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Nonstandard Hulls of C^*-Algebras and Their Applications

Stefano Baratella

Dipartimento di Matematica, Università di Trento, Via Sommarive 14, I-38123 Povo, TN, Italy; stefano.baratella@unitn.it

Abstract: For the sake of providing insight into the use of nonstandard techniques à la A. Robinson and into Luxemburg's nonstandard hull construction, we first present nonstandard proofs of some known results about C^*-algebras. Then we introduce extensions of the nonstandard hull construction to noncommutative probability spaces and noncommutative stochastic processes. In the framework of internal noncommutative probability spaces, we investigate properties like freeness and convergence in distribution and their preservation by the nonstandard hull construction. We obtain a nonstandard characterization of the freeness property. Eventually we provide a nonstandard characterization of the property of equivalence for a suitable class of noncommutative stochastic processes and we study the behaviour of the latter property with respect to the nonstandard hull construction.

1. Introduction

In this work we apply nonstandard techniques à la Abraham Robinson to C^*-algebras, C^*-probability spaces (also known as noncommutative probability spaces) and noncommutative stochastic processes.

Our starting point is the nonstandard hull construction due to Luxemburg [1]. For the sake of completeness, in Section 2 we briefly outline such a construction in the case of a C^*-algebra. Functional analysts are probably more familiar with the ultraproduct construction (see [2]), which is an important tool in the study of C^*-algebras (see also [3] or [4]). Actually, every ordinary ultraproduct of C^*-algebras can be realized as the nonstandard hull of some internal C^*-algebra. Therefore, we loosely say that we deal with ultraproducts of C^*-algebras.

Concerning the terminology that we adopt throughout this paper, we use the attribute "ordinary", rather than "standard", when referring to some mathematical notion which is familiar to most mathematicians. The reason being that the term "standard" has a precise technical meaning in the framework of nonstandard techniques. We refer the reader to [5] as a valuable reference for the relevant notions and for the construction of nonstandard universes. We also mention [6] [§1] for a concise axiomatic introduction to the subject.

We stress that we may almost completely rewrite this paper in ultraproduct language or, gearing towards logic, within the framework of continuous logic (see [7]). In our opinion we get slightly more generality by working with nonstandard hulls. Indeed, most of the results in this paper apply to the internal C^*-algebras and not just to the standard ones (the latter being the nonstandard extensions of ordinary C^*-algebras). Admittedly, our approach is motivated by our familiarity with the nonstandard techniques and by our belief that, in many cases, a nonstandard proof is simpler and more intuitive than a proof of the same result written in ultraproduct language.

As for the paper's contents, we begin by saying that a significant amount of the material that we present in the first part stems from questions or problems posed in [8]. Actually, we devote Section 3 to nonstandard proofs of three known results which are

related to the content of [8]. While not strictly pertinent to noncommutative probability, we present those proofs mostly to give insight on the use of nonstandard techniques in the nonstandard hull framework. In Section 3, we occasionally point out what seem to be inaccuracies or mistakes in [8].

In Section 4 we provide results about weights that are defined on nonstandard hulls and we prove a weak property of normality for a class of those weights, thus extending a result obtained in [8].

In Section 5 we deal with C^*-probability spaces. After some preliminary results about states, we show that the property of freeness of a family of subalgebras is preserved by forming the nonstandard hull of a C^*-probability space. We introduce the nonstandard notion of almost freeness and we show that it coincides with freeness on standard families of subalgebras of a standard C^*-probability space, thus obtaining a nonstandard characterization of the ordinary freeness property.

In Section 5 we also obtain a nonstandard characterization of the noncommutative notion of convergence in distribution and we provide an elementary nonstandard proof that the property of $*$-freeness is preserved by convergence in $*$-distribution. In the last part of Section 5, we investigate the behaviour of the free product of C^*-probability spaces with respect to the nonstandard hull construction.

