Schur apolarity and how to use it

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Introduction

The aim of this thesis is to introduce a useful tool to study the tensor decomposition of elements belonging to Schur modules which we call *Schur apolarity theory*.

The main context in which this theory is developed is the *tensor decomposition*, a recent and active multidisciplinary research topic, see e.g. [BCC$^+$18, Lan12, OR20, BDE19, BBG$^+$21, GM19, BB21, Chi19] and references therein for some recent papers, with a plenty of relations with applied mathematics and the other sciences, see e.g. [BCS13, Lan17, BBC$^+$19, BC19].
Let $V_1, \dots, V_d$ be vector spaces of finite dimension over a field $\mathbb{K}$. Throughout all this document we assume that the field $\mathbb{K}$ is algebraically closed and of characteristic 0. A tensor t is an element of the vector space $V_1 \otimes \cdots \otimes V_d$, i.e. the vector space generated by all the possible products $\{v_{i_1} \otimes \cdots \otimes v_{i_d}\}$, where each v_{i_j} runs through the elements of a basis of V_j. Therefore we may write t as

$$t = \sum_{i_1,\dots,i_d} t_{i_1,\dots,i_d} \cdot v_{i_1} \otimes \cdots \otimes v_{i_d}$$

where $t_{i_1,\dots,i_d} \in \mathbb{K}$. An element t which can be written as

$$t = w_1 \otimes \cdots \otimes w_d$$

for some $w_i \in V_i$ is said to have *tensor rank*, or simply *rank*, equal to 1. It is clear that not any tensor can be written as a rank 1 element. However, since any tensor can be written as a sum of rank 1 tensors, one may wonder which is the minimal number of rank 1 tensors whose sum is t. Therefore we say that a tensor t has *rank* r if

$$r = \min\left\{s \ : \ t \in \langle t_1, \dots, t_s \rangle, \ t_i \text{ of rank 1 for every } i\right\}.$$

Even though the problem seems to be pretty algebraic, there is also a big interplay with geometry. Indeed, since the tensor rank is invariant under scalar multiplication, one can consider the *Segre embedding*

$$\begin{aligned} \nu_{1,\ldots,1} : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_d) &\longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_d) \\ ([w_1], \ldots, [w_d]) &\longmapsto [w_1 \otimes \cdots \otimes w_d]. \end{aligned}$$

The image of this map is the *Segre variety* and it is clear that its points are the rank 1 tensors. On the other hand, tensors of higher rank are naturally parametrized by the generic elements of the so called *secant varieties* of Segre varieties. Given a non degenrate variety $X \subset \mathbb{P}^N$, in the sense that $\langle X \rangle = \mathbb{P}^N$, the r-th secant variety of X is

$$\sigma_r(X) := \overline{\{p \in \mathbb{P}^N : p \in \langle p_1, \ldots, p_r \rangle,\ p_i \in X\}},$$

where the overline denotes the Zariski closure. Secant varieties form a chain of strict inclusions

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \subset \mathbb{P}^N.$$

In particular there exists an integer g such that $\sigma_g(X) = \mathbb{P}^N$. The smallest g for which $\sigma_g(X) = \mathbb{P}^N$ is called *generic* rank. These secant varieties have an expected dimension

$$\exp.\dim \sigma_r(X) := \min\{N, r(\dim(X)+1)-1\},$$

which sometimes is not attained. In this last case X is said to be *r-defective*. The tensor decomposition of a tensor, the dimensions of secant varieties of Segre varieties and the generic rank are of great interest and they are not easy to be computed.

Hereinbefore we have talked about tensors without prescribed relations or symmetry. Assume that $V_1 = \cdots = V_d$ and consider the space $V \otimes \cdots \otimes V =: V^{\otimes d}$. Given a *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ of d, i.e. a non increasing sequence of positive integers whose sum is d, one can draw its *Young diagram* putting λ_1 boxes in a row, λ_2 boxes below it and so on, all left justified. Assign a filling to such diagram with distinct integers from the set $\{1, \ldots, d\}$ in such a way that any sequence from top to bottom or from left to right is strictly increasing. The result of the filling is called *standard tableau of shape* λ. Once such a standard tableau of shape λ is fixed, using the natural action of the symmetric group $\mathfrak{S}_d$ on $V^{\otimes d}$ that permutes the factors, one can define an endomorphism

$$c_\lambda : V^{\otimes d} \longrightarrow V^{\otimes d}$$

called *Young symmetrizer*. Its image $\mathbb{S}_\lambda V := c_\lambda(V^{\otimes d})$ is a space of tensors satisfying particular relations and it is called *Schur module*. It turns out that such modules are irreducible representations of the groups $GL(n)$ and $SL(n)$, i.e. they are spaces on which the groups act without splitting them in a direct sum of stable subspaces. Particular and famous examples are obtained when $\lambda = (d)$ or $\lambda = (1^d) := (1, \dots, 1)$, in which one gets

$$\mathbb{S}_{(d)}V = \operatorname{Sym}^d V \quad \text{and} \quad \mathbb{S}_{(1,\dots,1)}V = \bigwedge^d V,$$

the spaces of *symmetric* and *skew-symmetric* tensors respectively, i.e. tensors invariant under permutations of the factors, up to the sign of the permutation in the latter case.

In analogy with the Segre varieties and rank 1 tensors, the chosen-to-be elementary tensors inside $\mathbb{P}(\mathbb{S}_\lambda V)$ are parametrized by irreducible non degenerate algebraic varieties naturally contained in it. Explicitly they are *flag varieties*

$$X = (\mathbb{F}(n_1, \dots, n_s; V),\ \mathcal{O}(d_1, \dots, d_s)),$$

i.e. the variety parametrizing flags of subspaces of V of prescribed dimensions n_i, counted with multiplicities d_i. As tensors the points of this variety can be written as

$$p = \left[(v_1 \wedge \dots \wedge v_{n_s})^{\otimes d_s} \otimes \dots \otimes (v_1 \wedge \dots \wedge v_{n_1})^{\otimes d_1}\right] \in X$$

representing the flag

$$\langle v_1, \dots, v_{n_1} \rangle \subset \dots \subset \langle v_1, \dots, v_{n_s} \rangle \subset V.$$

We refer to such tensors as *λ-rank* 1 tensors to highlight the space $\mathbb{S}_\lambda V$ to which they belong. In analogy to the tensor case, we say that a tensor t has λ-rank r if it can be written as sum of r λ-rank 1 elements and such a sum is minimal on the number of addends. Also in this case it makes sense to talk about secant varieties and generic λ-rank.

When $\lambda = (d)$ or $\lambda = (1^d)$, it turns out that X is a Veronese variety $\nu_d(\mathbb{P}^n)$ or a Grassmann variety $\mathbb{G}(k, V)$. The tensor decomposition in these cases have been already studied, see e.g. [BCC+18, IK99, ABMM21, AOP12] and references therein. For both of them a specific tool called *apolarity theory* has been developed.

In this manuscript we refer to them as *classical* and *skew-symmetric* apolarity theory respectively. Starting from the basic contraction map $V \times V^* \longrightarrow \mathbb{K}$, where V^* denotes the dual vector space of V, the two apolarity theories have at their foundations the *apolarity actions* which are the maps

$$\mathrm{Sym}^d V \otimes \mathrm{Sym}^e V^* \longrightarrow \mathrm{Sym}^{d-e} V,$$

$$\bigwedge^d V \otimes \bigwedge^e V^* \longrightarrow \bigwedge^{d-e} V$$

that roughly can be though as *derivations* with respect the prescribed symmetric or exterior product respectively.

The main goal of this thesis is to introduce an apolarity theory for any Schur module $\mathbb{S}_\lambda V$. More explicitly, we seek for a theory that can be restricted to the symmetric or skew-symmetric case returning the already known apolarities. Moreover, we look for an apolarity action which is *global*, in the sense that it allows us to *evaluate* any element of any Schur module $\mathbb{S}_\mu V^*$ onto elements of any other space $\mathbb{S}_\lambda V$, even though such evaluation will result to be trivial in most of the cases. This theory has been developed in Chapters 3 and 4 and with a lack of imagination it has been called *Schur apolarity theory*. The evaluation map for any couple of partitions λ and μ is

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

and it is called *Schur apolarity action*, cf. Definition 3.2.1 and Proposition 3.2.2 of this document. The space $\mathbb{S}_{\lambda/\mu} V$ is called *skew Schur module* and it is defined in an analogous way to usual Schur modules, cf. Definition 3.1.8. The space in which the Schur apolarity takes place is

$$\begin{aligned}\mathbb{S}^\bullet V &:= \mathrm{Sym}^\bullet \left(V \oplus \bigwedge^2 V \oplus \cdots \oplus \bigwedge^{n-1} V\right) / I^\bullet \\ &\simeq \bigoplus_{(a_1,\dots,a_{n-1}) \in \mathbb{N}^{n-1}} \left(\mathrm{Sym}^{a_1}(V) \otimes \mathrm{Sym}^{a_2}\left(\bigwedge^2 V\right) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}\left(\bigwedge^{n-1} V\right)\right) / I^\bullet,\end{aligned}$$

where $I^\bullet$ is the two-sided ideal generated by the Plücker relations, cf. Formula (3.1.2). The advantage of using such space is to avoid the redundancy of isomorphic copies of some Schur modules $\mathbb{S}_\lambda V$ appearing in the tensor algebra. We do not regard $\mathbb{S}^\bullet V$ as a commutative algebra, but only as vector space, see cf. Remark 3.1.4.

In similarity with to the known apolarities, for a fixed $t \in \mathbb{S}_\lambda V$ the Schur apolarity action induces a linear map

$$\mathcal{C}_t^{\lambda,\mu} : \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

which we call *catalecticant map of λ and μ with respect to t*. In Chapter 4 we study deeply the relation between these maps and the λ-rank of a tensor t in $\mathbb{S}_\lambda V$. For a fixed $t \in \mathbb{S}_\lambda V$ we can define its *apolar set* as

$$t^\perp := \bigoplus_\mu \ker \mathcal{C}_t^{\lambda,\mu},$$

i.e. the subspace of $\mathbb{S}^\bullet V^*$ given by all the elements that kill t via the Schur apolarity action, cf. Definition 3.2.7.
The core result in both the classical and skew-symmetric apolarity theories is the *apolarity lemma*, cf. Theorem 2.1.8 and cf. Theorem 2.2.11. In both these cases the key fact is to associate an *ideal* to a tensor of symmetric or skew-symmetric rank 1. Hence we have

$$\nu_d(\mathbb{P}^n) \ni p \longleftrightarrow I(p) \subset \operatorname{Sym}^\bullet V^*,$$

$$\mathbb{G}(k,V) \ni p \longleftrightarrow I^\wedge(p) \subset \bigwedge\nolimits^\bullet V^*,$$

see Remark 2.1.6 and Definition 2.2.9 for their definitions respectively. However in our general context with $\mathbb{S}^\bullet V$ we are dealing only with vector spaces. Therefore we associate a subspace $I(p) \subset \mathbb{S}^\bullet V^*$, cf. Definition 3.4.1, to a tensor of λ-rank 1. With these ingredients one can state and prove the so called *Schur apolarity lemma*.

Theorem (Lemma of Schur apolarity, Theorem 3.4.9)**.** *Let λ be a partition of length less than n and let $\mathbb{S}_\lambda V$ be an irreducible representation of $SL(n)$ together with the respective flag variety $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$. Let $f \in \mathbb{P}(\mathbb{S}_\lambda V)$ and let also $p_1, \ldots, p_r \in X$ be tensors of λ-rank 1. Then the following are equivalent:*

(1) *there exists $c_1, \ldots, c_r \in \mathbb{K}$ such that $f = \sum_{i=1}^r c_i p_i$,*

(2) $I(p_1, \ldots, p_r) := \bigcap_{i=1}^r I(p_i) \subset f^\perp$.

The main applications of this new apolarity theory are the computation of the λ-rank of tensors together with their decompositions in some cases, and the study the secant varieties of flag varieties. So little is known on these two topics except for the famous instances of symmetric and skew-symmetric tensors.

Eventually Schur apolarity is not the first attempt of an apolarity theory for tensors related to flag varieties. Indeed it is worth citing the famous *Non-abelian apolarity*, introduced by J. M. Landsberg and G. Ottaviani in [LO13], which is a general apolarity theory that involves vector bundles techniques. All along the text we will often exhibit analogies and differences between the Non-abelian and the Schur apolarity.

The text is organized as follows. In Chapter 1 basic facts needed to develop the theory are set. Next, Chapter 2 illustrates the state of the art of the apolarity theories related to the Schur apolarity. The heart of the document is Chapter 3 in which the Schur apolarity theory is presented and theorems are proved. The closing part of the thesis, Chapter 4, is devoted to apply the theory to the computation of the λ-rank of tensors.

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Chapter 1

Preliminaries

This introductory chapter is devoted to recall basic facts of Representation theory and rational homogeneous varieties, and to set the notation we will use all along the manuscript. Starting from now until the end of the document the base field $\mathbb{K}$ will be algebraically closed and of characteristic 0.

1.1 Tensors, tensor algebra, symmetric algebra and exterior algebra

We recall in this section some classic facts about tensors in general. The material is taken from [Bou89].

Tensor algebra. We give here a brief introduction to the tensor product and the tensor algebra. The following construction can be performed also for R-modules, with R commutative ring. However we give the definitions only for the case of vector spaces. For more details see [Bou89, p. 484].

Definition 1.1.1. Let V and W be finite dimensional vector spaces over $\mathbb{K}$. The *tensor product of V and W* is a vector space $V \otimes W$ over $\mathbb{K}$ together with a bilinear map

$$\begin{aligned} \alpha : V \times W &\longrightarrow V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

which is *universal*; this means that given any other bilinear map $\beta : V \times W \longrightarrow U$, with U any vector space on the same base field, there exists a unique map $\gamma : V \otimes W \longrightarrow U$ which sends the element $v \otimes w$ in $\beta(v, w)$ such that the following diagram

$$\begin{tikzcd} V \times W \arrow[r, "\alpha"] \arrow[dr, "\beta"] & V \otimes W \arrow[d, "\gamma"] \\ & U \end{tikzcd}$$

commutes. Similarly one can define the tensor product of d vector spaces $V_1 \otimes \cdots \otimes V_d$. The elements of the tensor product of vector spaces are called *tensors*. The elements that can be written as

$$v_1 \otimes \cdots \otimes v_d \tag{1.1.1}$$

are called *decomposable* or *rank* 1 elements.

Remark that from the definition the tensor product is functorial. Indeed, if there are linear maps $V_1 \to V_2$ and $W_1 \to W_2$, then there is a linear map $V_1 \otimes V_2 \to W_1 \otimes W_2$ acting as the tensor product of the two given maps.

Lemma 1.1.2. *Let V and W be two vector spaces over $\mathbb{K}$ such that $\{v_i\}$ and $\{w_j\}$ are basis of V and W respectively. Then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.*

We omit the proof for which we refer to [Bou89].

When the vector spaces are all the same, i.e. $V = V_1 = \cdots = V_d$, we refer to $V \otimes \cdots \otimes V =: V^{\otimes d}$ ad the *d-fold tensor product of V*. We set $V^{\otimes 0} = \mathbb{K}$ and $V^{\otimes 1} = V$. For any couple of positive integers d and e we have a map

$$V^{\otimes d} \times V^{\otimes e} \longrightarrow V^{\otimes d+e} \tag{1.1.2}$$

such that

$$(v_1 \otimes \cdots \otimes v_d, v_{d+1} \otimes \cdots \otimes v_{d+e}) \mapsto v_1 \otimes \cdots \otimes v_d \otimes v_{d+1} \otimes \cdots \otimes v_{d+e}.$$

and then it is extended by linearity. It is easy to see that this product is associative but not commutative.

Definition 1.1.3. The set

$$V^{\otimes \bullet} := \bigoplus_{d \geq 0} V^{\otimes d}$$

equipped with the linear maps (1.1.2) is a graded associative algebra called *tensor algebra*. The graded pieces are exactly $\left(V^{\otimes \bullet}\right)_d = V^{\otimes d}$ for any $d \geq 0$.

There are also other aspects of the tensor product which are important to be recalled. First of all note that the natural action of the general linear group $GL(V)$ on V can be extended to $V^{\otimes d}$, i.e. there is a map

$$GL(V) \times V^{\otimes d} \longrightarrow V^{\otimes d} \tag{1.1.3}$$

which sends the couple $(g, v_1 \otimes \cdots \otimes v_d)$ to $(g \cdot v_1) \otimes \cdots \otimes (g \cdot v_d)$ for any $g \in GL(V)$ and is extended by linearity. There is also another natural action on $V^{\otimes d}$ given by the symmetric group $\mathfrak{S}_d$. It is described as the map

$$V^{\otimes d} \times \mathfrak{S}_d \longrightarrow V^{\otimes d} \tag{1.1.4}$$

which sends the couple $(v_1 \otimes \cdots \otimes v_d, \sigma)$ to $v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$ for any $\sigma \in \mathfrak{S}_d$ and is extended by linearity.

Secondarily, consider the vector space of all linear maps from a vector space V to W, which we denote with $\mathrm{Hom}(V, W)$. It can be proved that there is an isomorphism $\mathrm{Hom}(V, W) \simeq V^* \otimes W$, where V^* denotes the dual vector space to V. Analogously, the set of multilinear maps from $V_1 \times \cdots \times V_d$ to W, which we usually denote with $\mathrm{Hom}(V_1 \times \cdots \times V_d, W)$, is isomorphic to the tensor product $V_1^* \otimes \cdots \otimes V_d^* \otimes W$. This reflects some known facts. For instance the tensor product $V^* \otimes W$ is actually the vector space of $\dim(V) \times \dim(W)$ matrices. Moreover if in the above descriptions we set $W = \mathbb{K}$, we get that

$$\mathrm{Hom}(V, \mathbb{K}) \simeq V^* \otimes \mathbb{K} \simeq V^*$$

is just the vector space of linear functionals on V, and

$$\mathrm{Hom}(V_1 \times \cdots \times V_d, \mathbb{K}) \simeq V_1^* \otimes \cdots \otimes V_n^* \otimes \mathbb{K} \simeq V_1^* \otimes \cdots \otimes V_n^* \tag{1.1.5}$$

is the vector space of multilinear functionals on $V_1 \times \cdots \times V_d$.

Symmetric algebra. We give here a short description of symmetric tensors and of the symmetric algebra. For more details see [Bou89, p. 497].

Definition 1.1.4. Let V be a vector space over $\mathbb{K}$ and consider the d-fold tensor product $V^{\otimes d}$. The vector space

$$\operatorname{Sym}^d V := V^{\otimes d} / \langle \{ v_1 \otimes \cdots \otimes v_d - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}, \text{ for any } \sigma \in \mathfrak{S}_d \} \rangle$$

is called *d-th symmetric power of* V. Its elements are called *symmetric tensors* and they are naturally contained in $V^{\otimes d}$ via $i : \operatorname{Sym}^d \hookrightarrow V^{\otimes d}$ defined on decomposable elements as

$$i(v_1 \dots v_n) = \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

and which is extended by linearity. We set $\operatorname{Sym}^0 V = \mathbb{K}$ and $\operatorname{Sym}^1 V = V$. The image of the projection $\pi : V^{\otimes d} \longrightarrow \operatorname{Sym}^d V$ on decomposable elements is

$$v_1 \dots v_d := \pi(v_1 \otimes \cdots \otimes v_d)$$

where the v_i's naturally commutes.

Remark 1.1.5. Note that symmetric tensors can be realized also as symmetric multilinear functionals on $V^* \times \cdots \times V^*$, i.e. maps

$$\beta : V^* \times \cdots \times V^* \longrightarrow \mathbb{K}$$

as in (1.1.5) that satisfies $\beta(x_{\sigma(1)}, \dots, x_{\sigma(d)}) = \beta(x_1, \dots, x_d)$ for any $\sigma \in \mathfrak{S}_d$.

Remark 1.1.6. Like the tensor product, the symmetric product $\operatorname{Sym}^d V$ has a universal property. This means that given any other symmetric multilinear map $\alpha : V \times \cdots \times V \longrightarrow U$ with U vector space, then there exists a unique linear map $\gamma : \operatorname{Sym}^d V \longrightarrow U$ taking $v_1 \dots v_d$ to $\alpha(v_1, \dots, v_d)$ and making the following diagram

$$\begin{tikzcd}
V \times \cdots \times V \arrow[r, "\pi"] \arrow[dr, "\alpha"'] & \operatorname{Sym}^d V \arrow[d, "\gamma"] \\
& U
\end{tikzcd}$$

commute.

It is easy to see that if $\{v_i\}$ is a basis for V, then $\{v_{i_1} \dots v_{i_d},\ 1 \le i_1 \le \cdots \le i_d \le n\}$ is a basis for $\mathrm{Sym}^d V$.

Given any couple of positive integers d and e we have linear maps

$$\mathrm{Sym}^d V \times \mathrm{Sym}^e V \longrightarrow \mathrm{Sym}^{d+e} V \tag{1.1.6}$$

which sends the elements $(v_1 \dots v_d, w_1 \dots w_e)$ to $v_1 \dots v_d w_1 \dots w_e$ and it is extended by linearity. Note that by the universality of the symmetric product the domain of this map can be exchanged with $\mathrm{Sym}^d V \otimes \mathrm{Sym}^e V$. On the other hand, with the same integers as before we can define a map

$$\mathrm{Sym}^{d+e} V \longrightarrow \mathrm{Sym}^d \otimes \mathrm{Sym}^e V \tag{1.1.7}$$

such that it is acts as

$$v_1 \dots v_{d+e} \mapsto \sum_{\sigma \in \mathfrak{S}_{d+e}^{d,e}} v_{\sigma(1)} \dots v_{\sigma(d)} \otimes v_{\sigma(d+1)} \dots v_{\sigma(d+e)}$$

and it is extended by linearity, where

$$\mathfrak{S}_{d+e}^{d,e} := \{\sigma \in \mathfrak{S}_{d+e} : \sigma(1) < \cdots < \sigma(d),\ \sigma(d+1) < \cdots < \sigma(d+e)\}.$$

The maps (1.1.6) and (1.1.7) are called *multiplication maps* and *diagonal* or *comultiplication maps* of symmetric powers respectively.

Definition 1.1.7. The set

$$\mathrm{Sym}^\bullet V := \bigoplus_{d \ge 0} \mathrm{Sym}^d V$$

equipped with the maps (1.1.6) is a commutative associative graded algebra called *symmetric algebra*. The graded pieces are $(\mathrm{Sym}^\bullet V)_d = \mathrm{Sym}^d V$.

Remark 1.1.8. Note that the space of symmetric tensors of order d can be realized also as the vector space of homogeneous polynomials of degree d in $\dim(V)$ variables. Indeed there is an isomorphism

$$\mathrm{Sym}^d V \simeq \mathbb{K}[x_1, \dots, x_{\dim(V)}]_d \tag{1.1.8}$$

such that if $\{w_1, \ldots, w_{\dim(V)}\}$ is a basis of V, then it associates the decomposable element $w_1 \ldots w_{\dim(V)}$ to the monomial $x_1 \ldots x_{\dim(V)}$. The space on the right hand side denotes the vector space of homogeneous polynomials of degree d in the variables $x_1, \ldots, x_{\dim(V)}$ and we have the correspondence $v_i \leftrightarrow x_i$. The collection of the isomorphisms (1.1.8) for any $d \geq 0$ induce an isomorphism of graded algebras

$$\mathrm{Sym}^\bullet V \simeq \mathbb{K}[x_1, \ldots, x_{\dim(V)}] \tag{1.1.9}$$

of the symmetric algebra with the algebra of polynomials in the variables $x_1, \ldots, x_{\dim(V)}$.

In the following when we write $\mathrm{Sym}^\bullet V^*$, i.e. the symmetric algebra over the dual vector space, we will mean the *graded dual algebra* i.e.

$$\mathrm{Sym}^\bullet V^* := \bigoplus_{d \geq 0} \mathrm{Sym}^d V^*$$

equipped with analogues of the maps (1.1.6). As a last remark note that the symmetric powers of V inherits a $GL(V)$ action from (1.1.3)

$$g \cdot (v_1 \ldots v_d) := (g \cdot v_1) \ldots (g \cdot v_d),$$

for any $g \in GL(V)$ and d positive integer.

Exterior algebra. We give here a short description of skew-symmetric tensors and of the exterior algebra. For more details see [Bou89, p. 507].

Definition 1.1.9. Let V be a vector space over $\mathbb{K}$ and consider the d-fold tensor product $V^{\otimes d}$. The vector space

$$\bigwedge^d V := V^{\otimes d} / \langle \{v_1 \otimes \cdots \otimes v_d : \exists i, j \text{ such that } v_i = v_j\} \rangle \tag{1.1.10}$$

is called *d-th exterior power of* V. Its elements are called *skew-symmetric tensors* and they are naturally contained in $V^{\otimes d}$ via the inclusion $i : \bigwedge^d V \hookrightarrow V^{\otimes d}$ that on decomposable elements acts as

$$i(v_1 \wedge \cdots \wedge v_d) = \sum_{\sigma \in \mathfrak{S}_d} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

and is extended by linearity. We set $\bigwedge^0 V = \mathbb{K}$ and $\bigwedge^1 V = V$. The image of the projection $\pi : V^{\otimes d} \longrightarrow \bigwedge^d V$ on decomposable elements is

$$v_1 \wedge \cdots \wedge v_d =: \pi(v_1 \otimes \cdots \otimes v_d). \tag{1.1.11}$$

The elements of $\bigwedge^d V$ which can be written as in (1.1.11) are called skew-symmetric tensors *of skew-symmetric rank* 1. Note that by the relations appearing in (1.1.10), if $d > \dim(V)$, then $\bigwedge^d V = \langle 0 \rangle$, while if $d = \dim(V)$, then $\bigwedge^d V \simeq \mathbb{K}$.

Remark 1.1.10. Note that the space of skew-symmetric tensors of order d can be realized also as alternating multilinear functionals on $V^* \times \cdots \times V^*$ i.e. maps

$$\beta : V^* \times \cdots \times V^* \longrightarrow \mathbb{K}$$

as in (1.1.5) that satisfies $\beta(v_{\sigma(1)}, \ldots, v_{\sigma(d)}) = \operatorname{sgn}(\sigma)\beta(v_1, \ldots, v_d)$ for any $\sigma \in \mathfrak{S}_d$, where $\operatorname{sgn}(\sigma)$ stands for the sign of the permutation σ.

Remark 1.1.11. As in the previous paragraphs, the exterior power $\bigwedge^d V$ has a universal property. This means that given any other alternating multilinear map $\alpha : V \times \cdots \times V \longrightarrow U$ with U vector space, then there exists a unique linear map $\gamma : \bigwedge^d V \longrightarrow U$ taking $v_1 \wedge \cdots \wedge v_d$ to $\alpha(v_1, \ldots, v_d)$ and making the following diagram

$$\begin{tikzcd} V \times \cdots \times V \arrow[r, "\pi"] \arrow[dr, "\alpha"'] & \bigwedge^d V \arrow[d, "\gamma"] \\ & U \end{tikzcd}$$

commute.

It is easy to see that if $\{v_i\}$ is a basis of V, then $\{v_{i_1} \wedge \cdots \wedge v_{i_d} : 1 \leq i_1 < \cdots < i_d \leq \dim(V)\}$ is a basis of $\bigwedge^d V$.

Given any couple of positive integers d and e such that $d + e \leq \dim(V)$ we have linear maps

$$\bigwedge^d V \otimes \bigwedge^e V \longrightarrow \bigwedge^{d+e} V \tag{1.1.12}$$

which sends the decomposable elements $(v_1 \wedge \cdots \wedge v_d, w_1 \wedge \cdots \wedge w_e)$ to $v_1 \wedge \cdots \wedge v_d \wedge w_1 \wedge \cdots \wedge w_e$ and it is extended by linearity. On the other hand, with the same choice of integers we have a linear map

$$\bigwedge^{d+e} V \longrightarrow \bigwedge^d V \otimes \bigwedge^e V \tag{1.1.13}$$

such that it sends decomposable elements

$$v_1 \wedge \cdots \wedge v_{d+e} \mapsto \sum_{\sigma \in \mathfrak{S}_{d+e}^{d,e}} \operatorname{sgn}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(d)} \otimes v_{\sigma(d+1)} \wedge \cdots \wedge v_{\sigma(d+e)}$$

and is extended by linearity. Again, the subgroup $\mathfrak{S}_{d+e}^{d,e}$ is

$$\mathfrak{S}_{d+e}^{d,e} := \{\sigma \in \mathfrak{S}_{d+e} : \sigma(1) < \cdots < \sigma(d),\ \sigma(d+1) < \cdots < \sigma(d+e)\}.$$

The maps (1.1.12) and (1.1.13) are called *multiplication maps* and *diagonal* or *comultiplication maps* of exterior powers respectively.

Definition 1.1.12. The set

$$\bigwedge^{\bullet} V := \bigoplus_{d=0}^{\dim(V)} \bigwedge^{d} V$$

equipped with the maps (1.1.12) is an associative graded algebra called *exterior algebra*. The graded pieces are $(\bigwedge^{\bullet} V)_d = \bigwedge^d V$ when $0 \leq d \leq \dim(V)$, otherwise they are $\langle 0 \rangle$.

Contrary to the symmetric case, the exterior algebra is a finite dimensional algebra and hence it makes sense to talk about its dual algebra $\bigwedge^{\bullet} V^*$. In particular there is an isomorphism of algebras $\left(\bigwedge^{\bullet} V\right)^* \simeq \bigwedge^{\bullet} V^*$.

As a last remark note that exterior powers of V inherits a $GL(V)$ action from (1.1.3). Explicitly for any $d \leq \dim(V)$ positive integer and $g \in GL(V)$

$$g \cdot (v_1 \wedge \cdots \wedge v_d) := (g \cdot v_1) \wedge \cdots \wedge (g \cdot v_d).$$

1.2 Varieties parametrizing tensors

The general, symmetric and skew-symmetric tensors presented in Section 1.1 have a natural and deep connection with specific algebraic varieties. We present them here briefly. The material is taken from [Har13] and [Lan12].

Definition 1.2.1. Let $X \subset \mathbb{P}^n$ be a non degenerate algebraic variety. Given a point $p \in \mathbb{P}^n$, we define its *X-rank* as

$$r_X(p) = \min\{r \in \mathbb{N} : p \in \langle p_1, \ldots, p_r \rangle, \text{ for some } p_i \in X\}.$$

We define then the *r-th secant variety of X* as

$$\sigma_r(X) := \overline{\{p \in \mathbb{P}^n : r_X(p) = r\}}$$

where $\overline{Y}$ denotes the Zariski closure of Y. The smallest r such that $p \in \sigma_r(X)$ but $p \notin \sigma_{r-1}(X)$ is called *border* X-rank.

Remark 1.2.2. If X is irreducible, then also the secant varieties $\sigma_r(X)$ are irreducible for any r positive integer. Moreover it is clear by the definition that the secant varieties form a nested sequence of inclusions

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \cdots \subset \sigma_r(X) \subset \cdots \subset \mathbb{P}^n.$$

The least integer g for which there is the equality $\sigma_g(X) = \mathbb{P}^n$ is called *generic X-rank*. Note also that the generic X-rank highly depends on the dimension of the secant varieties. By a simple count of the parameters, we may define the *expected dimension* of the secant variety as

$$\operatorname{exp.dim}(\sigma_r(X)) := \min\{n, r(\dim(X) + 1) - 1\}.$$

The expected dimension is always an upper bound to the actual dimension. In literature there are several examples in which this expected dimension is not attained, i.e. where it happens that

$$\dim(\sigma_r(X)) < \min\{n, r(\dim(X) + 1) - 1\}. \tag{1.2.1}$$

Whenever (1.2.1) holds we say that X is *r-defective* or that $\sigma_r(X)$ is *defective*.

General tensors: Segre varieties. Let $V_1, \ldots, V_d$ be vector spaces over the field $\mathbb{K}$. Denote with $\mathbb{P}(V_i)$ the respective projective spaces. We can define a map

$$\begin{aligned} \nu_{1,\ldots,1} : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_d) &\longrightarrow \mathbb{P}(V_1 \otimes \cdots \otimes V_d) \\ ([v_1], \ldots, [v_d]) &\longmapsto [v_1 \otimes \cdots \otimes v_d]. \end{aligned}$$

It can be proved that the image of $\nu_{1,\ldots,1}$ is an algebraic variety called *Segre variety*. We refer to $\nu_{1,\ldots,1}$ as *Segre embedding*. Note that by definition the points of the

Segre variety are exactly the rank 1 tensors we have defined in (1.1.1). The indices of $\nu_{1,\ldots,1}$ indicates that the map is the embedding determined by the very ample line bundle $\mathcal{O}(1,\ldots,1)$ over $\mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_d)$. The generic element of the secant variety $\sigma_r(X)$ to a Segre variety can be written as the sum

$$\sum_{i=1}^{r} c_i[v_{i,1} \otimes \cdots \otimes v_{i,d}]$$

minimal on the number of addends, for $c_i \in \mathbb{K}$. In this particular instance where X is a Segre variety, we will refer to the X-rank as *tensor rank*.

Symmetric tensors: Veronese varieties. Let V be a vector space over $\mathbb{K}$ of dimension $n+1$. We may define the map

$$\begin{aligned} \nu_d : \mathbb{P}(V) &\longrightarrow \mathbb{P}(\operatorname{Sym}^d V) \\ [l] &\longmapsto [l^d] \end{aligned}$$

that sends the projective class of an element of V regarded as a linear form, to the projective class of its d-th power. The map ν_d is called *Veronese embedding* and the image $X_{n,d} := \nu_d(\mathbb{P}(V))$ is called *Veronese variety*. The Veronese embedding is the inclusion determined by the very ample line bundle $\mathcal{O}(d)$ over $\mathbb{P}(V)$.
It is clear that the points of X-rank 1 where X is a Veronese variety are all the elements of $\operatorname{Sym}^d V$ that can be written as a power of a linear form. Hence the generic element of $\sigma_r(X)$ is something like

$$c_1[l_1^d] + \cdots + c_r[l_r^d] \tag{1.2.2}$$

where $c_i \in \mathbb{K}$. When X is a Veronese variety we refer to the X-rank as *symmetric rank*. The decomposition (1.2.2) is called *Waring decomposition*. The classification of all the defective secant varieties of Veronese varieties is described by the so called *Alexander-Hirschowitz theorem* which can be found in [AH95].

Skew-symmetric tensors: Grassmann varieties. Let V be a vector space of dimension n over $\mathbb{K}$. Let us denote the collection of all the k-dimensional subspaces of V with $\mathbb{G}(k,V)$. Consider an element $W \in \mathbb{G}(k,V)$. If $\{v_1,\ldots,v_k\}$ is a basis of W, we may define a map

$$\begin{aligned} i : \mathbb{G}(k,V) &\longrightarrow \mathbb{P}\left(\bigwedge^k V\right) \\ W &\longmapsto [v_1 \wedge \cdots \wedge v_k] \end{aligned}$$

Note that given any other basis $\{w_1, \ldots, w_k\}$ of W, then

$$[v_1 \wedge \cdots \wedge v_k] = [w_1 \wedge \cdots \wedge w_k]$$

since they only differ by a constant given by the determinant of the matrix of change of coordinates. The map i is called *Plücker embedding* and its image is called *Grassmann variety*. The Plücker embedding is the inclusion determined by the very ample line bundle $\bigwedge^k \mathcal{U}^\vee$, where $\mathcal{U}$ is the tautological bundle over $\mathbb{G}(k,V)$. It is clear that the points of X-rank 1 are exactly the tensors of skew-symmetric rank 1. We will refer to the X-rank as skew-symmetric rank when X is a Grassmann variety. In this case the generic element of $\sigma_r(\mathbb{G}(k,V))$ is something like

$$\sum_{i=1}^{r} c_i [v_{i,1} \wedge \cdots \wedge v_{i,d}].$$

for some $c_i \in \mathbb{K}$. A classification theorem on the defectivity of Grassmann varieties is still missing. The most updated state of the art is in [Bor13].

1.3 Basic Representation Theory

We recall here the basic facts of representation theory needed to develop the central part of this document. The main references for this section are [FH13], [Ful97] and [Hum12a]. The theory is general for any reductive group G. We will use it only for the case of the special linear group.

Definition 1.3.1. A *complex (algebraic) Lie group G* is a group with a structure of complex manifold (of algebraic variety) such that the maps

$$\begin{aligned} G \times G &\longrightarrow G \\ (a,b) &\longmapsto a \cdot b \end{aligned}$$

and

$$\begin{aligned} G &\longrightarrow G \\ a &\longmapsto a^{-1} \end{aligned}$$

are both holomorphic (algebraic).

Complex algebraic groups and complex Lie groups have a clear connection which is the usual relation between algebraic varieties and complex manifolds. Moreover their representation theory is determined in both cases by the representation theory of Lie algebras. By virtue of this fact in this chapter we may sometimes confuse the terms Lie groups and algebraic groups.

Definition 1.3.2. Given a complex Lie group G, a *representation* of G is a vector space V together with a map of complex Lie groups

$$\rho : G \longrightarrow GL(V).$$

A representation is said to be *reducible* if it can be written as $V = V_1 \oplus V_2$, with V_1 and V_2 representations of G, otherwise it is called *irreducible*. A finite dimensional representation V of G is *completely reducible* if V can be written as a finite direct sum of irreducible representations.

Definition 1.3.3. Let $\rho_1 : G \longrightarrow GL(V_1)$ and $\rho_2 : G \longrightarrow GL(V_2)$ be two representations of a Lie group G and let $\varphi : V_1 \longrightarrow V_2$ be a linear map. Then φ is a *G-equivariant* morphism if

$$\varphi\left(\rho_1(g) \cdot v\right) = \rho_2(g) \cdot \varphi(v)$$

for any $g \in G$ and $v \in V_1$.

Since G is a complex manifold, it makes sense to compute the tangent space to G at some point. In particular, the tangent space $\mathfrak{g} = T_eG$ at the identity $e \in G$ is an example of a *Lie algebra*, i.e. a vector space equipped with a skew-symmetric bilinear map

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

called *Lie brackets* that satisfies the *Jacobi identity*

$$[X,[Y,Z]] - [Y,[X,Z]] + [Z,[X,Y]] = 0.$$

If $\mathfrak{g} = T_e G$ for some complex Lie group G, a particular instance of a representation of Lie groups is the *adjoint representation*. It is the map

$$\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$$

that sends an element g to the differential $(d\Psi_g)_e : \mathfrak{g} \longrightarrow \mathfrak{g}$ of the map $\Psi_g : G \longrightarrow G$ which acts as conjugation by g. Moreover in this case one can prove that the Lie brackets are defined as

$$[X, Y] := \mathrm{ad}(X)(Y)$$

where

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

is the differential at the identity of the adjoint representation and $\mathfrak{gl}(\mathfrak{g}) = \mathrm{End}(\mathfrak{g})$. See [FH13, p. 104–107].

Remark 1.3.4. In the case of $G = GL(V)$ or $G = SL(V)$, it can be easily seen that

$$[X, Y] = XY - YX$$

for any $X, Y \in GL(V)$ or $SL(V)$.

Definition 1.3.5. A *representation* of a Lie algebra $\mathfrak{g}$ is a vector space V together with a Lie algebra morphism

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

A representation V of a Lie algebra $\mathfrak{g}$ is said to be *reducible* if it can be written as $V = V_1 \oplus V_2$, with V_1 and V_2 representations of $\mathfrak{g}$, otherwise it is called *irreducible*. A finite dimensional representation V of $\mathfrak{g}$ is *completely reducible* if V can be written as a finite direct sum of irreducible representations.

Definition 1.3.6. Let $\rho_1 : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_1)$ and $\rho_2 : \mathfrak{g} \longrightarrow \mathfrak{gl}(V_2)$ be two representations of a Lie algebra $\mathfrak{g}$ and let $\varphi : V_1 \longrightarrow V_2$ be a linear map. Then φ is a $\mathfrak{g}$-*equivariant* morphism if

$$\varphi\left(\rho_1(X) \cdot v\right) = \rho_2(X) \cdot \varphi(v)$$

for any $X \in \mathfrak{g}$ and any $v \in V_1$.

Lemma 1.3.7 (Schur's Lemma)**.** *Let V_1 and V_2 be irreducible representations of a Lie group G (of a Lie algebra $\mathfrak{g}$). Let $\varphi : V_1 \longrightarrow V_2$ be a G-equivariant morphism ($\mathfrak{g}$-equivariant morphism). Then the following is true:*

1) Either $\varphi = 0$ or φ is an isomorphism.

2) If $V_1 = V_2$, then $\varphi = c \cdot \mathrm{id}_{V_1}$, for some $c \in \mathbb{K} \setminus \{0\}$.

We have the following fact.

Proposition 1.3.8 ([FH13], p. 109)**.** *Let H and G be Lie groups with G simply connected. Then the morphisms between G and H are in one-to-one correspondence with the morphisms of the respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. The correspondence is given associating to $\rho : G \longrightarrow H$ the differential at the identity $(d\rho)_e : \mathfrak{g} \longrightarrow \mathfrak{h}$.*

We omit the proof of this fact. We refer to [Ott95, Prop. 3.15, p. 14] for a proof. As a consequence of this proposition we have that if G is a simply connected Lie group and $H = GL(V)$, i.e. V is a representation of G, then we have that the representations of G are in one-to-one correspondence with the representations of the respective Lie algebra $\mathfrak{g}$.

Remark 1.3.9. Even though there is a link between the representations of a Lie group G and the respective Lie algebra $\mathfrak{g}$, the actions of the group and the algebra on the same representation V are slightly different. For instance if we consider the action of G, if $V = W_1 \oplus W_2$, then

$$g(w_1 \otimes w_2) = (g \cdot w_1) \otimes (g \cdot w_2),$$

where $w_1 \in W_1$, $w_2 \in W_2$, $g \in G$, and if $\rho_i : G \longrightarrow GL(W_i)$, then $g \cdot w_i$ denotes $\rho_i(g)w_i$ for $g \in G$. However, since the same representation of the Lie algebra is obtained by differentiation, we get that

$$X(w_1 \otimes w_2) = (X \cdot w_1) \otimes w_2 + w_1 \otimes (X \cdot w_2),$$

where $w_1 \in W_1$, $w_2 \in W_2$, $X \in \mathfrak{g}$, and if $(d\rho_i)_e : \mathfrak{g} \longrightarrow \mathfrak{gl}(W_i)$, then $X \cdot w_i$ denotes $\rho_i(X)w_i$ for $X \in \mathfrak{g}$.

Definition 1.3.10. A subspace I of a Lie algebra $\mathfrak{g}$ is called *ideal* if for every $X \in I$, $Y \in \mathfrak{g}$ we have that $[X, Y] \in I$. A Lie algebra that does not contain any ideal except itself and the zero ideal is called *simple*.

Definition 1.3.11. Given any Lie algebra $\mathfrak{g}$, we may define its *derived series*

$$\mathcal{D}^1(\mathfrak{g}) := [\mathfrak{g}, \mathfrak{g}],\ \mathcal{D}^n(\mathfrak{g}) := [\mathcal{D}^{n-1}(\mathfrak{g}), \mathcal{D}^{n-1}(\mathfrak{g})].$$

We say that $\mathfrak{g}$ is *solvable* if $\mathcal{D}^n(\mathfrak{g}) = 0$ for some n. A Lie algebra $\mathfrak{g}$ for which any non zero ideal is not solvable is called *semisimple*.

We have this important fact regarding semisimple Lie algebras.

Theorem 1.3.12 (Weyl). *Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be any finite dimensional representation of $\mathfrak{g}$. Then V is completely reducible, i.e. it can be written as a finite direct sum of irreducible representations.*

For a proof of this fact see [Hum12a, First Theorem in §6.3]. It follows that if G is a complex Lie group with a semisimple Lie algebra $\mathfrak{g}$ as tangent space at the identity, then every irreducible representation of G is an irreducible representation of $\mathfrak{g}$. Moreover it is clear that to study the representations of G, or $\mathfrak{g}$ equivalently, is enough to understand which are the irreducible representations.

We recall briefly the classification of finite dimensional semisimple Lie algebras. The main idea is to find a *Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$, i.e. a maximal abelian subalgebra of $\mathfrak{g}$ such that for any $x \in \mathfrak{h}$, the map $\mathrm{ad}(x) : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ is semisimple. The action of this commuting semisimple endomorphisms of $\mathfrak{g}$ allows to obtain the *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha} \mathfrak{g}_\alpha \right)$$

where the action of $\mathfrak{h}$ preserves any $\mathfrak{g}_\alpha$ which is the space

$$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \mathrm{ad}(H)(X) = \alpha(H)X\}$$

where $\alpha \in \mathfrak{h}^*$ is a common eigenvalue called *root* of the Lie algebra $\mathfrak{g}$. The space $\mathfrak{g}_\alpha$ is called *root space*. Note that also $\mathfrak{h}$ is a root space determined by the eigenvalue 0. In the following we assume that $\mathfrak{g}$ is semisimple. We have that

Proposition 1.3.13. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.*

A proof of this proposition can be obtained as a combination of [Hum12a, First proposition in §4.2] and [Hum12a, First lemma in §8.1].

Denote with $R \subset \mathfrak{h}^*$ the set of roots. We have the following facts.

Remark 1.3.14. (i) every root space $\mathfrak{g}_\alpha$ is one dimensional.

(ii) if $\alpha \in R$, then $k\alpha \in R$ if and only if $k = \pm 1$.

(iii) R generates a real vector space $\Lambda_R \subset \mathfrak{h}$ of rank the dimension of $\mathfrak{h}$.

Consider the lattice Λ_R generated by R and choose a real functional $l \in \mathfrak{h}^*$ irrational with respect to the lattice. This gives an ordering of the roots and a splitting of the set of roots

$$R = R^+ \cup R^-,$$

where $R^+ = \{\alpha : l(\alpha) > 0\}$ and analogously for R^-. The sets R^+ and R^- are called sets of *positive* and *negative* roots respectively. We say that a positive or negative root $\alpha \in R$ is *primitive* or *simple* if it cannot be written as sum of two positive or negative roots respectively.

The classification theorem of finite dimensional semisimple Lie algebras is based on the study of the simple roots and on the evaluation of a bilinear form on $\mathfrak{g}$ called *Killing form* on couples of positive roots. Such a study is implemented on a graph representation of the root system of $\mathfrak{g}$ called *Dynkin diagram*. The Killing form is defined as the map

$$B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$$

such that $B(X, Y) := \mathrm{Tr}\,(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$.

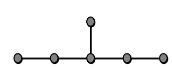
Proposition 1.3.15. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following facts hold.*

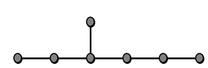
(i) (Cartan's criterion). Let $\mathfrak{g}$ be any semisimple Lie algebra. Then $\mathfrak{g}$ is semisimple if and only if B is non degenerate.

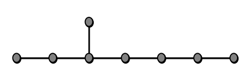
(ii) Let $\mathfrak{g}$ be semisimple and consider B as induced on $\mathfrak{g}^*$. Then B is positive definite on $\wedge_R$.

