$ is called a KMS$_\beta$-states if the following holds true:*

(1) it is positive, i.e. $\omega(a^*a)\geq 0$ for all $a\in\mathfrak{A}$;

(2) it is normalized, i.e. $\|\omega\| := \sup\{|\omega(a)| \mid \|a\| = 1,\ a\in\mathfrak{A}\} = 1$;

(3) it satisfies the KMS$_\beta$-condition: for all $a,b\in\mathfrak{A}$ there is a holomorphic function F_{ab} on the strip $S_\beta := \mathbb{R}\times i(0,\beta)\subset\mathbb{C}$ with a continuous extension to $\overline{S_\beta}$ such that

$$F_{ab}(t) = \omega(a\varphi_t(b)) \qquad \text{and} \qquad F_{ab}(t+i\beta) = \omega(\varphi_t(b)a)\,.$$

■

Theorem 5.3.3. *Let $\omega^{\mathfrak{A}}$ and $\omega^{\mathfrak{B}}$ be KMS$_\beta$-states for the C^*-dynamical systems $(\mathfrak{A},\varphi_{t_{\mathfrak{A}}},\mathbb{R})$ and $(\mathfrak{B},\phi_{t_{\mathfrak{B}}},\mathbb{R})$ respectively and denote with $\Phi_{t,s}$ an extension of $\varphi_t\otimes\phi_s$ to an automorphism of $\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$ such that*

$$\Phi_{t,s}(a\otimes b) = \varphi_t(a)\otimes\phi_s(b) \tag{5.3.4}$$

for any $a\otimes b\in\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$. Then there exists a KMS$_\beta$ state $\omega^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}$ for the C^*-dynamical system $(\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B},\Phi_{t,t},\mathbb{R})$ such that

$$\omega^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}(a\otimes b) = \omega^{\mathfrak{A}}(a)\,\omega^{\mathfrak{B}}(b)\,. \tag{5.3.5}$$

Remark 5.3.4. Before proving our claim, let us remark that the existence of $\Phi_{t,s}$ is guaranteed by [73, Proposition B13]. Furthermore, on account of [73, Corollary B12], the state $\omega^{\mathfrak{A}}\otimes\omega^{\mathfrak{B}}$ extends to a state $\omega^{\mathfrak{A}}\hat{\otimes}_\epsilon\omega^{\mathfrak{B}}$ on $\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$ which satisfies Equation (5.3.5). So to prove Theorem 5.3.3 it is enough to check that $\omega^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}$ satisfies the KMS$_\beta$ condition.

Let us also stress that this theorem can be proved using modular theory. ■

Proof of Theorem 5.3.3. We hereto denote by S_β the strip associated to the KMS$_\beta$-states $\omega^{\mathfrak{A}}$ and $\omega^{\mathfrak{B}}$, and by $F^{\mathfrak{A}} := F^{\mathfrak{A}}_{a_1,a_2}$ and $F^{\mathfrak{B}} := F^{\mathfrak{B}}_{b_1,b_2}$ the corresponding holomorphic functions for every $a_1,a_2\in\mathfrak{A}$, $b_1,b_2\in\mathfrak{B}$.

Consider now $c,d\in\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$. Since $\mathfrak{A}\otimes\mathfrak{B}$ is a dense $*$-sub algebra of $\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}$ there exist sequences of $c_i\in\mathfrak{A}\otimes\mathfrak{B}$ and $d_i\in\mathfrak{A}\otimes\mathfrak{B}$ which converge in the injective tensor norm to c and d respectively. In particular, we may write $c_i := \sum_{k_i} c_{k_i1}\otimes c_{k_i2}$ and $d_i := \sum_{l_i} d_{l_i1}\otimes d_{l_i2}$, with $c_{k_i1}\otimes c_{k_i2}, d_{j1}\otimes d_{j2}\in\mathfrak{A}\otimes\mathfrak{B}$. Using Equation (5.3.4) and (5.3.5) together with the linearity of $\omega^{\mathfrak{A}}$ and $\omega^{\mathfrak{B}}$, for any $t,s\in S_\beta$ it holds

$$\omega^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}(d_i\Phi_{t,s}(c_i)) = \Sigma_{k_il_i}\omega^{\mathfrak{A}}(d_{l_i1}\varphi_t(c_{k_i1}))\omega^{\mathfrak{B}}(d_{l_i2}\phi_s(c_{k_i2})).$$

Since $\omega^{\mathfrak{A}}$ and $\omega^{\mathfrak{B}}$ are β-KMS states, it follows that

$$\omega^{\mathfrak{A}\hat{\otimes}_\epsilon\mathfrak{B}}(d_i\Phi_{t,s}(c_i)) = \Sigma_{k_il_i}F^{\mathfrak{A}}_{d_{l_i1},c_{k_i1}}(t)F^{\mathfrak{B}}_{d_{l_i2},c_{k_i2}}(s),$$

where $F^{\mathfrak{A}}_{d_{l_i1},c_{k_i1}}$ and $F^{\mathfrak{B}}_{d_{l_i2},c_{k_i2}}$ are holomorphic functions for any k,l such that $F^{\mathfrak{A}}_{d_{l_i1},c_{k_i1}}$ and $F^{\mathfrak{B}}_{d_{l_i2},c_{k_i2}}$ are analytic on S_β, continuous and bounded on $\bar{S}_\beta$. Since for any i the sums in k_i and l_i are finite, and the product and sum of two analytic functions remains analytic, the above expression extends to a holomorphic function F_{d_i,c_i} analytic on $S_\beta\times S_\beta$, and bounded and continuous on the closure $\bar{S}_\beta\times\bar{S}_\beta$. This yields a sequence of holomorphic functions $F_i := F_{d_i,c_i}$

analytic on $S_\beta \times S_\beta$, and bounded and continuous on the closure $\bar{S}_\beta \times \bar{S}_\beta$. Moreover, we claim that the sequence $(F_i)_i$ converges uniformly on the boundary of $S_\beta \times S_\beta$ to some function. To verify our claim it suffices to check this for $\mathbb{R} \times \mathbb{R}$. Hereto we take $t \times s \in \mathbb{R} \times \mathbb{R}$ and compute

$$\lim_i |\omega^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}(d\Phi_{t,s}(c)) - \omega^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}(d_i\Phi_{t,s}(c_i)|^2 \leq \lim_i ||c - c_i||^2 + ||d - d_i||^2 = 0,$$

where we used that $\omega^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}$ is a state and that c_i and d_i converge to c and d, respectively. Since the limit does not depend on $t \times s$ the convergence is uniform. As a result of [20, Prop. 5.3.5] the functions F_i satisfy

$$\sup_{z \in \bar{S}_\beta \times \bar{S}_\beta} |F_i(z)| = \sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} |F_i(t,s)|.$$

It follows that

$$\sup_{z \in \bar{S}_\beta \times \bar{S}_\beta} |F_i(z) - F_j(z)| = \sup_{(t,s) \in \mathbb{R} \times \mathbb{R}} |F_i(t,s) - F_j(t,s)|. \tag{5.3.6}$$

Since (F_i) converges uniformly on the boundary of $S_\beta \times S_\beta$ to some function, in particular the sequence (F_i) is uniformly Cauchy on the boundary. Hence, the right hand side of (5.3.6) tends to zero as $i, j \to \infty$. This implies that (F_i) is uniformly Cauchy on $\bar{S}_\beta \times \bar{S}_\beta$ and hence the sequence (F_i) converges uniformly to some continuous function $F := F_{d,c}$ on $\bar{S}_\beta \times \bar{S}_\beta$. In particular, the sequence (F_i) also converges uniformly to F on every compact subset of $S_\beta \times S_\beta$, so F is analytic on $S_\beta \times S_\beta$ by [30, Proposition 3]. We conclude that the limiting function F is analytic on $S_\beta \times S_\beta$ and continuous and bounded on $\bar{S}_\beta \times \bar{S}_\beta$. Restricting to the diagonal, i.e. $t = s$, this function satisfies

$$F_{d,c}(t) = \omega^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}(d\Phi_{t,t}(c)).$$

By a similar argument as above one can show that it holds also

$$F_{d,c}(t + i\beta) = \omega^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}(\Phi_{t,t}(c)d)\,.$$

This conclude our proof. □

As a direct consequence of Theorem 5.2.3 and Theorem 5.3.3 we get the following result.

Corollary 5.3.5. *Assume the setup of Theorem 5.2.3 and Theorem 5.3.3. Let $\omega_\hbar^{\mathfrak{A}}$ and $\omega_\hbar^{\mathfrak{B}}$ be a sequence of (KMS$_\beta$-)states for $\mathfrak{A}_\hbar := \pi_\hbar(\mathfrak{A})$ and $\mathfrak{B}_\hbar := \pi_\hbar(\mathfrak{B})$. If $\omega_\hbar^{\mathfrak{A}}$ and $\omega_\hbar^{\mathfrak{B}}$ admit a classical limit, i.e. for every $f \in \tilde{\mathfrak{A}}_0$ and $g \in \tilde{\mathfrak{B}}_0$ there exist the limits*

$$\omega_0^{\mathfrak{A}}(f) = \lim_{\hbar \to 0} \omega_\hbar^{\mathfrak{A}}(Q_\hbar^{\mathfrak{A}}(f)) \qquad \textit{and} \qquad \omega_0^{\mathfrak{B}}(g) = \lim_{\hbar \to 0} \omega_\hbar^{\mathfrak{B}}(Q_\hbar^{\mathfrak{B}}(g))$$

then the sequence of (KMS$_\beta$-)state $\omega_\hbar^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}$ has a classical limit given by

$$\omega_0^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}(f \otimes g) = \lim_{\hbar \to 0} \omega_\hbar^{\mathfrak{A}\hat{\otimes}_\epsilon \mathfrak{B}}(Q_\hbar(f \otimes g))\,.$$

■

5.4 Examples

The above machinery will be applied to quantum spin systems and many-body particle systems in the context of Schrödinger operators.

5.4.1 Spin systems

In this section we show how quantum spin systems arise from classical spin systems starting from the quantization formalism defined in Section 2.2.2. Indeed, a single sphere S^2 is quantized onto the matrix algebra $\mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2)) \cong M_{N+1}(\mathbb{C})$ using quantization maps $Q'_{1/N}$ defined by (2.2.40). As notice first by Lieb in [60], and independently in [98], certain spin operators can be obtained by applying the quantization map $Q'_{1/N}$ to polynomials. To this end, we define $J := N/2$, the **spin quantum number** (or simply the **spin**), and consider the maps $Q^{(1)}_{1/J} := Q'_{1/N}$. It can be shown that (see also Chapter 8 and Table 8.1 for a general discussion)

$$\begin{aligned} (J+1)\cos(\theta) &\mapsto S_z \\ (J+1)\sin(\theta)\cos(\phi) &\mapsto S_x \\ (J+1)\sin(\theta)\sin(\phi) &\mapsto S_y \end{aligned} \tag{5.4.7}$$

where (θ, ϕ) (resp (x, y, z)) are spherical (resp. cartesian) coordinates on S^2 and S_x, S_y, S_z can be understood as a (unitary finite dimensional) irreducible representation of the Lie algebra $\mathfrak{su}(2)$ on the Hilbert space $\mathbb{C}^{2J+1}$. Furthermore these operators satisfy $[S_x, S_y] = iS_z$ cyclically. We refer to Section 8.1.2 for a more detailed discussion.

A general classical spin system is typically defined as a polynomial on the cartesian product of say d spheres S^2, denoted by $\times_d S^2$, where d indicates the number of classical spins. Therefore the classical algebra on which classical spin systems are defined is $C(\times_d S^2)$ or equivalently $C(S^2)^{\otimes_\epsilon d}$ (see Corollary 5.1.3). As a byproduct of Theorem 5.2.3, the quantization maps are given by linear extension of the following map

$$\begin{aligned} &Q^{(d)}_{1/J} : \tilde{\mathfrak{A}}_0^{\otimes_\epsilon d} \to \underbrace{M_{2J+1}(\mathbb{C}) \otimes \cdots \otimes M_{2J+1}(\mathbb{C})}_{d\ times}; \\ &Q^{(d)}_{1/J}(f_1, ..., f_d) = \underbrace{Q^{(1)}_{1/J}(f_1) \otimes \cdots \otimes Q^{(1)}_{1/J}(f_d)}_{d\ times}, \end{aligned} \tag{5.4.8}$$

where $Q^{(1)}_{1/J}$ is given by (2.2.40), and $\tilde{\mathfrak{A}}_0$ the dense subalgebra of $C(S^2)$ given by polynomials in three real variables restricted to the sphere S^2. Keeping this in mind, we now provide two illustrating examples where quantization theory and spin systems come together.

The Ising model

We consider the *classical Ising model* in a transverse magnetic field B. The corresponding function $h^{Is} \in C(\times_d S^2)$ is defined by

$$h^{Is}(e_1, ..., e_d) = -\sum_{j=1}^{d-1} z_j z_{j+1} - B\sum_{j=1}^{d} x_j, \quad (e_j = (x_j, y_j, z_j) \in S^2, \quad j = 1, ..., d).$$

Employing the identification $C(\times_d S^2) \simeq C(S^2)^{\otimes_\epsilon d}$, we obtain

$$h^{Is} := -\sum_{j=1}^{d-1} h_{z_j} \otimes h_{z_{j+1}} \otimes 1_{S^2} \otimes \cdots \otimes 1_{S^2} - B\sum_{j=1}^{d} h_{x_j} \otimes 1_{S^2} \otimes \cdots \otimes 1_{S^2},$$

where each $h_z, h_x \in C(S^2)$ are given respectively by $h_z(e_j) = z_j$ and $h_x(e_j) = x_j$ for all $j = 1, ..., d$.

In view of (5.4.7), we see that the coordinate function $(J+1)x$ is mapped to S_x, $(J+1)y$ is mapped to S_y, and $(J+1)z$ is mapped to S_z. Analogously to the work done in [60] let us now replace these coordinates e_j by $(J+1)e_j$. We then apply our quantization maps (5.4.8) to this function. It not difficult to see that this image yields the following operator

$$H_d^{Is} = -\sum_{j=1}^{d-1} S_z(j)S_z(j+1) - B\sum_{j=1}^{d} S_x(j),$$

where the operators $S_x(j)$ and $S_z(j)$ act as the operators S_x and S_z on $\mathcal{H}_j = \mathbb{C}^{2J+1}$ and as the unit matrix 1_{2J+1} elsewhere. This operator exactly corresponds to the *quantum Ising model* of

d immobile spin particles each with total angular momentum J under a ferromagnetic coupling, defined on the Hilbert space $\mathcal{H}^d = \bigotimes_{j=1}^d \mathcal{H}_j$, with $\mathcal{H}_j = \mathbb{C}^{2J+1}$. Hence,

$$Q_{1/J}^{(d)}(h_J^{Is}) = H_d^{Is},$$

where h_J^{Is} is defined on the scaled vectors $(J+1)e_j$. Note that the operator H_d^{Is} clearly depends on J since it is defined on the Hilbert space $\mathcal{H}^d = \bigotimes_{j=1}^d \mathbb{C}^{2J+1}$.

This shows the interplay between, on the one hand classical spin systems (also called symbols) on a product of spheres, and on the other hand the quantum Hamiltonian describing the quantum Ising model.

The Heisenberg model

We consider the classical Heisenberg spin model h^{Hei} on $\times_d S^2$ defined by

$$h^{Hei}(e_1, ..., e_d) := -\sum_{j=1}^{d-1} x_j x_{j+1} + y_j y_{j+1} + z_j z_{j+1}.$$

Applying the quantization maps (5.4.8) to h^{Hei} we obtain by a similar argument as in the previous example $Q_{1/J}^{(d)}(h_J^{Hei}) = H_d^{Hei}$, where the operator H_d^{Hei} denotes the *quantum Heisenberg model* on the Hilbert space $\mathcal{H}^d = \bigotimes_{j=1}^d \mathbb{C}^{2J+1}$,

$$H_d^{Hei} = -\sum_{j=1}^{d-1} S_j \cdot S_{j+1},$$

with each of the operators in $S_j = (S_j^x, S_j^y, S_j^z)$ acting on the Hilbert space $\mathcal{H}_J = \mathbb{C}^{2J+1}$ and as the identity elsewhere. As before, note that the function h_J^{Hei} is defined on the vectors $(J+1)e_j$.

Remark 5.4.1. As a result of the properties of the continuous bundle of C^*-algebras in all these examples it may be clear that in the classical limit $J \to \infty$ the norm of the quantum Hamiltonians correspond to the supremum norm of the corresponding classical functions, in the sense that

$$\lim_{J\to\infty} \|H_d^{Quantum}\|_J = \|h_d^{classical}\|_\infty$$

Of course, in view of the correspondence between functions and operators (viz. (5.4.7)), one should normalize the operators S_x, S_y, S_z appearing in the quantum Hamiltonians by a factor $1/(J+1)$ in order to make the above limit existing. ∎

In conclusion, we would like to emphasize that the strict deformation quantization of the d-fold tensor product of S^2 with itself provides a new perspective in order to study the thermodynamic limit (i.e. $d \to \infty$) and classical limit (i.e. $J \to \infty$) of the spin system of interest. The properties of the quantization maps can be extremely useful in order to study the above mentioned limits of for example the free energy, the possible convergence of Gibbs states, or for (algebraic) ground states induced by eigenvectors [60, 98]. Indeed, in a slightly different context Lieb [60] implicitly used the properties of the quantization maps (2.2.40) and (5.4.8) in order to prove the existence of such limits.

5.4.2 The resolvent algebra and Schrödinger operators

In this section we show that the resolvent of Schödinger operators for non-interacting particle systems can be given in terms of an integral of the tensor product of quantization maps. To achieve our goal, we shall benefit from [23, 70].

Let (X, σ) be a symplectic vector space admitting a complex structure and denote be $C_{\mathcal{R}}(X)$ the commutative C^*-algebra of functions on (X, σ). Similar to the case of the (non-commutative) resolvent algebra $\mathfrak{R}(X, \sigma)$ of Buchholz and Grundling ([23]), the algebra $C_{\mathcal{R}}(X)$

is the C^*-subalgebra of $C_b(X)$ (the algebra of continuous functions on X that are bounded with respect to the supremum norm) generated by the functions

$$h_x^\lambda(y) = 1/(i\lambda - x \cdot y),$$

for $x \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. The inner product $\cdot$ gives rise to a norm $||\cdot||$ and a topology (the standard ones for real pre-Hilbert spaces X), making h_x^λ a continuous function. We now define the space $\mathcal{S}_\mathcal{R}(X) \subset C_\mathcal{R}(X)$ consisting of so-called levees $g \circ p_x$

$$\mathcal{S}_\mathcal{R}(X) = \text{span}\{g \circ p_x \text{ levee} \mid g \in \mathcal{S}(\text{ran}(P)\},$$

where a levee $f : X \to \mathbb{C}$ is a composition $f = g \circ P$ of some finite dimensional projection P and some function $g \in C_0(\text{ran}(P))$, or in view of Weyl quantization, $g \in \mathcal{S}(\text{ran}(P))$. As shown in [70, Proposition 2.4] $\mathcal{S}_\mathcal{R}(X)$ is a dense $*$-Poisson subalgebra of $C_\mathcal{R}(X)$.

Now let us denote the resolvent algebra by $\mathfrak{R}(X, \sigma)$. This is the C^*-subalgebra of $\mathfrak{B}(\mathcal{F}(\bar{X}))$ generated by the resolvents $R(\lambda, x) := (i\lambda - \varphi(x))^{-1}$ for $\lambda \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}$, where $\mathcal{F}(\bar{X})$ denoted the bosonic Fock space (symmetric Hilbert space) of the completion of X with respect to its complex inner product. It can be shown that the fibers $\mathfrak{A}_0 := C_\mathcal{R}(X)$ ($\hbar = 0$) and the constant fiber $\mathfrak{A}_\hbar = \mathfrak{R}(X, \sigma)$ above $\hbar \neq 0$ entail a continuous bundle of C^*-algebras over $I = [0, \infty)$. In addition, in [70, Theorem 3.7] van Nuland showed that there exists a strict deformation quantization of the commutative resolvent algebra $\mathfrak{A}_0 = C_\mathcal{R}(X)$ over base space I with non-zero fibers given by the (non-commutative) resolvent algebra $\mathfrak{A}_\hbar = \mathfrak{R}(X, \sigma)$. The corresponding quantization maps (denoted by $Q_\hbar^W$) are defined in terms of Weyl-quantization on the dense Poisson $*$-subalgebra $\mathcal{S}_\mathcal{R}(X) \subset C_\mathcal{R}(X) = \mathfrak{A}_0$. Furthermore, it can be shown that these maps are surjective.

Since $C_\mathcal{R}(X)$ and the resolvent algebra $\mathfrak{R}(X, \sigma)$ are nuclear C^*-algebras (see e.g. [24, Proposition 3.4]), there exists a strict deformation quantization of $C_\mathcal{R}(X) \otimes C_\mathcal{R}(X)$ (cf. Theorem 5.2.3). In particular, the quantization maps are defined on the dense Poisson algebra $\mathcal{S}_\mathcal{R}(X) \otimes \mathcal{S}_\mathcal{R}(X) \subset \mathfrak{A}_0 \times \mathfrak{A}_0$.

Schrödinger operators affiliated with the resolvent algebra

From now on, we set $X = \mathbb{R}^2$ with its standard symplectic form σ and work in the Schrödinger representation π_0 of $\mathfrak{R}(\mathbb{R}^2, \sigma)$. We denote by Q, P the canonical position and momentum operators in the Schrödinger representation. Let $H = H(P, Q)$ be a self-adjoint operator. When its resolvent is contained in $\pi_0(\mathfrak{R}(\mathbb{R}^2, \sigma))$ we may consider its preimage

$$\tilde{R}_H(\lambda) = \pi_0^{-1}((i\lambda - H)^{-1}), \quad (\lambda \in \mathbb{R} \setminus \{0\}), \tag{5.4.9}$$

as long as λ is not in the spectrum of H. We then say that H is **affiliated** with $\mathfrak{R}(\mathbb{R}^2, \sigma)$. Since $\mathbb{R}^2$ is finite dimensional, Equation (5.4.9) holds for Schrödinger operators with compact resolvent or for Schrödinger operators with potential $V \in C_0(\mathbb{R})$ [23, Proposition 6.2].

Many particle systems

We consider ($\hbar$-dependent) Schrödinger operators H_i ($i = 1, ..., N$) each densely defined on some Hilbert space $\mathcal{H}_i$ and affiliated with $\mathfrak{R}(\mathbb{R}^2, \sigma)$. We then consider the tensor product of these operators

$$H := H_1 \otimes 1_2 \otimes \cdots \otimes 1_N + 1_1 \otimes H_2 \otimes \cdots \otimes 1_N + ... + 1_1 \otimes 1_2 \otimes \cdots \otimes H_N, \tag{5.4.10}$$

where 1_i denotes the identity operator on $\mathcal{H}_i$ for $i = 1, ..., N$. One can extend the operator H to a densely defined self-adjoint operator on $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$. By construction, the operators H_i now viewed as operators on $\mathcal{H}$ commute. The operator H therefore describes a system of N non-interacting particles. To simplify matters, let us restrict to the case when $N = 2$ and let us assume that the spectra of H_1 and H_2 are bounded from below. It can then be shown that the resolvent of H is given as a (operator valued) function of H_2 in terms of a Dunford integral [59], using the fact that $R_1 = 1_1 \otimes R_2$ obviously commutes with $R_2 = R_1 \otimes 1_2$. Concretely, this means that for any λ in the set $\rho(H)\bigcap_{i=1}^{2}\rho(H_i)$ (where ρ denotes the resolvent), we have

$$R_H(\lambda) = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\Gamma_k} dz(z+\lambda+H_1)^{-1}(z-H_1)^{-1}, \tag{5.4.11}$$

where Γ_k is a suitable contour crossing the real axis in some point $x_k\in\mathbb{R}$ where x_k increasing towards infinity as $k\to\infty$. We can rewrite (5.4.11) as

$$R_H(\lambda) = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\Gamma_k} dzR_1(z+\lambda)R_2(z),$$

where R_1 and R_2 denote the resolvent of $-H_1$ and H_2, respectively. Since each of them is affiliated with $\mathfrak{R}(\mathbb{R}^2,\sigma)$ we can consider their preimages under π_0 which we denote by $\tilde{R}_1$ and $\tilde{R}_2$. Since π_0 is a faithful representation we obtain

$$\tilde{R}_H(\lambda) = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\Gamma_k} dz\tilde{R}_1(z+\lambda)\tilde{R}_2(z).$$

The previous results in this section now imply the existence of two functions $f_1^{z+\lambda}, f_2^{z}\in\mathcal{C}_{\mathcal{R}}(\mathbb{R}^2)$ such that

$$\tilde{R}_1(z+\lambda) = Q_\hbar^W(f_1^{z+\lambda})\otimes 1_2;$$
$$\tilde{R}_2(z) = 1_1\otimes Q_\hbar^W(f_2^{z}).$$

Combining the above results yields

$$\tilde{R}_H(\lambda) = \lim_{k\to\infty}\frac{1}{2\pi i}\int_{\Gamma_k} dzQ_\hbar^W(f_1^{z+\lambda})\otimes Q_\hbar^W(f_2^{z}).$$

This implies that the resolvent of Schödinger operators for non-interacting particle system (as defined above) can be given in terms of an integral of the tensor product of quantization maps, quantizing functions in the commutative resolvent algebra.

6. Semi-classical properties of Berezin quantization maps

In this chapter we investigate the semi-classical properties of Berezin quantization maps. Unless specified differently, we hereto assume we are given a coherent pure state quantization $(\mathcal{H}_\hbar, \Psi_\hbar^\sigma, \mu_\hbar)_{\hbar \in I, \sigma \in S}$ of a symplectic manifold (S, ω_S). We denote by $Q_\hbar^B$ the associated Berezin quantization map and consider the rescaled measure $\mu_\hbar = c(\hbar)\mu_L$ and Hilbert space $L^2(S, d\mu_\hbar)$ (viz. Definition 2.2.2 and (2.2.27), or [55, Prop. II 1.5.2] for a general construction). We moreover assume that these maps satisfy the von Neumann (2.1.2) and Rieffel condition (2.1.1) for elements in the C^*-algebra $C_0(S)$, as typically happens in the additional case of a deformation quantization of S.

6.1 Spectral asymptotics for Berezin quantization maps

We first study the semi-classical behavior of the spectrum of Berezin quantization maps. This yields the main result of this section relating the spectrum of Berezin quantization to the range of the function that is quantized. We start with a definition.

Definition 6.1.1. *The* **distance** *between a bounded set $X \subset \mathbb{C}$ and a nonempty set $Y \subset \mathbb{C}$ is defined by*

$$dist(X, Y) = \sup_{x \in X} \inf_{y \in Y} |x - y|. \tag{6.1.1}$$

■

Using this definition we have the following theorem.

Theorem 6.1.2. *Given a coherent pure state quantization $(\mathcal{H}_\hbar, \Psi_\hbar^\sigma, \mu_\hbar)_{\hbar \in I, \sigma \in S}$ of a symplectic manifold (S, ω_S). Assume that the associated Berezin quantization maps, denoted by $Q_\hbar^B$, satisfy the von Neumann and Rieffel condition on $C_0(S)$. Then, for any $f \in C_c(S)$ one has*

$$\lim_{\hbar \to 0} dist\Big(ran(f), \sigma(Q_\hbar^B(f))\Big) = 0, \tag{6.1.2}$$

where $\sigma(Q_\hbar^B(f))$ denotes the spectrum of the operator $Q_\hbar^B(f)$, and dist is the distance function defined in Definition 6.1.1.

Proof. Let us assume by contradiction that the statement in the theorem is not true. Then, there exists $\delta > 0$, a function $f \in C_c(S)$, and a sequence $(\lambda_{\hbar_k})_k$ in ran(f) such that dist$(\lambda_{\hbar_k}, \sigma(Q_{\hbar_k}^B(f))) \geq \delta > 0$ for all k. Since supp(f) is compact (as f is continuous) also ran(f) is compact so that we can extract a subsequence $\lambda_{\hbar_{k'}}$ converging to a point $r \in$ ran(f). Hence, for all $\epsilon > 0$ there exists a K_ϵ such that $|r - \lambda_{\hbar_{k'}}| < \epsilon$ for all $k' \geq K_\epsilon$. This implies that $r \notin \sigma(Q_{\hbar_{k'}}(f))$ for $k' \geq K_\epsilon$, which means that the resolvent operator associated to r and denoted by $R_r(Q_{\hbar_{k'}}^B(f))$ exists for all $k' \geq K_\epsilon$.

Now, we can find an element $\sigma \in S$ such that $f(\sigma) = r$. By property (2) of Definition 2.2.2, we can always recover $f(\sigma)$ as

$$f(\sigma) = \lim_{\hbar \to 0} \langle \Psi_\hbar^\sigma, Q_\hbar^B(f) \Psi_\hbar^\sigma \rangle, \tag{6.1.3}$$

where $\Psi_\hbar^\sigma$ denotes the coherent state vector induced by the point $\sigma \in S$ on which f is defined.

For this σ and $k' \geq K_\epsilon$ let us now estimate

$$1 = |\langle \Psi^\sigma_{\hbar_{k'}}, R_r(Q^B_{\hbar_{k'}}(f))Q_{\hbar_{k'}}(f-r)\Psi^\sigma_{\hbar_{k'}}\rangle|^2 \leq$$
$$||R_r(Q^B_{\hbar_{k'}}(f))||^2 \cdot ||Q^B_{\hbar_{k'}}(f-r)\Psi^\sigma_{\hbar_{k'}}||^2, \tag{6.1.4}$$

using in the first equality the fact that $Q^B_{\hbar_{k'}}(f-r) = Q^B_{\hbar_{k'}}(f) - Q^B_{\hbar_{k'}}(1_X)r = Q^B_{\hbar_{k'}}(f) - r$. We now make the following estimation on

$$||Q^B_{\hbar_{k'}}(f-r)\Psi^\sigma_{\hbar_{k'}}||^2 = \langle \Psi^\sigma_{\hbar_{k'}}, (Q^B_{\hbar_{k'}}(f-r))^* Q^B_{\hbar_{k'}}(f-r)\Psi^\sigma_{\hbar_{k'}}\rangle =$$
$$\langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}((f-r)^*)Q^B_{\hbar_{k'}}(f-r)\Psi^\sigma_{\hbar_{k'}}\rangle, \tag{6.1.5}$$

where we used that the quantization maps preserve the adjoint. Then,

$$\left|\langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}((f-r)^*)Q^B_{\hbar_{k'}}(f-r)\Psi^\sigma_{\hbar_{k'}}\rangle - \langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}(|f-r|^2)\Psi^\sigma_{\hbar_{k'}}\rangle\right| \leq$$
$$\left|\left|Q^B_{\hbar_{k'}}((f-r)^*)Q^B_{\hbar_{k'}}(f-r) - Q^B_{\hbar_{k'}}(|f-r|^2)\right|\right|, \tag{6.1.6}$$

using the Cauchy-Schwarz inequality and the fact that $|\Psi^{(\sigma)}_{\hbar_{k'}}\rangle$ are unit vectors. As a result of the von Neumann condition (cf. (2.1.2)) we conclude that the above inequality uniformly converges to zero as $k' \to \infty$. This together with (6.1.4) and (6.1.5) implies that

$$1 \leq ||R_r(Q^B_{\hbar_{k'}}(f))||^2 \bigg(\langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}((f-r)^*)Q^B_{\hbar_{k'}}(f-r)|\Psi^\sigma_{\hbar_{k'}}\rangle - \langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}(|f-r|^2)\Psi_{\hbar_{k'}}r^\sigma\rangle\bigg)$$
$$+ ||R_r(Q^B_{\hbar_{k'}}(f))||^2 \langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}(|f-r|^2)\Psi^\sigma_{\hbar_{k'}}\rangle \leq$$
$$||R_r(Q^B_{\hbar_{k'}}(f))||^2 \cdot \left|\left|Q^B_{\hbar_{k'}}((f-r)^*)Q^B_{\hbar_{k'}}(f-r) - Q^B_{\hbar_{k'}}(|f-r|^2)\right|\right|$$
$$+ ||R_r(Q^B_{\hbar_{k'}}(f))||^2 \langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}(|f-r|^2)\Psi^\sigma_{\hbar_{k'}}\rangle.$$

By (6.1.3) it follows that $\lim_{k'\to\infty}\langle \Psi^\sigma_{\hbar_{k'}}, Q^B_{\hbar_{k'}}(|f-r|^2)\Psi^\sigma_{\hbar_{k'}}\rangle = |f(\Omega) - r|^2 = 0$. Since also $\lim_{k'\to\infty} ||Q^B_{\hbar_{k'}}((f-r)^*)Q^B_{\hbar_{k'}}(f-r) - Q^B_{\hbar_{k'}}(|f-r|^2)|| = 0$, it must follow that $||R_r(Q^B_{\hbar_{k'}}(f))||^2 \to \infty$ as $k' \to \infty$, which also implies that $||R_r(Q^B_{\hbar_{k'}}(f))|| \to \infty$ as $k' \to \infty$. In order to conclude we recall that that

$$||R_r(Q^B_{\hbar_{k'}}(f))||_{\hbar_{k'}} \leq \frac{1}{\text{dist}(r, \sigma(Q^B_{\hbar_{k'}}(f))}.$$

Combining the above inequalities yields for k' large enough the final inequality

$$0 < \delta \leq \text{dist}(\lambda_{\hbar_{k'}}, \sigma(Q^B_{\hbar_{k'}}(f))) \leq \frac{1}{||R_r(Q^B_{\hbar_{k'}}(f))||_{\hbar_{k'}}}. \tag{6.1.7}$$

By taking the limit $k' \to \infty$ of the above inequality we clearly arrive at a contradiction, since the right-hand side converges to zero. This proves the theorem. □

6.2 Classical limit for Berezin quantization maps

In this section we introduce the notion of the *classical limit*. In order to do so, let us denote by $H_\hbar$ some ($\hbar$-dependent) Hamiltonian encoding a quantum theory on some Hilbert space $\mathcal{H}_\hbar$, and by $\{\psi_\hbar\}_{\hbar>0}$ a sequence of corresponding normalized eigenvectors of $\{H_\hbar\}_{\hbar>0}$, of course, assuming they exist.[1] We want to investigate what happens to these eigenvectors in the regime $\hbar \to 0$. It is a hard problem to capture the behavior of the sequence $\{\psi_\hbar\}_{\hbar>0}$ in $\mathcal{H}_\hbar$, and in

[1]As already mentioned in the introduction, the parameter $\hbar$ might correspond to Planck's constant occurring in Schrödinger operators but it can for example also indicate the number of particles N described by a quantum spin system. The common idea is that the limit should reproduce a classical theory.

general it has not even a limit herein.[2]

However in a C^*-algebraic setting things become more smooth. The most common way is to consider the corresponding algebraic vector states $\{\omega_\hbar\}_{\hbar>0}$ on $\mathfrak{B}(\mathcal{H}_\hbar)$, defined by

$$\omega_\hbar(\cdot) = \langle\psi_\hbar, (\cdot)\psi_\hbar\rangle. \tag{6.2.8}$$

A natural question is to find a suitable set of physical observables for which the sequence of states $\{\omega_\hbar\}_{\hbar>0}$ converge (in some topology) to a state on a certain commutative algebra, establishing then a link between the quantum and classical theory. If it does, then the limit is also called the **classical limit** of the sequence of eigenvectors $\{\psi_\hbar\}_{\hbar>0}$. This strongly depends on this sequence which behavior is in general unknown, especially when $\hbar$ changes. One does therefore not escape making assumptions, but instead of imposing conditions on the eigenvectors $\psi_\hbar$ and the Hilbert space $\mathcal{H}_\hbar$ we impose conditions on the algebra of observables, as will become clear soon. For purpose of this chapter we are interested in the semi-classical behavior of $\hbar$-dependent eigenvectors $\{\psi_\hbar\}_{\hbar>0}$ in $\mathcal{H}_\hbar$ corresponding to operators of the form

- $Q_\hbar^B(e_h)$, where $e_h \in C_0(S)$ and $Q_\hbar^B$ is the Berezin quantizaton map on S (cf. Definition 2.2.3). The function e_h is in principle related to a classical Hamiltonian h on phase space S;

We will prove that under some hypotheses the vector state (6.2.8) associated to such sequences of eigenvectors admits a classical limit with respect to the observables $Q_\hbar^B(f)$, $f \in C_0(S)$ with $\overline{f} = f$.[3] As a result of Remark 2.2.4, the operators $Q_\hbar^B(f)$ $(f \in C_0(S))$ are compact and any compact operator on the pertinent Hilbert space can be can be approximated in this way, so that a relatively large class of physical observables is considered. Using these observables the statement that the sequence of eigenvectors $\{\psi_\hbar\}_{\hbar>0}$ admits a classical limit now means that

$$\omega_0(f) = \lim_{\hbar\to 0^+} \omega_\hbar(Q_\hbar^B(f)), \tag{6.2.9}$$

exists for all $f \in C_0(S)$ and and defines a state ω_0 on $C_0(S)$. The state ω_0 may be regarded as the classical limit of the sequence of vector states $\omega_\hbar$ defined in (6.2.8). From the mathematical side, by the Riesz Representation Theorem the statement in (6.2.9) means that for all $f \in C_0(S)$ one has

$$\mu_0(f) = \lim_{\hbar\to 0^+} \int_S f(\sigma) d\mu_{\psi_\hbar}(\sigma), \tag{6.2.10}$$

where μ_0 is the probability measure corresponding to the state ω_0 and $\mu_{\psi_\hbar}$, with $\hbar > 0$, is a probability measure on S with density $B_{\psi_\hbar}(\sigma) := |\langle\Psi_\hbar^\sigma, \psi_\hbar\rangle|^2$ (where $\Psi_\hbar^\sigma$ is the corresponding coherent state vector) also called the **Husimi density function** associated to the unit vector $\psi_\hbar$. In other words $\mu_{\psi_\hbar}$ is given by

$$d\mu_{\psi_\hbar}(\sigma) = |\langle\Psi_\hbar^\sigma, \psi_\hbar\rangle|^2 d\mu_\hbar(\sigma)\ , \quad \sigma \in S\ ; \tag{6.2.11}$$

and $\mu_\hbar$ is defined by (2.2.27). It still remains a challenging problem to work with (6.2.9) (or equivalently with the Husimi function) as there is a priori no information on the eigenvectors $\{\psi_\hbar\}_{\hbar>0}$ corresponding to the quantum Hamiltonians. Nonetheless, we will see that the semi-classical behavior of the sequence of eigenvectors $\{\psi_\hbar\}_{\hbar>0}$ above is encoded by the algebraic properties of the function e_h, which are relatively well manageable and allow us to prove the existence of the classical limit.

[2]Even the dimension of the Hilbert space itself may depend on $\hbar$. In such cases the limit $\hbar \to 0$ of the sequence $\{\psi_\hbar\}_\hbar$ (and clearly also of the Hilbert space in which it stays) is undefined. We will see an explicit example in Chapter 8.

[3]By (ii) in Definition 2.1.3 it follows that $Q_\hbar^B(f)$ is self-adjoint, as should be the case in order to define a physical observable.

6.2.1 Isometric embedding $\mathcal{H}_\hbar \subset L^2(S, d\mu_\hbar)$ and $Q_\hbar^B$-equivariant group representations

We first introduce some preparatory results necessary to study the classical limit. We start with a proposition on the coherent pure state quantization of the symplectic manifold (S, ω_S). This is actually based on [55, Proposition II 1.5.2]. We now prove the proposition.

Proposition 6.2.1. *Referring to the coherent state vectors $\Psi_\hbar^\sigma \in \mathcal{H}_\hbar$ used to construct the quantization Berezin map $Q_\hbar^B$, there exists an isometry $W : \mathcal{H}_\hbar \to L^2(S, d\mu_\hbar)$, completely defined by*

$$(W\phi)(\sigma) = \langle \Psi_\hbar^\sigma, \phi \rangle \,, \tag{6.2.12}$$

in particular $W^*W = I_{\mathcal{H}_\hbar}$ and $p := WW^* : L^2(S, d\mu_\hbar) \to L^2(S, d\mu_\hbar)$ is the orthogonal projector onto $ran(W) = \overline{ran(W)}$;

Proof. From now on

$$||\Phi||^2_{L^2(S,d\mu_\hbar)} = \int_S d\mu_\hbar(\sigma) |\langle \Psi_\hbar^\sigma, \phi \rangle|^2, \tag{6.2.13}$$

where we have introduced the notation $\Phi := W\phi$. By property (1) in Definition 2.2.2 applied to $q_\hbar(\sigma)$ lifted to the coherent state $\Psi_\hbar^\sigma$, for every $\phi \in \mathcal{H}_\hbar$ the associated function $S \ni \sigma \mapsto (W\phi)(\sigma) := \langle \Psi_\hbar^\sigma, \phi \rangle$ satisfies $W\phi \in L^2(S, d\mu_\hbar)$ and $||W\phi||_{L^2(S,d\mu_\hbar)} = ||\phi||_{\mathcal{H}_\hbar}$. Hence $W^*W = I$, the remaining part is a standard property of isometric maps in Hilbert spaces. □

If there moreover exists an action of a group G acting by symplectomorphisms on the manifold (S, ω_S), it follows that the quantization maps $Q_\hbar^B$ are equivariant under a suitable unitary representation of G in $\mathcal{H}_\hbar$. The precise statement is given in the proposition below. To this end for any $g \in G$ we denote the pullback of the action of G on functions $f : S \to \mathbb{C}$ by ζ_g, i.e.,

$$(\zeta_g f)(\sigma) = f(g^{-1}\sigma) \,, \quad (\sigma \in S \,, g \in G) \,.$$

Proposition 6.2.2. *Let G be a group acting by symplectomorphisms on the symplectic manifold (S, ω_S). Then there exists a unitary representation $U : G \to \mathfrak{B}(\mathcal{H}_\hbar)$ such that the maps $Q_\hbar^B$ are ζ-equivariant,*

$$U_g Q_\hbar^B(f) U_g^{-1} = Q_\hbar^B(\zeta_g f) \,, \quad g \in G, \ f \in L^\infty(S, d\mu_\hbar) \,. \tag{6.2.14}$$

The representation U is completely defined by the requirement

$$WU_g\psi = p\zeta_g(W\psi) \quad \textit{for every } \psi \in \mathcal{H}_\hbar \textit{ and } g \in G. \tag{6.2.15}$$

Proof. Let us write $\mathcal{H}^\hbar := L^2(S, d\mu_\hbar)$. By definition of W for any $\phi \in \mathcal{H}_\hbar$, as $\Phi := W\phi$ is a function on S, we can define the operator $u_g : L^2(S, d\mu_\hbar) \to L^2(S, d\mu_\hbar)$ by $(u_g\Phi)(\sigma) := \Phi(g^{-1}\sigma)$. This map is isometric, since the action of the group preserves the Liouville measure μ_L (and thus also $d\mu_\hbar$) as the action is made of symplectomorphisms, and it is finally surjective as the reader immediately proves since the action of each g is bijective. By construction we also have $u_g(\mathrm{ran}(W)) \subset \mathrm{ran}(W)$. By construction $u_1 = I$ and $u_g u_{g'} = u_{g \cdot g'}$. Next $U_g : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ is defined by $U_g := W^* u_g W : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$. This operator is unitary since u_g is unitary and $U_g^* U_g = W^* u_g^* W W^* u_g W = W^* u_g^* p u_g W = W^* u_g^* u_g W = W^* W = I$ together with $U_g U_g^* = W^* u_g W W^* u_g^* W = W^* u_g p u_g^* W = W^* u_g p u_{g^{-1}} W W^* u_g u_{g^{-1}} W = W^* W = I$. By construction, $U_1 = I$ and $U_g U_{g'} = U_{g \cdot g'}$ so that $G \ni g \mapsto U_g$ is a unitary representation. Let us finally prove the equivariance property. For any $\phi, \psi \in \mathcal{H}_\hbar$, we now compute

$$\begin{aligned}
\langle \phi, Q_\hbar^B(\zeta_{g^{-1}} f)\psi \rangle &= \int_S d\mu_\hbar(\sigma) \overline{(W\phi)(\sigma)} (W\psi)(\sigma) f(g\sigma) = \int_S d\mu_\hbar(\sigma) \overline{\Phi(\sigma)} \Psi(\sigma) f(g\sigma) \\
&= \int_S d\mu_\hbar(\sigma) \overline{\Phi(g^{-1}\sigma)} \Psi(g^{-1}\sigma) f(\sigma) = \int_S d\mu_\hbar(\sigma) \overline{(u_g\Phi)(\sigma)} (u_g\Psi)(\sigma) f(\sigma) \\
&= \int_S d\mu_\hbar(\sigma) \overline{(WU_g\phi)(\sigma)} (WU_g\psi)(\sigma) f(\sigma) = \langle U_g\phi, Q_\hbar^B(f) U_g\psi \rangle = \langle \phi, U_g^* Q_\hbar^B(f) U_g \psi \rangle,
\end{aligned}$$

where we used the fact that the measure $d\mu_\hbar$ is G-invariant. Since this holds for any $\phi, \psi \in \mathcal{H}_\hbar$, we can conclude that $U_g^* Q_\hbar^B(f) U_g = Q_\hbar^B(\zeta_{g^{-1}} f)$. replacing g for g^{-1} and noticing that $U_{g^{-1}} = (U_g)^{-1} = U_g^*$, we have the thesis. The last statement is a rephrasing of $U_g = W^* u_g W$, using $W^* W = I$ and $WW^* = p$. $\square$

Notice that both W and U generally depend on the value of $\hbar$.

6.2.2 Localization of eigenvectors

We establish a first important result concerning the localization of eigenvectors, which allows us to prove the existence of the classical limit in section 6.2.3. This topic is not new, several similar results have been achieved over the years [26, 28, 32, 98, 107] where different classes of neighborhoods of localization were exploited.[4] In such works one typically studies semi-classical defect measures induced by the relevant eigenvectors and uses techniques from pseudo-differential calculus. In this section, we will not go into such details, we however aim to provide a rather neat algebraic approach that forms the basis for spontaneous symmetry breaking as explained in the next sections.

Proposition 6.2.3. *Given a coherent pure state quantization $(\mathcal{H}_\hbar, \Psi_\hbar^\sigma, \mu_\hbar)_{\hbar \in I, \sigma \in S}$ of a symplectic manifold (S, ω_S). Assume that the associated Berezin quantization maps, denoted by $Q_\hbar^B$, satisfy the von Neumann and Rieffel condition on $C_0(S)$. Let $e \in C_0(S)$ be a real-valued function and $\{\phi_\hbar\}_\hbar \subset \mathcal{H}_\hbar$ be a sequence of eigenvectors of $Q_\hbar^B(e)$ with eigenvalues $\{\lambda_\hbar\}_\hbar$ such that, for some $\Lambda \in ran(e)$ is such that*

$$\lambda_\hbar \to \Lambda \quad \text{for } \hbar \to 0^+. \tag{6.2.16}$$

The following facts are true where $\Phi_\hbar := W\varphi_\hbar$ as before.

(1) Referring to any open neighborhood of the set $e^{-1}(\Lambda)$ of the form

$$\mathcal{V}_\epsilon := e^{-1}((\Lambda - \epsilon, \Lambda + \epsilon)), \tag{6.2.17}$$

for every given $\epsilon > 0$ one has

$$||\Phi_\hbar||_{L^2(S \setminus \mathcal{V}_\epsilon, d\mu_\hbar)} \to 0, \quad \text{for} \quad \hbar \to 0^+ . \tag{6.2.18}$$

(2) If $e^{-1}(\Lambda) = \{\sigma_0\} \in S$ and the family of sets $\{\mathcal{V}_\epsilon\}_{\epsilon>0}$ is a fundamental system of neighborhoods of σ_0 then $\langle \varphi_\hbar, Q_\hbar(f)\varphi_\hbar \rangle \to f(\sigma_0)$ as $\hbar \to 0^+$ for every $f \in C_0(S)$.