In Section 6 we apply the results from the previous section. After recalling the notion of stochastic process over a C^*-algebra given in [9], we extend the nonstandard hull construction to an internal noncommutative stochastic process. In this setting we deal with the notion of equivalence. We provide nonstandard versions of the reconstruction theorem in [9] and of other results therein. In this regard, we notice that, in light of the above-mentioned relationship between nonstandard hulls and ultraproducts, the nonstandard hull of an internal noncommutative stochastic process should be related to some sort of ultraproduct construction that applies to a family of ordinary noncommutative stochastic processes. We could not find any reference to such a construction in the literature. Eventually, we briefly discuss the adaptedness and the Markov properties in the framework of noncommutative stochastic processes, with special attention to the case of the nonstandard hull of an internal process.

In the mostly speculative Section 7, we try to make sense of the belief that a nonstandard universe does, or should have, physical significance on its own (see [10], for instance). We translate nonstandardly a result on the approximation of a Fock space by means of a sequence of so-called toy Fock spaces and we give a presentation of the nonstandard hull of an internal Fock space.

Finally, we point out that the ultraproduct construction is extensively used in [10]. In our opinion, the nonstandard techniques allow for simpler and more natural proofs of a large number of results given therein.

2. Preliminaries

We refer mostly to [11] for the basics of the theory of C^*-algebras. All C^*-algebras are assumed to be unital. We denote an algebra unit by 1. The term subalgebra always stands for C^*-subalgebra. Similarly, the term homomorphism of C^*-algebras always refers to a $*$-homomorphism.

As in [11], we use the term inner product rather than hermitian product.

We assume that the reader is familiar with the notions and the basic techniques of nonstandard analysis as introduced, for instance, in [5]. The reader who is interested in an axiomatic presentation of those techniques may refer to [6] [§1].

Here we just recall that a nonstandard universe allows to properly extend each infinite mathematical object X under consideration of an object $^*X$, in a way that X and $^*X$ satisfy the same properties which are definable by means of bounded quantifier formulas in the first order language of set theory. This property is referred to as the *Transfer Principle*.

We warn the reader that the notation $^*X$ for the nonstandard extension of an ordinary mathematical object X should not be confused with X^*, denoting the adjoint of some element X, whenever the latter makes sense.

Sets of type $^*X$ are called *standard*. An element of some standard set is called an *internal* set. If A, B are internal sets, by B^A we denote the internal set of all internal B-valued functions defined on A.

Relative to a nonstandard universe one can formulate the internal equivalents of all ordinary mathematical notions. Intuitively, to each property P which is bounded quantifier-definable (possibly with parameters) in the language of set theory there corresponds a property $^*P$ and the Transfer Principle ensures that a set X satisfies P if and only if its nonstandard extension $^*X$ satisfies $^*P$. Therefore we can consider, for instance, *continuity; *compactness, etc. For simplicity, we will omit the initial "star" when it is clear that the property under consideration applies to some internal set.

A nonstandard universe also contains sets which are not standard. This is ensured by the so-called κ-*saturation* property: for some uncountable cardinal κ which is sufficiently large for our purposes, we require that every family of cardinality smaller than κ of internal sets with the finite intersection property has nonempty intersection. It can be proved that, for every κ as above, there exists some κ-saturated nonstandard universe (see, for instance, [12]).

In the following we will also make use of Keisler's *Internal Definition Principle*: In every nonstandard universe, a bounded quantifier formula in the first order language of set theory with internal parameters defines an internal set.

The so-called *Overspill Lemma* is a straightforward consequence of ω_1-saturation. We formulate the former relative to $^*\mathbb{C}$: Any internal subset of $^*\mathbb{C}$ that contains arbitrarily large finite (in absolute value) hypercomplex numbers also contains some infinite hypercomplex (i.e., some infinite element in $^*\mathbb{C}\backslash\mathbb{C}$).

We assume that some sufficiently saturated nonstandard universe has been fixed throughout this paper and we briefly recall the nonstandard hull construction (see [1]). As we mentioned in the Introduction, the nonstandard hull is a slight generalization of the ultraproduct construction in functional analysis. In this paper we prefer the former construction because, assuming familiarity with the nonstandard techniques, it is much simpler than the ultraproduct.