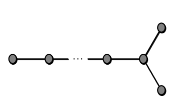
For a proof of this facts see [FH13, Proposition C.10, p. 480] for (i), and [FH13, Corollary 14.30, p. 208] for (ii). It can be proved that

Theorem 1.3.16. *The finite dimensional simple Lie algebras are classified. The four families $\mathfrak{sl}_n$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ are the Lie algebras associated to the Lie groups $SL(n)$, $SO(2n+1)$, $Sp(2n+1)$ and $SO(2n)$. The other are the exceptional cases.*

A_n, $n \geq 1$, $\mathfrak{sl}_{n+1}$, $\mathfrak{e}_6$, 

B_n, $n \geq 1$ $\mathfrak{so}_{2n+1}$, $\mathfrak{e}_6$, 

C_n, $n \geq 1$ $\mathfrak{sp}_{2n}$, $\mathfrak{e}_6$, 

D_n, $n \geq 1$  $\mathfrak{so}_{2n}$, $\mathfrak{f}_4$, ,

G_2,

For a proof of this theorem we refer to [FH13] and [Hum12a].

Definition 1.3.17. Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{g}$. For any $\lambda \in \mathfrak{h}^*$ we define the *weight space* as

$$V_\lambda := \{v \in V : \rho(H)v = \lambda(H)v, \text{ for all } H \in \mathfrak{h}\}.$$

The functional $\lambda \in \mathfrak{h}^*$ is called *weight*.

We have the following

Proposition 1.3.18. *Let $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be an irreducible representation. Then*

(i) the action of every root space $\mathfrak{g}_\alpha$ of the Cartan decomposition of $\mathfrak{g}$ on V via ρ gives a map

$$\rho(\mathfrak{g}_\alpha) : V_\lambda \longrightarrow V_{\lambda+\alpha}$$

for any weight space V_λ.

(ii) we get a direct sum

$$V = \bigoplus V_\lambda$$

where the sum runs over a finite set of $\lambda \in \mathfrak{h}^*$.

For a proof see [Hum12a, First lemma in §20.1]. The proof is just a direct computation using the definition of the root and weight spaces.

Definition 1.3.19. Let V be a representation of a semisimple Lie algebra $\mathfrak{g}$. A non zero vector $v \in V$ is called *highest weight vector* if $v \in V_\lambda$ for some non zero weight space V_λ and it belongs to the kernel of the action induced by every root space $\mathfrak{g}_\alpha$ for any $\alpha \in R^+$. The eigenvalue λ is called *highest weight*.

Proposition 1.3.20 (Proposition 14.13, p, 202–203, [FH13])**.** *Let V be a representation of a semisimple Lie algebra $\mathfrak{g}$. Then*

(i) there always exists a highest weight vector $v \in V$,

(ii) the subspace generated by the successive applications on v of the negative root spaces $\mathfrak{g}_\alpha$, i.e. $\alpha \in R^-$, is an irreducible representation of $\mathfrak{g}$,

(iii) the representation V is irreducible if and only if there is a unique highest weight vector.

For a proof we refer to [FH13, Proposition 14.13, p. 203]. We focus now to the case $G = SL(n)$. In this case $\mathfrak{sl}_n := T_e SL(n)$ is the vector space of $n \times n$ traceless matrices. In this case we can take as Cartan subalgebra

$$\mathfrak{h} = \{a_1 H_1 + \cdots + a_n H_n : a_1 + \cdots + a_n = 0\}$$

where H_i is the diagonal matrix with 1 in the position (i,i) and 0 in any other position. It can be proved that in this case the roots are all of the form $L_i - L_j$, where $L_i \in \mathfrak{h}^*$ is such that $L_i(H_j) = \delta_{i,j}$. Choose the linear functional l on the lattice generated by the roots such that

$$l(a_1 L_1 + \cdots + a_n L_n) = c_1 a_1 + \cdots + c_n a_n$$

with $c_1 + \cdots + c_n = 0$ and $c_1 > \cdots > c_n$. Then we have

$$R^+ = \{L_i - L_j : \ i < j\},\ R^- = \{L_i - L_j : \ j < i\}.$$

It is easy to see that the simple positive roots are $L_i - L_{i+1}$ for $i = 1, \ldots, n-1$. The root space $\mathfrak{g}_{L_i - L_j}$ associated to the root $L_i - L_j$ is generated by $E_{i,j}$, i.e. the matrix which has a 1 in the position (i,j) and 0 in any other position. In general we may find subalgebras $\mathfrak{s}_\alpha$ of $\mathfrak{g}$ which are isomorphic to $\mathfrak{sl}_2$ defined as

$$\mathfrak{s}_\alpha := \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

In our particular instance we have that if $\alpha = L_i - L_j$, then the three summands are generated by $E_{i,j}$, $E_{j,i}$ and $[E_{i,j}, E_{j,i}] = H_i - H_j$. Sometimes for brevity this last element is denoted with $H_{L_i - L_j}$. We have the following

Definition 1.3.21. The fundamental weights ω_j are the dual basis of $H_{L_i - L_{i+1}}$ where $L_i - L_{i+1}$ is a simple positive root, i.e. it holds

$$\omega_j(H_i - H_{i+1}) = \delta_{i,j}.$$

It can be proved that the $H_{L_i - L_j}$, with $L_i - L_j$ simple root, are a basis of $\mathfrak{h}$ so that the fundamental weights ω_j are finite and linearly independent. It is not hard to see that the fundamental weights are

$$\omega_j = L_1 + \cdots + L_j$$

for $j = 1, \ldots, n-1$. For more general facts of the theory, the fundamental weights span the *weight lattice*, i.e. all the functionals in $\mathfrak{h}^*$ which assume integer values on every H_α. It is not hard to see that the representation with highest weight the fundamental weight ω_i is $\bigwedge^{\omega_i} V$. In this case we call it *fundamental representation*. For every ω in the weight lattice there is a unique irreducible representation of highest weight ω. Note that since any ω in the weight lattice is an integer linear combination of the fundamental weights, we are able to compute any irreducible representation of $\mathfrak{g}$. However, the irreducible representation of highest weight ω will be not given as the tensor product of the respective fundamental representations, but it will be an irreducible representation contained in such product. In general the tensor product of representations is reducible. We adopt the notation

$$\Gamma_{a_1,\ldots,a_n}$$

to denote the irreducible representation of highest weight $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$. Such a representation must appear as an irreducible representation inside the product

$$\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}\left(\bigwedge^2 V\right) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}\left(\bigwedge^{n-1} V\right).$$

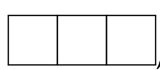
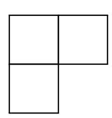
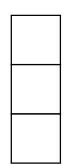
Indeed, it is easy to see that λ appears as the weight of $\Gamma_{a_1,\ldots,a_n}$ in this tensor product and that this is a highest weight vector.

Schur modules. We begin a short digression to introduce Schur modules and eventually to see that they are the irreducible representations of $SL(n)$. The material is taken mainly from [FH13] and [Ful97].

Let d be a positive integer and consider the d-fold tensor product $V^{\otimes d}$. Recall that via (1.1.4) the group $\mathfrak{S}_d$ acts on $V^{\otimes d}$.

Definition 1.3.22. A *partition* of d is a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geq \cdots \geq \lambda_k \geq 0$ and $\lambda_1 + \cdots + \lambda_k = d$. The *Young diagram* of λ is the diagram obtained drawing λ_1 boxes in a row, λ_2 boxes below it and so on, all left justified. The number k is the *length* of the partition. Sometimes we denote d with $|\lambda|$ and we write $\lambda \vdash d$. Only when dealing with partitions the power notation a^b means that a is repeated b times in the sequence.

Example 1.3.23. Consider $d = 3$. Then the possible partitions are (3), $(2, 1)$ and $(1, 1, 1)$. Their Young diagrams are

,  and 

respectively.

Definition 1.3.24. Let λ be a partition together with its diagram. A *Young tableau of shape* λ is the diagram of λ together with a filling of integers. A tableau of shape λ is called *semistandard* if any sequence of integers from left to right is weakly increasing, while from top to bottom any sequence is strictly increasing. We will abbreviate it with just *sstd* tableau. A sstd tableau of shape λ is called *standard* if also the row sequences are strictly increasing and each entry appears once. In this case we will just write *std* tableau.

Example 1.3.25. Let $d = 3$ and $\lambda = (2, 1)$. Then the following are sstd tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,,$$

while these are std tableaux

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Remark 1.3.26. Let d be a positive integer and λ a partition of it. Usually when we write sstd tableaux we pick integers from the set $[\dim(V)] := \{1, \ldots, \dim(V)\}$, while in the std case we pick them from $[d] := \{1, \ldots, d\}$. If T is a std tableau of shape λ, then it induces an action on the factors of the tensor product $V^{\otimes d}$ via (1.1.4). Specifically there are subgroups of R_λ^T, $C_\lambda^T \subset \mathfrak{S}_d$ generated by the permutations which preserves the rows and the columns of T respectively. For example if

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

we have $R_\lambda^T = \{id, (1, 2)\}$ and $C_\lambda^T = \{id, (1, 3)\}$. Sometimes we drop the T in the notation if the choice of the std tableau is clear.

Definition 1.3.27. Let d be a positive integer, λ a partition of d and T a std tableau of shape λ. The endomorphism

$$\begin{aligned} c_\lambda^T : V^{\otimes d} &\longrightarrow V^{\otimes d} \\ v_1 \otimes \cdots \otimes v_d &\longmapsto \sum_{\tau \in C_\lambda} \sum_{\sigma \in R_\lambda} \operatorname{sgn}(\tau) v_{\tau(\sigma(1))} \otimes \cdots \otimes v_{\tau(\sigma(d))} \end{aligned}$$

is called *Young symmetrizer* determined by λ and T. The image

$$\mathbb{S}_\lambda^T V := c_\lambda^T \left(V^{\otimes d} \right)$$

is called *Schur module* determined by λ and T.

Roughly Schur modules are the spaces of tensors obtained from $V^{\otimes d}$ firstly symmetrizing along the rows of T, and then skew-symmetrizing along the columns of T.

Remark 1.3.28. Note that in the literature Schur modules are called *Weyl modules*. As shown in [Wey03, Chapter 2, p.32], given a partition λ one can introduce two modules $L_\lambda V$ and $K_{\lambda'} V$ starting from $V^{\otimes d}$ called *Schur* and *Weyl modules* respectively. The partition λ' is the *conjugate* of λ. However if the base field is of characteristic zero, then these two modules are isomorphic. See [Wey03, Proposition 2.1.18] where it is written in a slightly more general form. Conformally with the notation of [FH13] we call the modules in Definition 1.3.27 Schur modules.

Remark 1.3.29. It is clear by the Definition 1.3.27 that any two Schur modules determined by two different std tableaux of the same shape are isomorphic. Hence we drop the T in the notation and write only $\mathbb{S}_\lambda V$.

Example 1.3.30. Let d be a positive integer and consider the partition $\lambda = (d)$. In this case there is only one std tableau of shape (d)

$$\begin{array}{|c|c|c|} \hline 1 & \dots & d \\ \hline \end{array}\,.$$

The subgroups are $R_\lambda = \mathfrak{S}_d$ and $C_\lambda = \{id\}$. By the Definition 1.3.27 we get that the Young symmetrizer is such that

$$c_{(d)}(v_1 \otimes \cdots \otimes v_d) = \sum_{\sigma \in \mathfrak{S}_d} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

and hence $\mathbb{S}_{(d)} V = \operatorname{Sym}^d V$ is the d-th symmetric power of V.

Example 1.3.31. Let $d \leq \dim(V)$ be a positive integer and consider the partition $\lambda = (1^d)$. In this case there is only one std tableau of shape (1^d)

$$\begin{array}{|c|} \hline 1 \\ \hline \vdots \\ \hline d \\ \hline \end{array}\,.$$

The subgroups are $R_\lambda = \{id\}$ and $C_\lambda = \mathfrak{S}_d$. By Definition 1.3.27 we get that the Young symmetrizer is such that

$$c_{(1^d)}(v_1 \otimes \cdots \otimes v_d) = \sum_{\tau \in \mathfrak{S}_d} \operatorname{sgn}(\tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(d)}$$

and hence $\mathbb{S}_{(1^d)} V = \bigwedge^d V$ is the d-th exterior power of V.

Example 1.3.32. The first instance of a Schur module which is not a symmetric or exterior power is given by $d = 3$ and the partition $\lambda = (2,1)$. In this case we have two std Young tableau of shape λ which are

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}, \quad T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}.$$

Choose the first std tableau. As described in Remark 1.3.26 we have that $R_\lambda^{T_1} = \{id, (1,2)\}$ and $C_\lambda^{T_1} = \{id, (1,3)\}$. If we choose the second tableau, the two subgroups of $\mathfrak{S}_d$ are swapped. The image of the Young symmetrizer $c_\lambda^{T_1}$ is the Schur module $\mathbb{S}_{(2,1)} V$ which is naturally contained in $\bigwedge^2 V \otimes V$.

Remark 1.3.33. Let d be a positive integer, λ a partition of d. By the definition of Young symmetrizer, the Schur module $\mathbb{S}_\lambda V$ is naturally contained in the product of exterior powers

$$\mathbb{S}_\lambda V \subset \bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_h} V. \tag{1.3.1}$$

The sequence $\lambda' := (\lambda'_1, \ldots, \lambda'_h)$ is again a partition of d called *conjugate partition to* λ. Its Young diagram can be obtained transposing the diagram of λ as if it would be a matrix. We denote the product of exterior powers appearing in (1.3.1) with $\bigwedge_{\lambda'} V$.

Now it is time to describe the irreducible representations of $\mathfrak{sl}_n$, and hence of $SL(n)$. We have seen that the irreducible representation $\Gamma_{a_1,\ldots,a_{n-1}}$ with highest weight $a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1}$ is a subspace of

$$\operatorname{Sym}^{a_1} V \otimes \operatorname{Sym}^{a_2}\left(\bigwedge^2 V\right) \otimes \cdots \otimes \operatorname{Sym}^{a_{n-1}}\left(\bigwedge^{n-1} V\right).$$

We have the following link between the $\Gamma_{a_1,\ldots,a_{n-1}}$ and Schur modules.

Proposition 1.3.34 (Prop. 15.14, p. 223 in [FH13])**.** *Let λ be a partition of length less then* $\dim(V)$*. Then the Schur module $\mathbb{S}_\lambda V$ is the irreducible representation $\Gamma_{a_1,\dots,a_{n-1}}$ of $\mathfrak{sl}_n$, where a_i is the number of columns of length i in the diagram of λ.*

Hence Schur modules are irreducible representations of $\mathfrak{sl}_n$ and of $SL(n)$. It is clear that for any sequence $(a_1, \dots, a_{n-1})$ one can find a partition λ such that $\mathbb{S}_\lambda V \simeq \Gamma_{a_1,\dots,a_{n-1}}$.

As already told, the tensor product of irreducible representations is not irreducible in general. However such representation must be completely reducible. In the specific instance of representations of $\mathfrak{sl}_n$, or $SL(n)$ equivalently, the tensor product of two irreducible representations is described by the *Littlewood-Richardson rule* .

Proposition 1.3.35 (Littlewood-Richardson rule)**.** *Let λ and μ be two partitions and consider the Schur modules $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V$. Then*

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \simeq \bigoplus_\nu N_\nu^{\lambda,\mu} \mathbb{S}_\nu V \tag{1.3.2}$$

where the sum runs over the partitions ν of $|\lambda| + |\mu|$ and the integers $N_\nu^{\lambda,\mu}$ are the Littlewood-Richardson coefficients .

For a proof of this proposition we refer to [FH13, Corollary 6.6, p. 78–79]. Roughly it can be seen that such modules are determined by their characters which are *Schur polynomials*. Since the product of two Schur polynomials is a linear combination of Schur polynomials which respects an analogous Littlewood-Richardson rule, see [Mac98, p. 142], one gets for free the decomposition in (1.3.2). Clearly not all Schur modules $\mathbb{S}_\nu V$ with $|\nu| = |\lambda| + |\mu|$ appear in (1.3.2). Notice also that via this rule one gets that the vector space of equivariant morphisms

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \longrightarrow \mathbb{S}_\nu V$$

is of dimension $N_\nu^{\lambda,\mu}$. The coefficients $N_\nu^{\lambda,\mu}$ can be obtained with an algorithmic procedure which is described in Chapter 3 of this document.

A basis for Schur modules. We would like to give a little more explicit insight of the elements of Schur modules. We recall briefly the construction given in [Stu08] of a basis of these spaces in terms of sstd tableaux.

The definition of Schur modules depends on the choice of a std tableau. In the cases of symmetric or exterior powers there is only one std tableau of the given shape. However in general there could be more than one std tableau of the assigned shape. For example if $\lambda = (2,1)$, then there are 2 std tableaux of shape λ

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \quad T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}. \tag{1.3.3}$$

Consider a std tableau T of shape λ, where $\lambda \vdash d$, and let $v_1, \ldots, v_n$ be a basis of V. The couple (T, S), where S is a sstd tableau of the same shape as T, is called *bitableau* and describes an element of $\mathbb{S}_\lambda V$ in the following way. At first consider the element

$$v_{(T,S)} := v_{i_1} \otimes \cdots \otimes v_{i_d} \in V^{\otimes d},$$

where $v_{i_j} = v_k$ if there is a k in the box of S corresponding to the box of T in which a j appears. We may drop the T in $v_{(T,S)}$ if the choice of T is clear. After that, applying the Young symmetrizer c_λ^T to $v_{(T,S)}$ we get an element of $\mathbb{S}_\lambda V$. For example if $\lambda = (2,1)$, consider the bitableau given by the first std tableau in (1.3.3) and the sstd tableau

$$S = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}.$$

Then

$$c_\lambda(v_S) = c_\lambda(v_1 \otimes v_1 \otimes v_2) = 2(v_1 \otimes v_1 \otimes v_2 - v_2 \otimes v_1 \otimes v_1) = 2 \cdot v_1 \wedge v_2 \otimes v_1.$$

We may represent it pictorially as

$$c_\lambda(v_S) = 2 \cdot \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} - \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right) \tag{1.3.4}$$

where in this particular instance the tableau represents tensor products of vectors with the order prescribed by T_1. In the text we use sometimes this notation.
As a consequence of [Stu08, Theorem 4.1.12, p. 142], one has the following result.

Proposition 1.3.36. *The set*

$$\{c_\lambda^T(v_{(T,S)}) \;:\; S \text{ is a semistandard tableau of shape } \lambda\}$$

is a basis of the module $\mathbb{S}_\lambda V$.

Let us see an example on how the choice of a standard tableau affects the construction of $\mathbb{S}_\lambda V$.

Example 1.3.37. Let $\lambda = (2,1)$ and consider the std tableaux

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \; T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and the sstd tableau

$$S = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}.$$

We get that

$$v_{(T_1,S)} = v_1 \otimes v_1 \otimes v_2, \; v_{(T_2,S)} = v_1 \otimes v_2 \otimes v_1.$$

Applying the respective Young symmetrizers we get that

$$c_\lambda^{T_1}(v_{(T_1,S)}) = 2 \cdot (v_1 \otimes v_1 \otimes v_2 - v_2 \otimes v_1 \otimes v_1) = (v_1 \wedge v_2) \otimes v_1,$$
$$c_\lambda^{T_2}(v_{(T_2,S)}) = 2 \cdot (v_1 \otimes v_2 \otimes v_1 - v_2 \otimes v_1 \otimes v_1) = (v_1 \wedge v_2) \otimes v_1.$$

It is clear that the only difference resides in how the two modules embeds in $V^{\otimes 3}$. As elements of $\bigwedge^2 V \otimes V$ the two images are actually the same.

1.4 The geometry of minimal orbits

In this section we recall briefly the geometry behind the representation theory of Lie groups. The material is taken from [FH13], [Ful97] and [Hum12b].

For a given Lie or algebraic group G, we denote with $\mathfrak{g}$ the respective Lie algebra. We start with some basic definitions.

Definition 1.4.1. Let G be an algebraic group and let X be an algebraic variety. The group G *acts on the variety* X if there exists an algebraic map

$$G \times X \longrightarrow X$$

such that $(g, x) \mapsto g \cdot x \in X$ and $(e_G, x) \mapsto e_G \cdot x = x$ for any $g \in G$ and $x \in X$, where e_G is the identity element of G. The variety X is *homogeneous* if the action is transitive, i.e. for any $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$.

Proposition 1.4.2. *Let an algebraic group G act on the variety X. Then each orbit is a smooth locally closed subset of X such that its boundary, i.e. the set-theoretic difference between its closure and itself, is union of orbits of strictly smaller dimension. In particular orbits of minimal dimension are closed.*

For a proof see [Hum12b, First Proposition in §8.3, p. 60]. Clearly in the instance of a homogeneous variety X, one gets only one orbit which is X itself. Note that in general the number of orbits may be infinite.

Proposition 1.4.3 (Chevalley's Theorem)**.** *Let G be an algebraic group and let H be a subgroup. Then there exists a representation $\rho : G \longrightarrow GL(V)$ and a line $L \subset V$ such that*

$$H = \{g \in G : \rho(g)V = V\}.$$

For a proof see [Hum12b, First Theorem in §11.2, p. 80]. As we see in a moment the study of minimal orbits is closely related to special subalgebras of $\mathfrak{g}$ and subgroups of G.

Definition 1.4.4. Let G be a Lie or an algebraic group with Lie algebra $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let R be the set of roots of $\mathfrak{g}$ and assume that after a choice of a real functional on the span of the lattice generated by the roots that $R = R^+ \cup R^-$. The subalgebra

$$\mathfrak{b} := \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

is called *Borel subalgebra*. Any connected subgroup $B \subset G$ whose Lie algebra is $\mathfrak{b}$ is called *Borel subgroup*.

It is easy to see that $\mathfrak{b}$ is a solvable Lie algebra and that it is maximal with this property. Hence one can deduce that Borel subgroups are connected solvable subgroups of G that are not contained in any other connected solvable subgroup except themselves. Note also that since B is not normal in general, the space G/B cannot be regarded as an algebraic group. However it inherits a structure of quasi-projective variety.

Remark 1.4.5. In the case of $G = SL(V)$ one can see that B is the group of upper triangular matrices, i.e. of the automorphisms which preserve a complete flag in V. By virtue of this fact, the quotient G/B can be seen as the set of all the complete flags in V known as *flag variety*.

Theorem 1.4.6 (Borel fixed point theorem)**.** *Let B be a connected solvable algebraic group and let X be a non-empty projective variety on which G acts. Then B has a fixed point of X.*

For a proof we refer to [Hum12b, First Theorem in §21.2, p. 134].

We may want to study more general quotients of an algebraic group G. A subgroup $P \subset G$ such that it contains a Borel subgroup is called *parabolic subgroup*. From the maximality of B, the subgroup P cannot be solvable. Borel subgroups are regarded as minimal parabolic subgroups.

Given an irreducible representation V of G with highest weight vector v, we may consider the projective space $\mathbb{P}(V)$ together with the induced action of G. The orbit $G \cdot [v]$ will be a quasi-projective variety by Proposition 1.4.2. If P is the stabilizer of $[v]$, i.e.

$$P = \{g \in G : \ g \cdot [v] = [v]\} = \{g \in G : \ g \cdot v \subset \langle v \rangle\}$$

the points of $G \cdot [v]$ are in one-to-one correspondence with the cosets of G/P. Since a Borel subgroup B fixes the line spanned by the highest weight vector, we must have that $B \subset P$ and hence P is parabolic. It can be proved the following crucial fact.

Proposition 1.4.7. *Let V be an irreducible representation of G and let v be its highest weight vector up to scalar multiplication. Let P be the parabolic subgroup given as the set of all elements of G fixing $[v] \in \mathbb{P}(V)$. Then the variety $G \cdot [v] \subset \mathbb{P}(V)$ is closed and hence projective.*

We give an idea of the proof for the case $G = SL(V)$. In general the strategy is similar.

We prove at first that the cosets set G/B is closed. Then since any parabolic subgroup P contains a Borel subgroup B, we get a surjective map $G/B \longrightarrow G/P$ which in turn will give us that G/P is closed.

Therefore, let $B \subset G$ be a Borel subgroup of G. We have already observed that a Borel subgroup $B \subset SL(V)$ can be seen as the set of automorphisms of V that preserve a complete flag in V and hence G/B can be seen as the variety of complete flags in V. Therefore G/B can be seen as the orbit of G acting on the fixed flag and hence it is a quasi-projective variety from Proposition 1.4.2. Consider the Borel subgroup B of maximal dimension among all Borel subgroups and consider the complete flag variety G/B. The orbit of any element of G/B can be realized as a quotient G/R where R denotes the stabilizer of that point. Since R preserves a flag, its elements can be written as upper triangular matrices and hence it is solvable. Since the dimension of R is less then the dimension of B, we get the inequality $\dim G/B \leq \dim G/R$ from which we get the minimality on the dimension. By Proposition 1.4.2 we get that G/B is closed. To prove that also G/P is closed it is enough to consider the surjective map $G/B \longrightarrow G/P$ and the fact that G/B is closed. As a corollary one gets that all Borel subgroups are conjugated and have the same dimension.

The previous construction has a description also in terms of Lie algebras. Indeed given a parabolic subgroup $P \subset G$, its Lie algebra $\mathfrak{p} \subset \mathfrak{g}$, called *parabolic subalgebra*, must contain the Borel subalgebra $\mathfrak{b}$ and up to conjugacy it can be written as

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in T} \mathfrak{g}_\alpha$$

where T is a set of roots that contains R^+. Since $\mathfrak{p}$ is a subalgebra, the set T must be closed under addition. Hence it can be seen that T must be generated by R^+ and the negative of the roots of a subset Σ of the set of simple roots. Therefore all parabolic subalgebras can be obtained in this way and are in one-to-one correspondence with subsets Σ of the set of simple roots. The same holds for the respective parabolic subgroups $P \subset G$ up to conjugacy. From this one gets a straightforward classification of all the rational homogeneous varieties G/P determined by subsets of the set of simple roots. For more details see [FH13, 382–395].

Remark 1.4.8. If $G = SL(V)$, up to conjugacy the parabolic subgroups are the ones given by block diagonal matrices and the varieties G/P are in general *partial* or complete flag varieties. For instance, if $G = SL(3)$, then B is the subgroup upper triangular matrices. A parabolic subgroup is

$$P = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

In this instance the variety G/P is the Grassmann variety, as a particular case of a flag variety, of two dimensional planes in $V \simeq \mathbb{C}^3$, i.e. $G/P \simeq (\mathbb{P}^2)^*$.

1.5 Varieties parametrizing tensors (reprise)

We give in this section an explicit description of the tensors parametrized by flag varieties. The main reference is [Ful97].

In Section 1.4 we have seen that there exists a classification of the rational homogeneous varieties of the group $G = SL(n)$. We give a better description of them here in terms of tensors. At first we need the following fact.

Lemma 1.5.1. *Let V be an n-dimensional vector space and λ a partition of length strictly less than n. Then up to scalar multiplication, the highest weight vector of the representation $\mathbb{S}_\lambda V$ is the image via the Young symmetrizer of the element associated to the sstd tableau $U(\lambda)$ of shape λ whose i-th row is filled with i's.*

For a proof see [Ful97, Lemma 4, p. 113]. We can consider then the orbit $G \cdot v$, where v is the highest weight vector of $\mathbb{S}_\lambda V$. As we have seen in the previous section, this orbit is isomorphic to the quotient G/P with P parabolic subgroup of G and hence it is a rational homogeneous variety.

If $\lambda = (\lambda_1^{a_1}, \ldots, \lambda_k^{a_k})$, where $\lambda_1 > \cdots > \lambda_k > 0$ and $a_1 + \cdots + a_k < \dim(V)$, the Flag variety embedded in $\mathbb{P}(\mathbb{S}_\lambda V)$ is

$$X = \mathbb{F}(n_1, \ldots, n_k; V) := \{(V_1, \ldots, V_k) : V_1 \subset \cdots \subset V_k \subset V,\ \dim(V_i) = n_i\}$$

embedded with $\mathcal{O}(d_1, \ldots, d_k)$. The integers k_i and d_i are such that

$$n_i = \sum_{j=1}^{i} a_j \quad \text{and} \quad d_i = \lambda_i - \lambda_{i+1}$$

setting $\lambda_{k+1} = 0$, for $i = 1, \ldots, k$. In other words, we have computed the conjugate partition to λ

$$\lambda' = (n_k^{d_k}, \ldots, n_1^{d_1}).$$

The points of X-rank 1 look like

$$(v_1 \wedge \cdots \wedge v_{n_k})^{\otimes d_k} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1}$$

representing the flag

$$\langle v_1, \ldots, v_{n_1} \rangle \subset \cdots \subset \langle v_1, \ldots, v_{n_k} \rangle.$$

In this case we refer to the X-rank as *λ-rank* to underline the module $\mathbb{S}_\lambda V$ we are dealing with. Note that the notation of the tensor and of the flag is inverted. We prefer this one since in this case the n_i's appear in the same order as they appear in λ'. Note also that in this familiy there are included also the Veronese and Grassmann varieties. In general we have this diagram of inclusions

$$\begin{array}{ccccc}
\mathbb{F}(n_1,\ldots,n_k) & \subset & \prod_{i=1}^k \mathbb{G}(n_{k-i+1},V) & \subset & \prod_{i=1}^k \mathbb{P}(\bigwedge^{n_{k-i+1}} V) \\
 & & & & \cap \\
\cap & & & & \prod_{i=1}^k \mathbb{P}(\mathrm{Sym}^{a_{k-i+1}}(\bigwedge^{n_{k-i+1}} V)) \\
 & & & & \cap \\
\mathbb{P}(\mathbb{S}_\lambda V) & & \subset & & \mathbb{P}\left(\bigotimes_{i=1}^k \mathrm{Sym}^{a_{k-i+1}}(\bigwedge^{n_{k-i+1}} V)\right)
\end{array}$$

where in rightmost column we have inclusions given by the composition of Veronese and Segre embeddings. The bottom inclusion follows naturally by the definition of Schur module. However it will be clearer in the third chapter in which we introduce the *Plücker relations*, cf. (3.1.2).

Remark 1.5.2. Rational homogeneous varieties are smooth, irreducible and non degenerate since any G-orbit in an irreducible representation of G spans V. Hence it makes sense to talk about the λ-rank of a point in $\mathbb{P}(\mathbb{S}_\lambda V)$.

A complete classification of the defective secant varieties of rational homogeneous varieties is missing except some particular cases. The most updated results about it can be found in [BDDG07] where the authors developed an algorithm to compute their secant dimensions. See also [BD04] for the secant dimensions of the *adjoint* varities, i.e. minimal orbits inside the projectivization of the adjoint representation, and [BD10] for secant dimensions of low-dimensional homogeneous varieties.

We conclude this section noting that these are not the only examples of tensors parametrized by varieties. For instance there exists also *partially symmetric tensors* whose parametrizing varieties are the *Segre-Veronese* varieties, i.e. images of the Segre-Veronese embedding of products of projective spaces. For a general treatment about tensors see [BCC+18].

Chapter 2

Apolarities

This chapter is devoted to the introduction of the known apolarity theories. They regards symmetric tensors, skew-symmetric tensors and tensors arising from other irreducible representations of the special linear group.

2.1 Symmetric apolarity theory

In this section we recall the symmetric apolarity theory for symmetric tensors. Sometimes we refer to it as the *classical* apolarity theory since it has been the first one to be developed. The main reference is [IK99] and [Ger96].

Let V be a vector space of dimension n over $\mathbb{K}$. Recall the isomorphism $\operatorname{Sym}^d V \simeq \mathbb{K}[x_1, \dots, x_n]_d$ described in (1.1.8). By virtue of this, we may confuse the terms symmetric tensors and homogeneous polynomials. Clearly we have an analogous isomorphism $\operatorname{Sym}^e V^* \simeq \mathbb{K}[\partial_1, \dots, \partial_n]_e$ for the dual vector space. The action of V^* on V is translated in terms of the x_i and ∂_j as $\partial_i(x_j) = \delta_{i,j}$.

Definition 2.1.1. Given a vector space V over $\mathbb{K}$ of dimension n there exists an equivariant map called *apolarity action*

$$\circ : \operatorname{Sym}^\bullet V \otimes \operatorname{Sym}^\bullet V^* \longrightarrow \operatorname{Sym}^\bullet V$$

such that when restricted to the tensor product $\operatorname{Sym}^d V \otimes \operatorname{Sym}^e V^*$ it is the zero map if $e > d$, otherwise with a slight abuse of notation it is the map

$$\circ : \operatorname{Sym}^d V \otimes \operatorname{Sym}^e V^* \longrightarrow \operatorname{Sym}^{d-e} V$$

defined in the following way. Given the monomials $x^{\underline{\alpha}} = x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \operatorname{Sym}^d V$ and $\partial^{\underline{\beta}} = \partial_1^{\beta_1} \dots \partial_n^{\beta_n} \in \operatorname{Sym}^e V^*$, we have that

$$\circ(x^{\underline{\alpha}} \otimes \partial^{\underline{\beta}}) := \begin{cases} \prod_{i=1}^n \frac{(\alpha_i)!}{(\alpha_i - \beta_i)!} x^{\underline{\alpha} - \underline{\beta}} & \text{if } \beta_i \leq \alpha_i \text{ for any } i, \\ 0 & \text{otherwise} \end{cases}$$

and is extended by linearity. We denote the image $\circ(x^{\underline{\alpha}} \otimes \partial^{\underline{\beta}})$ with $\partial^{\underline{\beta}} \circ x^{\underline{\alpha}}$ for simplicity.

Remark 2.1.2. There is an equivalent definition of the apolarity action. When it is restricted to $\operatorname{Sym}^d V \otimes \operatorname{Sym}^e V^*$ with $e < d$, it can be defined as the composition of maps

$$\operatorname{Sym}^d V \otimes \operatorname{Sym}^e V^* \longrightarrow \operatorname{Sym}^{d-e} V \otimes \operatorname{Sym}^e V \otimes \operatorname{Sym}^e V^* \longrightarrow \operatorname{Sym}^{d-e} V$$

where the first map is the product of the comultiplication map (1.1.7) of symmetric powers with the identity on $\operatorname{Sym}^e V^*$. The second map is the product of the identity map on $\operatorname{Sym}^{d-e} V$ with the pairing $\operatorname{Sym}^e V \otimes \operatorname{Sym}^e V^* \longrightarrow \mathbb{K}$.

Definition 2.1.3. Given a homogeneous polynomial $f \in \operatorname{Sym}^d V$, for any $1 \leq e \leq d$ the linear map induced by the apolarity action

$$\mathcal{C}_f^{d,e} : \operatorname{Sym}^e V^* \longrightarrow \operatorname{Sym}^{d-e} V$$

defined as $\mathcal{C}_f^{d,e}(\partial) := \partial \circ f$ for any $\partial \in \operatorname{Sym}^e V^*$ is called *catalecticant map* . Fixed an element $f \in \operatorname{Sym}^d V$, the set

$$f^\perp := \bigoplus_{e \geq 1} \ker \mathcal{C}_f^{d,e}$$

is called *apolar set to f*.

It can be proven that the apolar set $f^\perp$ is actually an ideal inside $\operatorname{Sym}^\bullet V^*$. Hence we will refer to it as the *apolar ideal*. Note that for any $e > d$ it follows by definition that $(f^\perp)_e = \operatorname{Sym}^e V^*$.

Remark 2.1.4. Recall that homogeneous polynomials can be decomposed via Waring decompositions into sums of powers of linear forms, see section 1.2. The minimal number of such linear forms is called symmetric rank. It can be seen that catalecticant maps can give information about the symmetric rank of a symmetric tensor. Indeed if $f \in \operatorname{Sym}^d V$, then

$$f = l^d \text{ if and only if } \operatorname{rk} \mathcal{C}_f^{d,e} = 1$$

for any $1 \leq e \leq d$ and some $l \in V$. Clearly if $f = l^d$, then since catalecticant maps act as derivations, we get that the image of $\mathcal{C}_f^{d,e}$ is spanned by l^{d-e} and hence it is one dimensional. On the contrary assume that $\operatorname{rk} \mathcal{C}_f^{d,e} = 1$ and consider the one dimensional generator of the annihilator of $\ker \mathcal{C}_f^{d,e}$ in $\operatorname{Sym}^d V$. If $e = 1$ we can assume that such a generator is a linear form l. For $e > 1$ it is easy to see that the generator of the annihilator of the kernel must be l^e. From this it follows directly that f must be equal to l^d.

We have the following fact borrowed from [OR20] characterizing the apolar ideal.

Lemma 2.1.5. *Let $f \in \operatorname{Sym}^d V$. The apolar ideal $f^\perp$ is determined by its socle, i.e. it holds*

$$(f^\perp)_e = \left[(f^\perp)_d : I^{d-e}\right]_e := \left\{g \in \operatorname{Sym}^e V^* \ : \ \forall h \in I^{d-e},\ (gh) \circ f = 0\right\}$$

where $I := (\partial_1, \ldots, \partial_n) \subset \operatorname{Sym}^\bullet V^*$ is the irrelevant ideal of the symmetric algebra.

Proof. We have to prove the two inclusions. Let $g \in (f^\perp)_e$. Then for any $h \in I^{d-e}$ it is obvious that $gh \circ f = h \circ (g \circ f) = 0$ and hence g belongs to the set on the right hand side.
On the contrary assume that $g \in \operatorname{Sym}^e V^*$ is such that $(gh) \circ f = h \circ (g \circ f) = 0$ for any $h \in I^{d-e}$. This must happen also for any monomial $h = \partial^{\underline{\alpha}} \in I^{d-e}$ with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + \cdots + \alpha_n = d - e$. This means that the coefficients of the polynomial $g \circ f$ are all equal to zero, i.e. $g \in (f^\perp)_e$. This concludes the proof. $\square$

Remark 2.1.6. Consider a power l^d of a linear form $l \in V$. As a point $p = [l]$ of $\mathbb{P}(V)$, we can consider its *ideal* $I(p)$. We will consider the ideal $I(p)$ also when dealing with the point $[l^d]$ of the Veronese variety. For instance if $V = \mathbb{C}^3$ and $l = x_1 - x_3$, the respective point is $p = [1 : 0 : -1]$ and its ideal will be $I(p) = (\partial_2, \partial_1 + \partial_3)$.

Remark 2.1.7. Consider a homogeneous polynomial of symmetric rank 1, i.e. $l^d \in \operatorname{Sym}^d V$. By Remark 2.1.6 we can consider its associated ideal $I(p)$. It can be proven that $I(p)_d = (l^d)^{\perp}_d$. Indeed let ∂_l be the dual element to l, i.e. $\partial_l \circ l^d = d \cdot l^{d-1}$, and fix a basis of V^* in which ∂_l appears between the generators and any other element of such a basis kills l. Then it is easy to see that $(l^d)^{\perp}_d$ is the hyperplane inside $\operatorname{Sym}^d V^*$ generated by all monomials of degree d different from ∂_l^d. On the other hand with this choice of coordinates, up to an order of the chosen basis of V the associated point p to l^d has coordinates $[1 : 0 : \cdots : 0]$. Hence its ideal $I(p)$ is generated by all linear elements different from ∂_l. From this it follows directly that $I(p)_d = (l^d)^{\perp}$. From the fact that $\operatorname{Sym}^d V \times \operatorname{Sym}^d V^* \longrightarrow \mathbb{K}$ is a perfect pairing, it follows that $I(p)_d$ can be seen as the space of homogeneous polynomials of degree d that vanish on p.

The following is a central result in the apolarity theory of symmetric tensors.

Theorem 2.1.8 (Apolarity Lemma, Lemma 1.15 in [IK99]). *Let $Z = \{p_1, \dots, p_r\} \subset \mathbb{P}(V)$ be a subscheme of closed reduced points whose associated linear forms, in the sense of Remark 2.1.6, are $l_1, \dots, l_r \in V$. Then for any $f \in \operatorname{Sym}^d V$,*

$$I(p_1, \dots, p_r) \subset f^{\perp} \text{ if and only if } f = c_1 l_1^d + \cdots + c_r l_r^d,$$

where $I(p_1, \dots, p_r)$ denotes the of the ideal of the scheme Z, for some $c_i \in \mathbb{K}$.

Proof. The proof we present is the proof found in [OR20, Theorem 6.8]. Let us prove at first that

$$f \in \langle l_1^d, \dots, l_r^d \rangle \text{ if and only if } I(p_1, \dots, p_r)_d \subset (f^{\perp})_d.$$

The element $f \in \operatorname{Sym}^d V$ is in $\langle l_1^d, \dots, l_r^d \rangle$ if and only if every linear form on $\mathbb{P}(\operatorname{Sym}^d V)$ that vanishes on $\langle l_1^d, \dots, l_r^d \rangle$, vanishes also on f. This is equivalent to say that every homogeneous polynomial of degree d that vanishes on $\langle l_1^d, \dots, l_r^d \rangle$, it vanishes also on f. This proves that $f \in \langle l_1^d, \dots, l_r^d \rangle$ is equivalent to the condition $I(p_1, \dots, p_r)_d \subset (f^{\perp})_d$. Now we have to prove that $I(p_1, \dots, p_r)_d \subset (f^{\perp})_d$ holds if and only if $I(p_1, \dots, p_r) \subset f^{\perp}$. Clearly this latter statement implies the former. On the contrary assume that the inclusion in degree d holds. Then

$$I(p_1, \dots, p_r)_e \subset \left[I(p_1, \dots, p_r)_d : I^{d-e} \right]_e \subset \left[(f^{\perp})_d : I^{d-e} \right]_e = (f^{\perp})_e$$

for any $e \geq 1$, where I is the irrelevant ideal of the symmetric algebra. The last equality follows from Lemma 2.1.5, while the inclusion in the centre follows from the hypothesis. Hence $I(p_1, \dots, p_r) \subset f^{\perp}$ and this concludes the proof. $\square$

Example 2.1.9. The usefulness of the Theorem 2.1.8 is clear. For instance consider $V = \mathbb{K}^2$ and the homogeneous polynomial $f = x_1^2 x_2$. One can see that rank of the catalecticant map $\mathcal{C}_f^{3,1}$ is equal to 2 and hence f must have symmetric rank at least equal to 2. Moreover one can check the equalities

$$(f^\perp)_1 = \langle 0 \rangle,\ (f^\perp)_2 = \langle \partial_2^2 \rangle \text{ and } (f^\perp)_3 = \langle \partial_1^3, \partial_1\partial_2^2, \partial_2^3 \rangle.$$

Consider the three linear forms $l_1 = x_1 + x_2$, $l_2 = x_1 - x_2$ and $l_3 = x_2$. The respective points in $\mathbb{P}^1$ are $p_1 = [1 : 1]$, $p_2 = [1 : -1]$ and $p_3 = [0 : 1]$ and the ideal of their union is $I(p_1, p_2, p_3) = (\partial_1^3 - \partial_1\partial_2^2)$. Hence we have the inclusion $I(p_1, p_2, p_3) \subset f^\perp$ and by Theorem 2.1.8 we get that f can be decomposed as a sum of 3 powers of linear forms. Solving a simple linear system one gets that

$$x^2 y = \frac{1}{6}(x_1 + x_2)^3 + \frac{1}{6}(x_1 - x_2)^3 - \frac{1}{3}x_2^3.$$

Remark 2.1.10. Two remarks has to be made. At first note that the Theorem 2.1.8 does not give in general the minimal decomposition of a symmetric tensor. However the ideal of the points associated to such minimal decomposition must be contained in the apolar ideal. Moreover the Theorem 2.1.8 does not give directly information about *how many* Waring decompositions of a given symmetric tensor exist. If a symmetric tensor admits a unique Waring decomposition it will be called *identifiable*. Note that the ideal of the points associated to different Waring decompositions of an unidentifiable symmetric tensor must all be contained in the apolar ideal. See [COV17] for identifiability problem of symmetric tensors.

2.2 Skew-symmetric apolarity theory

In this section we recall the skew-symmetric apolarity theory for skew-symmetric tensors. The main reference is [ABMM21]. In this section we assume that $d \leq n - d$, i.e. $2d \leq n$.

Definition 2.2.1. Let V be a vector space of dimension n over $\mathbb{K}$. The *skew-symmetric apolarity action* is the equivariant map

$$\lrcorner : \bigwedge^\bullet V \otimes \bigwedge^\bullet V^* \longrightarrow \bigwedge^\bullet V$$

such that when restricted to $\bigwedge^d V \otimes \bigwedge^e V^*$ it is zero map if $e > d$, otherwise it is defined in the following way. With a slight abuse of notation, for any $v_1 \wedge \cdots \wedge v_d \in \bigwedge^d V$ and any $\alpha_1 \wedge \cdots \wedge \alpha_e \in \bigwedge^e V^*$

$$\lrcorner(v_1 \wedge \cdots \wedge v_d \otimes \alpha_1 \wedge \cdots \wedge \alpha_e) := \sum_{R \subset \{1,\ldots,d\}} \operatorname{sgn}(R) \cdot \det(\alpha_i(v_j))_{(i=1,\ldots,e,\ j \in R)} \cdot v_{\overline{R}},$$

where the sum runs over all possible ordered subsets $R \subset \{1, \ldots, d\}$ of cardinality e and the symbol $\operatorname{sgn}(R)$ denotes the sign of the permutation that sends the list $(1, \ldots, k)$ to the list in which the integers of R appears first, preserving the order. The set $\overline{R}$ is the complementary set to R in $\{1, \ldots, k\}$ and $v_{\overline{R}}$ denotes the wedge product of the vector v_i's not chosen by R. We denote the image $\lrcorner(v_1 \wedge \cdots \wedge v_d \otimes \alpha_1 \wedge \cdots \wedge \alpha_e)$ with $\alpha_1 \wedge \cdots \wedge \alpha_e \lrcorner v_1 \wedge \cdots \wedge v_d$ for simplicity.

Remark 2.2.2. There is an equivalent definition of the skew-symmetric apolarity action. When it is restricted to $\bigwedge^d V \otimes \bigwedge^e V^*$ with $e < d$, it can be defined as the composition of maps

$$\bigwedge^d V \otimes \bigwedge^e V^* \longrightarrow \bigwedge^{d-e} V \otimes \bigwedge^e V \otimes \bigwedge^e V^* \longrightarrow \bigwedge^{d-e} V$$

where the first map is the product of the comultiplication map of exterior powers (1.1.13) with the identity on $\bigwedge^e V^*$. The second map is the product of the identity map on $\bigwedge^{d-e} V$ with the pairing $\bigwedge^e V \otimes \bigwedge^e V^* \longrightarrow \mathbb{K}$.

Definition 2.2.3. Given an element $f \in \bigwedge^k V$, the linear map induced by the skew-symmetric apolarity action

$$\mathcal{C}_f^{\wedge,d,e} : \bigwedge^e V^* \longrightarrow \bigwedge^{d-e} V$$

defined as $\mathcal{C}^{\wedge,d,e}(\alpha_1 \wedge \cdots \wedge \alpha_e) := \alpha_1 \wedge \cdots \wedge \alpha_e \lrcorner v_1 \wedge \cdots \wedge v_d$ is called *catalecticant map of skew-symmetric apolarity theory* for any $1 \leq e \leq d$. Fixed an element $f \in \bigwedge^k V$, the set

$$f^{\perp} := \bigoplus_{e \geq 1} \ker \mathcal{C}_f^{\wedge,d,e},$$

is called *apolar set to* f.

It can be proven that the apolar set $f^{\perp}$ is actually an ideal inside $\bigwedge^{\bullet} V^*$. Hence we will refer to it as the *apolar ideal*. Note that for any $d < e \leq n$ it follows by definition that $(f^{\perp})_e = \bigwedge^e V^*$.