Proof. (1) The thesis arises from the following fact we shall prove later

$$\langle \varphi_\hbar, Q_\hbar^B((e - \Lambda)^2)\varphi_\hbar \rangle \to 0 \quad \text{for } \hbar \to 0, \tag{6.2.19}$$

where we are using $Q_\hbar^B$ defined on $L^\infty(S, d\mu_\hbar)$. Indeed, (6.2.19) can be rephrased to

$$\int_S (e(\sigma) - \Lambda)^2 |\Phi_\hbar(\sigma)|^2 d\mu_\hbar \to 0 \quad \text{for } \hbar \to 0\,,$$

so that also

$$\int_{S \setminus \mathcal{V}_\epsilon} (e(\sigma) - \Lambda)^2 |\Phi_\hbar(\sigma)|^2 d\mu_\hbar \to 0 \quad \text{for } \hbar \to 0\,,$$

as well because the integrand is non-negative. However, by definition of $\mathcal{V}_\epsilon$, $|e(x) - \Lambda| \geq \epsilon$ if $x \in S \setminus \mathcal{V}_\epsilon$ and thus

$$0 \leq \int_{S \setminus \mathcal{V}_\epsilon} |\Phi_\hbar(x)|^2 d\mu_\hbar \leq \frac{1}{\epsilon^2} \int_{S \setminus \mathcal{V}_\epsilon} (e(\sigma) - \Lambda)^2 |\Phi_\hbar(\sigma)|^2 d\mu_\hbar \to 0,$$

[4] Our neighborhoods $\mathcal{V}_\epsilon$ are particularly adapted to the remaining proofs of our work.

which implies (6.2.18). To conclude, it is sufficient to prove (6.2.19). To this end we have, by using linearity of $Q^B_\hbar$, the fact that $||\varphi_\hbar|| = 1$ and $Q^B_\hbar(e)\varphi_\hbar = \lambda_\hbar\varphi_\hbar$ in particular,

$$\begin{aligned}\langle\varphi_\hbar, Q^B_\hbar((e-\Lambda)^2)\varphi_\hbar\rangle &= \langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle - 2\Lambda\langle\varphi_\hbar, Q^B_\hbar(e)\varphi_\hbar\rangle + \Lambda^2\langle\varphi_\hbar,\varphi_\hbar\rangle \\ &= \langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle - 2\lambda_\hbar\Lambda\langle\varphi_\hbar,\varphi_\hbar\rangle + \Lambda^2\langle\varphi_\hbar,\varphi_\hbar\rangle\,.\end{aligned}$$

That is, $\langle\varphi_\hbar, Q^B_\hbar((e-\Lambda)^2)\varphi_\hbar\rangle = \langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle + \Lambda^2 - 2\lambda_\hbar\Lambda$. On the other hand it also holds,

$$\langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle \to \Lambda^2\,. \tag{6.2.20}$$

In fact, by the von Neumann condition we know that $||Q^B_\hbar(e^2) - Q^B_\hbar(e)Q^B_\hbar(e)|| \to 0$ where the norms are the C^* ones. This, in turn, entails $||Q^B_\hbar(e^2)\varphi_\hbar - Q^B_\hbar(e)Q^B_\hbar(e)\varphi_\hbar||| \to 0$ referring to the Hilbert space norms. Using the hypothesis $\lambda_\hbar \to \Lambda$ and $Q^B_\hbar(e)\varphi_\hbar = \lambda_\hbar\varphi_\hbar$ we deduce $||Q^B_\hbar(e^2-\Lambda^2)\varphi_\hbar|| \le ||Q^B_\hbar(e^2)\varphi_\hbar - \lambda^2_\hbar\varphi_\hbar|| + |\lambda^2_\hbar - \Lambda^2|\,||\varphi_\hbar|| \to 0$, so that, from the Cauchy-Schwartz inequality, $\left|\langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle - \Lambda^2\right| = \left|\langle\varphi_\hbar, Q^B_\hbar(e^2-\Lambda^2)\varphi_\hbar\rangle\right| \le ||Q^B_\hbar(e^2-\Lambda^2)\varphi_\hbar|| \to 0$. Finally, by the triangle inequality we estimate

$$|\langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle + \Lambda^2 - 2\lambda_\hbar\Lambda| \le |\langle\varphi_\hbar, Q^B_\hbar(e^2)\varphi_\hbar\rangle - \Lambda^2| + |2\Lambda^2 - 2\lambda_\hbar\Lambda|,$$

which goes to zero by the previous observations, as $\hbar \to 0$. This establishes (6.2.19) ending the proof.

(2) For every given $m > 0$,

$$\left|\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle - f(\sigma_0)\right| = \left|\int_S |\Phi_\hbar(\sigma)|^2 f(\sigma)\frac{d\sigma}{(2\pi\hbar)^n} - f(\sigma_0)\right|$$

$$= \left|\int_S |\Phi_\hbar(\sigma)|^2(f(\sigma)-f(\sigma_0))d\mu_\hbar(\sigma)\right| \le \int_S |\Phi_\hbar(\sigma)|^2|f(\sigma)-f(\sigma_0)|d\mu_\hbar(\sigma)$$

$$\le \int_{S\setminus\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2|f(\sigma)-f(\sigma_0)|d\mu_\hbar(\sigma) + \int_{\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2|f(\sigma)-f(\sigma_0)|d\mu_\hbar(\sigma)\,.$$

In summary,

$$\left|\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle - f(\sigma_0)\right| \le 2||f||_\infty \int_{S\setminus\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma) + \sup_{\sigma\in\mathcal{V}_{1/m}} |f(\sigma)-f(\sigma_0)|\,.$$

Take $\epsilon > 0$. Since the sets $\mathcal{V}_{1/m}$ are a fundamental system of neighborhoods of σ_0 and f is continuous, there is $m_\epsilon \in \mathbb{N}$ such that $\sup_{\sigma\in\mathcal{V}_{1/m_\epsilon}} |f(\sigma)-f(\sigma_0)| < \epsilon/2$. Due to statement (1), we can also find $H_\epsilon > 0$ such that $0 < \hbar < H_\epsilon$ implies

$$\int_{S\setminus\mathcal{V}_{1/m_\epsilon}} |\Phi_\hbar(\sigma)|^2|f(\sigma)-f(\sigma_0)|d\mu_\hbar(\sigma) < \epsilon/2\,.$$

Summing up, for every $\epsilon > 0$, there is H_ϵ such that $\left|\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle - f(\sigma_0)\right| \le \epsilon$ if $0 < \hbar < H_\epsilon$. This is the thesis we wanted to prove. □

6.2.3 Classical limit for eigenvectors of Berezin quantization maps

We prove the existence of the classical limit for a sequence of eigenvectors corresponding to the operators $Q^B_\hbar(e)$, where $e \in C_0(S)$. In that case $Q^B_\hbar(e)$ is compact [55, Thm. II. 1.3.5] with a point spectrum except for 0 at most, and we focus on a sequence of eigenvalues $\lambda_\hbar$ and corresponding eigenvectors $\phi_\hbar$ of the maps $Q^B_\hbar(e)$ such that $\lambda_\hbar \to \Lambda = e(\sigma_0)$ $(\hbar \to 0^+)$, for some $\sigma_0 \in S$. Again, we assume the existence of a coherent pure state quantization $(\mathcal{H}_\hbar, \Psi^\sigma_\hbar, \mu_\hbar)_{\hbar\in I,\sigma\in S}$ of a symplectic manifold (S,ω_S) such that its associated Berezin maps $Q^B_\hbar$ satisfy the von Neumann and Rieffel condition on $C_0(S)$. Moreover, we require that the manifold (S,ω_S) is connected. In this case (S,ω_S) admits a complete Riemannian metric. We will work with this metric so that as a result of the Hopf–Rinow Theorem any closed and bounded subset of S is compact.

Theorem 6.2.4 (Classical limit without symmetry). *Consider $e \in C_0(S)$ and $\Lambda \neq 0$ such that*

$$\Lambda = e(\sigma_0) \quad \textit{for a unique point } \sigma_0 \in S. \tag{6.2.21}$$

Let $\{\varphi_\hbar\}_{\hbar>0}$ be a family of eigenvectors with eigenvalues $\{\lambda_\hbar\}_{\hbar>0}$ of $Q^B_\hbar(e)$ converging to Λ, as $\hbar \to 0$. With these assumptions,

$$\langle \varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar \rangle \to f(\sigma_0) \quad \textit{as } \hbar \to 0^+, \textit{ for every } f \in C_0(S).$$

Proof. Taking proposition 6.2.3 (2) into account, the only fact to be proved is that the family $\{\mathcal{V}_\epsilon\}_{\epsilon>0} = e^{-1}((\Lambda-\epsilon, \Lambda+\epsilon))$ forms a fundamental system of σ_0. Notice that σ_0 is the unique point in S such that $e(\sigma_0) = \max_{\sigma\in S} |e(\sigma)|$. Now suppose that there is an open ball B_r centered at σ_0 of radius $r > 0$ such that $\mathcal{V}_{1/m} \not\subset B_r$ for $m > m_0$ (i.e., $\{\mathcal{V}_\epsilon\}_{\epsilon>0}$ is *not* a fundamental system of neighborhoods of σ_0). As a consequence, for every $m > m_0$ there is $\sigma_m \in S \setminus B_r$ such that $\sigma_m \in \mathcal{V}_{1/m}$, i.e., $|e(\sigma_m) - \Lambda| < 1/m$. Since $\Lambda = e(\sigma_0) \neq 0$ and $e \in C_0(S)$, it follows that $\lim_{|\sigma|\to\infty} |\Lambda - e(\sigma)| = |\Lambda| > 0$ so that the sequence of σ_m must be bounded because $|e(\sigma_m) - \Lambda| \to 0 \neq \Lambda$. In summary, $\{\sigma_m \in S \setminus B_r \mid |\Lambda - e(\sigma_m)| \leq 1/m\} \subset S$ is contained in a compact set K, whenever $m > m_0$. Since $K \setminus B_r = K \cap (S \setminus B_r)$ is compact as well, we can extract a subsequence $\sigma'_{m_k} \to \sigma'_0 \in K \setminus B_r$. Since $|e(\sigma_{m_k}) - \Lambda| < 1/m_k \to 0$ for $k \to +\infty$, continuity of e implies $e(\sigma'_0) = \Lambda$. We observe that $\sigma'_0 \neq \sigma_0$ since $\sigma'_0 \in K \setminus B_r$ and $\sigma_0 \in B_r$. However, $e(\sigma_0) = \Lambda = e(\sigma'_0)$ so that two distinct points reach the same value. This is in contradiction with the fact that the value is attained in a unique point. □

We now pass to a more elaborated case where a symmetry of e is present. We consider a group G acting by symplectomorphism $G \ni g : S \ni \sigma \mapsto g\sigma \in S$ on (S, ω_S), and focus on the two cases: either G is a compact topological group or (b) G is a discrete group.

Differently from the simpler result established with Theorem 6.2.4, we now also assume that the eigenspaces associated to the sequence of eigenvalues $\lambda_\hbar$ of $Q^B_\hbar(e)$ have dimension 1. We will see in the Chapter 9 that this condition is satisfied if e assumes a specific form.

Theorem 6.2.5 (Classical limit with symmetry). *Consider a group G either finite or topological compact, $e \in C_0(S)$ and assume the following hypotheses.*

(a) G acts continuously in the topological-group case[5] *on (S, ω_S) in terms of symplectomorphisms.*

(b) e is invariant under G.

(c) The action of G is transitive on $e^{-1}(\{\Lambda\})$.

Then the following facts are valid for every chosen $\sigma_0 \in e^{-1}(\{\Lambda\})$ and a family of eigenvectors $\{\varphi_\hbar\}_{\hbar>0}$ of $Q^B_\hbar(e)$ with non-degenerate eigenvalues $\{\lambda_\hbar\}_{\hbar>0}$ converging to Λ as $\hbar \to 0$ for some $\Lambda \in ran(e) \setminus \{0\}$.

(1) If G is topological and compact and μ_G is the normalized Haar measure,

$$\lim_{\hbar\to 0^+} \langle \varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar \rangle = \int_G f(g\sigma_0)\, d\mu_G(g), \quad \textit{for every } f \in C_0(S). \tag{6.2.22}$$

(2) If G is finite and N_G is the number of elements of G,

$$\lim_{\hbar\to 0^+} \langle \varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar \rangle = \frac{1}{N_G} \sum_{g\in G} f(g\sigma_0), \quad \textit{for every } f \in C_0(S). \tag{6.2.23}$$

The right-hand sides of (6.2.22) and (6.2.23) are independent of the choice of σ_0.

[5]The action $G \times S \ni (g, \sigma) \mapsto g\sigma \in S$ is continuous.

Proof. We assume that G is a compact topological group and prove (1). Compactness guarantees in particular the existence of a unique normalized two-sided Haar measure μ_G. The case (2) has the same (actually easier) proof just by everywhere replacing the integral with $\frac{1}{N_G}\sum_{g\in G}$, which is the same as saying that the Haar measure is discrete.

To go on, choose $g \in G$ arbitrarily. As G leaves the function e invariant, it follows by Proposition 6.2.2 that U_g commutes with $Q^B_\hbar(e)$ and the eigenspace spanned by $\varphi_\hbar$ is one-dimensional. As a result, $U_g\varphi_\hbar = e^{ia}\varphi_\hbar$ for some real a and thus $\Phi_\hbar(g^{-1}\sigma) = pWU_g\varphi_\hbar = e^{ia}\Phi_\hbar(\sigma)$. Since the measure $d\mu_L$ is G-invariant, the measure $d\mu_{\phi_\hbar} = |\Phi_\hbar|^2 d\mu_\hbar$ is also G-invariant. We can therefore replace σ with $g^{-1}\sigma$ in both $|\Phi_\hbar(\sigma)|^2$ and $d\sigma$:

$$\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle = \int_S |\Phi_\hbar(\sigma)|^2 f(\sigma)d\mu_\hbar(\sigma) = \int_S |\Phi_\hbar(g^{-1}\sigma)|^2 f(\sigma)d\mu_\hbar(g^{-1}\sigma)\,.$$

Replacing again σ with $g\sigma$ in the complete integrand,

$$\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle = \int_S |\Phi_\hbar(\sigma)|^2 f(g\sigma)d\mu_\hbar(\sigma)\,.$$

Since the left-hand side is independent of g, we can integrate both sides with respect to the normalized Haar measure on G, obtaining

$$\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle = \int_G\int_S |\Phi_\hbar(\sigma)|^2 f(g\sigma)d\mu_\hbar(\sigma)d\mu_G(g) = \int_S |\Phi_\hbar(\sigma)|^2 \int_G f(g\sigma)d\mu_G(g)d\mu_\hbar(\sigma)\,.$$

Above, in order to interchange the two integrals, we have also used the fact that $S \times G \ni (\sigma, g) \mapsto |\Phi_\hbar(\sigma)|^2|f(g\sigma)|$ is integrable in the product measure since

$$\int_G\int_S |\Phi_\hbar(\sigma)|^2|f(g\sigma)|d\mu_\hbar(\sigma)d\mu_G(g) \le ||f||_\infty \int_G\int_S |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma)d\mu_G(g)$$

$$= ||f||_\infty \int_S |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma)\int_G 1 d\mu_G(g) = ||f||_\infty \int_S |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma) = 1\,,$$

and then we have used the Fubini-Tonelli theorem, exploiting the fact that the Liouville measure on S is σ- finite. In summary,

$$\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle = \int_S |\Phi_\hbar(\sigma)|^2 F(\sigma)d\mu_\hbar(\sigma), \quad \text{where}$$

$$F(\sigma) := \int_G f(g\sigma)d\mu_G(g), \quad (\sigma \in S). \tag{6.2.24}$$

Notice that this function is (i) bounded (since f is bounded and μ_G finite), (ii) continuous (as it immediately arises form Lebesgue's dominated convergence since, again, f is bounded and μ_G finite), (iii) constant on $e^{-1}(\Lambda)$, since $F(g'\sigma) = F(\sigma)$ because μ_G is G-invariant and the action of G on $e^{-1}(\{\Lambda\})$ is transitive. To go on, since $\int_S |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma) = 1$, we can write, $\left|\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle - F(\sigma_0)\right| = \left|\int_S |\Phi_\hbar(\sigma)|^2(F(\sigma) - F(\sigma_0))d\mu_\hbar(\sigma)\right|$ for any arbitrarily taken $\sigma_0 \in e^{-1}(\Lambda)$, Defining $\mathcal{V}_\delta := e^{-1}(\Lambda - \delta, \Lambda + \delta)$ and for every given $m \in \mathbb{N} \setminus \{0\}$, we can now estimate

$$\left|\langle\varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle - F(\sigma_0)\right|$$

$$\le \int_{\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2|F(\sigma) - F(\sigma_0)|d\mu_\hbar(\sigma) + \int_{S\setminus\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2|F(\sigma) - F(\sigma_0)|d\mu_\hbar(\sigma)$$

$$\le \sup_{\sigma\in\mathcal{V}_{1/m}} |F(\sigma) - F(\sigma_0)| \int_{\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma) + 2||F||_\infty \int_{S\setminus\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma)$$

$$\le \sup_{\sigma\in\mathcal{V}_{1/m}} |F(\sigma) - F(\sigma_0)| + 2||F||_\infty \int_{S\setminus\mathcal{V}_{1/m}} |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma)\,. \tag{6.2.25}$$

Let us now focus on the two terms in the last line of (6.2.25) separately. The following lemma is true.

Lemma 6.2.6. *Under the hypotheses of theorem 6.2.5 and with F defined in (6.2.24), for every $\epsilon > 0$, there is $m_\epsilon \in \mathbb{N}$ such that* $\sup_{\sigma\in\mathcal{V}_{1/m_\epsilon}} |F(\sigma) - F(\sigma_0)| < \epsilon/2$.

Proof. Let us define $\Gamma := e^{-1}(\Lambda)$. This set is G-invariant, i.e., $g(\Gamma) \subset \Gamma$ for every $g \in G$ because $e(g\sigma) = e(\sigma)$ for every $g \in G$ and $\sigma \in S$. Since the action of G on Γ is transitive, we also have that $\Gamma = \{g\sigma_0 \mid g \in G\}$ for every chosen $\sigma_0 \in \Gamma$. This identity has the important consequence that Γ is compact under our main hypotheses. Indeed, if G is finite, then Γ is made of a finite number of points and thus it is compact *a fortiori*. If G is a topological compact group and its action is continuous, as requested in the main hypotheses, then Γ is the image of a compact set under the continuous function $G \ni g \mapsto g\sigma_0$ and thus Γ is compact as well.
If $\delta > 0$ a δ-*covering* of Γ is a set of the form $C_\delta := \bigcup_{\sigma\in\Gamma} B_\delta(\sigma)$, where $B_\delta(\sigma)$ is an open ball of radius δ centered at σ.
Since Γ is compact, there is a closed ball B centered at the origin of finite positive radius such that Γ is completely contained in the interior of B. All other balls we shall consider in this proof will be assumed to be contained in the interior of B as well. Since $|e(\sigma)| \to 0$ for $|\sigma| \to +\infty$ and $\Lambda \neq 0$, we can always fix the radius of B such that $|\Lambda - e(\sigma)| > \eta$, for some $\eta > 0$, if $\sigma \notin B$.
Our next step consists of proving that, given a δ-covering C_δ of Γ, with $\delta > 0$ arbitrarily taken, there exists $m_\delta \in \mathbb{N}$ such that $\mathcal{V}_{1/m_\delta} \subset C_\delta$. Indeed, suppose that it is not the case for some $\delta > 0$. As a consequence, for every $m \in \mathbb{N}$, it must be $\mathcal{V}_{1/m} \not\subset C_\delta$, so that there is $\sigma_m \in \mathcal{V}_{1/m}$ not included in C_δ. Choosing $m > 1/\eta$ and keeping in mind that $|e(\sigma_m) - \Lambda| < 1/m$, we can also exclude that σ_m stays outside B. In other words, for every $m \in \mathbb{N}$ sufficiently large, we have a point $\sigma'_m \in B \setminus C_\delta$ with $|e(\sigma'_m) - \Lambda| < 1/m$. Since $B \setminus C_\delta$ is compact, we can extract a subsequence $\sigma'_{m_k} \to \sigma'_0 \in B \setminus C_\delta$ for $k \to +\infty$. As $1/m_k \to 0$ as $k \to +\infty$, continuity imposes that $e(\sigma'_0) = \Lambda$ and thus $\sigma'_0 \in \Gamma$. This is not possible because $\Gamma \subset C_\delta$ that is disjoint from $B \setminus C_\delta$. We have proved that every C_δ covering of Γ contains $\mathcal{V}_{1/m_\delta}$ if $m_\delta \in \mathbb{N}$ is sufficiently large.
Now take $\sigma_0 \in \Gamma$. Noticing that B is compact and F is continuous thereon, we can use its uniform continuity. Given $\epsilon > 0$, there is $\delta_\epsilon > 0$ such that $|F(\sigma) - F(\sigma')| < \epsilon/2$ if $|\sigma - \sigma'| < \delta_\epsilon$. With this remark, consider a C_{δ_ϵ} covering of Γ. If $\tau \in C_{\delta_\epsilon}$ we have $|F(\tau) - F(\sigma_0)| = |F(\tau) - F(\sigma_0^\tau)|$ where $\sigma_0^\tau \in \Gamma$ is the center of $B_{\delta_\epsilon}(\sigma_0^\tau)$ which contains τ. The identity above is valid because F is constant in Γ. Uniform continuity therefore implies that $|F(\tau) - F(\sigma_0)| < \epsilon/2$ if $\tau \in C_{\delta_\epsilon}$. In summary, given $\epsilon > 0$, if $m_\epsilon \in \mathbb{N}$ is sufficiently large to assure that $\mathcal{V}_{1/m_\epsilon} \subset C_{\delta_\epsilon}$, we have the thesis $\sup_{\sigma\in\mathcal{V}_{1/m_\epsilon}} |F(\sigma) - F(\sigma_0)| \leq \sup_{\sigma\in C_{\delta_\epsilon}} |F(\sigma) - F(\sigma_0)| < \epsilon/2$, concluding the proof. □

Keeping that m_ϵ and exploiting (1) in Proposition 6.2.3, we can find $H_\epsilon > 0$ such that

$$2||F||_\infty \int_{S\setminus\mathcal{V}_{1/m_\epsilon}} |\Phi_\hbar(\sigma)|^2 d\mu_\hbar(\sigma) < \epsilon/2,$$

for $0 < \hbar < H_\epsilon$. Looking again at the last line of (6.2.25), we conclude that for every $\epsilon > 0$, there is H_ϵ such that $0 < \hbar < H_\epsilon$ implies $\left|\langle\varphi_\hbar, Q_\hbar^B(f)\varphi_\hbar\rangle - F(\sigma_0)\right| < \epsilon$, concluding the proof. □

It is of great importance to notice that both right-hand sides can be recast to integrals with respect to G-invariant probability measures μ and ν on S with supports given by the whole orbit $G\sigma_0$. The following more general result holds for locally compact Hausdorff spaces.

Proposition 6.2.7. *Let G be a topological compact or finite group with an action (continuous in the first case) on a locally compact Hausdorff space S. Then there are two regular Borel probability measures on S, respectively μ and ν, such that*

$$\int_G f(g\sigma_0)d\mu_G(g) = \int_S f d\mu\,; \qquad \frac{1}{N_G}\sum_{g\in G} f(g\sigma_0) = \int_S f d\nu, \qquad \textit{for all } f \in C_0(S),$$

where μ_G is the normalized Haar measure on G in the first case and N_G is the number of elements of G in the second case. These measures are invariant under the action of G on S and each of their supports is the whole orbit $G\sigma_0$.

Proof. It is sufficient to prove the thesis for the former case, the latter being easier. Noticing that $G\sigma_0$ is compact, one has that linear map $C_0(S) \ni f \mapsto \int_G f(g\sigma_0)d\mu_G(g) \in \mathbb{C}$ is $||\cdot||_\infty$ continuous on $C_0(S)$. Riesz' theorem for complex measures implies that it the integral with respect to a (uniquely defined) regular Borel complex measure μ on S. This measure is actually positive, since the map is positive. Riesz' theorem for positive measures implies that the support of the measure is included in the closed set $G\sigma_0$, since the map vanishes when evaluated on $f \in C_c(S)$ whose support is included in an open set with empty intersection with $G\sigma_0$. The integral produces the value 1 if $f(\sigma) = 1$ on the compact orbit $G\sigma_0$, hence μ is a probability one. The invariance of the integral under the pullback action of G on its argument f proves that μ is also G invariant, again using the uniqueness property in Riesz' theorem for positive measures. Since $\mu(G\sigma_0) = 1$, its support cannot be empty. If σ belongs to the support also $g\sigma$ does for every $g \in G$. As the action of G is transitive on $G\sigma_0$ we conclude that $\text{supp}(\mu) = G\sigma_0$. □

The next proposition proves a result concerning coherent states as approximate eigenvectors of the quantization maps $Q_\hbar^B(e)$.

Proposition 6.2.8. *Given a coherent pure state quantization of S with coherent states $\Psi_\hbar^\sigma$ and associated Berezin quantization maps $Q_\hbar^B(e)$ defined by (2.2.40), where $e \in C_0(S)$. Given $\sigma \neq \sigma'$ two distinct points in S, we assume that the function $\hbar \mapsto c(\hbar)|\langle \Psi_\hbar^\sigma, \Psi_\hbar^{\sigma'}\rangle|$ tends to 0 as $\hbar \to 0$, where $c : I_0 \to \mathbb{R} \setminus \{0\}$ is a positive continuous function coming from the pure state quantization satisfying $\mu_\hbar = c(\hbar)\mu_L$ (cf. Definition 2.2.2). Given $E \in ran(e)$ and let $\sigma \in e^{-1}(E)$, the preimage of E under the function e. Then,*

$$\lim_{\hbar \to 0} ||Q_\hbar^B(e)\Psi_\hbar^\sigma - E\Psi_\hbar^\sigma|| = 0. \tag{6.2.26}$$

Proof. By uniform continuity of e, given $\epsilon > 0$ we can find $\delta > 0$ such that for all $\sigma' \in B_\delta(\sigma)$ one has $|e(\sigma') - e(\sigma)| < \epsilon/2$, where $B_\delta(\sigma) \subset S$ is equipped with the Riemannian metric associated to S.

$$\begin{aligned}
&\left|\left|Q_\hbar^B(e)\Psi_\hbar^\sigma - E\Psi_\hbar^\sigma\right|\right| = \left|\left|\int_S d\mu_\hbar(\sigma')(e(\sigma') - e(\sigma))\langle \Psi_\hbar^{\sigma'}, \Psi_\hbar^\sigma\rangle \Psi_\hbar^\sigma\right|\right| \\
&\leq \left|\left|\int_{B_\delta(\sigma)} d\mu_\hbar(\sigma')(e(\sigma') - e(\sigma))\langle \Psi_\hbar^{\sigma'}, \Psi_\hbar^\sigma\rangle \Psi_\hbar^\sigma\right|\right| \\
&+ \left|\left|\int_{S\setminus B_\delta(\sigma)} d\mu_\hbar(\sigma')(e(\sigma') - e(\sigma))\langle \Psi_\hbar^{\sigma'}, \Psi_\hbar^\sigma\rangle \Psi_\hbar^\sigma\right|\right| \\
&\leq \sup_{\sigma' \in B_\delta(\sigma)} |e(\sigma') - e(\sigma)| \int_{B_\delta(\sigma)} d\mu_\hbar(\sigma')|\langle \Psi_\hbar^{\sigma'}, \Psi_\hbar^\sigma\rangle| \\
&+ \int_{S\setminus B_\delta(\sigma)} d\mu_\hbar(\sigma')|e(\sigma') - e(\sigma)||\langle \Psi_\hbar^{\sigma'}, \Psi_\hbar^\sigma\rangle|.
\end{aligned}$$

The first term is bounded by $\epsilon/2$ using uniform continuity of e, and the fact that $1 = \int_S d\mu_\hbar(\sigma)|\Psi_\hbar^\sigma\rangle\langle\Psi_\hbar^\sigma|$. For the second addend, we use the assumption stating that

$$c(\hbar)|\langle \Psi_\hbar^\sigma, \Psi_\hbar^{\sigma'}\rangle| \to 0, \quad (\hbar \to 0). \tag{6.2.27}$$

Hence, for $\hbar$ sufficiently small, one can estimate

$$c(\hbar)\text{sup}_{\sigma' \in S\setminus B_\delta(\sigma)}|\langle \Psi_\hbar^\sigma, \Psi_\hbar^{\sigma'}\rangle| < \epsilon/2C. \tag{6.2.28}$$

where C is the integration constant given by $C := \int_{S\setminus B_\delta(\sigma} d\sigma|e(\sigma') - e(\sigma)|$. In summary, we conclude that

$$||Q_\hbar^B(e)\Psi_\hbar^\sigma - E\Psi_\hbar^\sigma|| \leq \epsilon. \tag{6.2.29}$$

□

7. Dynamical symmetry groups and spontaneous symmetry breaking

In this Chapter we introduce the notion of spontaneous symmetry breaking (SSB) in an algebraic framework. This approach particularly allows for commutative as well as non-commutative C^*-algebras, and therefore for classical and quantum theories. In the event that they are related by a continuous bundle of C^*-algebras (cf. Definition 2.1.1) this permits one to study SSB as a possibly emergent phenomenon by switching of a semi-classical parameter (e.g. Planck's constant occurring in some index set I). All that is needed are the continuity properties of the C^*-bundle specified by the continuous cross-sections. This approach is therefore perfectly suitable for studying emergent phenomena arising in the classical limit of an underlying quantum theory.

7.1 Spontaneous symmetry breaking: general concepts

For decades the natural phenomenon of spontaneous symmetry breaking (SSB) has been a topic of great interest in mathematical physics and theoretical physics. It lays the foundation of many physical phenomena, like phase transitions in condensed-matter systems, the formation of Bose-Einstein condensates, superconductivity of metals and it plays a role in the Higgs mechanism [86, 104] furnishing mass generation of W and Z bosons, i.e. the gauge bosons that mediate the weak interaction. Numerous studies have led to important results and insights concerning symmetry and its possible breakdown in a various number of physical models.

The general and common concept behind spontaneous symmetry breaking, originating in the field of condensed matter physics where one typically considers the infinite particle, or often called thermodynamic limit, is based on the idea that if a collection of quantum particles becomes larger, the symmetry of the system as a whole becomes more unstable against small perturbations [104, 100]. A similar statement can be made for quantum systems in their classical limit, where sensitivity against small perturbations now should be understood to hold in the relevant semi-classical regime, meaning that a semi-classical parameter (e.g. $\hbar$) approaches zero[1] at fixed system size [57, 66].

Showing the occurrence of SSB in a certain particle system can be done at various levels of rigour. Each particular method has its merits and drawbacks. In theoretical statistical physics and mathematical physics the main concept that the relevant system becomes sensitive to small perturbations is often taken into account by adding a so-called infinitesimal symmetry breaking field term in some cases induced by the so-called magnetization operator [52, 95, 103, 104]. Consequently, one aims to show that the limit in the growing number of particles or lattice sites (viz. thermodynamic limit) becomes "singular", at least at the level of states, e.g. the ground state. If this happens, one says that the symmetry of the limiting system is spontaneously broken. To get an idea what this means let us again consider the quantum Curie-Weiss Hamiltonian $H_{1/N}^{CW}$, i.e.

$$H_{1/N}^{CW} = -\frac{J}{2N}\sum_{i,j=1}^{N}\sigma_3(i)\sigma_3(j) - B\sum_{j=1}^{N}\sigma_1(j), \tag{7.1.1}$$

where $B \in (0,1)$ denotes the magnetic field and J a coupling constant that can be chosen to

[1] Mathematically, as already seen this parameter is an element of the base space corresponding to a C^*-bundle (see Definition 2.1.1). In the context of the classical limit, the zero-limit of this parameter corresponds to a classical theory, encoded by a commutative C^*-algebra. We stress again that the precise interpretation of this parameter depends on the physical situation.

be one. The symmetry-breaking term is typically taken to be

$$\delta^{CW}_{1/N} = \epsilon \sum_{x=1}^{N} \sigma_3(x). \tag{7.1.2}$$

In this approach originating with the ideas of Bogoliubov, one argues that the correct order of the limits should be $\lim \epsilon \to 0 \lim N \to \infty$ [100, 104], which gives SSB by one of the two pure classical ground states on the limit algebra $C(B^3)$, with B^3 the closed unit ball in $\mathbb{R}^3$, where the sign of ϵ determines the direction of symmetry breaking.[2] In contrast, the opposite order $\lim N \to \infty \lim \epsilon \to 0$ gives the symmetric but mixed (i.e. nonphysical) ground state on the limit algebra.[3] It is then said that the symmetry is broken spontaneously if there is a difference in the order of the limits, as exactly happens in this example.[4] In this sense, if SSB occurs, the limit $N \to \infty$ can indeed be seen as singular [9, 13, 104].

Instead, this thesis is based on an abstract definition of SSB characterized by a purely algebraic framework. It stands on an algebraic formulation of symmetries and ground states [19, 20, 57] and equally applies to finite and infinite systems and to classical and quantum systems, namely that the ground state, suitably defined of a system with G-invariant dynamics (where G is some group, typically a discrete group or a Lie group) is either pure (or more generally, extremal) but not G-invariant, or G-invariant but mixed. Even though at first sight both approaches to SSB might seem different, it turns out that the former approach is equivalent to the algebraic one (see e.g. [96, Prop. 1.6, Lemma 1.7] and [85, Ch. 6]). As already mentioned each approach to SSB has its pros and cons and which method is chosen depends on the specific purpose.

Remark 7.1.1. It may perhaps seem more natural to only require that the ground state fails to be G-invariant. However, since in the C^*-algebraic formalism ground states that are not necessarily pure are taken into account as well, this gives the possibility of forming G-invariant mixtures of non-invariant states that lose the purity properties one expects physical ground states to have. A similar statement holds for equilibrium states, where "pure" is replaced by "primary", which corresponds to a mathematical property of pure thermodynamic phases [19, 20, 57]. We refer to Section 7.2 for the precise definitions. ■

Accordingly, what is singular about the thermodynamic limit of systems with SSB is the fact that the exact pure ground state of a finite quantum system converges to a mixed state on the limit system, explained in detail in the previous chapter (see also Chapters 8–9 and [58, 66, 98]). In this algebraic approach the general physical idea that spontaneous symmetry breaking should be related to instability and sensitivity of the system against small perturbations in the relevant regime (see previous discussion) is elucidated in Section 7.4.

7.2 Dynamical symmetry groups, ground states and SSB in algebraic quantum theory

Let us remind the reader the general context where the notion of spontaneous symmetry breaking takes place.

Given a C^*-**dynamical system** $(\mathfrak{A}, \alpha)$ (cf. Definition 5.3.1) and an α-**invariant state** ω, i.e., $\omega(a) = \omega(\alpha_t(a))$ for every $a \in \mathfrak{A}$ and $t \in \mathbb{R}$, there is a unique one-parameter group of unitaries $U := \{U_t\}_{t\in\mathbb{R}}$ which implements α in the GNS representation, i.e., $\pi_\omega(\alpha_t(a)) = U_t^{-1}\pi_\omega(a)U_t$, and leaves fixed the cyclic vector $U_t\Psi_\omega = \Psi_\omega$ [64]. U is *at least* strongly continuous in $\mathfrak{B}(\mathcal{H}_\omega)$ as a consequence of the strong continuity of α in $\mathfrak{A}$ and the properties of the GNS construction. Indeed, we have that $||U_t^{-1}\pi_\omega(a)\Psi_\omega - \pi_\omega(a)\Psi_\omega|| = ||U_t^{-1}\pi_\omega(a)U_t\Psi_\omega - \pi_\omega(a)\Psi_\omega|| \leq ||U_t^{-1}\pi_\omega(a)U_t - \pi_\omega(a)|| = ||\alpha_t(a) - a||$.

Remark 7.2.1. The strong continuity of α in $\mathfrak{A}$ is not always compatible with *unbounded* self-adjoint generators of U (and bounded generator is equivalent to say that $\{U_t\}_{t\in\mathbb{R}}$ is continuous

[2] We refer to Chapter 3 and Chapter 8 for details on these states and the construction of this algebra.

[3] This mixture is precisely the one determined by Theorem 8.3.2.

[4] It can be shown that occurrence of SSB in the context of spin systems equivalently means the appearance of a non-trivial magnetization [95, 96].

also in the operator norm of $\mathfrak{B}(\mathcal{H}_\omega)$) as shown in Example 3.2.36 in [19]. Even if we shall deal with unbounded self-adjoint generators, the case we are about to discuss is safe. That is because the relevant C^*-algebra is made of compact operators and the following general result applies. ■

Proposition 7.2.2. *Let $A \in \mathfrak{A} := \mathfrak{B}_\infty(\mathcal{H})$ and $\{U_t\}_{t\in\mathbb{R}}$ be a one-parameter group of unitary operators in the Hilbert space $\mathcal{H}$ that is strongly continuous in $\mathfrak{B}(\mathcal{H})$. The one-parameter group of C^*-algebra automorphisms induced by U is strongly continuous in $\mathfrak{A}$: $||U_t^{-1}AU_t - U_u^{-1}AU_u|| \to 0$ if $t \to u$ (also if the selfajoint generator of U is unbounded).*

Proof. The group composition law permits us to prove the thesis just for $u = 0$ without loss of generality. A compact operator A is the norm-operator limit of $A_N = \sum_{k=1}^{N} \langle \psi_k \,, \cdot \rangle \phi_k$ for suitable ψ_k and ϕ_k. Writing $A = A_N + R_N$, where $||R_N|| \to 0$ as $N \to +\infty$, the following chain of inequalities hold $||U_tAU_{-t} - A|| \leq ||U_{-t}A_NU_t - A_N|| + ||U_{-t}R_NU_t - R_N|| \leq ||U_{-t}A_NU_t - A_N|| + ||U_{-t}R_NU_t|| + ||R_N|| = ||U_{-t}A_NU_t - A_N|| + 2||R_N||$. As $||R_N|| \to 0$ as $N \to +\infty$ independently of t, to conclude it is sufficient to prove that, for finite N, $||U_{-t}A_NU_t - A_N|| \to 0$ for $t \to 0$. In turn, exploiting the triangular inequality N times, the thesis is valid provided it holds for $N = 1$, i.e., for $A_{\psi,\phi} = \langle \psi \,, \cdot \rangle \phi$. Per direct inspection $||\langle \psi \,, \cdot \rangle \phi|| \leq ||\psi|| \, ||\phi||$ so that, defining $\psi_t := U_t\psi$ and $\phi_t := U_t\phi$, we have $||U_{-t}A_{\psi,\phi}U_t - A_{\psi,\phi}|| = ||\langle \psi_{-t} \,, \cdot \rangle \phi_{-t} - \langle \psi \,, \cdot \rangle \phi|| \leq ||\langle \psi_{-t} \,, \cdot \rangle (\phi_{-t} - \phi)|| + ||\langle \psi_{-t} - \psi \,, \cdot \rangle \phi|| \leq ||\psi|| \, ||\phi_{-t} - \phi|| + ||\psi - \psi_{-t}|| \, ||\phi|| \to 0$ for $t \to 0$. □

We now introduce the definition of a ground state of a C^*-dynamical system $(\mathfrak{A}, \alpha)$.

Definition 7.2.3. *A* **ground state** *of a C^*-dynamical system $(\mathfrak{A}, \alpha)$ is an algebraic state $\omega : \mathfrak{A} \to \mathbb{C}$ such that*

(a) the state is α-invariant, i.e, $\omega(\alpha_t(a)) = \omega(a)$ for all $t \in \mathbb{R}$ and all $a \in \mathfrak{A}$,

(b) the self-adjoint generator H of the strongly-continuous one-parameter unitary group $U_t = e^{-itH}$ which implements α in a given GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ under the requirement $U_t\Psi_\omega = \Psi_\omega$, has spectrum $\sigma(H) \subset [0, +\infty)$.

■

It is not difficult to prove that the set $S^{ground}(\mathfrak{A}, \alpha)$ of ground states of $(\mathfrak{A}, \alpha)$ is convex and $*$-weak closed (see e.g. [57]), so that it is also compact by the Banach-Alaoglu theorem. The Krein-Milman theorem implies that all ground states can be constructed out of limit points of convex combinations of *extremal ground states* in the $*$-weak topology. The relevance of the extremal ground states relies upon this property of them: they are the building blocks for constructing all other ground states exactly as **pure states**, i.e., extremal states in the convex weak-$*$ compact set of all algebraic states on $\mathfrak{A}$, which in turn are the building blocks for constructing all algebraic states. However the elements of $S^{ground}(\mathfrak{A}, \alpha)$ are not necessarily pure states. Nonetheless, in many cases of physical interest, extremal ground states are exactly the pure states which are also ground states [57].

When $(\mathfrak{A}, \alpha)$ is a C^*-dynamical system also endowed with a group G acting on $\mathfrak{A}$ with a group representation $\gamma : G \ni g \to \gamma_g$ in terms of C^*-automorphisms $\gamma_g : \mathfrak{A} \to \mathfrak{A}$, we say that G is a **dynamical symmetry group** if $\gamma_g \circ \alpha_t = \alpha_t \circ \gamma_g$ for all $g \in G$ and $t \in \mathbb{R}$.

According to [57], **spontaneous symmetry breaking (SSB)** occurs for a dynamical system $(\mathfrak{A}, \alpha)$ endowed with a dynamical symmetry group G if there are no G-invariant ground states which are extreme points in $S^{ground}(\mathfrak{A}, \alpha)$. Within the usual situation where the extremal points in $S^{ground}(\mathfrak{A}, \alpha)$ are the ground pure states of $\mathfrak{A}$, occurrence of SSB means that G-invariant ground states must be necessarily *mixed* states, and pure ground states cannot be G-invariant. A more frequent situation occurs when there is *at least* one extremal point in $S^{ground}(\mathfrak{A}, \alpha)$ that fails to be G-invariant (although invariant extreme ground states may exist). In this case one says that **weak SSB** takes place.

For the sake of shortness we can only stick to the succinctly illustrated relevant technical definitions, an exhaustive discussion on the physical importance of SSB in various contexts and on the different also inequivalent definitions of SSB is presented in [57, 100].

7.3 SSB of ground states as an emergent phenomenon in Berezin quantization on symplectic manifolds

The above definitions in particular apply to the commutative case where $\mathfrak{A} := C_0(S)$ endowed with the C^*-norm $||\cdot||_\infty$, referred to a symplectic manifold S and the associated Poisson structure $(C^\infty(S), \{\cdot,\cdot\})$. In this case the states ω are nothing but the *regular*[5] *Borel probability measures* μ_ω over S. More precisely, if $\omega : \mathfrak{A} \to \mathbb{C}$ is an algebraic state, the $C_0(S)$ version of Riesz's representation theorem of generally complex measures on locally compact Hausdorff spaces [84], taking continuity, positivity and $||\omega|| = 1$ into account, proves that $(\mathcal{H}_\omega, \pi_\omega, \Psi_\omega)$ has this form

$$\mathcal{H}_\omega = L^2(S, \mu_\omega), \quad (\pi_\omega(f)\psi)(\sigma) = f(\sigma)\psi(\sigma), \quad \Psi_\omega(\sigma) = 1, \quad \text{for all } f \in C_0(S),\ \psi \in \mathcal{H}_\omega \text{ and } \sigma \in S.$$

With this representation, the pure states are *Dirac measures* concentrated at any point $\sigma \in S$.

A C^*-dynamical system structure is constructed when the dynamical evolution is furnished by the pullback action of the Hamiltonian flow $\phi^{(h)}$, *provided it is complete*, generated by a (real) Hamiltonian function $h \in C^\infty(S)$, i.e., $\alpha_t^{(h)}(f) := f \circ \phi_t^{(h)}$ for every $f \in C_0(S)$ and $t \in \mathbb{R}$.

It is easy to demonstrate that $(C_0(S), \alpha^{(h)})$ is a C^*-dynamical system (in particular $\alpha^{(h)}$ leaves $C_0(S)$ invariant and is strongly continuous[6]). By direct inspection one sees that $\pi_\omega(C_c^\infty(S))$ is included in the domain of the self-adjoint generator of the unitary implementation of $\alpha^{(h)}$ in the GNS representation of a state ω. This generator acts as $-i\{h, \pi_\omega(f)\} = -i\{h, f\}$ if $f \in C_c^\infty(S)$, where $\{\cdot,\cdot\}$ is the Poisson bracket associated to the symplectic form.

Proposition 7.3.1. *The ground states of $(C_0(S), \alpha^{(h)})$ are all of the regular Borel probability measures on S whose support is contained in the closed set $N_h := \{\sigma \in S \mid dh(\sigma) = 0\}$.*

Proof. If the regular Borel probability measure μ on S has support in N_h, then it is invariant under the flow of h and in its GNS representation the action of that flow is trivial. Therefore its self-adjoint generator is the zero operator which fulfills the thesis. Suppose *vice versa* that a regular Borel probability measure μ on S defines a state invariant under the Hamiltonian flow, then μ itself must be $\phi^{(h)}$-invariant. Passing to the GNS representation, the condition of $\phi^{(h)}$ invariance implies $\langle\Psi_\mu, \{h, \pi_\omega(a)\}\Psi_\mu\rangle = 0$ for $a \in C_c^\infty(S)$, and the condition of positive self-adjoint generator implies $-i\langle\Psi_\mu, \pi_\mu(a)\{h, \pi_\omega(a)\}\Psi_\mu\rangle \geq 0$ for $a \in C_c^\infty(S)$. In other words, $\int_S \{h, a\} d\mu = 0$ and $-i\int_S \overline{a}\{h, a\} d\mu \geq 0$ must be valid for $a \in C_c^\infty(S)$. Decomposing $a = f + ig$ with f and g real valued and using the former condition, the latter yields $\int_S f\{h, g\} d\mu - \int_S g\{h, f\} d\mu \geq 0$ for all real valued $f, g \in C_c^\infty(S)$. Replacing f with $-f$, we conclude that actually the identity holds $\int_S f\{h, g\} d\mu - \int_S g\{h, f\} d\mu = 0$. Noticing that $g\{h, f\} = \{h, gf\} - f\{h, g\}$ and using again $\int_S \{h, a\} d\mu = 0$, we conclude that $\int_S f\{h, g\} d\mu = 0$ must be valid for every real valued $f, g \in C_c^\infty(S)$. Taking, e.g., $g \in C_c^\infty(S)$ such that $g(q, p) = q^1$ in an open set including the support of f, we have that $\int_S f \frac{\partial h}{\partial q^1} d\mu = 0$. In general $\int_S f(dh)_k d\mu = 0$ for every $k = 1, \ldots, \dim(S)$ and $f \in C_c^\infty(S)$ real valued. If, for a given k, $(dh)_k(\sigma_0) > 0$ (the case < 0 is analogous), then $(dh)_k(\sigma_0) > c > 0$ in an open neighborhood $O \ni \sigma_0$. Taking $f \in C_c^\infty(S)$ supported in O with $f \geq 0$, we have $0 = \int_S f(dh)_k d\mu > c\int_O f d\mu \geq 0$, so that $\int_O f d\mu = 0$. Since the functions of $C_c^\infty(O)$ are uniformly dense in $C_c(O)$, the result extends to $f \in C_c(O)$. Arbitrariness of f and the uniqueness part of Riesz' theorem for positive measures implies that $\mu(O) = 0$. In summary, the points σ with $(dh)_k(\sigma) \neq 0$ stay outside the support of μ. That is the thesis. □

If ω is a ground state of $(C_0(S), \alpha^{(h)})$, in view of the above discussion, $\alpha^{(h)}$ is trivially implemented: $U_t = I$ for every $t \in \mathbb{R}$ and the positivity condition on the spectrum of the generator of U_t is automatically fulfilled. In this case the extremal elements of $S^{ground}(\mathfrak{A}, \alpha)$ are the Dirac measures concentrated at the points $\sigma \in X$ such that $dh(\sigma) = 0$. In particular, *extremal* ground states are *pure* states.

[5] All positive Borel measures on S are automatically regular due to Theorem 2.18 in [84].

[6] To prove the continuity at $t = 0$, use in particular the fact that, if $f \in C_0(S)$ and $\delta > 0$, then $|f(\phi_t^{(h)}(\sigma))| < \epsilon$ for $|t| \leq \delta$ and $\sigma \notin K_{\epsilon,\delta}$, where the latter set is compact and $K_{\epsilon,\delta} := \Pi_S\Phi([-\delta, \delta] \times K_\epsilon)$ with $\Phi : (t, \sigma) \mapsto (t, \phi_t^{(h)}(\sigma))$, $\Pi_S : \mathbb{R} \times X \to X$ being the canonical projection, and the compact $K_\epsilon \subset X$ is such that $|f(\tau)| < \epsilon$ if $\tau \notin K_\epsilon$.

The accumulated results permit us to discuss the phenomenon of (weak) spontaneous symmetry breaking as an *emergent phenomenon* when passing from the quantum realm to the classical world by switching off $\hbar$ [57, 100]. We address the reader to the next two chapters where we will provide several examples in which the symmetry is typically (weakly) broken in the classical limit, whilst on the quantum side no form of symmetry breaking takes place. We in particular refer to Section 8.3.2, Section 9.5 or [57] for a wide discussion on the physical relevance of this viewpoint and the various implications in understanding the quantum-classical transition.

To put this concept in a general context, we assume the existence of a coherent pure state quantization $(\mathcal{H}_\hbar, \Psi_\hbar^\sigma, \mu_\hbar)_{\hbar\in I_0,\sigma\in S}$ of (S, ω_S) with Berezin quantization maps $Q_\hbar^B$, and a continuous bundle of C^*-algebras over I_0 with fibers $\mathfrak{A}_0 := C_0(S)$ and $\mathfrak{A}_\hbar := \mathfrak{B}_\infty(\mathcal{H}_\hbar)$ and consider the maps $Q_\hbar : C_0(S) \to \mathfrak{A}_\hbar$. We now focus on the (quantum) C^*-algebra $\mathfrak{A}_\hbar$

The dynamical evolution described by a $\hbar$-parametrized family of one-parameter group of C^*-authomorphisms $\mathbb{R} \ni t \mapsto \alpha_t^\hbar$ is induced by a corresponding $\hbar$-parametrized family of one-parameter unitary groups $\mathbb{R} \ni t \mapsto U_t^\hbar = e^{-itH_\hbar}$:

$$\alpha_t^\hbar(A) := U_{-t}^\hbar A U_t^\hbar, \quad (A \in \mathfrak{A}_\hbar\ ). \tag{7.3.3}$$

It follows that $\alpha^\hbar$ is strongly continuous in $\mathfrak{A}_\hbar$ due to Proposition 7.2.2, and thus $(\mathfrak{A}_\hbar, \alpha^\hbar)$ is a C^*-dynamical system. We assume that the generator $H_\hbar$ of $U^\hbar$ has compact resolvent and that its lowest eigenvalue is non-degenerate. In particular, $H_\hbar$ is affiliated with $\mathfrak{A}_\hbar$ (viewed as a von Neumann algebra) and thus, in particular, $H_\hbar$ is an observable of the physical system represented by $\mathfrak{A}_\hbar$.

Let us also assume the existence of a dynamical symmetry group G acting by symplectomorphisms on (S, ω_S), so that as a consequence of Prop. 6.2.2 the unitary action of G is given by the $Q_\hbar^B$-equivariant representation (6.2.15). In particular, $U_g Q_\hbar^B(f) U_g^* = Q_\hbar^B(\gamma_g f)$, where γ_g is given by $\gamma_g f = f \circ g^{-1}$. We furthermore require that $H_\hbar$ commutes with U_g for all $g \in G$.