Let A be an internal C^*-algebra. The nonstandard hull of A is the ordinary C^*-algebra $\widehat{A}$ defined by letting:

(1)　$\mathrm{Fin}(A) = \{a \in A \mid \|a\| < n \text{ for some } n \in \mathbb{N}\}$;

(2)　for $a, b \in A$, $a \approx b$ if $\|a - b\| \approx 0$;

(3)　for $a \in A$, $\widehat{a} = \{x \in A \mid x \approx a\}$;

(4)　$\widehat{A} = \left(\mathrm{Fin}(A)/_{\approx}\right) = \{\widehat{a} \mid a \in \mathrm{Fin}(A)\}$.

We define operations on $\widehat{A}$ as follows:

$$0 = \widehat{0}; \quad 1 = \widehat{1}; \quad r\widehat{a} + \widehat{b} = \widehat{(ra + b)}; \quad (\widehat{a})(\widehat{b}) = \widehat{ab}; \quad \widehat{a}^* = \widehat{(a^*)}$$

and norm by $\|\widehat{a}\| = {}^\circ\|a\|$, for all $a, b \in \mathrm{Fin}(A)$ and all $r \in \mathbb{C}$.

It can be easily verified that the operations on $\widehat{A}$ are well defined and that $\widehat{A}$ is a C^*-algebra. If $X \subset \mathrm{Fin}(A)$, we let $\widehat{X} = \{\widehat{x} : x \in X\}$. In particular, if B is a subalgebra of A then $\widehat{B}$ is a subalgebra of $\widehat{A}$.

Let A be an ordinary C^*-algebra. Under the assumption that the set of individuals of our nonstandard universe is a superset of A, we have that $A \subseteq \widehat{A}$. Furthermore, by identifying $a \in A$ with $\widehat{a}$, we have that A is a C^*-subalgebra of $\widehat{^*A}$. As is customary, we write $\widehat{A}$ for $\widehat{^*A}$.

If $\phi : A \to B$ is a homomorphism of ordinary C^*-algebras, we let

$$\begin{aligned} \widehat{\phi} : \quad \widehat{A} \quad &\to \quad \widehat{B} \\ \widehat{a} \quad &\mapsto \quad \widehat{^*\phi(a)} \end{aligned}$$

Since homomorphisms are norm-contracting, the map $\widehat{\phi}$ is well-defined. Furthermore, it is straightforward to verify that it is a homomorphism.

All the above assumptions and notations are in force throughout this paper.

Similarly to the above, one defines the nonstandard hull $\widehat{H}$ of an internal Hilbert space H. It is a straightforward verification that $\widehat{H}$ is an ordinary Hilbert space with respect to the standard part of the inner product of H. Furthermore, let $B(H)$ be the internal C^*-algebra of bounded linear operators on some internal Hilbert space H and let A be a subalgebra of $B(H)$. Each $\widehat{a} \in \widehat{A}$ can be regarded as an element of $B(\widehat{H})$ by letting $\widehat{a}(\widehat{x}) = \widehat{a(x)}$, for all $x \in H$ of finite norm. (Note that $\widehat{a}(\widehat{x})$ is well defined since a is norm–finite.) Therefore we can regard $\widehat{A}$ as a C^*-subalgebra of $B(\widehat{H})$.

3. Three Known Results

The results in this section can be rephrased in ultraproduct language and can be proved by using the theory of ultraproducts. The nonstandard proofs that we present below show how to apply the nonstandard techniques in combination with the nonstandard hull construction.

3.1. Infinite Dimensional Nonstandard Hulls Fail to Be von Neumann Algebras

In [8] [Corollary 3.26] it is proved that the nonstandard hull $\widehat{B(H)}$ of the in internal algebra $B(H)$ of bounded linear operators on some Hilbert space H over $^*\mathbb{C}$ is a von Neumann algebra if and only if H is (standard) finite dimensional. Actually, this result can be easily improved by showing that no infinite dimensional nonstandard hull is, up to isometric isomorphism, a von Neumann algebra. It is well-known that, in any infinite dimensional von Neumann algebra, there is an infinite sequence of mutually orthogonal non-zero projections. Hence one may want to apply [8] [Corollary 3.25]. Albeit the statement of the latter is correct, its proof in [8] is wrong in the final part. Therefore we begin by restating and reproving [8] [Corollary 3.25] in terms of increasing sequences of projections. We denote by $\mathrm{Proj}(A)$ the set of projections of a C^*-algebra A.