Remark 2.2.4. In a similar fashion to the apolarity action for symmetric tensors, we would like to show that the skew-symmetric apolarity theory can extract information about the skew-symmetric rank. Indeed if $f \in \bigwedge^d V$, then

$$f = v_1 \wedge \cdots \wedge v_d \text{ if and only if } \operatorname{rk}(\mathcal{C}_f^{\wedge,d,1}) = d$$

Clearly if $f = v_1 \wedge \cdots \wedge v_d$, one can check directly the rank of the skew-catalecticant map. On the other hand assume that $\operatorname{rk}(\mathcal{C}_f^{\wedge,d,1}) = d$. This means that the $\dim \ker \mathcal{C}_f^{\wedge,d,1} = n - d$. Hence we can find $n - d$ independent linear forms $\alpha_1, \dots, \alpha_{n-d} \in V^*$ that vanish on f via the skew-symmetric apolarity action. By the definition of skew-symmetric apolarity action, from the condition

$$\alpha_i \lrcorner f = 0$$

for any $i = 1, \dots, n - d$, we can conclude that $f \in \bigwedge^k V_i$ where V_i is the subspace of V given by vectors v such that $\alpha_i(v) = 0$. Hence we get that

$$f \in \bigwedge^d (V_1 \cap \cdots \cap V_{n-d})$$

and since the space on the right is d-dimensional, we get the thesis up to a constant.

Lemma 2.2.5. *Let $f \in \bigwedge^d V$. The apolar ideal $f^\perp$ is determined by its socle, i.e. it holds*

$$(f^\perp)_e = \left[(f^\perp)_d : J^{d-e}\right]_e = \left\{g \in \bigwedge^e V^* \ : \ \forall h \in J^{d-e},\ (g \wedge h) \lrcorner f = 0\right\}$$

where $J := (\alpha_1, \dots, \alpha_n) \subset \bigwedge^\bullet V^*$ is the irrelevant ideal of the exterior algebra.

Proof. Let $g \in (f^\perp)_e$. Then, for any $h \in J$ we have that $(g \wedge h) \lrcorner f = \varepsilon \cdot h \lrcorner (g \lrcorner f) = 0$, where $\varepsilon \in \{-1, +1\}$ is the sign due to the exchange of g and h. The reason why this equality on the contraction holds is described in the next Remark 2.2.6. Hence g belongs to the set on the right.
On the contrary assume that $g \in \bigwedge^e V^*$ is such that $(g \wedge h) \lrcorner f = h \lrcorner (g \lrcorner f) = 0$. Then this happens also for any $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_e}$, where $i_1, \dots, i_e \in \{1, \dots, n\}$ are all distinct. Consequently the element $(g \lrcorner f) \in \bigwedge^{d-e} V$ must be the zero element, i.e. $g \in (f^\perp)_e$. This concludes the proof. $\square$

Remark 2.2.6. In the proof of Lemma 2.2.5 we have used the fact that $(g \wedge h) \lrcorner f = \varepsilon \cdot h \lrcorner (g \lrcorner f)$. This equality is equivalent to the following fact.

Proposition 2.2.7. *Consider the composition*

$$\bigwedge^a V^* \otimes \bigwedge^b V^* \otimes \bigwedge^{a+b} V \longrightarrow \bigwedge^{a+b} V^* \otimes \bigwedge^{a+b} V \longrightarrow \mathbb{K}, \tag{2.2.1}$$

where the first map is the product of a multiplication map of exterior powers (1.1.12) *and the identity on* $\wedge^{a+b}V$*, while the second is the usual pairing of exterior powers. Consider then this second composition*

$$\bigwedge^a V^* \otimes \bigwedge^b V^* \otimes \bigwedge^{a+b} V \longrightarrow \bigwedge^b V^* \otimes \bigwedge^b V \longrightarrow \mathbb{K}, \tag{2.2.2}$$

up to a sign, where the first map is the product of the skew-symmetric apolarity action restricted to $\wedge^a V^* \otimes \wedge^{a+b} V$ *and the identity on* $\wedge^b V$*, while the last map is again the usual pairing of exterior powers. Then the maps* (2.2.1)*,* (2.2.2) *and* (2.2.2) *with a and b exchanged are the same up to a sign, for any a and b such that* $1 \leq 2a, 2b \leq n$*.*

Proof. We prove the equality of the maps (2.2.1) and (2.2.2) on decomposable elements. The general statement will follow from linearity. Consider the elements $\alpha_1 \wedge \cdots \wedge \alpha_a \in \bigwedge^a V^*$, $\alpha_{a+1} \wedge \cdots \wedge \alpha_{a+b} \in \bigwedge^b V^*$ and $v_1 \wedge \cdots \wedge v_{a+b} \in \bigwedge^{a+b} V$. The map (2.2.1) acts as

$$\begin{gathered}
(\alpha_1 \wedge \cdots \wedge \alpha_a) \otimes (\alpha_{a+1} \wedge \cdots \wedge \alpha_{a+b}) \otimes (v_1 \wedge \cdots \wedge v_{a+b}) \\
\downarrow \\
(\alpha_1 \wedge \cdots \wedge \alpha_a \wedge \alpha_{a+1} \wedge \cdots \wedge \alpha_{a+b}) \otimes (v_1 \wedge \cdots \wedge v_{a+b}) \\
\downarrow \\
\det \left(\alpha_i(v_j)\right)_{i,j=1,\ldots,a+b} .
\end{gathered} \tag{2.2.3}$$

Call H the matrix $\left(\alpha_i(v_j)\right)_{i,j=1,\ldots,a+b}$.

On the other hand let us study the map (2.2.2). We apply the skew-symmetric apolarity action as described by the Remark 2.2.2. We get that

$$\begin{gathered}
(\alpha_1 \wedge \cdots \wedge \alpha_a) \otimes (\alpha_{a+1} \wedge \cdots \wedge \alpha_{a+b}) \otimes (v_1 \wedge \cdots \wedge v_{a+b}) \\
\downarrow \\
(\alpha_{a+1} \wedge \cdots \wedge \alpha_{a+b}) \otimes (\sum_{\sigma \in \mathfrak{S}^{a,b}_{a+b}} \operatorname{sign}(\sigma) \cdot \det(\alpha_i(h_{\sigma(j)}))_{i,j=1,\ldots,a}) \cdot \left(v_{\sigma(a+1)} \wedge \cdots \wedge v_{\sigma(a+b)}\right) \\
\downarrow
\end{gathered}$$

$$\sum_{\sigma \in \mathfrak{S}_{a+b}^{a,b}} \operatorname{sign}(\sigma) \cdot \det(\alpha_i(h_{\sigma(j)}))_{i,j=1,\dots,a} \cdot \det(\alpha_k(h_{\sigma(l)}))_{k,l=a+1,\dots,a+b}, \tag{2.2.4}$$

where $\mathfrak{S}_{a+b}^{a,b}$ is the subgroup of $\mathfrak{S}_{a+b}$ involved in the description of the maps (1.1.13). Hence one has only to prove that (2.2.3) and (2.2.4) are the same. Remark that if $a = 1$, then (2.2.4) is the Laplace expansion of (2.2.3), so in this case the statement is true. If $a > 1$, then (2.2.4) is the generalized Laplace expansion by complementary minors of (2.2.3). The equality between (2.2.3) and (2.2.4) with a and b swapped follows with the same argument up to a sign. This concludes the proof. □

Example 2.2.8 (Generalized Laplace expansion of a matrix)**.** We illustrate here with an example how the generalized Laplace expansion of a matrix works. Note that the similarities with the contraction on the wedge products we are considering.

Suppose we want to compute the determinant of the matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}.$$

Hence in this instance we have $a = b = 2$. Choose the first two rows to perform the generalized Laplace expansion. At first note that there are six ways of picking two different columns up to the order. Consider the choice of columns (i, j), $1 \leq i < j \leq 4$. This corresponds to consider the minor

$$\begin{pmatrix} a_{1,i} & a_{1,j} \\ a_{2,i} & a_{2,j} \end{pmatrix}$$

and the complementary cofactor

$$\begin{pmatrix} a_{3,k} & a_{3,k} \\ a_{4,l} & a_{4,l} \end{pmatrix}$$

where $k \neq i$ and $l \neq j$. For instance if $(i, j) = (1, 3)$, then pictorially we have

$$A = \begin{pmatrix} \color{blue}{a_{1,1}} & a_{1,2} & \color{blue}{a_{1,3}} & a_{1,4} \\ \color{blue}{a_{2,1}} & a_{2,2} & \color{blue}{a_{2,3}} & a_{2,4} \\ a_{3,1} & \color{red}{a_{3,2}} & a_{3,3} & \color{red}{a_{3,4}} \\ a_{4,1} & \color{red}{a_{4,2}} & a_{4,3} & \color{red}{a_{4,4}} \end{pmatrix}.$$

Then the formula is

$$\det(A) = \sum_{1\le i<j\le 4} \varepsilon_{i,j,k,l} \cdot \det\begin{pmatrix} a_{1,i} & a_{1,j} \\ a_{2,i} & a_{2,j}\end{pmatrix} \cdot \det\begin{pmatrix} a_{3,k} & a_{3,k} \\ a_{4,l} & a_{4,l}\end{pmatrix},$$

where $\varepsilon_{i,j,k,l}$ is the sign of the permutation that sends the sequence $(1,2,3,4)$ in the sequence (i,j,k,l).

Definition 2.2.9. Let V be a vector space of dimension n over $\mathbb{K}$. Let $p = [v_1 \wedge \cdots \wedge v_d] \in \mathbb{G}(d,V)$ be a point of skew-symmetric rank 1. The up to scalar multiplication the point p represents a vector subspace $W \subset V$ of dimension d generated by $v_1,\dots,v_d$. Using duality, W is cut out by $n-d$ linear forms $\alpha_1,\dots,\alpha_{n-d} \in V^*$. The *ideal of p* is defined as

$$I^{\wedge}(p) := (\alpha_1,\dots,\alpha_{n-d}) \subset \bigwedge\nolimits^{\bullet} V^*.$$

For any $p_1,\dots,p_r \in \mathbb{G}(d,V)$, their ideal is defined as

$$I^{\wedge}(p_1,\dots,p_r) := \bigcap_{i=1}^{r} I^{\wedge}(p_i).$$

Remark 2.2.10. Consider a skew-symmetric tensor of skew-symmetric rank 1, i.e. $p = v_1 \wedge \cdots \wedge v_d \in \bigwedge^d V$. By Definition 2.2.9 we can consider its associated ideal $I^{\wedge}(p)$. It can be proven that $I^{\wedge}(p)_d = (p)^{\perp}_d$. Indeed consider the basis $\{v_1,\dots,v_d\}$ of the subspace associated to p and extend it to a basis $\{v_1,\dots,v_n\}$ of V. Let $\{\alpha_1,\dots,\alpha_n\}$ be the dual basis of V^*. Then it is easy to see that $(p)^{\perp}_d$ is the hyperplane generated by all elements $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_d}$ with $i_1,\dots,i_d \in \{1,\dots,n\}$ and different from $\alpha_1 \wedge \cdots \wedge \alpha_d$. In particular note that between the integers $i_1,\dots,i_d$ there must be at least one of them picked from the set $\{d+1,\dots,n\}$. On the other hand by Definition 2.2.9 we have that $I^{\wedge}(p) = (\alpha_{d+1},\dots,\alpha_n)$. From this it is clear that one must have that $I^{\wedge}(p)_d = (p)^{\perp}$. Via the perfect pairing $\bigwedge^d V \otimes \bigwedge^d V^* \longrightarrow \mathbb{K}$, this allows us to think $I^{\wedge}(p)_d$ as the space of linear forms in $\bigwedge^d V^*$ cutting out the space $\langle p \rangle$.

The following is an analogous result to Theorem 2.1.8 in the context of skew-symmetric tensors.

Theorem 2.2.11 (Skew-symmetric apolarity Lemma, Lemma 12 at p. 6 in [ABMM21])**.** *Let $p_1,\dots,p_r \in \mathbb{G}(d,V)$ be points of skew-symmetric rank 1. Then for any $t \in \bigwedge^d V$*

$$I^{\wedge}(p_1,\dots,p_r) \subset t^{\perp} \textit{ if and only if } t = c_1p_1 + \cdots + c_rp_r$$

for some $c_i \in \mathbb{K}$.

Proof. We give a different proof from the one in [ABMM21]. This one will the follow the idea of the proof the Theorem 2.1.8.
At first note that $t \in \langle p_1,\dots,p_r\rangle$ if and only if every element of $\wedge^d V^*$ vanishing on $\langle p_1,\dots,p_r\rangle$ via the skew-symmetric apolarity action vanishes also on t. Hence we get the equivalence

$$t \in \langle p_1,\dots,p_r\rangle \text{ if and only if } I^{\wedge}(p_1,\dots,p_r)_d \subset (t^{\perp})_d.$$

Let us prove now that the inclusion $I^{\wedge}(p_1,\dots,p_r)_d \subset (t^{\perp})_d$ is equivalent to the inclusion $I^{\wedge}(p_1,\dots,p_r) \subset t^{\perp}$. Clearly the latter implies the former. On the contrary assume that the inclusion in degree d holds. Then

$$I^{\wedge}(p_1,\dots,p_r)_e \subset \left[I^{\wedge}(p_1,\dots,p_r)_d : J^{d-e}\right]_e \subset \left[(t^{\perp})_d : J^{d-e}\right]_e = (t^{\perp})_e$$

for any $1 \le e \le n$, where J is the irrelevant ideal of the exterior algebra. The last equality follows from Lemma 2.2.5, while the inclusion in the centre follows from the hypothesis. Hence $I^{\wedge}(p_1,\dots,p_r) \subset t^{\perp}$. This concludes the proof. $\square$

Example 2.2.12. Let $V = \mathbb{K}^8$ and consider the skew-symmetric tensor

$$t = v_1 \wedge v_3 \wedge v_5 \wedge v_6 + v_1 \wedge v_4 \wedge v_5 \wedge v_6 + v_2 \wedge v_3 \wedge v_5 \wedge v_6 + \\ + v_2 \wedge v_4 \wedge v_5 \wedge v_6 + v_1 \wedge v_3 \wedge v_7 \wedge v_8 \in \bigwedge^4 V.$$

It is easy to see that $\mathrm{rk}(\mathcal{C}_t^{\wedge,d,1}) = 8$ and hence t must have skew-symmetric rank at least 2. Consider the two four dimensional vector subspaces $W_1 = \langle v_1+v_2, v_3+v_4, v_5, v_6\rangle$ and $W_2 = \langle v_1, v_3, v_7, v_8\rangle$ whose associated points of $\mathbb{G}(4,V)$ are $p_1 = [(v_1+v_2)\wedge(v_3+v_4)\wedge v_5\wedge v_6]$ and $p_2 = [v_1\wedge v_3\wedge v_7\wedge v_8]$. Their ideals are

$$I^{\wedge}(p_1) = (\alpha_1-\alpha_2, \alpha_3-\alpha_4, \alpha_7, \alpha_8) \text{ and } I^{\wedge}(p_2) = (\alpha_2, \alpha_4, \alpha_5, \alpha_6).$$

One can check that $I^{\wedge}(p_1,p_2)$ is generated in degree 2 and it is included in $t^{\perp}$. Hence by Theorem 2.2.11 we have that t can be written as a linear combination of p_1 and p_2 and hence it has skew-symmetric rank 2

$$t = (v_1+v_2)\wedge(v_3+v_4)\wedge v_5\wedge v_6 + v_1\wedge v_3\wedge v_7\wedge v_8 \in \bigwedge^4 V.$$

Remark 2.2.13. Analogously to the symmetric case, Theorem 2.2.11 provides a decomposition of an element but in general this decomposition may be not minimal or unique. Also in this case a skew-symmetric tensor which admits a unique decomposition is called identifiable. The most updated state of the art about identifiability of skew-symmetric tensors is in [BV18] and [Bor13].

2.3 Non-abelian apolarity theory

In this section we recall the non-abelian apolarity which is a general theory for any variety X via vector bundles. It has been introduced in [LO13] with the goal of finding equations of secant varieties of a variety X in terms of minors of certain linear maps.

Definition 2.3.1. Let $X \subset \mathbb{P}^n$ be a projective variety. Let L be a very ample line bundle on X which gives the embedding $X \subset \mathbb{P}^n = \mathbb{P}(H^0(X,L)^*)$. Consider also another vector bundle E on X of rank e such that $H^0(E^\vee \otimes L)$ is not trivial. Consider the natural contraction map

$$H^0(E) \otimes H^0(E^\vee \otimes L) \longrightarrow H^0(L).$$

It induces a morphism

$$A : H^0(X,E) \times H^0(X,L)^* \longrightarrow H^0(X,E^\vee \otimes L)^*$$

where $E^\vee$ denotes the dual vector bundle to E, such that for any $f \in H^0(X,L)^*$ it induces a linear map

$$A_f : H^0(X,E) \longrightarrow H^0(X,E^\vee \otimes L)^*$$

defined as $A_f(s) := A(s \otimes f)$ for any $s \in H^0(X,E)$. The map A is called *non-abelian apolarity action*.

In the hunt of equations of secant varieties there is the following result.

Proposition 2.3.2 (Prop. 5.1.1 at p. 588 of [LO13])**.** *Let $X \subset \mathbb{P}(H^0(X,L)^*)$ be a projective variety and let L be a very ample line bundle on X. Let E be a vector bundle of rank e on X. Then*

$$\sigma_r(X) \subset Rank_r(E) := \sigma_{er}\left(\nu_{1,1}\left(\mathbb{P}(H^0(X,E)^*) \times \mathbb{P}(H^0(X,E^\vee \otimes L)^*)\right)\right) \cap \mathbb{P}(H^0(X,L)^*)$$

i.e. the $er+1$ *minors of* A_f*, with* $f \in H^0(X,L)^*$*, give equations for* $\sigma_r(X)$*.*

We do not report a proof of this fact which can be found in [LO13, Prop. 5.1.1, p. 588]. However along the proof the following fact is highlighted. If $x = [v] \in X$, then there is the inclusion $H^0(I_x \otimes E) \subseteq \ker A_v$, where $H^0(I_x \otimes E)$ is the space of global sections of E vanishing on x. This statement has been generalized by the authors.

Proposition 2.3.3 (Prop. 5.4.1 at p. 590 of [LO13])**.** *Let* $f = \sum_{i=1}^k v_i \in H^0(X,L)^*$*, with* $[v_i] \in X$*, and let* $Z = \{[v_1], \dots, [v_r]\}$*. Then we have the inclusions*

$$H^0(I_Z \otimes E) \subseteq \ker A_f,$$
$$H^0(I_Z \otimes E^\vee \otimes L) \subseteq \operatorname{Im} A_f^\perp,$$

where $H^0(I_Z \otimes E)$ *is the space of global sections vanishing on* Z *and similarly for* $H^0(I_Z \otimes E^\vee \otimes L)$*.*

We do not report a proof here, see [LO13]. This proposition is regarded in the work [Ott13] of the same author of [LO13] as *non-abelian apolarity lemma*.

Even though the non-abelian apolarity seems such an abstract object, it is related to the previously introduced apolarities as the following remarks highlight.

Remark 2.3.4 (Classical apolarity)**.** In the instance of Definition 2.3.1, consider $X = \mathbb{P}^n = \mathbb{P}(V)$ and let $L = \mathcal{O}(d)$, for some integer $d \geq 1$. Hence L is a very ample line bundle and there is the embedding

$$\mathbb{P}^n \hookrightarrow \mathbb{P}\left(H^0(\mathbb{P}^n, \mathcal{O}(d))^*\right) = \mathbb{P}\left(\operatorname{Sym}^d V\right),$$

i.e. the Veronese embedding. Consider also the line bundle $E = \mathcal{O}(e)$. The space of global sections $H^0(X, E^\vee \otimes L)^* \simeq H^0(X, \mathcal{O}(d-e))^*$ is not trivial if and only if $e \leq d$. It is a known fact that these spaces are all isomorphic to symmetric powers of V. Explicitly the non-abelian apolarity action in this case is

$$\circ : \operatorname{Sym}^e V^* \times \operatorname{Sym}^d V \longrightarrow \operatorname{Sym}^{d-e} V$$

which is the classical apolarity action. Moreover note that Proposition 2.3.2, with X, L and E as chosen in these lines, is the Remark 2.1.4.

Remark 2.3.5 (Skew-symmetric apolarity)**.** Let $X = \mathbb{G}(d, V)$ be a Grassmann variety, with $2d \leq \dim(V)$. We can consider the universal sub-bundle $\mathcal{U}$ of the trivial bundle whose fibre at a point $p \in \mathbb{G}(d, V)$ is the d-dimensional vector subspace of V represented by p. It is a known fact that this bundle has no global sections while its dual $\mathcal{U}^\vee$ does. In particular the bundle $\wedge^d \mathcal{U}^\vee$ is linear and very ample and give rise to the embedding

$$X \hookrightarrow \mathbb{P}\left(H^0\left(X, \bigwedge^d \mathcal{U}^\vee\right)^*\right) \simeq \mathbb{P}\left(\bigwedge^d V\right),$$

i.e. the Plücker embedding. Consider then the bundle $E = \wedge^e \mathcal{U}^\vee$. For more general facts of the theory we have that $H^0(X, E) \simeq \wedge^e V^*$ and $H^0(X, E^\vee \otimes L)$ is not trivial if and only if $e \leq d$. Moreover in this case $H^0(X, E^\vee \otimes L)^* \simeq \wedge^{d-e} V$. Hence the non-abelian apolarity action in this case is

$$\lrcorner : \bigwedge^e V^* \times \bigwedge^d V \longrightarrow \bigwedge^{d-e} V.$$

which is the skew-symmetric apolarity action. Note that in this case the Proposition 2.3.2, with X, L as chosen in these lines and $E = \mathcal{U}^\vee$, is the Remark 2.2.4.

The isomorphisms of spaces of global sections and symmetric or exterior powers have actually a more general background which we use along the text. If X is a rational homogeneous variety of some simple algebraic group, i.e. $X = G/P$, we can consider a *homogeneous* bundle E over X, i.e. roughly a bundle where the group G acts appropriately. It turns out that homogeneous bundles are all classified. Indeed such result follows from the description of the irreducible representations of P. Moreover a classic result known as *Borel-Weil Theorem* allows to construct irreducible representations of G as spaces of global sections of the previously cited homogeneous bundles. We will use these facts more explicitly along this document.

There is another apolarity we have not mentioned called *multigraded apolarity*. It appeared firstly in [CGG02] and regards in general partially symmetric tensors. In the more recent paper [Gał16], a general apolarity for *toric* varieties that features also Cox rings is presented. We do not present here those works since we are not interested in those kind of structured tensors or varieties.

We would like to end the chapter with some final remarks. There is a greatest common idea between all the apolarities we have introduced which is the *evaluation*. It appeared disguised in different ways, as derivatives in the symmetric case, as evaluation of equations on k-dimensional planes in the skew-symmetric one, or as evaluation of global sections in the more general context of non-abelian apolarity. In the next chapter we introduce the core of this document, the *Schur apolarity action*, which is an apolarity theory for rational homogenous varieties of the group $SL(n)$. With this premise, it seemed right to carry on the idea of evaluation also in that extension. We will see also a connection with the non-abelian apolarity.

Chapter 3

Schur apolarity

This chapter is dedicated to the original introduction of the *Schur apolarity*, a general apolarity theory for any rational homogeneous variety of the group $SL(n)$. The first section is devoted to recall basic facts from representation theory and combinatorics. In the second one the Schur apolarity action is introduced and in the third it is compared with the non-abelian apolarity action. Eventually last section is devoted to prove a result analogous to the previous apolarity lemmata.

3.1 Representation theory, again

In this section we recall basic facts related to representation theory and combinatorics useful for the introduction of the Schur apolarity. The main reference for this section is [Ful97].

The space $\mathbb{S}^\bullet V$. By Definition 1.3.27 it is clear that the Schur module $\mathbb{S}_\lambda V$ is naturally contained in $V^{\otimes d}$, where $|\lambda| = d$. Recall the actions of the groups $GL(V)$ and $\mathfrak{S}_d$ on $V^{\otimes d}$ described in (1.1.3) and (1.1.4) respectively . As previously told, it is clear via that formulae that these two actions commute. A celebrated theorem called *Schur-Weyl duality* describes the decomposition of $V^{\otimes d}$ into $\mathfrak{S}_d \times GL(V)$-modules. Specifically

Theorem 3.1.1 (Schur-Weyl duality)**.** *Let V be a finite dimensional vector space over an algebraically closed field of characteristic 0 and consider the d-fold tensor product $V^{\otimes d}$. Then under the action of $\mathfrak{S}_d \times GL(V)$ we have the decomposition*

$$V^{\otimes d} \simeq \bigoplus_{\lambda \vdash d} V_\lambda \otimes \mathbb{S}_\lambda V \simeq \bigoplus_{\lambda \vdash d} (\mathbb{S}_\lambda V)^{\oplus m_\lambda}, \tag{3.1.1}$$

where V_λ is the irreducible representation of $\mathfrak{S}_d$ associated to λ and $\mathbb{S}_\lambda V$ is the Schur module associated to λ. The integer m_λ is the dimension of V_λ.

For a proof we refer to [FH13, Theorem 6.3, p. 77]. As told in the statement of the theorem, the spaces V_λ are irreducible representations of the group $\mathfrak{S}_d$. We do not thoroughly describe them here but let us say just a few words. Being $\mathfrak{S}_d$ a finite group, every finite dimensional representation is completely reducible in sum of irreducible representations, see [FH13, Proposition 1.5 and Corollary 1.6, p. 6]. Moreover, the number of irreducible representations of $\mathfrak{S}_d$ is exactly the number of partitions of d. To construct V_λ one has to consider the *group algebra* $\mathbb{C}\mathfrak{S}_d$, i.e. an algebra generated by elements e_σ with $\sigma \in \mathfrak{S}_d$ such that $e_\sigma \cdot e_\tau := e_{\sigma\cdot\tau}$, where the last product appearing at the index is the composition of permutations. Then given λ a partition together with a std tableau of shape λ, one can define the element $c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{C}\mathfrak{S}_d$, where

$$a_\lambda := \sum_{\sigma \in R_\lambda} e_\sigma,$$

$$b_\lambda := \sum_{\sigma \in C_\lambda} \operatorname{sgn}(\sigma) e_\sigma,$$

where R_λ and C_λ are the subgroups of $\mathfrak{S}_d$ as defined in Remark 1.3.26. The element c_λ is called *Young symmetrizer*. The reason why of this homonymy with the endomorphism in Definition 1.3.27 is due to the fact that one can define a right action

$$\begin{aligned} V^{\otimes d} \times \mathbb{C}\mathfrak{S}_d &\longrightarrow V^{\otimes d} \\ (v_1 \otimes \cdots \otimes v_d, e_\sigma) &\longmapsto (v_1 \otimes \cdots \otimes v_d) \cdot e_\sigma := v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \end{aligned}$$

for any $e_\sigma \in \mathbb{C}\mathfrak{S}_d$ and any $v_1 \otimes \cdots \otimes v_d \in V^{\otimes d}$, and then extend it by linearity. By [FH13, Theorem 4.3, p. 46] the image of $\mathbb{C}\mathfrak{S}_d$ via c_λ acting on it on the right with the obvious action is an irreducible representation of $\mathfrak{S}_d$. The choice of two different std tableaux of shape λ gives rise to two isomorphic representations. Eventually, let us state that m_λ is the number of possible std tableaux of shape λ with the integers in $[d]$.
Coming back to the Formula (3.1.1), one can see that the different number of isomorphic copies of $\mathbb{S}_\lambda V$ appearing inside $V^{\otimes d}$ is exactly m_λ. As we have seen in

(1.3.37), the elements of two isomorphic copies of $\mathbb{S}_\lambda V$ regarded as elements of $\bigwedge_{\lambda'} V$ via the obvious inclusion look like the same. On the other hand the difference resides on how these modules are embedded in $V^{\otimes d}$. Since in the following we are not interested in how they embed in $V^{\otimes d}$, we would like to get rid of the redundant copies and consider only one module. To this end we introduce the following space.

Definition 3.1.2. Let V be a vector space of finite dimension n. We denote with $\mathbb{S}^\bullet V$ the vector space

$$\mathbb{S}^\bullet V := \bigoplus_{\lambda \,:\, l(\lambda)<n} \mathbb{S}_\lambda V,$$

i.e. the space obtained as the direct sum of any Schur module, one copy for each λ such that $l(\lambda) < n$.

Remark 3.1.3. It can be seen that the space $\mathbb{S}^\bullet V$ can be obtained as the quotient

$$\begin{aligned}\mathbb{S}^\bullet V &:= \mathrm{Sym}^\bullet \left(V \oplus \bigwedge^2 V \oplus \cdots \oplus \bigwedge^{n-1} V \right) / I^\bullet \\ &\simeq \bigoplus_{(a_1,\dots,a_{n-1})\in\mathbb{N}^{n-1}} \left(\mathrm{Sym}^{a_1}(V) \otimes \mathrm{Sym}^{a_2}\left(\bigwedge^2 V\right) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}\left(\bigwedge^{n-1} V\right)\right) / I^\bullet,\end{aligned}$$

where $I^\bullet$ is the following two-sided ideal. For any $p \geq q \geq 0$, it is generated by the relations

$$\begin{aligned}&(v_1 \wedge \cdots \wedge v_p) \cdot (w_1 \wedge \cdots \wedge w_q) + \\ &\qquad - \sum_{i=1}^{p} (v_1 \wedge \cdots \wedge v_{i-1} \wedge w_1 \wedge v_{i+1} \wedge \cdots \wedge v_p) \cdot (v_i \wedge w_2 \wedge \cdots \wedge w_q).\end{aligned} \tag{3.1.2}$$

Remark 3.1.4. It is not hard to prove that the space $\mathbb{S}^\bullet V$ obtained as a quotient contains every Schur module $\mathbb{S}_\lambda V$ with multiplicity one. Indeed, observe at first that any partition λ can be written uniquely as a sequence $(a_1, \dots, a_{n-1})$ of positive integers where a_i is the number of columns in the diagram of λ of length i. With this notation, it is possible to split the Young symmetrizer in the product $c_\lambda = a_\lambda \cdot b_\lambda = a''_\lambda \cdot a'_\lambda \cdot b_\lambda$. The operator a'_λ is the element of the group algebra $\mathbb{C}\mathfrak{S}_d$ given by the sum of all the e_σ's where σ is a permutation of R_λ which permutes columns of the same length. Call the subgroup generated by these permutations with P. Note that the element $a'_\lambda \cdot b_\lambda$ acts on $V^{\otimes|\lambda|}$ returning

$$V^{\otimes|\lambda|} \cdot a'_\lambda \cdot b_\lambda = \mathrm{Sym}^{a_1}(V) \otimes \mathrm{Sym}^{a_2}\left(\bigwedge^2 V\right) \otimes \cdots \otimes \mathrm{Sym}^{a_{n-1}}\left(\bigwedge^{n-1} V\right). \quad (3.1.3)$$

Then a''_λ is the element of the group algebra given by the sum of all the e_σ's, where σ is any representative of a left coset in R_λ/P. Then we get that

$$\mathbb{S}_\lambda V = V^{\otimes|\lambda|} \cdot c_\lambda = V^{\otimes|\lambda|} \cdot a''_\lambda \cdot a'_\lambda \cdot b_\lambda$$

so that $\mathbb{S}_\lambda V$ is obtained projecting the addend of $\mathbb{S}^\bullet V$ appearing in Formula (3.1.3) via the quotient with $I^\bullet$, cf. [FH13, Proposition 15.55, p. 236]. The first appearance of the ring $\mathbb{S}^\bullet V$ has been made in [Tow77, Tow79]. See also [Ful97, p. 124–126] for other details about it also on the primality of $I^\bullet$.

Remark 3.1.5. The equations (3.1.2) are known as *Plücker relations* and they are the equations of the rational homogeneous varieties included as minimal orbits inside the projectivization $\mathbb{P}(\mathbb{S}_\lambda V)$.

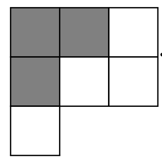
We want to underline the following fact. Even though $\mathbb{S}^\bullet V$ is a commutative algebra and hence it has a natural product, we are not going to regard it this way. Indeed we are going to use only its vector space structure and the fact that it contains every Schur module with multiplicity one. Anyway, to ease the construction of the theory we will imagine to build every Schur module using a fixed std tableau. If $\lambda \vdash d$, this one will be given by filling the diagram of λ from top to bottom, starting from the first column, with the integers $1, \ldots, d$. For instance if $\lambda = (3, 2, 1)$, then

$$T = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 1 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}.$$

Skew diagrams and skew Schur modules.

Definition 3.1.6. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_h)$ be two partitions. We say that $\mu \subset \lambda$ if $h \leq k$ and $\mu_i \leq \lambda_i$ for all i's, possibly setting some μ_i equal to 0. In this instance we say that the diagram of μ is a *subdiagram* of the one of λ. If $\mu \subset \lambda$, the *skew Young diagram of shape* λ/μ is the diagram associated to the sequence of integers $\lambda/\mu := (\lambda_1 - \mu_1, \ldots, \lambda_k - \mu_k)$ obtained removing the diagram of μ from the diagram of λ in the left upper corner. A *skew Young tableau* T of shape λ/μ is the diagram of λ/μ with a filling. The definitions of sstd and std tableau apply also in this context with the same choice of integers described in Remark 1.3.26.

Remark 3.1.7. The sequence λ/μ need not to be a partition in general and hence we do not regard it in this way. For instance, if $\lambda = (3,3,1)$ and $\mu = (2,1)$, then $\mu \subset \lambda$ and $\lambda/\mu = (1,2,1)$. The respective skew Young diagram of shape λ/μ is

$\lambda/\mu =$  .

Definition 3.1.8. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_h)$ be two partitions of the integers d and e respectively, such that $\mu \subset \lambda$. Let T be a std tableau of shape λ/μ. The endomorphism

$$\begin{aligned} c^T_{\lambda/\mu} : V^{\otimes d-e} &\longrightarrow V^{\otimes d-e} \\ v_1 \otimes \cdots \otimes v_d &\longrightarrow \sum_{\tau \in C_{\lambda/\mu}} \sum_{\sigma \in R_{\lambda/\mu}} \operatorname{sgn}(\tau) v_{\tau(\sigma(1))} \otimes \cdots \otimes v_{\tau(\sigma(d))} \end{aligned}$$

is called *Young symmetrizer* determined by λ/μ and T. Note that it is a generalization of Definition 1.3.27 which occurs when $\mu = (0)$. The image

$$\mathbb{S}_{\lambda/\mu} V := c^T_{\lambda/\mu}\left(V^{\otimes d-e}\right)$$

is called *skew Schur module*. Again, this module coincides with the one in Definition 1.3.27 whenever $\mu = (0)$.

Remark 3.1.9. Of course different choices of skew tableaux of the same shape give rise to isomorphic modules. Moreover, let ν and λ be two partitions such that $\lambda \subset \nu$. The vector space $\mathbb{S}_{\nu/\lambda} V$ is again a representation of $SL(n)$ but in general it may be reducible. However a general decomposition in sum of irreducible representations is known

$$\mathbb{S}_{\nu/\lambda} V \cong \bigoplus_{\mu} N_\nu^{\lambda,\mu} \mathbb{S}_\mu V \tag{3.1.4}$$

where $N_\nu^{\lambda,\mu}$ are the Littlewood-Richardson coefficients we have already met in (1.3.2). For instance if $\nu = (3,3,1)$ and $\lambda = (2,1)$, then we get

$$\mathbb{S}_{(3,3,1)/(2,1)} V \simeq \mathbb{S}_{(2,1,1)} V \oplus \mathbb{S}_{(2,2)} V \oplus \mathbb{S}_{(3,1)} V.$$

In particular it follows that if λ, μ and ν are a triplet of partitions such that $N_\nu^{\lambda,\mu} \neq 0$, then clearly $\lambda, \mu \subset \nu$ and $|\lambda| + |\mu| = |\nu|$ and we have the inclusions $\mathbb{S}_\nu V \subset \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ and $\mathbb{S}_\mu V \subset \mathbb{S}_{\nu/\lambda} V$, in both cases with multiplicity $N_\nu^{\lambda,\mu}$.

We will see later that skew Schur modules are useful in the definition of the Schur apolarity action.

Littlewood-Richardson coefficients. In Section 1.3 we have seen that the space of equivariant morphisms $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \longrightarrow \mathbb{S}_\nu V$ has dimension $N_\nu^{\lambda,\mu}$. We describe a basis of this morphisms. To this end we recall at first a classic computation of such coefficients.

Definition 3.1.10. Let ν/λ be any skew diagram and consider a sstd skew tableau T of shape ν/λ. The *word associated to* T is the string of integers obtained from T reading its entries from left to right, starting from the bottom row. The obtained word $w_1 \dots w_k$ is called both *Yamanouchi word* and *reverse lattice word* if for any s from 0 to $k-1$ and for any i, the sequence $w_k w_{k-1} \dots w_{k-s}$ contains the integer $i+1$ at most as many times as it contains the integer i. For short these words will be denominated *Y-words*. The *content* of T is the sequence of integers $\mu = (\mu_1, \dots, \mu_n)$ where μ_i is the number of i's in T. Note that this may be not a partition.

Example 3.1.11. For instance given

$$T_1 = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}\,, \qquad T_2 = \begin{array}{|c|c|c|} \hline & & 1 \\ \hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} \end{array}$$

their associated words are $w_{T_1} = 231221$ and $w_{T_2} = 231121$. Remark that only the second is a Y-word since in w_{T_1} the subsequence 122 has the integer 2 appearing more times than the integer 1. In the first instance the content is $(2,3,1)$, while in the second is $(3,2,1)$.

Definition 3.1.12. Let λ and ν be two partitions such that $\lambda \subset \nu$ and consider a skew sstd tableau T of shape ν/λ. The tableau T is a *Littlewood-Richardson skew tableau* if its associated word is a Y-word.

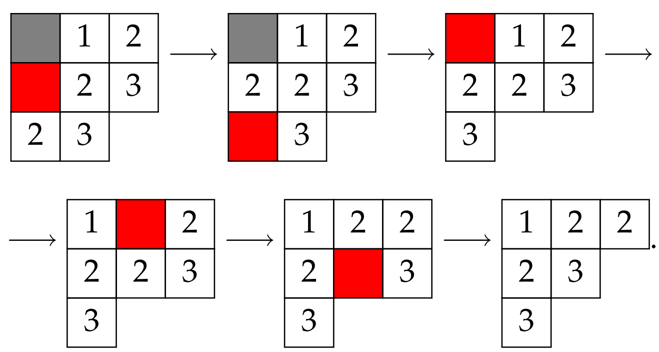
We recall now an algorithm introduced by *Schützenberger* in [Sch77] called *jeu de taquin* which allows to obtain a sstd tableau starting from a sstd skew tableau. Consider a sstd skew tableau T of shape ν/λ for some partitions $\lambda \subset \nu$. An *inside corner* is a removed box of the diagram ν/λ, i.e. a box of the diagram of λ contained in ν, with the following condition: below it and next to the right of it there is either a box of ν not in λ or there is no box. Obviously there may be several inside corners in a skew Young diagram. The main idea of the algorithm is to expel, after some steps, any inside corner. The steps of the algorithm are the following.

(1) Choose an inside corner and look at the box below it or next to it on the right, even if there is no box.

(2) Exchange the position of the box with the least integer and the inside corner. If the boxes have the same integer then move the box below the inside corner. If there is no box, move directly the other box. Apply again this step to the same inside corner until it is expelled from the diagram.

(3) If there is still an inside corner return to (1), else the algorithm ends.

For example



In this particular instance there was only one inside corner at the beginning of the procedure. We have highlighted in red the chosen inside corner until it gets expelled. A couple of facts can be proved.

Proposition 3.1.13 (Claim 2, p.15, [Ful97])**.** *The jeu de taquin algorithm produces a sstd tableau. Moreover, the construction does not depend on the choice of the inside corner performed at every step; the result will always be the same sstd tableau.*

Notice also that the construction can be reversed to come back to the original skew tableau. We state here other facts borrowed from [Ful97] which we are going to use.

Proposition 3.1.14 (Prop. 3, p. 64, [Ful97])**.** *Let μ, λ and ν be such that $\mu, \lambda \subset \nu$ and $|\mu| + |\lambda| = |\nu|$, and consider the skew diagram ν/λ. The number of Littlewood-Richardson skew tableau of shape ν/λ and content μ is exactly $N_\nu^{\lambda,\mu}$.*

Proposition 3.1.15 (Lemma 1, p. 65, [Ful97]). *Let μ, λ and ν be a triplet of partitions such that $N_{\nu}^{\lambda,\mu} \neq 0$. Then T is a Littlewood-Richardson tableau of shape ν/λ and content μ if and only if applying the jeu de taquin to T one gets the sstd tableau $U(\mu)$ of shape μ that have the i-th row filled only with the integer i.*

We omit the proofs of these facts for which we refer to [Ful97].

Multiplication maps. In order to develop our theory we need to use multiplication maps $\mathcal{M}_{\nu}^{\lambda,\mu} : \mathbb{S}_{\lambda}V^* \otimes \mathbb{S}_{\mu}V^* \longrightarrow \mathbb{S}_{\nu}V^*$. These maps will be non-trivial whenever $N_{\nu}^{\lambda,\mu} \neq 0$ and they will be the projections onto one of the addends $\mathbb{S}_{\nu}V^*$ appearing in the decomposition of $\mathbb{S}_{\lambda}V^* \otimes \mathbb{S}_{\mu}V^*$ into sum of irreducible representations as described by (1.3.2). We give now a brief description of these maps. Such a description follows from taking the transpose map of the inclusions $\mathbb{S}_{\nu}V \hookrightarrow \mathbb{S}_{\lambda}V \otimes \mathbb{S}_{\mu}V$ which are described in [ABW82] in terms of coSchur modules.

Remark 3.1.16. For a given partition λ such that $l(\lambda) \leq \dim(V)$, we write a bit more explicitly the action of a Young symmetrizer $c_{\lambda} \in \mathbb{C}\mathfrak{S}_d$ on $V^{\otimes|\lambda|}$. Recall that we can write it as the product $a_{\lambda} \cdot b_{\lambda}$. Given a partition $\mu = (\mu_1, \dots, \mu_k)$ we use the notation

$$\mathrm{Sym}_{\mu} V := \mathrm{Sym}^{\mu_1} V \otimes \cdots \otimes \mathrm{Sym}^{\mu_k} V,$$

which is similar to the one already adopted for products of exterior powers of V indexed by a partition. Then obviously $\mathrm{im}(a_{\lambda}) = \mathrm{Sym}_{\lambda} V$ after grouping some factors of $V^{\otimes|\lambda|}$. With a little abuse of notation we can see b_{λ} as the map

$$b_{\lambda} : \mathrm{Sym}_{\lambda} V \longrightarrow V^{\otimes|\lambda|} \longrightarrow \bigwedge\nolimits_{\lambda'} V$$

which first embeds $\mathrm{Sym}_{\lambda} V$ in $V^{\otimes|\lambda|}$ and then acts by skew-symmetrizing certain factors of $V^{\otimes|\lambda|}$. More specifically, we can exploit such an action with intermediate steps as

$$\mathrm{Sym}_{\lambda} V \longrightarrow \left(V \otimes \mathrm{Sym}^{\lambda_1 - 1} V\right) \otimes \cdots \otimes \left(V \otimes \mathrm{Sym}^{\lambda_k - 1} V\right) \longrightarrow \bigwedge\nolimits^{\lambda'_1} V \otimes \mathrm{Sym}_{\overline{\lambda}} V$$

where $\overline{\lambda} = (\lambda_1 - 1, \dots, \lambda_k - 1)$. The first map is the product of comultiplication maps of the symmetric algebra (1.1.7) applied on each symmetric power $\mathrm{Sym}^{\lambda_i} V$, while the second one is the multiplication map of the exterior algebra (1.1.12) applied on the factors V in the image of the previous map. Then one continues with the same technique on $\mathrm{Sym}_{\overline{\lambda}} V$ skew-symmetrizing column by column, arriving eventually on $\bigwedge_{\lambda'} V$. Notice that from this point of view we do not care of which factors have been symmetrized or skew-symmetrized. The image of b_λ is clearly the Schur module $\mathbb{S}_\lambda V$ as we know it from Definition 1.3.27.

Remark 3.1.17. We can perform a sort of inverse construction with respect the one in the previous remark. Indeed, consider a partition λ such that $\lambda_1 \leq \dim(V)$ and consider the products of exterior and symmetric powers $\mathrm{Sym}_{\lambda'} V$ and $\bigwedge_\lambda V$ respectively. We can consider the composition of maps

$$\bigwedge\nolimits_\lambda V \longrightarrow \mathrm{Sym}_{\lambda'} V,$$

given by intermediate steps of compositions of maps such as

$$\bigwedge\nolimits_\lambda V \longrightarrow \left(V \otimes \bigwedge\nolimits^{\lambda_1 - 1} V\right) \otimes \cdots \otimes \left(V \otimes \bigwedge\nolimits^{\lambda_k - 1} V\right) \longrightarrow \mathrm{Sym}^{\lambda'_1} V \otimes \bigwedge\nolimits_{\overline{\lambda}} V$$

where $\overline{\lambda} = (\lambda_1 - 1, \ldots, \lambda_k - 1)$. The first map is the product of comultiplications map of the exterior algebra (1.1.13) applied on each exterior power $\bigwedge^{\lambda_i} V$, while the second one is the multiplication map of the symmetric algebra (1.1.6) applied on the factors V in the image of the previous map. Then one continues with the same technique on $\bigwedge_{\overline{\lambda}} V$ symmetrizing column by column, arriving eventually on $\mathrm{Sym}_{\lambda'} V$.

Definition 3.1.18. Let V be a vector space of finite dimension n and let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition such that $\lambda_1 < n$. We denote the image of the map

$$\bigwedge\nolimits_\lambda V \longrightarrow \mathrm{Sym}_{\lambda'} V$$

described above with $W_\lambda V$ and we call it *coSchur* module.

Notice that we could define such modules analogously using a similar version of Young symmetrizer.

Remark 3.1.19. In [ABW82] the names they gave to Schur and coSchur modules are swapped with respect to our notation.

We have the following crucial result.

Proposition 3.1.20 (cf. Prop. II.4.1 in [ABW82])**.** *Let V be a vector space of finite dimension over a field $\mathbb{K}$ of characteristic zero. Then for any partition λ of length less than* $\dim(V)$ *and such that $\lambda_1 \leq \dim(V)$, there is an isomorphism between the Schur module $\mathbb{S}_\lambda V$ associated to λ and the coSchur module $W_{\lambda'} V$ associated to the conjugate partion λ'.*

For a proof we refer to [ABW82]. In particular remark that in [ABW82] the result is more general since V is a free R-module.
We recall now the definition of the inclusions $\mathbb{S}_\nu V \hookrightarrow \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ given in the section IV.2 of [ABW82]. Remark that in [ABW82] they actually define an inclusion map for coSchur modules. Here we describe the analogous construction for Schur modules. Before proceeding we need some combinatorial facts.

Remark 3.1.21. The set of std tableau of shape λ with entries in $\{1,\dots,|\lambda|\}$ is in 1 to 1 correspondence with the set of Y-words of length n and *content* λ, i.e. with λ_i times the integer i. This correspondence must not be confused with the definition of word associated to a tableau seen in Definition 3.1.10.