In this situation, as opposed to the classical ($\hbar = 0$) case, no SSB occurs. This fact should be physically evident since there is only one quantum "ground state" (in the sense of a vector state with minimal energy) which is G-invariant. However the algebraic notion of ground state given above seems to be more complex and it deserves a closer scrutiny. We have the following general result.

Proposition 7.3.2. *Consider the C^*-dynamical system $(\mathfrak{B}_\infty(\mathcal{H}_\hbar), \alpha^\hbar)$, the latter defined in (7.3.3), and with dynamical symmetry group G whose unitary and $Q_\hbar^B$-equivariant action is defined in (6.2.14)-(6.2.15). Assume the generator $H_\hbar$ has compact resolvent, a non-degenerate lowest eigenvalue and that $H_\hbar$ commutes with the unitary representation of G on a densely defined domain of $H_\hbar$. No SSB (or weak SSB) occurs for $\hbar > 0$.*

Proof. As is well known (see e.g., Theorem 7.75 in [64]), if $\mathcal{H}$ is any complex Hilbert space, the algebraic states on $\mathfrak{B}_\infty(\mathcal{H})$ are all normal and coincide with the *statistical operators*: trace class, unit trace, positive operators $\rho : \mathcal{H} \to \mathcal{H}$. Here $\omega_\rho(A) = Tr(\rho A)$ for every $A \in \mathfrak{B}_\infty(\mathcal{H})$. Let us assume the dynamical invariance property $\omega_\rho(\alpha_t^\hbar(A)) = \omega_\rho(A)$ for every $t \in \mathbb{R}$ and $A \in \mathfrak{B}_\infty(\mathcal{H}_\hbar)$. Taking $A = \langle \cdot, \psi_\hbar^{(n)} \rangle \psi_\hbar^{(m)}$ where $H_\hbar \psi_\hbar^{(n)} = E_\hbar^{(n)} \psi_\hbar^{(n)}$[7] it is easy to prove that ρ must commute with the PVM of $H_\hbar$ so that, in the strong-topology $\rho = \sum_{n=0}^{+\infty} p_n \langle \cdot, \psi_\hbar^{(n)} \rangle \psi_\hbar^{(n)}$. Where $p_n \geq 0$ and $\sum_n p_n = 1$. The eigenvectors $\psi_\hbar^{(n)}$ may be a rearrangement of the initial ones separately in each eigenspace of $H_\hbar$. The GNS representation of ρ takes this form (up to unitary equivalence) as the reader can prove by direct inspection

$$\mathcal{H}_\rho := \oplus_{p_n \neq 0} L^2(\mathbb{R}^n, dx)\,; \quad \pi_\rho(A) := \oplus_{p_n \neq 0} A\,; \quad \Psi_\rho := \oplus_{p_n \neq 0} \sqrt{p_n} \psi_\hbar^{(n)}\,.$$

It is not difficult to see that the strongly continuous one-parameter group of unitaries V which leaves Ψ_ρ invariant and implements $\alpha^\hbar$ is here $V_t\,(\oplus_{p_n \neq 0} \phi_n) = \oplus_{p_n \neq 0} e^{-it(H_\hbar - E_\hbar^{(n)})} \phi_n$. Its generator is $K = \oplus_{p_n \neq 0}(H_\hbar - E_\hbar^{(n)} I)$. It is obvious that $\sigma(K) \subset [0, +\infty)$ only if $p_n \neq 0$ for $n = 0$ so that $\rho = \langle \cdot, \psi_\hbar^{(0)} \rangle \psi_\hbar^{(0)}$. The unique algebraic ground state is therefore $\omega_\hbar^{(0)}(A) = \langle \psi_\hbar^{(0)}, A \psi_\hbar^{(0)} \rangle$

[7] According to (9.2.23), $n = 0, 1, \ldots$ labels the Hilbert basis of eigenvectors of $H_\hbar$ with non-decreasing eigenvalues.

for $A \in \mathfrak{B}_\hbar$. This state is also G-invariant since the eigenspace of $H_\hbar$ with minimal eigenvalue has dimension one and the PVM of $H_\hbar$ commutes with the unitary representation of G, since $H_\hbar$ does so by hypothesis. □

7.4 SSB in Nature

In this section we discuss the role of spontaneous symmetry breaking in Nature. The previous discussions in this chapter (see also Section 8.3.2 and Section 9.5) have shown that SSB is a natural phenomenon emerging in the limit of large number of particles $N \to \infty$ or similarly, in the context of Schrödinger operators, in the classical limit in $\hbar \to 0$. Indeed, a pure state typically converges to a mixed state as we have seen in detail in Chapter 6. However, as explained in Section 7.2 this is not the whole story, since theoretically speaking SSB also occurs whenever pure states are not invariant under the pertinent symmetry group. It is precisely the latter notion of SSB that occurs in Nature: the limiting ground state is typically observed to be *asymmetric*, in the sense that it is pure but not G- invariant. This state is therefore called the "physical" ground state.

7.4.1 "Flea mechanism"

At first sight, spontaneous symmetry breaking (SSB) is a paradoxical phenomenon: in *Nature*, finite quantum systems (such as ferromagnets) evidently display it, yet in *Theory* it seems forbidden in such systems. Indeed, for finite quantum systems the ground state of a generic Hamiltonian is unique and hence invariant under whatever symmetry group G it may have (and similarly for equilibrium states at positive temperature, which are always unique). Hence SSB, in the sense of having a family of asymmetric ground states (or equilibrium states) related by the action of G, seems possible only in infinite quantum systems or in classical systems (for both of which the arguments proving uniqueness, typically based on the Perron–Frobenius Theorem, break down).

Applied to SSB in infinite quantum systems, this means that some approximate and robust form of symmetry breaking should already occur in large but finite systems, *despite the fact that uniqueness of the ground state seems to forbid this.* Similarly, SSB in a classical system should be foreshadowed in the quantum system whose classical limit it is, at least for tiny but positive values of Planck's constant $\hbar$. To accomplish this, it must be shown that for finite N or $\hbar > 0$ the system is not in its ground state (or equilibrium state), but in some other state having the property that as $N \to \infty$ or $\hbar \to 0$ it converges (in a suitable sense, see Landsman [57], Chapters 7, 8) to a symmetry-broken ground state (i.e. the physical ground state, as observed in Nature) of the limit system which is either an infinite quantum system or a classical system. Since the symmetry of a state is preserved under the limits in question (provided these are taken correctly), this implies that the actual physical state at finite N or $\hbar > 0$ must already break the symmetry.[8] We refer to Jona-Lasinio, Martinelli, & Scoppola [50] for $1d$ Schrödinger operators with a symmetric double well potential in the classical limit, and (independently) Koma & Tasaki [52], van Wezel [103] and van Wezel & van den Brink [104] for (general) quantum spin systems in the thermodynamic limit.

In summary, we are seeking for states satisfying the following two properties:[9]

- in the limit ($\hbar \to 0$ or $N \to \infty$) the states are pure but not G- invariant and therefore correspond to the correct physical (symmetry broken) ground states (even though

[8] *Bogoliubov's method of quasi-averages* [15] looks superficially similar to this idea, see e.g. Wreszinski & Zagrebnov [106]. However, in Bogoliubov's work the symmetry-breaking term seems to be a purely formal device which is removed after taking the appropriate limit. In this thesis, as well as in the works just cited in the main text, the symmetry-breaking perturbations are physical and the essential point is that their importance *grows* in the limit, or, as e.g. phrased in van Wezel & van den Brink [104]: "The general idea behind spontaneous symmetry breaking is easily formulated: as a collection of quantum particles becomes larger, the symmetry of the system as a whole becomes more unstable against small perturbations." The same is true as $\hbar$ becomes smaller at fixed system size, and in fact the analogy between $N \to \infty$ and $\hbar \to 0$ is strong, as shown in the paper [100] and M.Sc. thesis [97].

[9] We stress that these properties are compatible with the general idea regarding SSB, namely that the symmetry of the system as a whole should become more unstable against small perturbations whenever the collection of quantum particles becomes large.

mathematically, the limiting state predicted by theory is invariant, but mixed.

- they should break the symmetry already before the pertinent limit, meaning that $1 << N < +\infty$ or $1 >> \hbar > 0$. However, for relatively large values of $\hbar$ or small values of N the state should be G-invariant.

A mathematical mechanism is therefore required for obtaining these physical ground states used to represent symmetry breaking in Nature.[10] Based on the aforementioned two properties, we introduce an approach originating in the field of perturbation theory: small perturbations of the quantum Hamiltonian should yield the right physical (symmetry-broken) state as described above already for finite, but large N, or small values of $\hbar$. This would explain the fact that symmetry breaking occurs in real (and thus finite) materials. The physical intuition behind this mechanism is that these tiny perturbations should arise naturally and might correspond either to imperfections of the material or contributions to the Hamiltonian from the (otherwise ignored) environment.

Let us now illustrate this idea with an example introduced by Jona-Lasinio et al (later called the "flea on the elephant" [93]), originally defined for the theory of Schrödinger operators but later also adapted to mean-field quantum spin systems [100]. We hereto consider the Schrödinger operator with symmetric double well, defined on suitable dense domain in $\mathcal{H} = L^2(\mathbb{R})$ by

$$H_\hbar = -\hbar^2 \frac{d^2}{dx^2} + \frac{1}{4}\lambda(x^2 - a^2)^2, \tag{7.4.4}$$

where $\lambda > 0$ and $a \neq 0$. For any $\hbar > 0$ the ground state of this Hamiltonian is unique and hence invariant under the $\mathbb{Z}_2$-symmetry $\psi(x) \mapsto \psi(-x)$; with an appropriate phase choice it is real, strictly positive, and doubly peaked above $x = \pm a$. Yet the associated classical Hamiltonian

$$h_0(q,p) = p^2 + \frac{1}{4}\lambda(q^2 - a^2)^2, \tag{7.4.5}$$

defined on the classical phase space $\mathbb{R}^2$, has a two-fold degenerate ground state, namely the Dirac measures $\omega_\pm$ defined by

$$\omega_\pm(f) = f(\pm a, 0)., \tag{7.4.6}$$

where the point(s) (q_0, p_0) are the (absolute) minima of h_0. These (pure) states are clearly not $\mathbb{Z}_2$-invariant. From these, one may construct the mixed *symmetric* state $\omega_0 = \frac{1}{2}(\omega_+ + \omega_-)$, which in fact is the classical limit (cf. Theorem 9.3.4, item (b)) of the algebraic ground state $\omega_\hbar$ of (7.4.4) as $\hbar \to 0$, viz.

$$\omega_\hbar(a) = \langle \psi_\hbar^{(0)}, a\psi_\hbar^{(0)} \rangle, \ (a \in \mathfrak{B}_\infty(L^2(\mathbb{R}))); \tag{7.4.7}$$

defined in terms of the usual ground state eigenfunction $\psi_\hbar^{(0)} \in L^2(\mathbb{R})$ of $H_\hbar$ (assumed to be a unit vector).

In view of the above discussion we need to find a quantum "ground-ish" state that converges to either one of the physical classical ground states ω_+ or ω_- rather than to the nonphysical mixture ω_0. To this end, we perturb (7.4.4) by adding an asymmetric term δV (i.e., the "flea"), which, however small it is, under reasonable assumptions localizes the ground state $\psi_\hbar^{(\delta)}$ of the perturbed Hamiltonian in such a way that $\omega_\hbar^{(\delta)} \to \omega_+$ or ω_-, depending on the sign and location of δV (see Figures 7.1–7.2).[11] Furthermore, these figures show that symmetry breaking occurs

[10] A traditional mechanism to accomplish this, originating with Anderson [4], is based on forming symmetry-breaking linear combinations of low-lying states (sometimes called "Anderson's tower of states") whose energy difference vanishes in the pertinent limit. This approach frequently used in quantum spin systems indeed yields the right "physical" symmetry broken state in the pertinent limit (see also Section 7.1). However, it still does not account for the fact that in Nature real and hence *finite* materials evidently display spontaneous symmetry breaking, since in Theory it seems forbidden in such systems (since, as we have seen, it allows SSB only in classical or infinite quantum systems).

[11] The ground state wave function of the perturbed Hamiltonian (which has two peaks for $\delta V = 0$) localizes in a direction which given that localization happens may be understood from energetic considerations. For example, if δV is positive and is localized to the right, then the relative energy in the left-hand part of the double well is lowered, so that localization will be to the left.

already for small, but positive values of $\hbar$. This is precisely the essence of the argument and the flea mechanism: SSB is already foreshadowed in quantum mechanics for small yet positive $\hbar$, if only approximately. We moreover stress that in this approach a single limit suffices (in this example this is $\hbar \to 0$, whilst for mean-field quantum spin systems this would be $N \to \infty$ [100]).

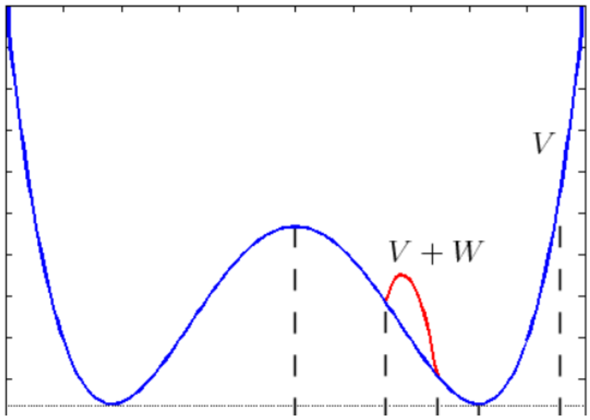

Figure 7.1: The $\mathbb{Z}_2$-symmetric double well potential V (blue) with assymetric perturbation indicated by W (red).

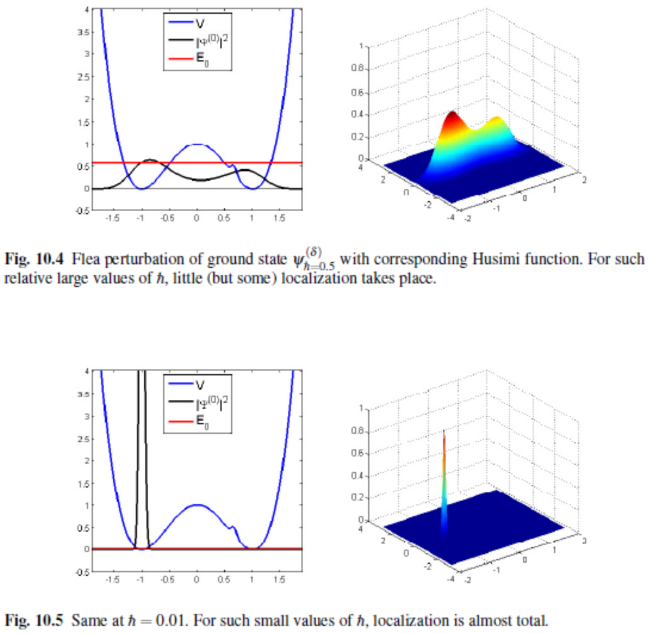

Figure 7.2: The ground state eigenfunction of the perturbed Hamiltonian (left) and the associated Husimi density (right) for two values of Planck's constant $\hbar$.

II Applications

In the second part we apply the previous results, especially those of Chapter 6 and Chapter 7, to concrete physical systems. We first consider mean-field quantum spin systems indexed by the number of spin particles N. In the limit $N \to \infty$ these systems have a classical counterpart on the closed unit three-ball $B^3 \subset \mathbb{R}^3$ and its smooth boundary, i.e. the two-sphere S^2. A detailed analysis is carried out and semi-classical results on such systems are proved. In particular the classical limit of the quantum Curie-Weiss model and its relation with symmetry breaking is discussed.

Finally, in Chapter 9 Schrödinger operators are studied. These operators are labeled by the parameter $\hbar$ and have a classical counterpart on $\mathbb{R}^{2n}$, if $\hbar$ is sent to zero in a suitable sense. As Schrödinger operators are unbounded one has to be more careful. We provide a method to prove, under certain conditions on the potential, the existence of the classical limit of a sequence of ground state eigenvectors. This is discussed for a certain class of potentials depending on their internal symmetry, and in this setting new light is shed on symmetry breaking in the classical regime $\hbar \to 0$.

8. Mean-field theories

In this chapter we focus on *homogeneous mean-field* theories, which are defined by a single-site Hilbert space $\mathcal{H}_x = \mathcal{H} = \mathbb{C}^n$ and local Hamiltonians of the type

$$H_\Lambda = |\Lambda|\tilde{h}(T_0^{(\Lambda)}, T_1^{(\Lambda)}, \cdots, T_{n^2-1}^{(\Lambda)}), \tag{8.0.1}$$

where $\tilde{h}$ is a polynomial on $M_n(\mathbb{C})$, and Λ denotes a finite lattice on which H_Λ is defined [57]. Here $T_0 = 1_{M_n(\mathbb{C})}$, and the matrices $(T_i)_{i=1}^{n^2-1}$ in $M_n(\mathbb{C})$ form a basis of the real vector space of traceless self-adjoint $n \times n$ matrices; the latter may be identified with i times the Lie algebra $\mathfrak{su}(n)$ of $SU(n)$, so that $(T_0, T_1, ..., T_{n^2-1})$ is a basis of i times the Lie algebra $\mathfrak{u}(n)$ of the unitary group $U(n)$ on $\mathbb{C}^n$. Finally, we define the macroscopic average spin operators

$$T_i^{(\Lambda)} = \frac{1}{|\Lambda|}\sum_{x\in\Lambda} T_i(x). \tag{8.0.2}$$

Example 8.0.1. Consider the quantum Curie-Weiss Hamiltonian on a chain (cf. Section 4.3)

$$H_{1/N}^{CW} : \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N\ times} \to \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N\ times};$$

$$H_{1/N}^{CW} = \left(-\frac{J}{2N}\sum_{i,j=1}^{N} \sigma_3(i)\sigma_3(j) - B\sum_{j=1}^{N}\sigma_1(j)\right). \tag{8.0.3}$$

Regarding (8.0.1) is it not difficult to see that

$$\tilde{h}^{CW}(T_1, T_2, T_3) = -2N(JT_3^2 + BT_1), \tag{8.0.4}$$

where $T_\mu = \frac{1}{N}\sum_{i=1}^N \sigma_\mu(i)/2,\ (\mu = 1, 2, 3)$. ■

8.1 Mean-field quantum spin systems, symbols and strict deformation quantization

As noticed by Landsman in [55, 57], a continuous bundle of C^*- algebras provides a natural setting to describe models in quantum statistical mechanics. By interpreting the semi-classical parameter as the number of particles of a system, namely $\hbar = 1/N \in 1/\mathbb{N} \cup \{0\}$, the limit $N \to \infty$ is often used to study the so-called *thermodynamic limit*, namely the density of the system N/V is kept fixed, and the volume V of the system sent to infinity, as well. This approach has been rigorously studied using operator algebras since the 1960s. The limiting system constructed in the limit $N = \infty$ is typically quantum statistical mechanics in infinite volume. In this setting the so-called *quasi-local* observables are studied: these give rise to a non-commutative continuous bundles of C^*-algebras (with C^*-algebra of the continuous bundle denoted by $\mathfrak{A}^{(q)}$) with fibers at $1/N$ given by a N-fold (projective) tensor product of a C^*-algebra with itself, the fiber at $N = \infty$, i.e., $1/N = 0$, is given by the infinite (projective) tensor product of this C^*-algebra, and similar as Section 2.2.2 the base space I is defined by

$$I = \{1/N \mid N \in \mathbb{N}\} \cup \{0\} \equiv (1/\mathbb{N}) \cup \{0\}, \tag{8.1.5}$$

with the topology inherited from $[0, 1]$. That is, we put $\hbar = 1/N$, where $N \in \mathbb{N}$ is interpreted a the number of sites of the model; the interest is the limit $N \to \infty$.

However, the limit $N \to \infty$ can also provide the relation between classical (spin) theories viewed as limits of quantum statistical mechanics. In this case the *quasi-symmetric* (or *macro-*

scopic) observables are studied and these induce a continuous bundle of C^*-algebras $\mathfrak{A}^{(c)}$ which is defined over the same base space $I := 1/\mathbb{N} \cup \{0\} \subset [0,1]$ with exactly the same fibers at $1/N$ as the bundle C^*- algebra $\mathfrak{A}^{(q)}$, but differ at $N = \infty$, in the sense that $\mathfrak{A}_0^{(c)}$ is commutative (viz. Section 3.2). It is precisely the algebra $\mathfrak{A}^{(c)}$ which relates these (spin) systems to strict deformation quantization, since macroscopic observables are defined by (quasi-) symmetric sequences which in turn are induced by the quantization maps (3.3.56). By Theorem 4.2.1 (i.e. for the case of (2×2)-matrices) these maps are related to Berezin quantization maps quantizing polynomials in three real variables restricted to S^2. Exactly these Berezin quantization maps can be used to prove properties of mean-field quantum spin systems in the limit as $N \to \infty$ as we see in this chapter.

8.1.1 Symbol and relation with quantization maps

Let us now introduce the notion of a **classical symbol**, i.e. a function

$$h_N := \sum_{k=0}^{M} N^{-k} h_k + O(N^{-(M+1)}), \tag{8.1.6}$$

for some $M \in \mathbb{N}$ and where each h_k is a smooth real-valued function on the manifold one considers. The first term h_0 is called the **principal symbol**.

For mean-field theories on a lattice it is precisely the commutative C^*-bundle with fibers (3.3.45)–(3.3.46) and quantization maps $Q_{1/N}$ defined by equations (3.3.56)–(3.3.57) that relates the corresponding quantum Hamiltonian to these symbols. The classical symbol h_N is typically a polynomial on $S(M_k(\mathbb{C}))$ and its image under the maps $Q_{1/N}$ yields the mean-field quantum Hamiltonian $H_{1/N}$ in question, i.e. h_N is said to be the *classical symbol* of $H_{1/N}$. However, as every spin interacts with every other spin the geometric configuration of the lattice is irrelevant [58, 100], without loss of generality we may restrict ourselves to mean-field quantum spin chains. For purpose of this thesis however we focus on one-dimensional spin chains, i.e. we consider tensor products of $M_2(\mathbb{C})$. As a result, the classical symbol becomes a polynomial in three real variables restricted to $S(M_2(\mathbb{C})) \cong B^3$.[1] Furthermore, we will see below that the associated principal symbol exactly plays the role of the polynomial $\tilde{h}$ defined in the very beginning of this chapter.

It is a known fact that mean-field Hamiltonians, initially defined on the Hilbert space $\bigotimes_{n=1}^N \mathbb{C}^2$, induce quasi-symmetric sequences. Such sequences typically leave the symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$ of dimension $N+1$ (the *symmetric* N-fold tensor product of $\mathbb{C}^2$ with itself) invariant. In what follows we consider mean-field quantum spin systems whose Hamiltonians $H_{1/N}$ are restricted to this subspace, since quantum spin systems arising in that way are typically of the form $Q_{1/N}^B(h_N|_{S^2})$ with $h_N|_{S^2}$ the classical symbol (cf. (8.1.6)) in three real variables restricted to $\partial B^3 = S^2$ and $Q_{1/N}^B$ is defined by (2.2.40) (we refer e.g. to Section 4.3 for an example). This fact follows from Theorem 4.2.1. Indeed, a mean-field quantum spin Hamiltonian $H_{1/N}$ restricted to the symmetric subspace is given by the operator $Q_{1/N}^B(h_0)+O(1/N)$ where h_0 is a polynomial (principal symbol) in three real variables restricted to S^2, and where $O(1/N)$ is meant in norm. Such spin systems are widely studied in condensed matter and theoretical physics, but also in mathematical physics they form an important field of research, especially in view of spontaneous symmetry breaking (SSB). One tries to calculate quantities like the free energy, or the entropy of the system in question and considers for example their thermodynamic limit as the number of sites N increases to infinity [60].

Remark 8.1.1. We stress that the parameter N at the same time may denote the number of spin $\frac{1}{2}$-particles described by the restricted mean-field Hamiltonian $H_N|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ as well as the total angular momentum $J = N/2$ of a single spin particle, already noticed by Ettore Majorana, proposing that pure (vector) states induced by a sequence of eigenvectors of an N-qubit system which are permutation-invariant correspond to pure spin $J = N/2$ quantum states which in turn are represented by N points on the Bloch sphere S^2 [62]. Comparing this with the result

[1] It is clear that for higher dimensional quantum spin chains, the classical symbol will be a polynomial in $k^2 - 1$ real variables restricted to $S(M_k(\mathbb{C}))$.

by Lieb [60], the "thermodynamic" limit $N \to \infty$ and the "classical" limit $J \to \infty$ are therefore taken in the same way. Note that this is based on the fact that the single-site algebra is fixed, i.e. we consider $M_2(\mathbb{C})$. If one instead considers a spin system on the N-fold tensor product of the the algebra $M_{2J+1}(\mathbb{C})$ with itself one can try to perform two limits $J, N \to \infty$, using the fact that $M_{2J+1}(\mathbb{C}) = Q^B_{1/J}(f)$ by surjectvity of (2.2.40). This goes beyond the scope of this thesis. ■

8.1.2 Examples

Example 8.1.2 (Example 1 revisited)**.** Let us go back to the example of the quantum Curie-Weiss model introduced in Section 4.3. We are interested in the classical symbol associated with this model. We already know from Section 4.3 that the Hamiltonian is a quantization of the *classical Curie-Weiss Hamiltonian* h_0^{CW} defined in (4.3.32). Let us perform the precise calculation. In order to do so we have to normalize the quantum Curie-Weis model of the previous example with a global factor $1/(N-1)$ [60, 98]. We now recall a result originally obtained by Lieb [60], namely that under the maps $Q^B_{1/N}$ given by (2.2.40) one has a correspondence between functions G (also called upper symbol) on the sphere S^2 and operators A_G on $\mathbb{C}^{N+1}$ such that they satisfy the relation $A_G = Q^B_{1/N}(G)$. For some spin operators, the functions G are determined (see Table 8.1 below). As usual each S_1, S_2, S_3 can be under-

Spin Operator	$G(\theta, \phi)$
S_3	$\frac{1}{2}(N+2)\cos(\theta)$
S_3^2	$\frac{1}{4}(N+2)(N+3)\cos(\theta)^2 - \frac{1}{4}(N+2)$
S_1	$\frac{1}{2}(N+2)\sin(\theta)\cos(\phi)$
S_1^2	$\frac{1}{4}(N+2)(N+3)\cos(\phi)\sin(\theta) - \frac{1}{4}(N+2)$
S_2	$\frac{1}{2}(N+2)\sin(\theta)\sin(\phi)$
S_2^2	$\frac{1}{4}(N+2)(N+3)\sin(\phi)\sin(\theta) - \frac{1}{4}(N+2)$

Table 8.1: Spin operators on $\mathrm{Sym}^N(\mathbb{C}^2) \simeq \mathbb{C}^{N+1}$ and their corresponding upper symbols G.

stood as a (unitary finite dimensional) irreducible representation of the Lie algebra $\mathfrak{su}(2)$ on the Hilbert space $\mathbb{C}^{N+1}$. Furthermore these operators satisfy $[S_1, S_2] = iS_3$ cyclically.[2] It is not difficult to see that $S_3 = \frac{1}{2}\sum_k \sigma_3(k)|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ and similarly for S_1 and S_2. Using these results, a straightforward computation based on a combinatorial argument shows that

$$H^{CW}_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)} = Q^B_{1/N}(h_0^{CW}) - \frac{3J}{N}Q^B_{1/N}(z^2) + \frac{1}{N}Q^B_{1/N}(1). \tag{8.1.7}$$

We write $h_N := h_0^{CW} + N^{-1}(-3Jz^2 + 1)$, so that by linearity of $Q^B_{1/N}$ one has $Q^B_{1/N}(h_N) = H^{CW}_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$. The function $h_N : S^2 \to \mathbb{R}$ is the classical symbol associated to the quantum spin operator $H^{CW}_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ in three real-variables restricted to the unit sphere S^2. The principal symbol h_0^{CW} indeed assumes the same form as $\tilde{h}^{CW}$ in (8.0.4). ■

Example 8.1.3. Let us consider the Lipkin-Meshkov-Glick (LMG) model. This model was first proposed to describe phase transitions in atomic nuclei [61, 38], and more recently it was found that the LMG model is relevant to many other quantum systems, such as cavity QED [64]. The (scaled) Hamiltonian of a general LMG model is given by

$$H^{LMG}_{1/N} = -\frac{\lambda}{N(N+2)}(S_1^2 + \gamma S_2^2) - \frac{B}{N+2}S_3, \tag{8.1.8}$$

where as before $S_1 = \frac{1}{2}\sum_k \sigma_1(k)$ is the total spin operator in direction 1 and so on. We are interested in $\lambda > 0$, standing for a ferromagnetic interaction, $\gamma \in (0, 1]$ describing the anisotropic in-plane coupling, and B is the magnetic field along z direction with $B \geq 0$. By a

[2] We recall from Chapter 5 that the number $J := N/2$ is also called the *spin* of the given irreducible representation.

similar computation as before, we find

$$H^{LMG}_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)} = Q^B_{1/N}(h^{LMG}_0) + \frac{1}{N}Q^B_{1/N}(1) - \frac{3}{2N}Q^B_{1/N}(1^2+\gamma 2^2), \tag{8.1.9}$$

where $h^{LMG}_0 := -\frac{1}{4}(x^2+\gamma y^2)-Bz$ denotes the principal symbol of $h_N := h^{LMG}_0+N^{-1}(-\frac{3}{2}(x^2+\gamma y^2)+1)$ is the classcial symbol in three real variables restricted to S^2. ■

8.2 Semi-classical properties of mean-field theories

We now apply the methods from Chapter 6 to mean-field quantum spin systems. We start with a result concerning the semi-classical behavior of the dynamics, followed by a subsection relating the asymptotic properties of the spectrum of a mean-field quantum spin Hamiltonian to the range of the principal symbol. Finally, we prove some fundamental properties of the eigenvectors of the corresponding Hamiltonian in the regime of large number of particles $N \to \infty$.

8.2.1 Classical limit of the dynamics

In this subsection we show that the *Heisenberg dynamics* on the algebra $\mathfrak{A}_{1/N} := \mathfrak{B}^{\otimes N}$ (in the specific case, when $\mathfrak{B} = M_k(\mathbb{C})$) defined by mean-field quantum spin Hamiltonians $H_{1/N}$ converges to the *classical dynamics* on the Poisson manifold $S(\mathfrak{B})$ generated by its corresponding principal symbol, i.e. a classical Hamiltonian h_0 defined on $S(M_k(\mathbb{C}))$. In contrast to the previous sections the theory based on the dynamics holds in a general setting where not only one-dimensional mean-field quantum spin systems (i.e. tensor products of $\mathfrak{B} = M_2(\mathbb{C})$) restricted to the symmetric subspace are considered. We henceforth focus on mean-field Hamiltonians defined on $\mathfrak{A}_{1/N}$. We shall mainly follow the approaches from [35, 36]. We start with a definition adapted from [35].

Definition 8.2.1. *Let $\tilde{\mathfrak{Y}}$ be the set of quasi-symmetric sequences introduced in Section 3.2. For each $N \in \mathbb{N}$ let $(T_{t,1/N} = \exp(tG_{1/N}))_{t\in\mathbb{R}}$ be a strongly continuous group of automorphisms of contractions on $\mathfrak{A}_{1/N}$ with generator $G_{1/N}$. We say that a sequence of such automorphisms has good mean-field properties if it is* **approximately symmetry preserving***, by which we mean that for all $x \in \tilde{\mathfrak{Y}}$ the sequence $N \mapsto T_{t,1/N}x_{1/N}$ is also in $\tilde{\mathfrak{Y}}$.* ■

For $x_{1/N} \in \mathfrak{A}_{1/N}$ we write $T.x.$ for the sequence $(T.x.)_{1/N} = T_{1/N}x_{1/N}$. It follows that if the above conditions are satisfied, $T_0 : x_0 \mapsto (T.x.)_0$ is well-defined. This yields the following theorem.

Theorem 8.2.2 ([35, Thm. 1.2])**.** *For each $N \in \mathbb{N}$ let $(T_{t,1/N} = \exp(tG_{1/N}))_{t\in\mathbb{R}}$ be a strongly continuous group of automorphisms of contractions on $\mathfrak{A}_{1/N}$. Then the following conditions are equivalent:*

(1) for each $t \in \mathbb{R}$, $T_{t,\cdot}$ is approximately symmetry preserving and the set of sequences x such that $x_{1/N} \in Dom(G_{1/N})$ and $||G_{1/N}x_{1/N}||$ is uniformly bounded, is dense in $\tilde{\mathfrak{Y}}$ in the seminorm defined by (3.2.38)*.*

(2) The operator G_0 defined on the domain $Dom(G_0) = \{x_0 \mid x. \in \tilde{\mathfrak{Y}} \text{ and } G.x. \in \tilde{\mathfrak{Y}}\}$ by $G_0x_0 = (G.x.)_0$ is well-defined, closed and generates a semigroup of contractions $T_{t,0}$ on $C(S(M_k(\mathbb{C})))$. This operator is defined by $T_{t,0}x_0 = (T_{t,\cdot}x.)_0$ where x_0 is given by (3.2.36) *or, equivalently by* (3.2.44)*.*

■

If these conditions are satisfied, $T_{t,0} = e^{tG_0}$ is the **mean-field limit** of $T_{t,\cdot}$. Moreover, $T_{t,0}$ is implemented by a weak*-continuous 2-sided flow $(F_t)_{t\in\mathbb{R}}$ on $S(\mathfrak{B})$, i.e., $\mathbb{R} \times S(\mathfrak{B}) \ni (t,\omega) \mapsto F_t(\omega)$ is jointly continuous, $F_t \circ F_s = F_{t+s}$ for all $t,s \in \mathbb{R}$, and $T_{t,0}(f) = f \circ F_t$. Equivalent conditions and more details can be found in [36, Theorem 2.3].

To move on our discussion we consider a specific class of generators, namely $G_{1/N}(.) = iN[H_{1/N}, .]$, for some $H_{1/N} \in \mathfrak{A}_{1/N}$, playing the role of a Hamiltonian.

We will show that under some assumptions on $H_{1/N}$, the group of automorphisms $T_{t,1/N} = \exp(itN\mathrm{ad}(H_{1/N}))$ has a mean field limit $T_{t,0}$ which is the group of automorphism of $C(S(\mathfrak{B}))$ generated by the vector field $\{h_0, .\}$, where $h_0 \in C(S(\mathfrak{B}))$. For this we need a notion of differentiability on the limiting space $C(S(\mathfrak{B}))$. We will see that the (quantum) commutators have limits as a certain Poisson-bracket for differentiable functions on $C(S(\mathfrak{B}))$. We follow the approach from [36, Definition 4.1].

Definition 8.2.3 ([35, Def. 2.1])**.** *We say that a function $f \in C(S(\mathfrak{B}))$ is differentiable if*

(1) for all $\omega \in S(\mathfrak{B})$ there exists an element $df(\omega)$ of $\mathfrak{B}$ such that for all $\sigma \in S(\mathfrak{B})$ the derivative

$$\langle \sigma - \omega, df(\omega)\rangle = \lim_{t\to 0} \frac{1}{t}(f((1-t)\omega + t\sigma) - f(\omega)) \tag{8.2.10}$$

exists as a weak-continuous affine functional of σ.*

(2) The maps $\omega \mapsto \langle \sigma - \omega, df(\omega)\rangle$ are weak-continuous, uniformly for $\sigma \in S(\mathfrak{B})$.*

■

For each ω we can fix a $df(\omega)$ as the unique element of $\mathfrak{B}$ such that $\langle \omega, df(\omega)\rangle = 0$, so that we can always write the derivative in the form $\langle \sigma, df(\omega)\rangle$. With this convention, item (2) above can be reformulated as saying that the map $\omega \mapsto df(\omega)$ is weak*-to-norm continuous.

Example 8.2.4. Let us consider the case $\mathfrak{B} = M_2(\mathbb{C})$. Then, for $\omega, \sigma \in S(\mathfrak{B}) \cong B^3$, and $f \in C(B^3)$:

$$\begin{aligned} &\lim_{t\to 0} \frac{1}{t}(f((1-t)\omega + t\sigma) - f(\omega)) = \\ &\lim_{t\to 0} \frac{1}{t}(f(\omega + t(\sigma - \omega)) - f(\omega)) = \\ &\langle \sigma - \omega, df(\omega)\rangle, \end{aligned} \tag{8.2.11}$$

provided f is the restriction to B^3 of some $\tilde{f} \in C^1(\mathbb{R}^3)$. Roughly speaking, this means that $\langle \sigma - \omega, df(\omega)\rangle = df(\omega) \cdot (\sigma - \omega)$.

The next step is to find the (2×2)-matrix associated to the differential $df(\omega)$. We first identify ω with a density matrix $\rho_{\mathbf{x}}$ parametrized by a unique point $\mathbf{x} \in B^3$, under the map $B^3 \ni \mathbf{x} \mapsto \frac{1}{2}(I + \mathbf{x} \cdot \overrightarrow{\sigma})$, with $\overrightarrow{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ and σ_k the three spin Pauli matrices. We denote by $a_{df(\mathbf{x})}$ the matrix associated to $df(\mathbf{x})$. Given now $\omega_1, \omega_2 \in S(M_2(\mathbb{C}))$, so that $\rho_{\mathbf{x_1}} = \frac{1}{2}(I + \mathbf{x_1} \cdot \overrightarrow{\sigma})$, and $\rho_{\mathbf{x_2}} = \frac{1}{2}(I + \mathbf{x_2} \cdot \overrightarrow{\sigma})$, where $\mathbf{x_1}$ and $\mathbf{x_2}$ uniquely correspond to ω_1 and ω_2, respectively. We can expand the matrix $a_{df(\mathbf{x_1})}$ as follows,

$$a_{df(\mathbf{x_1})} = f_0^{(\mathbf{x_1})} I + f^{(\mathbf{x_1})} \cdot \overrightarrow{\sigma},$$

for some coefficients $f_k^{(\mathbf{x_1})}$, $(k = 0, 1, 2, 3)$. By a computation it follows that

$$\langle \mathbf{x_2} - \mathbf{x_1}, df(\mathbf{x_1})\rangle = df(\mathbf{x_1}) \cdot (\mathbf{x_2} - \mathbf{x_1}) = f^{(\mathbf{x_1})} \cdot (\mathbf{x_2} - \mathbf{x_1})$$

If we set

$$f_k^{(\mathbf{x_1})} = df_k(\mathbf{x_1}); \; (k = 1, 2, 3),$$

we find

$$a_{df(\mathbf{x_1})} = df(\mathbf{x_1}) \cdot \overrightarrow{\sigma}.$$

■

Let x, y be sequences in $\tilde{\mathfrak{Y}}$. We denote by $[x, y]$ the sequence $n \mapsto [x_{1/N}, y_{1/N}]$. It is not true that $[x, y]$ is quasi-symmetric if x and y are quasi-symmetric. However, the next theorem gives a positive result in this direction when considering symmetric sequences. Here it also has been proven that $dx_0(\omega)$ exists for a symmetric sequence x [35].

Theorem 8.2.5. *Let x and y be symmetric sequences. Then $[x, y]$ is quasi-symmetric and for all $\omega \in S(\mathfrak{B})$ we have*

$$[x, y]_0(\omega) = N\langle\omega, [dx_0(\omega), dy_0(\omega)]\rangle. \tag{8.2.12}$$

■

Given now differentiable functions $f, g \in C(S(\mathfrak{B}))$, we define their bracket $\{f, g\}$ in $C(S(\mathfrak{B}))$ by

$$\{f, g\}(\omega) = i\langle\omega, [df(\omega), dg(\omega)]_{\mathfrak{B}}\rangle. \tag{8.2.13}$$

We now show that this bracket coincides with the one we has introduced in Definition 3.1.3. We fix $k \in \mathbb{N}$ arbitrary and consider $\mathfrak{B} = M_k(\mathbb{C})$.

Lemma 8.2.6. *The bracket* (8.2.13) *coincides (up to a sign) with the bracket defined in Definition 3.1.3.*

Proof. With abuse of notation, we denote $f \equiv \tilde{f} \in C^\infty(\mathbb{R}^{k^2-1})$ to be an arbitrary extension of the function $f \in S(M_k(\mathbb{C}))$ (see Section 3.1.3 for details). We then need to find the $(k \times k)$-matrix $a_{df(\mathbf{x})}$ associated to the differential $df(\mathbf{x})$. Given states $\omega, \sigma \in S(M_k(\mathbb{C}))$, then there exist unique density matrices ρ_ω and ρ_σ such that $\omega(a) = \mathrm{Tr}(\rho_\omega a)$, and $\sigma(a) = \mathrm{Tr}(\rho_\sigma a)$. The matrices ρ_σ and ρ_ω are characterized by the following expression

$$\rho_\omega = \frac{1}{k} I_k + \sum_{j=1}^{k^2-1} x_j b_j, \tag{8.2.14}$$

where $x_j \in \mathbb{R}$ and $b_j = iT_j$, for some basis $\{T_j\}_{j=1}^{k^2-1}$ of the Lie algebra of $SU(k)$, consisting of traceless anti-hermitian $(k \times k)$ matrices. A similar expression exists for ρ_σ, using the notation y_j for its coefficients. Expanding the matrix $a_{df(\mathbf{x})}$ in this basis yields

$$a_{df(\mathbf{x})} = f_0^{(\mathbf{x})} I_k + \sum_{j=1}^{k^2-1} f_j^{(\mathbf{x})} b_j, \tag{8.2.15}$$

where now the coefficients $f_j^{(\mathbf{x})}$ $(j = 0, ..., k^2 - 1)$ may be complex. We again compute (and find a similar expression)

$$\langle\sigma - \omega, df(\omega)\rangle = df(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) = f^{(\mathbf{x})} \cdot (\mathbf{y} - \mathbf{x}). \tag{8.2.16}$$

Therefore, we set

$$f_l^{(\mathbf{x})} = df_l(\mathbf{x}), \ (l = 1, , .., k^2 - 1), \tag{8.2.17}$$

so that

$$a_{df(\mathbf{x})} = \sum_{l=1}^{k^2-1} df_l(\mathbf{x}) b_l. \tag{8.2.18}$$

Finally, we have to show that the bracket is compatible with the one defined in Definition 3.1.3. We compute

$$\begin{aligned} [a_{df(\mathbf{x})}, a_{dg(\mathbf{x})}]_{M_k(\mathbb{C})} &= [df(\mathbf{x}) \cdot b, dg(\mathbf{x}) \cdot b]_{M_k(\mathbb{C})} \\ &= df_k(\mathbf{x}) dg_l(\mathbf{x}) [b_k, b_l]_{M_k(\mathbb{C})} \\ &= iC_{kl}^r df_k(\mathbf{x}) dg_l(\mathbf{x}) b_r. \end{aligned} \tag{8.2.19}$$

The coefficients C_{kl}^r are the structure constants coming from the Lie algebra of $SU(k)$. It follows that

$$i\langle\omega, [a_{df(\mathbf{x})}, a_{dg(\mathbf{x})}]_{M_k(\mathbb{C})}\rangle = -C^r_{kl} df_k(\mathbf{x}) dg_l(\mathbf{x}) \mathrm{Tr}(\rho_\omega \cdot \sigma_r) \\ = -C^r_{kl} df_k(\mathbf{x}) dg_l(\mathbf{x}) x_r, \tag{8.2.20}$$

which indeed (up to a minus sign) corresponds to the bracket defined in (3.1.23). □

A natural question is whether or not it is possible to extend the domain of (8.2.12) (and hence the domain of $\{\cdot,\cdot\}$ beyond the symmetric sequences. Otherwise we would be limited to strictly polynomial systems. These considerations lead to a less restrictive condition on the sequence $(H_{1/N})$. The precise result is states in the assumption below (see [35] for details and proof).

Assumption 8.2.7. *Let $H_{1/N} = \sum_{M\leq N} S_{M,N} H^{(N)}_{1/M}$, with $H_{1/M} \in \mathfrak{A}_{1/M}$, so that in particular $(H_{1/N})_N$ is quasi-symmetric. Assume*

(1) For all M the limit $\tilde{H}_{1/M} := \lim_{N\to\infty} H^{(N)}_{1/M}$ exists in the strong operator topology.

(2) $\sum_{m=1}^\infty m \sup_{N\geq M} ||H^{(N)}_{1/M}||_{1/M} < \infty$,

where the norm $||\cdot||_{1/M}$ is the operator norm on $\mathfrak{A}_{1/M}$.

These two conditions enable us to extend the bracket (8.2.12) beyond the symmetric sequences. We have the following theorem about the mean-field limit, taken from [35].

Theorem 8.2.8. *Given a sequence of self-adjoint Hamiltonians $(H_{1/N})_N$, satisfying the conditions of Assumption 8.2.7. Set, $T_{t,1/N} = \exp(itN ad(H_{1/N}))$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then,*

(1) $(T_{t,\cdot})_{t\in\mathbb{R}}$ has a mean-field limit $(T_{t,0})_{t\in\mathbb{R}}$ which is the group of automorphisms of $C(S(\mathfrak{B}))$ generated by $\{h_0, .\}$.

(2) $T_{t,0}$ is implemented by a flow F_t, a differential equation for $\omega_t = F_t\omega$ being $\langle \frac{d}{dt}\omega_t, a\rangle = i\langle\omega_t, [dh_0(\omega_t), a]\rangle$ for all $\omega \in S(\mathfrak{B})$, and $a \in \mathfrak{B}$.

■

Remark 8.2.9. *A general sequence $(y_{1/N})_N$ of self-adjoint Hamiltonians in $\tilde{\mathfrak{Y}}$ may not have a mean-field limit since the bracket (8.2.12) may not exists for such sequences.* ■

This machinery finally allows us to relate the quantum Heisenberg dynamics to the classical dynamics. To this end we focus on mean-field quantum spin Hamiltonians $H_{1/N}$ on $\mathfrak{A}_{1/N} = \mathfrak{B}^{\otimes N}$, (where $\mathfrak{B} = M_k(\mathbb{C})$ is fixed) satisfying Assumption 8.2.7. By Theorem 8.2.8 it follows that $(T_{t,1/N})_{t\in\mathbb{R}} = (\exp(itN\mathrm{ad}(H_{1/N})))_{t\in\mathbb{R}}$ has a mean field limit $(T_{0,t})_{t\in\mathbb{R}}$ on $C(S(\mathfrak{B}))$ generated by the vector field $\{h_0, .\}$. In particular, as a result of Theorem 8.2.2

$$T_{t,0} f = f \circ \phi^{h_0}_t =: \alpha^{h_0}_t(f), \tag{8.2.21}$$

where $\phi^{h_0}_t$ is the flow associated to the vector field $\{h_0, .\}$. Observe moreover that $T_{t,1/N}$ is nothing else than the time evolution à la Heisenberg:

$$T_{t,1/N} x_{1/N} = e^{itNH_N} x_{1/N} e^{-itNH_N}. \tag{8.2.22}$$

In what follows we apply these ideas to the quantization maps $Q_{1/N}$ defined by (3.3.56) on the domain $\tilde{A}_0$ given by (3.3.54). Hence, we let $x_{1/N} := Q_{1/N}(p)$, where $p \in \tilde{\mathfrak{A}}_0 \subset C(S(\mathfrak{B}))$, with $\mathfrak{B} = M_k(\mathbb{C})$. We wish to control the limit of $T_{t,1/N} Q_{1/N}(\alpha^{h_0}_t(p))$ (for p polynomial on $S(\mathfrak{B})$) as $N \to \infty$. However, for any polynomial p on $S(\mathfrak{B})$, $T_{t,0} \circ p \in C(S(\mathfrak{B}))$ is not necessarily a polynomial so that $Q_{1/N}(\alpha^{h_0}_t(p))$ may be not well-defined. In order to circumvent this problem one can try to extend the quantization maps (3.3.56) to all of $C(S(\mathfrak{B}))$. This is quite technical and will not be done in this thesis. Nonetheless, we can simply restrict to Hamiltonian systems for which the flow is polynomial. The precise statement is given in the following proposition.

Proposition 8.2.10. *Let $H_{1/N}$ be a mean-field quantum spin Hamiltonian satisfying Assumption 8.2.7, and denote by h_0 the limit given by* (3.2.36)*. Consider the Hamiltonian vector field $X^{h_0}(\cdot) = \{h_0, \cdot\}$ with Hamiltonian flow denoted by $\phi_t^{h_0}$. Assume there exist polynomials p in k^2-1 real variables restricted to $S(\mathfrak{B})$ such that the time evolution $\alpha_0^t(p) = p \circ \phi_t^{h_0}$ corresponding to the vector field X^{h_0} is a polynomial. Then for these p the following holds*

$$\lim_{N\to\infty} ||T_{t,1/N}(Q_{1/N}(p)) - Q_{1/N}(\alpha_t^{h_0}(p))||_N = 0. \tag{8.2.23}$$

In other words, the Heisenberg dynamics converges in norm to the classical dynamics.

Proof. Note that

$$T_{t,0}Q_0(p) = p \circ \phi_t^{h_0} = Q_0(\alpha_t^{h_0}(p)), \tag{8.2.24}$$

since $Q_0(f) = f$ by definition of the quantization maps. As $\tilde{\mathfrak{Y}}$ is an algebra, the sequences $(T_{t,1/N}(Q_{1/N}(p)))_N$ and $Q_{1/N}(\alpha_t^{h_0}(p))$ are (quasi) symmetric, also the sequence

$$\bigg(0, ..., T_{t,1/N}(Q_{1/N}(p)) - Q_{1/N}(\alpha_t^{h_0}(p)), ...,\bigg)_N \tag{8.2.25}$$

is (quasi) symmetric, and hence defines a continuous section of the bundle C^*-algebra $\mathfrak{A}^{(c)}$ defined by the fibers (3.3.45)–(3.3.46) in Chapter 3. It follows from the continuity properties of the bundle condition (cf. 2.1.1) that the norm limit equals zero. This proves the proposition. □

8.2.2 Semi-classical properties of the spectrum

In contrast to the quantization maps (3.3.56), the Berezin quantization maps $Q_{1/N}^B$ defined by (2.2.40) are expressed in terms of coherent states which allows us to prove semi-classical properties of mean-field quantum spin chains restricted to the symmetric subspace of dimension $N+1$. We have seen that the classical counterpart of such systems is the two-sphere S^2, and therefore we shall consider the symplectic manifold $(S^2, \sin\theta d\theta \wedge d\phi)$ where $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$ with associated Berezin quantization maps $Q_{1/N}^B$ defined for parameters varying in the discrete base space $(1/\mathbb{N}) \cup \{0\}$.