Lemma 1. *Let A be an internal C^*-algebra and let $(p_n)_{n \in \mathbb{N}}$ be an increasing sequence of projections in $\mathrm{Proj}(\widehat{A})$. Then there exists an increasing sequence of projections $(q_n)_{n \in \mathbb{N}}$ in $\mathrm{Proj}(A)$ such that, for all $n \in \mathbb{N}$, $p_n = \widehat{q}_n$.*

Proof. We recursively define $(q_n)_{n \in \mathbb{N}}$ as follows: As q_0 we pick any projection $r \in \mathrm{Proj}(A)$ such that $p_0 = \widehat{r}$. (See [8] [Theorem 3.22(vi)].) Then we assume that $q_0 < \cdots < q_n$ in $\mathrm{Proj}(A)$ are such that $p_i = \widehat{q}_i$ for all $0 \leq i \leq n$. Again by [8] [Theorem 3.22(vi)], we can further assume that $p_{n+1} = \widehat{r}$, for some $r \in \mathrm{Proj}(A)$. By [11] [II.3.3.1], we have $\widehat{rq_n} = \widehat{q}_n$, namely $rq_n \approx q_n$. Hence, by Transfer of [11] [II.3.3.5], for all $k \in \mathbb{N}^+$ there is $r_k \in \mathrm{Proj}(A)$ such that $q_n \leq r_k$ and $\|r - r_k\| < 1/k$. By Overspill, there is $q \in \mathrm{Proj}(A)$ such that $q_n \leq q$ and $q \approx r$. We let $q_{n+1} = q$. $\square$

Then we immediately get the following:

Corollary 1. *Let A be an internal C^*-algebra of operators and let $(p_n)_{n \in \mathbb{N}}$ be a sequence of non-zero mutually orthogonal projections in $\mathrm{Proj}(\widehat{A})$. Then $\widehat{A}$ is not a von Neumann algebra.*

Proof. From $(p_n)_{n \in \mathbb{N}}$, we get an increasing sequence $(p'_n)_{n \in \mathbb{N}}$ of projections in A by letting $p'_n = p_0 + \cdots + p_n$, for all $n \in \mathbb{N}$. By Lemma 1, there exists an increasing sequence $(q'_n)_{n \in \mathbb{N}}$ of projections in A. From the latter we get a sequence $(q_n)_{n \in \mathbb{N}}$ of non-zero mutually orthogonal projections, by letting $q_0 = q'_0$ and $q_{n+1} = q'_{n+1} - q'_n$, $n \in \mathbb{N}$. Finally, [8] [Proposition 3.22] applies. $\square$

Proposition 1. *The following are equivalent for an internal C^*-algebra of operators A:*

1. *A is (standard) finite dimensional;*

2.　　*$\widehat{A}$ is a von Neumann algebra.*

Proof. $(1) \Rightarrow (2)$ This is a straightforward consequence of the fact that A is isomorphic to a finite direct sum of internal matrix algebras of standard finite dimension over ${}^*\mathbb{C}$ and that the nonstandard hull of each summand is a matrix algebra over $\mathbb{C}$ of the same finite dimension.

$(2) \Rightarrow (1)$ Suppose $\widehat{A}$ is an infinite dimensional von Neumann algebra. Then in $\widehat{A}$ there is an infinite sequence of mutually orthogonal non-zero projections, contradicting Corollary 1. Therefore $\widehat{A}$ is finite dimensional and so is A. $\quad\square$

A straightforward consequence of the Transfer Principle and of Proposition 1 is that, for an ordinary C^*-algebra of operators A,

$$\widehat{A} \text{ is a von Neumann algebra} \iff A \text{ is finite dimensional.}$$

It is worth noticing that there is a construction known as tracial nostandard hull which, applied to an internal C^*-algebra equipped with an internal trace, returns a von Neumann algebra. See [8] [§3.4.2]. Not surprisingly, there is also an ultraproduct version of the tracial nostandard hull construction. See [13].

3.2. Real Rank Zero Nonstandard Hulls

The notion of real rank of a C^*-algebra is a non-commutative analogue of the covering dimension. Actually, most of the real rank theory concerns the class of real rank zero C^*-algebras, which is rich enough to contain the von Neumann algebras and some other interesting classes of C^*-algebras (see [11,14] [V.3.2]).