Idea of the proof. We can define two functions α and β

$$\{\text{std tableaux of shape } \lambda\ \} \xrightarrow{\ \alpha\ } \{\text{Y-words of length } n \text{ and content } \lambda\}$$

and

$$\{\text{std tableaux of shape } \lambda\ \} \xleftarrow{\ \beta\ } \{\text{Y-words of length } n \text{ and content } \lambda\}$$

that are inverses each other.
Let T be a std tableau of shape λ with entries in $\{1,\dots,|\lambda|\}$. Remark that each entry $l \in \{1,\dots,|\lambda|\}$ appears in a certain position (i,j), i.e. in the box in the i-th row and j-th column. Then consider the sequence $(a_1,\dots,a_{|\lambda|})$, where we set $a_l = i$ if l appears in the position (i,j). Roughly, starting with the smallest entry in $\{1,\dots,|\lambda|\}$ record the row in which it appears in T. We define $\alpha(T)$ to be the reversed sequence

$$\alpha(T) := \mathrm{rev}(a_1,\dots,a_{|\lambda|}) := (a_{|\lambda|},\dots,a_1),$$

where we consider rev as the involution that acts on the set of words reversing them. It is easy to see that $\alpha(T)$ is a Y-word since the sequences of integers are determined by the entries of T which already satisfy the non decreasing property.

Let $\underline{a}$ be a Y-word of content λ, so that λ_i is the number of times in which an integer i appears in $\underline{a}$. Consider its reversed sequence $\mathrm{rev}(\underline{a}) = (a_1,\dots,a_{|\lambda|})$. We define the std tableau $\beta(\underline{a})$ of shape λ in the following way. For any integer $i \in \{1,\dots,k\}$, where $k = l(\lambda)$, consider the subsequence

$$\mathrm{rev}(\underline{a})_i := (a_{k_1}, \ldots, a_{k_{\lambda_i}})$$

given by all i's with respect to the order with which they appear in $\mathrm{rev}(\underline{a})$. Then we set the (i,j)-th entry of $\beta(\underline{a})$ to be equal to k_j appearing in the subsequence $\mathrm{inv}(\underline{a})_i$. Even in this case it is clear that the image will be a std tableau of shape λ. For more details we refer to [ABW82, Definition IV.1.3, p. 252]. $\square$

Example 3.1.22. Consider $\lambda = (3,1)$ and let T be the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

We apply α to T. We get at first $(a_1, \ldots, a_4) = (1,1,2,1)$, so that $\alpha(T) = (1,2,1,1)$. Now we apply β to see that actually we come back to T. Consider the reversed sequence $\mathrm{rev}(\alpha(T)) = (1,1,2,1)$, which has content λ. Here we have only two subsequences

$$\mathrm{rev}(\alpha(T))_1 = (a_1, a_2, a_4) = (1,1,1) \text{ and } \mathrm{rev}(\alpha(T))_2 = (a_3) = (2).$$

Hence it follows that

$$\beta((\alpha(T))) = T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Definition 3.1.23 (Definition IV.2.2, p. 257 in [ABW82])**.** Let V be a Littlewood-Richardson skew tableau of shape ν/λ and content μ. We can define a new tableau V' of shape ν/λ and content μ' not necessarily sstd as following. Let $\underline{a}$ the word associated to V and consider $\underline{a}' := \alpha(\beta(\underline{a})')$, where with the notation $\beta(\underline{a})'$ we mean the conjugate tableau to $\beta(\underline{a})$, i.e. the one obtained from the latter transposing it as if it would be a matrix. Then define V' as the skew tableau of shape ν/λ whose associated word is $\underline{a}'$.

Example 3.1.24. Let $\nu = (3,2)$, $\lambda = (1)$ and consider

$$V = \begin{array}{|c|c|c|}\hline \blacksquare & 1 & 1 \\ \hline 1 & 2 \\ \cline{1-2}\end{array},$$

where in this case the content is $\mu = (3,1)$. The associated word is $\underline{a} = (1,2,1,1)$ which is a Y-word. Then

$$\beta(\underline{a}) = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} \quad \text{and} \quad \beta(\underline{a})' = \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}$$

so that $\alpha(\beta(\underline{a})') = (3,1,2,1)$. Therefore we get

$$V' = \begin{ytableau} *(gray) & 2 & 1 \\ 3 & 1 \end{ytableau}.$$

Definition 3.1.25 (Definition IV.2.4, p. 258)**.** Consider three partitions λ, μ and ν such that $N_\nu^{\lambda,\mu} \neq 0$ and consider the diagrams of ν/λ and μ. Let V be a Littlewood-Richardson skew tableau of shape ν/λ and content μ. When considering a diagram, we denote with (i,j) the entry of the diagram positioned in the i-th row and j-th column. We can define a map $\sigma_V : \nu/\lambda \longrightarrow \mu$ from the entries of the diagram of ν/λ to the entries of the diagram of μ such that

$$\sigma_V(i,j) := (V(i,j), V'(i,j)),$$

for every entry (i,j) of ν/λ.

Definition 3.1.26. Given a Littlewood-Richardson skew tableau of shape ν/λ and content μ, we can define a map $\eta : \mathrm{Sym}_{\nu/\lambda} V \longrightarrow \mathrm{Sym}_\mu V$. It is given by the composition of maps

$$\mathrm{Sym}_{\nu/\lambda} V \hookrightarrow \bigotimes^{\nu/\lambda} V \longrightarrow \bigotimes^{\mu} V \longrightarrow \mathrm{Sym}_\mu V$$

where the first map is the obvious inclusion. The notation $\bigotimes^{\nu/\lambda} V$ stands for the space $V^{\otimes(|\nu|-|\lambda|)}$ in which each copy of V is indexed by an entry (i,j) of the diagram of ν/λ, and similarly for $\bigotimes^\mu V$. Therefore the map in the middle associates to each copy of V in $\bigotimes^{\nu/\lambda} V$ index by (i,j), to the copy of V in $\bigotimes^\mu V$ indexed by $\sigma_V(i,j)$. The last map is given by the product of multiplication maps of the symmetric algebra (1.1.6).

Definition 3.1.27. We are now ready to describe the inclusion $\mathbb{S}_\nu V \hookrightarrow \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$. It is the composition of the maps

$$\mathbb{S}_\nu V \xrightarrow{\simeq} W_{\nu'} V \hookrightarrow \mathrm{Sym}_\nu V \xrightarrow{\theta} \mathrm{Sym}_\lambda V \otimes \mathrm{Sym}_\mu V \xrightarrow{b_\lambda \otimes b_\mu} \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$$

where the first map is the isomorphism of Proposition 3.1.20 and the second map is the natural inclusion of coSchur modules inside a product of symmetric powers of V. The fourth map involves the elements b_λ and b_μ needed in the definitions of the respective Young symmetrizers and the image of $b_\lambda \otimes b_\mu$ is clearly $\mathbb{S}_\lambda V \otimes \mathbb{S}_\nu V$. Now we are going to describe the map θ. It is defined as the following composition of maps

$$\theta : \mathrm{Sym}_\nu V \longrightarrow \mathrm{Sym}_\lambda V \otimes \mathrm{Sym}_{\nu/\lambda} V \xrightarrow{\mathrm{id}_{\mathrm{Sym}_\lambda V} \otimes \eta} \mathrm{Sym}_\lambda V \otimes \mathrm{Sym}_\mu V$$

where the first map is given by the product of comultiplication maps of the symmetric algebra (1.1.7):

$$\mathrm{Sym}^{\nu_i} V \longrightarrow \mathrm{Sym}^{\lambda_i} V \otimes \mathrm{Sym}^{\nu_i - \lambda_i} V,$$

while the second map is product of the identity of $\mathrm{Sym}_\lambda V$, with the map $\eta : \mathrm{Sym}_{\nu/\lambda} V \longrightarrow \mathrm{Sym}_\mu V$ seen in Definition 3.1.26.

Remark 3.1.28. Hitherto we have seen that fixed a choice of a Littlewood-Richardson skew tableau of shape ν/λ and content μ give rise to an inclusion $\mathbb{S}_\nu V \hookrightarrow \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$. This is tantamount to the fact that we can define $N_\nu^{\lambda,\mu}$ of such maps. Since their images are linearly independent, they actually are a basis of the space of equivariant morphisms from $\mathbb{S}_\nu V$ to $\mathbb{S}_\lambda V \otimes \mathbb{S}_\nu V$. For more details on this last fact we refer to [ABW82, Lemma IV.2.5, p. 259]. The transpose of the maps we have obtained, i.e the maps

$$\mathcal{M}_\nu^{\lambda,\mu} : \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\nu V^* \tag{3.1.5}$$

are the maps we are looking for.

Example 3.1.29. Some of the maps $\mathcal{M}_\nu^{\lambda,\mu}$ are easy to be defined. For instance let $\lambda = (d^k)$ and $\mu = (e^h)$ be two *rectangular* partitions, i.e. their diagrams are $k \times d$ and $h \times e$ rectangles respectively. Assume that $k + h \le \dim(V)$ and let $\nu = (d^k, e^h)$. Then $N_\nu^{\lambda,\mu} = 1$.
We describe $\mathcal{M}_\nu^{\lambda,\mu}$ on elements of the basis and then we extend it by linearity. Consider two sstd tableau S_1 and S_2 of shape λ and μ, we can consider the two elements $c_\lambda(v_{S_1})$ and $c_\mu(v_{S_2})$ of the basis of $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V$ respectively as described in Proposition 1.3.36. Then the image of $c_\lambda(v_{S_1}) \otimes c_\mu(v_{S_2})$ via $\mathcal{M}_\nu^{\lambda,\mu}$ is the image of the Young symmetrizer c_ν of the tensor associated to the not necessarily sstd

tableau given by putting S_2 right below S_1. It is clear that the map defined in this way is equivariant and linear.
For instance if $\lambda = (3)$, $\mu = (2)$ and $\nu = (3,2)$, we can consider the two elements

$$x_1^3 = c_\lambda(x_{S_1}) \quad \text{where} \quad S_1 = \boxed{1}\boxed{1}\boxed{1},$$

$$x_2^2 = c_\mu(x_{S_2}) \quad \text{where} \quad S_2 = \boxed{2}\boxed{2}.$$

Then we have

$$\mathcal{M}_{(3,2)}^{(3),(2)}(x_1^3 \otimes x_2^2) = 12 \cdot (x_1 \wedge x_2) \otimes (x_1 \wedge x_2) \otimes x_1 = c_\nu(x_{S_3})$$

where

$$S_3 = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

One can have analogous descriptions for the maps that add a row or a group of rows. We will use explicitly some of these maps in the next section.

Remark 3.1.30. We would like to close this section observing that the space $\mathbb{S}^\bullet V$ together with the maps $\mathcal{M}_\nu^{\lambda,\mu}$ is a graded ring. Th graded pieces are

$$(\mathbb{S}^\bullet V)_d := \bigoplus_{\lambda \,:\, |\lambda| = d} \mathbb{S}_\lambda V.$$

Indeed, ramark that if $\mathcal{M}_\nu^{\lambda,\mu}$ is not trivial with $|\lambda| = a$ and $|\mu| = b$, given $g \in \mathbb{S}_\lambda V$ and $h \in \mathbb{S}_\mu V$, then $\mathcal{M}_\nu^{\lambda,\mu}(g \otimes h) \in \mathbb{S}_\nu V$ so that the relation

$$(\mathbb{S}^\bullet V)_a \cdot (\mathbb{S}^\bullet V)_b \subset (\mathbb{S}^\bullet V)_{a+b}$$

is satisfied since $|\nu| = |\lambda| + |\mu| = a + b$.

3.2 The Schur apolarity action

In this section we introduce the Schur apolarity action showing its properties also via examples.

We would like to define an apolarity action whose domain is $\mathbb{S}^\bullet V \otimes \mathbb{S}^\bullet V^*$ and that can be restricted to $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$, with λ and μ suitably chosen. Moreover we would like that when $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V^*$ are either both symmetric or exterior powers of V, it returns the classical or skew-symmetric apolarity actions respectively. Naively it seems natural to require that $\mu \subset \lambda$ and that this map is

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

and to set φ equal to the zero map if $\mu \not\subset \lambda$.

Definition 3.2.1. Let $\mathbb{S}_\lambda V \subset \bigwedge_{\lambda'} V$ and $\mathbb{S}_\mu V^* \subset \bigwedge_{\mu'} V$ be two Schur modules. Then the *Schur apolarity action* is defined as the map

$$\varphi : \mathbb{S}^\bullet V \otimes \mathbb{S}^\bullet V^* \longrightarrow \mathbb{S}^\bullet V$$

such that when restricted to a product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$, with $\mu \subset \lambda$, it is the restriction of the map

$$\tilde{\varphi} : \bigwedge_{\lambda'} V \otimes \bigwedge_{\mu'} V^* \longrightarrow \bigwedge_{\lambda'/\mu'} V$$

where $\lambda'/\mu' = (\lambda'_i - \mu'_i)$, that acts as the product of skew-symmetric apolarity actions $\bigwedge^{\lambda'_i} V \otimes \bigwedge^{\mu'_i} V \longrightarrow \bigwedge^{\lambda'_i - \mu'_i} V$ as described in Definition 2.2.1, while it is the trivial map if $\mu \not\subset \lambda$. With an abuse of notation we denote both the Schur apolarity action and the one restricted to $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$ with the same symbol φ.

A priori we are not able to say that the image is contained in the skew Schur module $\mathbb{S}_{\lambda/\mu} V$. The following fact will clear our minds.

Proposition 3.2.2. *Let $\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \to \bigwedge_{\lambda'/\mu'} V$ be the Schur apolarity action restricted to $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$, where $\mu \subset \lambda$. Then the image of φ is contained in $\mathbb{S}_{\lambda/\mu} V$.*

Example 3.2.3. We report an example in preparation for the proof. Let $\lambda = (3, 2, 1)$ and $\mu = (2)$ and consider the restricted Schur apolarity

$$\varphi : \mathbb{S}_{(3,2,1)} V \otimes \mathbb{S}_{(2)} V^* \longrightarrow \mathbb{S}_{(3,2,1)/(2)} V$$

which is non trivial since $(2) \subset (3, 2, 1)$. Consider two elements of the basis associated to the sstd tableaux

$$U = \begin{ytableau} a & b & c \\ d & e \\ f \end{ytableau} \quad \text{and} \quad V = \begin{ytableau} \alpha & \beta \end{ytableau}$$

We use the pictorial notation we have introduced when generating a basis of all these spaces, for instance in (1.3.4). Hence only in this case when we write a tableau, actually we are considering the tensor product of the respective vectors in the order prescribed by the fixed std tableau used to build $\mathbb{S}_\lambda V$. Therefore for example we get that

$$c_{(2)}(e_V) = \begin{ytableau} \alpha & \beta \end{ytableau} + \begin{ytableau} \beta & \alpha \end{ytableau} = \alpha \otimes \beta + \beta \otimes \alpha.$$

With this notation performing Schur apolarity action means to contract the addends of $c_\mu(e_V)$ with the elements of the addends of $c_\lambda(e_U)$ that are in the left upper corner. For instance if we consider the two addends

$$\begin{ytableau} a & b & c \\ d & e \\ f \end{ytableau} \qquad \begin{ytableau} \alpha & \beta \end{ytableau}$$

of $c_\lambda(e_U)$ and $c_\mu(e_V)$ respectively, then the result of the contraction will be

$$\alpha(a)\beta(b) \cdot \begin{ytableau} *(gray) & *(gray) & c \\ d & e \\ f \end{ytableau}.$$

Now consider all the possible contractions performed between the addends using Schur apolarity all the addends. We group the surviving elements in the following way.
Consider at first all the addends with the same coefficient, for example $\alpha(a)\beta(b)$. Secondarily look at the skew diagrams that arise after the contraction. There may be *disjoint subdiagrams*, i.e. diagrams contained in the skew diagram which do not share any row or column. In our case we get two disjoint subdiagrams, $(2,1)$ and (1). Then within the collection of all the elements with the same coefficient, group the elements such that the respective disjoint subdiagrams share the same content. For example we may get the two collections

$$\alpha(a)\beta(b) \cdot \left(\begin{ytableau} *(gray) & *(gray) & c \\ d & e \\ f \end{ytableau} - \begin{ytableau} *(gray) & *(gray) & c \\ f & e \\ d \end{ytableau} + \begin{ytableau} *(gray) & *(gray) & c \\ e & d \\ f \end{ytableau} - \begin{ytableau} *(gray) & *(gray) & c \\ f & d \\ e \end{ytableau} \right). \tag{3.2.1}$$

It is clear now that the elements we have collected satisfy some symmetric relations. Indeed we can see that they satisfy the definition via Young symmetrizers of the skew Schur module. Moreover remark that they come with right sign due to the sign of the permutations inherited from the Young symmetrizations performed by $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V^*$. Therefore the element in (3.2.1) is the image via the Young symmetrizer of the element e_T where

$$T = \begin{ytableau} *(gray) & *(gray) & c \\ d & e \\ f \end{ytableau}.$$

Obviously this can be said for any collection of elements after the contraction. Hence the image will be contained in $\mathbb{S}_{(3,2,1)/(2)}V$.

Proof. We now prove the statement in general. For this purpose, consider two partitions λ and μ such that $\mu \subset \lambda$. Consider two elements of the basis of $\mathbb{S}_\lambda V$ and $\mathbb{S}_\mu V^*$ as described in Proposition 1.3.36. In the following the latin letters $a, b, \dots$ denote the elements of the filling of the tableau of shape λ, while greek letters $\alpha, \beta, \dots$ denote the elements of the filling of the tableau of shape μ. We use the pictorial notation as in the Example 3.2.3. With this notation, performing Schur apolarity means erasing the diagram of μ in the diagram of λ in the top left corner contracting the respective letters coming from the elements of V with those of V^*. Every such skew tableau comes with a coefficient $\alpha(b)\beta(d)\dots$ given by the contraction. The image of Schur apolarity between two elements of the basis is then given by a sum of skew tableaux of shape λ/μ with proper fillings and coefficients.

Remark that skew tableaux may contain *disjoint* subdiagrams, i.e. diagrams which do not share neither a row nor a column. Let us group the addends in the image in such a way that they all have the same coefficients and the respective disjoint subdiagrams share the same contents.

It is possible to find such elements for two reasons. At first, if we consider permutations of the bigger diagram which send the erased elements in the right positions, we can find disjoint subdiagrams which share the same fillings. Moreover, if the elements fixed are contracted by a single addend of the element in $\mathbb{S}_\mu V^*$, the coefficients are always the same.

After collecting the elements, every single group of skew tableaux is such that the fillings of every disjoint subdiagram are permuted accordingly to the symmetrization rules of a std skew tableau of shape λ/μ. Hence, every such group is the image via $c_{\lambda/\mu}$ of some element of $V^{\otimes|\lambda|-|\mu|}$.

In particular, remark that the signs due to permutations come in the right way since permutations along rows of the skew tableau come from permutations along rows of the bigger diagram. The same happens for exchanges along columns with the proper sign of the permutations. This proves that the image is contained in $\mathbb{S}_{\lambda/\mu}V$. $\square$

Remark 3.2.4 (Symmetric and skew-symmetric apolarity actions)**.** Consider the partitions $\lambda = (d)$ and $\mu = (e)$ with $e \leq d$, and consider the spaces $\mathbb{S}_{(d)}V = \operatorname{Sym}^d V$ ad $\mathbb{S}_{(e)}V^* = \operatorname{Sym}^e V^*$. Then the Schur apolarity action restricted to $\operatorname{Sym}^d V \otimes \operatorname{Sym}^e V^*$ is the map

$$\varphi : \operatorname{Sym}^d V \otimes \operatorname{Sym}^e V^* \longrightarrow \operatorname{Sym}^{d-e} V$$

since $\mathbb{S}_{(d)/(e)}V \simeq \operatorname{Sym}^{d-e} V$. Hence in this case the Schur apolarity action coincides with the classical apolarity action.
If we transpose all the diagrams, i.e. if we consider $\lambda' = (1^d)$ instead of $\lambda = (d)$ and $\mu' = (1^e)$ instead of $\mu = (e)$, and we assume that $e \leq d < \dim(V)$, then $\mathbb{S}_{(1^d)}V = \bigwedge^d V$ and $\mathbb{S}_{(1^e)}V^* = \bigwedge^e V^*$. In this case the Schur apolarity action restricted to $\bigwedge^d V \otimes \bigwedge^e V^*$ is the map

$$\varphi : \bigwedge^d V \otimes \bigwedge^e V^* \longrightarrow \bigwedge^{d-e} V$$

since $\mathbb{S}_{(1^d)/(1^e)}V \simeq \bigwedge^{d-e} V$. Hence in this case the restriction of the Schur apolarity action returns the skew-symmetric apolarity action.

Example 3.2.5. Consider $\lambda = (2,2)$ and $t = ((v_1 + v_2) \wedge (v_2 + v_3))^{\otimes 2} \in \mathbb{S}_{(2,2)}\mathbb{K}^4$, which has $(2,2)$-rank 1. We may wonder which elements $s = \alpha \in \mathbb{S}_{(1)}(\mathbb{K}^4)^*$ vanish on t via the Schur apolarity action. We get

$$\begin{aligned}\varphi(t \otimes s) =& \alpha(v_1 + v_2) \cdot (v_1 + v_3) \otimes ((v_1 + v_2) \wedge (v_2 + v_3)) + \\ & -\alpha(v_2 + v_3) \cdot (v_1 + v_2) \otimes ((v_1 + v_2) \wedge (v_2 + v_3)).\end{aligned}$$

Consequently $\varphi(t \otimes s) = 0$ is equivalent to $\alpha(v_1 + v_2) = \alpha(v_2 + v_3) = 0$. Hence the space of elements of $\mathbb{S}_{(1)}(\mathbb{K}^4)^*$ which annihilates t via Schur apolarity is

$$\langle x_4, x_1 - x_2 + x_3 \rangle.$$

Remark that the generators of this space are the linear equations of the subspace $\langle v_1 + v_2, v_2 + v_3 \rangle \subset \mathbb{K}^4$.

Example 3.2.6. Another example is $\lambda = (2,1)$ and $\mu \subset \lambda$. For instance if $\mu = (2)$, consider the elements

$$t = v_1 \wedge v_2 \otimes v_1 \in \mathbb{S}_{(2,1)}V \quad \text{and} \quad s = \alpha\beta \in \mathbb{S}_{(2)}V.$$

Then we get that

$$\varphi(t \otimes s) = \alpha(v_1)\beta(v_1) \cdot v_2 - \alpha(v_2)\beta(v_1) \cdot v_1 + \beta(v_1)\alpha(v_1) \cdot v_2 - \beta(v_2)\alpha(v_1) \cdot v_1.$$

We can collect some summands to get

$$\varphi(t \otimes s) = \big(\alpha(v_1)\beta(v_1) + \beta(v_1)\alpha(v_1)\big) \cdot v_2 - \big(\alpha(v_2)\beta(v_1) + \beta(v_2)\alpha(v_1)\big) \cdot v_1$$

and hence $\varphi(t \otimes s) = 0$ if and only if $\alpha(v_1)\beta(v_1) + \beta(v_1)\alpha(v_1) = \alpha(v_2)\beta(v_1) + \beta(v_2)\alpha(v_1) = 0$. Therefore we get that all the elements s in $\mathbb{S}_{(2)}V^*$ such that $\varphi(t \otimes s) = 0$ belong to

$$\langle x_2^2,\ x_i x_j,\ \text{where } j = 3, \dots, n \rangle.$$

Definition 3.2.7. Let λ and μ be two partitions such that $\mu \subset \lambda$ and consider the Schur apolarity action restricted to $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^*$. For any fixed $f \in \mathbb{S}_\lambda V$, there is an induced linear map

$$\mathcal{C}_f^{\lambda,\mu} : \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

such that $\mathcal{C}_f^{\lambda,\mu}(g) := \varphi(f \otimes g)$. We call the map $\mathcal{C}_f^{\lambda,\mu}$ *catalecticant map of* λ *and* μ *with respect to* f, or simply *catalecticant map* if no specification is needed. We call the set

$$f^\perp := \bigoplus_\mu \ker \mathcal{C}_f^{\lambda,\mu}$$

apolar set of f.

We have already seen in Remarks 2.1.4 and 2.2.4 that catalecticant maps have a role on detecting bounds on the symmetric or skew-symmetric rank respectively. In the next chapter we will see an analogue with respect to the general case.

3.3 Non-abelian apolarity vs. Schur apolarity

This section is devoted to understand connections between the non-abelian apolarity and Schur apolarity.

Let us begin with the most obvious difference. In our case X is a flag variety $\mathbb{F}(n_1, \dots, n_k; V)$ and the line bundle L will be

$$L = \left(\bigwedge^{n_1} \mathcal{U}_1^\vee\right)^{\otimes d_1} \otimes \cdots \otimes \left(\bigwedge^{n_k} \mathcal{U}_k^\vee\right)^{\otimes d_k} = \mathcal{O}(d_1, \dots, d_k),$$

where $\mathcal{U}_i$ is the tautological bundle over X such that the fibre over a point $(V_1, \dots, V_k) \in X$ is the subspace V_i of the flag. Such a choice gives rise to the embedding of X with $\mathcal{O}(d_1, \dots, d_k)$ into $\mathbb{P}(H^0(X, L)^*) = \mathbb{P}(\mathbb{S}_\lambda V)$, where λ is the partition such that it has d_i columns of length n_i. The choice of E is more free. For instance we may take

$$E = \left(\bigwedge^{n_1} \mathcal{U}_1^\vee\right)^{\otimes d'_1} \otimes \cdots \otimes \left(\bigwedge^{n_k} \mathcal{U}_k^\vee\right)^{\otimes d'_k}$$

where $d'_i \leq d_i$ for any i. This choice gives rise to a space of global sections which is isomorphic to the module $\mathbb{S}_\mu V^*$ such that μ has diagram with d'_i columns of length n_i. Observe that $\mu \subset \lambda$. Then it turns out that

$$L \otimes E^\vee \simeq \left(\bigwedge^{n_1} \mathcal{U}_1^\vee\right)^{\otimes (d_1 - d'_1)} \otimes \cdots \otimes \left(\bigwedge^{n_k} \mathcal{U}_k^\vee\right)^{\otimes (d_k - d'_k)}.$$

The last isomorphism follows from the isomorphism $\bigwedge^a \mathcal{F}^\vee \otimes \bigwedge^b \mathcal{F} \simeq \bigwedge^{a-b} \mathcal{F}^\vee$, for a given vector bundle $\mathcal{F}$ and $b \leq a$. Hence we get that $H^0(X, L \otimes E^\vee)^*$ is isomorphic to the module $\mathbb{S}_\nu V$ where ν is the partition whose diagram has $d_i - d'_i$ columns of length n_i. Hence we get a map like

$$A : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\nu V.$$

On the other hand restricting the Schur apolarity action to the same domain we get a map

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{\lambda/\mu} V$$

where in general $\mathbb{S}_{\lambda/\mu} V \neq \mathbb{S}_\nu V$. However we can state the following fact.

Proposition 3.3.1. *Let X be a rational homogeneous variety of $SL(V)$. Suppose that via the non-abelian apolarity action constructed using vector bundles on X there is a non trivial map*

$$A : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\nu V. \tag{3.3.1}$$

Then the map (3.3.1) *is the composition of the restricted Schur apolarity action with a surjective map* $\mathbb{S}_{\lambda/\mu} V \longrightarrow \mathbb{S}_\nu V$.

Proof. First of all remark that clearly the non-abelian apolarity action is an equivariant map whenever it involves homogeneous bundles over rational homogeneous varieties and representations. We prove that if such a map (3.3.1) exists, then $N_\lambda^{\nu,\mu} \neq 0$. This implies that $\mathbb{S}_\nu V$ appears in the decomposition into sum of irreducible modules of $\mathbb{S}_{\lambda/\mu} V$. Hence one has only to compose the restricted Schur apolarity action with a map $\mathbb{S}_{\lambda/\mu} V \longrightarrow \mathbb{S}_\nu V$ to conclude the proof.
Consider the map (3.3.1) which is non trivial by hypothesis. We may consider its transpose

$$A^t : \mathbb{S}_\nu V^* \longrightarrow \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V$$

and we may perform the tensor product with $\mathbb{S}_\mu V^*$ on the right and then compose it with the map $\mathbb{S}_\mu V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{K}$:

$$\mathbb{S}_\nu V^* \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V \otimes \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_\lambda V^*.$$

Transposing again we get the non trivial map

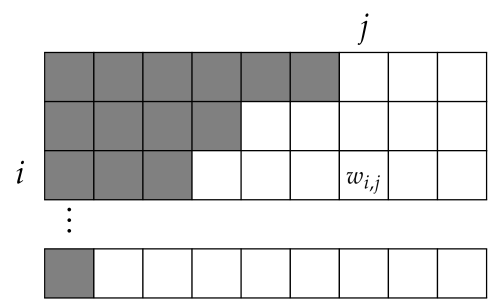
$$\mathbb{S}_\lambda V \longrightarrow \mathbb{S}_\nu V \otimes \mathbb{S}_\mu V$$

which is injective by Schur's Lemma. Since $\mathbb{S}_\nu V \otimes \mathbb{S}_\mu V$ can be decomposed into a sum of irreducible modules via (1.3.2), using Schur's Lemma one more time we get that $\mathbb{S}_\nu V$ appears in the decomposition of the tensor product. This means that $N_\lambda^{\nu,\mu} \neq 0$. Consequently, since $N_\lambda^{\nu,\mu} \neq 0$ we have that $\mathbb{S}_\nu V$ is contained in $\mathbb{S}_{\lambda/\mu} V$ by (3.1.4). This concludes the proof. □

Despite Proposition 3.3.1, there are several instances in which the Schur apolarity and the non-abelian apolarity actions coincide. To begin with, by Remarks 2.3.4 and 2.3.5 and Example 3.2.4, both this general apolarities can be restricted to obtain the classical apolarity action and the skew-symmetric apolarity action. In general this happens also for at least any Grassmann variety $\mathbb{G}(k, V)$ embedded with $\mathcal{O}(d)$, $d \geq 1$. Remark that this family includes also the usual Veronese varieties and Grassmann varieties embedded with $\mathcal{O}(1)$. Before proving this fact we have the following.

Lemma 3.3.2. *Consider the partition (d^k) and let $\mu = (\mu_1, \dots, \mu_k)$ be any partition such that $\mu \subset (d^k)$. Then $\mathbb{S}_{(d^k)/\mu} V \simeq \mathbb{S}_\nu V$ is irreducible, where ν is the partition $\nu = (d - \mu_k, d - \mu_{k-1}, \dots, d - \mu_1)$.*

Proof. Let μ be a partition such that $\mu \subset (d^k)$. We may assume that $\mu = (\mu_1, \dots, \mu_k)$, with $\mu_1 \le d$. We prove that $\mathbb{S}_{(d^k)/\mu} V$ is irreducible by showing that $N^{\mu,\nu}_{(d^k)} = 0$ for any partition ν of $dk - |\mu|$, except for $\nu = (d - \mu_k, d - \mu_{k-1}, \dots, d - \mu_1)$ for which $N^{\mu,\nu}_{(d^k)} = 1$. In this last case we must show that there exists only one skew Littlewood-Richardson tableau of shape $(d^k)/\mu$ and content ν.
The partition μ is uniquely determined by the sequence of positive integers $(a_1, \dots, a_k)$, where a_i is the number of columns of length i. Consider then the skew Young diagram of $(d^k)/\mu$. In its diagram we will have a_i columns of length $k - i$, for any $1 \le i < k$, and $d - (a_1 + \dots + a_k)$ columns of length k. In order to compute $N^{\mu,\nu}_{(d^k)}$ for any partition ν of $dk - |\mu|$ we must fill the diagram of $(d^k)/\mu$ with the integers in $[j]$ where $j \le dk$. Let us consider the skew diagram as a matrix. Hence we call the entry of the box of the i-th row and j-th column with $w_{i,j}$.



In order to get a Littlewood-Richardson tableau we have some constraints to respect. Reading from left to right all the sequences must be weakly increasing:

$$w_{i,\mu_j+1} \le \cdots \le w_{i,d}. \tag{3.3.2}$$

for any $1 \le i \le k$. From top to bottom all the sequences must be strictly increasing:

$$w_{1,j} < \cdots < w_{k,j} \tag{3.3.3}$$

for any $1 \le j \le d$. A disclaimer has to be made. Some of the $w_{i,j}$ may not appear since the respective box belongs to a box of μ that has been removed. If this is the situation, every other box in position (k, l) is also removed, where $k \le i$ and $l \le j$, i.e. every other box on the left or above of $w_{i,j}$. In virtue of this, the sequences (3.3.2) and (3.3.3) must start with the appropriate $w_{i,j}$ on the left. We have not written this explicitly to not make the notation too cumbersome. The third constraint we need is that the sequence

$$w_{k,\mu_k+1}\dots w_{k,d}\; w_{k-1,\mu_{k-1}+1}\dots w_{k-1,d}\dots w_{1,\mu_1+1}\dots w_{1,d} \tag{3.3.4}$$

must be a Y-word. The proof is executed row by row of $(d^k)/\mu$ starting from the one at the top. For simplicity we assume that the first row of μ is different from the first row of (d^k). In general one has only to adapt the indexes appearing in the following discussion.
The sequence (3.3.3) tells us that the rightmost column must be filled with a sequence of strictly increasing integers from top to bottom. Moreover by the third constraint (3.3.4) the integer in the right upper corner of the diagram must be a 1. Consequently the constraint (3.3.2) implies that the first row must be filled with all 1's.
Now move to the second row and look at the rightmost entry $w_{2,d}$. Obviously $w_{2,j} > 1$ and hence by (3.3.2) the entries on its left must be less or equal of it. If $w_{2,d} \neq 2$, then the word (3.3.4) would not be a Y-word. Therefore we must have $w_{2,d} = 2$ and hence all the integers appearing on the second row must all be equal either to 1 or 2. Note that there must be at least as many 2's as many 1's in the first rows by (3.3.2). Actually there cannot be more 2's in the second row then the number of 1's in the first since the word (3.3.4) would not be a Y-word. Hence the remaining integers must all be equal to 1.
The proof proceed with the same tricks we have already applied. In the i-th row we must put as many i's as the number of $i-1$'s in the $(i-1)$-th row. Then in the boxes on the left of the sequence of i's we will have as many $(i-1)$'s as the number of $(i-2)$'s in the $(i-1)$-th row. The filling of the i-th row continue in this way until one arrives to add the integer 1 if above there is no box of $(d^k)/\mu$ in order to make (3.3.4) a Y-word. Remark that this is the only possibility due to the constraints (3.3.2), (3.3.3) and (3.3.4).

						1	1	1
				1	1	2	2	2
			1	2	2	3	3	3
$\vdots$								
	1	$\dots$	$k-2$	$k-1$	$k-1$	k	k	k

So far we have proved that there is a unique Littlewood-Richardson tableau of shape $(d^k)/\mu$. Its content can be computed easily. The number of 1's is equal to

$$(d-\mu_1)+(\mu_1-\mu_2)+\cdots+(\mu_{k-1}-\mu_k)=d-\mu_k$$

and in general the number of i's is equal to $d-\mu_i$. Hence the content of this Littlewood-Richardson skew tableau of shape $(d^k)/\mu$ is ν. The fact that this is the unique Littlewood-Richardson skew tableau of shape $(d^k)/\mu$ implies that $N^{\mu,\nu}_{(d^k)}=1$, while $N^{\mu,\eta}_{(d^k)}=0$ for any other partition η of dk. This concludes the proof. $\square$

Corollary 3.3.3. *Let $X=(\mathbb{G}(k,V),\mathcal{O}(d))$, with $d\geq 1$ and $2k\leq n$, embedded in $\mathbb{P}(\mathbb{S}_{(d^k)}V)$. Suppose that via the non-abelian apolarity action one gets the non trivial map*

$$A:\mathbb{S}_{(d^k)}V\otimes\mathbb{S}_\mu V^*\longrightarrow\mathbb{S}_\nu V.$$

Then it coincides with the Schur apolariy action restricted to $\mathbb{S}_\mu V^*\otimes\mathbb{S}_{(d^k)}V$, i.e.

$$\varphi:\mathbb{S}_{(d^k)}V\otimes\mathbb{S}_\mu V^*\longrightarrow\mathbb{S}_{(d^k)/\mu}V.$$

Proof. By hypothesis we have a non trivial map

$$A:\mathbb{S}_{(d^k)}V\otimes\mathbb{S}_\mu V^*\longrightarrow\mathbb{S}_\nu V.$$

Applying Proposition 3.3.1, we get that the module $\mathbb{S}_\nu V$ appears in the decomposition of $\mathbb{S}_{\lambda/\mu}V$ and in particular the map A is the composition of the map φ and the surjective map $\mathbb{S}_{\lambda/\mu}V\longrightarrow\mathbb{S}_\nu V$. By Lemma 3.3.2 this last surjective map is just the identity map and hence the maps A and φ coincides. This concludes the proof. $\square$

Example 3.3.4. So far we have seen cases in which both the non-abelian and Schur apolarity coincide. However there are also instances in which they do not coincide. For instance if $X=\mathbb{F}(1,2;V)$, with $V\simeq\mathbb{K}^3$, embedded with $\mathcal{O}(1,1)$ in $\mathbb{S}_{(2,1)}V$. The line bundle which gives the embedding is

$$L=\bigwedge^2\mathcal{U}_2^\vee\otimes\mathcal{U}_1^\vee=\mathcal{O}(1,1).$$

Consider the line bundle $E=\mathcal{U}_1^\vee$. We have that $H^0(X,E)\simeq\mathbb{S}_{(1)}V^*$ and $E^\vee\otimes L\simeq\bigwedge^2\mathcal{U}_2^\vee$. Moreover $H^0(X,E^\vee\otimes L)^*\simeq\mathbb{S}_{(1,1)}V$. Hence we get the non-abelian apolarity action

$$A:\mathbb{S}_{(2,1)}V\otimes\mathbb{S}_{(1)}V^*\longrightarrow\mathbb{S}_{(1,1)}V.$$

On the other hand the Schur apolarity action restricted to the same domain is

$$\varphi : \mathbb{S}_{(2,1)}V \otimes \mathbb{S}_{(1)}V^* \longrightarrow \mathbb{S}_{(1,1)}V \oplus \mathbb{S}_{(2)}V$$

since $\mathbb{S}_{(2,1)/(1)}V \simeq \mathbb{S}_{(1,1)}V \oplus \mathbb{S}_{(2)}V$ by (3.1.4). Hence the two maps does not coincide in this case.

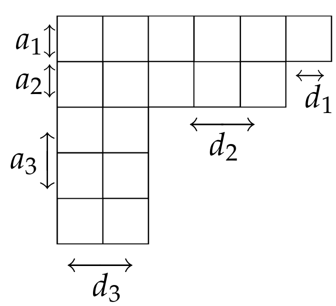
3.4 Schur apolarity lemma

We state and prove here the main result of Schur apolarity theory which is the Schur apolarity lemma. Before that we need a key definition.

Definition 3.4.1. Let $\lambda = (\lambda_1^{a_1}, \dots, \lambda_k^{a_k})$ be a partition of length $l(\lambda) = a_1 + \cdots + a_k < n = \dim(V)$. The minimal orbit $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$ is the flag variety $\mathbb{F}(n_1, \dots, n_k; V)$ embedded with $\mathcal{O}(d_1, \dots, d_k)$ where

$$n_i = \sum_{j=1}^{i} a_i, \text{ and } d_i = \lambda_i - \lambda_{i+1}, \text{ setting } \lambda_{k+1} = 0.$$

For instance, if $\lambda = (6, 5, 2^3)$, then $(d_1, d_2, d_3) = (1, 3, 2)$ and $(n_1, n_2, n_3) = (1, 2, 5)$. Pictorially we have



A point $p \in X$ is of the form

$$p = \left[(v_1 \wedge \cdots \wedge v_{n_k})^{\otimes d_k} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1}\right]$$

and it represents the flag

$$V_1 = \langle v_1, \dots, v_{n_1} \rangle \subset \cdots \subset V_k = \langle v_1, \dots, v_{n_k} \rangle.$$

We may assume that their annihilators are generated by

$$V_1^{\perp} = \langle x_{n_1+1}, \dots, x_n \rangle \supset \cdots \supset V_k^{\perp} = \langle x_{n_k+1}, \dots, x_n \rangle.$$

Consider the spaces

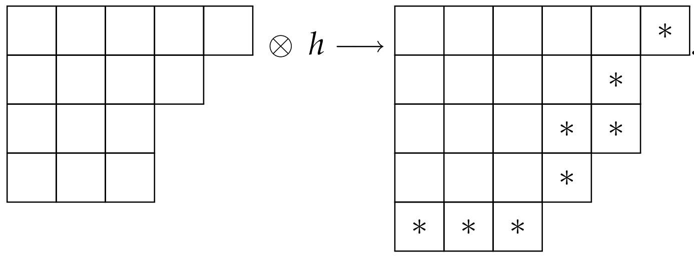
$$\operatorname{Sym}^1 V_k^{\perp},\ \operatorname{Sym}^{d_k+1} V_{k-1}^{\perp}, \dots, \operatorname{Sym}^{d_2+1} V_1^{\perp}, \tag{3.4.1}$$

which we will refer to as "generators". Then we define the *ideal $I(p)$ associated to the point p* as the the smallest vector subspace of $\mathbb{S}^\bullet(V^*)$ that contains the generators and is closed under all multiplication maps.

Remark 3.4.2. Observe that given $p \in \mathbb{S}_\eta V$ of η-rank 1, then the elements of $I(p)$ kill p via the Schur apolarity action. We give an idea of such a fact. To begin with, once p is fixed we can consider the generators of $I(p)$. With an easy computation one can see that any element belonging to one of the spaces appearing between the generators kills p. This is due to the fact that such spaces are given by the equations of some subspace of the flag associated to p, and such that via the Schur apolarity action such elements are evaluated in the part of p representing the chosen subspace.
Assume then to have an element $g \in \mathbb{S}^\bullet V^*$ such that $\varphi(g \otimes p) = 0$, where φ denotes the Schur apolarity action according to Definition 3.2.1. We can assume that $p \in \mathbb{S}_\eta V^*$ and $g \in \mathbb{S}_\lambda V^*$ for some η and λ. If $\lambda \not\subset \eta$, then the Schur apolarity action restricted to $\mathbb{S}_\lambda V^* \otimes \mathbb{S}_\eta V$ is trivial. Therefore, given any $h \in \mathbb{S}_\mu V^*$, for any μ such that $N_\nu^{\lambda,\mu} \neq 0$, then $\mathcal{M}_\nu^{\lambda,\mu}(g \otimes h)$ is still apolar to p since $\lambda \subset \nu \not\subset \eta$ and the restricted Schur apolarity action will be trivial.
Assume then that $\lambda \subset \nu$. Consider now an element $h \in \mathbb{S}_\mu V^*$ and consider the element

$$g \cdot h := \mathcal{M}_\nu^{\lambda,\mu}(g \otimes h) \tag{3.4.2}$$

for some multiplication map $\mathcal{M}_\nu^{\lambda,\mu}$ as defined in the first section of this chapter, which we assume to be non trivial, i.e. $N_\nu^{\lambda,\mu} \neq 0$. If $\nu \not\subset \eta$, then $\varphi(g \cdot h \otimes p) = 0$ since the restricted Schur apolarity action is trivial.
Assume then that $\nu \subset \eta$. We want to prove that $g \cdot h$ kills p via the restricted Schur apolarity action. At first we need to spend a word on the multiplication maps (3.1.5). Even though the definition of such maps is a bit cumbersome, it can be seen that their action is such that pictorially the element h is rearranged in a skew shape and positioned below the part of the tensor dedicated to g



Therefore when computing $\varphi(g \cdot h \otimes p)$, we can see that g is always applied to the same position as it was applied when compuitng $\varphi(g \otimes p)$. Hence we get that $\varphi(g \cdot h \otimes p) = 0$ as well. This is tantamount to the following fact. Consider the diagram

$$\begin{tikzcd}
 & \mathbb{S}_\lambda V^* \otimes \mathbb{S}_\mu V^* \otimes \mathbb{S}_\eta V \arrow[dl] \arrow[dr] & \\
\mathbb{S}_\nu V^* \otimes \mathbb{S}_\eta V \arrow[ddr] & & \mathbb{S}_\mu V^* \otimes \mathbb{S}_{\eta/\lambda} V \arrow[d] \\
 & & \mathbb{S}_{\nu/\lambda} V^* \otimes \mathbb{S}_{\eta/\lambda} V \arrow[dl] \\
 & \mathbb{S}_{\eta/\nu} V &
\end{tikzcd}$$

where on the right hand side we have that

$$g \otimes h \otimes p \mapsto h \otimes \varphi(g \otimes p) \mapsto \varphi(h \otimes \varphi(g \otimes p)) \in \mathbb{S}_{\eta/\nu} V,$$

while on the left

$$g \otimes h \otimes p \mapsto g \cdot h \otimes p \mapsto \varphi(g \cdot h \otimes p) \in \mathbb{S}_{\eta/\nu} V.$$

Then the fact that $\varphi(h \otimes \varphi(g \otimes p)) = 0$ for any $h \in \mathbb{S}_\mu V^*$, it implies that $\varphi(g \cdot h \otimes p) = 0$.

Remark 3.4.3 (Restriction to the symmetric case)**.** Let $p = [l^d] \in \nu_d(\mathbb{P}^n)$ be a point of a Veronese variety, i.e. a point of symmetric rank 1. Regarding $\mathbb{P}^n$ as a Grassmannian, the point p can be seen as the projective class of $l^{\otimes d}$, so that it represents the line $\langle l \rangle$. Hence the respective annihilator $\langle l \rangle^\perp$ is generated by n linear forms. Applying Definition 3.4.1 we get the ideal $I(p) \subset \mathbb{S}^\bullet V^*$. In particular remark that also the multiplication maps $\mathrm{Sym}^d V^* \otimes \mathrm{Sym}^e V^* \longrightarrow \mathrm{Sym}^{d+e} V^*$ are

involved in the definition. They are unique up to scalar multiplication. Hence $I(p) \cap \mathrm{Sym}^\bullet V \subset \mathrm{Sym}^\bullet V^*$ is the ideal of a point of symmetric rank 1 we have introduced in Remark 2.1.6.
Remark that we cannot obtain elements of $I(p) \cap \mathrm{Sym}^e V^*$ for any e from the images of some other multiplication map $\mathcal{M}_\nu^{\lambda,\mu}$ apart from the multiplication maps of the symmetric algebra. This is due to the fact that when we consider the map $\mathcal{M}_\nu^{\lambda,\mu} : \mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V \longrightarrow \mathbb{S}_\nu V$, then we have that $\nu \supset \lambda, \mu$. Therefore if ν is not a partition related to a symmetric power, then when ν is involved in a multiplication map, the image cannot be contained in any symmetric power.

Remark 3.4.4 (Restriction to the skew-symmetric case)**.** The same happens when we consider $p = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V$, a point of the cone over a Grassmannian $\mathbb{G}(k, V)$. Since p represents a k-dimensional space, its annihilator has dimension $n-k$. Applying Definition 3.4.1 we can construct the ideal $I(p)$. In particular, as with the symmetric case, the multiplication maps $\bigwedge^d V^* \otimes \bigwedge^e V^* \longrightarrow \bigwedge^{d+e} V^*$ are involved in the definition of $I(p)$ and again, they are unique up to scalar multiplication. Hence the intersection $I(p) \cap \bigwedge^\bullet V^* = I^\wedge(p)$ is the ideal of the point p as defined in Definition 2.2.9.

Remark 3.4.5. The integer d_1 does not appear between the integers involved in the symmetric powers determining the generators. However this is not a big problem since as Example 3.4.7 will show, such powers are needed to include between the generators some special elements of $\mathbb{S}^\bullet V^*$. On the other hand it is sufficient to keep in mind the case of the Veronese variety, i.e. $X = (\mathbb{F}(1;V), \mathcal{O}(d))$. In such an instance, $d = d_1 \geq 1$ but both with the classical apolarity and the Schur apolarity, the integer d is not involved in the defintion of $I(p)$.