As a result, Theorem 6.1.2 applies to the manifold $(S^2, \sin\theta d\theta \wedge d\phi)$. It yields an important corollary, relating the spectrum of a mean-field quantum spin system with Hamiltonian $H_{1/N}$ to the range of its principal S^2. Indeed, as already mentioned in the beginning of this chapter, the operator $H_{1/N}$ leaves $\mathrm{Sym}^N(\mathbb{C}^2)$ invariant, and we therefore consider its restriction $H_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ which typically assumes the form $Q_{1/N}^B(h_N)$, where h_N is a classical symbol of the type $h = h_0 + \sum_{k=1}^M N^{-k}h_k$, with each h_i $(i = 0, ..., M)$ a polynomial in three real variables restricted to S^2. The precise result is stated in the following proposition.

Proposition 8.2.11. *Given a mean-field quantum spin system with Hamiltonian $H_{1/N}|_{Sym^N(\mathbb{C}^2)} = Q_{1/N}^B(h_N)$, where $Q_{1/N}^B$ is defined by* (2.2.40)*, h_N is a classical symbol of the type $h = h_0 + \sum_{k=1}^M N^{-k}h_k$, and each h_i $(i = 0, ..., M)$ is a real-valued polynomial in three real variables restricted to S^2. With a slight abuse of notation, let us write $H_{1/N} := H_{1/N}|_{Sym^N(\mathbb{C}^2)}$. Then, the spectrum of $H_{1/N}$ is related to the range of the principal symbol h_0 in the following way,*

$$\lim_{N\to\infty} dist(ran(h_0), \sigma(H_{1/N})) = 0. \tag{8.2.26}$$

Proof. This follows for example from Weyl's Theorem (see e.g. [102] for details) applied to the hermitian matrices $Q_{1/N}^B(h_0)$ and $Q_{1/N}^B(\sum_{k=1}^M N^{-k}(h_k)) = \sum_{k=1}^M N^{-k}Q_{1/N}^B(h_k)$, stating that if $\lambda_N^{(i)}$ is the i^{th} eigenvalue of $H_{1/N} = Q_{1/N}^B(h_0) + \sum_{k=1}^M N^{-k}Q_{1/N}^B(h_k)$, and $\epsilon_N^{(i)}$ is the i^{th}

eigenvalue of $Q_{1/N}(h_0)$, then

$$|\lambda_N^{(i)} - \epsilon_N^{(i)}| \leq ||Q_{1/N}(\sum_{k=1}^{M} N^{-k} h_k)|| \leq \frac{1}{N} \max_{1 \leq k \leq M} ||h_k||_\infty \to 0 \ \ (N \to \infty),$$

where in the last step we used that $||Q_{1/N}^B(h_k)|| \leq ||h_k||_\infty$. In particular, we conclude that $\lim_{N\to\infty} \mathrm{dist}(\sigma(Q_{1/N}^B(h_0)), \sigma(H_{1/N})) = 0$. By the triangle inequality applied to the distance function in Theorem 6.1.2, the result follows. □

8.2.3 Classical limit of eigenvectors

We prove the existence of the classical limit associated to a sequence of eigenvectors corresponding to a mean-field quantum spin Hamiltonian. We start with a result (Lemma 8.2.12) that is a specific case of Proposition 6.2.8 applied to $(S^2, \sin\theta d\theta \wedge d\phi)$.

Lemma 8.2.12. *Given a strict deformation quantization of S^2 with associated quantization maps $Q_{1/N}^B(h_0)$ defined by spin coherent states (cf.* (2.2.40)*), where h_0 is a continuous function on S^2. Given $E \in ran(h_0)$ and suppose that $\Omega \in h_0^{-1}(E)$, the preimage of E under h_0. Then it holds*

$$\lim_{N\to\infty} ||Q_{1/N}^B(h_0)\Psi_N^\Omega - E\Psi_N^\Omega|| = 0, \tag{8.2.27}$$

where $Q_{1/N}^B$ is the extension of the maps (2.2.40) *to $C(S^2)$ (cf. Lemma 2.1.7).*

Proof. On account of Proposition 6.2.8 we just have to prove that that function $N \mapsto c(1/N)|\langle\Psi_N^\Omega, \Psi_N^{\Omega'}\rangle|$, for $c(1/N) = \frac{N+1}{4\pi}$, goes to zero as $N \to \infty$. To this end we note that the overlap between two coherent spin states[3] is given by

$$|\langle\Psi_N^\Omega, \Psi_N^{\Omega'}\rangle| = \left(\frac{1+t}{2}\right)^{N/2}, \tag{8.2.28}$$

where $t \in [-1, 1)$ denotes the cosine of the angle between the both (different) coherent states. As a result,

$$\lim_{N\to\infty} \frac{N+1}{4\pi}\left(\frac{1+t}{2}\right)^{N/2} = 0. \tag{8.2.29}$$

□

Remark 8.2.13. We stress that the same result holds when we replace the principal symbol h_0 by h_N. Indeed, since such h_N is typically given by $h = h_0 + \sum_{k\geq 1}^M N^{-k} h_k$ for some $M \in \mathbb{N}$, by linearity $Q_{1/N}(h_N) = Q_{1/N}(h_0) + \sum_{k\geq 1}^M N^{-k} Q_{1/N}(h_k)$. Since each $||Q_{1/N}(h_k)|| \leq ||h_k||_\infty$, clearly one has $||\sum_{k\geq 1}^M N^{-k} Q_{1/N}(h_k)|| \leq \frac{1}{N}\max_{1\leq k\leq M} ||h_k||_\infty$, and hence $||Q_{1/N}(h_N)v^{(\Omega)} - Ev^{(\Omega)}|| \leq ||Q_{1/N}(h_0)v^{(\Omega)} - Ev^{(\Omega)}|| + O(1/N)$, so that by the lemma also $\lim_{N\to\infty} ||Q_{1/N}(h_N)v^{(\Omega)} - Ev^{(\Omega)}|| = 0$. This in particular implies that coherent spin states are "approximate" eigenvectors of restricted mean-field quantum spin Hamiltonians $H_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$. ■

In what follows we study the classical limit $N \to \infty$ with respect to the set of physical observables by $\{Q_{1/N}^B(f) \mid f \in C(S^2)\}$ where the quantization maps $Q_{1/N}^B(\cdot) \in \mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$ are defined by extension of the maps (2.2.40) to all of $C(S^2)$ which is possible by Lemma 2.1.7. To this end, we assume the following data of the corresponding Hamiltonians $H_{1/N}$ with principal symbol h_0 restricted to S^2.

Assumption 8.2.14. *We assume:*

[3]We particularly refer to equation (A.1.7) and the general construction on coherent spin states in Appendix A.1.

(a) G is a finite group and acts on $(S^2, \sin\theta d\theta \wedge d\phi)$ in terms of symplectomorphisms.

(b) $H_{1/N}$ restricted to $Sym^N(\mathbb{C}^2)$ commutes with the unitary representation U_g constructed by Proposition 6.2.2.

(c) The action of G leaves invariant $h_0^{-1}(\{\Lambda\})$ and is transitive on it.

■

This yields the following proposition proving the classical limit of a sequence of eigenvectors corresponding to $H_{1/N}$. We do not require the restriction of $H_{1/N}$ to the symmetric subspace, instead we require permutation invariance of the eigenvectors. We stress that the proposition is a special case of Theorem 6.2.5 specified to a finite group G and the manifold $(S^2, \sin\theta d\theta \wedge d\phi)$.

Proposition 8.2.15. *Let us consider a mean-field quantum spin system $H_{1/N}$ with principal symbol h_0 on S^2 such that the previous assumptions hold. Then the following fact is valid for every chosen $\Omega_0 \in h_0^{-1}(\{\Lambda\})$ and a family of permutation-invariant eigenvectors $\{\varphi_N\}_{N\in\mathbb{N}}$ of $H_{1/N}$ with non-degenerate eigenvalues $\{\lambda_N\}_{N\in\mathbb{N}}$ converging to Λ as $N \to \infty$ for some $\Lambda \in ran(h_0)$.*

$$\lim_{N\to\infty} \langle \varphi_N, Q^B_{1/N}(f)\varphi_N\rangle = \frac{1}{N_G}\sum_{g\in G} f(g\Omega_0), \quad \textit{for every } f \in C(S^2), \tag{8.2.30}$$

where N_G denotes the number of elements of G. Moreover, the right-hand side of (8.2.30) is independent of the choice of Ω_0.

Note that by the assumptions the eigenvectors are permutation-invariant and hence restrict to $\mathrm{Sym}^N(\mathbb{C}^2)$, so that the expression $\langle \varphi_N, Q^B_{1/N}(f)\varphi_N\rangle$ makes sense. Let us now prove the proposition.

Proof. This is a direct application of the techniques used in the proof of Theorem 6.2.5. We shall provide an alternative more direct proof exploiting the fact that G is finite. Let us define $\Gamma := h_0^{-1}(\{\Lambda\})$ which is a finite set consisting of at most N_G elements. Indeed, given $\Omega \in \Gamma$ this set satisfies $\Gamma = \{g\Omega_0 \mid g \in G\}$ for every chosen $\Omega_0 \in \Gamma$ due to G-invariance of h_0 and transitivity of G on Γ (Assumption 8.2.14 (c)).

Let us now label the elements $\Omega \in h_0^{-1}(\{\Lambda\})$ by Ω_i where $i = 0, .., n-1$. Obviously $n \leq N_G$. We define $A_i := \{g \in G \mid g\Omega_0 = \Omega_i\}$. By uniform continuity of f, given $\epsilon > 0$, we can find $\delta > 0$ such that for all $\Omega \in B_\delta(\Omega_i) = \{\Omega \in S^2 | \ d_{S^2}(\Omega, \Omega_i) < \delta\}$ one has $|f(\Omega) - f(\Omega_i)| < \epsilon/2n$, for all $i = 0, ..., n-1$. Let us fix one such neighborhood $U_0 := B_\delta(\Omega_0)$. As all points in Γ are distinct we can choose $g_i \in A_i$ $(i = 0, ..., n-1)$ so that the sets $U_i := g_i^{-1}U_0$ are neighborhoods of Ω_i, using the fact that the map $\Omega \mapsto g\Omega$ is a homeomorphism. Since the points Ω_i are distinct the n neighborhoods can be picked to be pairwise disjoint. As U_g commutes with $H_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ and the eigenspace spanned by $\varphi_\hbar$ is one-dimensional, it follows that $U_g\varphi_N = e^{ia}\varphi_N$ for some real a and thus $\Phi_N(g^{-1}\Omega) = e^{ia}\Phi_N(\Omega)$. Since the spherical measure $d\mu_{S^2}$ is G-invariant, the measure $d\mu_{\varphi_N} = \frac{N+1}{4\pi}|\Phi_N|^2 d\mu_{S^2}$ is also G-invariant. Writing $d\Omega := d\mu_{S^2}$ and $\Phi(\Omega) \equiv \Phi_N(\Omega) = \langle \varphi_N, \Psi_N^{(\Omega)}\rangle$ we find

$$\frac{N+1}{4\pi}\int_{U_0} d\Omega|\Phi(\Omega)|^2 = \frac{N+1}{4\pi}\int_{U_1} d\Omega|\Phi(\Omega)|^2 = ... = \frac{N+1}{4\pi}\int_{U_{n-1}} d\Omega|\Phi(\Omega)|^2. \tag{8.2.31}$$

Under stereographic projection the neighborhoods U_i correspond to open sets in $\mathbb{C}$ which we will denote by the same name, and the points $S^2 \ni \Omega_i$ will be denoted by $z_i \in \mathbb{C}$. Given now the sequence of normalized eigenstates $\varphi \equiv (\varphi_N)_N$ of $H_{1/N} = Q_{1/N}(h_N)$ (with $Q_{1/N}$ defined by (3.3.56) and where h_N is the classical symbol of the form (8.1.6) corresponding to $H_{1/N}$) and defining $\Psi^\Omega \equiv \Psi_N^\Omega$ we compute

$$\langle \varphi, Q^B_{1/N}(f)\varphi\rangle_{\mathbb{C}^{N+1}} = \frac{N+1}{4\pi}\int_{S^2} d\Omega f(\Omega)|\langle\varphi, \Psi^\Omega\rangle|^2 =$$
$$\frac{N+1}{\pi}\int_{\mathbb{C}} \frac{dz^2}{(1+|z|^2)^2} f(z)|\Phi(z)|^2. \tag{8.2.32}$$

We then split this integral as follows:

$$\int_{\cup_i U_i} + \int_{\mathbb{C}\setminus \cup_i U_i} \frac{N+1}{\pi} \frac{dz^2}{(1+|z|^2)^2} f(z)|\Phi(z)|^2. \tag{8.2.33}$$

In what follows we use the fact that

$$1 = \langle \varphi, \varphi \rangle = \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{dz^2}{(1+|z|^2)^2} |\Phi(z)|^2. \tag{8.2.34}$$

We then write

$$\begin{aligned}
&\lim_{N\to\infty} \left| \langle \varphi, Q_{1/N}(f)\varphi \rangle - \frac{1}{N_G} \sum_{g\in G} f(gz_0) \right| \\
&= \lim_{N\to\infty} \left| \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{dz^2}{(1+|z|^2)^2} f(z)|\Phi(z)|^2 - \frac{1}{N_G} \sum_{i=0}^{n-1} \sum_{g\in A_i} \frac{N+1}{\pi} \int_{\mathbb{C}} \frac{dz^2}{(1+|z|^2)^2} f(gz_0)|\Phi(z)|^2 \right| \\
&\leq \lim_{N\to\infty} \left| \frac{N+1}{\pi} \int_{\mathbb{C}\setminus \cup U_i} \frac{dz^2}{(1+|z|^2)^2} \left(f(z) - \frac{1}{N_G} \sum_{i=0}^{n-1} \sum_{g\in A_i} f(gz_0) \right) |\Phi(z)|^2 \right| \\
&\quad + \left| \frac{N+1}{\pi} \int_{\cup U_i} \frac{dz^2}{(1+|z|^2)^2} \left(f(z) - \frac{1}{N_G} \sum_{i=0}^{n-1} \sum_{g\in A_i} f(gz_0) \right) |\Phi(z)|^2 \right|
\end{aligned}$$

We now estimate the first integral:

$$\begin{aligned}
&\left| \frac{N+1}{\pi} \int_{\mathbb{C}\setminus \cup U_i} \frac{dz^2}{(1+|z|^2)^2} \left(f(z) - \frac{1}{N_G} \sum_{i=0}^{n-1} \sum_{g\in A_i} f(gz_0) \right) |\Phi(z)|^2 \right| \\
&\leq (N_G+1)||f||_\infty^2 \frac{N+1}{\pi} \int_{\mathbb{C}\setminus \cup U_i} \frac{dz^2}{(1+|z|^2)^2} |\Phi(z)|^2 \\
&= (N_G+1)||f||_\infty^2 ||\varphi||^2_{\mathbb{C}\setminus \cup U_i}.
\end{aligned} \tag{8.2.35}$$

Let us now consider the set $\mathcal{V}_\gamma := h_0^{-1}((\Lambda-\gamma, \Lambda+\gamma))$. By playing with γ we can obtain $\mathcal{V}_\gamma \subset \cup_i U_i$, so that the norm $||\varphi||^2_{\mathbb{C}\setminus \cup U_i}$ can be made bounded by the norm $||\varphi||_{S\setminus \mathcal{V}_\gamma}$. As a result of Proposition 6.2.3 applied to φ_N and $Q^B_{1/N}(h_0|_{S^2})$, the latter norm tends to zero as N becomes sufficiently large, so that (8.2.35) can be made bounded by $\epsilon/2$ as N is sufficiently large.[4] We are thus done if we can show that

$$\left| \frac{N+1}{\pi} \int_{\cup U_i} \frac{dz^2}{(1+|z|^2)^2} f(z)|\Phi(z)|^2 - \frac{1}{N_G} \sum_{i=0}^{n-1} \sum_{g\in A_i} \frac{N+1}{\pi} \int_{\cup U_i} \frac{dz^2}{(1+|z|^2)^2} f(gz_0)|\Phi(z)|^2 \right| < \epsilon/2,$$

whenever N sufficiently large. Since $U_i \cap U_j = \emptyset$ if $i \neq j$, we can write

$$\begin{aligned}
&\frac{N+1}{\pi} \int_{\cup U_i} \frac{dz^2}{(1+|z|^2)^2} f(z)|\Phi(z)|^2 \\
&= \sum_{i=0}^{n-1} \frac{N+1}{\pi} \int_{U_i} \frac{dz^2}{(1+|z|^2)^2} f(z)|\Phi(z)|^2.
\end{aligned}$$

[4]Since $H_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)} = Q_{1/N}(h_N)|_{\mathrm{Sym}^N(\mathbb{C}^2)} = Q^B_{1/N}(h_0|_{S^2}) + O(1/N)$, the vector φ_N is strictly speaking not an eigenvector of $Q^B_{1/N}(h_0|_{S^2})$, but only asymptotically. Nonetheless, with a slight adaptation of Proposition 6.2.3 one can still obtain the bound on the norm of φ.

Making use of (8.2.31) we can estimate

$$\left|\frac{N+1}{\pi}\int_{\cup U_i}\frac{dz^2}{(1+|z|^2)^2}f(z)|\Phi(z)|^2-\frac{1}{N_G}\sum_{i=0}^{n-1}\sum_{g\in A_i}\frac{N+1}{\pi}\int_{\cup U_i}\frac{dz^2}{(1+|z|^2)^2}f(gz_0)|\Phi(z)|^2\right|$$
$$\leq\left|\sum_{i=0}^{n-1}\int_{U_i}\frac{N+1}{\pi}\frac{dz^2}{(1+|z|^2)^2}f(z)|\Phi(z)|^2-\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\frac{|A_j|}{N_G}\frac{N+1}{\pi}\int_{U_j}\frac{dz^2}{(1+|z|^2)^2}f(gz_0)|\Phi(z)|^2\right|$$
$$\leq\sum_{i=0}^{n-1}\left|\frac{N+1}{\pi}\int_{U_i}\frac{dz^2}{(1+|z|^2)^2}(f(z)-f(gz_0))|\Phi(z)|^2\right|. \tag{8.2.36}$$

Since $\frac{N+1}{\pi}\int_{\mathbb{C}}\frac{dz^2}{(1+|z|^2)^2}|\Phi(z)|^2=1$, we find that each term is bounded by $\sup_{z\in U_i}|f(z)-f(gz_0)|=\sup_{z\in U_i}|f(z)-f(z_i)|$. By uniform continuity of f, each supremum is bounded by $\epsilon/2n$, so that (8.2.36) is bounded by $\epsilon/2$. This shows that

$$\left|\langle\varphi,Q^B_{1/N}(f)\varphi\rangle-\frac{1}{N_G}\sum_{g\in G}f(g\Omega_0)\right|<\epsilon, \tag{8.2.37}$$

whenever N sufficiently large, which concludes the proof of the theorem. □

8.3 Applications to Curie-Weiss model and SSB

In this section we first consider the manifold $(S^2,\sin\theta d\theta\wedge d\phi)$ and apply our findings to the Curie-Weiss model $H^{CW}_{1/N}$ introduced in Section 4.3. We show that Proposition 8.2.15 applies to the ground state eigenvector, yielding the classical limit of the Curie-Weiss model in terms of a mixed state on the phase space S^2. To this end let us show that Assumption 8.2.14 and the additional assumptions in Proposition 8.2.15 are met.

The relevant symmetry group acting on S^2 is $\mathbb{Z}_2$ which consists of two elements, and the non-trivial element ζ_{-1} acts as follows on functions f on the sphere S^2,

$$(\zeta_{-1}f)(\theta,\phi)=f(\pi-\theta,-\phi),\quad(\theta\in(0,\pi),\phi\in(-\pi,\pi)). \tag{8.3.38}$$

It is not difficult to see that this action leaves the Poisson bracket (2.2.32) invariant, and therefore it acts as symplectomorphisms.

Moreover, it is an easy exercise to see that the unitary representation U_{-1} on $\mathfrak{B}(\mathrm{Sym}^N(\mathbb{C}^2))$ predicted by Proposition 6.2.2 is implemented by the operator $U_{-1}:=\otimes_{n=1}^N\sigma_1|_{\mathrm{Sym}^N(\mathbb{C}^2)}$ (which makes sense since $\otimes_{n=1}^N\sigma_1$ definitely leaves invariant the subspace $\mathrm{Sym}^N(\mathbb{C}^2)\subset\bigotimes_{n=1}^N\mathbb{C}^2$. As $H^{CW}_{1/N}$ commutes with $\otimes_{n=1}^N\sigma_1$ and the operators $H^{CW}_{1/N}$ and $\otimes_{n=1}^N\sigma_1$ both leave the subspace $\mathrm{Sym}^N(\mathbb{C}^2)$ invariant, it follows that

$$[U_{-1},H^{CW}_{1/N}|_{\mathrm{Sym}^N(\mathbb{C}^2)}]=0. \tag{8.3.39}$$

We assume $B\in(0,1)$ and $J=1$, in which regime the CW model exhibits spontaneous symmetry breaking (SSB) as we shall see shortly. For this choice of parameters we consider the absolute minima of the classical CW Hamiltonian h_0^{CW} (cf. (4.3.32)) restricted to S^2 which plays the role of the principal symbol as explained in the previous section. That is, we put $\Lambda_0=\min_{\Omega\in S^2}\{h_0^{CW}(\Omega)\}$.[5] By a simple calculation these minima are attained in $\{x_\pm=(B,0,\pm\sqrt{1-B^2})\}$, or equivalently in spherical coordinates, in

$$\{(\theta,\phi)_\pm=(\cos^{-1}(\pm\sqrt{1-B^2}),0)\}. \tag{8.3.40}$$

The action ζ_{-1} obviously leaves $(h_0^{CW})^{-1}(\Lambda_0)$ invariant and acts transitively on it. In summary, we have shown that the conditions in Assumption 8.2.14 hold.

[5] We point out that both minima of h_0^{CW}, initially defined as points on B^3, lie on its boundary S^2.

On the quantum side one can prove (see Appendix C.1) that for each $N = 1, 2, 3, \dots$ the ground-state vector $\Psi_N^{(0)}$ of $H_{1/N}^{CW}$ is *unique* (up to phase factors and normalization) and belongs to the symmetric space

$$\mathrm{Sym}^N(\mathbb{C}^2) = \underbrace{\mathbb{C}^2 \otimes_s \cdots \otimes_s \mathbb{C}^2}_{N\ times} \cong \mathbb{C}^{N+1}. \tag{8.3.41}$$

In particular, the associated sequence of ground state eigenvalues $\lambda_N^{(0)}$ converge to Λ_0 as $N \to \infty$. This is proved in the following Lemma.

Lemma 8.3.1. *The sequence of ground state eigenvalues $\lambda_N^{(0)}$ corresponding to the operator $H_{1/N}^{CW}$ converge to $\Lambda_0 = min(h_0^{CW})$ as $N \to \infty$.*

Proof. By Theorem 6.1.2 we can immediately conclude that

$$\mathrm{dist}\bigg(\mathrm{ran}(h_0^{CW}), \sigma(Q_{1/N}^B(h_0^{CW}))\bigg) = \sup_{x \in \mathrm{ran}(h_0^{CW})} \inf_{y \in \sigma(Q_{1/N}^B(h_0^{CW}))} |x - y| \to 0, \ (N \to \infty). \tag{8.3.42}$$

We now show that the spectrum of $Q_{1/N}^B(h_0^{CW})$ is contained in the (connected and closed set) $\mathrm{ran}(h_0^{CW}) \subset \mathbb{R}$. Indeed, let $f := h_0^{CW} - \Lambda_0$. Then $f \geq 0$ so that by positivity of the Berezin map $Q_{1/N}^B$ also $Q_{1/N}^B(f) \geq 0$. By construction, the spectrum of $Q_{1/N}^B(f)$ satisfies

$$\sigma(Q_{1/N}^B(f)) = \sigma(Q_{1/N}^B(h_0^{CW})) - \Lambda_0 \subset [0, \infty)$$

Consequently,

$$\epsilon_{1/N}^{(0)} \geq \Lambda_0,$$

where $\epsilon_{1/N}^{(0)}$ is the minimum eigenvalue of $Q_{1/N}^B(h_0^{CW})$. Since by Rieffels condition also

$$|\epsilon_{1/N}^{(max)}| = ||Q_{1/N}^B(h_0^{CW})|| \leq ||h_0^{CW}||_\infty,$$

with $\epsilon_{1/N}^{(max)}$ the maximum eigenvalue of $Q_{1/N}^B(h_0^{CW})$, it follows that

$$\sigma(Q_{1/N}^B(h_0^{CW})) \subset \mathrm{ran}(h_0^{CW}), \tag{8.3.43}$$

as desired.

Combining (8.3.42) and (8.3.43) we obtain

$$\lim_{N\to\infty} |\epsilon_{1/N}^{(0)} - \Lambda_0| = 0;$$
$$\lim_{N\to\infty} |\epsilon_{1/N}^{(max)} - ||h_0^{CW}||_\infty| = 0.$$

Since $H_{1/N}^{CW} = Q_{1/N}(h_0^{CW}) + O(1/N)$ the same result applies to $\lambda_N^{(0)}$ as follows for example from the proof of Proposition 8.2.11. We conclude that

$$\lim_{N\to\infty} |\lambda_{1/N}^{(0)} - \Lambda_0| = 0,$$

as we needed to prove. ☐

We can finally apply Proposition 8.2.15 and find that for any $f \in C(S^2)$

$$\lim_{N\to\infty} \langle \Psi_N^{(0)}, Q_{1/N}^B(f)\Psi_N^{(0)} \rangle = \frac{1}{2}\bigg(f((\cos^{-1}(\sqrt{1-B^2}), 0)) + f((\cos^{-1}(-\sqrt{1-B^2}), 0))\bigg). \tag{8.3.44}$$

In other words,

$$\lim_{N\to\infty}\omega_{1/N}^{(0)}(Q_{1/N}^B(f)) = \omega^{(0)}(f), \tag{8.3.45}$$

where

$$\omega_{1/N}^{(0)}(Q_{1/N}^B(f)) := \langle\Psi_N^{(0)}, Q_{1/N}^B(f)\Psi_N^{(0)}\rangle \tag{8.3.46}$$

and

$$\omega^{(0)}(f) := \frac{1}{2}(f((\cos^{-1}(\sqrt{1-B^2}), 0)) + f((\cos^{-1}(-\sqrt{1-B^2}), 0))). \tag{8.3.47}$$

We now extend the previous result to polynomials on B^3, i.e. we consider $\tilde{\mathfrak{A}}_0$ (the set of polynomials in three real variables restricted to B^3). Indeed, we show that a similar result has been verified for all $f \in \tilde{\mathfrak{A}}_0$ using the quantization maps $Q_{1/N}$ defined by (3.3.56). This is proved in the following theorem.

Theorem 8.3.2. *Let $Q_{1/N} : \tilde{\mathfrak{A}}_0 \to M_2(\mathbb{C})^N$ be the quantization maps defined by linear extension of* (3.3.56) - (3.3.57)*, cf. Theorem 3.3.4, and let $\Psi_N^{(0)}$ be the (unit) ground state vector of the Hamiltonian* (4.3.28) *of the quantum Curie–Weiss model. Then*

$$\lim_{N\to\infty}\omega_{1/N}^{(0)}(Q_{1/N}(f)) = \omega^{(0)}(f), \tag{8.3.48}$$

for all $f \in \tilde{\mathfrak{A}}_0$, where $\omega_{1/N}^{(0)}$ and $\omega^{(0)}$ are defined in (8.3.46) *and* (8.3.47)*, respectively.*

Unfolding (8.3.48) on the basis of (8.3.46) and (8.3.47), the theorem therefore states that

$$\lim_{N\to\infty}\langle\Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle = \tfrac{1}{2}(f(\mathbf{x}_+) + f(\mathbf{x}_-)), \tag{8.3.49}$$

for any polynomial function f on B^3 (parametrizing the state space of $M_2(\mathbb{C})$), where the points $\mathbf{x}_\pm \in B^3$ are given by (8.3.40).

Proof. The proof of Theorem 8.3.2 is easy. Indeed, on account of Theorem 4.2.1 it follows that for any $f \in \tilde{\mathfrak{A}}_0$

$$|\omega_{1/N}^{(0)}(Q_{1/N}(f)) - \omega_{1/N}^{(0)}(Q_{1/N}^B(f|_{S^2}))| \leq ||Q_{1/N}(f)|_{\mathrm{Sym}^N(\mathbb{C}^2)} - Q_{1/N}^B(f|_{S^2})|| \to 0, \ (N\to\infty), \tag{8.3.50}$$

using the fact that $\Psi_N^{(0)} \in \mathrm{Sym}^N(\mathbb{C}^2)$ which is an invariant subspace for the operators $Q_{1/N}(f)$. □

Therefore, at least for polynomials f on B^3, the existence of the classical limit $\omega^{(0)}(f|_{S^2}) = \lim_{N\to\infty}\omega_{1/N}^{(0)}(Q_{1/N}^B(f|_{S^2}))$ predicted by Proposition 8.2.15 also implies that existence of the classical limit stated in Theorem 8.3.2 (cf. (8.3.48)).

Remark 8.3.3. The existence of the limit in Theorem 8.3.2 has been independently verified using numerical simulations, yielding convincing numerical evidence about the large N behavior of $\Psi_N^{(0)}$, summarized in Assumption 8.3.7 in Section 8.3.1 below. Even though a mathematical proof has been given above, for completeness of this thesis we provide a comprehensive overview of techniques used and numerical evidence obtained in order to arrive at the same result. ■

8.3.1 Alternative proof of Theorem 8.3.2

Our proof relies on the large-N behavior of the components of $\Psi_N^{(0)}$. By permutation symmetry of the Hamiltonian and uniqueness of the ground state we know that $\Psi_N^{(0)}$ lies in the symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2)$ of $(\mathbb{C}^2)^{N\otimes}$. To this end we use the Dicke basis of that subspace with respect to which the asymptotics of $\Psi_N^{(0)}$ will be studied. We refer to Appendix C.1 for details.

With the help of a good numerical evidence, we are in a position to verifiy (8.3.48) where, for convenience we fixed $B = 1/2$ and $J = 1$. We will take advantage of some preparatory results we are going to discuss. The first one is a pivotal proposition whose proof is a bit technical.[6]

Proposition 8.3.4. *Let $h : S^2 \to \mathbb{C}$ be a bounded measurable function that is $C^1(A)$ for some open set $A \subset S^2$. Then the following properties hold for every $\Omega' \in A$:*

(a) *If $\ell > 0$, then*

$$h(\Omega') = \lim_{N\to\infty} \frac{\ell(N+1)}{4\pi} \int_{S^2} h(\Omega)|\langle \Psi_N^{\Omega'}, \Psi_N^{\Omega}\rangle|^{2\ell} d\Omega. \tag{8.3.51}$$

(b) *In particular,*

$$\left| h(\Omega') - \frac{\ell(N+1)}{4\pi} \int_{S^2} h(\Omega)|\langle \Psi_N^{\Omega'}, \Psi_N^{\Omega}\rangle|^{2\ell} d\Omega \right| \leq \frac{B_\ell \|h\|_\infty + C_\ell^{(A)} \|dh\|_\infty^{(A)}}{\sqrt{N}}, \tag{8.3.52}$$

where

$$\|dh\|_\infty^{(A)} = \sup_{\Omega\in A} \sqrt{\mathbf{g}_\Omega(d\overline{h}, dh)}, \tag{8.3.53}$$

in which $\mathbf{g}_\Omega$ is the inner product on $T^*_\Omega S^2$ induced from $\mathbb{R}^3$, and $B_\ell, C_\ell^{(A)} \geq 0$ are constants independent of h and Ω' (but $C_\ell^{(A)}$ depends on A).

Proof. See Appendix B.3. □

Remark 8.3.5.

(1) Here $\|dh\|_\infty^{(A)}$ could be infinite and, in that case, (8.3.52) is trivially valid for every choice of $C_\ell^{(A)}$. It is, however, always possible to restrict A to a smaller open set with compact closure included in the initial set A where h is C^1. In that case, $\|dh\|_\infty^{(A)}$ is finite. This observation applies to all similar statements we will establish in the rest of the work.

(2) The apparently cumbersome formulation of Proposition 8.3.4, where A does not coincide with S^2, is really necessary, since we will use this and similar results exactly where the functions in question are not everywhere C^1.

(3) In view of Definition 2.2.2 (2) applied to the manifold $(S^2, \sin\theta d\theta \wedge d\phi)$ property (a) automatically holds . ■

Another crucial building block of the proof of Theorem 8.3.2 is good numerical evidence about the behavior of the coherent components of $\langle \Psi_N^{(0)}, \Omega\rangle$ for large N (see Appendix C.3). Namely, for sufficiently large N, we have for $\ell = 1$ and $\ell = 1/2$,

$$\frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^{2\ell} \approx \frac{N+1}{4\pi 2^\ell}|\langle \Psi_N^{\Omega_+}, \Psi_N^{\Omega}\rangle|^{2\ell} + \frac{N+1}{4\pi 2^\ell}|\langle \Psi_N^{\Omega_-}, \Psi_N^{\Omega}\rangle|^{2\ell}, \tag{8.3.54}$$

where $\Omega_\pm$ define a pair of corresponding unit vectors $\mathbf{x}_\pm$ as in (8.3.47), always assuming $J = 1$ and $B = 1/2$. In terms of polar angles θ, ϕ, these read

$$(\theta_+, \phi_+) = (\pi/6, 0), \quad (\theta_-, \phi_-) = (5\pi/6, 0). \tag{8.3.55}$$

Remark 8.3.6. The practical meaning of (8.3.54) is that, as N increases, the map $\Omega \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^{2\ell}$ increasingly accurately approximates a linear combination of two functions, each of which, in turn, tends to a Dirac delta distribution centered at Ω_+ and Ω_- respectively, in accordance with part (a) in Proposition 8.3.4. In particular, the set of points Ω where $\frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^{2\ell}$ is appreciably different from zero tends to concentrate around Ω_+ and Ω_-. ■

[6]Here, and henceforth in similar statements, when dealing with differentiable functions defined on S^2 we always refer to the differentiable structure induced on S^2 by $\mathbb{R}^3$.

We now focus on the behaviour of the ground state eigenvector $\Psi_N^{(0)}$. To this end, we occasionally write $\Omega_{\theta,\phi} \equiv \Psi_N^{\Omega}$ to emphasize the explicit dependence on the angles $\Omega \equiv (\theta, \phi)$.

In Figure 8.1 and Figure 8.2 below the function $(\theta, \phi) \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\theta,\phi}\rangle|^2$ is computed for $N = 150$; the peaks at the values $(\theta, \phi) = (\pi/6, 0)$ and $(\theta, \phi) = (5\pi/6, 0)$ are clearly visible.

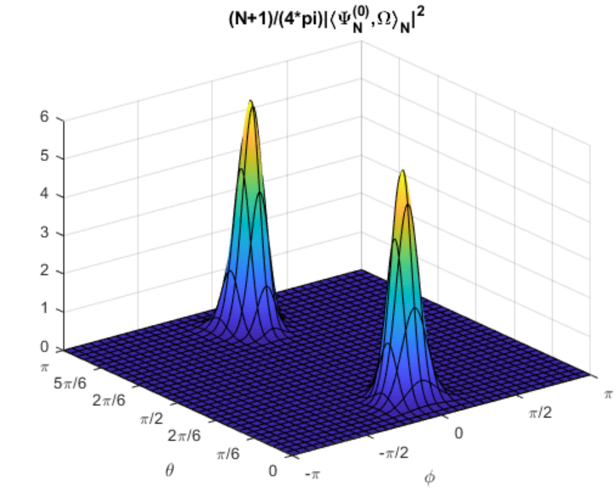

Figure 8.1: $\frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\theta,\phi}\rangle|^2$ as a function of θ and ϕ, for $N = 150, J = 1, B = 1/2$.

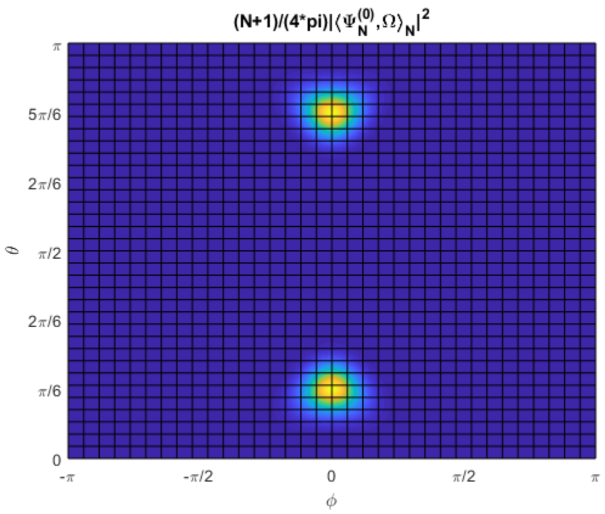

Figure 8.2: Top view of the previous plot.

In Figure 10 (Appendix C.3), the angle $\phi = 0$ is fixed and a plot of the two functions

$$\theta \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\theta,0}\rangle|^2; \tag{8.3.56}$$

$$\theta \mapsto \frac{N+1}{8\pi}|\langle \Omega_+, \Omega_{\theta,0}\rangle|^2 + \frac{N+1}{8\pi}|\langle \Omega_-, \Omega_{\theta,0}\rangle|^2 \tag{8.3.57}$$

is given. It is evident that the two graphs are almost indistinguishable and this fact becomes more and more evident as N increases. Similarly, in Figure 11, the angle $\theta = \pi/6$ is fixed and a plot of the two functions

$$\phi \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\pi/6,\phi}\rangle|^2; \tag{8.3.58}$$

$$\phi \mapsto \frac{N+1}{8\pi}|\langle \Omega_+, \Omega_{\pi/6,\phi}\rangle|^2 + \frac{N+1}{8\pi}|\langle \Omega_-, \Omega_{\pi/6,\phi}\rangle|^2 \tag{8.3.59}$$

is displayed. It is once again evident that the two graphs are almost indistinguishable and this fact becomes the more evident the N increases. We repeated the same analysis for the point $5\pi/6$, but omitted this plot as its graph looks similar due to symmetry. Moreover, in the appendix we produce similar plots for $\ell = 1/2$.

Concerning assumptions (a) and (b) below, we will employ an L^2 interpretation of (8.3.54) for $\ell = 1$ partially suggested by Remark 8.3.6, and an even weaker interpretation for $\ell = 1/2$. As a matter of fact, the proof of Theorem 8.3.2 directly uses the three requirements in Assumption 8.3.7 below which are supported by numerical evidence (Appendix C.3), independently of (8.3.54).

To state item (c) in these assumptions, we define, for $\Omega_0 \in S^2$ and $r > 0$,

$$D_r(\Omega_0) = \{\Omega \in S^2 \mid \Phi(\Omega, \Omega_0) < r\}, \tag{8.3.60}$$

where Φ denotes the angle between Ω and Ω_0. It is clear that $D_r(\Omega_0)$ is a geodesic disk on S^2 centered at Ω_0 with radius r.

Assumption 8.3.7. *On numerical evidence (Appendix C.3), we assume the following properties:*

(a) $\lim_{N\to\infty}$

$$\int_{S^2} \left(\frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^2 - \frac{N+1}{8\pi} |\langle \Psi_N^{\Omega_+}, \Psi_N^{\Omega} \rangle|^2 - \frac{N+1}{8\pi} |\langle \Psi_N^{\Omega_-}, \Psi_N^{\Omega} \rangle|^2 \right) d\Omega = 0. \tag{8.3.61}$$

(b) *There is a constant $G \geq 0$ such that for every $N \in \mathbb{N}$ and $\ell = 1/2, 1$,*

$$\int_{S^2} \left| \frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2\ell} - \frac{N+1}{4\pi 2^{\ell}} |\langle \Psi_N^{\Omega_+}, \Psi_N^{\Omega} \rangle|^{2\ell} - \frac{N+1}{4\pi 2^{\ell}} |\langle \Psi_N^{\Omega_-}, \Psi_N^{\Omega} \rangle|^{2\ell} \right| d\Omega \leq G. \tag{8.3.62}$$

(c) *For every $n \in \mathbb{N}$ and $\ell = 1/2, 1$, the sequence of maps*

$$S^2 \setminus D_{1/n}(\Omega_+) \cup D_{1/n}(\Omega_-) \ni \Omega \mapsto \frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2\ell} \tag{8.3.63}$$

is bounded by some constant $K_n \geq 0$ and pointwise converges to 0.

■

Remark 8.3.8.
(a) Using Lebesgue's dominated convergence theorem, item (c) implies in particular that, if $A \subset S^2$ is a given open set containing Ω_+ and Ω_-, then

$$\lim_{N\to\infty} \int_{S^2 \setminus A} \frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2\ell} = 0. \tag{8.3.64}$$

(b) For given $\ell = 1/2$ or 1, the class of functions

$$S^2 \ni \Omega \mapsto \frac{N+1}{4\pi 2^{\ell}} |\langle \Psi_N^{\Omega_+}, \Psi_N^{\Omega} \rangle|^{2\ell} - \frac{N+1}{4\pi 2^{\ell}} |\langle \Psi_N^{\Omega_-}, \Psi_N^{\Omega} \rangle|^{2\ell} \tag{8.3.65}$$

also satisfies (c), as is clear from the proof of Proposition 8.3.4. ■

Together with Proposition 8.3.4 and the elementary facts about the states $|\Omega\rangle$ presented in Appendix A.1, these properties of $\Psi_N^{(0)}$ (assumed valid on the basis of their numerical evidence) are the source of the following two lemmas:

Lemma 8.3.9. *Let $h : S^2 \to \mathbb{C}$ be a bounded measurable function that is $C^1(A)$ for some open set $A \subset S^2$ containing both Ω_+ and Ω_-. On Assumption 8.3.7, where (b) and (c) are required* *only for $\ell = 1$, one has*

$$\lim_{N\to\infty} \frac{(N+1)}{4\pi} \int_{S^2} h(\Omega)|\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 d\Omega = \tfrac{1}{2}h(\Omega_+) + \tfrac{1}{2}h(\Omega_-). \tag{8.3.66}$$

Proof. See Appendix B.3. □

Notation 8.3.10. From now on, S denotes the **south pole** of S^2 determined by $\theta = \pi$ in standard spherical polar coordinates. ■

Lemma 8.3.11. *Let $h : S^2 \to \mathbb{C}$ be a bounded measurable function that is $C^1(A)$ for some open set $A \subset S^2$ that does not contain S. On Assumption 8.3.7, where (b) and (c) are required only for $\ell = 1$, for any $\Omega' \in A$, $M \in \mathbb{N}$, and $N > M$ one has*

$$\begin{aligned}\left|\int_{S^2} \frac{N+1}{4\pi} \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle h(\Omega) \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'}\rangle d\Omega - \langle \Psi_N^{(0)}, \Psi_N^{\Omega'}\rangle h(\Omega')\right| \\ \leq \frac{K^{(A)}\|h\|_\infty + \sqrt{C\|f\|_\infty^2 + D^{(A)}\|dF\|_\infty^{(A)}}}{(N-M)^{1/4}},\end{aligned} \tag{8.3.67}$$

where the constants $C, K^{(A)}, D^{(A)} \geq 0$ may depend on M, and $K^{(A)}$ and $D^{(A)}$ may also depend on A, but $C, K^{(A)}, D^{(A)}$ are independent of Ω', h, and F, where

$$F(\Omega) = |h(\Omega) - h(\Omega')|^2. \tag{8.3.68}$$

Proof. See Appendix B.3. □

After these preparations we are finally in a position to prove Theorem 8.3.2.