Remark 3.4.6. In the case $\lambda = (\lambda_1^{a_1})$ and $X = (\mathbb{G}(a_1, V), \mathcal{O}(\lambda_1))$, Definition 3.4.1 still applies. Indeed in this case we have only one space between the generators. Namely, if $p \in X$ represent a a_1-dimensional linear space $W \subset V$, then we must take into account only the space $\mathrm{Sym}^1 W^\perp$.

Example 3.4.7. Let $V = \mathbb{C}^3$ and $\lambda = (2,1)$. The minimal orbit inside $\mathbb{P}(\mathbb{S}_{(2,1)}\mathbb{C}^3)$ is the Flag variety $X = \big(\mathbb{F}(1,2;\mathbb{C}^3), \mathcal{O}(1,1)\big)$. Let $\{v_1, v_2, v_3\}$ be a basis of $\mathbb{C}^3$ and $\{x_1, x_2, x_3\}$ be the respective dual basis of $(\mathbb{C}^3)^*$. Let $p = v_1 \wedge v_2 \otimes v_1 \in \mathbb{S}_{(2,1)}\mathbb{C}^3$ be a point of $(2,1)$-rank 1. We may assume that p is the element associated to the sstd tableau

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

and it represents the flag

$$V_1 = \langle v_1 \rangle \subset V_2 = \langle v_1,\ v_2 \rangle.$$

Hence the annihilators are

$$V_1^{\perp} = \langle x_2,\ x_3 \rangle \supset V_2^{\perp} = \langle x_3 \rangle.$$

With this notation the generators are

$$\mathrm{Sym}^1 V_2^{\perp} = V_2^{\perp} \text{ and } \mathrm{Sym}^2 V_1^{\perp} = \langle x_2^2,\ x_2x_3,\ x_3^2 \rangle.$$

For any $\mu \subset (2,1)$ we can check that the ideal $I(p)$ associated to p is such that

$$\begin{gathered} I(p) \cap \mathbb{S}_{(1)}(\mathbb{C}^3)^* = \langle x_3 \rangle, \\ I(p) \cap \mathbb{S}_{(1,1)}(\mathbb{C}^3)^* = \langle x_1 \wedge x_3,\ x_2 \wedge x_3 \rangle, \\ I(p) \cap \mathbb{S}_{(2)}(\mathbb{C}^3)^* = \langle x_1x_3,\ x_2x_3,\ x_3^2,\ x_2^2 \rangle, \\ I(p) \cap \mathbb{S}_{(2,1)}(\mathbb{C}^3)^* = \langle c_{(2,1)}(e_S)\ :\ S \text{ is a sstd tableau in which an entry is } 3 \rangle. \end{gathered}$$

Remark that if we would have chosen $\lambda = (3,1)$, a point of λ-rank 1 has the form

$$v_1 \wedge v_2 \otimes (v_1)^{\otimes 2}$$

and we may assume that it represents the same flag of the previous point. Therefore we may think that p is the element associated to the sstd tableau

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array}.$$

In consequence we get the same annihilators of the flag and also the same generators for $I(p)$. This is no longer true if we count two times the plane instead of the line, i.e. if $\lambda = (3,2)$. Indeed, the point of λ-rank 1 will look like

$$(v_1 \wedge v_2)^{\otimes 2} \otimes v_1.$$

It represents the same flag with the same annihilators, and the associated sstd tableau is

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}.$$

However the generators of $I(p)$ in this case will be the spaces $V_2^{\perp}$ and $\mathrm{Sym}^3 V_1^{\perp}$. In this example it is clear the reason why we pick symmetric powers of the annihilators. They allow us to evaluate the equations belonging to the fixed annihilator to the respective vector subspace of the flag.

Before stating and proving the Schur apolarity lemma, cf. Theorem 3.4.9, we need a preparatory fact.

Lemma 3.4.8. *Let λ be a partition of length less then n. Let $p \in X \subset \mathbb{P}(\mathbb{S}_\lambda V)$ be a point of λ-rank 1. Then we have the equality*

$$I(p)_\lambda = (p^{\perp})_\lambda$$

where $I(p)_\lambda$ and $(p^{\perp})_\lambda$ denotes $I(p) \cap \mathbb{S}_\lambda V^*$ and $(p^{\perp}) \cap \mathbb{S}_\lambda V^*$ respectively.

Proof. The proof is performed step by step. At first we treat the case of λ being a rectangular partition. Then we move to the case in which λ is union of two rectangles and eventually the general case.
Let us begin with λ being a rectangular partition, i.e. something like $\lambda = (d^k)$. Then we may assume that

$$p = (v_1 \wedge \cdots \wedge v_k)^{\otimes d}$$

and also we may assume that it is the highest weight vector of the irreducible representation $\mathbb{S}_{(d^k)}V$. Hence p is associated to the sstd tableaux of shape λ

$$U = \begin{array}{|c|c|c|c|}\hline 1 & \dots & \dots & 1 \\ \hline \vdots & \vdots & \vdots & \vdots \\ \hline k & \dots & \dots & k \\ \hline \end{array}\,.$$

In this case the minimal orbit X is the Grassmann variety $(\mathbb{G}(k,V), \mathcal{O}(d))$. The point p represents the subspace $\langle v_1, \ldots, v_k \rangle \subset V$ and its annihilator is $W = \langle x_{k+1}, \ldots, x_n \rangle$. By the Definition 3.4.1, the ideal $I(p)$ is determined by

$$\mathrm{Sym}^1 W = \langle x_{k+1}, \ldots, x_n \rangle.$$

Using the description of a basis of $\mathbb{S}_\lambda V^*$ given in Proposition 1.3.36, it is easy to see that all those elements whose associated sstd tableau of shape λ is different from U kills p via the Schur apolarity action and hence they generate the hyperplane $(p^{\perp})_\lambda$. Now we prove that all these elements belong to $I(p)$ and this will give us

the inclusion $I(p)_\lambda \supset (p^\perp)_\lambda$. The equality follows for dimensional reasons since the only element missing is the one whose associated sstd tableau is U.
Fix an element of $(p^\perp)_\lambda$ determined by a sstd tableau S of shape λ different from U. The construction of Schur modules via Young symmetrizers shows that this element is built by first symmetrizing along rows and then skew-symmetrizing along columns a particular tensor in $V^{\otimes dk}$. This construction can be written applying consecutively the maps

$$\mathcal{M}^{(d^i),(d)}_{(d^{i+1})} : \mathbb{S}_{(d^i)}V^* \otimes \operatorname{Sym}^d V^* \longrightarrow \mathbb{S}_{(d^{i+1})}V^*,$$

i.e. adding row by row of S, for i from 1 to $k-1$. Recall that such maps are unique up to scalar multiplication since the respective Littlewood-Richardson coefficient is 1. Note that, up to choosing the correct sign, one can add the rows in any order. Remark also that since S is different from U, at least one integer from $k+1$ to n appears in S and it represents one of the equations $x_{k+1}, \ldots, x_n$ which we put as generators of the ideal of a point. Moreover, these integers appear at least in the last row since the i-th row of a sstd tableaux has entries greater or equal than i by definition. Hence look for the first row of S in which an integer from $k+1$ to n appears and consider the row R above it. Then we can see the element $c_{(d^k)}(e_S)$ of $(p^\perp)_\lambda$ as image via the map

$$\mathcal{M}^{(d^{k-1}),(d)}_{(d^k)} : \mathbb{S}_{(d^{k-1})}V^* \otimes \mathbb{S}_{(d)}V^* \longrightarrow \mathbb{S}_{(d^k)}V^*$$

of the elements $c_{(d^{k-1})}(e_{S_1})$ and $c_{(d)}(e_{S_2})$ of $\mathbb{S}_{(d^{k-1})}V^*$ and $\mathbb{S}_{(d)}V^*$, where S_1 is S without R and $S_2 = R$, possibly with a minus sign. Let us make an example to clear the ideas. If the tableau S is

$$S = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \hline 3 & 3 & 4 \\ \hline \end{array}$$

then the respective element is

$$\begin{aligned} c_{(3^3)}(e_S) &= v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_3 \wedge v_4 + \\ &\quad v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_3 \wedge v_4 \otimes v_1 \wedge v_2 \wedge v_3 + \\ &\quad v_1 \wedge v_3 \wedge v_4 \otimes v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \wedge v_3, \end{aligned}$$

and it can be seen as product via $\mathcal{M}^{(3^2),(3)}_{(3^3)}$ of the elements $-c_{(3^2)}(e_{S_1})$ and $c_{(3)}(e_{S_2})$, where

$$S_1 = \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 3 & 3 & 4 \\ \hline \end{array}$$

and

$$S_2 = \begin{array}{|c|c|c|} \hline 2 & 2 & 3 \\ \hline \end{array}.$$

Applying the same argument to the element associated to S_1, one can conclude via induction on the rows that the element associated to S_1 is an element of $I(p)$. Hence the element associated to S belongs to the $I(p)$ by definition. This concludes the case in which λ is a rectangular partition.

Assume now that λ is given by the union of two rectangles. This means that $\lambda = ((d+e)^k, e^{h-k})$, where $d, e > 0$ and $0 < h < k$. The minimal orbit X is the Flag variety $(\mathbb{F}(k, h; V), \mathcal{O}(d, e))$ and a point $p \in X$ has the form

$$p = (v_1 \wedge \cdots \wedge v_h)^{\otimes e} \otimes (v_1 \wedge \cdots \wedge v_k)^{\otimes d}$$

and we may also assume that it is the highest weight vector of the irreducible representation $\mathbb{S}_\lambda V$. This time the sstd tableau U representing p is

$$U = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & \dots & \dots & \dots & \dots & \dots & 1 \\ \hline \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline k & \dots & \dots & \dots & k & \dots & k \\ \hline \vdots & \vdots & \vdots & \vdots \\ \cline{1-4} h & \dots & \dots & h \\ \cline{1-4} \end{array}.$$

Arguing as in the previous case, it is easy to see that $(p^\perp)_\lambda$ is the hyperplane in $\mathbb{S}_\lambda V^*$ generated by all those elements whose associated sstd tableau of shape λ is different from U. Fix one of these elements and call S the associated sstd tableau. We prove again that $I(p)_\lambda \supset (p^\perp)_\lambda$. The equality follows for dimensional reasons. We discriminate some cases. At first assume that S contains an integer from $h+1$ to n, meaning that in the respective element some of the linear forms $x_{h+1}, \dots, x_n$ appears. This integer can appear either in $h \times e$ rectangle on the left, or on the $k \times d$ rectangle on the right. Suppose that such integer appears in the first $h \times e$ rectangle. Consider then the map

$$\mathcal{M}_\lambda^{(e^h),(d^k)} : \mathbb{S}_{(e^h)} V^* \otimes \mathbb{S}_{(d^k)} V^* \longrightarrow \mathbb{S}_\lambda V^*.$$

In this case we can see that the element $c_\lambda(e_S)$ is the image via this map of the elements $c_{(e^h)}(e_{S_1})$ and $c_{(d^k)}(e_{S_2})$. The sstd tableaux S_1 and S_2 have shape (e^h) and (d^k) respectively, and they are given by the rectangle $h \times e$ on the left of S_1 and the rectangle $k \times d$ on the right of S_1 respectively. This happens because

$$N_\lambda^{(e^h),(d^k)} = 1,\ \mathbb{S}_{\lambda/(e^h)}V^* \simeq \mathbb{S}_{(d^k)}V^*$$

and the only Littlewood-Richardson skew tableau of shape $\lambda/(e^h)$ and content (d^k) is the one given by all i's in the i-th row. Arguing as in the rectangular case at the beginning of the proof, one can see that $c_{(e^h)}(e_{S_1})$ is an element of $I(p)$.
Suppose then that the integer from $h+1$ to n appears in the second $k \times d$ rectangle. Consider then the map

$$\mathcal{M}_\lambda^{((d+e)^k),(e^{h-k})} : \mathbb{S}_{((d+e)^k)}V^* \otimes \mathbb{S}_{(e^{h-k})}V^* \longrightarrow \mathbb{S}_\lambda V^*.$$

Again the element $c_\lambda(e_S)$ can be seen as image via this map of the elements $c_{((d+e)^k)}(e_{S_1})$ and $c_{(e^{h-k})}(e_{S_2})$. The sstd tableaux S_1 and S_2 have shape $((d+e)^k)$ and (e^{h-k}) respectively, and they are given by the top $k \times (d+e)$ rectangle of S and the bottom $(h-k) \times e$ rectangle of S respectively. Again, this happens because

$$N_\lambda^{((d+e)^k),(e^{h-k})} = 1,\ \mathbb{S}_{\lambda/((d+e)^k)}V^* \simeq \mathbb{S}_{(e^{h-k})}V^*$$

and the only Littlewood-Richardson skew tableau of shape $\lambda/((d+e)^k)$ and content (e^{h-k}) is the one with all i's in the i-th row. Using another time the rectangular case discussed at the beginning of this proof, one can see that $c_{((d+e)^k)}(e_{S_1})$ is an element of $I(p)$.
Suppose now that none of the integers $h+1,\dots,n$ appears in S. This means that the $h \times e$ rectangle of S on the left is equal to the one of U. On the other hand, the $k \times d$ rectangle on the right must differ from the same rectangle contained in U. Hence at least one integer from $k+1$ to h appears in the $k \times d$ rectangle on the right. Observe now that following the construction of this element via the Young symmetrizer, some symmetrizations along rows are prescribed. Indeed note that permutations that fix the rectangle $k \times d$ on the right, fix also the other rectangle since this one has the i-th row filled with i's. Moreover, if a permutation exchange an element from the rectangle on the right with the one on the left we get only two situations. If we exchange two equal numbers, we get the same picture we started with; otherwise after skew-symmetrizing along columns we get the zero element. This happens since we will get two equal numbers in the same column due to the fact that all the integers of the second rectangle are contained in the first one. For example consider the sstd tableau

$$S = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&2&2&3\\ \hline 3&3&\multicolumn{2}{c}{}\\ \cline{1-2} 4&4&\multicolumn{2}{c}{}\\ \cline{1-2}\end{array},$$

where we have coloured in different ways the rectangles we have referred to so far. Then whenever we exchange integers along rows between the two rectangles we obtain either a diagram with an integer repeated at least twice in a column, or we get that the first rectangle is the same. Then using the Young symmetrizer, the first one goes to zero. This allows us to perform the following construction. Select an integer that appears in the $k \times d$ rectangle on the right of S which is not in the $k \times d$ rectangle of U. For instance if S is as above, in that case U is

$$U = \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\ \hline 2&2&2&2\\ \hline 3&3&\multicolumn{2}{c}{}\\ \cline{1-2} 4&4&\multicolumn{2}{c}{}\\ \cline{1-2}\end{array},$$

and we may consider the number 3 since it appears in the 2×2 rectangle on the right of S but not in the same rectangle of U. Coming back to S, we can see the element $c_\lambda(e_S)$ as image of a series of products. At first we have to trim the rows of S which do not contain the chosen integer in the $h - k$ bottom rows of the $h \times e$ rectangle on the left. This gives us that $c_\lambda(e_S)$ is image of the product

$$S = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&2&2&3\\ \hline 3&3&\multicolumn{2}{c}{}\\ \cline{1-2} 4&4&\multicolumn{2}{c}{}\\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\ \hline 2&2&2&3\\ \hline 3&3&\multicolumn{2}{c}{}\\ \cline{1-2}\end{array} \otimes \begin{array}{|c|c|}\hline 4&4\\ \hline\end{array},$$

where we are omitting the symbol of the multiplication map and of the Young symmetrizer to not complicate the notation. This means that when we read the tableaux above we are actually considering the image via the suitable multiplication map, of the images via the respective Young symmetrizers of the elements associated to those tableaux on the right hand side of the equality. Let us call S_1 the tableau on the left in the product. We can now notice it can be written as the product

$$S_1 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 3 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 3 & 3 & 3 \\ \hline\end{array} \otimes \begin{array}{|c|c|}\hline 2 & 2 \\ \hline\end{array} \tag{3.4.3}$$

where we are again omitting the various symbols to not complicate the notation. It is easy to see that this holds also in general. Let us call S_2 the first tableau on the right hand side of the equality in (3.4.3). Then proceeding as in the rectangular case on can see that

$$S_2 = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 2 & 3 \\ \hline\end{array} = -\begin{array}{|c|c|c|c|}\hline 2 & 3 & 3 & 3 \\ \hline\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline\end{array} \tag{3.4.4}$$

where we are again omitting the various symbols. Let us call S_3 the first tableau on the right hand side of the equality (3.4.4). One can see that $c_{(4)}(e_{S_3})$ is an element of $I(p)$ by its definition and we can conclude that $c_\lambda(e_S)$ belongs to $I(p)$.
The general case, i.e. when more than 2 subspaces are involved follows exactly as the cases discussed above. Also in this case we may choose p as the highest weight vector of the representation, whose associated sstd tableau of shape λ is $U(\lambda)$, and $(p^\perp)_\lambda$ is generated by all $c_\lambda(e_S)$ where S is a sstd tableau of shape λ different from U. Then one proves the inclusion $I(p)_\lambda \supset (p^\perp)_\lambda$ and the equality will follow for dimensional reasons. Eventually one has to discuss several cases divided on which integers appear in S. This concludes the proof. □

We are now ready to state the main result of the theory.

Theorem 3.4.9 (Lemma of Schur apolarity)**.** *Let λ be a partition of length less than n and let $\mathbb{S}_\lambda V$ be an irreducible representation of $SL(n)$ together with the respective minimal orbit $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$. Let $f \in \mathbb{P}(\mathbb{S}_\lambda V)$ and let also $p_1, \dots, p_r \in X$. Then the following are equivalent:*

(1) *there exists $c_1, \dots, c_r \in \mathbb{K}$ such that $f = \sum_{i=1}^r c_i p_i$,*

(2) *$I(p_1, \dots, p_r) := \bigcap_{i=1}^r I(p_i) \subset f^\perp$.*

Proof. We prove the two implications separately. Assume at first that $f = c_1 p_1 + \cdots + c_r p_r$. Since the elements of the subspace $I(p_i)$ kill p_i via Schur apolarity action for every p_i, we get that every element $I(p_i, \dots, p_r) = \bigcap_{i=1}^r I(p_i)$ kills every element in $\langle p_1, \dots, p_r \rangle$, and hence kills f.

Assume now that the inclusion $I(p_1, \dots, p_r) \subset f^\perp$ holds. Using the notation of Lemma 3.4.8, this clearly implies that $I(p_1, \dots, p_r)_\lambda \subset (f^\perp)_\lambda$. By Lemma 3.4.8 this means that

$$I(p_1, \dots, p_r)_\lambda = \bigcap_{i=1}^{r} I(p_i)_\lambda = \bigcap_{i=1}^{r} (p_i^\perp)_\lambda \subset (f^\perp)_\lambda. \tag{3.4.5}$$

We may consider then the restricted Schur apolarity action

$$\varphi : \mathbb{S}_\lambda V \otimes \mathbb{S}_\lambda V^* \longrightarrow \mathbb{K}$$

which is clearly a perfect pairing. Moreover the induced catalecticant map $\mathcal{C}_g^{\lambda,\lambda}$ is such that $\ker \mathcal{C}_g^{\lambda,\lambda} = (g^\perp)_\lambda$, for any $g \in \mathbb{S}_\lambda V$. Hence we may interpret the hyperplane $(g^\perp)_\lambda$ as the set of linear forms which vanish on the line $\langle g \rangle$. Hence the last inclusion in (3.4.5), i.e.

$$\bigcap_{i=1}^{r} (p_i^\perp)_\lambda \subset (f^\perp)_\lambda$$

can be translated as the fact that the linear forms vanishing on the span $\langle p_1, \dots, p_r \rangle$ vanish also on the line $\langle f \rangle$. This clearly implies that $\langle f \rangle \subset \langle p_1, \dots, p_r \rangle$. This concludes the proof. $\square$

Example 3.4.10. Consider $\lambda = (2,1)$ and $t_1 = v_1 \wedge v_2 \otimes v_1$, $t_2 = v_3 \wedge v_4 \otimes v_4 \in \mathbb{S}_{(2,1)}\mathbb{K}^4$ points of $(2,1)$-rank 1. Let $t = t_1 + t_2$. We can compute that

$$\begin{aligned} t^\perp = & \langle 0 \rangle \oplus \langle x_1x_3,\ x_1x_4,\ x_2^2,\ x_2x_3,\ x_2x_4,\ x_3^2 \rangle \oplus \langle x_1 \wedge x_3,\ x_1 \wedge x_4,\ x_2 \wedge x_3,\ x_2 \wedge x_4 \rangle \oplus \\ & \oplus \left\langle \begin{array}{|c|c|} \hline i & j \\ \hline k \\ \cline{1-1} \end{array},\ \text{with } i \leq j,\ i < k, \text{ except } (i,j,k) = (1,1,2),\ (3,4,4),\ \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} - \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 4 \\ \cline{1-1} \end{array} \right\rangle \oplus \\ & \oplus \bigoplus_{\mu \not\subset (2,1)} \mathbb{S}_\mu(\mathbb{K}^4)^* \end{aligned}$$

where the non-trivial spaces are the ones determined by the partitions (1), (2), $(1,1)$ and $(2,1)$ with this order. In the last explicit linear span the notation with the sstd tableau stands for the element of the basis determined by such tableau according to Proposition 1.3.36. We have omitted the symbol of Young symmetrizer and tensor products to simplify the notation.

The first thing we can remark is that t has $(2,1)$-rank 1. Indeed this is due to the fact that $\ker \mathcal{C}_t^{(2,1),(1)}$ does not contain any non-zero element. Moreover, we may compute that

$$\begin{aligned} I(t_1) =& \langle x_3,\ x_4\rangle \oplus \langle x_1x_3,\ x_1x_4,\ x_2^2,\ x_2x_3,\ x_2x_4,\ x_3^2,\ x_3x_4,\ x_4^2\rangle \oplus \\ & \oplus \langle x_1\wedge x_3,\ x_1\wedge x_4,\ x_2\wedge x_3,\ x_2\wedge x_4,\ x_3\wedge x_4\rangle \oplus \\ & \oplus \left\langle \begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array},\ \text{with } i\leq j,\ i<k,\ \text{except } (i,j,k)=(1,1,2) \right\rangle \oplus \dots \end{aligned}$$

and

$$\begin{aligned} I(t_2) =& \langle x_1,\ x_2\rangle \oplus \langle x_1^2,\ x_1x_2,\ x_1x_3,\ x_1x_4,\ x_2^2,\ x_2x_3,\ x_2x_4,\ x_3^2\rangle \oplus \\ & \oplus \langle x_1\wedge x_2,\ x_1\wedge x_3,\ x_1\wedge x_4,\ x_2\wedge x_3,\ x_2\wedge x_4\rangle \oplus \\ & \oplus \left\langle \begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array},\ \text{with } i\leq j,\ i<k,\ \text{except } (i,j,k)=(3,4,4) \right\rangle \oplus \dots \end{aligned}$$

and hence

$$\begin{aligned} I(t_1,t_2) =& \langle 0\rangle \oplus \langle x_1x_3,\ x_1x_4,\ x_2^2,\ x_2x_3,\ x_2x_4,\ x_3^2\rangle \oplus \\ & \oplus \langle x_1\wedge x_3,\ x_1\wedge x_4,\ x_2\wedge x_3,\ x_2\wedge x_4\rangle \oplus \\ & \oplus \left\langle \begin{array}{|c|c|}\hline i & j \\ \hline k \\ \cline{1-1}\end{array},\ \text{with } i\leq j,\ i<k,\ \text{except } (i,j,k)=(1,1,2),\ (3,4,4) \right\rangle \oplus \dots \end{aligned}$$

which can be readily seen to be contained in $t^\perp$.

Example 3.4.11. Consider the complete flag variety $\mathbb{F}(1,2,3,\mathbb{K}^4)$ embedded with $\mathcal{O}(1,1,1)$ in $\mathbb{P}(\mathbb{S}_{(3,2,1)}\mathbb{K}^4)$. We would like to compute the $(3,2,1)$-rank of the element

$$t = v_1\wedge v_2\wedge v_3\otimes v_1\wedge v_2\otimes v_3 - v_1\wedge v_2\wedge v_3\otimes v_2\wedge v_3\otimes v_1 \in \mathbb{S}_{(3,2,1)}\mathbb{K}^4.$$

Remark at first that t cannot have $(3,2,1)$-rank 1. Indeed in general a point p of $(3,2,1)$-rank 1 is such that the rank of the catalecticant map $\mathcal{C}_p^{(3,2,1),(2)}$ is 5, while in this case $\mathcal{C}_t^{(3,2,1),(2)}$ has rank 6.

Consider the tensors t_1 and t_2 of $(3,2,1)$-rank 1 such that they represent the flags

$$\langle v_1 + v_3 \rangle \subset \langle v_1 + v_3,\ v_2 \rangle \subset \langle v_1 + v_3,\ v_2,\ v_3 \rangle$$

and

$$\langle v_1 - v_3 \rangle \subset \langle v_1 - v_3,\ v_2 \rangle \subset \langle v_1 - v_3,\ v_2,\ v_3 \rangle$$

respectively. Let us call these spaces (V_1, V_2, V_3) and (W_1, W_2, W_3) respectively. The respective annihilators are

$$\langle x_4, x_1 - x_3, x_2 \rangle \supset \langle x_4, x_1 - x_3 \rangle \supset \langle x_4 \rangle$$

and

$$\langle x_4, x_1 + x_3, x_2 \rangle \supset \langle x_4, x_1 + x_3 \rangle \supset \langle x_4 \rangle.$$

Call these spaces $(V_1^\perp, V_2^\perp, V_3^\perp)$ and $(W_1^\perp, W_2^\perp, W_3^\perp)$ respectively. We check the inclusions $I(t_1, t_2) \cap \mathrm{Sym}^i(\mathbb{K}^4)^* \subset t^\perp \cap \mathrm{Sym}^i(\mathbb{K}^4)^*$ for $i = 1, 2, 3$, i.e. only for the generators of $I(t_1)$ and $I(t_2)$. We have that

$$(I(t_1) \cap I(t_2)) \cap \mathrm{Sym}^1(\mathbb{K}^4)^* = V_3^\perp \cap W_3^\perp = \langle x_4 \rangle = \ker \mathcal{C}_t^{(3,2,1),(1)},$$

$$\begin{aligned}(I(t_1) \cap I(t_2)) \cap \mathrm{Sym}^2(\mathbb{K}^4)^* = \mathrm{Sym}^2 V_2^\perp \cap \mathrm{Sym}^2 W_2^\perp = \langle x_4^2 \rangle \subset \\ \subset \ker \mathcal{C}_t^{(3,2,1),(2)} = \langle x_1x_4,\ x_2x_4,\ x_3x_4,\ x_4^2 \rangle\end{aligned}$$

$$\begin{aligned}&(I(t_1) \cap I(t_2)) \cap \mathrm{Sym}^3(\mathbb{K}^4)^* = \mathrm{Sym}^3 V_1^\perp \cap \mathrm{Sym}^3 W_1^\perp = \langle x_2^2x_4,\ x_2^3,\ x_2x_4^2,\ x_4^3 \rangle \subset \\ &\subset \langle x_1^3, x_1^2x_4, x_1x_2x_4, x_1x_3x_4, x_1x_4^2, x_2^3, x_2^2x_4, x_2x_3x_4, x_2x_4^2, x_3^3, x_3^2x_4, x_3x_4^2, x_4^3 \rangle\end{aligned}$$

where the last linear span is $\ker \mathcal{C}_t^{(3,2,1),(2)}$. Hence from Theorem 3.4.9 we get that $t \in \langle t_1, t_2 \rangle$. Solving a simple linear system we can write

$$\begin{aligned}t = \frac{1}{2}(v_1 + v_3) \wedge v_2 \wedge v_3 \otimes (v_1 + v_3) \wedge v_2 \otimes (v_1 + v_3) + \\ - \frac{1}{2}(v_1 - v_3) \wedge v_2 \wedge v_3 \otimes (v_1 - v_3) \wedge v_2 \otimes (v_1 - v_3)\end{aligned}$$

where the first and second summands are t_1 and t_2 respectively.

Chapter 4

Some computations with Schur apolarity

This chapter is devoted to give an insight on the most direct application of the Schur apolarity theory, i.e. the computation of the λ-rank of a tensor. Some of the results below have been extracted in the work [Sta21].

The first section is dedicated to make explicit the relation between λ-rank and the rank of catalecticant maps. In the second and third section we investigate the second secant variety of the flag variety $X = (\mathbb{F}(1,k;V), \mathcal{O}(1,1))$ and of any Grassmann variety $X = (\mathbb{G}(k,V), \mathcal{O}(d))$, with $d > 1$, respectively.

4.1 Catalecticants and λ-rank

This section is devoted to exploit the link between λ-rank 1 tensors and the rank of catalecticant maps. In particular we will see that if a tensor of $\mathbb{S}_\lambda V$ has λ-rank 1, then this will be equivalent to have specific ranks for some catalecticant maps. The following obvious remark occurs.

Remark 4.1.1. Of course one can use the equations of the minimal orbit $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$ to test if the chosen point has λ-rank 1. However, if for a point $p \in X$ we get $r = \operatorname{rk}\left(\mathcal{C}_p^{\lambda,\mu}\right)$, then by the homogeneity of X and by the subadditivity of the rank of linear maps we get that if $t = t_1 + \cdots + t_s$, with t_i of λ-rank 1, then

$$\operatorname{rk}\left(\mathcal{C}_t^{\lambda,\mu}\right) = \operatorname{rk}\left(\mathcal{C}_{t_1+\cdots+t_s}^{\lambda,\mu}\right) \leq \sum_{i=1}^{s} \operatorname{rk}\left(\mathcal{C}_{t_i}^{\lambda,\mu}\right) = r \cdot s.$$

for $t = t_1 + \cdots + t_s$, with $t_i \in \mathbb{S}_\lambda V$ of λ-rank 1. This means that given any $t \in \mathbb{S}_\lambda V$, if $\mathrm{rk}\left(\mathcal{C}_t^{\lambda,\mu}\right) > r \cdot s$, then t has λ-rank equal at least to $s+1$. We will use this trick in this chapter.

4.1.1 An example with $k = d = 2$ and $n = 4$.

Let us begin with the case $X = (\mathbb{G}(k,V), \mathcal{O}(d))$ included in $\mathbb{P}\left(\mathbb{S}_{(d^k)} V\right)$. As described in Section 1.5 the points of X represent k-dimensional linear subspaces of V repeated d times, i.e.

$$[(v_1 \wedge \cdots \wedge v_k)]^{\otimes d} \in X$$

for some $v_1, \ldots, v_k \in V$. As observed in the beginning of this section, to start our study we can pick any point $p \in X$. To ease our computations consider the highest weight vector of the representation. This implies that we get also a description of p in terms of a std Young tableau. By Definition 3.2.7, whenever $\mu \subset (d^k)$ we get a non-trivial catalecticant map

$$\mathcal{C}_p^{(d^k),\mu} : \mathbb{S}_\mu V^* \longrightarrow \mathbb{S}_{(d^k)/\mu} V.$$

Even for small values of d and k there are plenty of these maps and an analysis of all of them is not advisable. Therefore we start with a couple of examples to make the situation clearer and after that we will generalize.

Consider the variety $X = \left(\mathbb{G}(2,\mathbb{K}^4), \mathcal{O}(2)\right)$ embedded in $\mathbb{P}(\mathbb{S}_{(2,2)}\mathbb{K}^4) \simeq \mathbb{P}^{19}$. The partitions $\mu \subset (2,2)$ are only $\mu = (1)$, $(1,1)$, (2), $(2,1)$ and $(2,2)$. We describe some of the related catalecticant maps. We start with $\mu = (1)$.

Proposition 4.1.2. *Let $p \in \mathbb{P}(\mathbb{S}_{(2,2)}\mathbb{K}^4)$. Then p has $(2,2)$-rank 1 if and only if $\mathrm{rk}(\mathcal{C}_p^{(2,2),(1)}) = 2$.*

Proof. Assume that p has $(2,2)$-rank 1, i.e. $p = \left[(v_1 \wedge v_2)^{\otimes 2}\right] \in X$, for some $v_i \in \mathbb{K}^4$. Assume that $\{v_1, \ldots, v_4\}$ is a basis of $\mathbb{K}^4$, and that $\{x_1, \ldots, x_4\}$ is the respective dual basis of $(\mathbb{K}^4)^*$. Then

$$\mathcal{C}_p^{(2,2),(1)}(x) = x(v_1)v_2 \otimes v_1 \wedge v_2 - x(v_2)v_1 \otimes v_1 \wedge v_2 \in \mathbb{S}_{(2,2)/(1)}\mathbb{K}^4.$$

Imposing the conditions $x(v_1) = x(v_2) = 0$, it follows that $\ker \mathcal{C}_p^{(2,2),(1)} = \langle x_3, x_4 \rangle$. Hence the rank of the map is 2.

On the other hand, assume that $\mathrm{rk}\left(\mathcal{C}_p^{(2,2),(1)}\right) = 2$. For dimensional reasons this implies that in the kernel of the catalecticant map there are two independent linear forms of $(\mathbb{K}^4)^*$ which vanish on p. Assume then that p is a sum of $(2,2)$-rank 1 tensors $p_1, \ldots, p_r$. We want to prove that $r = 1$. We can assume that the p_i's are linearly independent. Therefore the respective vector subspaces represented by them must be different.
Since the map $\mathcal{C}_p^{(2,2),(1)}$ evaluates linear forms on these subspaces, if two points or more are independent we should get strictly less then 2 linear forms belonging to the kernel. This is tantamount to the research of linear forms that vanish on the sum of the respective vector subspaces represented by the p_i. Being different subspaces, their sum must be at least of dimension 3. Hence in the kernel we should find at most 1 linear form if $r > 1$. Hence the only possibility is that all the p_i's are multiple each other and hence $p = p_1$ which is what we wanted. □

Remark 4.1.3. As the previous proof has lightly showed, if $p \in X$, then the generating elements of $\ker \mathcal{C}_p^{(2,2),(1)}$ are exactly the equations whose vanishing locus is the subspace of $\mathbb{K}^4$ associated to p. Moreover, applying Remark 4.1.1 we get that given any $t \in \mathbb{S}_{(2,2)}\mathbb{K}^4$, then if $\mathrm{rk}\, \mathcal{C}_t^{(2,2),(1)} = r$, then t has $(2,2)$-rank at least $\lceil \frac{r}{2} \rceil$. Unfortunately, since the dimension of the domain of $\mathcal{C}_t^{(2,2),(1)}$ is 4 it is evident that the trick can give information up to $(2,2)$-rank 2.

Proposition 4.1.4. *Let $p \in \mathbb{P}(\mathbb{S}_{(2,2)}\mathbb{K}^4)$. Then p has $(2,2)$-rank 1 if and only if* $\mathrm{rk}\left(\mathcal{C}_p^{(2,2),(1,1)}\right) = 1$.

Proof. Suppose that $p \in \mathbb{S}_{(2,2)}\mathbb{K}^4$ has $(2,2)$-rank 1, i.e. $p = \left[(v_1 \wedge v_2)^{\otimes 2}\right] \in X$ for some $v_i \in \mathbb{K}^4$. Let us denote with $\{v_1, \ldots, v_4\}$ a basis of $\mathbb{K}^4$ and let $\{x_1, \ldots, x_4\}$ be the respective dual basis of $(\mathbb{K}^4)^*$. Then

$$\mathcal{C}_p^{(2,2),(1,1)}(x_1 \wedge x_2) = \det \begin{pmatrix} x_1(v_1) & x_1(v_2) \\ x_2(v_1) & x_2(v_2) \end{pmatrix} \cdot v_1 \wedge v_2.$$

Imposing the vanishing of the determinant we get that $\ker \mathcal{C}_p^{(2,2),(1,1)} = \langle x_i \wedge x_j : i < j,\ j = 3, 4 \rangle$. Hence the rank of the map is 1.
Assume now that $\mathrm{rk}\left(\mathcal{C}_p^{(2,2),(1,1)}\right) = 1$. Suppose that p is a sum of $(2,2)$-rank 1 elements $p_1, \ldots, p_r$. We want to prove that $r = 1$. We may write $p_i = (v_{1,i} \wedge v_{2,i})^{\otimes 2}$ with $i = 1, \ldots, r$, for some vectors $v_{j,i}$.
By action of the catalecticant map, it is clear that $p = c_1 p_1 + \cdots + c_r p_r$ if and only if $\mathrm{im}(\mathcal{C}_t^{(2,2),(1,1)}) \subset \langle v_{1,i} \wedge v_{2,i},\ i = 1, \ldots, r \rangle$. Suppose by contradiction that p has

$(2,2)$-rank 2 or more. Then we can find at least two points p_i and p_j such that $v_{1,i} \wedge v_{2,i}$ and $v_{1,i} \wedge v_{2,i}$ are independent. Without loss of generality, up to rename the vectors we may assume that $p_i = p_1 = (v_1 \wedge v_2)^{\otimes 2}$. On the other hand we may assume that $p_j = p_2$. Being p_1 and p_2 independent, we must have that $p_2 \neq p_1$ and hence up to rename the vectors we can have only two cases for p_2. The first is $p_2 = v_1 \wedge v_3$, while the second is $p_2 = v_3 \wedge v_4$, for some vectors v_3 and v_4. Remark that their corresponding subspaces intersects the subspace represented by p_1 in dimension either 1 or 0 respectively. Using the dual basis, it is easy to see that the image of the catalecticant map must contain either $v_1 \wedge v_2$ and $v_1 \wedge v_3$, or $v_1 \wedge v_2$ and $v_3 \wedge v_4$. This implies that the rank of the map is at least 2, in contradiction with the hypothesis. Hence p must have $(2,2)$-rank 1 and this concludes the proof. $\square$

Example 4.1.5. The previous proposition together with Remark 4.1.1 can be applied to compute the $(2,2)$-rank in some easy cases. For instance consider

$$t = (v_1 \wedge v_2)^{\otimes 2} + (v_1 \wedge v_3)^{\otimes 2} + (v_1 \wedge v_4)^{\otimes 2} + (v_2 \wedge v_3)^{\otimes 2} + (v_2 \wedge v_4)^{\otimes 2} + (v_3 \wedge v_4)^{\otimes 2}.$$

Obviously by this presentation we have that the $(2,2)$-rank of t is at most 6. One can easily compute that $\mathcal{C}_t^{(2,2),(1,1)}$ is the 6×6 identity matrix whose rank is obviously 6. Using Remark 4.1.1 we can say that t has $(2,2)$-rank at least 6. Since t is written with a $(2,2)$-rank 6 decomposition, we can immediately conclude that the scripture is minimal and t has $(2,2)$-rank 6.

The cases $\mu = (2), (2,1)$ can be treated with analogous arguments and hence we omit them. The case $\mu = (2,2)$ is trivial since the catalecticant matrix is a vector and has always rank 1 if computed on non zero elements. In Table 4.1 we collect the results of this study applied to all the $\mu \subset (2,2)$. The cases (1) and $(2,1)$ have been put together since the respective catalecticant maps are one the transpose of the other. In the second column the rank of the catalecticant is computed on a $(2,2)$-rank 1 element. The column on right displays the order of the linear map.

μ	$\mathrm{rk}(\mathcal{C}_t^{(2,2),\mu})$	order of the matrix
$(1), (2,1)$	2	4×20 / 20×4
(2)	3	10×10
$(1,1)$	1	6×6

Table 4.1: Ranks of the catalecticants on $(2,2)$-rank 1 elements.

Looking at the Table 4.1 it is evident that the catalecticant map which gives more information about the $(2,2)$-rank is the one determined by $\mu = (1,1)$. Note that roughly this is the catalecticant map which chops a column of the diagram of $(2,2)$ and splits it in two equal halves.

4.1.2 The case $X = (\mathbb{G}(k,V), \mathcal{O}(d))$, with any k, n and d.

We generalize now to the case of any $X = (\mathbb{G}(k,V), \mathcal{O}(d))$, $d \geq 1$. In this case the respective partition is $\lambda = (d^k)$ for any d and k positive integers. By the duality of Grassmannians we can assume that $2k \leq \dim(V)$. The diagram of λ is a $k \times d$ rectangle.
In the case $k = d = 2$ we have seen that the catalecticant map which erases one column give the best bound. We would like to check in general if the catalecticant maps that erase a certain number of columns give good bounds on the (d^k)-rank of a tensor. Since we cannot check the rank of every existing catalecticant map, we will only focus on this kind of catalecticant maps. Of course we will not be able to say that the bound determined by these maps is the best one achievable via catalecticants.

Let $p \in X$ be a point of (d^k)-rank 1. As already remarked we can take p as the highest weight of the representation and write

$$p = \left[(v_1 \wedge \cdots \wedge v_k)^{\otimes d}\right].$$

Let $h \leq d$ and consider the partition (h^k). Pictorially it is the $k \times h$ rectangle and it is obviously contained in the left upper corner of the diagram of λ. We can consider the map

$$\mathcal{C}_p^{(d^k),(h^k)} : \mathbb{S}_{(h^k)} V^* \longrightarrow \mathbb{S}_{(d^k)/(h^k)} V.$$

By Proposition 3.3.2 we have that $\mathbb{S}_{(d^k)/(h^k)} V \simeq \mathbb{S}_{(d-h)^k} V$. Remark also that

$$\mathcal{C}_p^{(d^k),(h^k)} = \left(\mathcal{C}_p^{(d^k),((d-h)^k)}\right)^t$$

and hence it makes sense to focus only to the case $h \leq \lfloor \frac{d}{2} \rfloor$.

Proposition 4.1.6. *Let $p \in \mathbb{P}(\mathbb{S}_{(d^k)} V)$. Then p has (d^k)-rank 1 if and only if $\mathrm{rk}(\mathcal{C}_p^{(d^k),(h^k))}) = 1$ for any $h \leq \lfloor \frac{d}{2} \rfloor$.*

Proof. Suppose that p has (d^k)-rank 1. We may write

$$p = \left[(v_1 \wedge \cdots \wedge v_k)^{\otimes d}\right].$$

Then it follows that

$$\operatorname{im}\left(\mathcal{C}_p^{(d^k),(h^k)}\right) = \langle (v_1 \wedge \cdots \wedge v_k)^{\otimes d-h} \rangle$$

and hence the rank of the map is 1.
Suppose now that $\operatorname{rk}\left(\mathcal{C}_p^{(d^k),(h^k)}\right) = 1$ and that p can be written as sum of $p_1, \dots, p_r$ of (d^k)-rank 1. We can write each p_i as $p_i = (v_{1,i} \wedge \cdots \wedge v_{k,i})^{\otimes d}$. If p has (d^k)-rank either 2 or more, then there are at least two linearly independent points p_i and p_j in its decomposition. In particular the respective subspaces will be different. Hence we will have that $(v_{1,i} \wedge \cdots \wedge v_{k,i})^{\otimes d-h}$ and $(v_{1,j} \wedge \cdots \wedge v_{k,j})^{\otimes d-h}$ are contained in the image of the catalecticant map. Since these last elements are linearly independent the rank of the map will be at least 2. This cannot happen since by hypothesis the rank of the map must be 1. Hence $r = 1$ and p must have (d^k)-rank 1. This concludes the proof. □

Remark 4.1.7. Given a tensor $t \in \mathbb{S}_{(d^k)}\mathbb{K}^n$, by Remark 4.1.1 we get that if the rank of $\mathcal{C}_t^{(d^k),(h^k)}$ is equal to r, then t has (d^k)-rank at least r. Since by a simple count

$$\dim \mathbb{S}_{(h^k)}\mathbb{K}^n = \left(\prod_{j=k}^{n-1} \frac{h+j}{j}\right)\left(\prod_{j=k-2}^{n-2} \frac{h+j}{j}\right)\cdots\left(\prod_{j=1}^{n-k} \frac{h+j}{j}\right)$$

we have that $\dim \mathbb{S}_{(a^k)}\mathbb{K}^n > \mathbb{S}_{(b^k)}\mathbb{K}^n$ if $a > b$. Hence the most square catalecticant map is the one which gives more information. It is achieved when $h = \lceil \frac{d}{2} \rceil$.

4.1.3 The case of flag varieties

We discuss now the case of any flag variety. The situation here is a bit different with respect to the Grassmannian case as the following example shows.

Example 4.1.8. Consider the complete flag variety $\mathbb{F}(1,2,3;\mathbb{K}^4)$ embedded with $\mathcal{O}(1,1,1)$ in $\mathbb{P}(\mathbb{S}_{(3,2,1)}\mathbb{K}^4)$. Consider the element $t \in \mathbb{S}_{(3,2,1)}\mathbb{K}^4$

$$t = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \otimes v_1 + v_1 \wedge v_2 \wedge v_3 \otimes v_2 \wedge v_3 \otimes v_3. \tag{4.1.1}$$

The formula (4.1.1) is a decomposition of t as a sum of two $(3,2,1)$-rank 1 elements. Hence the $(3,2,1)$-rank of t is at most 2. The $(3,2,1)$-rank 1 elements represent the flags

$$\langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, v_3 \rangle, \quad \langle v_3 \rangle \subset \langle v_2, v_3 \rangle \subset \langle v_1, v_2, v_3 \rangle.$$

Consider for a moment only the first addend

$$t_1 = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \otimes v_1$$

and consider the catalecticant map $\mathcal{C}_{t_1}^{(3,2,1),(2)} : \mathbb{S}_{(2)}(\mathbb{K}^4)^* \longrightarrow \mathbb{S}_{(3,2,1)/(2)}\mathbb{K}^4$. We have that

$$\begin{aligned}
&\mathcal{C}_{t_1}^{(3,2,1),(2)}(\alpha\beta) = \\
&= \sum_{i=1}^{3}(-1)^{i+1}\left[\alpha(v_i)\beta(v_1) + \alpha(v_1)\beta(v_i)\right] \cdot v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_3 \otimes v_2 \otimes v_1 + \\
&- \sum_{i=1}^{3}(-1)^{i+1}\left[\alpha(v_i)\beta(v_2) + \alpha(v_2)\beta(v_i)\right] \cdot v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_3 \otimes v_1 \otimes v_1.
\end{aligned}$$

It is easy to see that

$$\ker \mathcal{C}_{t_1}^{(3,2,1),(2)} = \langle x_1x_4, x_2x_4, x_3x_4, x_4^2, x_3^2 \rangle$$

and hence $\mathrm{rk}(\mathcal{C}_{t_1}^{(3,2,1),(2)}) = 10$. As already remarked, this rank number must be the same for any $t \in \mathbb{F}(1,2,3;\mathbb{K}^4)$. Coming back to t, if we consider the same catalecticant map we get

$$\ker \mathcal{C}_{t}^{(3,2,1),(2)} = \langle x_1x_4, x_2x_4, x_3x_4, x_4^2 \rangle$$

which means that $\mathrm{rk}(\mathcal{C}_{t_1}^{(3,2,1),(2)}) = 11$. This implies that t cannot have $(3,2,1)$-rank 1 and hence the decomposition (4.1.1) is minimal.

In contrast with the case of Grassmann varieties, erasing a column of the diagram of $(3,2,1)$ may not give the right information about the $(3,2,1)$-rank 1. Indeed if we consider the partition $(1,1,1) \subset (3,2,1)$ together with the respective catalecticant map we have that

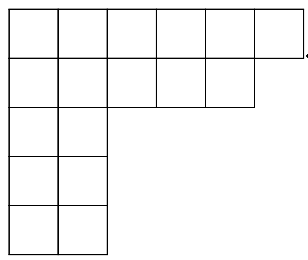
$$\ker \mathcal{C}_{t}^{(3,2,1),(1,1,1)} = \langle x_1 \wedge x_2 \wedge x_4, x_1 \wedge x_3 \wedge x_4, x_2 \wedge x_3 \wedge x_4 \rangle$$

and hence $\mathrm{rk}(\mathcal{C}_t^{(3,2,1),(1,1,1)}) = 1$ which happens also for any $t \in \mathbb{F}(1,2,3;\mathbb{K}^4)$. This is due to the fact that the two flags involved in the decomposition have in common the same biggest subspace $\langle v_1, v_2, v_3 \rangle$.

Despite the strange behaviour of the rank of catalecticant maps and the λ-rank of tensors, we are able to give a lower bound on the λ-rank of a tensor. Let $X = (\mathbb{F}(n_1, \ldots, n_s; \mathbb{K}^n), \mathcal{O}(d_1, \ldots, d_s))$ in $\mathbb{P}(\mathbb{S}_\lambda V)$. Hence we have that

$$\lambda = \left((d_1 + \cdots + d_s)^{n_1}, (d_2 + \cdots + d_s)^{n_2 - n_1}, \ldots, d_s^{n_s - n_{s-1}}\right).$$

The respective Young diagram is given by d_s columns of length n_s, then d_{s-1} columns of length n_{s-1} and so on up to d_1 columns of length n_1. For example if $X = \left(\mathbb{F}(1,2,5;\mathbb{K}^6), \mathcal{O}(1,3,2)\right)$, then $\lambda = (6,5,2^3)$ and its Young diagram is



Proposition 4.1.9. *Let λ be a partition such that the longest column in its diagram has length n_s and it is repeated d_s times. Consider the catalecticant map determined by $(e^{n_s}) \subset \lambda$ with $e \leq d_s$, i.e. the one that removes the first e columns of length n_s from the diagram of λ. Then*

$$\mathbb{S}_{\lambda/(e^{n_s})} V \simeq \mathbb{S}_{\mu_e} V$$

where

$$\mu_e = ((d_1 + \cdots + d_{s-1} + (d_s - e))^{n_1},\ (d_2 + \cdots + d_{s-1} + (d_s - e))^{n_2 - n_1}, \ldots \\ \ldots, (d_{s-1} + (d_s - e))^{n_{s-1} - n_{s-2}}, (d_s - e)^{n_s}) \tag{4.1.2}$$

i.e. λ with the first e columns of length n_s removed.

The proof can be done performing verbatim the same steps of the proof of the Proposition 3.3.2 and hence we do not report it here.

Algorithm 4.1.10. In order to get a bound on the λ-rank using catalecticants, we give an algorithm that via the computation of a sequence of ranks of "consecutive"

catalecticant maps returns a lower bound on the λ-rank of the given tensor. The last registered rank will be a lower bound on the λ-rank of t.
Before going along with the description of the procedure, we need a preparatory fact.

Proposition 4.1.11. *Let λ be a partition with d_i columns of length n_i, with $i = 1, \ldots, s$, and let $t \in \mathbb{S}_\lambda V$. If*

$$\operatorname{rk} \mathcal{C}_t^{\lambda,\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}} = 1, \text{ then for any } 1 \leq e \leq d_s, \text{ we have that } \operatorname{rk} \mathcal{C}_t^{\lambda,(e^{n_s})} = 1.$$

Proof. Let us prove the contraposition, i.e. if there exists an $e \in \{1, \ldots, d_s\}$ such that

$$\operatorname{rk} \mathcal{C}_t^{\lambda,(e^{n_s})} > 1, \text{ then } \operatorname{rk} \mathcal{C}_t^{\lambda,\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}} > 1.$$

Assume that for a certain $e \in \{1, \ldots, d_s\}$ it happens that $\operatorname{rk} \mathcal{C}_t^{\lambda,(e^{n_s})} > 1$. From the hypothesis we can assume that the λ-rank of t is at least 2, otherwise it can be easily seen that the rank of $\mathcal{C}_t^{\lambda,(e^{n_s})}$ would be equal to 1 which is against the hypothesis. Assume then that $t = t_1 + \cdots + t_r$ has λ-rank $r \geq 2$, where every t_i has λ-rank 1 and it is written as

$$t_i = (v_{1,i} \wedge \cdots \wedge v_{n_s,i})^{\otimes d_s} \otimes \cdots \otimes (v_{1,i} \wedge \cdots \wedge v_{n_1,i})^{\otimes d_1}$$

for some vectors $v_{1,i}, \ldots, v_{n_s,i} \in V$, for all $i = 1, \ldots, r$. The catalecticant map $\mathcal{C}_t^{\lambda,(e^{n_s})}$ clearly acts on the first products $(v_{1,i} \wedge \cdots \wedge v_{n_s,i})^{\otimes d_s}$ of every t_i. Since the rank of such map is at least 2, we can find at least two points t_i and t_j such that

$$(v_{1,i} \wedge \cdots \wedge v_{n_s,i}) \neq (v_{1,j} \wedge \cdots \wedge v_{n_s,j})$$

and also such that the images via the catalecticant map of the respective duals $(x_{1,i} \wedge \cdots \wedge x_{n_s,i})^{\otimes e}$ and $(x_{1,j} \wedge \cdots \wedge x_{n_s,j})^{\otimes e}$ are linearly independent. Consider now the catalecticant map

$$\mathcal{C}_t^{\lambda,\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}} : \mathbb{S}_{\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}} V^* \longrightarrow \mathbb{S}_{\mu_{\lceil \frac{d_s}{2} \rceil}} V$$

and the elements $(x_{1,i} \wedge \cdots \wedge x_{n_s,i})^{\otimes \lceil \frac{d_s}{2} \rceil}$ and $(x_{1,j} \wedge \cdots \wedge x_{n_s,j})^{\otimes \lceil \frac{d_s}{2} \rceil}$. It is clear that they are linearly independent and that their images via $\mathcal{C}_t^{\lambda,\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}}$ are also linearly independent. Therefore we get that the rank of $\mathcal{C}_t^{\lambda,\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}}$ is at least 2. By contraposition we get that if $\operatorname{rk} \mathcal{C}_t^{\lambda,\left(\lceil \frac{d_s}{2} \rceil\right)^{n_s}} = 1$, then $\operatorname{rk} \mathcal{C}_t^{\lambda,(e^{n_s})} = 1$ for any $1 \leq e \leq d_s$. This concludes the proof. □

The procedure of the algorithm we propose for a lower bound of the λ-rank of a given $t \in \mathbb{S}_\lambda V$ is defined in this way. Let $t \in \mathbb{S}_\lambda V$ and consider the catalecticant map that removes half of the d_s columns of maximal length from the diagram of λ, possibly rounded up to the next integer if needed. Compute the rank of such a catalecticant map

$$\mathcal{C}_t^{\lambda,(\lceil \frac{d}{2} \rceil^{n_s})} : \mathbb{S}_{(\lceil \frac{d}{2} \rceil^{n_s})} V^* \longrightarrow \mathbb{S}_{\mu_{\lceil \frac{d}{2} \rceil}} V,$$

where $\mu_{\lceil \frac{d}{2} \rceil}$ denotes the partition as in (4.1.2). If the rank of the map is strictly greater than 1, the algorithm stops and outputs this number. Such a number is a lower bound on the λ-rank of t. Otherwise, by Proposition 4.1.11 we get that

$$\operatorname{rk} \mathcal{C}_t^{\lambda,e^{n_s}} = 1,$$

for any $1 \leq e \leq d_s$. Therefore we can consider the catalecticant with $e = d_s$ and its image will be generated by only one element up to scalar multiplication. Such image is contained in $\mathbb{S}_{\mu_{d_s}} V$, where μ_{d_s} denotes the partition with d_i columns of length n_i, with $i = 1, \ldots, s-1$, accordingly with the notation used in (4.1.2). Choose a generator t_1 of the image of the chosen catalecticant map. Then set $\lambda = \mu_{d_s}$ and $t = t_1$ and repeat the previous steps.

In general at the i-th step we set $\lambda = \lambda_i$, where λ_i is the diagram with d_j columns of length n_j for $j = 1, \ldots, s-i+1$, and we set $t = t_i$, where $t_i \in \mathbb{S}_{\lambda_i} V$ is computed as the generator of the one dimensional image from the previous step of the algorithm. Then compute

$$\operatorname{rk}\left(\mathcal{C}_{t_i}^{\lambda_i,\left\lceil \frac{d_{s-i+1}}{2} \right\rceil^{n_{s-j}}} \right).$$

If it is strictly greater than 1 the algorithm outputs this number and it stops. Otherwise consider the catalecticant map that removes all the columns of length d_{s-i+1}. Compute the generator $t_{i+1} \in \mathbb{S}_{\lambda_{i+1}} V$ of the image of this last catalecticant map, set $\lambda = \lambda_{i+1}$ and $t = t_{i+1}$, and move to the $(i+1)$-th step.

In the case that the rank of every catalecticant map we meet along the procedure is 1, the algorithm eventually outputs 1. Obviously the output of the algorithm is an integer greater or equal than 1 and it is a lower bound on the λ-rank of t. Indeed, preliminary we have

Proposition 4.1.12. *Consider the flag variety $\mathbb{F}(n_1, \dots, n_s; V)$ embedded with $\mathcal{O}(d_1, \dots, d_s)$ in $\mathbb{P}(\mathbb{S}_\lambda V)$. Let $t \in \mathbb{S}_\lambda V$ be any point. If t has λ-rank 1, then the Algorithm 4.1.10 outputs a 1. The converse is true if $d_i \geq 2$ for any $i = 1, \dots, s$.*

Proof. Assume that t has λ-rank 1, i.e.

$$t = (v_1 \wedge \cdots \wedge v_{n_s})^{\otimes d_s} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1}$$

for some $v_i \in V$. The image of the first catalecticant map of the algorithm, i.e. the one determined by $e = 1$ and the partition (1^{n_s}), is the span of

$$(v_1 \wedge \cdots \wedge v_{n_s})^{\otimes d_s - 1} \otimes (v_1 \wedge \cdots \wedge v_{n_{s-1}})^{\otimes d_{s-1}} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1}$$

and hence the map has rank 1 since t is non zero. The same happens for the next steps until $e = \lfloor \frac{d_s}{2} \rfloor$. Therefore consider the catalecticant map that removes all the first d_s columns and consider the only generator t_{s-1} of its image

$$t_{s-1} = (v_1 \wedge \cdots \wedge v_{n_{s-1}})^{\otimes d_{s-1}} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1}.$$

At this point set $t = t_{s-1}$ and $\lambda = \mu_{d_s}$ and repeat the previous steps. It is obvious that the algorithm will not stop when computing the rank of any catalecticant map since any such number is equal to 1. Therefore it eventually outputs 1.
For the converse part, assume that $d_i \geq 2$ for any i and suppose that the output of the algorithm is 1. Suppose t have a λ-rank r decomposition $t = t_1 + \cdots + t_r$, with t_i of λ-rank 1. We may suppose that

$$t_i = (v_{1,i} \wedge \cdots \wedge v_{n_s,i})^{\otimes d_s} \otimes \cdots \otimes (v_{1,i} \wedge \cdots \wedge v_{n_1,i})^{\otimes d_1}$$

for some vectors $v_{1,i}, \dots, v_{n_s,i} \in V$, for all $i = 1, \dots, r$. For any $1 \leq e \leq d_s$, the image of the catalecticant map is one dimensional by Proposition 4.1.11 and it is contained in $\langle t_1^e, \dots, t_r^e \rangle$, where t_i^e denotes

$$t_i = (v_{1,i} \wedge \cdots \wedge v_{n_s,i})^{\otimes d_s - e} \otimes \cdots \otimes (v_{1,i} \wedge \cdots \wedge v_{n_1,i})^{\otimes d_1}$$

for all $i = 1, \dots, r$. We can assume that the element $(x_{1,1} \wedge \cdots \wedge x_{n_s,1})^{\otimes e}$, defined taking the dual elements $x_{i,j}$ to the vectors appearing in the biggest subspace associated to t_1, is not apolar to t. Hence its image is a generator of the one dimensional image of the respective catalecticant map. If such an element of $\mathbb{S}_{(e^{n_s})} V^*$

is the only one with this property, then we can already conclude that the biggest subspace must be the same for any t_i. Otherwise, if we could find another element defined in the same way as $(x_{1,1} \wedge \cdots \wedge x_{n_s,1})^{\otimes e}$ but using this time the other t_i's, then its image must be a scalar multiple of the image we have already obtained. Hence a certain linear combination of these two elements is apolar to t. On the other hand in the respective images of the two selected elements of $\mathbb{S}_{(e^{n_s})}V^*$ there are also the tensors t_1^e and t_i^e which are linearly independent, unless $e = d_s$ and $t_1^{d_s} = t_i^{d_s}$ which happens only if the points t_1 and t_i share the same remaining part of the flag. However, since we are assuming $d_s > 1$, and since the algorithm is settled to pick $\lceil \frac{d_s}{2} \rceil$, we are always considering $e < d_s$ that allows to avoid such a problem. Hence, if the rank of the catalecticant is 1, we get that all the subspaces are the same. Therefore the image of the catalecticant map is one dimensional and is generated by $t' = t_1^{d_s} + \cdots + t_r^{d_s}$. At this point the proof is just a repetition of the previous reasoning until one arrives to get $t_1 = \cdots = t_r$, i.e. t has λ-rank 1. This concludes the proof. □

Corollary 4.1.13. *Let $t \in \mathbb{S}_\lambda V$ and suppose that the output of the Algorithm 4.1.10 applied to t is r. Then the λ-rank of t is greater or equal than r.*

Proof. The corollary follows directly from the combination of the Proposition 4.1.12 and the Remark 4.1.1. Indeed, we have seen in Proposition 4.1.12 that if $p \in \mathbb{S}_\lambda V$ has λ-rank 1, then the output of Algorithm 4.1.10 is 1. This in particular means that at every step of the algorithm the rank of the selected catalecticant map is equal to 1. Then if for a $t \in \mathbb{S}_\lambda V$ the Algorithm 4.1.10 applied to t outputs $r \geq 1$, it follows that this number has been reached before computing all the catalecticant maps involved in the procedure. Using Remark 4.1.1, we can already conclude that t will have λ-rank at least $\lceil \frac{r}{1} \rceil = r$. □

We resume briefly the Algorithm 4.1.10 in general for any $t \in \mathbb{S}_\lambda V$.

Algorithm 4.1.10.

Input: A partition λ and an element $t \in \mathbb{S}_\lambda V$, where the related minimal orbit inside the projectivization of the space is $X = (\mathbb{F}(n_1, \ldots, n_s; V), \mathcal{O}(d_1, \ldots, d_s))$.
Output: A lower bound of the λ-rank of t.

1) set $i = s$;
2) set $r = 0$;
3) **if** $i = 0$, **then**
4) **print** 1 **and exit**;
5) set $r = \operatorname{rk} \mathcal{C}_t^{\lambda,(\lceil \frac{d_i}{2} \rceil^{n_i})}$;

6) **if** $r > 1$, **then**
7) **print** r and **exit**;
8) consider the map $\mathcal{C}_t^{\lambda,(d_i^{n_i})}$ and compute the only generator t' of the image;
9) set $i = i - 1$, $t = t'$ and $\lambda = \lambda_{i-1}$ where this last partition is the one with d_j columns of length n_j, for $j = 1, \dots, i-1$. Come back to 3);

Remark 4.1.14. Let $t \in \mathbb{S}_\lambda V$ and suppose that λ has diagram with d_i columns of length n_i, with $i = 1, \dots, s$ and such that some d_i is equal to 1. Then if the Algorithm 4.1.10 outputs 1, obviously this does not imply that t has λ-rank 1. Indeed, consider as an example of this phenomenon the partition $\lambda = (5,3,1,1)$ and the tensor $t \in \mathbb{S}_\lambda V$

$$t = v_1 \wedge v_2 \wedge v_3 \wedge v_4 \otimes v_1 \wedge v_2 \otimes v_1 + v_1 \wedge v_2 \wedge v_5 \wedge v_6 \otimes v_1 \wedge v_2 \otimes v_1.$$

The output of the Algorithm 4.1.10 in this case is 1. This is due to the fact that the two points share the same partial flag $\langle v_1 \rangle \subset \langle v_1,\ v_2 \rangle$. Hence in the kernel of the first catalecticant map of the algorithm we will find the element $x_1 \wedge x_2 \wedge x_3 \wedge x_4$ $-x_1 \wedge x_2 \wedge x_5 \wedge x_6$. On the other hand it is clear that t has λ-rank 2.

Example 4.1.15 (Example 4.1.8 reprise)**.** Consider the tensor

$$t = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \otimes v_1 + v_1 \wedge v_2 \wedge v_3 \otimes v_2 \wedge v_3 \otimes v_3 \in \mathbb{P}\left(\mathbb{S}_{(3,2,1)}\mathbb{K}^4\right).$$

Following the notation of the Algorithm 4.1.10, when $i = s = 3$, then $\lambda = (3,2,1)$ and the first catalecticant map is

$$\mathcal{C}_t^{(3,2,1),(1,1,1)} : \mathbb{S}_{(1,1,1)}\left(\mathbb{K}^4\right)^* \longrightarrow \mathbb{S}_{(2,1)}\mathbb{K}^4.$$

As we have already seen in Example 4.1.8 this map has rank 1. Hence we are at the step 11) of the algorithm. Up to scalar multiplication, the generator of the image of $\mathcal{C}_t^{\lambda,(1^3)}$ is

$$t' = v_1 \wedge v_2 \otimes v_1 + v_2 \wedge v_3 \otimes v_3.$$

Set $\lambda = (2,1)$, $i = 2$ and $t = t'$. This time we have the map

$$\mathcal{C}_t^{(2,1),(1,1)} : \mathbb{S}_{(1,1)}\left(\mathbb{K}^4\right)^* \longrightarrow \mathbb{S}_{(1)}\mathbb{K}^4.$$

It is easy to see that its rank is 2. Hence the algorithm stops and the output is 2. As Corollary 4.1.13 shows, we get that t has λ-rank at least 2.

Corollary 4.1.16. *Let λ be a partition with d_i columns of length n_i, with $i = 1, \dots, s$ such that $d_i \geq 2$ for any i. Then the 2×2 minors of the catalecticant maps $\mathcal{C}_t^{\lambda_j,(\lceil \frac{d_j}{2} \rceil^{n_j})}$ give set-theoretic equations of the minimal orbit contained in $\mathbb{P}(\mathbb{S}_\lambda V)$, where λ_j is the partition with d_i columns of length n_i, with $i = 1, \dots, s - j + 1$.*

Remark 4.1.17. There is a geometric interpretation behind Example 4.1.8 and Remark 4.1.14. Indeed, in general we may consider the map of varieties

$$\mathbb{F}(n_1, \dots, n_k; V) \longrightarrow \mathbb{F}(n_1, \dots, n_{k-1}; V)$$

that sends $(V_1, \dots, V_{k-1}, V_k)$ to $(V_1, \dots, V_{k-1})$, i.e. it forgets the biggest subspace of the flag. This morphism is surjective; indeed the variety $\mathbb{F}(n_1, \dots, n_k; V)$ is a vector bundle over the smaller flag variety. This remark allows one to describe tensors starting from smaller flag varieties. For instance consider the flag variety $\mathbb{F}(1, 2, \mathbb{K}^4)$ and the $(2, 1)$-rank 2 tensor

$$t = v_1 \wedge v_2 \otimes v_1 + v_2 \wedge v_3 \otimes v_3.$$

Call the addends t_1 and t_2 respectively. Then we may consider the map

$$\pi : \mathbb{F}(1, 2, 3; \mathbb{K}^4) \longrightarrow \mathbb{F}(1, 2; \mathbb{K}^4)$$

and consider the fibres

$$\pi^{-1}(t_1) = \{(\langle v_1 \rangle, \langle v_1, v_2 \rangle, W) \ : \ \langle v_1 \rangle \subset \langle v_1, v_2 \rangle, \subset W \subset \mathbb{K}^4\}$$

and

$$\pi^{-1}(t_2) = \{(\langle v_3 \rangle, \langle v_2, v_3 \rangle, W) \ : \ \langle v_3 \rangle \subset \langle v_2, v_3 \rangle, \subset W \subset \mathbb{K}^4\}.$$

It is not difficult to see that we can find a 3-dimensional subspace W of $\mathbb{K}^4$ that contains both the flags represented by t_1 and t_2, namely $W = \langle v_1, v_2, v_3 \rangle$. For this it is enough to consider the projection

$$\rho : \mathbb{F}(1, 2, 3; \mathbb{K}^4) \longrightarrow \mathbb{G}(3, \mathbb{K}^4)$$

that sends (V_1, V_2, V_3) to V_3, and to compute $\rho(\pi^{-1}(t_1)) \cap \rho(\pi^{-1}(t_2))$. Hence we can describe two elements $\tilde{t}_1$ and $\tilde{t}_2$ of $\mathbb{S}_{(3,2,1)} V$ that represent the same flags of t_1 and t_2 both contained in W respectively. Namely they are

$$\tilde{t}_1 + \tilde{t}_2 = v_1 \wedge v_2 \wedge v_3 \otimes v_1 \wedge v_2 \otimes v_1 + v_1 \wedge v_2 \wedge v_3 \otimes v_2 \wedge v_3 \otimes v_3$$

i.e. the same element in Example 4.1.8. This trick allows one to build several examples of tensors with different λ-ranks starting from tensors of "smaller" formats.
Call λ_1-rank and λ_2-rank the X-ranks with respect the "smaller" and "bigger" flag variety obtained adding a subspace to the flag respectively . It is evident that the λ_2-rank of the resulting tensor is the λ_1-rank of the previous tensor. Moreover it is clear that also any non identifiability is preserved via this process. A similar procedure to get Remark 4.1.14 can be done as well.

4.1.4 The concise space of a tensor

We end this section with a classical study of the minimal space in which a specific tensor live. Such a study turns out to be useful to find a minimal decomposition of a tensor.

Definition 4.1.18. Let $t \in \mathbb{S}_\lambda V$ be any tensor. The minimal vector subspace $V' \subset V$ such that $t \in \mathbb{S}_\lambda V'$ is called the *concise space of t* .

Let us recall at first the following basic fact

Proposition 4.1.19 (See p.80 of [FH13]). *Let V be a vector space such that $V = V' \oplus V''$ for some vector subspaces $V', V'' \subset V$. Then the following equality holds*

$$\mathbb{S}_\nu V = \mathbb{S}_\nu \left(V' \oplus V''\right) \simeq \bigoplus_{\lambda,\, \mu \,:\, |\lambda|+|\mu|=|\nu|} N_\nu^{\lambda,\mu} \left(\mathbb{S}_\lambda V' \otimes \mathbb{S}_\mu V''\right). \tag{4.1.1}$$

Proposition 4.1.20. *Let $t \in \mathbb{S}_\lambda V$ be a tensor of λ-rank r whose concise space is $V' \subsetneq V$. If $t = t_1 + \cdots + t_s$ is any other decomposition of t such that there exists an i for which $t_i \notin \mathbb{S}_\lambda V'$, then $s > r$.*

Proof. Assume that $t = t_1 + \cdots + t_s$ is such that its concise space is $V' \subset V$ and suppose that there exists an $i \in \{1, \ldots, s\}$ such that $t_i \in \mathbb{S}_\lambda V'$. We want to prove that $s > r$.
Assume by contradiction that $s = r$. Fix a subspace $V'' \subset V$ such that $V = V' \oplus V''$. Suppose that

$$t_j = (v_{1,j} \wedge \cdots \wedge v_{n_k,j})^{\otimes d_k} \otimes \cdots \otimes (v_{1,j} \wedge \cdots \wedge v_{n_1,j})^{\otimes d_1}$$

for some vectors $v_{i,j} \in V$, for $i = 1, \ldots, n_k$ and $j = 1, \ldots, r$. From the decomposition $V = V' \oplus V''$ we can uniquely decompose any $v_{i,j} = v'_{i,j} + v''_{i,j}$, with $v'_{i,j} \in V'$

and $v''_{i,j} \in V''$, for $i = 1, \dots, n_k$ and $j = 1, \dots, r$. Therefore from (4.1.1) we get a decomposition $t_j = t'_j + t''_j$ for any $j = 1, \dots, r$, where

$$t_j = (v'_{1,j} \wedge \cdots \wedge v'_{n_k,j})^{\otimes d_k} \otimes \cdots \otimes (v'_{1,j} \wedge \cdots \wedge v'_{n_1,j})^{\otimes d_1} \in \mathbb{S}_\lambda V' = \mathbb{S}_\lambda V' \otimes \mathbb{S}_{(0)} V'',$$

has λ-rank 1, and $t'' \in \mathbb{S}_\lambda(V' \oplus V'') \setminus \mathbb{S}_\lambda V'$. Since t is sum of the t_i's, we get that

$$t - t'_1 - \cdots - t'_r = t''_1 + \cdots + t''_r.$$

Since the left hand side and the right hand side of the equality live in two spaces which are in direct sum from (4.1.1), we have that both must vanish. Hence we get $t = t'_1 + \cdots + t'_r$ and that

$$0 = t''_1 + \cdots + t''_r \in \bigoplus_{\nu,\ \mu:\ N_\lambda^{\nu,\mu} \neq 0,\ \nu \neq \lambda} \mathbb{S}_\nu V' \otimes \mathbb{S}_\mu V''. \tag{4.1.2}$$

From (4.1.1) we can decompose any t''_j as

$$t''_j = \sum_{\nu,\ \mu:\ N_\lambda^{\nu,\mu} \neq 0,\ \nu \neq \lambda} t_j^{\nu,\mu},$$

where $t_j^{\nu,\mu} \in \mathbb{S}_\nu V' \otimes \mathbb{S}_\mu V''$. Since the sum of the t''_j must vanish according to (4.1.2), we get that

$$\sum_{j=1}^r t''_j = \sum_{j=1}^r \left(\sum_{\nu,\ \mu:\ N_\lambda^{\nu,\mu} \neq 0,\ \nu \neq \lambda} t_j^{\nu,\mu} \right) = \sum_{\nu,\ \mu:\ N_\lambda^{\nu,\mu} \neq 0,\ \nu \neq \lambda} \left(\sum_{j=1}^r t_j^{\nu,\mu} \right) = 0.$$

Since we are adding elements which come from vector spaces that are in direct sum, they are linearly independent, so we get the condition

$$t_1^{\nu,\mu} + \cdots + t_r^{\nu,\mu} = 0$$

for any ν and μ such that $\nu \neq \lambda$ and $N_\lambda^{\nu,\mu} \neq 0$. Since $\sum_{j=1}^r t_j^{\nu,\mu} \in \mathbb{S}_\nu V' \otimes \mathbb{S}_\mu V''$, we can see such a sum as a matrix. Therefore we may write

$$t_1^{\nu,\mu} + \cdots + t_r^{\nu,\mu} = \sum_k w'_k \otimes w''_k \tag{4.1.3}$$

for some $w'_k \in \mathbb{S}_\nu V'$ and $w''_k \in \mathbb{S}_\mu V''$, which must be equal to the zero matrix. In particular remark that this holds also for $\mu = (1)$, i.e. when the elements w''_k are exactly the vectors $v''_{i,j}$. In this instance the elements w'_k are given by the tensors t'_i with a vector removed.
Fix a basis of $\mathbb{S}_{(1)}V'' = V''$ and decompose the elements w''_k with respect to such basis. Then collect the addends with respect to the elements of the fixed basis of V''. We get several linear combinations of the w'_k that must vanish. On the other hand all the w'_ks are linearly indepedent since they are given by the t'_is removing only one vector. Therefore it follows that all the vectors w''_k, and consequently all the $v''_{i,j}$, must vanish. This implies that the t''_i's are all equal to 0 so that $t_i \in \mathbb{S}_\lambda V'$ for any $i = 1, \ldots, r$. This is in contradiction with the hypothesis that there exists at least one t_i such that $t_i \notin \mathbb{S}_\lambda V'$. Therefore we must have $s > r$. This concludes the proof. □

Remark 4.1.21. If $t \in \mathbb{S}_\lambda V'$ and $V' \subset V$ is minimal with this property, then it is possible to find a basis of V' and an extension of it to a basis of V such that all the elements of the basis of V' are used to write t in its decomposition. This is the reason of the name *concise*.
Moreover, in this instance it is easy to see that if $t = t_1 + \cdots + t_r$, with t_i of λ-rank 1 for every i, and t of λ-rank r, if we denote with V_i the biggest subspace of the flag associated to t_i, then the dimension of the concise space is equal to

$$\dim(V_1 + \cdots + V_r).$$

This number can be computed directly. Indeed it is the rank of the catalecticant map $\mathcal{C}_t^{\lambda,(1)}$. To see this we have to make a couple of remarks. Firstly if for some i and j it happens that $V_i = V_j$, then the remaining parts of the respective flags must be distinct, otherwise we can sum t_i and t_j to a point of λ-rank 1 and the λ-rank of t would drop. Secondarily, it cannot happen that V_i and V_j intersect in a hyperplane and the remaining parts of the flags are equal and contained in such hyperplane since in this case we can sum t_i and t_j to an element of λ-rank 1 and hence the λ-rank of t would decrease. With these two facts in mind we get that once we have fixed a basis $\{x_1, \ldots, x_n\}$ dual to a basis of V with which t is written, then the non zero images of every x_i are all linearly independent. Hence they allow us to compute directly which vectors are needed to write down t.

4.2 The secant variety of Flag varieties $(\mathbb{F}(1,k;V),\mathcal{O}(1,1))$

This section is devoted to the study of the λ-rank appearing on the second secant variety to a Flag variety $(\mathbb{F}(1,k;V),\mathcal{O}(1,1))$. An algorithm which discriminates the elements on such secant variety will be provided.

Definition 4.2.1. Given a non degenerate variety $X\subset\mathbb{P}^N$, we use the following definition for the *tangential variety* of X

$$\tau(X):=\bigcup_{p\in X}T_pX$$

where T_pX denotes the tangent space to X at p.

Remark 4.2.2. By the definition of secant variety we have that for any variety X

$$\sigma_2(X)=\tau(X)\cup\sigma_2^0(X),$$

where $\sigma_2^0(X)$ is given by the elements of X-rank 2. Since in this case the X-rank is always almost 2, we have to understand which X-ranks appear on the tangential variety of X. If $X=\mathbb{F}(1,k;V)$ is homogeneous, the λ-rank is invariant under the action of $SL(V)$ and we may reduce to investigate a single tangent space. By virtue of this fact, choose p as the highest weight vector of this representation, i.e.

$$p=v_1\wedge\cdots\wedge v_k\otimes v_1 \tag{4.2.1}$$

for some $v_i\in\mathbb{K}^n$. This element may be represented with the sstd tableau

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,.$$

Recall the following classic result.

Lemma 4.2.3 (Tangent space to a Grassmannian)**.** *Let $p=[v_1\wedge\cdots\wedge v_k]$ be a point of $\mathbb{G}(k,V)\subset\mathbb{P}(\bigwedge^kV)$. Then*

$$\widehat{T_p\mathbb{G}(k,V)}=\sum_{i=1}^k v_1\wedge\cdots\wedge v_{i-1}\wedge V\wedge v_{i+1}\wedge\cdots\wedge v_k. \tag{4.2.2}$$

The proof of this Lemma is classical and we do not report it here. For more see [CGG05, p. 635–636].
For the computation of the tangent space of a Flag variety we adopt the following notation.

Notation 4.2.4. Let $\{v_1, \ldots, v_{n_k}\}$ be a basis of V. We can consider multi-indexes $I = (i_1, \ldots, i_k)$, where $i_1 < \cdots < i_k$. Denote

$$v_I := v_{i_1} \wedge \cdots \wedge v_{i_k},$$

and analogously

$$v_I^{\otimes d} = (v_{i_1} \wedge \cdots \wedge v_{i_k})^{\otimes d}.$$

With this notation a point of the Flag variety $\mathbb{F} = (\mathbb{F}(n_1, \ldots, n_k), \mathcal{O}(d_1, \ldots, d_k))$

$$p = (v_1 \wedge \cdots \wedge v_{n_k})^{\otimes d_k} \otimes \cdots \otimes (v_1 \wedge \cdots \wedge v_{n_1})^{\otimes d_1} \in \mathbb{F}$$

is denoted with

$$p = v_{I_k}^{\otimes d_k} \otimes \cdots \otimes v_{I_1}^{\otimes d_1}$$

where $I_j = (1, \ldots, n_j)$.

Corollary 4.2.5 (Tangent space to a Flag variety)**.** *Let $X \subset \mathbb{P}(\mathbb{S}_\lambda V)$ be the variety $(\mathbb{F}(n_1, \ldots, n_k; V), \mathcal{O}(d_1, \ldots, d_k))$ and let $p = v_{I_k}^{\otimes d_k} \otimes \cdots \otimes v_{I_1}^{\otimes d_1} \in X$. Using the description in Notation 4.2.4, the cone over the tangent space to X at p is the intersection of the span*

$$\langle v_{I_k}^{\otimes i_k} \otimes \sum_{j=1}^{n_k} v_1 \wedge \cdots \wedge v_{j-1} \wedge V \wedge v_{j+1} \wedge \cdots \wedge v_{n_k} \otimes v_{I_k}^{\otimes d_k - i_k - 1} \otimes v_{I_{k-1}}^{\otimes d_{k-1}} \otimes \cdots \otimes v_{I_1}^{\otimes d_1},$$

$$\ldots$$

$$v_{n_k}^{\otimes d_k} \otimes \cdots \otimes v_{I_2}^{\otimes d_2} \otimes v_{I_1}^{\otimes i_1} \otimes \sum_{j=1}^{n_1} v_1 \wedge \cdots \wedge v_{j-1} \wedge V \wedge v_{j+1} \wedge \cdots \wedge v_{n_1} \otimes v_{I_1}^{\otimes d_k - i_1 - 1} \rangle$$

with $\mathbb{S}_\lambda V$, where $i_j = 1, \ldots, d_j$.

This corollary follows directly from the lemma on the computation of the tangent space to a Grassmannian. Indeed it is enough to consider the Flag variety as an

incidence variety inside a product of Grassmann varieties. Then one has to compute the tangent space to the product of Grassmann varieties and to intersect with the respective Schur module.
We would like to give a more explicit description in the case of the Flag variety $\mathbb{F}$.

Corollary 4.2.6. *Let $p \in \mathbb{F}(1,k;V)$ be the highest weight vector of $\mathbb{S}_{(2,1^{k-1})}V$ as in* (4.2.1). *The cone over the tangent space $T_p\mathbb{F}(1,k;V)$ to $\mathbb{F}(1,k;V)$ at p is the subspace*

$$\begin{aligned}&\langle v_1 \wedge \cdots \wedge v_k \otimes v_1,\ v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1,\ i \in \{2,\ldots,k\},\\ &\ v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1,\ h \in \{1,\ldots,n\}\rangle \subset \mathbb{S}_{(2,1^{k-1})}V.\end{aligned}$$

Proof. By definition we have the inclusion $\mathbb{F}(1,k;V) \subset \mathbb{G}(1,V) \times \mathbb{G}(k,V)$. By Corollary 4.2.5, we have the equality

$$\widehat{T_p\mathbb{F}(1,k;V)} = \left(T_p\widehat{\mathbb{G}(1,V) \times \mathbb{G}(}k,V)\right) \cap \mathbb{S}_{(2,1^{k-1})}V$$

where with the notation $\hat{Y}$ we denote the affine cone over the projective variety Y. By Lemma 4.2.3 we see that $T_p\,(\widehat{\mathbb{G}(1,V) \times \mathbb{G}(}k,V))$ is the subspace

$$\begin{aligned}&\langle v_1 \wedge \cdots \wedge v_k \otimes v_1,\ v_1 \wedge \cdots \wedge v_k \otimes v_2, \ldots, v_1 \wedge \cdots \wedge v_k \otimes v_n,\\ &\quad v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1,\ h \in \{k+1,\ldots,n\},\ i \in \{1,\ldots,k\}\rangle.\end{aligned}$$

It is easy to see that if $i \neq 1$, the elements

$$v_1 \wedge \cdots \wedge v_k \otimes v_1,\ v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1$$

with $h \in \{k+1,\ldots,n\}$ satisfy the relations (3.1.2). Indeed, they are, up to the sign, the elements of the Schur module determined by the sstd tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} k & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} \hat{\imath} & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} k & \multicolumn{1}{c}{} \\ \cline{1-1} h & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \tag{4.2.3}$$

respectively, where $\hat{\imath}$ means that i is not appearing in the list. We can see also that the elements

$$v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1$$

satisfy the equations (3.1.2) for any $h = 2, \ldots, n$ and hence they belong to the module. Indeed they are the elements associated to the sstd tableaux

$$\begin{array}{|c|c|}\hline 1 & h \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} k & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}. \tag{4.2.4}$$

Consider now the space generated by the elements of $\mathbb{S}_{(2,1^{k-1})}V$ whose associated sstd tableaux is either in (4.2.3) or in (4.2.4). Since they are all different, by Proposition 1.3.36 the respective elements of $\mathbb{S}_{(2,1^{k-1})}V$ are linearly independent. The number of elements in (4.2.3) is $-k^2 + kn - n + k + 1$, and those in (4.2.4) are $n-1$, for a total of $-k^2 + kn + k$. Since the variety $\mathbb{F}(1, k; V)$ is smooth and of dimension $-k^2 + kn + k - 1$, we can conclude that $\widehat{T_p\mathbb{F}(1, k; V)}$ is the subspace

$$\begin{aligned} &\langle v_1 \wedge \cdots \wedge v_k \otimes v_1,\ v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1,\ i \in \{2, \ldots, k\}, \\ &\ v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1,\ h \in \{1, \ldots, n\}\rangle. \end{aligned}$$

□

To simplify the study of the $(2, 1^{k-1})$-rank appearing on the tangent space we split the generators of $T_p\mathbb{F}(1, k; V)$ in the three following sets:

(F.1) $v_1 \wedge \cdots \wedge v_k \otimes v_1$,

(F.2) $v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1$, for $i = 2, \ldots, k$ and $h = k+1, \ldots, n$,

(F.3) $v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1$ for $h = 2, \ldots, n$.

The tensors from (F.1) and (F.2) have $(2, 1^{k-1})$-rank 1 and they represent the flags

$$\langle v_1 \rangle \subset \langle v_1, \ldots, v_k \rangle$$

and

$$\langle v_1 \rangle \subset \langle v_1, \ldots, v_{i-1}, v_h, v_{i+1}, v_k \rangle$$

respectively.

Proposition 4.2.7. *The elements t from (F.3) have all $(2,1^{k-1})$-rank 1 for $h = 2, \dots, k$ and they represent the flags*

$$\langle v_h \rangle \subset \langle v_1, \dots, v_k \rangle.$$

If $h = k+1, \dots, n$ then the corresponding tensor has $(2,1^{k-1})$-rank 2 and it decomposes as

$$t = -\frac{1}{2}(v_1 - v_h) \wedge v_2 \wedge \cdots \wedge v_k \otimes (v_1 - v_h) + \frac{1}{2}(v_1 + v_h) \wedge v_2 \wedge \cdots \wedge v_k \otimes (v_1 + v_h).$$

Proof. Suppose that $h = 2, \dots, k$. Then t has the form

$$v_1 \wedge \cdots \wedge v_h \wedge \cdots \wedge v_k \otimes v_h$$

and hence it has $(2,1^{k-1})$-rank 1. It represents the flag

$$\langle v_h \rangle \subset \langle v_1, \dots, v_k \rangle.$$

If $h = k+1, \dots, n$, we can compute the kernel of the map $\mathcal{C}_t^{(2,1^{k-1}),(1)}$

$$\ker \mathcal{C}_t^{(2,1^{k-1}),(1)} = \langle x_{k+1}, \dots, \hat{x_h}, \dots, x_n \rangle$$

where $\hat{x_h}$ means that x_h does not appear among the generators. Moreover, observe that in general if t has $(2,1^{k-1})$-rank 1, then the catalecticant map $\mathcal{C}_t^{(2,1^{k-1}),(1)}$ has rank k. It is just the number of elements of V^* that cut out the biggest subspace of the flag associated to t. By Remark 4.1.1, since $\mathrm{rk}(\mathcal{C}_t^{(2,1^{k-1}),(1)}) = k+1$, we can already conclude that t has not $(2,1^{k-1})$-rank 1. Hence the decomposition

$$t = -\frac{1}{2}(v_1 - v_h) \wedge v_2 \wedge \cdots \wedge v_k \otimes (v_1 - v_h) + \frac{1}{2}(v_1 + v_h) \wedge v_2 \wedge \cdots \wedge v_k \otimes (v_1 + v_h)$$

is minimal and t has $(2,1^{k-1})$-rank 2. □

Remark 4.2.8. Since in the proof we obtained that $\mathrm{rk}(\mathcal{C}_t^{(2,1^{k-1}),(1)}) = k+1$ when $h = k+1, \dots, n$, using Remark 4.1.21 we can restrict our attention to the flag $\mathbb{F}(1,k;V')$ with $V' \simeq \mathbb{K}^{k+1}$, which is an adjoint variety. In this restricted case, in order to find a $(2,1^{k-1})$-rank 2 decomposition of the tensor, we should find at least one product of two distinct linear forms inside $\ker \mathcal{C}_t^{(2,1^{k-1}),(2)}$. Such linear forms are the equations of the two k-dimensional linear spaces in $V' \subset V$ of the two flags associated to the decomposition. We can see that

$$(x_1 - x_h)(x_1 + x_h) \in \ker \mathcal{C}^{(2,1^{k-1}),(2)}$$

are the linear forms we are looking for and the respective k-dimensional linear spaces are

$$v(x_1 - x_h) = \langle v_1 + v_h, v_2, \ldots, v_k \rangle$$

and

$$v(x_1 + x_h) = \langle v_1 - v_h, v_2, \ldots, v_k \rangle.$$

Now we have to study the possible sums of elements from the three different sets.

Remark 4.2.9. The sum of two elements from $(F.1)$ and $(F.2)$ is

$$\begin{aligned} a \cdot v_1 \wedge \cdots \wedge v_k \otimes v_1 + & b \cdot v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1 = \\ & v_1 \wedge \cdots \wedge v_{i-1} \wedge (a \cdot v_i + b \cdot v_h) \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1 \end{aligned}$$

which has $(2, 1^{k-1})$-rank 1.

Remark 4.2.10. The sum of two elements from $(F.1)$ and $(F.3)$ with $h = 2, \ldots, k$ turns out to be

$$a \cdot v_1 \wedge \cdots \wedge v_k \otimes v_1 + b \cdot v_1 \wedge \cdots \wedge v_k \otimes v_h = (v_1 + v_h) \wedge \cdots \wedge v_k \otimes (v_1 + v_h)$$

which has $(2, 1^{k-1})$-rank 1, while if $h = k+1, \ldots, n$ is

$$\begin{aligned} & a \cdot v_1 \wedge \cdots \wedge v_k \otimes v_1 + b \cdot (b \cdot v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1) = \\ & b \cdot v_1 \wedge \cdots \wedge v_k \otimes (a \cdot v_1 + b \cdot v_h) + (a \cdot v_1 + b \cdot v_h) \wedge v_2 \cdots \wedge v_k \otimes v_1 \end{aligned}$$

which is again an element of the set $(F.3)$.

Now we focus on the case $(F.2) + (F.3)$.

Proposition 4.2.11. *For any $v_j \in \mathbb{K}^n$, the sum of two elements from $(F.2)$ and $(F.3)$, i.e. the tensor*

$$\begin{aligned} t = a \cdot v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1 + & \\ + b \cdot (v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1), & \end{aligned} \tag{4.2.5}$$

has

(i) $(2,1^{k-1})$*-rank* 2 *if* v_j *and* v_h *are not multiple each other and* $v_h \in \langle v_2, \ldots, v_k \rangle$

(ii) $(2,1^{k-1})$*-rank* 3 *if* v_j *and* v_h *are not multiple each other and* $v_h \notin \langle v_2, \ldots, v_k \rangle$

(iii) $(2,1^{k-1})$*-rank* 2 *if* v_j *and* v_h *are multiple each other.*

Proof. Assume at first that v_j and v_h are not multiple each other. If $v_h \in \langle v_2, \ldots, v_k \rangle$, the sum (4.2.5) reduces to

$$a \cdot v_1 \wedge \cdots \wedge v_{i-1} \wedge v_j \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1 + b \cdot v_1 \wedge \cdots \wedge v_k \otimes v_h. \tag{4.2.6}$$

We claim that this element has $(2,1^{k-1})$-rank 2. Indeed if the tensor in (4.2.6) has $(2,1^{k-1})$-rank 1, then the map

$$\mathcal{C}_t^{(2,1^{k-1}),(1)} : \mathbb{S}_{(1)}(\mathbb{K}^n)^* \to \mathbb{S}_{(2,1^{k-1})/(1)}\mathbb{K}^n$$

has rank k. Since the catalecticant map in this case has rank $k+1$, by Remark 4.1.1 we can already conclude that the decomposition in (4.2.6) is minimal. This proves (i).
Assume now that $v_h \notin \langle v_2, \ldots, v_k \rangle$. In this case we use the catalecticant map

$$\mathcal{C}_t^{(2,1^{k-1}),(2)} : \mathbb{S}_{(2)}(\mathbb{K}^n)^* \to \mathbb{S}_{(2,1^{k-1})/(2)}\mathbb{K}^n$$

to compute the $(2,1^{k-1})$-rank of the element. At first remark that if t is a tensor of $(2,1^{k-1})$-rank 1, then the rank of this catalecticant map is k. Indeed for instance if $t = p$ as in (4.2.1), then the only elements of the basis of $\mathbb{S}_{(2)}(\mathbb{K}^n)^*$ which does not kill p and whose images via the catalecticant map are linearly independent are

$$x_1^2, x_1x_2, \ldots, x_1x_k$$

which are exactly k. In the instance of a tensor t like (4.2.5) one gets that the kernel of $\mathcal{C}_t^{(2,1^{k-1}),(2)}$ is the subspace

$$\ker \mathcal{C}_t^{(2,1^{k-1}),(2)} = \langle x_m x_n, \text{ where either } (m,n) = (h,h) \text{ or } m,n \neq 1,h \rangle$$

i.e. the elements of $\mathbb{S}_{(2)}(\mathbb{K}^n)^*$ not killing t are all the ones in the span

$$\langle x_1x_h, \ldots, x_kx_h, x_1x_2, \ldots, x_1x_k, x_1x_j, x_1^2 \rangle.$$

This means that $\operatorname{rk} \mathcal{C}_t^{(2,1^{k-1}),(2)} = 2k+1$. This implies by Remark 4.1.1 that t has $(2,1^{k-1})$-rank at least 3. By Proposition 4.2.7, the element t is written as a sum of 3 $(2,1^{k-1})$-rank 1 elements and hence its $(2,1^{k-1})$-rank is 3.
Finally assume that v_j and v_h are multiples. The element in (4.2.5) reduces to

$$\begin{aligned} t = a \cdot v_1 \wedge \cdots \wedge v_{i-1} \wedge v_h \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes v_1 + \\ + b \cdot (v_1 \wedge \cdots \wedge v_k \otimes v_h + v_h \wedge v_2 \cdots \wedge v_k \otimes v_1). \end{aligned}$$

We obtain again that $\operatorname{rk} \mathcal{C}_t^{(2,1^{k-1}),(1)} = k+1$ and hence t has not $(2,1^{k-1})$-rank 1 by Remark 4.1.1. One can see that t can be written as

$$\begin{aligned} t =& (v_1 - v_h) \wedge \cdots \wedge v_{i-1} \wedge (v_i - v_h) \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes (v_1 - v_h) + \\ &+ (v_1 + v_h) \wedge \cdots \wedge v_{i-1} \wedge (v_i + v_h) \wedge v_{i+1} \wedge \cdots \wedge v_k \otimes (v_1 + v_h) \end{aligned}$$

and hence t has $(2,1^{k-1})$-rank 2. Note that up to change of coordinates this is the tensor described in Remark 4.2.8. This concludes the proof. □

We collect the elements we found in a table.