Alternative proof of Theorem 8.3.2. Let us start the analysis of the large-N behavior of the expectation value $\langle \Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle$ for some fixed polynomial $f = f(\mathbf{x})$ in the components x_1, x_2, x_3 of $\mathbf{x} \in B^3$ (always supposing $J = 1$, $B = 1/2$). From (A.1.6) we have

$$\langle \Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle = \frac{N+1}{4\pi}\int_{S^2} d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle \langle \Psi_N^{\Omega}, Q_{1/N}(f)\Psi_N^{(0)}\rangle. \tag{8.3.69}$$

We argue that the above limit for $N \to \infty$ can be computed by restricting the integration set to $S^2 \setminus E$, where E is the closure of an open neighborhood of S such that E does not include Ω_+ and Ω_-. Indeed,

$$\begin{aligned}\left|\frac{N+1}{4\pi}\int_E d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle \langle \Psi_N^{\Omega}, Q_{1/N}(f)\Psi_N^{(0)}\rangle\right| \leq \\ \frac{N+1}{4\pi}\int_E d\Omega |\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle| \, \|\Psi_N^{\Omega}\| \|\Psi_N^{(0)}\| \|Q_{1/N}(f)\|,\end{aligned} \tag{8.3.70}$$

where $\|\Psi_N^{\Omega}\|^2 = \|\Psi_N^{(0)}\|^2 = 1$, and $\|Q_{1/N}(f)\|_N \to \|f\|_\infty$ as $N \to \infty$. Shrinking E if necessary, assumption (c) and Remark 8.3.8 part (a) therefore imply that

$$\left|\frac{N+1}{4\pi}\int_E d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle \langle \Psi_N^{\Omega}, Q_{1/N}(f)\Psi^{(0)}\rangle\right| \to 0. \tag{8.3.71}$$

In summary, decomposing the integration set in (8.3.69) as $S^2 = E \cup (S^2 \setminus E)$, we conclude that

$$\begin{aligned} L &= \lim_{N\to\infty} \langle \Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle \\ &= \lim_{N\to\infty} \frac{N+1}{4\pi}\int_{S_E^2} d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle \langle \Psi_N^{\Omega}, Q_{1/N}(f)\Psi_N^{(0)}\rangle,\end{aligned} \tag{8.3.72}$$

where we have defined $S^2_E = S^2 \setminus E$. Taking this result into account and exploiting (A.1.6) again, our final task just consists of computing the limit

$$L = \lim_{N\to\infty} \frac{(N+1)^2}{(4\pi)^2} \int_{S^2_E} d\Omega' \int_{S^2} d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle \langle \Psi_N^{\Omega}, Q_{1/N}(f) \Psi_N^{\Omega'} \rangle \langle \Psi_N^{\Omega'}, \Psi_N^{(0)} \rangle. \tag{8.3.73}$$

In view of the definitions of $Q_{1/N}$ and $\tilde{A}_0$, and taking advantage of linearity, it is sufficient to prove the claim for polynomials of the form

$$f(\mathbf{x}) = x_{j_1} \cdots x_{j_M}, \quad j_r \in \{1,2,3\}, \quad r = 1, \dots, M. \tag{8.3.74}$$

In this case, if $N \geq M$, we have

$$Q_{1/N}(f) = S_{M,N}(\sigma_{j_1} \otimes \cdots \otimes \sigma_{j_M} \otimes I_2 \otimes \cdots \otimes I_2). \tag{8.3.75}$$

The decisive observation for applying the technical results we have accumulated is that, as the states Ψ_N^{Ω} are factorized as in (2.2.43), we must have

$$\langle \Psi_N^{\Omega}, Q_{1/N}(f) \Psi_N^{\Omega'} \rangle_N = \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'} \rangle_{N-M} \langle \Psi_M^{\Omega}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle_M, \tag{8.3.76}$$

where

$$\langle \Psi_M^{\Omega}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle_M = \langle \Psi_M^{\Omega}, \sigma_{j_1} \otimes \cdots \otimes \sigma_{j_M} \Psi_M^{\Omega'} \rangle_M. \tag{8.3.77}$$

Note that we have explicitly written the the dependence of the inner product on the number of tensor factors. This entails

$$L = \lim_{N\to\infty} \frac{(N+1)^2}{(4\pi)^2} \int_{S^2_E} d\Omega' \int_{S^2} d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle_N \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'} \rangle_{N-M} \langle \Psi_M^{\Omega}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle_M \langle \Psi_N^{\Omega'}, \Psi_N^{(0)} \rangle_N. \tag{8.3.78}$$

The idea is now to apply Lemma 8.3.11 to the inner integral

$$\frac{N+1}{4\pi} \int_{S^2} d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle_N \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'} \rangle_{N-M} \langle \Psi_M^{\Omega}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle_M, \tag{8.3.79}$$

where the function $h \equiv f$ in the hypotheses of the lemma is now specialized to

$$S^2 \ni \Omega \mapsto k(\Omega, \Omega') = \langle \Psi_M^{\Omega}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle, \tag{8.3.80}$$

which depends also *parametrically* on Ω'. The map $S^2 \times S^2 \ni (\Omega, \Omega') \mapsto k(\Omega, \Omega')$ is trivially bounded and measurable (also in each variable separately). Furthermore, for every fixed $\Omega' \in S^2_E$, the restriction $S^2 \ni \Omega \mapsto k(\Omega, \Omega')$ is $C^1(A)$ with $A = S^2_E = S^2 \setminus E$ and the Ω-derivatives of $k(\Omega, \Omega')$ are jointly continuous on $A \times A$. If necessary we can redefine E as a smaller set, in order that the continuity of those derivatives remains still valid on the compact set $\overline{A}$. In this way, we obtain

$$\| d_\Omega K(\cdot, \cdot) \|_\infty^{(A\times A)} = \sup_{\Omega, \Omega' \in A} \mathbf{g}_\Omega (d_\Omega K(\Omega, \Omega'), d_\Omega K(\Omega, \Omega')) < \infty, \tag{8.3.81}$$

where $K(\Omega, \Omega') = |\langle \Psi_N^{\Omega}, Q_{1/M}(f) \Psi_N^{\Omega'} \rangle - \langle \Psi_N^{\Omega'}, Q_{1/M}(f) \Psi_N^{\Omega'} \rangle|^2$. For every fixed $\Omega' \in S^2_E$, we can apply Lemma 8.3.11 with the open set $A = S^2_E$ in common for all Ω'. Thus we obtain a first Ω'-dependent bound

$$\left| \frac{N+1}{4\pi} \int_{S^2} d\Omega \langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle_N \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'} \rangle_{N-M} \langle \Psi_M^{\Omega}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle_M - \langle \Psi_N^{(0)}, \Psi_N^{\Omega'} \rangle_N \langle \Psi_M^{\Omega'}, Q_{1/M}(f) \Psi_M^{\Omega'} \rangle_M \right|$$
$$\leq \frac{K^{(A)} \| k(\cdot, \Omega') \|_\infty + \sqrt{C \| k(\cdot, \Omega') \|_\infty^2 + D^{(A)} \| d_\Omega K(\cdot, \Omega') \|_\infty^{(A)}}}{(N-M)^{1/4}}. \tag{8.3.82}$$

where according to Lemma 8.3.11 the constants $K^{(A)}, C, D^{(A)}$ do not depend on the function $k(\cdot,\Omega')$, i.e., they do not depend on Ω' (the constants $K^{(A)}, D^{(A)}$ do depend on the set A which, however, is the same for all choices of Ω'). Finally, since

$$\|k(\cdot,\Omega')\|_\infty \le \|k(\cdot,\cdot)\|_\infty \quad \text{and} \quad \|d_\Omega K(\cdot,\Omega')\|_\infty^{(A)} \le \|d_\Omega K(\cdot,\cdot)\|_\infty^{(A\times A)}, \tag{8.3.83}$$

for sufficiently large N we also have a Ω'-uniform bound:

$$\left|\frac{N+1}{4\pi}\int_{S^2} d\Omega \langle \Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N \langle \Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}\langle \Psi_M^{\Omega}, Q_{1/M}(f)\Psi_M^{\Omega'}\rangle_M - \langle \Psi_N^{(0)},\Psi_N^{\Omega'}\rangle_N\langle \Psi_M^{\Omega'}, Q_{1/M}(f)\Psi_M^{\Omega'}\rangle_M\right|$$
$$\le \frac{K^{(A)}\|k(\cdot,\cdot)\|_\infty + \sqrt{C\|k(\cdot,\cdot)\|_\infty^2 + D^{(A)}\|d_\Omega K(\cdot,\cdot)\|_\infty^{(A\times A)}}}{(N-M)^{1/4}} = \frac{C^{(A)}}{(N-M)^{1/4}}. \tag{8.3.84}$$

Plugging this result in the right-hand side of (8.3.78), we immediately have

$$L = \lim_{N\to\infty}\frac{N+1}{4\pi}\int_{S_E^2} d\Omega' \langle \Psi_N^{(0)},\Psi_N^{\Omega'}\rangle_N \langle \Psi_M^{\Omega'}, Q_{1/M}(f)\Psi_M^{\Omega'}\rangle_M \langle \Psi_N^{\Omega'},\Psi_N^{(0)}\rangle$$
$$+ \lim_{N\to\infty}\frac{N+1}{4\pi}\int_{S_E^2} d\Omega' R_N(\Omega')\langle \Psi_N^{\Omega'},\Psi_N^{(0)}\rangle, \tag{8.3.85}$$

where $R_N(\Omega')$ is given by the expression

$$\frac{N+1}{4\pi}\int_{S^2}\langle \Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N \langle \Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}\langle \Psi_M^{\Omega}, Q_{1/M}(f)\Psi_M^{\Omega'}\rangle_M d\Omega - \langle \Psi_N^{(0)},\Psi_N^{\Omega'}\rangle_N\langle \Psi_M^{\Omega'}, Q_{1/M}(f)\Psi_M^{\Omega'}\rangle_M. \tag{8.3.86}$$

Let us focus on the second limit in (8.3.85). First of all, observe that (b) in Assumption 8.3.7, together with (a) in Proposition 8.3.4 with $\ell = 1/2$ and $f = 1$ constant, imply that the integral $\int_{S^2}(N+1)|\langle \Psi_N^{\Omega'},\Psi_N^{(0)}\rangle| d\Omega'$ is bounded when N increases, so that the corresponding integral over S_E^2 must be bounded as well. Since

$$|R_N(\Omega')| \le C^{(A)}/(N-M)^{1/4}, \tag{8.3.87}$$

where $C^{(A)}$ from (8.3.84) does not depend on Ω', we conclude that the second limit in (8.3.85) is 0. In summary,

$$L = \lim_{N\to\infty}\frac{N+1}{4\pi}\int_{S^2\setminus E} d\Omega' |\langle \Psi_N^{(0)},\Psi_N^{\Omega'}\rangle_N|^2 \langle \Psi_M^{\Omega'}, Q_{1/M}(f)\Psi_M^{\Omega'}\rangle_M. \tag{8.3.88}$$

We can rearrange the above integral into

$$L = \lim_{N\to\infty}\frac{N+1}{4\pi}\int_{S^2} d\Omega' Z(\Omega') |\langle \Psi_N^{(0)},\Psi_N^{\Omega'}\rangle_N|^2, \tag{8.3.89}$$

where $Z(\Omega') = \langle \Psi_N^{\Omega'}, Q_{1/M}(f)\Psi_N^{\Omega'}\rangle_M$ if $\Omega' \in S^2\setminus E$ and $Z(\Omega') = 0$ otherwise. With this change, we may apply Lemma 9.2.3 to the function Z, because it satisfies all requirements, finding

$$\lim_{N\to\infty}\langle \Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle = L = \tfrac{1}{2}(Z(\Omega_+) + Z(\Omega_-)). \tag{8.3.90}$$

However, since $\Omega_\pm \in S^2 \setminus E$, the very definition of Z yields

$$\lim_{N\to\infty}\langle \Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle = \frac{1}{2}\langle \Psi_M^{\Omega_+}, Q_{1/M}(f)\Psi_M^{\Omega_+}\rangle_M + \frac{1}{2}\langle \Psi_M^{\Omega_-}, Q_{1/M}(f)\Psi_M^{\Omega_-}\rangle_M. \tag{8.3.91}$$

From (3.3.50) - (3.3.51), (3.3.56), (3.3.57), (2.2.43), and (8.3.77) we have

$$\langle \Psi_M^{\Omega_\pm}, Q_{1/M}(f)\Psi_M^{\Omega_\pm}\rangle_M = \omega_\pm^{(0)}(f), \tag{8.3.92}$$

so that finally,

$$\lim_{N\to\infty}\langle\Psi_N^{(0)}, Q_{1/N}(f)\Psi_N^{(0)}\rangle = \tfrac{1}{2}\omega_+^{(0)}(f) + \tfrac{1}{2}\omega_-^{(0)}(f) = \omega^{(0)}(f), \tag{8.3.93}$$

and the proof is complete. □

8.3.2 Spontaneous symmetry breaking in the CW model

In the last part of this chapter we explain how symmetry breaking plays a role in the Curie-Weiss model in the parameter regime that $B \in (0,1)$ and $J = 1$. We recall the quantum Curie-Weiss model $H_{1/N}^{CW} \in M_2(\mathbb{C})^{\otimes N}$ defined in (4.3.28) and its classical counterpart $h_0^{CW} \in C(B^3)$ (cf. (4.3.32)).

Let us start on the classical side, i.e. we consider the Poisson manifold B^3 with Poisson bracket (4.0.2). As explained in Section 8.3 the relevant symmetry group is $\mathbb{Z}_2$, acting on $C(B^3)$. The non-trivial element (-1) that implements the symmetry $\gamma : G \to \mathrm{Aut}(C(B^3))$ is given by the map[7]

$$(\gamma_{-1}f)(x,y,z) = f(x,-y,-z),\ (x,y,z) \in B^3. \tag{8.3.94}$$

The C^*-dynamical system is constructed in terms of the time evolution, i.e. by the pullback action of the Hamiltonian flow $\phi^{h_0^{CW}}$ which is clearly complete on the compact manifold B^3. In other words,

$$\alpha_t^{h_0^{CW}}(f) = f \circ \phi_t^{h_0^{CW}},\ (f \in C(B^3)). \tag{8.3.95}$$

It is not difficult to see that γ commutes with the time evolution α.[8] Let us prove this in the following lemma.

Lemma 8.3.12. *There holds*

$$\alpha_t \circ \gamma_g = \gamma_g \circ \alpha_t,\quad (g \in \mathbb{Z}_2, t \in \mathbb{R}), \tag{8.3.96}$$

and hence G defines a dynamical symmetry group of the C^*-dynamical system $(C(B^3), \alpha_t^{h_0^{CW}})$.

Proof. It suffices to show that the Hamiltonian vector field $X^{h_0^{CW}}$ defined as

$$X^{h_0^{CW}}(f) = \{f, h_0^{CW}\} \tag{8.3.97}$$

satisfies

$$X^{h_0^{CW}}(\gamma_g^*(f)) = \gamma_g^*(X^{h_0^{CW}}(f)), \tag{8.3.98}$$

with γ^* the pullback of γ, since then using the identity $d/dt f(\phi_t^{h_0^{CW}}(x,y,z)) = (X^{h_0^{CW}}f)(\phi_t^{h_0^{CW}}(x,y,z))$ one can conclude by integration. In order to do so, we are done if we can show that

$$\{f_1, f_2\} \circ \gamma_g = \{f_1 \circ \gamma_g, f_2 \circ \gamma_g\},\quad (f_1, f_2 \in C(B^3); \tag{8.3.99}$$

because then, indeed

$$\gamma_g^*(X^{h_0^{CW}}(f)) = X^{h_0^{CW}}(f) \circ \gamma_g = \{f, h_0^{CW}\} \circ \gamma_g = \{f \circ \gamma_g, h_0^{CW}\} = X^{h_0^{CW}}(\gamma_g^*(f)), \tag{8.3.100}$$

using the invariance of h_0^{CW} under γ.

By the very definition of the Poisson bracket defined in (3.1.23) (specialized to the case $M_2(\mathbb{C})$) the thesis follows by a direct computation using the chain rule. □

[7] This map compared to the one defined in (8.3.38) is nothing else than the restriction of the action to S^2.

[8] As opposed to the case of the two-sphere S^2 where one typically uses the fact that the action of G is a symplectomorphism in order to prove the result, this does not hold now as B^3 is not a symplectic manifold.

Despite the fact that the manifold B^3 is not symplectic, Proposition 7.3.1 still applies. The ground states are convex combinations of Dirac measures concentrated at certain points in B^3. Indeed, these points have support in the set $\{x \in B^3 \mid dh_0^{CW}(x) = 0\} = \{(B, 0, \pm\sqrt{1-B^2}), (1, 0, 0)\}$. In particular, extremal ground states are pure states and it follows that the associated Dirac measures $\mu_{\pm}^{(0)}$ localized at $\mathbf{x}_{\pm} := (B, 0, \pm\sqrt{1-B^2})$, or the corresponding functionals $\omega_{\pm}^{(0)}$ on $C(S^2)$, given by $\omega_{\pm}^{(0)}(f) = f(\mathbf{x}_{\pm})$ are not invariant under the symmetry (8.3.94). Instead, $\mathbf{x}_{\pm}$ is mapped to $\mathbf{x}_{\mp}$. However, the mixture

$$\omega^{(0)} = \tfrac{1}{2}(\omega_{+}^{(0)} + \omega_{-}^{(0)}), \tag{8.3.101}$$

which also qualifies as a ground state in the algebraic sense, is invariant *but not pure*. At least in the language of algebraic quantum theory this is the essence of spontaneous symmetry breaking:

Extremal ground states are not invariant, whilst invariant ground states are not extremal.

We stress that strictly speaking the $\mathbb{Z}_2$-symmetry in the classical CW model is not spontaneous but weakly broken since the Dirac measure concentrated on $(1, 0, 0)$ is invariant. Furthermore, we point out that the choice of the magnetic field parameter B in the set $(0, 1)$ precisely yields (weak) spontaneous symmetry breaking (as just seen), whilst for $B \geq 1$ the $\mathbb{Z}_2$-symmetry is absent.

On the quantum side instead we know that for any $N < \infty$ the extremal (or in this case also the pure) ground state $\omega_{1/N}^{(0)}(\cdot) := \langle \Psi_N^{(0)}, (\cdot)\Psi_N^{(0)} \rangle$ induced by the vector $\Psi_N^{(0)}$ is unique (Appendix C.1). The relevant $\mathbb{Z}_2$-symmetry is given by the N-fold tensor of the automorphism on $M_2(\mathbb{C})$ defined by[9]

$$a \mapsto \sigma_1 a \sigma_1. \tag{8.3.102}$$

If ζ is the nontrivial element (-1) of $\mathbb{Z}_2$, we denote the automorphism of $M_2(\mathbb{C})^{\otimes N}$ induced by (8.3.102) by $\zeta^{1/N}$, i.e.

$$\zeta^{1/N}(A) = V_N^* A V_N, \tag{8.3.103}$$

where $V_N := \otimes_{n=1}^{N} \sigma_1$.

The time evolution on $M_2(\mathbb{C})^{\otimes N}$ is defined by

$$\alpha_t(A) = U_{-t} A U_t, \tag{8.3.104}$$

with $U_t = e^{-itH_{1/N}^{CW}}$. Since locally, for each $N \in \mathbb{N}$

$$[V_N, H_{1/N}^{CW}] = 0, \tag{8.3.105}$$

it follows that that $G = \mathbb{Z}_2$ defines a dynamical symmetry group. Uniqueness of $\Psi_N^{(0)}$ now implies that the state $\omega_{1/N}^{(0)}$ is strictly invariant under $\mathbb{Z}_2$, i.e.

$$\omega_{1/N}^{(0)} \circ \zeta^{1/N} = \omega_{1/N}^{(0)}. \tag{8.3.106}$$

In addition, it is easy to see that the algebraic state $\omega_{1/N}^{(0)}$ also qualifies a ground state in algebraic sense [99]. In summary, no SSB occurs in the C^*-dynamical system $(M_2(\mathbb{C})^{\otimes N}, \alpha_t^N)$. As a result of Proposition 6.2.7 the classical limit predicted by Theorem 8.3.2 can be recast to integrals with respect to a $\mathbb{Z}_2$-invariant regular Borel probability measure μ on B^3 with support given by the two point set $\{\sigma_{\pm}\}$, where $\sigma_{\pm}$ denotes the minima of the classical CW Hamiltonian h_0^{CW} defined on B^3. By definition, the measure μ is a $\mathbb{Z}_2$-invariant ground state of the C^*-dynamical system $(C(B^3), \alpha^{h_0^{CW}})$ and is therefore not concentrated on single points. This is thus a case of a non-extremal G-invariant ground state (i.e. a mixed state) responsible for

[9] See also Appendix C.1 for more details.

weak SSB. Hence, at least at the level of ground states we observe that spontaneous symmetry breaking shows up as emergent phenomenon when passing from the quantum to the classical world by sending $N \to \infty$.

9. Berezin quantization maps on $\mathbb{R}^{2n}$ and Schrödinger operators

9.1 Semi-classical properties of Berezin quantization maps on $\mathbb{R}^{2n}$

In Chapter 6 the existence of the classical limit of a sequence of eigenvectors for operators of the form $Q_\hbar^B(e)$ where e is some function in $C_0(S)$ and (S, ω_S) is a symplectic manifold has been proved. In this chapter we focus on the symplectic manifold $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$ which particularly admits a coherent pure state quantization as shown in Section 2.2.1. Entirely analogous to the propositions and theorems in Sections 6.1 and 6.2, for sake of completeness, we first recall and state some preliminaries on the associated Berezin quantization maps followed by localization results and existence of the classical limit (with and without symmetry). As these results are special cases of the propositions and theorems in Section 6.2 we omit the proofs.

We now state the analog of Theorem 6.1.2 relating the spectrum of the Berezin map to the range of the quantized function on $\mathbb{R}^{2n}$.

Proposition 9.1.1. *Given the coherent pure state quantization* $(\mathcal{H}_\hbar = L^2(\mathbb{R}^n, dx), \Psi_\hbar^\sigma, \mu_\hbar = \frac{dpdq}{(2\pi\hbar)^n})_{\hbar \in I, \sigma \in \mathbb{R}^{2n}}$ *with* $\Psi_\hbar^\sigma$ *given by* (2.1.11), *defined in Section 2.2.1. Then, for any* $f \in C_c(\mathbb{R}^{2n})$ *one has*

$$\lim_{\hbar \to 0} dist\Big(ran(f), \sigma(Q_\hbar^B(f))\Big) = 0, \tag{9.1.1}$$

where $\sigma(Q_\hbar^B(f))$ *denotes the spectrum of the operator* $Q_\hbar^B(f)$, *and dist is the distance function defined in Definition 6.1.1.* ■

Let us now move on to a proposition on the Berezin quantization specialized to the symplectic manifold $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$. As before, we denote by

$$||\Phi||_{L^2(\mathbb{R}^{2n}, d\mu_\hbar)} = \int_{\mathbb{R}^{2n}} d\mu_\hbar(\sigma) |\langle \Psi_\hbar^\sigma, \phi \rangle|^2 \tag{9.1.2}$$

the norm of the vector $\Phi \in L^2(\mathbb{R}^{2n}, d\mu_\hbar)$.

Proposition 9.1.2. *Referring to the coherent state vectors* (2.1.11) $\Psi_\hbar^\sigma \in L^2(\mathbb{R}^n, dx)$ *used to construct the quantization Berezin map (2.1.13),*

(a) there exists an isometry $W : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^{2n}, d\mu_\hbar)$, *completely defined by*

$$(W\phi)(q,p) = \langle \Psi_\hbar^{(q,p)}, \phi \rangle = \int_{\mathbb{R}^n} e^{\frac{ip \cdot q}{2\hbar}} e^{-\frac{ip \cdot x}{\hbar}} e^{-\frac{(x-q)^2}{2\hbar}} \phi(x) \frac{dx}{(\pi\hbar)^{n/2}}, \tag{9.1.3}$$

in particular $W^*W = I_{L^2(\mathbb{R}^n, dx)}$ *and* $p := WW^* : L^2(\mathbb{R}^{2n}, d\mu_\hbar) \to L^2(\mathbb{R}^{2n}, d\mu_\hbar)$ *is the orthogonal projector onto* $ran(W) = \overline{ran(W)}$;

(b) It holds $W(\mathcal{S}(\mathbb{R}^n)) \subset \mathcal{S}(\mathbb{R}^{2n})$.

Proof. (a) This is a direct application of the proof of Proposition 6.2.1.
(b) With trivial manipulations, $(W\psi)(q,p) = \frac{1}{(\pi\hbar)^{n/4}} \int_{\mathbb{R}^n} e^{-ip \cdot z/\hbar} e^{-(2z-x)^2/(2\hbar)} \psi(z + q/2) dz$, so that, where a, b, c, d, k are multi indices, passing the derivatives under the integration symbol

(for dominated convergence theorem) and using integration by parts since, in particular, $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$p^a q^c \partial_p^b \partial_q^d (W\psi)(q,p) = \int_{\mathbb{R}^n} e^{-ip\cdot z/\hbar} e^{-(2z-x)^2/(2\hbar)} \sum_k p_k^{abcd}(z,q) \partial_z^k \psi(z+q/2) dz,$$

where the sum is finite and $p_k^{abcd}(z,q)$ are polynomials in z and q. Hence, for some constant $K_{abcd} > 0$, it holds $|\sum_k p(z,q)_k \partial_z^k \psi(z+q/2)| \leq K_{abcd}$, so that

$$|p^a q^c \partial_p^b \partial_q^d (W\psi)(q,p)| \leq K_{abcd} \int_{\mathbb{R}^n} e^{-(2z-x)^2/(2\hbar)} dz = C_{abcd} < +\infty \quad \text{for all } (q,p) \in \mathbb{R}^{2n}.$$

Arbitrariness of a, b, c, d implies the thesis. □

The next proposition is the analog of Proposition 6.2.2.

Proposition 9.1.3. *Let G be a group acting by symplectomorphisms on the symplectic manifold $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$. Then there exists a unitary representation $U : G \to \mathfrak{B}(L^2(\mathbb{R}^n, dx))$ such that the maps $Q_\hbar^B$ are ζ-equivariant,*

$$U_g Q_\hbar^B(f) U_g^{-1} = Q_\hbar^B(\zeta_g f)\,, \quad g \in G,\ f \in L^\infty\left(\mathbb{R}^{2n}, d\mu_\hbar\right). \tag{9.1.4}$$

The representation U is completely defined by the requirement

$$WU_g\psi = p\zeta_g(W\psi) \quad \textit{for every } \psi \in L^2(\mathbb{R}^n, dx) \textit{ and } g \in G. \tag{9.1.5}$$

■

Let us now recall Proposition 6.2.3 concerning the localization of eigenvectors.

Proposition 9.1.4. *Consider the manifold $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$ with associated Berezin quantization maps $Q_\hbar^B$. Given a real-valued $e \in C_0(\mathbb{R}^{2n})$, let $\{\phi_\hbar\}_\hbar \subset L^2(\mathbb{R}^n)$ be a sequence of eigenvectors of $Q_\hbar^B(e)$ with eigenvalues $\{\lambda_\hbar\}_\hbar$ such that, for some $\Lambda \in ran(e)$ is such that*

$$\lambda_\hbar \to \Lambda \quad \textit{for } \hbar \to 0^+. \tag{9.1.6}$$

The following facts are true where $\Phi_\hbar := W\varphi_\hbar$ as before.

(1) Referring to any open neighborhood of the set $e^{-1}(\Lambda)$ of the form

$$\mathcal{V}_\epsilon := e^{-1}((\Lambda - \epsilon, \Lambda + \epsilon)), \tag{9.1.7}$$

for every given $\epsilon > 0$ one has

$$||\Phi_\hbar||_{L^2(\mathbb{R}^{2n} \setminus \mathcal{V}_\epsilon, d\mu_\hbar)} \to 0, \quad \textit{for} \quad \hbar \to 0^+\,. \tag{9.1.8}$$

(2) If $e^{-1}(\Lambda) = \{\sigma_0\} \in \mathbb{R}^{2n}$ and the family of sets $\{\mathcal{V}_\epsilon\}_{\epsilon > 0}$ is a fundamental system of neighborhoods of σ_0 then $\langle \varphi_\hbar, Q_\hbar(f)\varphi_\hbar \rangle \to f(\sigma_0)$ as $\hbar \to 0^+$ for every $f \in C_0(\mathbb{R}^{2n})$.

■

This finally leads to the existence of the classical limit for a sequence of eigenvectors corresponding to the operators $Q_\hbar^B(e)$, where $e \in C_0(\mathbb{R}^{2n})$. The function e is a classical observable and might be related to some Hamiltonian function, describing a classical mechanical system. Since standard Hamiltonians of the form $h(q,p) = p^2 + V(q)$ are unbounded, they stay outside the domain of the quantization map $Q_\hbar^B : C_0(\mathbb{R}^{2n}) \to \mathfrak{B}(L^2(\mathbb{R}^n, dx))$ and we cannot take advantage of the C^*-algebra formalism. A possibility is to interpret

$$e(q,p) = e^{-t(p^2 + V(q))} \tag{9.1.9}$$

for $t > 0$. Under mild and physically natural conditions on V, like $V \in C^\infty(\mathbb{R}^n)$ with $V(q) \to +\infty$ for $|q| \to +\infty$, the map e belongs to $C_0(\mathbb{R}^{2n})$ and in particular $Q_\hbar^B(e) \in \mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx))$

and also it is a positive operator. In this case the spectrum $\sigma(Q^B_\hbar(e))$ is a pure point spectrum made of non-negative eigenvalues with 0 as the unique, possibly accumulation point (possibly in the continuous spectrum).

For the sake of generality, in this section we shall not assume that precise form of the function e as in (9.1.9), we only assume that $e \in C_0(\mathbb{R}^{2n})$, so that $Q^B_\hbar(e)$ is compact with a point spectrum except for 0 at most, and we focus on a sequence of eigenvalues $\lambda_\hbar$ and corresponding eigenvectors $\psi_\hbar$ of the maps $Q^B_\hbar(e)$ such that $\lambda_\hbar \to \Lambda = e(\sigma_0)$ for $\hbar \to 0^+$. The precise form (9.1.9) will be adopted when discussing the interplay with Schrödinger operators.

Though our results assume the existence of such sequences of eigenvectors as we shall prove in the next section (cf. Corollary 9.2.8), if e is of the form (9.1.9) and under suitable hypotheses on V, the said sequence does exist when referring to the sequence of the maximal eigenvalues of $Q^B_\hbar(e)$. In that case $\Lambda = \max_{\sigma\in\mathbb{R}^{2n}} e(\sigma)$.

Let us not state the analog of Theorem 6.2.4.

Theorem 9.1.5 (Classical limit without symmetry)**.** *Consider $e \in C_0(\mathbb{R}^{2n})$ and $\Lambda \neq 0$ such that*

$$\Lambda = e(\sigma_0) \quad \textit{for a unique point } \sigma_0 \in \mathbb{R}^{2n}. \tag{9.1.10}$$

Let $\{\varphi_\hbar\}_{\hbar>0}$ be a family of eigenvectors with eigenvalues $\{\lambda_\hbar\}_{\hbar>0}$ of $Q^B_\hbar(e)$ converging to Λ, as $\hbar \to 0$. With these assumptions,

$$\langle\varphi_\hbar, Q_\hbar(f)\varphi_\hbar\rangle \to f(\sigma_0) \quad \textit{as } \hbar \to 0^+, \textit{ for every } f \in C_0(\mathbb{R}^{2n}).$$

■

Let us now consider the case when e admits an internal symmetry. To this end, we assume the group G to act by symplectomorphism $G \ni g : \mathbb{R}^{2n} \ni (q,p) \mapsto g(q,p) \in \mathbb{R}^{2n}$ on $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$. We focus on the case that (a) G is a compact topological group or (b) G is a discrete group.

Example 9.1.6. When assuming e of the form (9.1.9), two typical examples are

(i) the *double well* system on $\mathbb{R}^2$ with $G = \mathbb{Z}_2 = \{\pm 1\}$, so that $g(q,p) = (\pm q, \pm p)$;

(ii) the *Mexican hat* system on $\mathbb{R}^{2n}$, with $n > 1$, and $SO(n)$ as symmetry group G so that $g(q,p) = (gq, gp)$, where gz is the standard rotation of $z \in \mathbb{R}^n$ according to $g \in SO(n)$.

In both cases

$$h(q,p) = p^2 + (q^2-1)^2 \,, \tag{9.1.11}$$

where $(q,p) \in \mathbb{R}\times\mathbb{R}$ or $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$ respectively. ■

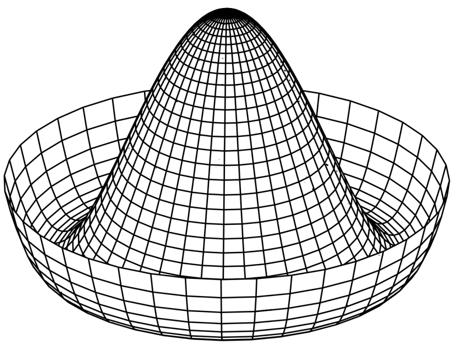

Figure 9.1: Mexican hat potential on $\mathbb{R}^2$ for $V(q) = (q^2-1)^2$ and $G = SO(2)$.

Differently from the simpler result established with Theorem 9.1.5, we now also assume that the eigenspaces associated to the sequence of eigenvalues $\lambda_\hbar$ of $Q^B_\hbar(e)$ have dimension one.[1] This gives the following theorem proving the classical limit of a sequence of eigenvectors corresponding to operators $Q_\hbar^B(e)$ in presence of a symmetry.

Theorem 9.1.7 (Classical limit with symmetry)**.** *Consider a group G either finite or topological compact, $e \in C_0(\mathbb{R}^{2n})$ and assume the following hypotheses.*

(a) G acts continuously in the topological-group case[2], on $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$ in terms of symplectomorphism.

(b) e is invariant under G.

(c) The action of G is transitive on $e^{-1}(\{\Lambda\})$.

Then the following facts are valid for every chosen $\sigma_0 \in e^{-1}(\{\Lambda\})$ and a family of eigenvectors $\{\varphi_\hbar\}_{\hbar>0}$ of $Q_\hbar^B(e)$ with non-degenerate eigenvalues $\{\lambda_\hbar\}_{\hbar>0}$ converging to Λ as $\hbar \to 0$ for some $\Lambda \in ran(e) \setminus \{0\}$.

(1) If G is topological and compact and μ_G is the normalized Haar measure,

$$\lim_{\hbar\to 0^+} \langle \varphi_\hbar, Q_\hbar^B(f)\varphi_\hbar\rangle = \int_G f(g\sigma_0)d\mu_G(g), \quad \textit{for every } f \in C_0(\mathbb{R}^{2n}). \tag{9.1.12}$$

(2) If G is finite and N_G is the number of elements of G,

$$\lim_{\hbar\to 0^+} \langle \varphi_\hbar, Q_\hbar^B(f)\varphi_\hbar\rangle = \frac{1}{N_G}\sum_{g\in G} f(g\sigma_0), \quad \textit{for every } f \in C_0(\mathbb{R}^{2n}). \tag{9.1.13}$$

The right-hand sides of (9.1.12) and (9.1.13) are independent of the choice of σ_0. ■

9.2 Schrödinger operators versus Berezin quantization maps

In this section we discuss the interplay between Schrödinger operators and Berezin quantization maps $Q_\hbar^B(e^{-th})$. We will see that under some assumptions on the potential appearing both in the Schrödinger operator and in $h = p^2 + V$, the operators $e^{-tH_\hbar}$ and $Q_\hbar^B(e^{-th})$ are related. We shall provide several theorems making this relation precise.

9.2.1 General setting

We consider $\hbar$-dependent (unbounded) Schrödinger operators $H_\hbar$ defined on some dense domain of $\mathcal{H} = L^2(\mathbb{R}^n, dx)$. Such operators are typically given by

$$H_\hbar := \overline{-\hbar^2\Delta + V}, \quad \hbar > 0\,, \tag{9.2.14}$$

where Δ denotes the Laplacian on $\mathbb{R}^n$, and V denotes multiplication by some real-valued function on $\mathbb{R}^n$, playing the role of the potential. One should impose conditions on the potential V in order to make $H_\hbar$ self-adjoint when $-\hbar^2\Delta + V$ is initially defined on $C_c^\infty(\mathbb{R}^n)$. Our general hypotheses will be the following ones.

(V1) V is a real-valued $C^\infty(\mathbb{R}^n)$ function.

(V2) $V(x) \to +\infty$ for $|x| \to +\infty$ (i.e., for every $M > 0$, there is $R_M > 0$ such that $V(x) > M$ if $|x| > R_M$).

(V3) $e^{-tV} \in \mathcal{S}(\mathbb{R}^n)$ for $t > 0$.

[1] Again this condition is satisfied if e is of the form (9.1.9) and V satisfies suitable hypotheses, when referring to the maximal eigenvalues of $Q_\hbar^B(e)$ and with $\Lambda = \max_{\sigma\in\mathbb{R}^{2n}} e(\sigma)$, as we shall see in the next section (cf. Corollary 9.2.8).

[2] The action $G \times \mathbb{R}^{2n} \ni (g, \sigma) \mapsto g\sigma \in \mathbb{R}^{2n}$ is continuous.

An elementary example consists of a real polynomial satisfying (V2). It also satisfies (V1) and (V3) automatically.

It is known that when $V \in L^2_{loc}(\mathbb{R}^n, dx)$ and $V(x) \geq 0$ pointwise then the operator $-\hbar^2\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ in view of Theorem X.28 [76] and that $H_\hbar \geq 0$. Referring to our general hypotheses, we observe that evidently (V1) implies $V \in L^2_{loc}(\mathbb{R}^n, dx)$. Furthermore, (V1) and (V2) immediately imply that

$$\inf_{x\in\mathbb{R}^n} V(x) = \min_{x\in\mathbb{R}^n} V(x) = c > -\infty \,. \tag{9.2.15}$$

Since $(H_\hbar + aI)^* = H_\hbar^* + aI$, the domain being the one of $\overline{H_\hbar} = H_\hbar^*$ for both sides, the essential self-adjointness property still holds when relaxing $V \geq 0$ to (V2), also obtaining $H_\hbar \geq cI$. Hence (V1)-(V2) imply that $H_\hbar$ is self-adjoint and

$$H_\hbar \geq (\min V) I \,. \tag{9.2.16}$$

Remark 9.2.1. We would like to point out that these conditions on the potential V (which typically plays the role of a "trapping" potential) and the domain of $H_\hbar$ generally apply to non-interacting particle systems. Of course, this is a limitation if one is interested in more complex problems where frequently non-smooth potentials are considered (e.g. semi-classical tunneling). ■

9.2.2 Comparison of $e^{-tH_\hbar}$ and $Q_\hbar^B(e^{-th})$

The physical idea is now that, in some sense $Q_\hbar^B(p^2+V)$ is a good approximation of $H_\hbar$ as $\hbar \to 0^+$. We wish to implement this idea in the framework of C^*-algebras of operators. Since the above operators are unbounded an appealing idea is to consider bounded functions of p^2+V and $H_\hbar$ in place of themselves. An apparent promising idea seems to be the use of the *resolvent algebra* of operators [23]. However this approach reveals to be awkward, essentially because the classical counterpart of the C^*-algebra of resolvents is too involved to be profitably handled. A different approach, already adopted in the previous section, relies on the use of (real) exponential functions and the associated quantum counterpart consisting of contraction semigroups. This idea indeed works and it is based upon the following list of comparison results.[3]

Proposition 9.2.2. *Let V satisfy (V1)-(V3) and consider the associated family of operators in $\mathfrak{B}(L^2(\mathbb{R}^n, dx))$ indexed by $\hbar > 0$ and defined by $e^{-tH_\hbar}$ according to (9.2.14), where $t > 0$ is given. It holds*

$$Q_\hbar^B(e^{-t(p^2+V)}) - e^{-tH_\hbar} \to 0 \quad \textit{in strong sense for } \hbar \to 0^+ \,. \tag{9.2.17}$$

More precisely both

$$e^{-tH_\hbar} \quad \textit{and} \quad Q_\hbar^B(e^{-t(p^2+V)}) \to e^{-tV} \quad \textit{in strong sense for } \hbar \to 0^+ \,. \tag{9.2.18}$$

Proof. As already observed in the main text, $H_\hbar$ is self-adjoint. Since $H_\hbar$ is also bounded below, it is also the generator of a strongly-continuous one-parameter semigroup in $L^2(\mathbb{R}^n, dx)$ as immediately arises from spectral calculus (see, e.g., [64]).

The Weyl and Berezin quantization procedures are equivalent when applied to $e^{-t(p^2+V)}$ since it is an element of the Schwartz space and $||Q_\hbar^W(f) - Q_\hbar^B(f)|| \to 0$ for $\hbar \to 0^+$ and a given $f \in \mathcal{S}(\mathbb{R}^n)$ (cf. Theorem 2.1.12 (2)). Since $e^{-t(p^2+V(x))}$ defines a function of $\mathcal{S}(\mathbb{R}^{2n})$ for every given $t > 0$, the proof consists of establishing the thesis with $Q_\hbar^B$ replaced for $Q_\hbar^W$. With this replacement, the main thesis (9.2.17) immediately arises collecting the theses of a pair of lemmata (Lemma 9.2.3 - Lemma 9.2.4) whose hypotheses are fulfilled if V satisfies (V1)-(V3). By direct inspection one sees that these lemmata just prove (9.2.18).

[3]This strongly depends on the regularity of the potential V. In general, the result may be even wrong!

Indeed, given $\psi \in L^2(\mathbb{R}^n)$ by the triangle inequality we have

$$\begin{aligned}&||(Q_\hbar^B(e^{-t(p^2+V)}) - e^{-tH_\hbar})\psi|| \leq \\ &||(Q_\hbar^B(e^{-t(p^2+V)}) - Q_\hbar^W(e^{-t(p^2+V)}))\psi|| + ||(Q_\hbar^W(e^{-t(p^2+V)}) - e^{-tH_\hbar})\psi|| \leq \\ &||Q_\hbar^B(e^{-t(p^2+V)}) - Q_\hbar^W(e^{-t(p^2+V)})|| + ||(Q_\hbar^W(e^{-t(p^2+V)}) - e^{-tH_\hbar})\psi||.\end{aligned} \tag{9.2.19}$$

The first term on the right hand side of (9.2.19) goes to zero as a result of Theorem 2.1.12 (2). For the second term we again apply the triangle inequality and obtain

$$||(Q_\hbar^W(e^{-t(p^2+V)}) - e^{-tH_\hbar})\psi|| \leq ||(Q_\hbar^W(e^{-t(p^2+V)}) - e^{-tV})\psi|| + ||(e^{-tV} - e^{-tH_\hbar})\psi||. \tag{9.2.20}$$

The first term on the right hand side of (9.2.20) goes to zero by Lemma 9.2.3 whilst the second term goes to zero by Lemma 9.2.4, both of them proved below. As already mentioned the second thesis (9.2.18) follows directly from these two lemmata.

Lemma 9.2.3. *Let $V \in L^2_{loc}(\mathbb{R}^n, dx)$ satisfy $\inf_{x\in\mathbb{R}^n} V(x) = c > -\infty$ for $x \in \mathbb{R}^n$. If $e^{-tV} \in \mathcal{S}(\mathbb{R}^n)$ for some $t > 0$, then $Q_\hbar^W(e^{-t(p^2+V)}) \to e^{-tV}$ for $\hbar \to 0^+$ in strong sense.*

Proof of Lemma 9.2.3. Notice that, with the said hypotheses, $e^{-t(p^2+V(x))}$ defines a function of $\mathcal{S}(\mathbb{R}^{2n})$ for every $t > 0$. Using (2.1.9) with $f(x,p) = e^{-t(p^2+V(x))} = e^{-tV(x)}e^{-tp^2} = f_1(x)f_2(p)$, we find, choosing $\psi \in L^2(\mathbb{R}^n, dx)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle\psi, Q_\hbar^W(e^{-t(p^2+V)})\phi\rangle = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} \widehat{f_2}(b)\left\langle\psi, e^{ib\cdot P}e^{-tV\left(x+\frac{\hbar}{2}bI\right)}\phi\right\rangle\, db$$

$$= \int_{\mathbb{R}^n} \frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}\left\langle\psi, e^{-tV\left(x-\frac{\hbar}{2}bI\right)}e^{ib\cdot P}\phi\right\rangle\, db = \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}\overline{\psi(x)}e^{-tV\left(x-\frac{\hbar}{2}b\right)}\phi\left(x+\hbar b\right) dxdb$$

$$= \int_{\mathbb{R}^n}\overline{\psi(x)}\int_{\mathbb{R}^n}\frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}e^{-tV\left(x-\frac{\hbar}{2}b\right)}\phi\left(x+\hbar b\right) dxdb\,,$$

where we have used the Fubini-Tonelli theorem since the function in the argument of the $\mathbb{R}^n\times\mathbb{R}^n$ integral is absolutely integrable. Indeed, $|\phi(x)| \leq C_k(1+|x|)^{-k}$ for every $k \in \mathbb{N}$ and some $C_k \geq 0$, and we can take $0 \leq \hbar \leq \delta$ for some $\delta > 0$. As a consequence we can find $\phi_0 \in L^2(\mathbb{R}^n, dx)$ such that $|\phi\left(x+\hbar b\right)| \leq |\phi_0(x)|$ when $0 \leq \hbar \leq \delta$. Now observe that $\psi\phi_0 \in L^1(\mathbb{R}^n, dx)$ since $\psi, \phi_0 \in L^2(\mathbb{R}^n, dx)$, and $e^{-tV\left(x-\frac{\hbar}{2}b\right)} \leq e^{-tc}$. Summing up, since $\widehat{f_2}$ is $L^1(\mathbb{R}^n, db)$ by construction, the argument of the $\mathbb{R}^n\times\mathbb{R}^n$ integral is bounded by the $L^1(\mathbb{R}^n\times\mathbb{R}^n, dx\otimes db)$ map $|e^{-tc}\widehat{f_2}\psi\phi_0|$ and thus it is absolutely integrable as well. We have so far established that, for $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n, dx)$,

$$\langle\psi, Q_\hbar^W(e^{-t(p^2+V)})\phi\rangle = \int_{\mathbb{R}^n}\overline{\psi(x)}\int_{\mathbb{R}^n}\frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}e^{-tV\left(x-\frac{\hbar}{2}b\right)}\phi\left(x+\hbar b\right) dxdb\,. \tag{9.2.21}$$

Since $\psi \in L^2(\mathbb{R}^n, dx)$ is arbitrary, we conclude that

$$\left(Q_\hbar^W(e^{-t(p^2+V)})\phi\right)(x) = \int_{\mathbb{R}^n}\frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}e^{-tV\left(x-\frac{\hbar}{2}b\right)}\phi\left(x+\hbar b\right) db\,. \tag{9.2.22}$$

Explicitly, $\widehat{f_2}(b) = \frac{e^{-b^2/(4t^2)}}{(2t)^{n/2}}$. At this juncture, with our choice of ϕ and $\hbar$, exploiting the same argument used for applying Fubini-Tonelli theorem, Lebesgue's dominated convergences implies that, for $\hbar \to 0$,

$$\left(Q_\hbar^W(e^{-t(p^2+V)})\phi\right)(x) \to \int_{\mathbb{R}^n}\frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}e^{-tV(x)}\phi\left(x\right) db$$

$$= \int_{\mathbb{R}^n} e^{ib\cdot 0}\frac{\widehat{f_2}(b)}{(2\pi)^{n/2}}db\, e^{-tV(x)}\phi\left(x\right) = f_2(0)\, e^{-tV(x)}\phi\left(x\right) = e^{-tV(x)}\phi\left(x\right)\ .$$

To conclude, consider $\psi \in L^2(\mathbb{R}^n, dx)$. We have

$$||(Q^W_\hbar(e^{-t(p^2+V)})-e^{-tV})\psi|| \leq ||(Q^W_\hbar(e^{-t(p^2+V)})-e^{-tV})(\psi-\phi)||+||(Q^W_\hbar(e^{-t(p^2+V)})-e^{-tV})\phi||$$
$$\leq (||(Q^W_\hbar(e^{-t(p^2+V)})|| + ||e^{-tV}||)||\psi-\phi|| + ||(Q^W_\hbar(e^{-t(p^2+V)}) - e^{-tV})\phi||$$
$$\leq (C+1)||\psi-\phi|| + ||(Q^W_\hbar(e^{-t(p^2+V)}) - e^{-tV})\phi|| \,.$$

where C arises form (2.1.8). In summary,

$$||(Q^W_\hbar(e^{-t(p^2+V)}) - e^{-tV})\psi|| \leq (C+1)||\psi-\phi|| + ||(Q^W_\hbar(e^{-t(p^2+V)}) - e^{-tV})\phi|| \,.$$

The found inequality permits us to prove the thesis of the lemma for the said ψ. Indeed, for every $\epsilon > 0$ we can first take $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $(C+1)||\psi-\phi|| < \epsilon/2$. With that choice of ϕ, we can next take $\delta > 0$ such that $0 \leq \hbar < \delta$ implies, for the first part of the proof, $||(Q^W_\hbar(e^{-t(p^2+V)}) - e^{-tV})\phi|| < \epsilon/2$. Hence $||(Q^W_\hbar(e^{-t(p^2+V)}) - e^{-tV})\psi|| < \epsilon$ if $0 \leq \hbar < \delta$ that is the thesis. □

Lemma 9.2.4. *If $V \in L^2_{loc}(\mathbb{R}^n, dx)$ satisfies $\inf_{x\in\mathbb{R}^n} V(x) = c > -\infty$ for $x \in \mathbb{R}^n$, $\psi \in L^2(\mathbb{R}^n, dx)$, and $T > 0$, then $\sup_{t\in[0,T]} ||(e^{-t\overline{(-\hbar^2\Delta+V)}} - e^{-tV})\psi|| \to 0$ for $\hbar \to 0$, where $-\hbar^2\Delta + V$ is defined on $C^\infty_c(\mathbb{R}^n)$.*

Proof of Lemma 9.2.4. Let us first assume $c = 0$. If $\lambda > 0$, from Theorem X.28 [76], we have that $\overline{(-\lambda\Delta + V)}$ is self-adjoint and positive and thus defines the generator of a semigroup. We take advantage of Corollary 3.18 in [31]. Now the relevant semigroups are $T^\lambda_t := e^{-t\overline{(-\lambda\Delta+V)}}$ and $T_t := e^{-tV}$ where $\lambda = \hbar^2$. Evidently $||T^\lambda_t||$, $||T_t|| \leq 1$ since the generators are positive. Next consider $\mathcal{D} := C^\infty_c(\mathbb{R}^n)$. It easy to prove that this dense subspace is a core of V viewed as a self-adjoint multiplicative operator (it is sufficient to observe that $V|_\mathcal{D}$ is symmetric and the defect spaces of $V|_\mathcal{D}$ are trivial[4]). With our definitions, if $f \in \mathcal{D}$, then $f \in D(\overline{-\lambda\Delta + V})$ and

$$\lim_{\lambda\to 0^+} \overline{-\lambda\Delta + V} f = \lim_{\lambda\to 0^+} (-\lambda\Delta f + Vf) = Vf,$$

holds trivially. All that means that the hypotheses of of Corollary 3.18 in [31] are true and thus the thesis is valid, that is nothing but the thesis of our lemma when $c = 0$. If $c \neq 0$, let us define $V_0 := V - c$. The thesis is true if replacing V for V_0 using the above proof. The thesis is also valid for V just because $e^{-t\overline{(-\hbar^2\Delta+V)}} = e^{-tc}e^{-t\overline{(-\hbar^2\Delta+V_0)}}$ so that

$$\sup_{t\in[0,T]} ||(e^{-t\overline{(-\hbar^2\Delta+V)}} - e^{-tV})\psi|| = \sup_{t\in[0,T]} |e^{-ct}|\, ||(e^{-t\overline{(-\hbar^2\Delta+V_0)}} - e^{-tV_0})\psi|| \to 0,$$

because $|e^{-ct}| < k < +\infty$ for $t \in [0,T]$ no matter the sign of c. □

We now pass to a comparison of the ground state of the Hamiltonian operator $H_\hbar$ and the corresponding of $Q^B_\hbar(e^{-t(p^2+V)})$. More precisely, we compare the maximal eigenvalues of $e^{-tH_\hbar}$ and $Q^B_\hbar(e^{-t(p^2+V)})$ for $t > 0$ given. Assuming (V1)-(V3), $V \in L^1_{loc}(\mathbb{R}^n, dx)$ is bounded from below and $V(x) \to +\infty$ for $|x| \to +\infty$ and thus the resolvent of $H_\hbar$ is compact due to Theorem XIII.67 in [77]. According to standard results on positive compact operators (see, e.g., [64]), if $\hbar > 0$,

(a) the spectrum of $H_\hbar$ is a pure point spectrum and there is a corresponding Hilbert basis of eigenvectors $\{\psi^{(j)}_\hbar\}_{j=0,1,\ldots}$ with corresponding eigenvalues

$$\sigma(H_\hbar) = \{E^{(j)}_\hbar\}_{j=0,1,2,\ldots} \quad \text{with } 0 \leq E^{(j)}_\hbar \leq E^{(j+1)}_\hbar \to +\infty \text{ as } j \to +\infty, \tag{9.2.23}$$

where every eigenspace has finite dimension;

(b) $e^{-tH_\hbar}$ is compact with spectrum $\sigma(e^{-tH_\hbar}) = \{0\} \cup \{e^{-tE^{(j)}_\hbar}\}_{j=0,1,2,\ldots}$, 0 being the unique point of the continuous spectrum, and the eigenspaces of $H_\hbar$ and $e^{-tH_\hbar}$ coincide;

[4] $(V|_\mathcal{D})^*\psi = \pm i\psi$ implies in particular that $\int_{\mathbb{R}^n}(V(x) \pm i)g(x)\psi(x)dx = 0$ for every $g \in \mathcal{D} = C^\infty_c(\mathbb{R}^n)$ and thus $(V(x) \pm i)\psi(x) = 0$ a.e.. Since $(V(x) \pm i) \neq 0$, it must be $\psi = 0$ a.e..

(c) the minimal eigenvalue $E_\hbar^{(0)}$ of $H_\hbar$ corresponds to the maximal eigenvalue of $e^{-tH_\hbar}$ according to

$$e^{-tE_\hbar^{(0)}} = ||e^{-tH_\hbar}|| \,. \tag{9.2.24}$$

Under the said hypotheses on V, we also know that $Q_\hbar^B(e^{-t(p^2+V)})$ is a positive compact operator due to Theorems 2.1.10 and 2.1.11 when $\hbar > 0$. These facts permit us to focus on interplay of the maximal eigenvalues of $e^{-tH_\hbar}$ and $Q_\hbar^B(e^{-t(p^2+V)})$ with the following proposition.

Proposition 9.2.5. *Assuming the hypotheses (V1)-(V3), given $t > 0$, let $\lambda_\hbar^{(0)}$ be the maximal eigenvalue of $Q_\hbar^B(e^{-t(p^2+V)})$. The following facts are true referring to $H_\hbar$ ($\hbar > 0$) as in proposition 9.2.2 with eigenvalues as in (9.2.23).*

(1) Both $\lambda_\hbar^{(0)}$ and $e^{-tE_\hbar^{(0)}} \to e^{-t\min V}$ as $\hbar \to 0^+$.

(2) Both $||Q_\hbar^B(e^{-t(p^2+V)})||$ and $||e^{-tH_\hbar}|| \to e^{-t\min V}$ as $\hbar \to 0^+$.