$(2,1^{k-1})$-rank	$\operatorname{rk} \mathcal{C}_t^{(2,1^{k-1}),(1)}$	$\operatorname{rk} \mathcal{C}_t^{(2,1^{k-1}),(2)}$	Notes
1	k	k	sets (1), (2) and (3)
2	$k+1$	$2k-1$	set (3)
3	$k+2$	$2k+1$	(2) + (3)

Table 4.2: The ranks appearing on the tangential variety to $\mathbb{F}$.

Remark that the only elements not in $\mathbb{F}(1,k;V)$ or $\sigma_2^0(X)$ are those of the third line of the table. Hence they are the only contribution of the tangential variety to $\sigma_2(X)$. Let us study now the elements of $\mathbb{F}$ lying on a secant line to $\mathbb{F}$. Such elements can be written as

$$v_1 \wedge \cdots \wedge v_k \otimes v_1 + w_1 \wedge \cdots \wedge w_k \otimes w_1$$

depicting that their $(2,1^{k-1})$-rank is at most 2. Remark that letting the group $SL(V)$ act on this element, the $(2,1^{k-1})$-rank and the numbers $\dim\langle v_1,\ldots,v_k\rangle \cap \langle w_1,\ldots,w_k\rangle$ and $\dim\langle v_1,\ldots,v_k\rangle + \langle w_1,\ldots,w_k\rangle$ are preserved. In particular we may assume that after having applied an element of $SL(V)$ we get that

$$t = v_1 \wedge \cdots \wedge v_h \wedge v_{h+1} \wedge \cdots \wedge v_k \otimes v_i + v_1 \wedge \cdots \wedge v_h \wedge v_{k+1} \wedge \cdots \wedge v_{2k-h} \otimes v_j \tag{4.2.7}$$

where the intersection of the k-dimensional spaces of the flags can be read directly on the tensor, i.e.

$$\langle v_1, \dots, v_k \rangle \cap \langle v_1, \dots, v_h, v_{k+1}, \dots, v_{2k-h} \rangle = \langle v_1, \dots, v_h \rangle.$$

The vectors v_i and v_j appearing after the tensor products in the first and second addend are one of the generators of the spaces $\langle v_1, \dots, v_k \rangle$ and $\langle v_1, \dots, v_h, v_{k+1}, \dots, v_{2k-h} \rangle$ respectively. Note that if both v_i and v_j belong to the intersection of the k-dimensional subspaces of the flags, then they can be the same vector up to scalar multiplication. Therefore we discriminate the elements on secant lines to $\mathbb{F}$ using only two invariants: the dimension of the intersection of the k-dimensional subspaces of the two flags and whether it holds $\langle v_i \rangle = \langle v_j \rangle$ or not. In terms of Schur apolarity action, we can use the catalecticant maps to discriminate the previous two properties. In more details the map

$$\mathcal{C}_t^{(2,1^{k-1}),(1)} : \mathbb{S}_{(1)} V^* \longrightarrow \mathbb{S}_{(2,1^{k-1})/(1)} V$$

will give information about the dimension of the intersection. Indeed consider a $(2, 1^{k-1})$-rank 2 element t as in (4.2.7) and denote with $\{x_i,\ i = 1, \dots, n\}$ the dual basis of the $\{v_i,\ i = 1, \dots, n\}$. Then the image of x_i is 0 if $x_i = x_{2k-h+1}, \dots, x_n$. Otherwise it is different from zero if $x_i = x_1, \dots, x_{2k-h}$ and one can see that all the images of these x_i's are linearly independent as elements of $\mathbb{S}_{(2,1^{k-1})/(1)} V$. Hence the rank of the catalecticant map is equal to the dimension of the sum of the two k-dimensional subspaces of the two flags involved. Once that this number is fixed, the rank of the map

$$\mathcal{C}_t^{(2,1^{k-1}),(2)} : \mathbb{S}_{(2)} V^* \longrightarrow \mathbb{S}_{(2,1^{k-1})/(2)} V$$

will help us to determine whenever $\langle v_i \rangle = \langle v_j \rangle$ is true or not. Indeed by the definition of Schur apolarity action, the element $x_p x_q \in \mathbb{S}_{(2)} V^*$, which can be written as $x_p \otimes x_q + x_q \otimes x_p$, is applied in both the factors of the tensor product $\bigwedge^k V \otimes \bigwedge^1 V$ that contains $\mathbb{S}_{(2,1^{k-1})} V$. Hence the equality or inequality between $\langle v_i \rangle$ and $\langle v_j \rangle$ will change the rank of this catalecticant map. Moreover note that if $\langle v_i \rangle \neq \langle v_j \rangle$, the rank of $\mathcal{C}^{(2,1^{k-1})}$ will change by 1 depending on whether both v_i and v_j belong or not to the intersection of the two k-dimensional subspaces.
We give in Table 4.3 a classification of all the possible classes depending on the invariants we have mentioned.

Remark 4.2.12. Suppose that we want to check if a tensor t has $(2, 1^{k-1})$-rank 2 and is such that the two respective flags share the same line and the biggest subspaces

$\dim V \cap W$	$\langle v_1 \rangle = \langle w_1 \rangle$	$v_1, w_1 \in V \cap W$	rank	(1)	(2)	(1^k)
k	True/False	Yes/No	1	k	k	1
$k-1$	True	Yes	1	k	k	1
$k-1$	False	No	2	$k+1$	$k+1$	2
$k-1$	False	Yes	2	$k+1$	k	2
$\vdots$						
h	True	Yes	2	$2k-h$	$2k-h$	1
h	False	No	2	$2k-h$	$2k$	2
h	False	Yes	2	$2k-h$	$2k-1$	2
$\vdots$						
0	False	No	2	$2k$	$2k$	2

Table 4.3: Classes of points on a secant line to $\mathbb{F}(1,k;V)$, listed in the first column by the dimension of $V \cap W$. We denote only in this table $\dim\langle v_1, \ldots, v_k\rangle \cap \langle w_1, \ldots, w_k\rangle$ with $\dim V \cap W$. The partitions $\mu = (1)$, (2) and (1^k) denotes instead the rank of the catalecticant $\mathcal{C}^{(2,1^{k-1}),\mu}$. The column "rank" stands for the $(2,1^{k-1})$-rank.

meet in a subspace of dimension h. A sufficient condition according to Table 4.3 is that $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(1^k)}\right) = 1$, $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(1)}\right) = 2k-h$ and $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(2)}\right) = 2k-h$. Obviously checking the ranks of these catalecticant maps is not enough to say that t is the tensor we are looking for. However to get some hint if t is such a tensor we can proceed as follows.
Consider the tensor of $(2,1^{k-1})$-rank 2 such that the respective catalecticant maps have that specific ranks. Call t_1 and t_2 the two $(2,1^{k-1})$-rank 1 tensors appearing in the decomposition of t. Since $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(1)}\right) = 2k-h$, using Remark 4.1.21 on the concise space of a tensor we can reduce to work in the space $\mathbb{S}_{(2,1^{k-1})}\mathbb{K}^{2k-h}$.
We can assume that $\{v_1, \ldots, v_{2k-h}\}$ is a basis of $\mathbb{K}^{2k-h}$ and that $\{x_1, \ldots, x_{2k-h}\}$ is the respective dual basis. Then we have that the k-dimensional subspaces of t_1 and t_2 are given by the equations $x_{k+1}, \ldots, x_{2k-h}$ and $x_{h+1}, \ldots, x_k$ respectively. On the other hand in both cases the line of the flag is cut out by the equations $x_2, \ldots, x_{2k-h}$. In order to have that a tensor $t \in \mathbb{S}_{(2,1^{k-1})}\mathbb{K}^{2k-h}$ admits a decomposition as $t_1 + t_2$, from the Theorem 3.4.9 we must have $I(t_1, t_2) \subset t^{\perp}$. In particular we should find

$$I(t_1,t_2)_{(2)} = \langle x_2^2,\ x_2x_3,\dots,x_2x_{2k-h},\ x_3^2,\dots,x_{2k-h}^2\rangle = \mathrm{Sym}^2\langle x_2,\dots,x_{2k-h}\rangle \subset (t^\perp)_{(2)}$$

and

$$I(t_1,t_2)_{(1,1)} = \langle x_i \wedge x_j \ :\ i \in \{h+1,\dots,k\},\ j \in \{k+1,\dots,2k-h\}\rangle \subset t^\perp.$$

Note that the dimension of the spaces $I(t_1,t_2)_{(2)}$ and $I(t_1,t_2)_{(1,1)}$ is exactly the dimension of $(t^\perp)_{(2)}$ and $(t^\perp)_{(1,1)}$ respectively after the reduction from V to $\mathbb{K}^{2k-h}$. The same discussion applies also to tensors $t \in \mathbb{S}_{(2,1^{k-1})}V$ of $(2,1^{k-1})$-rank 2 such that $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(1^k)}\right) = 2$, $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(1)}\right) = 2k-h$ and $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(2)}\right) = 2k$ or $\mathrm{rk}\left(\mathcal{C}_t^{(2,1^{k-1}),(2)}\right) = 2k-1$ in Table 4.3, i.e. when the flags associated to the respective $(2,1^{k-1})$-rank 1 elements do not share the same line.

Algorithm 4.2.13. The following algorithm distinguishes the $(2,1^{k-1})$-rank of the elements of border rank at most 2 in $\mathbb{S}_{(2,1^{k-1})}V$.

Input: An element $t \in \widehat{\sigma_2(\mathbb{F}(1,k;V))} \subset \mathbb{S}_{(2,1^{k-1})}V$.
Output: The $(2,1^{k-1})$-rank of t and which class the tensor belongs to.

compute $(r_1,r_2,r_3) = \left(\mathrm{rk}\,\mathcal{C}_t^{(2,1^{k-1}),(1^k)}, \mathrm{rk}\,\mathcal{C}_t^{(2,1^{k-1}),(1)}, \mathrm{rk}\,\mathcal{C}_t^{(2,1^{k-1}),(2)}\right)$
if $r_1 = 1$ and $r_2 = k$ **then**
    return t has $(2,1^{k-1})$-rank equal to 1, **exit**;
if $r_1 = 1$ but $r_2 > k$ **then**
    $r_2 = r_3 = 2k-h$ for some $0 \le h < k$. **Return** t has $(2,1^{k-1})$-rank and is given by two flags which share the same line, **exit**;
else if $r_1 = 2$ **then**
    if $(r_1,r_2,r_3) = (2,k+2,2k+1)$ **then**
    return t has $(2,1^{k-1})$-rank 3 and it is the element given by $(F.2)$+$(F.3)$ in Table 4.2, **exit**;
    else if $(r_1,r_2,r_3) = (2,2k-h,2k)$ **then**
    return t has $(2,1^{k-1})$-rank 2 and it is in the orbit with $\dim V \cap W = h$ and $\langle v_i\rangle \neq \langle v_j\rangle$ and either v_i or v_j is not in $V \cap W$, **exit**;
    else if $(r_1,r_2,r_3) = (2,2k-h,2k-1)$ **then**
    return t has $(2,1^{k-1})$-rank 2 and it is in the orbit with $\dim V \cap W = h$ and $\langle v_i\rangle \neq \langle v_j\rangle$ but $v_i, v_j \in V \cap W$, **exit**;

```
    end if
end if
end
```

Remark 4.2.14. Note that this is not a complete classification of the orbits appearing on $\sigma_2(X)$ by the action of $SL(V)$. To this end more distinctions have to be made. In particular one has to distinguish three cases depending on the fact that the lines $\langle v_i \rangle$ and $\langle v_j \rangle$ both belong to the intersection of the two k-dimensional subspaces, only one of them belongs to this intersection and eventually none of them belong to it. Since such a study does not regard the aim of this thesis, we skip the description of such orbits.

To conclude the study of this section, we obtain the following result.

Corollary 4.2.15. *Let $X = \mathbb{F}(1, k; V)$ embedded with $\mathcal{O}(1, 1)$ in $\mathbb{P}(\mathbb{S}_{(2,1^{k-1})}V)$. Then we have*

$$\sigma_2(X) \setminus X = \sigma_{2,2}(X) \cup \sigma_{2,3}(X),$$

where $\sigma_{r,s}(X)$ denotes the set $\{p \in \sigma_r(X) : r_X(p) = s\}$.

4.3 The secant variety of Grassmann varieties $(\mathbb{G}(k, V), \mathcal{O}(d))$

This section focuses on the study of the (d^k)-rank of points of the secant variety $\sigma_2(X)$ of the Grassmann variety $X = (\mathbb{G}(k, V), \mathcal{O}(d)) \subset \mathbb{P}\left(\mathbb{S}_{(d^k)}V\right)$ when $d > 1$. All along this section we denote this variety with $X_{k,n,d}$ for brevity, where $n = \dim V$.

4.3.1 The case $2 \leq 2k \leq n$ and $d \geq 1$.

Let $p = [(v_1 \wedge \cdots \wedge v_k)^{\otimes d}] \in X_{k,n,d}$ be a point of (d^k)-rank 1. We can assume that p is the highest weight vector of the representation. As in Notation 4.2.4 write $p = v_I^{\otimes d}$, with $I = (1, \ldots, k)$. We need a more explicit description of the tangent space to the Grassmann variety $X_{k,n,d}$.

Proposition 4.3.1. *Let $p = \left[v_I^{\otimes d}\right] \in X_{k,n,d}$, with $I = (1, \dots, k)$. The cone over the tangent space to $X_{k,n,d}$ at p is the space*

$$\left\langle \sum_{i=0}^{d-1} v_I^{\otimes i} \otimes \left(\sum_{j=1}^{k} v_1 \wedge \cdots \wedge v_{j-1} \wedge v_h \wedge v_{j+1} \wedge \cdots \wedge v_k \right) \otimes v_I^{\otimes d-i-1} \right\rangle$$

with $j = 1, \dots, k$ and $h = j$ or $h = k+1, \dots, n$.

Proof. Let $p = (v_1 \wedge \cdots \wedge v_k)^{\otimes d}$ be the highest weight vector of $\mathbb{S}_{(d^k)}V$. Assume that a basis for V is given by $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$. To compute the tangent space we use the Lemma 4.2.3. In this way we get the tangent space to the product of d Grassmannians $\mathbb{G}(k, V)$, and then we have to intersect with $\mathbb{S}_{(d^k)}V$. The tangent space to the product is the span

$$\langle v_I^{\otimes i} \otimes \left(v_1 \wedge \cdots \wedge v_{j-1} \wedge v_h \wedge v_{j+1} \wedge \cdots \wedge v_k\right) \otimes v_I^{\otimes d-1-i}, \qquad i = 0, \dots, d-1,\ j = 1, \dots, k\rangle$$

where either $h = j$ or $h = k+1, \dots, n$. Whenever there is $v_I^{\otimes 0}$, it means that the factor of the tensor product does not appear.
If $h = j$ we get the point p, which already belongs to $\mathbb{S}_{(d^k)}V$. If $h = k+1, \dots, n$, and note that no other cases can occur, we get that for any fixed j the sum

$$\sum_{i=0}^{d-1} v_I^{\otimes i} \otimes \left(v_1 \wedge \cdots \wedge v_{j-1} \wedge v_h \wedge v_{j+1} \wedge \cdots \wedge v_k\right) \otimes v_I^{\otimes d-1-i} \tag{4.3.1}$$

satisfies the relations (3.1.2) and hence belongs to $\mathbb{S}_{(d^k)}V$. We can see that for any $h = k+1, \dots, n$ the tensor (4.3.1) is the element of the basis associated to the sstd tableau of shape (d^k)

1	...	1	1
$\vdots$		$\vdots$	$\vdots$
$\vdots$		$\vdots$	$\hat{j}$
$\vdots$		$\vdots$	$\vdots$
k	...	k	h

where in the last column we have the sequence $(1, \dots, \hat{j}, \dots, k, h)$. Remark that h has been put at the end since it is the greatest integer appearing in that column.

The symbol $\hat{\jmath}$ means that j has been removed. Since for any $j = 1, \dots, k$ and $h = k+1, \dots, n$ all these sstd tableaux of shape (d^k) are different, by Proposition 1.3.36 the respective elements of $\mathbb{S}_{(d^k)}V$ are all linearly independent. Since they are exactly $k(n-k)+1$ counting also the point p, and since $X_{k,n,d}$ is smooth, we get that the cone $\widehat{T_p X_{k,n,d}}$ is spanned by the elements (4.3.1) for any $j = 1, \dots, k$ and $h = k+1, \dots, n$, and p. This concludes the proof. □

In order to study which (d^k)-ranks appear in the tangent space to $X_{k,n,d}$, we distinguish some families inside $T_p X_{k,n,d}$, where $p = v_I^{\otimes d}$ is as above. For the moment assume $d > 1$. The case $d = 1$ will be treated later.

$(G.1)$ $p = v_I^{\otimes d}$,

$(G.2)$ for any fixed j, consider the element in (4.3.1) with any $v_h \in V$ such that the respective element is non zero.

Remark that in $(G.2)$ we have exactly k families depending only on the value of j. Obviously the element in $(G.1)$ has (d^k)-rank 1.

Remark 4.3.2. The sum of any two elements in $(G.2)$ both determined by the same j is still an element of the same family $(G.2)$ determined by the same j. Indeed, without loss of generality we can assume $j = 1$. Two elements of this family can be written as

$$\sum_{i=0}^{d-1} v_I^{\otimes i} \otimes \left(v_{h_1} \wedge v_2 \wedge \cdots \wedge v_k\right) \otimes v_I^{\otimes d-1-i}$$

and

$$\sum_{i=0}^{d-1} v_I^{\otimes i} \otimes \left(v_{h_2} \wedge v_2 \wedge \cdots \wedge v_k\right) \otimes v_I^{\otimes d-1-i}$$

for some v_{h_1}, v_{h_2} in $\langle v_{k+1}, \dots, v_n \rangle$. Their sum will be

$$\sum_{i=0}^{d-1} v_I^{\otimes i} \otimes \left((v_{h_1} + v_{h_2}) \wedge v_2 \wedge \cdots \wedge v_k\right) \otimes v_I^{\otimes d-1-i}$$

which is still obviously and element of the family $(G.2)$. The same happens also for the other values of $j = 2, \dots, k$.

Proposition 4.3.3. *Let t be an element of the family (G.2) for some fixed j and h. Then its (d^k)-rank is d.*

Proof. Without loss of generality we can assume that $j = 1$ and $h = k + 1$. Hence up to a sign t looks like

$$t = \sum_{i=0}^{d-1} v_I^{\otimes i} \otimes (v_2 \wedge \cdots \wedge v_k \wedge v_{k+1}) \otimes v_I^{\otimes d-1-i}.$$

Roughly we can imagine that the subspace $\langle v_1, \ldots, v_k \rangle$ appears $d - 1$ times while the subspace $\langle v_2, \ldots, v_k, v_{k+1} \rangle$ dances in every position along the summation. We can consider the catalecticant map

$$\mathcal{C}_t^{(d^k),(1)} : \mathbb{S}_{(1)} V^* \longrightarrow \mathbb{S}_{(d^{k-1},d-1)} V$$

where the codomain $\mathbb{S}_{(d^k)/(1)} V \simeq \mathbb{S}_{(d^{k-1},d-1)} V$ is irreducible by Proposition 3.3.2. It is easy to see that the rank of $\mathcal{C}_t^{(d^k),(1)}$ is $k + 1$. Indeed all the elements in V^* that annihilate the subspace $\langle v_1, \ldots, v_k, v_{k+1} \rangle$ via $V \times V^* \longrightarrow \mathbb{K}$ are in the kernel of the catalecticant map. The elements dual to the v_i in $\{v_1, \ldots, v_{k+1}\}$, i.e. all the $\alpha_i \in V^*$ such that $\alpha_i(v_j) = \delta_{i,j}$ for any $i, j = 1, \ldots, k + 1$, give rise to non zero linearly independent images. Hence using Remark 4.1.21 on the concise space of a tensor we can reduce to work in a smaller dimensional space, i.e. with the subvariety

$$X_{k,k+1,d} = \left(\mathbb{G}(k, \mathbb{K}^{k+1}, \mathcal{O}(d)\right) \simeq \left((\mathbb{P}^k)^*, \mathcal{O}(d)\right) \simeq \nu_d((\mathbb{P}^n)^*)$$

which is a Veronese variety. Under this light the element t can be rewritten as the monomial

$$x^{d-1}y$$

where $x = (v_1 \wedge \cdots \wedge v_k)$ and $y = (v_2 \wedge \cdots \wedge v_k \wedge v_{k+1})$. The symmetric rank of this monomial, i.e. the rank with respect to $\nu_d((\mathbb{P}^n)^*)$, is d. This allows us to conclude that t has (d^k)-rank d with respect to the concise space. By Proposition 4.1.20 we can conclude that such a decomposition is the minimal one. This concludes the proof. □

Remark 4.3.4. The sum of p with an element in (G.2) is again an element of the family (G.2), for any $j = 1, \ldots, k$. Indeed without loss of generality and up to a sign we can assume that $j = 1$ and $v_h = v_{k+1}$, and consider the elements p and, up to a sign,

$$t = \sum_{i=0}^{d-1} v_I^{\otimes i} \otimes (v_2 \wedge \cdots \wedge v_k \wedge v_{k+1}) \otimes v_I^{\otimes d-1-i}.$$

Their sum is

$$t + p = \sum_{i=0}^{d-1} v_I^{\otimes i} \otimes ((v_1 + (-1)^k v_{k+1}) \wedge v_2 \wedge \cdots \wedge v_k) \otimes v_I^{\otimes d-1-i}$$

which is again an element of the family $(G.2)$. The same holds for any other value of j and h. From this observation and by Remark 4.3.2, it follows that we may reduce to study the (d^k)-rank of the sum of elements in the family $(G.2)$. Remark that we may distinguish k families inside obtained adding elements from $(G.2)$ and depending on the number of addends, each of them determined by distinct values of j. Unfortunately we have not been able to compute the (d^k)-rank of those elements except in specific cases.

4.3.2 The case $k = d = 2$.

In this paragraph we restrict our attention to the case $X_{2,n,2} = (\mathbb{G}(2, V), \mathcal{O}(2))$ embedded in $\mathbb{P}(\mathbb{S}_{(2,2)}V)$.
In this particular instance we have that the elements in the family $(G.2)$ have $(2,2)$-rank 2. Moreover since $k = 2$, the j in $(G.2)$ can only be either 1 or 2. Hence we have only to classes of tensors from $(G.2)$. The first given by only one element of $(G.2)$, and the second one given by the sum of two elements in $(G.2)$, the first with $j = 1$, and the second one with $j = 2$. We want to investigate the $(2,2)$-rank of this element. Remark that by the subadditivity of X-rank in general, we have that its $(2,2)$-rank is between 2 and 4. The $(2,2)$-rank 1 case is excluded easily using catalecticant maps.

To fix ideas, consider the element

$$\begin{aligned} t = (v_1 \wedge v_2) \otimes (v_2 \wedge v_3) + (v_2 \wedge v_3) \otimes (v_1 \wedge v_2) + \\ + (v_1 \wedge v_2) \otimes (v_1 \wedge v_4) + (v_1 \wedge v_4) \otimes (v_1 \wedge v_2) \end{aligned}$$

which we can write compactly as

$$t = (v_1 \wedge v_2) \otimes (v_2 \wedge v_3 + v_1 \wedge v_4) + (v_2 \wedge v_3 + v_1 \wedge v_4) \otimes (v_1 \wedge v_2). \tag{4.3.1}$$

If $\{v_1, \dots, v_n\}$ and $\{x_1, \dots, x_n\}$ are basis dual to each other of V and V^* respectively, it is easy to see that

$$\ker \mathcal{C}_t^{(2,2),(1)} = \langle x_5, \dots, x_n \rangle. \tag{4.3.2}$$

Hence we can reduce the number of variables and work with the variety $\tilde{X} = X_{2,4,2} = (\mathbb{G}(2, \mathbb{K}^4), \mathcal{O}(2))$.
We claim that t has $(2,2)$-rank 4. The trick to compute such $(2,2)$-rank is the following. A first we classify $(2,2)$-rank 2 and 3 tensors in $\mathbb{S}_{(2,2)}\mathbb{K}^n$ and then we prove that t cannot appear in this list using catalecaticant methods.

Proposition 4.3.5. *The $(2,2)$-rank 2 tensors in $\mathbb{S}_{(2,2)}\mathbb{K}^4$ can only be of either two types. The classification follows arguing the relative position of the 2-dimensional spaces represented by the points of $(2,2)$-rank 1:*

(2.1) the two subspaces does not intersect, i.e.

$$(u \wedge v)^{\otimes 2} + (w \wedge s)^{\otimes 2},$$

where $\langle u,\ v,\ w,\ s \rangle \simeq \mathbb{K}^4$.

(2.2) the two subspaces intersect on a line, i.e.

$$(v \wedge u)^{\otimes 2} + (v \wedge w)^{\otimes 2},$$

where $\langle u,\ v,\ w \rangle \simeq \mathbb{K}^3$.

Proof. Let t_1 and t_2 be two elements of $\mathbb{S}_{(2,2)}\mathbb{K}^4$ of $(2,2)$-rank 1 and consider their sum. We may assume that

$$t_1 + t_2 = (v_1 \wedge v_2)^{\otimes 2} + (w_1 \wedge w_2)^{\otimes 2}.$$

We can have only three cases. Firstly if $\langle v_1,\ v_2 \rangle = \langle w_1,\ w_2 \rangle$, then we get that

$$t_1 + t_2 = c \cdot (v_1 \wedge v_2)^{\otimes 2}$$

where $c \in \mathbb{K}$. Hence its $(2,2)$-rank is 1. The remaining two cases reflect the fact that $\dim(\langle v_1,\ v_2 \rangle \cap \langle w_1,\ w_2 \rangle)$ can be either 1 or 0, returning the cases (2.1) and (2.2). $\square$

Proposition 4.3.6. *The $(2,2)$-rank 3 tensors can be of five types. Let us call the three subspace related to the $(2,2)$-rank 1 elements in the decomposition of t as U_1, U_2 and U_3 of $\mathbb{K}^4$ involved. Denote with $(n_1, n_2, n_3) = (\dim(U_1 \cap U_2), \dim(U_1 \cap U_3), \dim(U_2 \cap U_3))$. Then a $(2,2)$-rank 3 tensor is partly determined by this sequence up to permutation. The cases are:*

(3.1) every couple intersects on a line but there is no common intersection between all the three subspaces $(1,1,1)$, i.e.

$$(v \wedge u)^{\otimes 2} + (u \wedge w)^{\otimes 2} + (v \wedge w)^{\otimes 2},$$

where $\langle u,\ v,\ w\rangle \simeq \mathbb{K}^3$.

(3.2) the subspaces intersect all in a common line $(1,1,1)$ and altogether they span $\mathbb{K}^4$, i.e.

$$(v \wedge u)^{\otimes 2} + (v \wedge w)^{\otimes 2} + (v \wedge s)^{\otimes 2},$$

where $\langle u,\ v,\ w,\ s\rangle \simeq \mathbb{K}^4$.

(3.3) two couples meet on a line but the third one is disjoint $(1,1,0)$, i.e.

$$(v \wedge u)^{\otimes 2} + (v \wedge w)^{\otimes 2} + (u \wedge s)^{\otimes 2},$$

where $\langle u,\ v,\ w,\ s\rangle \simeq \mathbb{K}^4$.

(3.4) one couple meets on a line, the other two are disjoint $(1,0,0)$, i.e.

$$(v \wedge u)^{\otimes 2} + (v \wedge w)^{\otimes 2} + ((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w))^{\otimes 2},$$

where $\langle u,\ v,\ w,\ s\rangle \simeq \mathbb{K}^4$, for some $a_1, \dots, a_4, b_1, b_2, b_3 \in \mathbb{K}$ such that $a_4 \neq 0$ and $b_1, b_3 \neq 0$.

(3.5) all the couples are disjoint $(0,0,0)$, i.e.

$$(u \wedge v)^{\otimes 2} + (w \wedge s)^{\otimes 2} + ((u+w) \wedge (v+s))^{\otimes 2},$$

where $\langle u,\ v,\ w,\ s\rangle \simeq \mathbb{K}^4$.

Proof. The proof proceeds arguing the mutual position of the involved vector subspaces of $\mathbb{K}^4$ represented by the three $(2,2)$-rank 1 tensors. Let us name the tensors t_1, t_2 and t_3 and their associated subspaces U_1, U_2 and U_3 respectively.

Let us start with the case in which every couple of the three subspaces intersect along a line but no common intersection between all of them occurs. We can assume $U_1 \cap U_2 = \langle v \rangle$, $U_1 \cap U_3 = \langle u \rangle$ and $U_2 \cap U_3 = \langle w \rangle$ for some vectors u, w and v in $\mathbb{K}^4$. Remark that u, v and w must be linearly independent. Otherwise the intersection of the three subspaces would be a line. This means that we can write $U_1 = \langle v,\ w \rangle$, $U_2 = \langle v,\ w \rangle$ and $U_3 = \langle u,\ w \rangle$ returning the case (3.1).

Consider then the case in which all the three subspaces meet along a common line, i.e. $U_1 \cap U_2 \cap U_3 = \langle v \rangle$, for some $v \in \mathbb{K}^4$. Let us call the generators of the three subspaces with $U_1 = \langle v,\ u \rangle$, $U_2 = \langle v,\ w \rangle$ and $U_3 = \langle v,\ s \rangle$. Since all the three subspaces must be distinct, we must have that $\langle u \rangle \neq \langle w \rangle$, and that either $s \in \langle u,\ w \rangle$ with $\langle s \rangle \neq \langle u \rangle, \langle w \rangle$, or $s \notin \langle u,\ w \rangle$. However one can see easily that if $s \in \langle u,\ w \rangle$, then the tensor has $(2,2)$-rank 2. Hence we must exclude this case, i.e. $s \notin \langle u,\ w \rangle$ and therefore we get the case (3.2).

Suppose now that, up to permutation, the couples of subspaces (U_1, U_2), and (U_1, U_3) meet along two distinct lines. We can assume that $U_1 \cap U_2 = \langle v \rangle$ and $U_1 \cap U_3 = \langle u \rangle$. If $U_2 \cap U_3$ is a line, we are back to the case (3.1). Suppose then $U_2 \cap U_3 = \langle 0 \rangle$. We may write $U_1 = \langle u,\ v \rangle$, $U_2 = \langle v,\ w \rangle$ and $U_3 = \langle u,\ s \rangle$ where $\langle u,\ v,\ w,\ s \rangle \simeq \mathbb{K}^4$. However one can see easily that if $s \in \langle u,\ w \rangle$, then the tensor has $(2,2)$-rank 2. Hence we get (3.3).

Assume now that, up to permutation, only the subspaces U_1 and U_2 meet along a line while the other couples are disjoint. Suppose that such a line is spanned by some $v \in \mathbb{K}^4$. We can assume that $U_1 = \langle u,\ v \rangle$ and $U_2 = \langle v,\ w \rangle$ such that $\langle u,\ v,\ w \rangle \simeq \mathbb{K}^3$. Complete the set $\{u,\ v,\ w\}$ to a basis of $\mathbb{K}^4$ and call s the missing vector. For the third subspace, the two generators must be of the form

$$a_1 u + a_2 v + a_3 w + a_4 s \quad \text{and} \quad b_1 u + b_2 v + b_3 w,$$

where in order to not get elements of U_1 and U_2 via linear combinations of the generators of U_3, we have to impose that $a_4 \neq 0$ and $b_1, b_3 \neq 0$. Indeed, if $a_4 = 0$, then for some linear combination one gets that U_3 has non zero intersection with either U_1 or U_2. If $b_1 = 0$, then if $b_2 \neq 0$ we get that the second generator belongs to U_2, hence we have to exclude this case. Otherwise if $b_2 = 0$, then we get that the second generator belongs to U_2 again and hence we have to exclude also this case. The discussion for $b_3 = 0$ is analogous. The same discussion applies also for the last condition. Remark that the element s can appear also in the second generator. However, after a reduction we can assume that in the second generator s does not appear. Hence we get the element in (3.4).

Last but not least we have the case in which the three subspaces are disjoint. Assume that $U_1 = \langle u,\ v\rangle$ and $U_2 = \langle w,\ s\rangle$, where $\langle u,\ v,\ w,\ s\rangle \simeq \mathbb{K}^4$. For dimensional reasons we have that the spaces $\langle U_1,\ w\rangle \cap U_3$ and $\langle U_1,\ s\rangle \cap U_3$ are two linearly independent lines. Indeed if they are linearly dependent, then they must be multiple each other. This means that when we write such vector with respect to the basis $\{u,\ v,\ w,\ s\}$, the coordinates of w and s are both zero. This implies that such a vector of U_3 belong also to U_1, which is in contrast with the hypothesis that U_1 and U_3 do not intersect. Up to choosing different generators of U_1 and U_2, we can assume that the two lines are generated by the vectors $u + w$ and $v + s$ respectively. Hence we get (3.5). This concludes the proof. $\square$

Remark 4.3.7. Consider the tensor t as in (4.3.1). Assuming that $\{v_1,\ v_2,\ v_3,\ v_4\}$ is a basis of $\mathbb{K}^4$ and $\{x_1,\ x_2,\ x_3,\ x_4\}$ is the respective dual basis of $(\mathbb{K}^4)^*$, we get that

$$\begin{aligned}
\ker \mathcal{C}_t^{(2,2),(1)} &= \langle 0\rangle,\\
\ker \mathcal{C}_t^{(2,2),(2)} &= \langle x_3^2,\ x_3x_4,\ x_4^2,\ x_1x_4 + x_2x_3\rangle,\\
\ker \mathcal{C}_t^{(2,2),(1,1)} &= \langle x_1 \wedge x_3,\ x_2 \wedge x_4,\ x_3 \wedge x_4,\ x_1 \wedge x_4 - x_2 \wedge x_3\rangle.
\end{aligned}$$

Hence $\mathrm{rk}\left(\mathcal{C}_t^{(2,2),(1)}\right) = 4$, $\mathrm{rk}\left(\mathcal{C}_t^{(2,2),(1)}\right) = 6$ and $\mathrm{rk}\left(\mathcal{C}_t^{(2,2),(1,1)}\right) = 2$.

Proposition 4.3.8. *Let $t \in \mathbb{S}_{(2,2)}\mathbb{K}^4$ as in* (4.3.1)*. Then t cannot be written with a decomposition as in (2.2), (3.1), (3.2), (3.3) and (3.4).*

Proof. The proof is performed as follows. For each case in the statement of this proposition, we assume that t admits such a decomposition. Then we show that we get a contradiction using the ranks of the catalecticant maps computed in Remark 4.3.7.

Firstly assume that t can be written as in (2.2). It is easy to see that in this case there exists a non zero element $\alpha \in \mathbb{S}_{(2,2)}(\mathbb{K}^4)^*$ such that $\alpha \in \ker \mathcal{C}_t^{(2,2),(1)}$. However by Remark 4.3.7, since $\mathrm{rk}\left(\mathcal{C}_t^{(2,2),(1)}\right) = 4$ we can conclude that t cannot admit such a decomposition. The same happens for the case (3.1).

Next, assume that t is as in (3.2). In this case, choosing as a basis of $\mathbb{K}^4$ the set $\{u,\ v,\ w,\ s\}$ and taking the respective dual basis for $(\mathbb{K}^4)^*$, one can see that the rank of the map $\mathcal{C}_t^{(2,2),(1,1)}$ should be 3. This is again in contrast with the hypothesis by Remark 4.3.7. The same holds for the case (3.3).

Eventually assume that t can be written as in (3.4). In particular assume that it is written as

$$(v \wedge u)^{\otimes 2} + (v \wedge w)^{\otimes 2} + ((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w))^{\otimes 2},$$

with $\langle u,\ v,\ w,\ s\rangle \simeq \mathbb{K}^4$, for some $a_1, \dots, a_4,\ b_1, b_2, b_3 \in \mathbb{K}$ such that $a_4 \neq 0$ and $b_1,\ b_3 \neq 0$. This time the discussion is a bit more involved. Assume that $\{u^*,\ v^*,\ w^*,\ s^*\}$ is a dual basis of $\{u,\ v,\ w,\ s\}$, and consider the canonical basis of $\bigwedge^2(\mathbb{K}^4)^*$ induced by the chosen basis. We compute the image of every element of such basis of $\bigwedge^2(\mathbb{K}^4)^*$ via the catalecticant map $\mathcal{C}_t^{(2,2),(1,1)}$ with respect to t.

$$\varphi(t \otimes (u^* \wedge v^*)) = u \wedge v + (b_1 a_2 - b_2 a_1)((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w)),$$

$$\varphi(t \otimes (u^* \wedge w^*)) = (b_1 a_3 - b_3 a_1)((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w)),$$

$$\varphi(t \otimes (u^* \wedge s^*)) = (b_1 a_4)((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w)),$$

$$\varphi(t \otimes (v^* \wedge w^*)) = v \wedge w + (b_2 a_3 - b_3 a_2)((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w)),$$

$$\varphi(t \otimes (v^* \wedge s^*)) = (b_2 a_4)((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w)),$$

$$\varphi(t \otimes (w^* \wedge s^*)) = (b_3 a_4)((a_1 v + a_2 u + a_3 w + a_4 s) \wedge (b_1 u + b_2 v + b_3 w)).$$

Decompose the six elements above of $\bigwedge^2(\mathbb{K}^4)^*$ with respect the basis induced by the basis $\{u^*,\ v^*,\ w^*,\ s^*\}$ of $(\mathbb{K}^4)^*$, and write a matrix with the coefficients of such decompositions. The resulting matrix M is the sum of $r^t r + E_{1,1} + E_{4,4}$, where $E_{i,j}$ is the matrix filled of 0's except in position (i, j) in which there is a 1, while r is the vector

$$r = \begin{pmatrix} b_1 a_2 - b_2 a_1 & b_1 a_3 - b_3 b_1 & b_1 a_4 & b_2 a_3 - b_3 a_2 & b_2 a_4 & b_3 a_4 \end{pmatrix}.$$

We have computed the 3×3 minors of M using `Macaulay2`, see [GS]. Such equations in the coefficients a_i's and b_i's describe the conditions for which the rank of the catalecticant map $\mathcal{C}_t^{(2,2),(1,1)}$ with respect to t is equal to 2. We get:

```
i1 : R = QQ[a_1..a_4,b_1..b_3];

i2 : V = matrix{{b_1*a_2-b_2*a_1,b_1*a_3-b_3*a_1,b_1*a_4,b_2*a_3-b_3*a_2,b_2*a_4,b_3*a_4}};

              1       6
o2 : Matrix R  <--- R

i3 : M = (transpose V)**V;

              6       6
o3 : Matrix R  <--- R

i4 : M = mutableMatrix M;

i5 : M_(0,0) = M_(0,0) + 1;

i6 : M_(3,3) = M_(3,3) + 1;

i7 : M = matrix M;

              6       6
o7 : Matrix R  <--- R

i8 : I = minors(3,M);

o8 : Ideal of R

i9 : mingens I

o9 = | a_4^2b_3^2 a_4^2b_2b_3 a_4^2b_1b_3 a_3a_4b_1b_3-a_1a_4b_3^2 a_4^2b_2^2 a_4^2b_1b_2
     ------------------------------------------------------------------------------------------
     a_3a_4b_1b_2-a_1a_4b_2b_3 a_4^2b_1^2 a_3a_4b_1^2-a_1a_4b_1b_3
     ------------------------------------------------------------------------------------------
     a_3^2b_1^2-2a_1a_3b_1b_3+a_1^2b_3^2 |

              1       10
o9 : Matrix R  <--- R
```

Note that between the generators of the ideal there are the elements $a_4^2b_3^2$ and $a_4^2b_1^2$ which cannot vanish since we have assumed that a_4, b_1 and b_3 are all different from 0. Hence the rank of the map $\mathcal{C}^{(2,2),(1,1)}$ is three and this is in contradiction with the hypothesis by Remark 4.3.7. This concludes the proof. □

It remains to discuss whether t admits a decomposition like (2.1) and (3.5). In this two cases we have no incompatibility conditions with the ranks of catalecticant maps with respect to t given in Remark 4.3.7.
In the first place assume that t can be written as in (2.1). We may write t as

$$(u_1 \wedge u_2)^{\otimes d} + (u_3 \wedge u_4)^{\otimes d}$$

where $\langle u_1,\ u_2,\ u_3,\ u_4\rangle \simeq \mathbb{K}^4$ and the addends are p_1 and p_2 respectively. Denote with $\{y_1, y_2, y_3, y_4\}$ the respective dual basis of $(\mathbb{K}^4)^*$. We get that the "generators" of $I(p_1)$ and $I(p_2)$ are

$$\langle y_3, y_4 \rangle \qquad \langle y_1, y_2 \rangle$$

respectively. Hence $I(t_1, t_2)_{(1)} = \langle 0 \rangle$ while $I(t_1, t_2)_{(2)} = \langle y_1y_3,\ y_1y_4,\ y_2y_3,\ y_2y_4 \rangle$. This means that we must find four products of two linear forms inside $\ker \mathcal{C}_t^{(2,2),(2)}$ such that the four linear forms are linearly independent. Suppose that

$$y_1 = \sum_{i=1}^{4} b_i x_i,\ y_2 = \sum_{i=1}^{4} c_i x_i,\ y_3 = \sum_{i=1}^{4} d_i x_i \text{ and } y_4 = \sum_{i=1}^{4} e_i x_i.$$

Using Remark 4.3.7 we impose four equations

$$\begin{aligned}
a_{1,1}x_3^2 + a_{1,2}x_3x_4 + a_{1,3}x_4^2 + a_{1,4}(x_1x_3 + x_2x_4) &= y_1y_3, \\
a_{2,1}x_3^2 + a_{2,2}x_3x_4 + a_{2,3}x_4^2 + a_{2,4}(x_1x_3 + x_2x_4) &= y_1y_4, \\
a_{3,1}x_3^2 + a_{3,2}x_3x_4 + a_{3,3}x_4^2 + a_{3,4}(x_1x_3 + x_2x_4) &= y_2y_3, \\
a_{4,1}x_3^2 + a_{4,2}x_3x_4 + a_{4,3}x_4^2 + a_{4,4}(x_1x_3 + x_2x_4) &= y_2y_4
\end{aligned}$$

and look for a solution in terms of the $a_{i,j}$ and b_k, c_k, d_k and e_k all in $\mathbb{K}$, assuming also that the two matrices

$$A = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$$

have both rank 4. For this purpose we have considered the ideal generated by all those equations inside the polynomial ring with variables the $a_{i,j}$ and b_k, c_k, d_k and e_k, and two auxiliary variables u and v. We have performed the computation using `Macaulay2`.

```
i1 : R = QQ[b_1..b_4,c_1..c_4,d_1..d_4,e_1..e_4,x_1..x_4,y_1..y_4,z_1..z_4,t_1..t_4,u,v];

i2 : A = matrix{{b_1,b_2,b_3,b_4},{c_1,c_2,c_3,c_4},{d_1,d_2,d_3,d_4},{e_1,e_2,e_3,e_4}}

o2 = | b_1 b_2 b_3 b_4 |
     | c_1 c_2 c_3 c_4 |
     | d_1 d_2 d_3 d_4 |
     | e_1 e_2 e_3 e_4 |

             4       4
o2 : Matrix R  <--- R

i3 : detA = det(A);

i4 : B = matrix{{x_1,x_2,x_3,x_4},{y_1,y_2,y_3,y_4},{z_1,z_2,z_3,z_4},{t_1,t_2,t_3,t_4}}

o4 = | x_1 x_2 x_3 x_4 |
     | y_1 y_2 y_3 y_4 |
     | z_1 z_2 z_3 z_4 |
     | t_1 t_2 t_3 t_4 |

             4       4
o4 : Matrix R  <--- R

i5 : detB = det(B);

i6 : L1 = {b_1*c_1,b_2*c_2,b_3*c_3-x_1,b_4*c_4-x_3,b_1*c_2+b_2*c_1,b_1*c_3+b_3*c_1,b_1*c_4+b_4*c_1
           -x_4,b_2*c_3+b_3*c_2-x_4,b_2*c_4+b_4*c_2,b_3*c_4+b_4*c_3-x_2}

o6 = {b c , b c , b c  - x , b c  - x , b c  + b c , b c  + b c , b c  + b c  - x , b c  + b c  -
       1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4    4   3 2    2 3
     ------------------------------------------------------------------------------------------
     x , b c  + b c , b c  + b c  - x }
      4   4 2    2 4   4 3    3 4    2

o6 : List

i7 : L2 = {b_1*d_1,b_2*d_2,b_3*d_3-y_1,b_4*d_4-y_3,b_1*d_2+b_2*d_1,b_1*d_3+b_3*d_1,b_1*d_4+b_4*d_1
           -y_4,b_2*d_3+b_3*d_2-y_4,b_2*d_4+b_4*d_2,b_3*d_4+b_4*d_3-y_2}

o7 = {b d , b d , b d  - y , b d  - y , b d  + b d , b d  + b d , b d  + b d  - y , b d  + b d  -
       1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4    4   3 2    2 3
     ------------------------------------------------------------------------------------------
     y , b d  + b d , b d  + b d  - y }
      4   4 2    2 4   4 3    3 4    2

o7 : List

i8 : L3 = {e_1*c_1,e_2*c_2,e_3*c_3-z_1,e_4*c_4-z_3,e_1*c_2+e_2*c_1,e_1*c_3+e_3*c_1,e_1*c_4+e_4*c_1
           -z_4,e_2*c_3+e_3*c_2-z_4,e_2*c_4+e_4*c_2,e_3*c_4+e_4*c_3-z_2}

o8 = {c e , c e , c e  - z , c e  - z , c e  + c e , c e  + c e , c e  + c e  - z , c e  + c e  -
       1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4    4   3 2    2 3
     ------------------------------------------------------------------------------------------
     z , c e  + c e , c e  + c e  - z }
      4   4 2    2 4   4 3    3 4    2

o8 : List
```

```
i9 : L4 = {d_1*c_1,d_2*c_2,d_3*c_3-t_1,d_4*c_4-t_3,d_1*c_2+d_2*c_1,d_1*c_3+d_3*c_1,d_1*c_4+d_4*c_1
           -t_4,d_2*c_3+d_3*c_2-t_4,d_2*c_4+d_4*c_2,d_3*c_4+d_4*c_3-t_2}

o9 = {c d , c d , c d  - t , c d  - t , c d  + c d , c d  + c d , c d  + c d  - t , c d  + c d  -
       1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4    4   3 2    2 3
     --------------------------------------------------------------------------------------------
     t , c d  + c d , c d  + c d  - t }
      4   4 2    2 4   4 3    3 4    2

o9 : List

i10 : L = L1 | L2 | L3 | L4 | {u*detA-1} | {v*detB -1}

o10 = {b c , b c , b c  - x , b c  - x , b c  + b c , b c  + b c , b c  + b c  - x , b c  + b c
        1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4    4   3 2    2 3
      --------------------------------------------------------------------------------------------
      - x , b c  + b c , b c  + b c  - x , b d , b d , b d  - y , b d  - y , b d  + b d , b d  +
         4   4 2    2 4   4 3    3 4    2   1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1
      --------------------------------------------------------------------------------------------
      b d , b d  + b d  - y , b d  + b d  - y , b d  + b d , b d  + b d  - y , c e , c e , c e  -
       1 3   4 1    1 4    4   3 2    2 3    4   4 2    2 4   4 3    3 4    2   1 1   2 2   3 3
      --------------------------------------------------------------------------------------------
      z , c e  - z , c e  + c e , c e  + c e , c e  + c e  - z , c e  + c e  - z , c e  + c e ,
       1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4    4   3 2    2 3    4   4 2    2 4
      --------------------------------------------------------------------------------------------
      c e  + c e  - z , c d , c d , c d  - t , c d  - t , c d  + c d , c d  + c d , c d  + c d  -
       4 3    3 4    2   1 1   2 2   3 3    1   4 4    3   2 1    1 2   3 1    1 3   4 1    1 4
      --------------------------------------------------------------------------------------------
      t , c d  + c d  - t , c d  + c d , c d  + c d  - t , b c d e u - b c d e u - b c d e u +
       4   3 2    2 3    4   4 2    2 4   4 3    3 4    2   4 3 2 1     3 4 2 1     4 2 3 1
      --------------------------------------------------------------------------------------------
      b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
       2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2     4 1 3 2     1 4 3 2
      --------------------------------------------------------------------------------------------
      b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u -
       3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3     1 4 2 3     2 1 4 3
      --------------------------------------------------------------------------------------------
      b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u - 1,
       1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4     2 1 3 4     1 2 3 4
      --------------------------------------------------------------------------------------------
      x y z t v - x y z t v - x y z t v + x y z t v + x y z t v - x y z t v - x y z t v +
       4 3 2 1     3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2
      --------------------------------------------------------------------------------------------
      x y z t v + x y z t v - x y z t v - x y z t v + x y z t v + x y z t v - x y z t v -
       3 4 1 2     4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3
      --------------------------------------------------------------------------------------------
      x y z t v + x y z t v + x y z t v - x y z t v - x y z t v + x y z t v + x y z t v -
       4 1 2 3     1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4
      --------------------------------------------------------------------------------------------
      x y z t v - x y z t v + x y z t v - 1}
       1 3 2 4     2 1 3 4     1 2 3 4

o10 : List

i11 : I = ideal(L);

o11 : Ideal of R
```

```
i12 : I == R

o12 = true
```