Proof. According to the discussion before proposition 9.2.5, it is sufficient to prove (1) since (2) immediately arises form the fact that the norm of $Q_\hbar^B(e^{-t(p^2+V)})$ coincides to $\lambda_\hbar^{(0)}$. Furthermore, since $||Q_\hbar^B(e^{-t(p^2+V)})|| \to ||e^{-t(p^2+V)}||_\infty = 1$ for $\hbar \to 0^+$ (that is nothing but the Rieffel condition which is valid for theorem 2.1.12 since $e^{-t(p^2+V)} \in C_0(\mathbb{R}^n)$), to prove the whole proposition it is enough to demonstrate that $E_\hbar^{(0)} \to 0^+$ as $\hbar \to 0^+$. Let us prove this fact. The spectral decomposition of the operator $H_\hbar \geq 0$ with pure point spectrum implies that $E_\hbar^{(0)} = \inf_{\psi\in D(H_\hbar)\,,||\psi||=1}\langle\psi, H_\hbar\psi\rangle$. Since $C_c^\infty(\mathbb{R}^{2n})$ is a core for $H_\hbar$, it is not difficult to prove that $E_\hbar^{(0)} = \inf_{\psi\in C_c^\infty(\mathbb{R}^{2n})\,,||\psi||=1}\langle\psi, H_\hbar\psi\rangle$. Let us prove that

$$\inf_{\psi\in C_c^\infty(\mathbb{R}^{2n})\,,||\psi||=1}\langle\psi, H_\hbar\psi\rangle \to 0 \quad \text{if } \hbar \to 0^+, \tag{9.2.25}$$

to conclude the whole proof. From the definition of $H_\hbar$ it immediately arises that, for $\psi \in C_c^\infty(\mathbb{R}^n)$ and $\hbar \geq \hbar' > 0$, $\langle\psi, H_\hbar\psi\rangle = -\hbar^2\langle\psi, \Delta\psi\rangle + \langle\psi, V\psi\rangle \geq -\hbar'^2\langle\psi, \Delta\psi\rangle + \langle\psi, V\psi\rangle = \langle\psi, H_{\hbar'}\psi\rangle \geq 0$. As a consequence

$$\exists \lim_{\hbar\to0^+} \inf_{\psi\in C_c^\infty(\mathbb{R}^{2n})\,,||\psi||=1}\langle\psi, H_\hbar\psi\rangle = \inf_{\hbar>0} \inf_{\psi\in C_c^\infty(\mathbb{R}^{2n})\,,||\psi||=1}\langle\psi, H_\hbar\psi\rangle \geq 0 \,.$$

Therefore (9.2.25) is true if, for every $\epsilon > 0$, there is $\psi_\epsilon \in C_c^\infty(\mathbb{R}^{2n})$ with $||\psi_\epsilon|| = 1$ and such that

$$\langle\psi_\epsilon, H_\hbar\psi_\epsilon\rangle = -\hbar^2\langle\psi_\epsilon, \Delta\psi_\epsilon\rangle + \langle\psi_\epsilon, V\psi_\epsilon\rangle < \epsilon, \tag{9.2.26}$$

if $\hbar > 0$ is sufficiently small. It is not difficult to find this ψ_ϵ. Let $x_0 \in \mathbb{R}^n$ be such that $V(x_0) = 0$ according to (V2), and, since V is continuous, let $B \ni x_0$ be an open ball of finite radius such that $|V(x)| < \epsilon/2$ if $x \in B$. Finally define $\psi_\epsilon \in C_c^\infty(\mathbb{R}^{2n})$ with $||\psi_\epsilon|| = 1$ and $supp(\psi_\epsilon) \subset B$. By construction, $0 \leq -\hbar^2\langle\psi_\epsilon, \Delta\psi_\epsilon\rangle + \langle\psi_\epsilon, V\psi_\epsilon\rangle < -\hbar^2\langle\psi_\epsilon, \Delta\psi_\epsilon\rangle + \epsilon/2$. To conclude it is sufficient to choose $\hbar > 0$ such that, for the said ψ_ϵ, we have $-\hbar^2\langle\psi_\epsilon, \Delta\psi_\epsilon\rangle < \epsilon/2$. Therefore, (9.2.26) holds and we have established (9.2.25). This proves the case $c = 0$. The case $c \neq 0$ immediately arises from the case $c = 0$ when defining $V_0 := V - c$, by noticing that the thesis is therefore valid for V replaced by V_0 and observing that $e^{-tH_\hbar} = e^{-tc}e^{-t(\overline{-\hbar\Delta+V_0})}$ and $Q_\hbar^B(e^{-t(p^2+V)}) = e^{-ct}Q_\hbar^B(e^{-t(p^2+V_0)})$. □

Corollary 9.2.6. *Under the hypotheses (V1)-(V3), the minimal eigenvalues of $H_\hbar := -\hbar\Delta + V$ satisfy*

$$E_\hbar^{(0)} \to \min_{x\in\mathbb{R}^n}\{V(x)\} = \min_{(q,p)\in\mathbb{R}^{2n}}\{p^2 + V(q)\} \quad \textit{for} \quad \hbar \to 0^+ \,.$$

Let us finally pass to prove that the dimension of the eigenspaces of the maximal eigenvalues of respectively $Q_\hbar^B(e^{-t(p^2+V)})$ and $e^{-tH_\hbar}$ is 1 in both cases.

Proposition 9.2.7. *Under the hypotheses (V1)-(V3) and for $t > 0$, the eigenspaces of $Q^B_\hbar(e^{-t(p^2+V)})$ and $H_\hbar$ associated to the respective maximal and minimal eigenvalues $\lambda^{(0)}_\hbar$ and $E^{(0)}_\hbar$ have both dimension 1.*

Proof. Concerning $e^{-tH_\hbar}$ the thesis is just the statement of Theorem XIII.47 in [77] since (V1)-(V3) imply the validity of the hypotheses of that theorem. Regarding $Q^B_\hbar(e^{-t(p^2+V)})$, the thesis arises from Theorem XIII.43 of [77] if we prove that $Q^B_\hbar(e^{-t(p^2+V)}) \in L^2(\mathbb{R}^n, dx)$ is (a) positive preserving and (b) ergodic. To this end we take advantage of (2.1.14). In particular we have that, if henceforth $e(x,p) := e^{-t(p^2+V(x))}$, we then have that

$$\mathcal{E}(x,p) := \left(e^{-\hbar\Delta_{2n}/4} e\right)(x,p) = \mathcal{E}_1(x)\mathcal{E}_2(p)\,, \quad \text{where}$$

$$\mathcal{E}_1(x) := \int_{\mathbb{R}^n} \frac{e^{-(x-y)^2/\hbar} e^{-tV(y)}}{(\pi\hbar)^{n/2}} dy\,, \quad \mathcal{E}_2(p) := \int_{\mathbb{R}^n} \frac{e^{-(p-r)^2/\hbar} e^{-tr^2}}{(\pi\hbar)^{n/2}} dr\,.$$

Evidently $\mathcal{E}_1(x), \mathcal{E}_2(p) > 0$ for every $x, p \in \mathbb{R}^n$ since the integrands of the integrals above are continuous and strictly positive. Furthermore $\mathcal{E} \in \mathcal{S}(\mathbb{R}^{2n})$ since it is the product of functions in $\mathcal{S}(\mathbb{R}^n)$ in the variable x and p respectively. We can exploit (9.2.22) obtaining

$$\left(Q^B_\hbar(e^{-t(p^2+V)})\phi\right)(x) = \int_{\mathbb{R}^n} \frac{\widehat{\mathcal{E}_2}(b)}{(2\pi)^{n/2}} \mathcal{E}_1\left(x - \frac{\hbar}{2}b\right) \phi\left(x + \hbar b\right) db\,.$$

Observe that $\widehat{\mathcal{E}_2}(b) > 0$ for every $b \in \mathbb{R}^n$ since, up to positive factors, it is the product of two Gaussian functions, because $\mathcal{E}_2$ is the convolution of such pair of functions by definition. We conclude that, if $\phi(x) \geq 0$ a.e. with $\phi \in L^2(\mathbb{R}^n, dx)$, then $\left(Q^B_\hbar(e^{-t(p^2+V(x))})\phi\right)(x) \geq 0$ a.e.. In summary, $Q^B_\hbar(e^{-t(p^2+V)})$ is positive preserving provided that $||\phi|| \neq 0$ for $\phi(x) \geq 0$ a.e., implies $Q^B_\hbar(e^{-t(p^2+V(x))})\phi \neq 0$. We prove this property simultaneously to the ergodicity property just by establishing that, for $\psi(x), \phi(x) \geq 0$ a.e. such that $\psi, \phi \in L^2(\mathbb{R}^n, dx) \setminus \{0\}$, then $\langle \psi, Q^B_\hbar(e^{-t(p^2+V(x))})\phi\rangle \neq 0$. Taking advantage of (9.2.21), we find for $\phi, \psi \in L^2(\mathbb{R}^n, dx)$ with $\psi(x), \phi(x) \geq 0$ a.e., and $||\psi||, ||\phi|| \neq 0$,

$$\langle \psi, Q^B_\hbar(e^{-t(p^2+V)})\phi\rangle = \int_{\mathbb{R}^{2n}} \frac{\widehat{\mathcal{E}_2}(b)}{(2\pi)^{n/2}} \mathcal{E}_1\left(x - \frac{\hbar}{2}b\right) \psi(x)\phi\left(x + \hbar b\right) dxdb\,.$$

If, in addition to $\psi(x), \phi(x) \geq 0$ a.e., it holds $||\psi||, ||\psi|| \neq 0$, then there must exist $k_1, k_2 > 0$ such that, defining the measurable sets $E_1 := (\phi^2)^{-1}(k_1, +\infty)$ and $E_2 := (\psi^2)^{-1}(k_2, +\infty)$, it holds $m(E_1), m(E_2) > 0$, where m is the Lebesgue measure on $\mathbb{R}^n$. Let us change coordinates on $\mathbb{R}^n \times \mathbb{R}^n$ from x, b to $x, z := x + \hbar b$. With this transformation,

$$\langle \psi, Q^B_\hbar(e^{-t(p^2+V)})\phi\rangle = \frac{1}{(2\pi\hbar^2)^{n/2}} \int_{\mathbb{R}^{2n}} \widehat{\mathcal{E}_2}\left(\frac{z-x}{\hbar}\right) \mathcal{E}_1\left(\frac{3x-z}{2}\right) \psi(x)\phi\left(z\right) dxdz$$

$$\geq \frac{\sqrt{k_1 k_2}}{(2\pi\hbar^2)^{n/2}} \int_{E_1 \times E_2} \widehat{\mathcal{E}_2}\left(\frac{z-x}{\hbar}\right) \mathcal{E}_1\left(\frac{3x-z}{2}\right) dxdz > 0$$

where the last integral is well defined since, up to a non-singular linear change of coordinates, the integrand is the product of two $\mathcal{S}(\mathbb{R}^n)$ functions which is therefore a $\mathcal{S}(\mathbb{R}^{2n})$. The integral is eventually strictly positive because the integrand is everywhere strictly positive and $E_1 \times E_2$ has strictly positive measure[5]: $m_{\mathbb{R}^{2n}}(E_1 \times E_2) = m(E_1)m(E_2) > 0$. □

In summary, if (V1)-(V3) are valid, then the eigenspaces of maximal eigenvalues of $Q^B_\hbar(e^{-t(p^2+V)})$ and $e^{-tH_\hbar}$ are spanned by corresponding unit eigenvectors $\varphi^{(0)}_\hbar$ and $\psi^{(0)}_\hbar$, respectively, defined up to phases. The vector $\psi^{(0)}_\hbar$ also defines the *ground state* of $H_\hbar$. It would nice to to study how these eigenvectors are related. We expect that they should coincide as

[5]If $m(F) > 0$ and $g : F \to \mathbb{R}$ is measurable and strictly positive, then $F = \cup_{n\in\mathbb{N}} g^{-1}(1/n, +\infty)$. It must be $m(g^{-1}(1/n_0, +\infty)) = m_0 > 0$ for some n_0 otherwise $m(F) = 0$ for internal regularity. Hence $\int_E g dx \geq \int_{g^{-1}(1/n_0, +\infty)} g dx \geq m_0/n_0 > 0$.

algebraic states when acting on the observables $Q^B_\hbar(f)$ in the limit of small $\hbar$. We postpone this discussion to Corollary 9.3.7, where we shall prove that it is in fact the case for suitable choices of V.

We finally have an important corollary regarding the validity of the hypotheses assumed in stating Theorem 9.1.5 and Theorem 9.1.7.

Corollary 9.2.8. *If $e \in C_0(\mathbb{R}^{2n})$ takes the form (9.1.9) and (V1)-(V3) are satisfied for V, the following facts are valid.*

(1) If $\min e$ is achieved in a unique point $\sigma_0 = (q_0, 0) \in \mathbb{R}^{2n}$, then all the hypotheses of Thm. 9.1.5 are valid with $\Lambda = e^{-tV(q_0)}$, $\lambda_\hbar := \lambda^{(0)}_\hbar$, $\varphi_\hbar = \varphi^{(0)}_\hbar$.

(2) If $\Lambda = e^{-t\min V}$, and the hypothesas (a)-(c) of Thm. 9.1.7 hold, then the theorem is valid with $\Lambda = e^{-t\min V}$, $\lambda_\hbar := \lambda^{(0)}_\hbar$, $\varphi_\hbar = \varphi^{(0)}_\hbar$.

In both cases $\varphi_\hbar = \varphi^{(0)}_\hbar$ is the unique (up to phases) unit eigenvector corresponding to eigenvalue $\lambda^{(0)}_\hbar$.

9.3 Semi-classical properties of Schrödinger operators

In this section we prove the existence of the classical limit of a sequence of eigenvectors of minimal eigenvalues corresponding to Schrödinger operators $H_\hbar$. As before, we use the Berezin quantization maps on $C_0(\mathbb{R}^{2n})$, i.e. compact operators on $L^2(\mathbb{R}^n)$, as set of physical observables. Similar as in Section 9.1, we first prove a localization result (Thm. 9.3.1) of such sequences, followed by two main theorems (Thm. 9.3.3 and Thm. 9.3.4) where again distinction is made between the presence of a symmetry or not. We would like to point out to the reader that localization of eigenvectors and semi-classical tunneling have been extensively studied for Schrödinger operators, see e.g. [6, 46, 47, 93], [45, Sect. 3.4] and [107, Sect. 7.2]. In such works very precise semi-classical estimates and bounds have been obtained for e.g. eigenvectors, quasi-modes, symbols, energies, etc.

9.3.1 Localization of eigenvectors

As Schrödinger operators are unbounded, proposition 9.1.4 cannot directly be applied to eigenvectors of such operators. We hereto prove a similar result yielding localization of ground state eigenvectors $\{\psi^{(0)}_\hbar\}_\hbar$, with minimal eigenvalue $E^{(0)}_\hbar$, of the Schrödinger operators $H_\hbar = \overline{-\hbar\Delta + V}$ on $L^2(\mathbb{R}^n, dx)$ where V satisfies conditions (V1)-(V3).

For $h(q,p) := p^2 + V(q)$ and some given $t > 0$, let us focus on the preimage $e^{-1}(\{\max e^{-th}\})$, where $e(q,p) := e^{-th(q,p)}$. We stress that this set generally contains more than one point and also it coincides with the preimage $h^{-1}(\{\min h\}) = \{p = 0\} \times V^{-1}(\{\min V\})$.

If $e_1(p) := e^{-tp^2}$ and $e_2(q) := e^{-tV(q)}$, consider a class of open neighborhoods $\mathcal{U}_\epsilon$ of $e^{-1}(\{\max e^{-th}\})$

$$\mathcal{U}_\epsilon := \mathcal{U}^1_\epsilon \times \mathcal{U}^2_\epsilon, \tag{9.3.27}$$

$$\mathcal{U}^1_\epsilon := e_1^{-1}((1-\epsilon, 1+\epsilon)) \quad \text{and} \quad \mathcal{U}^2_\epsilon = e_2^{-1}((e^{-t\min V} - \epsilon, e^{-t\min V} + \epsilon)), \quad \epsilon > 0.$$

If we assume $\epsilon > 0$ is sufficiently small, it easy to prove that there is $C > 0$, independent of ϵ, such that

$$\sigma \in \mathcal{U}_\epsilon \quad \text{implies} \quad |e^{-th(\sigma)} - \max e^{-th}| < C\epsilon. \tag{9.3.28}$$

This yields the following proposition.

Proposition 9.3.1. *Let $H_\hbar$ be as in* (9.2.1)*, where V satisfies (V1)-(V3). Let $\{\psi^{(0)}_\hbar\}_\hbar$ be a sequence of eigenvectors of a Schrödinger operators $\{H_\hbar\}_\hbar$ with minimal eigenvalues $\{E^{(0)}_\hbar\}_\hbar$ such that, according to corollary 9.2.6*

$$E^{(0)}_\hbar \to \min V = \min h, \ \textit{for } \hbar \to 0^+. \tag{9.3.29}$$

(1) If $\Psi_\hbar^{(0)} := W\psi_\hbar^{(0)}$ *and the open neighborhood* $\mathcal{U}_\epsilon$ *of* $\min h$ *is defined as in (9.3.27) for every (sufficiently small)* $\epsilon > 0$, *then*

$$||\Psi_\hbar^{(0)}||_{L^2(\mathbb{R}^{2n}\setminus\mathcal{U}_\epsilon, \frac{d^n p d^n q}{(2\pi\hbar)^n})} \to 0, \quad \text{for} \quad \hbar \to 0^+ \,. \tag{9.3.30}$$

(2) If $V^{-1}(\{\min V\}) = \{q_0\} \in \mathbb{R}^{2n}$ *and the family of sets* $\{\mathcal{U}_\epsilon\}_{\epsilon>0}$ *is a fundamental system of neighborhoods of* $\sigma_0 := (0, q_0)$, *then*

$$\langle\varphi_\hbar, Q_\hbar(f)\varphi_\hbar\rangle \to f(\sigma_0) \quad \text{as } \hbar \to 0^+ \text{ for every } f \in C_0(\mathbb{R}^{2n}).$$

Proof. (1) Take $t > 0$. As already done in previous proofs, without loss of generation we assume that that $\min_{x\in\mathbb{R}^n} V(x) = 0$. Since $E_\hbar^{(0)} \to 0$, clearly $e^{-tE_\hbar^{(0)}} \to 1$, as $\hbar \to 0$ (and $t > 0$). A suitable use of Jensen's inequality for probability measures leads to the following result stated in the auxiliary lemma below.

Lemma 9.3.2. *Under the hypotheses of proposition 9.3.1 with* $\min V = 0$,

$$\lim_{\hbar\to 0}\langle\psi_\hbar^{(0)}, e^{-tV}\psi_\hbar^{(0)}\rangle = 1 \quad \text{and} \quad \lim_{\hbar\to 0}\langle\psi_\hbar^{(0)}, e^{t\hbar^2\Delta}\psi_\hbar^{(0)}\rangle = 1\,. \tag{9.3.31}$$

Proof. Take $\varepsilon > 0$. Given $\phi \in C_c^\infty(\mathbb{R}^{2n})$, and using that $H_\hbar$ and in particular V are positive, for any $\hbar > 0$ we observe

$$\langle\phi, (H_\hbar + \varepsilon I)\phi\rangle = -\hbar^2\langle\phi, \Delta\phi\rangle + \langle\phi, (V + \varepsilon I)\phi\rangle \geq \langle\phi, (V + \varepsilon I)\phi\rangle \geq 0.$$

Since $\phi \in D(A)$ implies $\phi \in D(\sqrt{A})$ for a self-adjoint operator $A \geq 0$, we conclude that, if $\phi \in C_c^\infty(\mathbb{R}^b)$ then

$$||\sqrt{H_\hbar + \varepsilon I}\phi||^2 \geq ||\sqrt{V + \varepsilon}\phi||^2\,. \tag{9.3.32}$$

Since $C_c^\infty(\mathbb{R}^n)$ is a core for $H_\hbar$, it is also a core for $H_\hbar + \varepsilon I$. In particular, if $\psi \in D(H)$, there is a sequence $C_c^\infty(\mathbb{R}^n) \ni \phi_n \to \psi$ such that $(H_\hbar + \varepsilon I)\phi_n \to (H_\hbar + \varepsilon I)\psi$. Applying the bounded operator $(H_\hbar + \varepsilon I)^{-1/2}$ on both sides we also have

$$\sqrt{H_\hbar + \varepsilon I}\phi_n \to \sqrt{H_\hbar + \varepsilon I}\psi\,. \tag{9.3.33}$$

Since $||\sqrt{H_\hbar + \varepsilon I}(\phi_n - \phi_m)||^2 \geq ||\sqrt{V + \varepsilon}(\phi_n - \phi_m)||^2$, we conclude that the sequence of the $\sqrt{V + \varepsilon}\phi_n$ is Cauchy and thus it must converge in the Hilbert space when $\phi_n \to \psi$. As $\sqrt{V + \varepsilon}$ is closed (it is self-adjoint), we have that $\sqrt{V + \varepsilon}\phi_n \to \sqrt{V + \varepsilon}\psi$ so that $D(H_\hbar + \varepsilon I) \subset D(\sqrt{V + \varepsilon})$. From (9.3.32),

$$||\sqrt{H_\hbar + \varepsilon I}\psi||^2 \geq ||\sqrt{V + \varepsilon}\psi||^2. \tag{9.3.34}$$

Specializing the result to the normalized ground state eigenvector $\psi_\hbar^{(0)}$ of $H_\hbar$, we have

$$\lambda_\hbar^{(0)} + \varepsilon = ||\sqrt{H_\hbar + \varepsilon I}\psi_\hbar^{(0)}||^2 \geq ||\sqrt{V + \varepsilon}\psi_\hbar^{(0)}||^2 = \int_{[0+\infty)} s^2 d\mu^\varepsilon_{\psi_\hbar^{(0)}}(s)\,,$$

where $\mu^\varepsilon_{\psi_\hbar^{(0)}}$ is the spectral probability measure $\mu^\varepsilon_{\psi_\hbar^{(0)}}(F) := \langle\psi_\hbar^{(0)}, P_F^{\sqrt{(V+\varepsilon I)}}\psi_\hbar^{(0)}\rangle$ defined on $[0, +\infty)$. The function e^{-tx} is convex on $[0, +\infty)$. We can therefore apply *Jensen's inequality* for probability measures, obtaining

$$e^{-t(\lambda_\hbar^{(0)}+\varepsilon)} \leq e^{-t\int_{[0,+\infty)} s^2 d\mu^\varepsilon_{\psi_\hbar^{(0)}}(s)} \leq \int_{[0,+\infty)} e^{-ts^2} d\mu^\varepsilon_{\psi_\hbar^{(0)}}(s) = \langle\psi_\hbar^{(0)}, e^{-t(V+\varepsilon I)}\psi_\hbar^{(0)}\rangle.$$

As a consequence $e^{-t(\lambda_\hbar^{(0)}+\varepsilon)} \leq e^{-t\varepsilon}\langle\psi_\hbar^{(0)}, e^{-tV}\psi_\hbar^{(0)}\rangle \leq 1$, for every $\varepsilon > 0$, so that $e^{-t\lambda_\hbar^{(0)}} \leq \langle\psi_\hbar^{(0)}, e^{-tV}\psi_\hbar^{(0)}\rangle \leq 1$. Using the fact that $\lim_{\hbar\to 0}\lambda_\hbar^{(0)} = 0$, we obtain the former identity in (9.3.31) for every given $t > 0$, $\lim_{\hbar\to 0}\langle\psi_\hbar^{(0)}, e^{-tV}\psi_\hbar^{(0)}\rangle = 1$. With a strictly analogous procedure we also find the latter identity in (9.3.31) for every given $t > 0$. □

The next step is to show that

$$\langle\psi_\hbar^{(0)}, Q_\hbar^B((e^{-tV}-1)^2)\psi_\hbar^{(0)}\rangle \to 0 \quad \text{if } \hbar \to 0, \tag{9.3.35}$$

using the former in (9.3.31). We proceed as in the proof of Proposition 9.1.4. Hereto, we first observe

$$\langle\psi_\hbar^{(0)}, Q_\hbar^B((e^{-tV}-1)^2)\psi_\hbar^{(0)}\rangle = \langle\psi_\hbar^{(0)}, Q_\hbar^B(e^{-2tV})\psi_\hbar^{(0)}\rangle + 1 - 2\langle\psi_\hbar^{(0)}, Q_\hbar^B(e^{-tV})\psi_\hbar^{(0)}\rangle. \tag{9.3.36}$$

Proposition 2.1.13 particularly yields, $||Q_\hbar^B(e^{-tV}) - e^{-tV}|| \to 0$ as $\hbar \to 0^+$ in $\mathfrak{B}(L^2(\mathbb{R}^n, dx))$ for every chosen $t > 0$. This together with the former in (9.3.31) and $||\psi_\hbar|| = 1$, they imply that $\langle\psi_\hbar, Q_\hbar^B(e^{-2tV})\psi_\hbar\rangle$ and $\langle\psi_\hbar, Q_\hbar^B(e^{-tV})\psi_\hbar^{(0)}\rangle$ both converge to 1 as $\hbar \to 0$. Directly from (9.3.36), we conclude that (9.3.35) holds. With a strictly analogous procedure, exploiting the second identity in (9.3.31) and using again Proposition 2.1.13, for every given $t > 0$, $||Q_\hbar^B(e^{-tp^2}) - e^{t\hbar^2\Delta}|| \to 0$ as $\hbar \to 0^+$ in $\mathfrak{B}(L^2(\mathbb{R}^n, dx))$ one also obtains,

$$\lim_{\hbar\to 0}\langle\psi_\hbar^{(0)}, Q_\hbar^B((e^{t\hbar^2p^2}-1)^2)\psi_\hbar^{(0)}\rangle = 0. \tag{9.3.37}$$

To conclude, we have

$$\begin{aligned}||\Psi_\hbar^{(0)}||^2_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\times\mathcal{U}_\epsilon^2)} &= \int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\times\mathcal{U}_\epsilon^2)} |\Psi_\hbar^{(0)}(x)|^2 d\mu_\hbar \le \frac{1}{\epsilon^2}\int_{\mathbb{R}^{2n}\setminus(\mathbb{R}^n\times\mathcal{U}_\epsilon^2)} (1-e^{-V(q)})^2|\Psi_\hbar^{(0)}(x)|^2 d\mu_\hbar \\ &\le \frac{1}{\epsilon^2}\int_{\mathbb{R}^{2n}} (1-e^{-V(q)})^2|\Psi_\hbar(x)|^2 d\mu_\hbar = \frac{1}{\epsilon^2}\langle\psi_\hbar^{(0)}, Q_\hbar^B((e^{-tV}-1)^2)\psi_\hbar^{(0)}\rangle \to 0\,,\end{aligned}$$

for $\hbar \to 0^+$. A similar procedure relying on (9.3.37) proves that $||\Psi_\hbar^{(0)}||^2_{\mathbb{R}^{2n}\setminus(\mathcal{U}_\epsilon^1\times\mathbb{R}^n)} \to 0$ if $\hbar \to 0^+$. Just taking the union of the two considered sets

$$\mathbb{R}^{2n}\setminus(\mathcal{U}_\epsilon^1\times\mathbb{R}^n)\bigcup\mathbb{R}^{2n}\setminus(\mathbb{R}^n\times\mathcal{U}_\epsilon^2) = \mathbb{R}^{2n}\setminus\mathcal{U}_\epsilon,$$

and using the established properties, we eventually arrive at (9.3.30).
(2) The proof is essentially identical to that of (2) in proposition 9.1.4, just exploiting part (1) and everywhere replacing $\varphi_\hbar$ for $\psi_\hbar^{(0)}$, taking in particular $\Phi_\hbar = W\psi_\hbar^{(0)}$. □

9.3.2 Classical limit of ground states of Schrödinger operators

We now prove that the classical limit of ground states of Schrödinger operators exists. We start with the simple case when no symmetry is present in the potential.

Theorem 9.3.3 (Classical limit without symmetry)**.** *Let $H_\hbar$ be as in* (9.2.1)*, where V satisfies (V1)-(V3) and such that* $\min_{q\in\mathbb{R}^n} V(q) = V(q_0)$ *for a unique point $q_0 \in \mathbb{R}^n$ and define $\sigma_0 :=$* $(q_0, 0)$*.*

If $\{\psi_\hbar^{(0)}\}_{\hbar>0}$ is a family of eigenvectors with minimal eigenvalues $\{E_\hbar^{(0)}\}_{\hbar>0}$ of $H_\hbar$, then

$$\langle\psi_\hbar^{(0)}, Q_\hbar(f)\psi_\hbar^{(0)}\rangle \to f(\sigma_0) \quad \textit{as } \hbar \to 0^+, \textit{ for every } f \in C_0(\mathbb{R}^{2n}).$$

Proof. The proof is obtained by straightforwardly rephrasing the proof of Theorem 9.1.5, by replacing $\varphi_\hbar$ for $\psi_\hbar^{(0)}$, taking in particular $\Phi_\hbar := W\psi_\hbar^{(0)}$, and by exploiting the localization properties established in proposition 9.3.1, and using (9.3.28) in particular. To fulfill the hypotheses of theorem 9.1.5, observe that the eigenvalues $\lambda_\hbar := e^{-tE_\hbar^{(0)}}$ converge to $\Lambda :=$ $e^{-t\min V} = \max e^{-th} \neq 0$ and this value is reached at a unique point σ_0 as in theorem 9.1.5. □

Note that this theorem in particular proves the classical limit of the simple quantum harmonic oscillator.

Similar as before we now focus on more complex systems and consider the case when a symmetry is present. Let us again take a group G (a compact topological group or a discrete group) acting by symplectomorphism on $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k\wedge dq^k)$, $G \ni g : \mathbb{R}^{2n} \ni (q,p) \mapsto g(q,p) \in$

$\mathbb{R}^{2n}$. We know from proposition 9.1.3, that G admits a unitary representation $G \ni g \mapsto U_g \in \mathfrak{B}(L^2(\mathbb{R}^n, dx))$ which acts equivariantly (6.2.14) on the quantization map $Q^B_\hbar$.

Exploiting these notions we are finally in the position to prove our main result concerning the classical limit of a sequence of ground-state eigenvectors of $H_\hbar = \overline{\hbar^2\Delta + V}$. In a complete analogous way as in the previous section we consider the strict deformation quantization of the Poisson manifold $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$ (in the sense of Definition 2.1.3) associated with Berezin quantization maps $Q^B_\hbar$ acting either on the space $C_0(\mathbb{R}^{2n})$ or $C^\infty_c(\mathbb{R}^{2n})$, and prove the existence of the classical limit with respect to the observables $Q^B_\hbar(f)$. The following theorem contains the precise statement.

Theorem 9.3.4 (Classical limit with symmetry)**.** *Consider a group G either finite or topological compact, a self-adjoint Schrödinger operator on $L^2(\mathbb{R}^n, dx)$ $H_\hbar := \overline{-\hbar^2\Delta + V}$, as in (9.2.14) where $V : \mathbb{R}^n \to \mathbb{R}$ satisfies (V1)-(V3), and assume the following hypotheses.*

(a) G acts, continuously in the topological-group case[6]*, on $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$ in terms of symplectomorphism.*

(b) Defining $h(q,p) := p^2 + V(q)$, the action of G is leaves invariant $h^{-1}(\{\min h\})$ and is transitive on it.

(c) The unitary representation U of G on $L^2(\mathbb{R}^n, dx)$ constructed according to proposition 9.1.3 (whose action is equivariant on $Q^B_\hbar$) leaves $H_\hbar$ invariant as in (9.3.41) for every given $\hbar > 0$.

Then the following facts are valid for every chosen $\sigma_0 \in h^{-1}(\{\min h\})$ and for a family $\{\psi^{(0)}_\hbar\}_{\hbar>0}$ of (normalized) eigenvectors of $H_\hbar$ with (non-degenerate) minimal eigenvalues $\{E^{(0)}_\hbar\}_{\hbar>0}$ converging to $\min_{q\in bR^n} V(q) = \min_{(q,p)\in\mathbb{R}^{2n}} h(q,p)$ as $\hbar \to 0$. [7]

(1) If G is topological and compact,

$$\lim_{\hbar\to 0^+} \langle \psi^{(0)}_\hbar, Q^B_\hbar(f)\psi^{(0)}_\hbar \rangle = \int_G f(g\sigma_0) d\mu_G(g), \quad \textit{for every } f \in C_0(\mathbb{R}^{2n}); \tag{9.3.38}$$

where μ_G is the normalized Haar measure of G.

(2) If G is finite,

$$\lim_{\hbar\to 0^+} \langle \psi^{(0)}_\hbar, Q^B_\hbar(f)\psi^{(0)}_\hbar \rangle = \frac{1}{N_G} \sum_{g\in G} f(g\sigma_0), \quad \textit{for every } f \in C_0(\mathbb{R}^{2n}); \tag{9.3.39}$$

where N_G is the number of elements of G.

The right-hand sides of (9.3.38) and (9.3.39) are independent of the choice of σ_0.

Proof. We only consider the case of G topological, since the finite case is similar and easier as already discussed in the proof of Theorem 9.1.7. Let us define $\Psi_\hbar(\sigma) = W\psi^{(0)}_\hbar$. Since U_g commutes with $H_\hbar$ for (b) and the eigenspace of $\psi^{(0)}_\hbar$ has dimension 1, it must be $U_g\psi^{(0)}_\hbar = e^{ia_g}\psi^{(0)}_\hbar$ for some real a_g. (6.2.15) implies that $\Psi_\hbar(g^{-1}\sigma) = e^{ia_g}\Psi_\hbar(\sigma)$ so that, using also (a), the probability measure $|\Psi_\hbar(\sigma)|^2 \frac{d\sigma}{(2\pi\hbar)^n}$ turns out to be G-invariant. This result permits out to straightforwardly follow the proof of theorem 9.1.7, finding

$$\langle \psi^{(0)}_\hbar, Q^B_\hbar(f)\psi^{(0)}_\hbar \rangle = \int_{\mathbb{R}^{2n}} |\Psi_\hbar(\sigma)|^2 F(\sigma) \frac{d\sigma}{(2\pi\hbar)^n},$$

where F was defined in (6.2.24). Exactly as in the proof of theorem 9.1.7, F turns out to be (i) bounded, (ii) continuous and (iii) constant on $h^{-1}(\{\min h\})$. Going on as in the proof of Theorem 9.1.7, if $\sigma_0 \in h^{-1}(\{\min h\})$ is any point, we end up with the estimate

$$\left|\langle \varphi_\hbar, Q^B_\hbar(f)\varphi_\hbar\rangle - F(\sigma_0)\right| = \left| \int_{\mathbb{R}^{2n}} |\Phi_\hbar(\sigma)|^2 (F(\sigma) - F(\sigma_0)) \frac{d\sigma}{(2\pi\hbar)^n} \right|.$$

[6]The action $G \times \mathbb{R}^{2n} \ni (g,\sigma) \mapsto g\sigma \in \mathbb{R}^{2n}$ is continuous.

[7]This family always exists as seen in Corollary 9.2.6.

Rephrasing the proof of theorem 9.1.7, with $\mathcal{U}_\delta$ now defined as in (9.3.27) with $e = e^{-th} = e^{-tp^2}e^{-tV}$ for some $t > 0$ (notice that $e^{-1}(\{\max e\}) = h^{-1}(\{\min h\})$), we find

$$\left|\langle\psi_\hbar^{(0)}, Q_\hbar^B(f)\psi_\hbar^{(0)}\rangle - F(\sigma_0)\right| \leq \sup_{\sigma\in\mathcal{U}_{1/m}} |F(\sigma) - F(\sigma_0)| + 2||F||_\infty \int_{\mathbb{R}^{2n}\setminus\mathcal{U}_{1/m}} |\Psi_\hbar(\sigma)|^2 \frac{d\sigma}{(2\pi\hbar)^n}\,, \tag{9.3.40}$$

for every given $m \in \mathbb{N}\setminus\{0\}$. Analogously to lemma 6.2.6, we now have the following result.

Lemma 9.3.5. *Under the hypotheses of Thm 9.3.4, (b) in particular, and F defined in* (6.2.24), *for every $\epsilon > 0$, there is $m_\epsilon \in \mathbb{N}$ such that $\sup_{\sigma\in\mathcal{U}_{1/m_\epsilon}} |F(\sigma) - F(\sigma_0)| < \epsilon/2$, where $\mathcal{U}_\delta$ is defined in (9.3.27).*

Proof. The proof is very similar to the one of Lemma 6.2.6. If $e(q,p) := e^{-t(p^2+V)}$ for some $t > 0$, let us define $\Gamma := e^{-1}(\{\max e\}) = h^{-1}(\{\min h\})$. This set satisfies $\Gamma = \{g\sigma_0 \mid g \in G\}$ for every chosen $\sigma_0 \in \Gamma$ due to G-invariance of h and transitivity of G on Γ (hypothesis (b)), and is compact as said in the proof of Lemma 6.2.6. If $\delta > 0$ a δ-*covering* of Γ is defined as in the proof of Lemma 6.2.6. Since Γ is compact, there is a closed ball B centered at the origin of finite positive radius such that Γ is completely contained in the interior of B. All other balls we shall consider in this proof will be assumed to be contained in the interior of B as well. Since $|e(\sigma)| \to 0$ for $|\sigma| \to +\infty$ and $\max e \neq 0$, we can always fix the radius of B such that $|\max e - e(\sigma)| > \eta$, for some $\eta > 0$, if $\sigma \notin B$. The next step consists of proving that, given a δ-covering C_δ of Γ, with $\delta > 0$ arbitrarily taken, there exists $m_\delta \in \mathbb{N}$ such that $\mathcal{U}_{1/m_\delta} \subset C_\delta$, where $\mathcal{U}_\delta$ is defined as in (9.3.27), so that (9.3.28) is in particular valid. The proof is the same as in the proof of Lemma 6.2.6 with Λ replaced for $\max e$ and using (9.3.28) where necessary. Now take $\sigma_0 \in \Gamma$. Noticing that B is compact and F is continuous thereon, we can use its uniform continuity. Given $\epsilon > 0$, there is $\delta_\epsilon > 0$ such that $|F(\sigma) - F(\sigma')| < \epsilon/2$ if $|\sigma - \sigma'| < \delta_\epsilon$. With this remark, consider a C_{δ_ϵ} covering of Γ. If $\tau \in C_{\delta_\epsilon}$ we have $|F(\tau) - F(\sigma_0)| = |F(\tau) - F(\sigma_0^\tau)|$ where $\sigma_0^\tau \in \Gamma$ is the center of $B_{\delta_\epsilon}(\sigma_0^\tau)$ which contains τ. The identity above is valid because F is constant in Γ. Uniform continuity therefore implies that $|F(\tau) - F(\sigma_0)| < \epsilon/2$ if $\tau \in C_{\delta_\epsilon}$. In summary, given $\epsilon > 0$, if $m_\epsilon \in \mathbb{N}$ is sufficiently large to assure that $\mathcal{U}_{1/m_\epsilon} \subset C_{\delta_\epsilon}$, we have the thesis $\sup_{\sigma\in\mathcal{U}_{1/m_\epsilon}} |F(\sigma) - F(\sigma_0)| \leq \sup_{\sigma\in C_{\delta_\epsilon}} |F(\sigma) - F(\sigma_0)| < \epsilon/2$, concluding the proof. □

Keeping that m_ϵ and exploiting (1) in Proposition 9.3.1, we can find $H_\epsilon > 0$ such that

$$2||F||_\infty \int_{\mathbb{R}^{2n}\setminus\mathcal{U}_{1/m_\epsilon}} |\Phi_\hbar(\sigma)|^2 \frac{d\sigma}{(2\pi\hbar)^n} < \epsilon/2,$$

for $0 < \hbar < H_\epsilon$. Looking at (9.3.40), we conclude that for every $\epsilon > 0$, there is H_ϵ such that $0 < \hbar < H_\epsilon$ implies $\left|\langle\psi_\hbar^{(0)}, Q_\hbar^B(f)\psi_\hbar^{(0)}\rangle - F(\sigma_0)\right| < \epsilon$, concluding the proof. □

We stress that, in general, G does not necessarily define a *simultaneous* symmetry group for $h := p^2 + V$ and $H_\hbar = \overline{-\hbar^2\Delta + V}$ if $\hbar > 0$ and this requirement is however unnecessary for the validity of the theorem above. Nevertheless this does for example happen in the special physically relevant cases introduced in Example 9.1.6 and in these cases *both Theorem 9.1.7 and Theorem 9.3.4 are valid.*

Proposition 9.3.6. *Let the Hamiltonian in $L^2(\mathbb{R}^n, dx)$, $n \geq 1$, be of the form*

$$H_\hbar = \overline{-\hbar^2\Delta + V}$$

with V satisfying (V1)-(V3). Consider the natural action of a group of (some) isometries of $\mathbb{R}^n$, $G \ni (R,a) : \mathbb{R}^n \ni x \mapsto a + Rx \in \mathbb{R}^n$, with $R \in O(n)$ ($R \in \mathbb{Z}_2$ if $n = 1$) and $a \in \mathbb{R}^n$, in terms of symplectomorphisms on $\mathbb{R}^{2n}$,

$$(R,a) : \mathbb{R}^{2n} \ni (q,p) \mapsto (a + Rq, Rp) \in \mathbb{R}^{2n}\,, \quad (R,a) \in G\,.$$

If V is G-invariant, then the unitary representation $G \ni g \mapsto U_g \in \mathfrak{B}(L^2(\mathbb{R}^n, dx))$ constructed *according to proposition 9.1.3 leaves $H_\hbar$ invariant for every given $\hbar > 0$:*

$$U_g H_\hbar U_g^{-1} = H_\hbar \,. \tag{9.3.41}$$

Proof. Per direct inspection from $U_g := W^* u_g W$ and (9.1.3) we have that $(U_{(R,a)}\psi)(x) = \psi((R,a)^{-1}x)$. (Notice that, in this case, $U_{(R,a)}$ is unitary just because (R,a) is an isometry of $\mathbb{R}^n$ and thus leaves dx invariant.) The said action of U_g leaves $\mathcal{S}(\mathbb{R}^n)$ invariant. Furthermore $U_g\Delta|_{\mathcal{S}(\mathbb{R}^n)}U_g^{-1} = \Delta|_{\mathcal{S}(\mathbb{R}^n)}$ (again because the action of g is an isometry of $\mathbb{R}^n$) and $U_g V|_{\mathcal{S}(\mathbb{R}^n)}U_g^{-1} = V|_{\mathcal{S}(\mathbb{R}^n)}$ by hypothesis, so that $U_g H_\hbar|_{\mathcal{S}(\mathbb{R}^n)}U_g^{-1} = H_\hbar|_{\mathcal{S}(\mathbb{R}^n)}$. Taking the closure of both sides, we have (9.3.41). □

The result clearly applies to the cases $V(x) := (x^2-1)^2$ with $G := \mathbb{Z}_2$ for $n = 1$ and $G := SO(n)$ if $n > 1$, in particular.

Taking Corollary 9.2.8 into account, this discussion eventually produces a direct comparison of the classical limits referred to the eigenvectors $\varphi_\hbar^{(0)}$ and $\psi_\hbar^{(0)}$ respectively of $Q_\hbar^B(e^{-t(p^2+V)})$ and $e^{-t\overline{(-\hbar^2\Delta+V)}}$ and referred to the maximal eigenvalues (i.e. the minimal if focusing on $\overline{-\hbar^2\Delta+V}$).

Corollary 9.3.7. *Assume that one of the two cases is valid.*

(1) The hypotheses Theorem 9.3.3 are valid (so that Thm. 9.1.5 holds with $\Lambda = \max_{(q,p)\in\mathbb{R}^{2n}} e^{-t(p^2+V(q))}$),

(2) The hypotheses of Theorem 9.3.4 and Thm. 9.1.7, with $\Lambda = \max_{(q,p)\in\mathbb{R}^{2n}} e^{-t(p^2+V(q))}$, are simultaneously valid for the same group G (this happens in particular for the case of Proposition 9.3.6).

Then

$$\lim_{\hbar\to 0^+}\left(\langle\psi_\hbar^{(0)}, Q_\hbar^B(f)\psi_\hbar^{(0)}\rangle - \langle\varphi_\hbar^{(0)}, Q_\hbar^B(f)\varphi_\hbar^{(0)}\rangle\right) = 0, \quad \textit{for every } f \in C_0(\mathbb{R}^{2n}).$$

9.4 Classical limit of Gibbs states

We again consider the Schrödinger operator $H_\hbar$ defined by (9.2.14) with potential V satisfying (V1)-(V3) of hypothesis 9.2.1. It follows that $H_\hbar$ is self adjoint and by [76, Thm X. 28] the operator $-\hbar^2\Delta + V$ is essentially self-adjoint on $C_0(\mathbb{R}^n)$. By an easy computation one can show that $\mathcal{S}(\mathbb{R}^n) \subset D(H_\hbar)$, where $D(H_\hbar)$ denotes the domain of $H_\hbar$. Moreover, we require the property that $e^{-tH_\hbar}$ is trace-class for any $t > 0$, as typically happens when the potential V is a real polynomial blowing up at $+\infty$ (and therefore satisfying $(V2)$). To see this, we first observe that as a result of [90, Chapter 8] the bound

$$Tr[e^{-\beta H_\hbar}] \leq \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} d^n p d^n q e^{-\beta h(q,p)}, \tag{9.4.42}$$

with $h(q,p) = p^2 + V(q)$ automatically holds. The right-hand side is nothing else than $Tr[Q_\hbar^B(e^{-\beta h(q,p)})]$ which is finite if $e^{-\beta h(q,p)} \in L^1 \cap L^\infty$ as a result Proposition 2.1.10. Clearly, if V is a real polynomial satisfying hypothesis $(V2)$ the function $e^{-\beta(p^2+V(q))} \in L^1 \cap L^\infty$. It follows that $Tr[|e^{-\beta H_\hbar}|] = Tr[e^{-\beta H_\hbar}] < \infty$.

This yields the following result proving the classical limit of Gibbs states at inverse temperature $\beta < \infty$.

Proposition 9.4.1. *Let $H_\hbar$ be a Schrödinger operator of the above type such that $e^{-\beta(p^2+V(q))} \in L^1 \cap L^\infty$. Consider the β- Gibbs state $\omega_\hbar^\beta$ ($\beta < \infty$) given by*

$$\omega_\hbar^\beta(\cdot) = \frac{Tr[\,\cdot\, e^{-\beta H_\hbar}]}{Tr[e^{-\beta H_\hbar}]}. \tag{9.4.43}$$

Then the following limit exists for any $a \in C_0(\mathbb{R}^{2n})$,

$$\lim_{\hbar\to 0}\left|\omega_\hbar^\beta(Q_\hbar^B(a)) - \frac{\int_{\mathbb{R}^{2n}} d\sigma a(\sigma)e^{-\beta h(\sigma)}}{\int_{\mathbb{R}^{2n}} d\sigma e^{-\beta h(\sigma)}}\right| = 0, \tag{9.4.44}$$

where $\sigma = (q,p)$, $d\sigma$ is the Liouville measure on $\mathbb{R}^{2n}$ (which coincides with the standard $2n$-dimensional Lebesgue measure d^nqd^np), $h(\sigma) = p^2 + V(q)$ and $Q_\hbar^B$ is the Berezin quantization map defined by (2.1.13).

Remark 9.4.2. Before going to the proof we stress that, as a result of [18] any KMS state at inverse temperature β on a finite-dimensional connected symplectic manifold assumes, up to normalization the form $\int_{\mathbb{R}^{2n}} d\sigma a(\sigma)e^{-\beta h(\sigma)}$. It is therefore not surprising that (9.4.44) is expected. ■

In order to prove the proposition we start with a lemma relating the classical and quantum partition functions. This result can be seen as a corollary of the so-called Berezin-Lieb inequality (we refer to the books [41] and [28] for details). For sake of completeness we give a detailed proof in the following lemma.

Lemma 9.4.3. *It holds*

$$\lim_{\hbar\to 0}\left|(2\pi\hbar)^n Tr[e^{-\beta H_\hbar}] - \int_{\mathbb{R}^{2n}} d^npd^nqe^{-\beta h(q,p)}\right| = 0,$$

where $h(q,p) = p^2 + V(q)$.

Proof. As we have seen before, by [90, Chapter 8] the bound

$$Tr[e^{-\beta H_\hbar}] \leq \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} d^npd^nqe^{-\beta h(q,p)},$$

automatically holds. Since $e^{-tH_\hbar}$ is trace-class we can use the resolution of the identity of coherent state vectors $\Psi_\hbar^{(q,p)}$ [28, Prop. 6] and obtain

$$Tr[e^{-\beta H_\hbar}] = \int_{\mathbb{R}^{2n}} \frac{dqdp}{(2\pi\hbar)^n}\langle\Psi_\hbar^{(q,p)}, e^{-\beta H_\hbar}\Psi_\hbar^{(q,p)}\rangle.$$

By the spectral theorem,

$$\langle\Psi_\hbar^{(q,p)}, e^{-\beta H_\hbar}\Psi_\hbar^{(q,p)}\rangle = \int_0^\infty e^{-\beta\lambda}d\nu_\hbar^{(q,p)}(\lambda),$$

where $\nu_\hbar^{(q,p)}(F) = \langle P_F^{H_\hbar}\Psi_\hbar^{(q,p)}, \Psi_\hbar^{(q,p)}\rangle$, and $P_F^{H_\hbar}$ denotes the spectral probability measure on $[0,\infty)$ associated to the operator $H_\hbar$. Since the function $x \mapsto e^{-\beta x}$ is convex on $[0,\infty)$ we can apply Jensen's inequality for probability measures, obtaining

$$e^{-\beta\int_0^\infty \lambda d\nu_\hbar^{(q,p)}(\lambda)} \leq \int_0^\infty e^{-\beta\lambda}d\nu_\hbar^{(q,p)}(\lambda).$$

Since $\Psi_\hbar^{(q,p)} \in \mathcal{S}(\mathbb{R}^n) \subset D(H_\hbar)$ $((q,p) \in \mathbb{R}^{2n})$ it follows that

$$\int_0^\infty \lambda d\nu_\hbar^{(q,p)}(\lambda) = \langle\Psi_\hbar^{(q,p)}, H_\hbar\Psi_\hbar^{(q,p)}\rangle.$$

using the definition of the vectors $\Psi_\hbar^{(q,p)}$ is not difficult to see that[8]

$$\langle\Psi_\hbar^{(q,p)}, H_\hbar\Psi_\hbar^{(q,p)}\rangle = p^2 + \frac{n\hbar}{2} + V(q).$$

[8]We refer i.e. to [91] for an extensive calculation, but it can be done by hand easily.

Combining the above results yields, and integrating over the phase space $\mathbb{R}^{2n}$ with respect to the measure $d\mu_\hbar(q,p) = \frac{1}{(2\pi\hbar)^n} d^nqd^nq$ yields the inequality

$$\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} d^nqd^npe^{-\beta(p^2+\frac{n\hbar}{2}+V(q))} \leq Tr[e^{-\beta H_\hbar}] \leq \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} d^npd^nqe^{-\beta h(q,p)},$$

which implies

$$\int_{\mathbb{R}^{2n}} d^nqd^npe^{-\beta(p^2+\frac{n\hbar}{2}+V(q))} \leq (2\pi\hbar)^n Tr[e^{-\beta H_\hbar}] \leq \int_{\mathbb{R}^{2n}} d^npd^nqe^{-\beta h(q,p)}.$$

Since $p^2 + \frac{n\hbar}{2} + V(q)$ converges pointwise to $p^2 + V(q)$ an application of dominated convergence theorem now proves the thesis. □

We finally prove the proposition below.

Proof of Proposition 9.4.1. The proof is based on the work by [60] and a C^*-version of the *Peierls-Bogolyubov Inequality* [87, Thm 7], namely

$$\frac{Tr(Be^A)}{Tr(e^A)} \leq \ln\left[\frac{Tr(e^{A+B})}{Tr(e^A)}\right], \tag{9.4.45}$$

whenever B is bounded and self-adjoint, A is self-adjoint and bounded above such that $Tr(e^A) < \infty$. For $\lambda > 0$ we apply this inequality to $B = -\beta\lambda Q_\hbar^B(a)$ (with $a \in C_0(\mathbb{R}^{2n})$ real-valued so that $Q_\hbar^B(a)$ is self-adjoint and in particular compact), and to $A = -\beta H_\hbar$ which is self-adjoint, bounded above by e.g. zero and, as seen before its exponential $e^{-\beta H_\hbar}$ has finite trace. Let us define

$$f_\hbar^Q(\lambda) := -\beta^{-1}\ln\left[(2\pi\hbar)^n Tr(e^{-\beta(H_\hbar+\lambda Q_\hbar^B(a))})\right]. \tag{9.4.46}$$

Hence, with $\lambda > 0$ we see

$$[f_\hbar^Q(0) - f_\hbar^Q(-\lambda)]/\lambda \geq \omega_\hbar^\beta(Q_\hbar^B(a)) \geq [f_\hbar^Q(\lambda) - f_\hbar^Q(0)]/\lambda. \tag{9.4.47}$$

An application of the previous lemma yields the inequality

$$Z_\hbar^{cl}(\delta_\hbar) \leq Z_\hbar^Q \leq Z_\hbar^{cl}(0),$$

where $Z_\hbar^Q$ denotes the quantum partition function, i.e. $Z_\hbar^Q = Tr[e^{-\beta H_\hbar}]$, $Z_\hbar^{cl}(0) = Tr[Q_\hbar^B(e^{-\beta h})]$, and $Z_\hbar^{cl}(\delta_\hbar) = Tr[Q_\hbar^B(e^{-\beta(h+\delta_\hbar)})]$, with $\delta_\hbar = n\hbar/2$. It follows that

$$f^{cl}(0,0) \leq f_\hbar^Q(0) \leq f^{cl}(0,\delta_\hbar),$$

where, similarly as before, we defined for any $\lambda \in \mathbb{R}$ the functions

$$f^{cl}(\lambda,0) := -\beta^{-1}\ln\left[\int_{\mathbb{R}^{2n}} e^{-\beta(h(\sigma)+\lambda a(\sigma))}d\sigma\right];$$

$$f^{cl}(\lambda,\delta_\hbar) := -\beta^{-1}\ln\left[\int_{\mathbb{R}^{2n}} e^{-\beta(h(\sigma)+\lambda a(\sigma)+\delta_\hbar)}d\sigma\right].$$

We point out to the reader that, if A and B are operators, then $(A+B)^* = A^* + B^*$ if A is densely defined and $B \in \mathfrak{B}(\mathcal{H})$ [64]. As a result the operator $H_\hbar + \lambda Q_\hbar^B(a)$ is self adjoint on $D(H_\hbar)$. Moreover, as a corollary of [87, Thm 4] using that $e^{-\beta H_\hbar}$ is trace-class, we observe,

$$Tr[|e^{-\beta(H_\hbar+\lambda Q_\hbar^B(a))}|] = Tr[e^{-\beta(H_\hbar+\lambda Q_\hbar^B(a))}] \leq Tr[e^{-\beta H_\hbar}e^{-\beta\lambda Q_\hbar^B(a))}] < \infty,$$

so that in particular $e^{-\beta(H_\hbar+\lambda Q_\hbar^B(a))}$ is trace-class. Repeating the same argument as in the lemma applied to $e^{-\beta(H_\hbar+\lambda Q_\hbar^B(a))}$ we obtain

$$\int_{\mathbb{R}^{2n}} e^{-\beta(h(\sigma)+\lambda a(\sigma)+\delta_\hbar)}d\sigma \leq (2\pi\hbar)^n Tr[e^{-\beta(H_\hbar+\lambda Q_\hbar^B(a))}] \leq \int_{\mathbb{R}^{2n}} e^{-\beta(h(\sigma)+\lambda a(\sigma))}d\sigma.$$

This combined with (9.4.47) yields

$$[f^{cl}(\lambda,0) - f^{cl}(0,\delta_\hbar)]/\lambda \leq \omega_\hbar^\beta(Q_\hbar^B(a)) \leq [f^{cl}(0,\delta_\hbar)] - f^{cl}(-\lambda,0)]/\lambda.$$

Since the logarithmic function is continuous on the positive real axis, we observe $f^{cl}(0,\delta_\hbar) \to f^{cl}(0,0)$ as $\hbar \to 0$. We can therefore drop the second 0 and we write $f^{cl}(\lambda) := f^{cl}(\lambda,0)$. Hence,

$$[f^{cl}(\lambda) - f^{c}(0)]/\lambda \leq \limsup_{\hbar\to 0} \omega_\hbar^\beta(Q_\hbar^B(a)) \leq [f^{cl}(0) - f^{cl}(-\lambda)]/\lambda.$$

Since f^{cl} is differentiable in $\lambda \in \mathbb{R}$ we must have $\lim_{\lambda\to 0^+}[f^{cl}(\lambda) - f^{cl}(0)]/\lambda = \lim_{\lambda\to 0^+}[f^{cl}(0) - f^{cl}(-\lambda)]/\lambda = \frac{d}{d\lambda}|_{\lambda=0} f^{cl}(\lambda)$. It is not difficult to see that the derivative equals

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} f^{cl}(\lambda) = \frac{\int_{\mathbb{R}^{2n}} d\sigma a(\sigma)e^{-\beta h(\sigma)}}{\int_{\mathbb{R}^{2n}} d\sigma e^{-\beta h(\sigma)}}.$$

We conclude that

$$\lim_{\hbar\to 0} \omega_\hbar^\beta(Q_\hbar^B(a)) = \frac{\int_{\mathbb{R}^{2n}} d\sigma a(\sigma)e^{-\beta h(\sigma)}}{\int_{\mathbb{R}^{2n}} d\sigma e^{-\beta h(\sigma)}}.$$

This proves the proposition. □

9.5 SSB of ground states as an emergent phenomenon in Berezin quantization on $\mathbb{R}^{2n}$ with Schrödinger Hamiltonians

We start this section with a proposition concerning weak symmetry breaking of a classical system. More precisely, we consider the function $h = p^2 + V$ on $\mathbb{R}^{2n}$ playing the role of the classical Schrödinger Hamiltonian and show that the corresponding ground states exhibit weak symmetry breaking.

Proposition 9.5.1. *Consider the C^*-dynamical system $(C_0(\mathbb{R}^{2n}), \alpha^{(h)})$ where $\alpha^{(h)}$ is generated by the Hamiltonian $h = p^2 + V$ with*

$$V(q) = (q^2 - 1)^2\,. \tag{9.5.48}$$

Consider the natural action $g : (q,p) \mapsto (gq, gp)$ of $g \in G$ in terms of sympectomorphisms of $(\mathbb{R}^{2n}, \sum_{k=1}^n dp_k \wedge dq^k)$ as in proposition 9.3.6 with $G := \mathbb{Z}_2$ if $n = 1$ or $G := SO(n)$ if $n > 1$. Then G is a dynamical symmetry group of $(C_0(\mathbb{R}^{2n}), \alpha^{(h)})$ with action

$$\gamma_g f := f \circ g^{-1} \quad \textit{for all } g \in G \textit{ and } f \in C_0(\mathbb{R}^{2n}), \tag{9.5.49}$$

and weak SSB occurs.

Proof. The Hamiltonian flow $\alpha^{(h)}$ is complete since the level sets of h are compact and every solution of Hamilton equations is contained in one such set as h is dynamically conserved. Since the action of G is given by symplectomorphisms and every γ_g leaves h invariant and thus it commutes with the Hamiltonian flow, G is a dynamical symmetry group of the C^*-dynamical system $(C_0(\mathbb{R}^{2n}), \alpha^{(h)})$. From Lemma 7.3.1, the extremal ground states are defined by the Dirac measures concentrated at the set of zeros of dh. In all cases the only G-invariant extremal ground state is located at $(q_0, p_0) = (0, 0)$. There is however a plethora of non-G invariant extremal ground states located at the points $(q, 0)$ with $|q| = 1$ if $n > 2$ and exactly two non-G invariant extremal ground states $(\pm 1, 0)$ if $n = 1$. □

We now recall the commutative C^*-algebra $\mathfrak{A}_0 := C_0(\mathbb{R}^{2n})$ which is the $\hbar = 0$ fiber of the continuous C^*-bundle with Berezin deformation quantization maps $Q^B_\hbar : C_0(\mathbb{R}^{2n}) \to \mathfrak{A}_\hbar := \mathfrak{B}_\infty(L^2(\mathbb{R}^n dx))$. The dynamical evolution described by a $\hbar$-parametrized family of one-parameter group of C^*-automorphisms $\mathbb{R} \ni t \mapsto \alpha^\hbar_t$ is provided by a corresponding $\hbar$-parametrized family of one-parameter unitary groups $\mathbb{R} \ni t \mapsto U^\hbar_t := e^{-itH_\hbar}$:

$$\alpha^\hbar_t(A) := U^\hbar_{-t} A U^\hbar_t, \quad (A \in \mathfrak{A}_\hbar\,). \tag{9.5.50}$$

Notice that $\alpha^\hbar$ is strongly continuous in $\mathfrak{B}(\mathfrak{A}_\hbar)$ due to proposition 7.2.2 and thus $(\mathfrak{A}_\hbar, \alpha^\hbar)$ is a C^*-dynamical system. We assume that $H_\hbar := -\hbar^2\Delta + V$ where V satisfies (V1)-(V3). In this case $H_\hbar$ is affiliated to $\mathfrak{A}_\hbar$ because the resolvent of $H_\hbar$ is compact (Theorem XIII.67 in [77]) and thus, in particular, $H_\hbar$ is a physical observable of the system represented by $\mathfrak{A}_\hbar$.

That is not the whole story because, if choosing as before V as in (9.5.48), it turns out that $G = \mathbb{Z}_2$, for $n = 1$, or $G := SO(n)$, if $n > 1$, becomes a dynamical symmetry group as a consequence of Prop. 9.3.6 when the unitary action of G is given by the $Q^B_\hbar$-equivariant representation (9.1.5), so that $U_g Q^B_\hbar(f) U^*_g = Q^B_\hbar(\gamma_g f)$, where γ_g is the same as in (9.5.49). The next proposition is the analog of Proposition 7.3.2.

Proposition 9.5.2. *Consider the C^*-dynamical system $(\mathfrak{B}_\infty(L^2(\mathbb{R}^n, dx)), \alpha^\hbar)$, the latter defined in (9.5.50), and with dynamical symmetry group G whose unitary and $Q^B_\hbar$-equivariant action is defined in (9.1.4)-(9.1.5). No SSB (or weak SSB) occurs for $\hbar > 0$.*

Coming back to the potential $V(q) = (q^2-1)^2$ in particular, the existence of the classical limits established in theorem 9.3.4 now specializes to

(1) if $n > 1$,

$$\lim_{\hbar\to 0^+} \langle \psi^{(0)}_\hbar, Q^B_\hbar(f)\psi^{(0)}_\hbar \rangle = \int_{SO(n)} f(g\sigma_0) d\mu_{SO(n)}(g), \quad \text{for every } f \in C_0(\mathbb{R}^{2n}); \tag{9.5.51}$$

where $\mu_{SO(n)}$ is the normalized Haar measure of $SO(n)$ and $\sigma_0 = (q_0, 0)$ with $|q_0| = 1$,

(2) if $n = 1$,

$$\lim_{\hbar\to 0^+} \langle \psi^{(0)}_\hbar, Q^B_\hbar(f)\psi^{(0)}_\hbar \rangle = \frac{1}{2}(f(1,0) + f(-1,0)), \quad \text{for every } f \in C_0(\mathbb{R}^2). \tag{9.5.52}$$

It is of utmost relevance to notice that both right-hand sides can be reproduced as integrals with respect to $SO(n)/\mathbb{Z}_2$-invariant probability measures μ and ν on $\mathbb{R}^{2n}$ with supports given by the whole orbit $G\sigma_0$, where $G := SO(n)$ or $\mathbb{Z}_2$ respectively. For sake of completeness we state Proposition 6.2.7 specified to the manifold $\mathbb{R}^{2n}$.

Proposition 9.5.3. *Let G be a topological compact or finite group with an action (continuous in the first case) on $\mathbb{R}^{2n}$. Then there are two regular Borel probability measures on $\mathbb{R}^{2n}$, respectively μ and ν, such that*

$$\int_G f(g\sigma_0) d\mu_G(g) = \int_{\mathbb{R}^{2n}} f d\mu\,; \qquad \frac{1}{N_G}\sum_{g\in G} f(g\sigma_0) = \int_{\mathbb{R}^{2n}} f d\nu, \qquad \textit{for all } f \in C_0(\mathbb{R}^{2n}),$$

where μ_G is the normalized Haar measure on G in the first case and N_G is the number of elements of G in the second case. These measures are invariant under the action of G on $\mathbb{R}^{2n}$ and each of their supports is the whole orbit $G\sigma_0$.

These measures, which by definition are $SO(n)/\mathbb{Z}_2$-invariant ground states of the dynamical system $(\mathfrak{A}_0, \alpha^0) = (C_0(\mathbb{R}^{2n}), \alpha^{(h)})$ with $h(q,p) = p^2 + (q^2-1)^2$, are therefore not concentrated on single points, but they are concentrated on the set of points where h attains its minimum value: the orbit $\{(p = 0, q) \mid |q| = 1\}$ in the $SO(n)$ case and the set $\{(p = 0, q = \pm 1)\}$ in the $\mathbb{Z}_2$ case. Hence they are exactly a case of non-extremal G-invariant ground states responsible for the weak SSB discussed above.

All that proves that (weak) SSB of $SO(n)/\mathbb{Z}_2$ occurs in the classical limit $\hbar \to 0^+$, when achieving the theory in $\mathfrak{A}_0 = C_0(\mathbb{R}^{2n})$ with classical hamiltonian $h(q,p) = p^2 + V(q)$, where

$V(q) = (q^2 - 1)^2$, from the quantum theory in $\mathfrak{A}_\hbar = B_\infty(L^2(\mathbb{R}^n, dx))$ with Hamiltonian $H_\hbar = \overline{-\hbar^2\Delta + V}$. In this sense SSB shows up here as an *emergent phenomenon* [57] when passing from the quantum to the classical realm.

Remark 9.5.4. We stress that, this emergent phenomenon is quite general in the framework of Berezin quantization on $\mathfrak{A}_0 := C_0(\mathbb{R}^{2n})$ when we use the Berizin map $Q_\hbar^B(f)$ (with $Q_0^B(f) := f$) to describe the observables in $\mathfrak{A}_\hbar = B_\infty(L^2(\mathbb{R}^n, dx))$. By collecting Theorem 9.3.4, Prop. 7.3.1, Prop. 9.5.2, and Prop. 9.5.3, we see that, when dealing with a classical Hamiltonian $h(q,p) = p^2 + V(q)$ and its quantum companion $H_\hbar = \overline{-\hbar^2\Delta + V}$, where V satisfies (V1)-(V3)[9], emergent weak SSB shows up for some group G provided that the following requirements hold.

(a) G is compact or finite (its action on $\mathbb{R}^{2n}$ is continuous in the former case) and leaves invariant both h and $H_\hbar$. The action of G in the quantum Hilbert space is the $Q_\hbar^B$-equivariant action induced from the action of G on $\mathbb{R}^{2n}$ by symplectomorhisms (Prop. 9.1.3);

(b) $h^{-1}(\{\min h\})$ (equivalently, $V^{-1}(\{\min V\})$) includes more than one point;

(c) the action of G is transitive on $h^{-1}(\{\min h\})$.

According to Prop. 9.3.6, condition (a) is in particular valid when G is made of some isometries of $\mathbb{R}^n$, $(R,a) : \mathbb{R}^n \ni x \mapsto a + Rx \in \mathbb{R}^n$ ($a \in \mathbb{R}^n$, $R \in O(n)$) which leave V invariant and their action in terms of symplectomorphisms on $\mathbb{R}^{2n}$ is the standard one $(R,a) : \mathbb{R}^{2n} \ni (q,p) \mapsto (a + Rq, Rp) \in \mathbb{R}^{2n}$. ■

[9] Conditions (V1)-(V3), with $V(q) \to +\infty$ if $|q| \to +\infty$ in particular, imply that the flow of $h = p^2 + V$ is complete.

10. Perspectives and open problems

The work presented in this thesis raises several questions. First of all, we have focused on the classical limit of several quantum Hamiltonians, e.g. the limit $\hbar \to 0$ of n-particle Schrödinger operators or the limit $N \to \infty$ of mean-field quantum spin Hamiltonians defined on N lattice sites. Even though we obtained numerous rigorous results, they are established for a fixed finite-dimensional phase space, characterizing finite systems, and therefore they do not completely describe Nature since in Nature the number of particles is typically very large, i.e. the phase space has a very large dimension allowing for (approximately) emergent phenomena.[1] In Theory, this is accomplished by taking the limit of large number of particles $n \to \infty$ in a suitable sense.[2] A classical theory of an infinite system is now obtained by means of a double limit: the limit of large number of particles followed by the classical limit [60]. In the context of Schrödinger operators this means the limit $n \to \infty$ followed by the limit $\hbar \to 0$. For a general quantum spin system this translates to the thermodynamic limit $N \to \infty$ (with N the number of particles) and classical limit $J \to \infty$ respectively, where J denotes the spin and relates to the dimension $2J + 1$ of the single site algebra. We remind the reader that in case of mean-field quantum spin systems, $J = N/2$, so that the classical limit and thermodynamic limit are basically the same thing.

In Chapter 5 a framework is sketched to study such double limits. Definitely, more research should be carried out to obtain rigorous results. Another promising approach is outlined in the next section.

10.1 Resolvent algebra

The non-commutative resolvent algebra $\mathcal{R}(X)$ over some symplectic (possible infinite dimensional) manifold X, firstly introduced by Buchholz and Grundling [23], provides an excellent setting to study unbounded operators. In contrast to the well-known Weyl algebra, the resolvent algebra moreover provides a convenient framework for the study of finite as well as infinite particle systems, and therefore includes a lots of physics [23, 22]. Recently, new developments have been achieved concerning quantization of the so-called *commutative resolvent algebra* [70], representing a classical analog of $\mathcal{R}(X)$. The resolvent algebra (with its classical analog) is therefore a promising way to study the limit of large number of particles or thermodynamic limit as well as the classical limit and can therefore be used in the search of spontaneous symmetry breaking in large quantum systems and their classical counterparts. In addition, since infinite degrees of freedom are allowed, also important phenomena as Bose Einstein Condensation (BEC) and phase transitions (i.e. the non-uniqueness of KMS states at a given temperature) might be understood exploiting the algebraic properties of the resolvent algebra.

10.2 Infinite systems, phase transitions and SSB

Let us sketch some perspectives where both quantization theory and physics in infinite particle systems come together. Physical properties of interest are spontaneous symmetry breaking and phase transitions, both seen as emergent phenomena when passing to an infinite system (be it a classical or quantum one). In the algebraic approach used in this thesis, such properties

[1] In this thesis we have already encountered an important example of emergence, namely spontaneous symmetry breaking. In contrast to (classical) phase transitions, which are more common for infinite systems, we have seen that SSB is typical for finite classical theories, i.e. theories described on a finite dimensional phase space, as well.

[2] The mathematical approach for obtaining this limit exists under the name *inductive* limit, also denoted by thermodynamic limit in the mathematical physics literature.

are encoded by states. Two important classes of such states are the so-called Kubo-Martin-Schwinger (KMS) states and ground states. KMS states describe thermal equilibrium at fixed temperature, and are generally believed to have a great importance in physics especially with emphasis to phase transitions, that is, the occurrence of more than one KMS state (non-uniqueness) for a given dynamics. Ground states in turn correspond to zero temperature and play a crucial role in SSB [57]. It is widely accepted that most physical properties are encoded in terms of these states [19, 20].

Symmetry and its possibly breakdown might also play a role in phase transitions in infinite quantum systems. The question to be answered is to find out which KMS states are responsible for quantum phase transitions, in particular those KMS states obeying diverse symmetries for distinct values of β, as in this way the presence of symmetry breaking phases in quantum systems can be detected. Since this is in general a problem of great difficulty, the idea is to proceed in the following way aiming to reduce the question in terms of classical KMS states on a commutative C^*-algebra. A possibility is to consider the *commutative resolvent algebra* $\mathcal{A}_C := C_{\mathcal{R}}(X, \sigma)$. Analogously to the non-commutative resolvent algebra $\mathcal{R}(X, \sigma)$ the algebra $\mathcal{A}_C$ is the C^*-inductive limit of the net of its subalgebras $C_{\mathcal{R}}(Y, \sigma)$ where $Y \subset X$ ranges over all finite dimensional non-degenerate subspaces of X. This allows to study approximations of infinite by finite classical systems. This is a definitely a great advantage: physical features encoded by algebraic states corresponding to infinite systems can therefore be analyzed from the relatively well-known finite systems.

Classical KMS states on a C^*-algebra $\mathcal{A}$ are then defined as follows (we refer to [1, 40] for a more complete definition of classical KMS states). Let V be a smooth complete vector field $V \in \Gamma(TX)$. The flow associated to V induces a one-parameter group of $\alpha^V : \mathbb{R} \to \text{Aut}(\mathcal{A})$ of $*$-automorphisms on $\mathcal{A}$. A state $\omega : \mathcal{A} \to \mathbb{C}$ is said to be a (β, τ^V)-KMS state if for any $f, g \in \tilde{A}_0$ one has $\omega(\sigma(df, dg)) = -\beta\omega(gV(f))$, where $\tilde{A}_0$ is a suitable dense $*$ Poisson subalgebra of $\mathcal{A}$. For our purpose we should take $\mathcal{A} = \mathcal{A}_C$ with $\mathcal{A}_C$ the commutative resolvent algebra.

Several studies have shown the existence of phase transitions in infinite classical theories, in particular those related by symmetry broken phases [19, 57]. The goal now is to investigate the class of classical KMS states related to the existence of this kind of phase transitions. A detailed analysis of the pertinent symplectic manifold should, of course, be carried out, using the powerful method of reduction of symplectic manifolds with symmetry.

Eventually, one has to map back to the quantum case. To this end, quantization techniques should be exploited and one should seek for a quantization of these classical KMS states in terms of quantum KMS states. In this setting, the theory of *formal deformation quantization* (FDQ) is crucial as this quantization scheme shares the convenient property that no explicit quantization maps have to be constructed and classical states can be directly deformed into quantum states [101].

10.3 Explicit symmetry breaking in real materials

Finally, as already indicated in Section 7.4 the study of explicit symmetry breaking in real matter is not well understood, not even for $SO(n)$-invariant potentials, except in the case of $n = 1$ as this gives $\mathbb{Z}_2$-symmetry and the "flea" mechanism as discussed in Section 7.4 or in [93]. A first question to be answered is how to generalize this mechanism for Schrödinger operators with $SO(n)$-invariant potentials. A small asymmetric perturbation should be added to the $SO(n)$-symmetric potential in order the break the internal symmetry. The idea is the same: due to this asymmetry in the model the exact ground state localizes in a single point in phase space, in such a way that for finite, but small values of $\hbar > 0$ the exact ground state already localizes in a single point. Since point measures are not invariant and therefore break the symmetry, this would suggest that explicit symmetry breaking takes place for non-zero but small $\hbar$ and therefore the fact that in Nature (describing finite systems) one of the extreme symmetry-breaking states is found, rather than the nonphysical mixture as originally predicted by the theory.

Having an idea on how to define such perturbation, it should be proved mathematically. A possible strategy is based on the existence of the classical limit of the unperturbed models. Indeed, proving the existence of the classical limit (viz. Section 6.2) of the ground state of the

Hamiltonian in question and exploiting the notion of SSB in this limit, leads to the following conjecture whose details are based on the ideas of Section 7.4.

Conjecture 10.3.1. *Let $H_\hbar$ be a quantum Hamiltonian indexed by some parameter $\hbar$ (which might be discrete in the case of spin systems) and consider the corresponding sequence of ground states $\omega_\hbar^{gr}$. Assume there is a group G such that $\omega_\hbar^{gr}$ is G-invariant for any $\hbar \neq 0$ and that the classical limit of the sequence of ground states converges to some G- invariant mixed state as $\hbar \to 0$. Then, there exists a perturbation δ defined independently of $\hbar$, such that the ground state of the perturbed Hamiltonian $H_\hbar + \delta$ converges to an extreme ground state which is not invariant under the action of G. Moreover, the perturbation can be chosen in such a way that for finite, but small $\hbar$ the perturbed ground state already breaks the symmetry, whilst for relatively large values of $\hbar$ the state remains invariant.* ■

Appendices

1. Elementary facts on coherent spin states

A.1 Coherent spin states and Dicke basis in $\mathrm{Sym}^N(\mathbb{C}^2)$

Let $|\uparrow\rangle, |\downarrow\rangle$ denote the eigenvectors of σ_3 in $\mathbb{C}^2$, so that $\sigma_3|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_3|\downarrow\rangle = -|\downarrow\rangle$. If $\Omega \in S^2$, with polar angles $\theta_\Omega \in (0,\pi)$, $\phi_\Omega \in (-\pi,\pi)$, we define[1]

$$|\Omega\rangle_1 = \cos\frac{\theta_\Omega}{2}|\uparrow\rangle + e^{i\phi_\Omega}\sin\frac{\theta_\Omega}{2}|\downarrow\rangle. \tag{A.1.1}$$

Writing $\underline{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, it is easy to prove that

$$\Omega \cdot \underline{\sigma}|\Omega\rangle_1 = |\Omega\rangle_1. \tag{A.1.2}$$

If $N \in \mathbb{N}$, the associated N**-coherent spin state** $|\Psi_N^\Omega\rangle \in \mathrm{Sym}^N(\mathbb{C}^2)$, equipped with the usual scalar product $\langle\cdot,\cdot\rangle_N$ inherited from $(\mathbb{C}^2)^N$, is defined as follows [71] :

$$|\Psi_N^\Omega\rangle = \underbrace{|\Omega\rangle_1 \otimes \cdots \otimes |\Omega\rangle_1}_{N\ times}. \tag{A.1.3}$$

We occasionally also adopt the alternative notation $|\Omega_{\theta,\phi}\rangle_N$, which emphasizes the dependence of Ω on the angles (θ,ϕ), and we also use ordinary notation and simply write Ψ_N^Ω.

An explicit expression of Ψ_N^Ω can be presented through the so-called **Dicke basis** of $\mathrm{Sym}^N(\mathbb{C}^2)$, given by

$$\{|k, N-k\rangle \mid k = 0, 1 \ldots, N\}, \tag{A.1.4}$$

where $|k, N-k\rangle$ is the normalized vector obtained by symmetrization of a tensor product of N vectors in $\mathbb{C}^2$ whose k factors are of type $|\uparrow\rangle$ and the remaining $N-k$ factors are of type $|\downarrow\rangle$. A simple computation relying upon (A.1.1) and (2.2.43) yields

$$|\Omega_{\theta,\phi}\rangle_N = \sum_{k=0}^{N}\sqrt{\binom{N}{k}}\cos(\theta/2)^k \sin(\theta/2)^{N-k} e^{i(N-k)\phi}|k, N-k\rangle. \tag{A.1.5}$$

Coherent spin states form an *overcomplete set of vectors* for $\mathrm{Sym}^N(\mathbb{C}^2)$, in that

$$\langle\Xi,\Phi\rangle_N = \frac{N+1}{4\pi}\int_{S^2}\langle\Xi,\Psi_N^\Omega\rangle\langle\Psi_N^\Omega,\Phi\rangle d\Omega, \quad \text{for all } \Xi,\Phi \in \mathrm{Sym}^N(\mathbb{C}^2). \tag{A.1.6}$$

Here $d\Omega$ indicates the unique $SO(3)$-invariant Haar measure on S^2 with $\int_{S^2} d\Omega = 4\pi$, which, in turn, coincides with the measure generated by the metric induced to the embedded submanifold S^2 from $\mathbb{R}^3$. Another property relevant for our computations, which straightforwardly follows from (A.1.1) - (2.2.43), is

$$|\langle\Psi_N^\Omega, \Psi_N^{\Omega'}\rangle|^2 = \left(\frac{1+\cos\Phi(\Omega,\Omega')}{2}\right)^N, \tag{A.1.7}$$

where

$$\cos\Phi(\Omega_{\theta,\phi}, \Omega_{\theta',\phi'}) = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi') \tag{A.1.8}$$

[1] In the literature there are some inequivalent definitions of the overall non-constant phase affecting $|\Omega\rangle_1$ [71, 11], but all choices have the same important properties listed here.

is the cosine of the angle Φ between $\Omega_{\theta,\phi}$ and $\Omega_{\theta',\phi'}$.

2. Proofs of auxiliary results

B.1 Proofs Chapter 3

Proof of Lemma 3.3.5. The definition (3.2.32) of S_N implies

$$S_N(a_1 \otimes \cdots \otimes a_N) S_N(a'_1 \otimes \cdots \otimes a'_N) = \frac{1}{N!^2} \sum_{\sigma \in \mathcal{P}(N)} \sum_{\pi \in \mathcal{P}(N)} a_{\sigma(1)} a'_{\pi(1)} \otimes \cdots \otimes a_{\sigma(N)} a'_{\pi(N)}$$
$$= \frac{1}{N!^2} \sum_{\sigma} \sum_{\pi} a_{\sigma(1)} a'_{\sigma\circ\pi(1)} \otimes \cdots \otimes a_{\sigma(N)} a'_{\sigma\circ\pi(N)}, \tag{B.1.1}$$

since, for any given $\sigma \in \mathcal{P}(N)$, the map $\pi \mapsto \sigma \circ \pi$ is a bijection of the permutation group $\mathcal{P}(N)$. Exploiting the definition of S_N once again yields

$$\frac{1}{N!^2} \sum_{\sigma} \sum_{\pi} a_{\sigma(1)} a'_{\sigma\circ\pi(1)} \otimes \cdots \otimes a_{\sigma(N)} a'_{\sigma\circ\pi(N)} = \frac{1}{N!} \sum_{\pi \in \mathcal{P}(N)} S_N\left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right), \tag{B.1.2}$$

so that

$$S_N(a_1 \otimes \cdots \otimes a_N) S_N(a'_1 \otimes \cdots \otimes a'_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{P}(N)} S_N\left(a_1 a'_{\pi(1)} \otimes \cdots \otimes a_N a'_{\pi(N)}\right). \tag{B.1.3}$$

A similar arguments gives

$$S_N(a'_1 \otimes \cdots \otimes a'_N) S_N(a_1 \otimes \cdots \otimes a_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{P}(N)} S_N\left(a'_{\pi(1)} a_1 \otimes \cdots \otimes a'_{\pi(N)} a_N\right), \tag{B.1.4}$$

proving the claim. □

Proof of Equation (3.3.92). We have to compute the number of all possible bijective maps f_π (corresponding to permutations π^{-1} when $\pi \in \mathcal{P}(N)_K$) whose domain consists of the following N elements: L elements $\{b_{j_1}, \ldots, b_{j_L}\}$ together with $N-L$ identities I_k. All those elements are viewed as *distinct objects*. The codomain of f_π consists of N elements: M elements $\{b_{i_1}, \ldots, b_{i_M}\}$ together with $N-M$ identities I_k. Again, all those elements are viewed as distinct objects. We assume $L \leq M$ and the maps we want to count are those that map *exactly* K elements among those in $\{b_{j_1}, \ldots, b_{j_L}\}$ to distinct elements of the subset $\{b_{i_1}, \ldots, b_{i_M}\}$ of the codomain.

We start by choosing K couples whose first element is chosen from the set $\{b_{j_1}, \ldots, b_{j_L}\}$ and the corresponding second element (the image of the former according to f_π) is from the set $\{b_{i_1}, \ldots, b_{i_M}\}$. We can do this in

$$\frac{L(L-1)\cdots(L-K+1)M(M-1)\ldots(M-K+1)}{K!} \tag{B.1.5}$$

different ways, where the factor $1/K!$ is needed because the order we use to select the said K couples does not matter. This number can be rewritten as

$$\frac{L!}{(L-K)!} \frac{M!}{(M-K)!} \frac{1}{K!}. \tag{B.1.6}$$

We have now to assign the images via f_π of the remaining $L-K$ elements of the set $\{b_{j_1}, \ldots, b_{j_L}\}$ in the domain (having removed the K elements as above), which must be injectively mapped to the subset of the codomain consisting of $N-M$ unit elements I_k. Keeping the initial order of those $L-K$ elements, the image of the first one can be taken in $(N-M)$ ways, the image of the second one in $(N-M-1)$ ways, and so on. This leads to a number of

$$(N-M)(N-M-1)\cdots(N-M-(L-K)+1)=\frac{(N-M)!}{(N-L-M+K)!} \tag{B.1.7}$$

choices. The total number of choices is the product of (B.1.6) and (B.1.7). To conclude, we have to injectively assign the values of the remaining $N-L$ elements I_k of the domain of f_π into the set of remaining $N-L$ values of the codomain: this gives $(N-L)!$ choices. The total amount of choices is then identical to (3.3.92):

$$\frac{1}{K!}\frac{L!}{(L-K)!}\frac{M!}{(M-K)!}\frac{(N-L)!(N-M)!}{(N-L-M+K)!}. \tag{B.1.8}$$

□

B.2 Proofs Chapter 4

Lemma B.2.1. *Eq. (4.2.17) is true.*

Proof. As is well known (see [58] for a summary of those properties and technical references),

$$\Omega\cdot\sigma|\Omega\rangle_1=|\Omega\rangle_1\ ,$$

where $|\Omega\rangle_1$ stands for Ψ_1^Ω (viz. Section 2.2.2): Applying U to both sides gives

$$\Omega\cdot U\sigma U^*U|\Omega\rangle_1=U|\Omega\rangle_1.$$

Namely, from (4.2.13) we obtain

$$\Omega\cdot(R_U^{-1}\sigma)U|\Omega\rangle_1=U|\Omega\rangle_1,$$

that is

$$(R_U\Omega)\cdot\sigma\,Ur|\Omega\rangle_1=U|\Omega\rangle_1\ .$$

We also know that

$$(R_U\Omega)\cdot\sigma\,|R_U\Omega\rangle_1=|R_U\Omega\rangle_1\ .$$

Since the eigenspace of $(R^{-1}\Omega)\cdot\sigma$ with eigenvalue 1 is one-dimensional, for some real $\beta_{\Omega,U}$, we must have

$$U|\Omega\rangle_1=e^{i\beta_{\Omega,U}}|R_U\Omega\rangle_1\ .$$

Taking advantage of $|\Psi_N^\Omega\rangle=\underbrace{|\Omega\rangle_1\otimes\cdots\otimes|\Omega\rangle_1}_{N\ times}$ and $V=\underbrace{U\otimes\cdots\otimes U}_{N\ times}|_{\mathrm{Sym}^N(\mathbb{C}^2)}$, we immedialtey achieve (4.2.17) with $\alpha_{\Omega,U}=N\beta_{\Omega,U}$. □

Lemma B.2.2. $\lim_{N\to+\infty}C_N^{(j)}$ *exists and is finite.*

Proof. Since the left-hand side of (4.2.23) does not depend on N and the integral in the right-hand side tends to $p_m^{(j)}(\Omega')$, the only possibility that the limit $\lim_{N\to+\infty}C_N^{(j)}$ prevents from existing (or that makes it infinite) is $p_m^{(j)}(\Omega')=0$. This result should be true for all Ω', since $\lim_{N\to+\infty}C_N$ is independent of Ω'. However the polynomial $p_m^{(j)}$ (restricted to S^2) is not the zero function since it is an element of a basis. □

Proposition B.2.3. *Eq. (4.2.25) is true.*

Proof. We use the canonical (Dicke) basis introduced in Appendix A.1, given by the vectors $|n,N-n\rangle$ for $\mathrm{Sym}^N(\mathbb{C}^2)$ $(n=0,...,N)$, and first show that the matrix elements with respect to this basis are zero:

$$\langle n|Q_{1/N}(q(x,y,z))Q_{1/N}(x^2+y^2+z^2-1)|k\rangle = 0, \quad (k,n=0,...,N). \tag{B.2.9}$$

In order to do so, consider a basis vector $|k, N-k\rangle$. We expand $|k, N-k\rangle$ in the standard basis vectors β_i $(i = 1,...,2^N)$ spanning the Hilbert space $\bigotimes^N \mathbb{C}^2$. We denote by $\mathscr{O}^k$ the orbit consisting of $\binom{N}{k}$-basis vectors β_i with the same number of occurrence of the vectors e_2 and e_1, the two basis vectors of $\mathbb{C}^2$. By convention, we take e_1 such that $\sigma_3 e_1 = e_1$, and $\sigma_3 e_2 = -e_2$. It is not difficult to show that [97, 100]

$$|k, N-k\rangle = \frac{1}{\sqrt{\binom{N}{k}}}\sum_{l=1}^{\binom{N}{k}} \beta_{k,l}$$

where the subindex l in $\beta_{k,l}$ labels the basis vector $\beta_{k,l} \in \beta$ within the same orbit $\mathscr{O}^k$. Since we have $\binom{N}{k}$ such vectors per orbit, the sum in the above equation indeed is from $l = 1, ..., \binom{N}{k}$. By definition $Q_{1/N}(x_i^2) = S_{2,N}(\sigma_i \otimes \sigma_i)$ for $i = 1, 2, 3$. Using a combinatorial argument and the fact that all $|k\rangle$ are symmetric it follows that

$$S_{2,N}(\sigma_2 \otimes \sigma_2)|k\rangle = \frac{1}{\sqrt{\binom{N}{k}}}\sum_{l=1}^{\binom{N}{k}}(\sigma_2 \otimes \sigma_2 \otimes 1 \cdots \otimes 1)\beta_{k,l} =$$
$$\frac{1}{\sqrt{\binom{N}{k}}}\Big(-\binom{N-2}{k-2}\beta_{k-2,l} + 2\binom{N-2}{k-1}\beta_{k,l} - \binom{N-2}{k}\beta_{k+2,l}.$$

Similarly,

$$S_{2,N}(\sigma_1 \otimes \sigma_1)|k\rangle =$$
$$\frac{1}{\sqrt{\binom{N}{k}}}\Big(\binom{N-2}{k-2}\beta_{k-2,l} + 2\binom{N-2}{k-1}\beta_{k,l} + \binom{N-2}{k}\beta_{k+2,l};$$

and

$$S_{2,N}(\sigma_3 \otimes \sigma_3)|k\rangle =$$
$$\frac{1}{\sqrt{\binom{N}{k}}}\Big(\binom{N-2}{k-2}\beta_{k,l} - 2\binom{N-2}{k-1}\beta_{k,l} + \binom{N-2}{k}\beta_{k,l}.$$

In view of Definition 2.1.3 (property 3 (*iii*)) the cross-section $0 \to f$ and $1/N \to Q_{1/N}(f)$ defines a continuous cross-section of the bundle (cf. Theorem 3.3.4) implying that the following condition is automatically satisfied:[1]

$$\lim_{N\to\infty} ||Q_{1/N}(f)Q_{1/N}(f) - Q_{1/N}(fg)||_N = 0. \tag{B.2.10}$$

We apply this with $f = q(x, y, z)$ and $g(x, y, z) = x^2 + y^2 + z^2 - 1$. We first show that

$$\langle n|Q_{1/N}(q(x,y,z))Q_{1/N}(x^2+y^2+z^2-1))|k\rangle = 0,$$

[1]This is just the von Neumann condition (2.1.2).

for all basis vectors $|n\rangle$ and $|k\rangle$ in $\mathrm{Sym}^N(\mathbb{C}^2)$. Indeed, using the above identities one finds

$$\langle n|Q_{1/N}(q(x,y,z))Q_{1/N}(x^2+y^2+z^2-1)|k\rangle =$$
$$\frac{1}{\sqrt{\binom{N}{n}}}\frac{1}{\sqrt{\binom{N}{k}}}\sum_{l=1}^{\binom{N}{n}}\sum_{r=1}^{\binom{N}{k}}\langle\beta_{n,l}, Q_N(q(x,y,z))\Big(S_{2,N}(\sigma_1\otimes\sigma_1)+S_{2,N}(\sigma_2\otimes\sigma_2)+S_{2,N}(\sigma_3\otimes\sigma_3)\Big)\beta_{k,r}\rangle -$$
$$\frac{1}{\sqrt{\binom{N}{n}}}\frac{1}{\sqrt{\binom{N}{k}}}\sum_{l=1}^{\binom{N}{n}}\sum_{r=1}^{\binom{N}{k}}\langle\beta_{n,l}, Q_{1/N}(q(x,y,z))\beta_{k,r}\rangle =$$
$$\frac{1}{\sqrt{\binom{N}{n}}}\frac{1}{\sqrt{\binom{N}{k}}}\sum_{l=1}^{\binom{N}{n}}\langle\beta_{n,l}, Q_N(q(x,y,z))\left(\binom{N-2}{k-2}+\binom{N-2}{k}+2\binom{N-2}{k-1}-\binom{N}{k}\right)\beta_{k,r}\rangle$$
$$\frac{1}{\sqrt{\binom{N}{n}}}\frac{1}{\sqrt{\binom{N}{k}}}\sum_{l=1}^{\binom{N}{n}}\langle\beta_{n,l}, Q_N(q(x,y,z))\left(\binom{N}{k}-\binom{N}{k}\right)\beta_{k,r}\rangle = 0.$$

Since this holds for all basis vectors and $\mathrm{Sym}^N(\mathbb{C}^2)$ is invariant under $Q_{1/N}(q(x,y,z))$ and $Q_{1/N}(x^2+y^2+z^2-1)$, we conclude

$$\Big(Q_{1/N}(q(x,y,z))Q_{1/N}(x^2+y^2+z^2-1)\Big)|_{\mathrm{Sym}^N(\mathbb{C}^2)} = 0. \tag{B.2.11}$$

Therefore, for any symmetric unit vector $\phi\in\mathrm{Sym}^N(\mathbb{C}^2)$ we compute

$$||Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1))\phi||_N =$$
$$\left|\left|\Big(Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1)) - Q_{1/N}(q(x,y,z))Q_{1/N}(x^2+y^2+z^2-1)\Big)\phi\right|\right|_N$$
$$\leq ||Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1)) - Q_{1/N}(q(x,y,z))Q_{1/N}(x^2+y^2+z^2-1)||_N\,.$$

As a consequence of (B.2.10), for every $\epsilon>0$ there is N_ϵ such that

$$||Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1))\phi||_N < \epsilon \quad \text{if } N > N_\epsilon$$

the crucial observation is that due to (B.2.10) the number N_ϵ does not depend on the unit vector $\phi\in\mathrm{Sym}^N(\mathbb{C}^2)$. Therefore the above bound is uniform, and

$$||Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1))|_{\mathrm{Sym}^N(\mathbb{C}^2)}||_N$$
$$= \sup_{||\phi||=1\ ,\phi\in\mathrm{Sym}^N(\mathbb{C}^2)} ||Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1))\phi||_N \leq \epsilon \quad \text{if } N>N_\epsilon,$$

which means

$$\lim_{N\to\infty}||Q_{1/N}(q(x,y,z)(x^2+y^2+z^2-1))|_{\mathrm{Sym}^N(\mathbb{C}^2)}||_N = 0.$$

□

B.3 Proofs Chapter 8

Proof of Proposition 8.3.4.
From now on, S^2 is viewed as an embedded submanifold of $\mathbb{R}^3$ endowed with the differentiable structure, the metric and the associated measure (which coincides with $d\Omega$) induced by $\mathbb{R}^3$.