This proves that over the field of rational numbers $\mathbb{Q}$, the element 1 belong to the ideal. In particular this implies also that 1 belongs to the same ideal generated with complex coefficients. Hence we can conclude that our tensor t does not admit a decomposition as (2.1) since the respective system of polynomials does not admit a solution.
Consider now the case in which t can be written as (3.5). We may write t as

$$(u_1 \wedge u_2)^{\otimes 2} + (u_3 \wedge u_4)^{\otimes 2} + ((u_1 + u_3) \wedge (u_2 + u_4))^{\otimes 2},$$

where $\langle u_1,\ u_2,\ u_3,\ u_4\rangle \simeq \mathbb{K}^4$. Denote the respective dual basis of $(\mathbb{K}^4)^*$ with $\{y_1,\ y_2,\ y_3,\ y_4\}$ and call the respective points p_1, p_2 and p_3. The "generators" of the respective subspaces $I(p_1)$, $I(p_2)$ and $I(p_3)$ are

$$\langle y_3,\ y_4\rangle \quad \langle y_1,\ y_2\rangle \quad \langle y_1 - y_3,\ y_2 - y_4\rangle.$$

Hence we get that $I(p_1, p_2, p_3)_{(1)} = \langle 0\rangle$ and

$$I(p_1, p_2, p_3) = \langle y_2 \wedge y_4,\ y_2 \wedge y_3 + y_1 \wedge y_4,\ y_1 \wedge y_3\rangle.$$

This means that in the kernel $\ker \mathcal{C}_t^{(2,2),(1,1)}$ we must find such generators in the previous formula. Suppose that

$$y_1 = \sum_{i=1}^{4} b_i x_i,\ y_2 = \sum_{i=1}^{4} c_i x_i,\ y_3 = \sum_{i=1}^{4} d_i x_i \text{ and } y_4 = \sum_{i=1}^{4} e_i x_i.$$

As with the case (2.1), using Remark 4.3.7 we can impose a system

$$a_{1,1}x_1 \wedge x_3 + a_{1,2}x_2 \wedge x_4 + a_{1,3}x_3 \wedge x_4 + a_{1,4}(x_1 \wedge x_4 - x_2 \wedge x_3) = y_2 \wedge y_4,$$

$$a_{2,1}x_1 \wedge x_3 + a_{2,2}x_2 \wedge x_4 + a_{2,3}x_3 \wedge x_4 + a_{2,4}(x_1 \wedge x_4 - x_2 \wedge x_3) = y_2 \wedge y_3 + y_1 \wedge y_4,$$

$$a_{3,1}x_1 \wedge x_3 + a_{3,2}x_2 \wedge x_4 + a_{3,3}x_3 \wedge x_4 + a_{3,4}(x_1 \wedge x_4 - x_2 \wedge x_3) = y_1 \wedge y_3$$

and look for a solution in terms of the $a_{i,j}$ and b_k, c_k, d_k and e_k all in $\mathbb{K}$, assuming also that the two matrices

$$A = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \end{pmatrix}$$

have both full rank. To this end we have considered the ideal generated by all those equations inside the polynomial ring with variables the $a_{i,j}$ and b_k, c_k, d_k and e_k, and two auxiliary variables u and v. We have performed the computation using `Macaulay2`.

```
i1 : R = QQ[b_1..b_4,c_1..c_4,d_1..d_4,e_1..e_4,x_1..x_4,y_1..y_4,z_1..z_4,u,v];

i2 : A = matrix{{b_1,b_2,b_3,b_4},{c_1,c_2,c_3,c_4},{d_1,d_2,d_3,d_4},{e_1,e_2,e_3,e_4}}

o2 = | b_1 b_2 b_3 b_4 |
     | c_1 c_2 c_3 c_4 |
     | d_1 d_2 d_3 d_4 |
     | e_1 e_2 e_3 e_4 |

             4       4
o2 : Matrix R  <--- R

i3 : detA = det(A);

i4 : B1 = matrix{{x_2,x_3,x_4},{y_2,y_3,y_4},{z_2,z_3,z_4}} -- matrix w/ the first column removed

o4 = | x_2 x_3 x_4 |
     | y_2 y_3 y_4 |
     | z_2 z_3 z_4 |

             3       3
o4 : Matrix R  <--- R

i5 : B2 = matrix{{x_1,x_3,x_4},{y_1,y_3,y_4},{z_1,z_3,z_4}} --matrix w/ the second column removed

o5 = | x_1 x_3 x_4 |
     | y_1 y_3 y_4 |
     | z_1 z_3 z_4 |

             3       3
o5 : Matrix R  <--- R

i6 : B3 = matrix{{x_1,x_2,x_4},{y_1,y_2,y_4},{z_1,z_2,z_4}} --matrix w/ the third column removed

o6 = | x_1 x_2 x_4 |
```

```
     | y_1 y_2 y_4 |
     | z_1 z_2 z_4 |

             3       3
o6 : Matrix R  <--- R

i7 : B4 = matrix{{x_1,x_2,x_3},{y_1,y_2,y_3},{z_1,z_2,z_3}} --matrix w/ the fourth column removed

o7 = | x_1 x_2 x_3 |
     | y_1 y_2 y_3 |
     | z_1 z_2 z_3 |

             3       3
o7 : Matrix R  <--- R

i8 : L1 = {c_1*e_2-c_2*e_1,c_1*e_3-c_3*e_1-x_1,c_1*e_4-c_4*e_1-x_4,c_2*e_3-c_3*e_2+x_4,c_2*e_4
      -c_4*e_2-x_2,c_3*e_4-c_4*e_3-x_3}

o8 = {- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  + c e  -
         2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2    2 3    4     4 2    2 4
     ------------------------------------------------------------------------------------------
     x , - c e  + c e  - x }
      2     4 3    3 4    3

o8 : List

i9 : L2 = {b_1*d_2-b_2*d_1,b_1*d_3-b_3*d_1-y_1,b_1*d_4-b_4*d_1-y_4,b_2*d_3-b_3*d_2+y_4,b_2*d_4
      -b_4*d_2-y_2,b_3*d_4-b_4*d_3-y_3}

o9 = {- b d  + b d , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  + y , - b d  + b d  -
         2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2    2 3    4     4 2    2 4
     ------------------------------------------------------------------------------------------
     y , - b d  + b d  - y }
      2     4 3    3 4    3

o9 : List

i10 : L3 = L1 + L2 + {0,x_1+y_1-z_1,x_4+y_4-z_4,-x_4-y_4+z_4,x_2+y_2-z_2,x_3+y_3-z_3}

o10 = {- b d  + b d  - c e  + c e , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e
          2 1    1 2    2 1    1 2     3 1    1 3    3 1    1 3    1     4 1    1 4    4 1    1 4
      ------------------------------------------------------------------------------------------
      - z , - b d  + b d  - c e  + c e  + z , - b d  + b d  - c e  + c e  - z , - b d  + b d  -
         4     3 2    2 3    3 2    2 3    4     4 2    2 4    4 2    2 4    2     4 3    3 4
      ------------------------------------------------------------------------------------------
      c e  + c e  - z }
       4 3    3 4    3

o10 : List

i11 : Cond1 = L1 | L2 | L3 | {u*det(A)-1} | {v*det(B1)-1}

o11 = {- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  + c e  -
          2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2    2 3    4     4 2    2 4
      ------------------------------------------------------------------------------------------
      x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , - b d  +
       2     4 3    3 4    3     2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2
      ------------------------------------------------------------------------------------------
```

```
     b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , - b d  +
      2 3    4     4 2    2 4    2     4 3    3 4    3     2 1    1 2    2 1    1 2     3 1
     ----------------------------------------------------------------------------------------
     b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  +
      1 3    3 1    1 3    1     4 1    1 4    4 1    1 4    4     3 2    2 3    3 2    2 3
     ----------------------------------------------------------------------------------------
     z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
      4     4 2    2 4    4 2    2 4    2     4 3    3 4    4 3    3 4    3  4 3 2 1
     ----------------------------------------------------------------------------------------
     b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
      3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
     ----------------------------------------------------------------------------------------
     b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
      4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
     ----------------------------------------------------------------------------------------
     b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
      1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
     ----------------------------------------------------------------------------------------
     b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1}
      2 1 3 4     1 2 3 4             4 3 2       3 4 2       4 2 3       2 4 3       3 2 4       2 3 4

o11 : List

i12 : I1 = ideal(Cond1)

o12 = ideal (- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  +
                 2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2    2 3    4     4 2
     ----------------------------------------------------------------------------------------
     c e  - x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , -
      2 4    2     4 3    3 4    3     2 1    1 2     3 1    1 3    1     4 1    1 4    4
     ----------------------------------------------------------------------------------------
     b d  + b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , -
      3 2    2 3    4     4 2    2 4    2     4 3    3 4    3     2 1    1 2    2 1    1 2
     ----------------------------------------------------------------------------------------
     b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  +
      3 1    1 3    3 1    1 3    1     4 1    1 4    4 1    1 4    4     3 2    2 3    3 2
     ----------------------------------------------------------------------------------------
     c e  + z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
      2 3    4     4 2    2 4    4 2    2 4    2     4 3    3 4    4 3    3 4    3  4 3 2 1
     ----------------------------------------------------------------------------------------
     b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
      3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
     ----------------------------------------------------------------------------------------
     b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
      4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
     ----------------------------------------------------------------------------------------
     b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
      1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
     ----------------------------------------------------------------------------------------
     b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1)
      2 1 3 4     1 2 3 4             4 3 2       3 4 2       4 2 3       2 4 3       3 2 4       2 3 4

o12 : Ideal of R

i13 : I1 == R

o13 = true
```

```
i14 : Cond2 = L1 | L2 | L3 | {u*det(A)-1} | {v*det(B2)-1}

o14 = {- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  + c e  -
         2 1    1 2      3 1    1 3    1      4 1    1 4    4      3 2    2 3    4      4 2    2 4
      ------------------------------------------------------------------------------------------
      x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , - b d  +
       2     4 3    3 4    3     2 1    1 2      3 1    1 3    1      4 1    1 4    4      3 2
      ------------------------------------------------------------------------------------------
      b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , - b d  +
       2 3    4      4 2    2 4    2      4 3    3 4    3      2 1    1 2    2 1    1 2      3 1
      ------------------------------------------------------------------------------------------
      b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  +
       1 3    3 1    1 3    1      4 1    1 4    4 1    1 4    4      3 2    2 3    3 2    2 3
      ------------------------------------------------------------------------------------------
      z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
       4     4 2    2 4    4 2    2 4    2      4 3    3 4    4 3    3 4    3  4 3 2 1
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
       3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
       4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
      ------------------------------------------------------------------------------------------
      b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
       1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
      ------------------------------------------------------------------------------------------
      b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1}
       2 1 3 4     1 2 3 4           4 3 1       3 4 1       4 1 3       1 4 3       3 1 4       1 3 4

o14 : List

i15 : I2 = ideal(Cond2)

o15 = ideal (- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  +
                2 1    1 2      3 1    1 3    1      4 1    1 4    4      3 2    2 3    4      4 2
      ------------------------------------------------------------------------------------------
      c e  - x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , -
       2 4    2      4 3    3 4    3      2 1    1 2      3 1    1 3    1      4 1    1 4    4
      ------------------------------------------------------------------------------------------
      b d  + b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , -
       3 2    2 3    4      4 2    2 4    2      4 3    3 4    3      2 1    1 2    2 1    1 2
      ------------------------------------------------------------------------------------------
      b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  +
       3 1    1 3    3 1    1 3    1      4 1    1 4    4 1    1 4    4      3 2    2 3    3 2
      ------------------------------------------------------------------------------------------
      c e  + z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
       2 3    4      4 2    2 4    4 2    2 4    2      4 3    3 4    4 3    3 4    3  4 3 2 1
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
       3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
       4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
      ------------------------------------------------------------------------------------------
      b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
       1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
      ------------------------------------------------------------------------------------------
      b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1)
```

```
      2 1 3 4    1 2 3 4        4 3 1    3 4 1    4 1 3    1 4 3    3 1 4    1 3 4

o15 : Ideal of R

i16 : I2 == R

o16 = true

i17 : Cond3 = L1 | L2 | L3 | {u*det(A)-1} | {v*det(B3)-1}

o17 = {- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  + c e  -
          2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2    2 3    4     4 2    2 4
      ------------------------------------------------------------------------------------------
      x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , - b d  +
       2     4 3    3 4    3     2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2
      ------------------------------------------------------------------------------------------
      b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , - b d  +
       2 3    4     4 2    2 4    2     4 3    3 4    3     2 1    1 2    2 1    1 2     3 1
      ------------------------------------------------------------------------------------------
      b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  +
       1 3    3 1    1 3    1     4 1    1 4    4 1    1 4    4     3 2    2 3    3 2    2 3
      ------------------------------------------------------------------------------------------
      z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
       4     4 2    2 4    4 2    2 4    2     4 3    3 4    4 3    3 4    3  4 3 2 1
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
       3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
       4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
      ------------------------------------------------------------------------------------------
      b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
       1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
      ------------------------------------------------------------------------------------------
      b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1}
       2 1 3 4     1 2 3 4          4 2 1    2 4 1    4 1 2    1 4 2    2 1 4    1 2 4

o17 : List

i18 : I3 = ideal(Cond3)

o18 = ideal (- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  +
                2 1    1 2     3 1    1 3    1     4 1    1 4    4     3 2    2 3    4     4 2
      ------------------------------------------------------------------------------------------
      c e  - x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , -
       2 4    2     4 3    3 4    3     2 1    1 2     3 1    1 3    1     4 1    1 4    4
      ------------------------------------------------------------------------------------------
      b d  + b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , -
       3 2    2 3    4     4 2    2 4    2     4 3    3 4    3     2 1    1 2    2 1    1 2
      ------------------------------------------------------------------------------------------
      b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  +
       3 1    1 3    3 1    1 3    1     4 1    1 4    4 1    1 4    4     3 2    2 3    3 2
      ------------------------------------------------------------------------------------------
      c e  + z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
       2 3    4     4 2    2 4    4 2    2 4    2     4 3    3 4    4 3    3 4    3  4 3 2 1
      ------------------------------------------------------------------------------------------
      b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
       3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
```

```
     ------------------------------------------------------------------------------------
     b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
      4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
     ------------------------------------------------------------------------------------
     b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
      1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
     ------------------------------------------------------------------------------------
     b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1)
      2 1 3 4     1 2 3 4           4 2 1       2 4 1       4 1 2       1 4 2       2 1 4       1 2 4

o18 : Ideal of R

i19 : I3 == R

o19 = true

i20 : Cond4 = L1 | L2 | L3 | {u*det(A)-1} | {v*det(B4)-1}

o20 = {- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  + c e  -
           2 1   1 2     3 1   1 3   1       4 1   1 4   4       3 2   2 3   4       4 2   2 4
     ------------------------------------------------------------------------------------
     x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , - b d  +
      2     4 3   3 4   3     2 1   1 2      3 1   1 3   1      4 1   1 4   4      3 2
     ------------------------------------------------------------------------------------
     b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , - b d  +
      2 3   4      4 2   2 4   2      4 3   3 4   3      2 1   1 2   2 1   1 2      3 1
     ------------------------------------------------------------------------------------
     b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  +
      1 3   3 1   1 3   1      4 1   1 4   4 1   1 4   4      3 2   2 3   3 2   2 3
     ------------------------------------------------------------------------------------
     z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
      4     4 2   2 4   4 2   2 4   2      4 3   3 4   4 3   3 4   3   4 3 2 1
     ------------------------------------------------------------------------------------
     b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
      3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
     ------------------------------------------------------------------------------------
     b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
      4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
     ------------------------------------------------------------------------------------
     b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
      1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
     ------------------------------------------------------------------------------------
     b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1}
      2 1 3 4     1 2 3 4           3 2 1       2 3 1       3 1 2       1 3 2       2 1 3       1 2 3

o20 : List

i21 : I4 = ideal(Cond4)

o21 = ideal (- c e  + c e , - c e  + c e  - x , - c e  + c e  - x , - c e  + c e  + x , - c e  +
                 2 1   1 2     3 1   1 3   1       4 1   1 4   4       3 2   2 3   4       4 2
     ------------------------------------------------------------------------------------
     c e  - x , - c e  + c e  - x , - b d  + b d , - b d  + b d  - y , - b d  + b d  - y , -
      2 4   2      4 3   3 4   3      2 1   1 2      3 1   1 3   1      4 1   1 4   4
     ------------------------------------------------------------------------------------
     b d  + b d  + y , - b d  + b d  - y , - b d  + b d  - y , - b d  + b d  - c e  + c e , -
      3 2   2 3   4      4 2   2 4   2      4 3   3 4   3      2 1   1 2   2 1   1 2
     ------------------------------------------------------------------------------------
```

```
      b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  +
       3 1   1 3   3 1   1 3   1     4 1   1 4   4 1   1 4   4     3 2   2 3   3 2
      ---------------------------------------------------------------------------------------
      c e  + z , - b d  + b d  - c e  + c e  - z , - b d  + b d  - c e  + c e  - z , b c d e u -
       2 3   4     4 2   2 4   4 2   2 4   2     4 3   3 4   4 3   3 4   3   4 3 2 1
      ---------------------------------------------------------------------------------------
      b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u + b c d e u +
       3 4 2 1     4 2 3 1     2 4 3 1     3 2 4 1     2 3 4 1     4 3 1 2     3 4 1 2
      ---------------------------------------------------------------------------------------
      b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u - b c d e u +
       4 1 3 2     1 4 3 2     3 1 4 2     1 3 4 2     4 2 1 3     2 4 1 3     4 1 2 3
      ---------------------------------------------------------------------------------------
      b c d e u + b c d e u - b c d e u - b c d e u + b c d e u + b c d e u - b c d e u -
       1 4 2 3     2 1 4 3     1 2 4 3     3 2 1 4     2 3 1 4     3 1 2 4     1 3 2 4
      ---------------------------------------------------------------------------------------
      b c d e u + b c d e u - 1, - x y z v + x y z v + x y z v - x y z v - x y z v + x y z v - 1)
       2 1 3 4     1 2 3 4           3 2 1     2 3 1     3 1 2     1 3 2     2 1 3     1 2 3

o21 : Ideal of R

i22 : I4 == R

o22 = true
```

As before, this proves that over the field of rational numbers $\mathbb{Q}$ the tensor t does not admit a $(2,2)$-rank 3 decomposition since 1 belongs to the ideal. As with the previous computation we can conclude that our tensor t does not admit a decomposition as (3.5). All this long discussion allows us to conclude that

Corollary 4.3.9. *The* $(2,2)$*-rank of* $t \in \mathbb{S}_{(2,2)}\mathbb{K}^4$ *as in* (4.3.1) *is* 4*. In particular we have that if* $X_{2,n,2} = (\mathbb{G}(2,V), \mathcal{O}(2))$*, then*

$$\sigma_2(X_{2,n,2}) \setminus X_{2,n,2} = \sigma_{2,2}(X_{2,n,2}) \cup \sigma_{2,4}(X_{2,n,2}),$$

where $\sigma_{r,s}(X) := \{p \in \sigma_r(X) \ : \ r_X(p) = s\}$*.*

Remark that the locus $\sigma_{2,4}(X_{2,n,2})$ corresponds to the tangential variety of $X_{2,n,2}$. Therefore we have computed that the $(2,2)$-rank of a general element belonging to a tangential line to $X_{2,n,2}$ is 4. This result has some similarity with the case in which X is a Veronese variety. Analogous computations for this case have been made in [BGI11].

4.4 On the maximum rank of a tensor

In this section we show a small computation regarding the *maximum* X-rank of a point. Such a paragraph has been extracted in the paper [BS20].

In the end of the previous section we have seen that if $X_{2,n,2} = (\mathbb{G}(2,V), \mathcal{O}(2))$, then there are points of border $(2,2)$-rank 2 and $(2,2)$-rank 4, i.e. points whose X-rank is strictly greater than their border X-rank. This is a well known phenomenon classically studied in the case of X being a Veronese variety, that lead to trying to classify the X ranks of the points of fixed border X-rank, and to describe how these X-ranks are arranged with respect to the nesting of the secant varieties. In extreme this lead to discover which is the *maximum* X-rank and where it is attained.

It follows

Question 4.4.1. *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth and non-degenerate variety. Which is the maximum X-rank for a point $p \in \mathbb{P}^N$?*

Even though the question looks like simple, to find an answer is in general difficult and only in few cases the maximum X-rank has been computed explicitly. However some upper bounds are present in the literature. We report here in chronological order some of the principal results we are aware of. We denote with r_{max} the maximum X-rank.

Proposition 4.4.2 (Prop. 5.1, p. 348, [LT10])**.** *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth and non-degenerate variety of dimension n. Then $r_{max} \leq N + 1 - n$.*

Proposition 4.4.3 (Theorem 1, p. 1022, [BT15])**.** *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth and non-degenerate variety and denote with r_{gen} the generic X-rank. Then we have the inequality $r_{max} \leq 2r_{gen}$.*

Proposition 4.4.4 (Theorem 6, p. 1024, [BT15])**.** *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth and non-degenerate variety such that X is not itself a hypersurface. Suppose that $\sigma_{r_{gen}-1}(X)$ is a hypersurface. Then $r_{max} \leq 2r_{gen} - 1$. If the hypersurface $\sigma_{r_{gen}-1}(X)$ has no points of multiplicity equal to* $\deg(\sigma_{r_{gen}-1}(X)) - 1$*, then $r_{max} \leq 2r_{gen} - 2$.*

Proposition 4.4.5 (Proposition 7, p. 1025, [BT15])**.** *Let $X \subset \mathbb{P}^N$ be an irreducible, smooth and non-degenerate variety and suppose that $\sigma_r(X)$ has codimension c. Let $r_{max,r}$ be the maximum X-rank appearing in $\sigma_r(X)$. Then it holds*

$$r_{max} \leq \max\{r_{max,s}, (c+1)r\}.$$

Proposition 4.4.6 (Theorem 3.7, p. 118, [BHMT18])**.** *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate curve in $\mathbb{P}^N$. Then $r_{max} \leq 2r_{gen} - 1$. Moreover, if $\sigma_{r_{gen}-1}(X)$ is a hypersurface, then $r_{max} \leq 2r_{gen} - 2$.*

Proposition 4.4.7 (Theorem 3.9, p. 118, [BHMT18]). *Let G be a connected algebraic group, V an irreducible representation of G, and $X = G/P \subset \mathbb{P}(V)$ a rational homogeneous variety. Then $r_{max} \leq 2r_{gen} - 1$. Moreover, if $\sigma_{r_{gen}-1}(X)$ is a hypersurface, then $r_{max} \leq 2r_{gen} - 2$.*

Remark that several of the previous bounds are improved when some secant variety is a hypersurface. There are numerous instances of such varieties. For example there are the Segre varieties $\nu_{1,1}\left(\mathbb{P}(V) \times \mathbb{P}(V)\right)$ parametrizing rank 1 square matrices, Veronese varieties $\nu_2(\mathbb{P}^n)$ of rank 1 symmetric matrices, Grassmann varieties $\mathbb{G}(2, V)$ of rank 1 skew-symmetric matrices, and eventually Flag varieties $\mathbb{F}(1, n-1; \mathbb{K}^n)$ of rank 1 square traceless matrices. In all these instances the last non filling secant variety is cut out by determinant or the Pfaffian equation.
Apart from the cases of matrices in which the maximum X-rank is linked to the maximum rank of the associated matrix, other instances of varieties X for which a secant variety is a hypersurface are not so common. We will compute a new example which is the variety $X = \left(\mathbb{G}(2,\mathbb{K}^4), \mathcal{O}(2)\right)$. To prove it, we recall at first a lower bound on the dimension of the secant varieties to X.

Conjecture 4.4.8 (Conjecture 4.7, case 2.(d), p. 248 in [BDDG07]). *Let $X = \left(\mathbb{G}(2,\mathbb{K}^4), \mathcal{O}(2)\right) \subset \mathbb{P}\left(\mathbb{S}_{(2,2)}\mathbb{K}^4\right) \simeq \mathbb{P}^{19}$. Then X has the following secant dimensions*

$$\dim \sigma_2(X) = 9,\ \dim \sigma_3(X) = 13,\ \dim \sigma_4(X) = 16 \text{ and } \dim \sigma_5(X) = 18.$$

In particular the generic $(2,2)$-rank is 6.

Remark that for $r = 3, 4$ and 5 the secant varieties are suspected to be defective and that the $\sigma_5(X)$ is a hypersurface. Since these dimensions have been achieved computing the span of the tangent spaces at r random points of X, these numbers are only lower bounds for the actual dimension of such secant varieties. We now give an upper bound for such dimensions which is equal to the provided lower bounds implying the equality of the dimensions. Before we need the following remark.

Remark 4.4.9 (Section 5.3 of [LO13]). Let X be a variety embedded in $\mathbb{P}\left(H^0(X,L)^*\right)$ via the very ample line bundle L, and let $E \longrightarrow X$ be a rank e bundle over X. Then recall that

$$\sigma_r(X) \subseteq Rank_r(E).$$

Moreover, if for a given smooth point $[v] \in Rank_r(E)$, the map $A_v : H^0(X,E) \longrightarrow H^0(X, E^\vee \otimes L)$ is the respective non-abelian apolarity action, then we have that the affine conormal space of $Rank_r(E)$ at $[v] \in Rank_r(E)$ is given as the image of the map

$$\ker A_v \otimes \operatorname{im} A_v^{\perp} \longrightarrow H^0(X, L) = V^*.$$

We now prove Conjecture 4.4.8.

Proof. In [BDDG07] the authors provide the lower bounds on the secant dimensions of $X = \left(\mathbb{G}(2, \mathbb{K}^4), \mathcal{O}(2)\right) \subset \mathbb{P}^{19}$, explicitly

$$\dim \sigma_2(X) \geq 9,\ \dim \sigma_3(X) \geq 13,\ \dim \sigma_4(X) \geq 16 \text{ and} \dim \sigma_5(X) \geq 18.$$

Except the case $r = 2$ in which the lower bound is the expected dimension of $\sigma_2(X)$, in all the other cases the lower bounds do not tell us anything about the defectiveness or not of the respective secant varieties. However for some $t \in \mathbb{S}_{(2,2)}\mathbb{K}^4$ consider the catalecticant map

$$\mathcal{C}_t^{(2,2),(1,1)} : \bigwedge^2 (\mathbb{K}^4)^* \longrightarrow \bigwedge^2 \mathbb{K}^4.$$

Note that this is a case in which the Non-abelian apolarity and the Schur apolarity coincide by Corollary 3.3.3. Specifically choosing $L = \left(\wedge^2 \mathcal{U}^\vee\right)^{\otimes 2}$ and $E = \wedge^2 \mathcal{U}^\vee$ of rank 1, where $\mathcal{U}^\vee$ is the dual of the tautological bundle, we get that

$$H^0(X, E) \simeq \bigwedge^2 (\mathbb{K}^4)^*,\ H^0(X, L)^* \simeq \mathbb{S}_{(2,2)}\mathbb{K}^4 \text{ and } H^0(X, E^\vee \otimes L)^* \simeq \bigwedge^2 \mathbb{K}^4.$$

Now we use in combination the Proposition 4.1.4 and the Remark 4.4.9 to compute the dimension of $Rank_r(E)$ for $r = 3, 4$ and 5.

Let $r = 3$ and consider $v = p_1 + p_2 + p_3$, with $p_i \in X$ generic points. Remark at first that the map A_v is symmetric. Therefore, in order to compute the affine conormal space to $Rank_r(E)$ it is enough to compute the image of

$$\operatorname{Sym}^2 (\ker A_v) \longrightarrow \mathbb{S}_{(2,2)}(\mathbb{K}^4)^*,$$

which is the restriction of the quotient map

$$\operatorname{Sym}^2 \left(\bigwedge^2 (\mathbb{K}^4)^*\right) \longrightarrow \mathbb{S}_{(2,2)}(\mathbb{K}^4)^*$$

given by the relation

$$\langle (x_1 \wedge x_2) \cdot (x_3 \wedge x_4) - (x_1 \wedge x_3) \cdot (x_2 \wedge x_4) + (x_2 \wedge x_3) \cdot (x_1 \wedge x_4) \rangle.$$

Since v is sum of three generic points of X, its $(2,2)$-rank is 3. Hence the kernel of A_v is 3 dimensional. More explicitly, by the genericity we may assume that p_1, p_2 and p_3 are as in (3.5), i.e.

$$v = (u \wedge v)^{\otimes 2} + (w \wedge s)^{\otimes 2} + ((u+w) \wedge (v+s))^{\otimes 2}$$

for some u, v, w and s linearly independent vectors of $\mathbb{K}^4$. Choosing the respective dual basis of $(\mathbb{K}^4)^*$ and considering the induced basis of $(\bigwedge^2(\mathbb{K}^4)^*)$, we can see that A_v has matrix

$$A_v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and that the kernel is

$$\ker A_v = \langle x_1 \wedge x_3 - x_1 \wedge x_4,\ x_1 \wedge x_3 - x_2 \wedge x_3,\ x_1 \wedge x_3 - x_2 \wedge x_4 \rangle.$$

It is easy to see that the space $\mathrm{Sym}^2(\ker A_v)$, whose dimension is 6, projects injectively onto $\mathbb{S}_{(2,2)}(\mathbb{K}^4)^*$. Indeed, it is enough to check that the generator of the quotient space does not belong to the space $\mathrm{Sym}^2(\ker A_v)$. Therefore we get that

$$6 = \dim \left(\widehat{T_v Rank_3(E)} \right)^{\perp} = \operatorname{codim} Rank_r(E) = 19 - \dim Rank_3(E)$$

so that $\dim Rank_3(E) = 13$. Since $Rank_3(E)$ contains $\sigma_3(X)$, we get that $\dim \sigma_3(X) \leq 13$. Therefore, since the lower bound in [BDDG07] is exactly 13 we can conclude that $\dim \sigma_3(X) = 13$. In particular it is defective with defect equal to 1.

Let $r = 4$. We proceed as in the case $r = 3$. In this case pick $v = p_1 + \cdots + p_4$, where every $p_i \in X$ is generic. In particular from genericity assumption we can suppose that p_1, p_2 and p_3 are as in (3.5), so that

$$\begin{aligned} v = (u \wedge v)^{\otimes 2} + (w \wedge s)^{\otimes 2} + ((u+w) \wedge (v+s))^{\otimes 2} + \\ + ((a_1 u + \cdots + a_4 s) \wedge (b_1 u + \cdots + b_4 s))^{\otimes 2} \end{aligned}$$

where every pair picked from the two dimensional spaces appearing above is disjoint. It is easy to see that

$$A_v = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + v^t v$$

where $v \in (\mathbb{K}^6)$ is the vector

$$v = (a_1 b_2 - a_2 b_1 \quad a_1 b_3 - a_3 b_1 \quad a_1 b_4 - a_4 b_1 \quad a_2 b_3 - a_3 b_2 \quad a_2 b_4 - a_4 b_2 \quad a_3 b_4 - a_4 b_3).$$

Obviously A_v is again symmetric and its rank is 4. Therefore its kernel has dimension 2, so that $\mathrm{Sym}^2(\ker A_v)$ has dimension 3. Even in this case it is easy to check that the space $\mathrm{Sym}^2(\ker A_v)$ projects injectively onto $\mathbb{S}_{(2,2)}(\mathbb{K}^4)^*$. Therefore we get

$$3 = \dim\left(\widehat{T_v Rank_4(E)}\right)^{\perp} = \operatorname{codim} Rank_4(E) = 19 - \dim Rank_4(E)$$

so that $\dim Rank_4(E) = 16$. Since $Rank_4(E)$ contains $\sigma_4(X)$, we get $\dim \sigma_4(X) \leq 16$. Since this number is equal to lower bound given at the beginning of the proof we can conclude that $\dim \sigma_4(X) = 16$.

Eventually the case $r = 5$ can be discussed as previous ones. Choosing $v = p_1 + \cdots + p_5$ with every $p_i \in X$ generic, we can again assume that the first three elements are as in (3.5). Again we can see that the matrix A_v will be symmetric of rank 5. Therefore $\ker A_v$ has dimension 1 and so $\mathrm{Sym}^2(\ker A_v)$. Since also in this case such space projects injectively in $\mathbb{S}_{(2,2)}(\mathbb{K}^4)^*$, we get that

$$1 = \dim\left(\widehat{T_v Rank_5(E)}\right)^{\perp} = \operatorname{codim} Rank_5(E) = 19 - \dim Rank_5(E)$$

so that $Rank_5(E)$ is an hypersurface. Hence one gets $\dim \sigma_5(X) \leq 18$. By the remark made at the beginning of the proof we can conclude that $\dim \sigma_5(X) = 18$. It follows that $\sigma_6(X) = \mathbb{P}^{19}$. This concludes the proof. $\square$

Therefore we have proved the following

Proposition 4.4.10. *Let* $X = \left(\mathbb{G}(2, \mathbb{K}^4), \mathcal{O}(2)\right) \subset \mathbb{P}\left(\mathbb{S}_{(2,2)}\mathbb{K}^4\right) \simeq \mathbb{P}^{19}$. *Then X has the following secant dimensions*

$$\dim \sigma_2(X) = 9,\ \dim \sigma_3(X) = 13,\ \dim \sigma_4(X) = 16 \textit{ and } \dim \sigma_5(X) = 18.$$

In particular the varieties $\sigma_3(X)$, $\sigma_4(X)$ and $\sigma_5(X)$ are defective and the generic $(2,2)$-rank is 6. Remark also that the $\sigma_5(X)$ is the determinantal hypersurface of degree 6.

In the case in which $X \subset \mathbb{P}^N$ is irreducible, smooth and non-degenerate such that one of its secant varieties is a hypersurface, we have been able to provide a new bound on the maximum X-rank. The work has been extracted in [BS20].

Proposition 4.4.11. *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate projective variety of dimension n and let W be a hypersurface strictly containing X. Let $P \in \mathbb{P}^N \setminus W$ be such that $r_X(P) \neq 2$, and let*

$$Y_P := \mathrm{cone}(P, X) = \bigcup_{x \in X} \langle x, P \rangle.$$

Then $Y_P \cap W$ is reducible of dimension n. Moreover there exists a line in Y_P through P that meets X in one point only and $W \setminus X$ in at least one other point.

Proof. Remark at first that the cone Y_P is irreducible of dimension $n+1$. Hence the components of the intersection $Y_P \cap W$ have dimension at least $n+1+N-1-N = n = \dim(X)$ (cf. [Sha13, Thm.1.24, p. 75]). Since Y_P is not contained in W, those components have dimension exactly n. Otherwise Y_P must be equal to one of the irreducible components of the intersection, and hence contained in W which is absurd.
Clearly $X \subset Y_P \cap W$. Assume for the sake of contradiction that $Y_P \cap W = X$. If this is the case, then every line contained in Y_P and passing through P meets W only on X and moreover such intersection is made by one point only, otherwise the point P would have X-rank 2 which is against our hypothesis. Now we show that the fact that every line $l \subset Y_P$ through P meets W only in one point of X, i.e.:

$$l \cap W = Q \in X \tag{4.4.1}$$

leads to a contradiction.
Since W is a hypersurface, it is cut out by a single homogeneous equation of degree $d > 1$, $W = V(f)$. As just shown every line $\overline{PQ} \subset Y_P$, with $Q \in X$, meets W only in Q. We can parametrize $\overline{PQ}$ as

$$\overline{PQ} = \{ sP + tQ \ : \ [s:t] \in \mathbb{P}^1 \} \cong \mathbb{P}^1$$

where in this notation the point Q is represented by the point $[0:1] \in \mathbb{P}^1$. Substituting the coordinates $sP + tQ$ of $\overline{PQ}$ in the equation f of the hypersurface we get a homogeneous equation of degree d in two variables which must vanish only at the point Q, i.e. we have an equation of the form

$$f(sP + tQ) = k_Q s^d \tag{4.4.2}$$

for some constant $k_Q \in \mathbb{K}$. We show that if this happens for every line $\overline{PQ}$, with $Q \in X$, then we will get a contradiction. Indeed, suppose that f can be written as the polynomial

$$f(x_0, \dots, x_N) = \sum_{(b_0,\dots,b_N) \in \mathbb{N}^{N+1},\ b_0+\dots+b_N=d} a_{(b_0,\dots,b_N)} x_0^{b_0} \dots x_N^{b_N}.$$

The coefficient of the monomial $s^{d-1}t$ after the substitution (4.4.2) has to be zero. Moreover it turns out to be the directional derivative of f at P in the direction of Q:

$$\begin{aligned} Df_P(Q) = a_{(b_0,\dots,b_N)}[b_0 \cdot p_0^{b_0-1} p_1^{b_1} \dots p_N^{b_N} \cdot q_0 + b_1 \cdot p_0^{b_0} p_1^{b_1-1} p_2^{b_2} \dots p_N^{b_N} \cdot q_1 + \dots \\ \dots + b_N \cdot p_0^{b_0} \dots p_{N-1}^{b_{N-1}} p_N^{b_N-1} \cdot q_N]. \end{aligned} \tag{4.4.3}$$

Since for a fixed P the form Df_P is linear in Q, then X is contained in the hyperplane $Df_P = 0$, which is non-trivial by the Euler formula and the fact that the evaluation of f at P is different from zero. This is in contradiction with non-degeneracy hypothesis on X. Hence there exists a line l inside Y_P containing P that intersects W in at least another point in $W \setminus X$. This concludes the proof. □

A direct consequence of this fact is the following

Theorem 4.4.12. *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety of dimension n and let $g = r_{gen}$ be the generic X-rank. If $\sigma_{g-1}(X)$ is a hypersurface, $g > 2$, then*

$$r_{max} \leq r_{max,g-1} + 1$$

where $r_{max,r_{gen}-1}$ is the maximum X-rank achieved on $\sigma_{r_{gen}-1}(X)$.

Proof. Let $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$ and $Y_P := \text{cone}(P, X) = \bigcup_{x \in X} \langle x, P \rangle$ as in Proposition 4.4.11 with $W = \sigma_{g-1}(X)$, $g > 2$, where we have shown that there exists a line of Y_P through P which intersects $\sigma_{g-1}(X)$ in at least two distinct points, say Q_1, Q_2 such that $Q_1 \in X$ and $Q_2 \in \sigma_{g-1}(X) \setminus X$. Therefore $P \in \langle Q_1, Q_2 \rangle$. If $Q_2 \in \sigma^0_{g-1}(X)$

then $r_X(P) \leq g$, while if the components of $Y_P \cap \sigma_{g-1}(X)$ different from X are all contained in $\sigma_{g-1}(X) \setminus \sigma^0_{g-1}(X)$ we can only say that $r_X(P) \leq r_{max,g-1} + 1$. □

We would like to point out some brief interesting consequences of this result.

Remark 4.4.13. Suppose that $X \subset \mathbb{P}^N$ is a non-degenerate irreducible variety such that the last non filling secant variety $\sigma_{g-1}(X)$ is a hypersurface, with $g > 2$, and let $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$. The intersection $Y_P \cap \sigma_{g-1}(X)$ cannot be contained in the sets $\sigma^0_s(X)$ for all $s \leq g-2$, otherwise the point $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$ must lie in $\sigma_s(X) \subset \sigma_{g-1}(X)$ which is impossible.

Remark 4.4.14. Suppose that $X \subset \mathbb{P}^N$ is a non-degenerate irreducible variety such that the last non filling secant variety $\sigma_{g-1}(X)$ is a hypersurface, with $g > 2$, and let $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$. The rank of the point $P \in \mathbb{P}^N \setminus \sigma_{g-1}(X)$ is g if and only if the intersection $Y_P \cap \sigma_{g-1}(X)$ contains at least one point of $\sigma^0_{g-1}(X)$ but we were not able to distinguish whether there exist points P for which $(Y_P \cap \sigma_{g-1}(X)) \subset (\sigma_{g-1}(X) \setminus \sigma^0_{g-1}(X))$. Of course if P is generic it is obvious that $Y_P \cap \sigma^0_{g-1}(X) \neq \emptyset$, and indeed the generic rank is g.

This leads us to the following conjecture.

Conjecture 4.4.15. *Let $X \subset \mathbb{P}^N$ be a smooth non-degenerate projective variety and let $g > 2$ be the generic X-rank. If $\sigma_{g-1}(X)$ is a hypersurface, then*

$$r_{max} = \max\{r_{max,g-1}, g\}$$

where $r_{max,g-1}$ is the maximum X-rank achieved on $\sigma_{g-1}(X)$.

4.4.1 Comparison

In this subsection we compare our result with already exisitng bounds on the maximum X-rank in the cases in which the maximum X-rank is known.

Veronese varieties. Let us denote with $X_{n,d}$ the Veronese variety given by the embedding of $\mathbb{P}^n$ via $\mathcal{O}(d)$. We report here the best bounds on the maximum symmetric rank to the best of our knowledge.

Remark 4.4.16. There is a finer notion of the X-rank when X is a Veronese variety called *open Waring rank*. Being a more subtle notion of rank, the open Waring rank of a form is always greater or equal than the symmetric rank. In [Jel13, Corollary 6, p. 331] J. Jelisiejew provides the bound

$$r_{max} \leq \binom{n+d-1}{n} - \binom{n+d-5}{n-2} \tag{4.4.4}$$

for $n \geq 2, d \geq 3$, while in [BDP17, Remark 4.18, p. 913] the authors E. Ballico and A. De Paris provide the bound

$$r_{max} \leq \binom{n+d-1}{n} - \binom{n+d-5}{n-2} - \binom{n+d-6}{n-2} \tag{4.4.5}$$

for $n \geq 2$, $d \geq 4$. Eventually another bound not linked to the open rank notion is given by A. De Paris in [DP16, Prop. 3.3, p. 28] and it applies only to homogeneous polynomials of any degree in three variables:

$$r_{max} \leq \left\lfloor \frac{d^2+6d+1}{4} \right\rfloor . \tag{4.4.6}$$

Remark 4.4.17. By the *Alexander-Hirschowitz theorem* [AH95] it is known that the Veronese varieties $X_{2,2}$, $X_{2,3}$ and $X_{2,4}$ have a secant variety $\sigma_r(X_{n,d})$ which is a hypersurface, namely for $r = 3$, 4 and 6 respectively. In [BGI11, Theorem 40, p. 19] and [BGI11, Theorem 44, p. 21] the authors have computed all the symmetric ranks appearing in the secant hypersurfaces of $X_{2,3}$ and $X_{2,4}$ which are 5 and 7 respectively. Moreover, in the case of $X_{2,3}$ they also proved that the maximum symmetric rank is 5 while a proof that the maximum symmetric rank with respect to $X_{2,4}$ is 7 has been computed in [DP15]. Remark that $X_{2,2}$ is a case of symmetric matrices so that the maximum symmetric rank appearing in the hypersurface is 2.

We are now ready to give a table with a comparison of the known bounds with the one we have provided.

Grassmann varieties. Consider $X = \mathbb{G}(k, V) \subset \mathbb{P}\left(\bigwedge^k V\right)$, where we assume that $2k \leq \dim(V)$. Among the cases in which the secant dimensions of Grassmann varieties are known, there is only one case, namely $\mathbb{G}(3, \mathbb{K}^7)$, in which a secant variety is a hypersurface, explicitly the one given by $r = 3$ as shown by [CGG05]. Moreover this a case in which $\bigwedge^3\mathbb{K}^7$ has a finite number of orbits under the action of $SL(\mathbb{K}^7)$ as it was illustrated by [Sch31]. Moreover in [ABMM21] the maximum

	Prop. 4.4.4	Prop. 4.4.7	(4.4.6)	(4.4.5)	(4.4.4)	our bound	$\mathbf{r_{max}}$
$X_{2,2}$	5	4	4			3	**3**
$X_{2,3}$	7	6	7		9	6	**5**
$X_{2,4}$	11	10	10	17	18	8	**7**

Table 4.4: Comparison of bounds on the maximum Waring rank when a secant variety of a Veronese variety is a hypersurface and where the maximum rank is known, apart from the cases of symmetric matrices.

skew-symmetric rank appearing in $\sigma_3(\mathbb{G}(3,\mathbb{K}^7))$ has been computed and it is equal to 3. Therefore we have the following comparison:

	Prop. 4.4.4	Prop. 4.4.7	our bound	$\mathbf{r_{max}}$
$\mathbb{G}(3,\mathbb{K}^7)$	7	6	4	**4**

Table 4.5: Comparison of bounds on the maximum skew-symmetric rank for the case $X = \mathbb{G}(3,\mathbb{K}^7)$.

Segre varieties. In the case of Segre varieties we highlight the instance of $X = \nu_{1,1,1}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$ in which the fourth secant variety is defective and it is an hypersurface. In this case the work [BH13] shows that the maximum tensor rank is 5, while the article [BL14] shows that there are rank 5 tensors appearing in the $\sigma_3(X) \subset \sigma_4(X)$. Therefore we get the following comparison

	Prop. 4.4.4	Prop. 4.4.7	our bound	$\mathbf{r_{max}}$
$\nu_{1,1,1}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$	9	8	6	**5**

Table 4.6: Comparison of bounds on the maximum tensor rank for the case $X = \nu_{1,1,1}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$.

Moreover remark that this is a case in which Conjecture 4.4.15 holds.

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