Proof of (a). Since the measure $d\Omega$ and $\cos\Phi(\Omega,\Omega')$ are both rotational invariant, we assume without loss of generality that Ω' coincides with $\mathbf{e}_z$ and we only demonstrate the claim for this choice. Writing $N' = N+1$, for $\ell > 0$ we have

$$I_N = \frac{\ell N'}{4\pi}\int_{S^2} h(\Omega)|\langle \Psi_N^{\Omega}, \Psi_N^{\Omega'}\rangle|^{2\ell} d\Omega = \frac{\ell N'}{2^{\ell N}4\pi}\int_{[0,\pi)\times(-\pi,\pi]} h(\theta,\phi)(1+\cos\theta)^{\ell N}\sin\theta d\theta d\phi. \quad \text{(B.3.12)}$$

Notice that the integral is well defined because $|\langle \Psi_N^{\Omega}, \Psi_N^{\Omega'}\rangle|^2$ is smooth and bounded by some constant when Ω ranges in S^2, h is L^1 with respect to $d\Omega$ because it is measurable and bounded, and S^2 has finite measure. The same argument applies to the integrals appearing in the rest of the proof. To go on, we decompose

$$h(\Omega) = h(\Omega') + h(\Omega) - h(\Omega') \quad \text{(B.3.13)}$$

so that

$$I_N = h(\Omega')\frac{\ell N'}{2^{\ell N}4\pi}\int_{S^2}(1+\cos\theta)^{\ell N}\sin\theta d\theta d\phi + \frac{\ell N'}{2^{\ell N}4\pi}\int_{S^2}[h(\Omega)-h(\Omega')](1+\cos\theta)^{\ell N}d\Omega. \quad \text{(B.3.14)}$$

A direct computation leads to

$$\frac{\ell N' h(\Omega')}{2^{\ell N}4\pi}\int_{S^2}(1+\cos\theta)^{\ell N}\sin\theta d\theta d\phi = h(\Omega')\frac{\ell(N+1)}{2^{\ell N+1}}\frac{2^{\ell N+1}}{\ell N+1} \to h(\Omega'), \quad \text{(B.3.15)}$$

as $N\to\infty$. To conclude the proof, we need to show that

$$\frac{\ell N'}{2^{\ell N}4\pi}\int_{S^2}[h(\Omega)-h(\Omega')](1+\cos\theta)^{\ell N}d\Omega \to 0 \quad \text{for } N\to\infty. \quad \text{(B.3.16)}$$

Actually, it is sufficient to establish that

$$\frac{\ell N'}{2^{\ell N}4\pi}\int_{A}|h(\Omega)-h(\Omega')||1+\cos\theta|^{\ell N}d\Omega \to 0 \quad \text{for } N\to\infty, \quad \text{(B.3.17)}$$

where $A\subset S^2$ is an open neighborhood of Ω', in particular the one appearing in the hypothesis where f is C^1. In fact, on $S^2\setminus A$ we have $\left|\frac{1+\cos\theta}{2}\right| \leq K < 1$ for some $K\in(0,1)$ so that $\ln K < 0$ and

$$\frac{\ell N'}{4\pi}\left|\frac{1+\cos\theta}{2}\right|^{\ell N} \leq \frac{\ell(N+1)}{4\pi}e^{\ell N\ln K} \to 0 \quad \text{for } N\to\infty. \quad \text{(B.3.18)}$$

Therefore,

$$\lim_{N\to\infty}\frac{\ell N'}{2^{\ell N}4\pi}\int_{S^2\setminus A}|h(\Omega)-h(\Omega')||1+\cos\theta|^{\ell N}d\Omega \leq 2\|h\|_\infty \ell N' e^{\ell N\ln K} = 0. \quad \text{(B.3.19)}$$

Restricting the initial *setA* if necessary, let us equip A with a local chart (of the differentiable structure induced from $\mathbb{R}^3$) obtained by the canonical projection onto the x,y plane (we use this chart because the chart of the coordinates θ,ϕ is singular at Ω', here coinciding with the north pole). It is not difficult to see that, in this coordinate patch where we can safely assume $\cos\theta > 0$, we have

$$\int_A |h(\Omega)-h(\Omega')|\cdot|1+\cos\theta|^{\ell N}d\Omega = \int_A [h(x,y)-h(0,0)]\frac{(1+\sqrt{1-x^2-y^2})^{\ell N}}{\sqrt{1-x^2-y^2}}dxdy \quad \text{(B.3.20)}$$

where we exploited the fact that the induced measure from $\mathbb{R}^3$ is $dxdy/\sqrt{1-x^2-y^2}$ in that coordinate patch. Assuming f of class C^1 in coordinates x,y on A, if necessary redefine again A as a smaller open neighborhood of $(0,0)$ whose closure (which is compact) is contained in the initial A. Lagrange's theorem applied to the segment joining (x,y) and $(0,0)$ then leads to the estimate

$$|h(x,y) - h(0,0)| = \left|\frac{\partial h}{\partial x}|_{(x',y')}x + \frac{\partial h}{\partial y}|_{(x',y')}y\right| \leq L_f^{(A)} r \tag{B.3.21}$$

where (x', y') is a point in A depending on (x, y), and

$$L_h^{(A)} = \sup_A \sqrt{\left|\frac{\partial h}{\partial x}\right|^2 + \left|\frac{\partial h}{\partial y}\right|^2} < \infty, \tag{B.3.22}$$

which exists because f is C^1 on the compact set $\overline{A}$, and where we adopted plane polar coordinates $x = r\cos\vartheta$, $y = r\sin\vartheta$ with $r = \sqrt{x^2+y^2}$. Collecting all results, using $z = \cos\theta = \sqrt{1-x^2-y^2} = \sqrt{1-r^2}$, we have

$$\frac{\ell N'}{2^N 4\pi}\int_A |h(\Omega) - h(\Omega')||1+\cos\theta|^{\ell N} d\Omega \leq \frac{L_h^{(A)}\ell N'}{2^{\ell N}4\pi}\int_{\{(r,\vartheta)\,|\,0\leq r\leq 1\}} \frac{(1+\sqrt{1-r^2})^{\ell N}}{\sqrt{1-r^2}} r^2 dr d\vartheta. \tag{B.3.23}$$

Integrating with respect to ϑ, (B.3.17) holds. This ends the proof of **(a)**, provided

$$J_N = \frac{\ell N'}{2^{\ell N+1}}\int_0^1 \frac{(1+\sqrt{1-r^2})^{\ell N}}{\sqrt{1-r^2}} r^2 dr \to 0 \quad \text{for } N \to \infty. \tag{B.3.24}$$

Changing variable to $x = \sqrt{1-r^2}$ and next to $t = \frac{1+x}{2}$, we find

$$\begin{aligned} J_N &= 2\ell N' \int_{1/2}^1 t^{\ell N+1/2}\sqrt{1-t}dt \leq 2\ell N' \int_0^1 t^{\ell N+1/2}\sqrt{1-t}dt \\ &= 2\ell(N+1)\frac{\Gamma(3/2)\Gamma(\ell N+3/2)}{\Gamma(\ell N+3)} \end{aligned} \tag{B.3.25}$$

Stirling's estimate then yields $|J_N| \leq L/\sqrt{\ell N}$ for some constant $L > 0$. With the previous discussion, this gives the key to (B.3.17) and hence to Assumption **(a)**, viz.

$$\left|\frac{\ell N'}{2^{\ell N}4\pi}\int_A [h(\Omega) - h(\Omega')](1+\cos\theta)^{\ell N} d\Omega\right| \leq L L_h^{(A)}/\sqrt{\ell N}. \tag{B.3.26}$$

Proof of (b). From (B.3.14), the identity in (B.3.15), (B.3.19), and (B.3.26) we have

$$\begin{aligned} &\left|h(\Omega') - \frac{\ell N'}{4\pi}\int_{S^2} h(\Omega)|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega'}\rangle|^{2\ell} d\Omega\right| \\ &\leq |h(\Omega')|\left|1 - \frac{\ell N+\ell}{\ell N+1}\right| + 2\|h\|_\infty \ell N' e^{\ell N \ln K} + \ell^{-1/2} L L_h^{(A)}/\sqrt{N} \\ &\leq \|h\|_\infty \frac{|1-\ell|}{\ell N+1} + \|h\|_\infty 2\ell N' e^{\ell N\ln K} + \ell^{-1/2} L L_h^{(A)}/\sqrt{N}, \end{aligned} \tag{B.3.27}$$

where $K \in (0,1)$ does not depend on h. With a standard argument one proves that, for some constant $C^{(A)} \geq 0$ independent of h, the constant $L_h^{(A)}$ in (B.3.22) satisfies

$$L_h^{(A)} \leq C^{(A)}\|dh\|_\infty^{(A)}, \tag{B.3.28}$$

where, if $\mathbf{g}_\Omega$ is the natural inner product on $T_\Omega^* S^2$ induced from $\mathbb{R}^3$,

$$\|dh\|_\infty = \sup_{\Omega\in A}\sqrt{\mathbf{g}_\Omega(d\overline{h}, dh)}. \tag{B.3.29}$$

Inequality (8.3.52) is therefore true defining $C_\ell^{(A)} = \ell^{-1/2} L C^{(A)}$, since

$$\|h\|_\infty\left(\frac{|1-\ell|}{\ell N+1} + 2\ell(N+1)e^{\ell N\ln K}\right) \leq B_\ell \|h\|_\infty/\sqrt{N}. \tag{B.3.30}$$

Recalling that $\ln K < 0$, we finally obtain

$$B_\ell = \sup_{N\in\mathbb{N}} \sqrt{N}\left(\frac{|1-\ell|}{\ell N+1} + 2\ell(N+1)e^{\ell N \ln K}\right) < \infty \tag{B.3.31}$$

Notice that, by construction B_ℓ and $C_\ell^{(A)}$ do not depend on Ω'. □

Proof of Lemma 8.3.9. We prove the claim for a real-valued h, the extension the the complex case being trivial. In the rest of the proof we always assume that A is sufficiently small according to Remark 8.3.5.(1), keeping the requirement $A \ni \Omega_\pm$. In particular, we suppose that $A = A_+ \cup A_-$ where A_+ and A_- are sufficiently small open neighborhoods of Ω_+ and Ω_- respectively.

We start the proof by observing that, taking advantage of a finite partition of unit, we can decompose $h = h_+ + h_-$ where $h_\pm$ are measurable, bounded and C^1 in A and satisfy $h_+ = 0$ in a neighborhood of Ω_-, and $h_- = 0$ in a neighborhood of Ω_+. If the claim is valid for each of these functions, by linearity it is also valid for h. Therefore, in the rest of the proof we assume that h also vanishes in a neighborhood of Ω_- in addition to satisfying the hypotheses in the statement of the lemma (the other case can be treated similarly).

As a second observation, we notice that (c) in Assumption 8.3.7 and Remark 8.3.8.(a), and the proof of Proposition 8.3.4 with (B.3.18), immediately imply that

$$\begin{aligned} &\frac{N+1}{4\pi}\int_{S^2\setminus A} |\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 d\Omega \to 0; \\ &\frac{N+1}{4\pi}\int_{S^2\setminus A} |\langle\Psi_N^{\Omega_\pm}, \Psi_N^{\Omega}\rangle|^2 d\Omega \to 0, \end{aligned} \tag{B.3.32}$$

respectively, for every open set A containing $\Omega_\pm$. In view of those remarks and using

$$\limsup_n (a_n + b_n) = \limsup_n a_n + \limsup_n b_n; \tag{B.3.33}$$

$$\liminf_n (a_n + b_n) = \liminf_n a_n + \liminf_n b_n, \tag{B.3.34}$$

if either $\{a_n\}_{n\in\mathbb{N}}$ or $\{b_n\}_{n\in\mathbb{N}}$ has a limit in $\mathbb{R}$, we can write

$$\begin{aligned} &\limsup_N \int_{S^2} \frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_+}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_-}\rangle|^2\right) h(\Omega) d\Omega \\ &= \limsup_N \int_{A} \frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_+}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_-}\rangle|^2\right) h(\Omega) d\Omega \\ &= \limsup_N \int_{A_+} \frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_+}\rangle|^2\right) h(\Omega) d\Omega, \end{aligned} \tag{B.3.35}$$

since the limit of the integration over $S^2 \setminus A$ is zero, and in the last line we exploited the fact that h vanishes around Ω_-. We can now decompose

$$\begin{aligned} &\int_{A_+} \frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_+}\rangle|^2\right) h(\Omega) d\Omega \\ &= h(\Omega_+)\int_{A_+} \frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_+}\rangle|^2\right) d\Omega \\ &+ \int_{A_+} \frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^2 - \tfrac{1}{2}|\langle\Psi_N^{\Omega}, \Psi_N^{\Omega_+}\rangle|^2\right) (h(\Omega) - h(\Omega_+)) d\Omega. \end{aligned} \tag{B.3.36}$$

Taking advantage of (C.1.11) and of the identity

$$|\langle\Omega_{\pi-\theta,-\phi}, \Omega_\pm\rangle_N| = |\langle\Omega_{\theta,\phi}, \Omega_\mp\rangle_N| \tag{B.3.37}$$

arising from (C.1.7), and choosing A_- as the image of A_+ under the symmetry

$$\theta \to \pi - \theta, \;\; \phi \to -\phi$$

that swaps Ω_+ and Ω_-, the first integral on the right-hand side can be rewritten as

$$\tfrac{1}{2}h(\Omega_+)\int_{A_+\cup A_-}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2\right)d\Omega. \tag{B.3.38}$$

Since $A_+\cup A_-=A$, the limit for $N\to\infty$ of the integral above vanishes because it is the difference of the limit of the analogous integral extended to the whole S^2, which vanishes due to the assumption (a), and the analogous limit when integrating over $S^2\setminus A$, which vanishes as well, as already observed. Hence

$$\begin{aligned}
&\limsup_N\int_{S^2}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2\right)h(\Omega)d\Omega\\
&=\limsup_N\int_{A_+}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle_N|^2\right)(h(\Omega)-h(\Omega_+))d\Omega\\
&\le\limsup_N\left|\int_{A_+}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2\right)(h(\Omega)-h(\Omega_+))d\Omega\right|\\
&\le\limsup_N\int_{A_+}\frac{N+1}{4\pi}\left||\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2\right||h(\Omega)-h(\Omega_+)|d\Omega\le G\epsilon,
\end{aligned} \tag{B.3.39}$$

where we exploited assumption (b) and the continuity of h at Ω_+, choosing the open set $A_+\ni\Omega_+$ such that $|h(\Omega)-h(\Omega_+)|<\epsilon$ is guaranteed if $\Omega\in A_+$. In summary,

$$\overline{I}=\limsup_N\int_{S^2}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2\right)h(\Omega)d\Omega\le G\epsilon. \tag{B.3.40}$$

This entire reasoning can be repeated changing the sign in the integrand from scratch, i.e., referring to

$$\limsup_N\int_{S^2}\frac{N+1}{4\pi}\left(\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2+\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2-|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2\right)h(\Omega)d\Omega, \tag{B.3.41}$$

finding

$$\limsup_N\int_{S^2}-\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Omega\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2\right)h(\Omega)d\Omega\le G\epsilon. \tag{B.3.42}$$

Since $\limsup_n(-a_n)=-\liminf_n a_n$, we conclude that

$$-G\epsilon\le\underline{I}=\liminf_N\int_{S^2}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2\right)h(\Omega)d\Omega. \tag{B.3.43}$$

In summary,

$$-G\epsilon\le\underline{I}\le\overline{I}\le G\epsilon\quad\text{for every }\epsilon>0 \tag{B.3.44}$$

and where $G\ge0$ is given. Therefore,

$$\lim_{N\to\infty}\int_{S^2}\frac{N+1}{4\pi}\left(|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_+}\rangle|^2-\tfrac{1}{2}|\langle\Psi_N^{\Omega},\Psi_N^{\Omega_-}\rangle|^2\right)h(\Omega)d\Omega=0. \tag{B.3.45}$$

Using Proposition 8.3.4, we conclude that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{(N+1)}{4\pi}\int_{S^2}h(\Omega)|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle|^2d\Omega\\
&=\lim_{N\to\infty}\frac{(N+1)}{8\pi}\int_{S^2}h(\Omega)|\langle\Psi_N^{\Omega_+},\Psi_N^{\Omega}\rangle|^2d\Omega+\lim_{N\to\infty}\frac{(N+1)}{8\pi}\int_{S^2}h(\Omega)|\langle\Psi_N^{\Omega_-},\Psi_N^{\Omega}\rangle|^2d\Omega\\
&=\tfrac{1}{2}h(\Omega_+)+\tfrac{1}{2}h(\Omega_-),
\end{aligned} \tag{B.3.46}$$

ending the proof. □

Proof of Lemma 8.3.11. First of all, notice that the absolute value in the left-hand side of (8.3.67) can be rearranged into a more useful form:

$$\begin{aligned}&\int_{S^2}\frac{N+1}{4\pi}\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N h(\Omega)\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}d\Omega-\langle\Psi_N^{(0)},\Psi_N^{\Omega'}\rangle_N h(\Omega')\\&=\frac{N+1}{4\pi}\int_{S^2}d\Omega\left(\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}h(\Omega)-\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N\langle\Psi_N^{\Omega},\Psi_N^{\Omega'}\rangle_N h(\Omega')\right),\end{aligned}\tag{B.3.47}$$

where we exploited (A.1.6) in the second summand of the first line. We intend to prove the claim with this rearranged form. Let us start by establishing the claim in the simplest case $f=1$, defining

$$I_N=\frac{N'}{4\pi}\int_{S^2}d\Omega\left(\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}-\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N\langle\Psi_N^{\Omega},\Psi_N^{\Omega'}\rangle_N\right),\tag{B.3.48}$$

where $N'=N+1$. The Cauchy-Schwartz' inequality implies

$$|I_N|\leq\sqrt{\frac{N'}{4\pi}\int|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N|^2d\Omega}\sqrt{\frac{N'}{4\pi}\int|\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}-\langle\Psi_N^{\Omega},\Psi_N^{\Omega'}\rangle_N|^2d\Omega}.\tag{B.3.49}$$

Here, eq. (A.1.5) gives rise to

$$\langle\Psi_N^{\Omega},\Psi_N^{\Omega'}\rangle_L=\Big(\cos(\theta/2)\cos(\theta'/2)+e^{i(\phi-\phi')}\sin(\theta/2)\sin(\theta'/2)\Big)^L=\langle\Psi_1^{\Omega},\Psi_1^{\Omega'}\rangle_1^L,\tag{B.3.50}$$

so that

$$\begin{aligned}|\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}-\langle\Psi_N^{\Omega},\Psi_N^{\Omega'}\rangle_N|^2&=|\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}|^2|1-\langle\Psi_M^{\Omega},\Psi_M^{\Omega'}\rangle_M|^2\\&=|\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}|^2|1-\langle\Psi_1^{\Omega},\Psi_1^{\Omega'}\rangle_1^M|^2.\end{aligned}\tag{B.3.51}$$

Inserting this result in (B.3.49), we find

$$\begin{aligned}|I_N|\leq&\sqrt{\frac{N'}{N'-M}}\sqrt{\frac{N'}{4\pi}\int|\langle\Psi_N^{(0)},\Psi_N^{\Omega}\rangle_N|^2d\Omega}\\&\times\sqrt{\frac{N'-M}{4\pi}\int|\langle\Psi_{N-M}^{\Omega},\Psi_{N-M}^{\Omega'}\rangle_{N-M}|^2|1-\langle\Psi_1^{\Omega},\Psi_1^{\Omega'}\rangle_1^M|^2d\Omega}.\end{aligned}\tag{B.3.52}$$

From a direct computation, we see that the map $S^2\times S^2\ni(\Omega,\Omega')\mapsto\langle\Psi_N^{\Omega},\Psi_N^{\Omega'}\rangle_1$ is nothing but the restriction to the unit sphere S^2 of the map

$$\mathbb{R}^3\times\mathbb{R}^3\ni(x,y,z,x',y',z')\mapsto\frac{(1+z+z'+zz'+xx'+yy'+ixy'+ix'y)}{2\sqrt{(1+z)(1+z')}},\tag{B.3.53}$$

where (x,y,z) and (x',y',z') are the Cartesian coordinates of Ω and Ω' respectively. From that, it is straightforward to establish that, for $\Omega'\neq S$, the function

$$S^2\ni\Omega\mapsto h_{\Omega'}(\Omega)=|1-\langle\Psi_M^{\Omega},\Psi_M^{\Omega'}\rangle_M|^2=1+|\langle\Psi_1^{\Omega},\Psi_1^{\Omega'}\rangle_1|^{2M}-2Re\langle\Psi_1^{\Omega},\Psi_1^{\Omega'}\rangle_1^M\tag{B.3.54}$$

vanishes for $\Omega=\Omega'$, and is measurable and bounded. Referring to the atlas on S^2 consisting of the 6 local charts given by the canonical projections onto the 3 coordinate 2-planes, it is finally obvious that $h_{\Omega'}$ is everywhere smooth with respect to the differentiable structure induced from $\mathbb{R}^3$, except for $\Omega=S$ (where $z=-1$). We may therefore apply (8.3.52) to the special case $h(\Omega)=h_{\Omega'}(\Omega)$ – which satisfies $h(\Omega')=0$ – in (B.3.52). Exploiting also (8.3.66) with $g=1$ to handle the large-N behavior of the first integral on the right-hand side of (B.3.52), which is bounded by some constant $H \geq 0$ when N increases, we conclude that, if $N > M$,

$$|I_N| \leq \frac{K^{(A)}}{(N-M)^{1/4}}, \tag{B.3.55}$$

for the constant

$$K^{(A)} = H\sqrt{C \sup_{\Omega,\Omega' \in A} |h_{\Omega'}(\Omega)| + D^{(A)} \sup_{\Omega,\Omega' \in A} \sqrt{\mathbf{g}_\Omega(d_\Omega h_{\Omega'}(\Omega), d_\Omega h_{\Omega'}(\Omega))}}. \tag{B.3.56}$$

Notice that with these definitions, C and $D^{(A)}$ do not depend on the choice of the function used here (viz. $h_{\Omega'}$), whereas $D^{(A)}$ only depends on A, which is the same for all possible choices of $\Omega' \in A$. Hence, no dependence on Ω' takes place.

Let us now turn attention to the general case where now h is a generic bounded measurable function that is $C^1(A)$, defining

$$J_N = \frac{N'}{4\pi}\int_{S^2} d\Omega \left(\langle \Psi_N^{(0)}, \Omega\rangle_N \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'}\rangle_{N-M} h(\Omega) - \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle_N \langle \Psi_N^{\Omega}, \Psi_N^{\Omega'}\rangle_N h(\Omega')\right). \tag{B.3.57}$$

Inserting a vanishing term

$$0 = \frac{N'}{4\pi}\int_{S^2} d\Omega \left(\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle_N \langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'}\rangle_{N-M} h(\Omega') - \langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle_N \langle \Psi_N^{\Omega}, \Psi_N^{\Omega'}\rangle_N h(\Omega')\right) \tag{B.3.58}$$

between the two summands on the right-hand side, the triangle inequality, the fact that $h(\Omega')$ is constant with respect to Ω, and the definition of I_N yield

$$|J_N| \leq \|h\|_\infty |I_N| + \frac{N'}{4\pi}\int_{S^2} d\Omega |\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle_N| \cdot |\langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'}\rangle_{N-M}||h(\Omega) - h(\Omega')|. \tag{B.3.59}$$

Applying the Cauchy-Schwartz inequality, we end up with

$$|J_N| \leq \|h\|_\infty |I_N| + \sqrt{\frac{N'}{4\pi}\int_{S^2} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle_N|^2 d\Omega}\sqrt{\frac{N'}{4\pi}\int_{S^2} |\langle \Psi_{N-M}^{\Omega}, \Psi_{N-M}^{\Omega'}\rangle_{N-M}|^2 |h(\Omega) - h(\Omega')|^2 d\Omega}. \tag{B.3.60}$$

As in the previous case, in particular taking advantage of (8.3.52) to estimate the last integral and using (B.3.55) and noticing that $1/(N-M) > 1/N$, we end up with

$$|J_N| \leq \frac{K^{(A)}\|h\|_\infty + \sqrt{C\|h\|_\infty^2 + D^{(A)}\|dF\|_\infty^{(A)}}}{(N-M)^{1/4}}, \tag{B.3.61}$$

for some constants $K^{(A)}, C, D^{(A)} \geq 0$, generally depending on M, but independent of Ω', h and F, where $F(\Omega) = |h(\Omega) - h(\Omega')|^2$ that is C^1 where h is C^1. □

3. Dicke components and numerical evidence

This part of the appendix provides numerical evidence for equations (8.3.61) - (8.3.63) and other useful properties are discussed in detail. We start with some results on the ground state eigenvector of the quantum Curie-Weiss model.

C.1 Dicke components of the ground state eigenvector $\Psi_N^{(0)}$ of $H_{1/N}^{CW}$

For any $N \in \mathbb{N}$, the ground state eigenvector $\Psi_N^{(0)}$ lives in the symmetric subspace $\mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes_{n=1}^N \mathbb{C}^2$. This non-obvious fact arises from the uniqueness of the CW-ground state vector (up to phases and normalization), which is ultimately a consequence of the *Perron–Frobenius Theorem*, and the fact that $H_{1/N}^{CW}$ is invariant under the natural action of permutation group of N elements [49], [97, §5.3], [100].

In order to do computations with $\Psi_N^{(0)}$, it therefore suffices to represent this vector in an $(N+1)$-dimensional basis for $\mathrm{Sym}^N(\mathbb{C}^2)$. This is a big numerical advantage: diagonalizing a $(N+1)$-dimensional matrix is much more efficient for a computer rather than diagonalizing a 2^N-dimensional matrix. The Dicke basis (A.1.4) we already introduced for this subspace therefore allows the expansion

$$\Psi_N^{(0)} = \sum_{k=0}^{N} c_N(k)|k, N-k\rangle, \tag{C.1.1}$$

where the coefficients $c_N(k)$ depend on N and, again from the Perron–Frobenius Theorem, the usual arbitrary phase affecting $\Psi_N^{(0)}$ can be chosen in order that

$$c_N(k) > 0, \quad k = 0, 1, \dots, N. \tag{C.1.2}$$

Both analytic asymptotics [49] and numerical computations [97] of the coefficients $c_N(k)$ are known, but no analytic expression has been found so far. To compute the expression $|\langle\Psi_N^{(0)}, \Omega_{\theta,\phi}\rangle|^{2l}$ popping up in equations (8.3.61) (for $l = 1/2$) and (8.3.63) (for $l = 1$) we use eqs. (C.1.1) and (A.1.5) for $\Psi_N^{(0)}$ and $|\Omega_{\theta,\phi}\rangle_N$ in terms of the Dicke basis. This way, the relevant inner product will be computed again in terms of the $N+1$ numerically favorable Dicke states, instead of 2^N basis vectors for $\bigotimes_{n=1}^N \mathbb{C}^2$.

Let us first focus on the $\mathbb{Z}_2$-action $\zeta^{1/N}$ on $M_2(\mathbb{C})^N$, already introduced in Section 8.3.2. This automorphism is unitarily implemented by the following unitary operator:

$$V_N = \underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{N\ times} \in M_2(\mathbb{C})^N, \tag{C.1.3}$$

$$\zeta^{1/N}(a) = V_N a V_N^{-1}, \tag{C.1.4}$$

where $a \in M_2(\mathbb{C})^N$. Since $V_1 = \sigma_1$, which swaps $|\uparrow\rangle$ and $|\downarrow\rangle$, we clearly have

$$V_N|k, N-k\rangle = |N-k, k\rangle. \tag{C.1.5}$$

Passing to the coherent spin state basis, this gives

$$V_N|\Omega_{\theta,\phi}\rangle_N = e^{-iN\phi}|\Omega_{\pi-\theta,-\phi}\rangle_N; \tag{C.1.6}$$
$$V_N|\Omega_{\pm}\rangle_N = |\Omega_{\mp}\rangle_N. \tag{C.1.7}$$

As we already saw, the (algebraic) CW-ground state $\omega_{1/N}^{(0)}$ defined by (8.3.46)) is invariant under the automorphism (C.1.4). The unit vector $\Psi_N^{(0)}$ of $\omega_{1/N}^{(0)}$ must therefore satisfy

$$V_N\Psi_N^{(0)} = \pm\Psi_N^{(0)}, \tag{C.1.8}$$

since $V_N^2 = I$. By (C.1.5), for the components (C.1.1), eq. (C.1.8) can be rephrased as

$$c_N(N-k) = \pm c_N(k), \tag{C.1.9}$$

where the sign does not depend on k. However, because $c_N(k) > 0$ only the + sign can actually occur. Thus the $\mathbb{Z}_2$-invariance of the ground state is equivalent to

$$c_N(k) = c_N(N-k), \quad k = 0, 1, \ldots, N, \tag{C.1.10}$$

and from (C.1.6) we also have

$$|\langle\Psi_N^{(0)}, \Omega_{\theta,\phi}\rangle|^2 = |\langle\Psi_N^{(0)}, \Omega_{\pi-\theta,-\phi}\rangle|^2. \tag{C.1.11}$$

C.2 Coefficients $c_N(k)$ for $N \geq 80$

We computed the components $c_N(k)$ of $\Psi_N^{(0)}$ using **Matlab**. However, from $N = 80$ onwards our program was not able to numerically distinguish anymore between the lowest eigenvalue $\epsilon_0^{(N)}$ of $H_{1/N}^{CW}$ and its first excited level $\epsilon_1^{(N)} > \epsilon_0^{(N)}$ in $\mathrm{Sym}^N(\mathbb{C}^2)$. As a consequence, within this numerical approximation, the $\epsilon_0^{(N)}$-eigenspace of $h_{1/N}^{CW}$ appears as a two-dimensional subspace $K^{(N)}$ of $\mathrm{Sym}^N(\mathbb{C}^2)$ and one needs to extract the actual ground state from the span of the pair of apparent degenerate eigenvectors $\Psi_N^{(0)\mathrm{matlab}}$ and $\Psi_N^{(1)\mathrm{matlab}}$ of $h_{1/N}^{CW}$ with the common eigenvalue $\epsilon_0^{(N)}$ computed by **Matlab**, which form an orthonormal basis of $K^{(N)}$. This can indeed be done, because $K^{(N)}$ is invariant under the unitary representation V_N (C.1.5) of the element -1 of $\mathbb{Z}_2$, which turns out to be *non-trivial* when restricted to that subspace. Hence

$$V_N|_{K^{(N)}} \neq I, \tag{C.2.12}$$

and since $V_N|_{K^{(N)}}$ is simultaneously unitary and self-adjoint, its spectrum consists only of two points ± 1. In other words, $K^{(N)}$ contains *exactly one* (up to phases) unit vector $\Phi^{(N)}$ such that $V_N\Phi^{(N)} = \Phi^{(N)}$. Since the true ground state of $h_{1/N}^{CW}$ satisfies the same condition and belongs to the same (approximate) subspace, we must have

$$\Psi_N^{(0)} = \Phi^{(N)}. \tag{C.2.13}$$

Therefore,[1] *$\Psi_N^{(0)}$ is the unique unit eigenvector of V_N with eigenvalue* 1. **Matlab** proposes a pair of orthonormal vectors $\Psi_N^{(0)\mathrm{matlab}}$ and $\Psi_N^{(1)\mathrm{matlab}}$, forming an orthonormal basis of $K^{(N)}$ which can be assumed to be of the form represented in the following picture, up to a change of the overall sign and the action of V_N (which simply reflects the function around the vertical axis localized at $N/2$).

If r_N denotes the ratio $r_N = H_L^{(N)}/H_R^{(N)}$, where $H_R^{(N)} \geq H_L^{(N)}$ is the height of the peak in the left part of the figure representing $\Psi_N^{(0)\mathrm{matlab}}$, and $H_L^{(N)}$ is defined analogously for the peak in the right part, it is not difficult to prove that the unique (up to phases) unit eigenvector $\Psi_N^{(0)}$

[1]Of course, with phases chosen such that the Perron–Frobenius condition (C.1.2) holds.

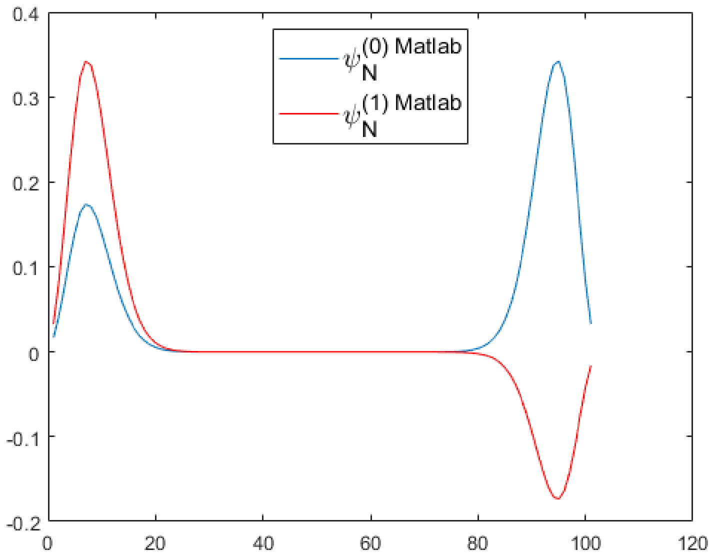

Figure C.1: Plot of $\Psi_N^{(0)\mathrm{matlab}}$ (in blue) and $\Psi_N^{(1)\mathrm{matlab}}$ (in red) for $N = 100, J = 1, B = 1/2$.

of U_N with eigenvalue 1 takes the form

$$\Psi_N^{(0)} = \frac{1}{\sqrt{2}} \left(\frac{1 + r_N}{\sqrt{1 + r_N^2}} \Psi_N^{(0)\mathrm{matlab}} + \frac{1 - r_N}{\sqrt{1 + r_N^2}} \Psi_N^{(1)\mathrm{matlab}} \right). \tag{C.2.14}$$

That is the desired ground state for $N \geq 80$. Notice that, with $\Psi_N^{(0)\mathrm{matlab}}$ and $\Psi_N^{(1)\mathrm{matlab}}$ as computed by **Matlab**, the components $c_N(k)$ of $\Psi_N^{(0)}$ also satisfy $c_N(k) \geq 0$ (instead of $c_N(k) > 0$ valid in the non-degenerate case).

Remark C.2.1. When $N < 80$, within our available computational precision **Matlab** *is* able to distinguish $\epsilon_0^{(N)}$ from $\epsilon_1^{(N)}$ and the computed vector $\Psi_N^{(0)\mathrm{matlab}}$ is such that $r_N = 1$. Therefore, as expected, (C.2.14) furnishes the actual ground state

$$\Psi_N^{(0)} = \Psi_N^{(0)\mathrm{matlab}}. \tag{C.2.15}$$

In the opposite direction, for $N > 150$ we obtain $r_N = 0$, so that (C.2.14) reduces to

$$\Psi_N^{(0)} = \frac{1}{\sqrt{2}} \left(\Psi_N^{(0)\mathrm{matlab}} + \Psi_N^{(1)\mathrm{matlab}} \right). \tag{C.2.16}$$

■

C.3 Numerical evidence for (a),(b),(c) in Assumption 8.3.7

We computed the integrals in (8.3.61) and (8.3.62) for increasing values of N: see Table 1 and Table 2 below, respectively.

Table 1. Numerical values of the left-hand side on (8.3.61) for increasing N.

N	Value of (8.3.61).
10	0.0060
20	$4.0922 \cdot 10^{-4}$
30	$3.8941 \cdot 10^{-5}$
60	$-1.4394 \cdot 10^{-5}$
90	$-2.7404 \cdot 10^{-6}$
120	$-4.2139 \cdot 10^{-7}$
150	$-6.0988 \cdot 10^{-8}$
180	$-8.6073 \cdot 10^{-9}$

Table 2. Numerical values of the left-hand side on (8.3.62) for increasing N.

N	Value of (8.3.62) for $l = 1$	Value of (8.3.62) for $l = 1/2$
10	0.2559	0.4185
20	0.1065	0.2095
30	0.0868	0.1860
40	0.0765	0.1731
50	0.0707	0.1649
60	0.0666	0.1590
70	0.0636	0.1547
80	0.0614	0.1514
90	0.0596	0.1488
100	0.0582	0.1469
110	0.0570	0.1452
120	0.0561	0.1439
130	0.0552	0.1427
140	0.0546	0.1418
150	0.0540	0.1409

From this table, it is clear that for $l = 1$ as well as $l = 1/2$, eq. (8.3.62) is decreasing in N, and therefore uniformly bounded in N. In fact, from this table it appears that

$$A_N = \int_{S^2} \frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2l} d\Omega \approx$$
$$B_N = \int_{S^2} \left(\frac{N+1}{4\sqrt{2}\pi} |\langle \Psi_N^{\Omega}, \Psi_N^{\Omega_+} \rangle|^{2l} + \frac{N+1}{4\sqrt{2}\pi} |\langle \Psi_N^{\Omega}, \Psi_N^{\Omega_-} \rangle|^{2l} \right) d\Omega, \tag{C.3.17}$$

as N becomes large. To be even more precise, we numerically computed the values of A_N and B_N for increasing values of N (see Table 3).

Table 3. A_N and B_N (as defined above) from (8.3.62) for increasing N.

N	A_N for $l = 1/2$	B_N for $l = 1/2$	A_N for $l = 1$	B_N for $l = 1$
10	2.3357	2.5471	0.9831	0.9772
20	2.6489	2.6846	0.9950	0.9946
30	2.7285	2.7330	0.9983	0.9982
40	2.7598	2.7574	0.9993	0.9993
50	2.7759	2.7719	0.9997	0.9997
60	2.7858	2.7816	0.9999	0.9999
70	2.7926	2.7884	0.9999	0.9999
80	2.7977	2.7935	1.0000	1.0000
90	2.8015	2.7974	1.0000	1.0000
100	2.8046	2.8005	1.0000	1.0000
110	2.8071	2.8031	1.0000	1.0000
120	2.8092	2.8052	1.0000	1.0000
130	2.8109	2.8070	1.0000	1.0000
140	2.8124	2.8085	1.0000	1.0000

This clearly suggests that for $l = 1/2$ both integrals converge to $2\sqrt{2} \approx 2.828$. Therefore, since the integral in (8.3.62) is bounded by $A + B$, there is strong numerical evidence that (8.3.62) is valid for some constant G, for example given by the sum of A_N and B_N, i.e., $G = 4\sqrt{2}$. A similar result holds for the case $l = 1$.

Furthermore, the validity of part (c) in Assumption 8.3.7 has been checked by comparing the graphs of the function

$$S^2 \ni \Omega \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Psi_N^{\Omega}\rangle|^{2\ell} \tag{C.3.18}$$

with the graphs of the function

$$S^2 \ni \Omega \mapsto \frac{N+1}{4\pi 2^{\ell}}|\langle \Psi_N^{\Omega_+}, \Psi_N^{\Omega}\rangle|^{2\ell} + \frac{N+1}{4\pi 2^{\ell}}|\langle \Psi_N^{\Omega_-}, \Psi_N^{\Omega}\rangle|^{2\ell}, \tag{C.3.19}$$

since the latter satisfies (c) ((b) Remark 8.3.8) and the graph of the former becomes more and more indistinguishable from the graph of the latter as N increases. We display various plots of the graphs of both functions for two typical different values of N. In order to make a clear comparison we avoid single $3d$ plots, but instead plot two $2d$ plots, one as a function of θ for fixed $\phi = 0$, and the other as a function of ϕ for fixed $\theta = \pi/6$. [2] These pairs of $2d$ plots (for $l = 1/2$ and $l = 1$) are depicted in the next pages for $N = 30$ and $N = 250$, and as always $J = 1, B = 1/2$.

[2]Note that due to symmetry we also could have chosen the point $\theta = 5\pi/6$. We indeed checked this numerically, but omitted the plots.

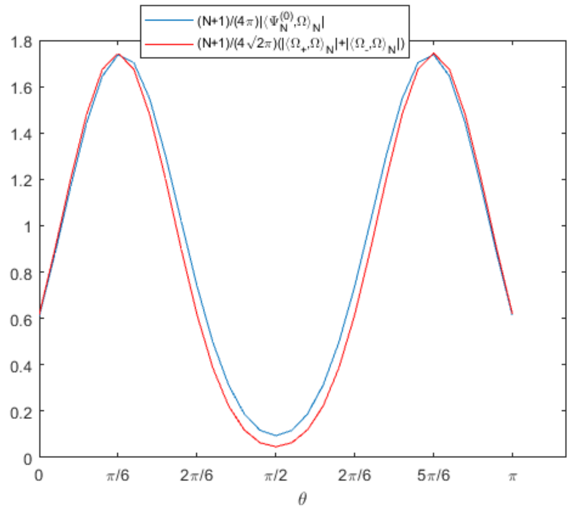

Figure 4: Plot for $N = 30$ of the functions, in blue and in red, respectively,

$$\theta \mapsto \frac{N+1}{4\pi}|\langle\Psi_N^{(0)}, \Omega_{\theta,0}\rangle|;$$
$$\theta \mapsto \frac{N+1}{4\sqrt{2}\pi}|\langle\Omega_+, \Omega_{\theta,0}\rangle| + \frac{N+1}{4\sqrt{2}\pi}|\langle\Omega_-, \Omega_{\theta,0}\rangle|$$

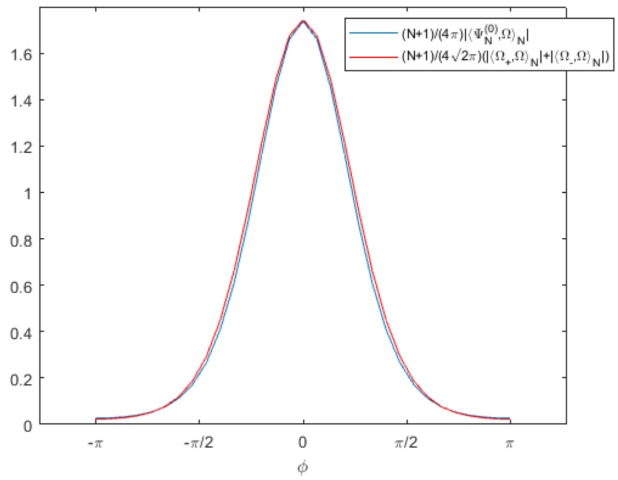

Figure 5: Plot for $N = 30$ of the functions, in blue and in red, respectively,

$$\phi \mapsto \frac{N+1}{4\pi}|\langle\Psi_N^{(0)}, \Omega_{\pi/6,\phi}\rangle|;$$
$$\phi \mapsto \frac{N+1}{4\sqrt{2}\pi}|\langle\Omega_+, \Omega_{\pi/6,\phi}\rangle| + \frac{N+1}{4\sqrt{2}\pi}|\langle\Omega_-, \Omega_{\pi/6,\phi}\rangle|$$

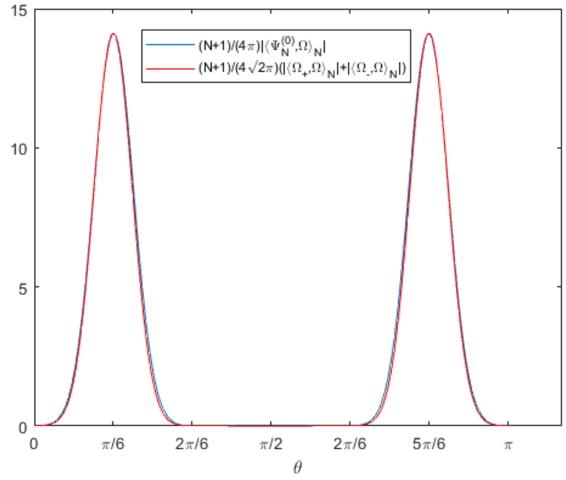

Figure 6: Plot for $N = 250$ of the functions, in blue and in red, respectively,

$$\theta \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\theta,0}\rangle|;$$
$$\theta \mapsto \frac{N+1}{4\sqrt{2}\pi}|\langle \Omega_+, \Omega_{\theta,0}\rangle| + \frac{N+1}{4\sqrt{2}\pi}|\langle \Omega_-, \Omega_{\theta,0}\rangle|$$

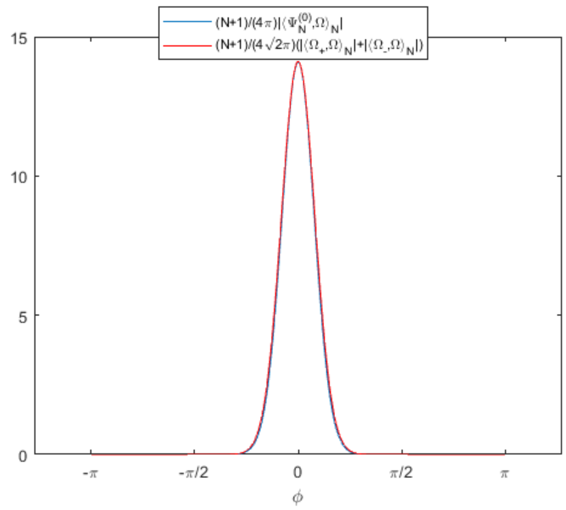

Figure 7: Plot for $N = 250$ of the functions, in blue and in red, respectively,

$$\phi \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\pi/6,\phi}\rangle|;$$
$$\phi \mapsto \frac{N+1}{4\sqrt{2}\pi}|\langle \Omega_+, \Omega_{\pi/6,\phi}\rangle| + \frac{N+1}{4\sqrt{2}\pi}|\langle \Omega_-, \Omega_{\pi/6,\phi}\rangle|$$

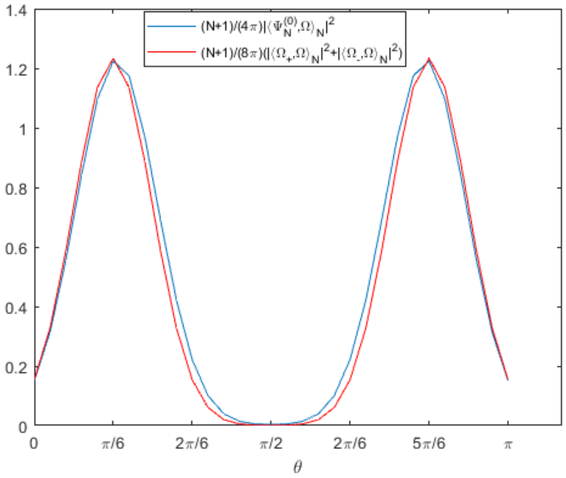

Figure 8: Plot for $N = 30$ of the functions, in blue and in red, respectively,

$$\theta \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\theta,0}\rangle|^2;$$
$$\theta \mapsto \frac{N+1}{8\pi}|\langle \Omega_+, \Omega_{\theta,0}\rangle|^2 + \frac{N+1}{8\pi}|\langle \Omega_-, \Omega_{\theta,0}\rangle|^2$$

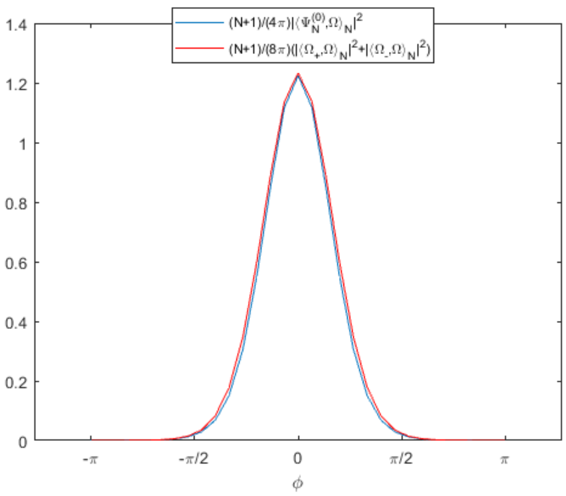

Figure 9: Plot for $N = 30$ of the functions, in blue and in red, respectively,

$$\phi \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\pi/6,\phi}\rangle|^2;$$
$$\phi \mapsto \frac{N+1}{8\pi}|\langle \Omega_+, \Omega_{\pi/6,\phi}\rangle|^2 + \frac{N+1}{8\pi}|\langle \Omega_-, \Omega_{\pi/6,\phi}\rangle|^2$$

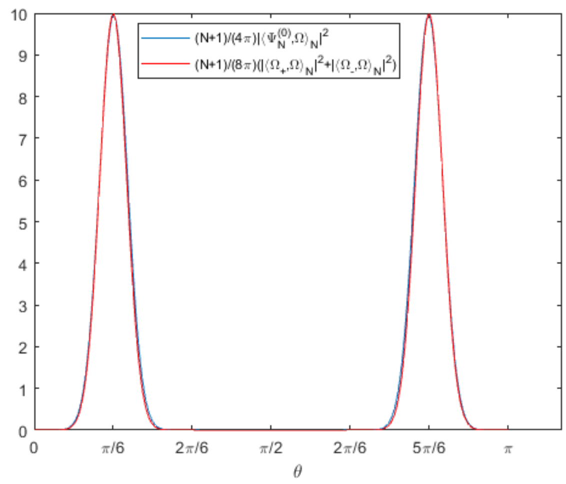

Figure 10: Plot for $N = 250$ of the functions, in blue and in red, respectively,

$$\theta \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\theta,0}\rangle|^2;$$
$$\theta \mapsto \frac{N+1}{8\pi}|\langle \Omega_+, \Omega_{\theta,0}\rangle|^2 + \frac{N+1}{8\pi}|\langle \Omega_-, \Omega_{\theta,0}\rangle|^2$$

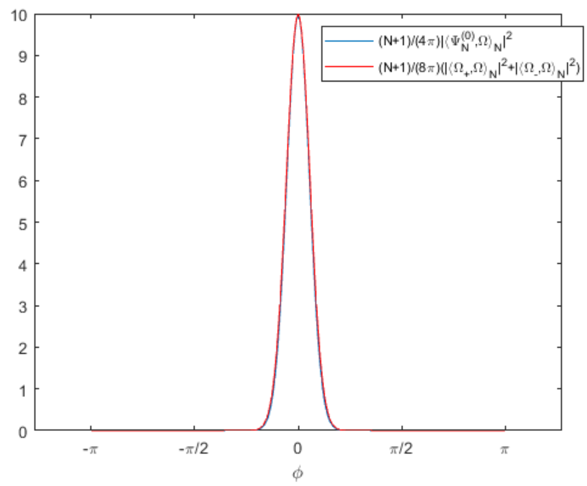

Figure 11: Plot for $N = 250$ of the functions, in blue and in red, respectively,

$$\phi \mapsto \frac{N+1}{4\pi}|\langle \Psi_N^{(0)}, \Omega_{\pi/6,\phi}\rangle|^2;$$
$$\phi \mapsto \frac{N+1}{8\pi}|\langle \Omega_+, \Omega_{\pi/6,\phi}\rangle|^2 + \frac{N+1}{8\pi}|\langle \Omega_-, \Omega_{\pi/6,\phi}\rangle|^2$$

Finally, we give another numerical fact corroborating Assumption 8.3.7, namely that the full width at half maximum (fwhm) of the function

$$N \mapsto \frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2\ell} \tag{C.3.20}$$

vanishes as $N \to \infty$, so that the function $\Omega \mapsto \frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2\ell}$ indeed behaves like a sum of two delta distributions in the regime $N \to \infty$.

To this end, we discretized θ and ϕ uniformly in N points on $(0, \pi/6)$ and $(-\pi, \pi)$ respectively, so that (C.3.20) becomes a $2d$ array of N^2 points. We then computed the number of points $a(N, \pi/6)$ at half height of the array $\frac{N+1}{4\pi} |\langle \Psi_N^{(0)}, \Psi_N^{\Omega} \rangle|^{2\ell}$ at fixed $\pi/6$, but varying the discrete values of ϕ. Then we repeated this step but now varying θ at fixed $\phi = 0$. Similarly as before, we now define $b(N, 0)$ to be the number of points at half maximum for $\phi = 0$. This basically means that we count the number of points in a rectangle at half maximum of the total array. It is clear that the area of the rectangle spanned by $a(N, \pi/6)$ and $b(N, 0)$ includes all points of the function at half maximum. Some of the values are given in the graph below:[3]

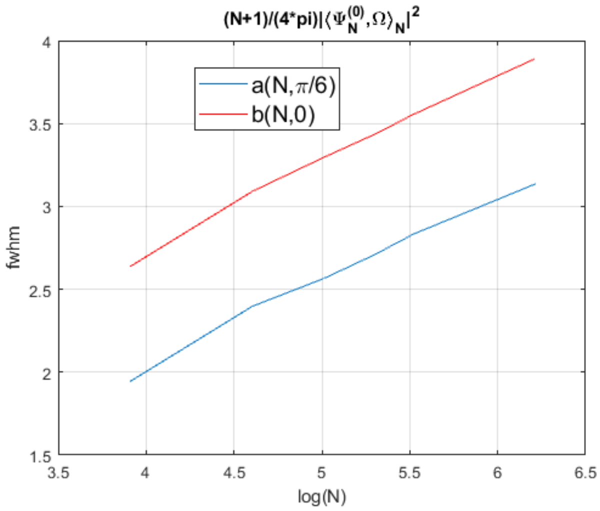

Figure 12: Full width at half maximum for the function (C.3.20), for $N = 50, 150, 200, 250, 500$ on a log scale . The red line corresponds to $\theta \in (0, \pi/2)$ and $\phi = 0$, whilst the blue line corresponds to $\theta = \pi/6$ and $\phi \in (-\pi, \pi)$.

It may be clear that the slope of both lines is about 0.5, which means that the fwhm goes like $\sqrt{N}$. It is also clear that $b(N, 0)$ seems to be translated with respect to $a(N, \pi/6)$ be a factor 2. We conclude that the number of points in the rectangle is approximately given by $\sqrt{N} \cdot \sqrt{N} \cdot 2 = 2N = O(N)$. Using the above discretization, we then have about $\sqrt{N}$ steps of $\pi/2N$ each, and about $2\sqrt{N}$ steps of $2\pi/N$ so that in particular the spanned rectangle has a width of $2\pi^2/N = O(1/N)$. This means that the fwhm of the function (C.3.20) indeed vanishes as $N \to \infty$.

[3]We display this for the case $\ell = 1$, but numerically checked that the same holds for $\ell = 1/2$.

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