In this section we prove that the property of being real rank zero is preserved by the nonstandard hull construction and, in case of a standard C^*-algebra, it is also reflected by that construction. Then we discuss a suitable interpolation property for elements of a real rank zero algebra.

Eventually we show that the P^*-algebras introduced in [8] [§3.5.2] are exactly the real rank zero C^*-algebras and we briefly mention further preservation results.

We recall the following (see [14]):

Definition 1. *An ordinary C^*-algebra A is of real rank zero (briefly: $\mathrm{RR}(A) = 0$) if the set of its invertible self-adjoint elements is dense in the set of self-adjoint elements.*

In the following we make essential use of the equivalents of the real rank zero property stated in [14] [Theorem 2.6].

Proposition 2. *The following are equivalent for an internal C^*-algebra A:*

(1) $\mathrm{RR}(\widehat{A}) = 0$;

(2) for all $\widehat{a}, \widehat{b}$ orthogonal elements in $(\widehat{A})_+$ there exists $\widehat{p} \in \mathrm{Proj}(\widehat{A})$ such that $(1 - \widehat{p})\widehat{a} = 0$ and $\widehat{p}\,\widehat{b} = 0$.

Proof. $(1) \Rightarrow (2)$: Let $\widehat{a}, \widehat{b}$ be orthogonal elements in $(\widehat{A})_+$. By [14] [Theorem 2.6(v)], for all $0 < \epsilon \in \mathbb{R}$ there exists a projection $\widehat{q} \in \widehat{A}$ such that $\|(1 - \widehat{q})\,\widehat{a}\| < \epsilon$ and $\|\widehat{q}\,\widehat{b}\| < \epsilon$. By [8] [Theorem 3.22], we can assume $q \in \mathrm{Proj}(A)$. Being $0 < \epsilon \in \mathbb{R}$ arbitrary, from $\|(1 - q)a\| < 2\epsilon$ and $\|qb\| < 2\epsilon$, by saturation we get the existence of some projection $p \in A$ such that $(1 - p)a \approx 0$ and $pb \approx 0$. Hence $(1 - \widehat{p})\widehat{a} = 0$ and $\widehat{p}\,\widehat{b} = 0$.

$(2) \Rightarrow (1)$: Follows from (v) $\Rightarrow$ (i) in [14] [Theorem 2.6]. $\quad\square$

Proposition 3. *Let A be an internal C^*-algebra such that $\mathrm{RR}(A) = 0$. Then $\mathrm{RR}(\widehat{A}) = 0$.*

Proof. Let $\widehat{a}, \widehat{b}$ be orthogonal elements in $(\widehat{A})_+$. By [8] [Theorem 3.22(iv)], we can assume that $a, b \in A_+$ and $ab \approx 0$. Hence $\|ab\| < \epsilon^2$, for some positive infinitesimal ϵ. By Transfer

of [14] [Theorem 2.6 (vi)], there is a projection $p \in A$ such that $\|(1-p)a\| < \epsilon$ and $\|pb\| < \epsilon$. Therefore $(1 - \widehat{p})\widehat{a} = 0$ and $\widehat{p}\,\widehat{b} = 0$ and we conclude by Proposition 2. $\square$

Proposition 4. *Let A be an ordinary C^*-algebra. The following are equivalent:*

(1) $\mathrm{RR}(A) = 0$;
(2) $\mathrm{RR}(^*A) = 0$;
(3) $\mathrm{RR}(\widehat{^*A}) = 0$.

Proof. $(1) \Rightarrow (2)$ holds by Transfer and $(2) \Rightarrow (3)$ holds by Proposition 3. Therefore it remains to prove $(3) \Rightarrow (1)$. As usual, we assume that A is a subalgebra of $\widehat{^*A}$ and we identify $a \in A$ with $\widehat{^*a} \in \widehat{^*A}$. We show that [14] [Theorem 2.6 (vi)] is satisfied. Let $a, b \in A_+$ and $0 < \epsilon \in \mathbb{R}$ be such that $\|ab\| < \epsilon^2$. Then $\|ab\| < \delta^2$, for some $\delta < \epsilon$. By assumption there is $\widehat{p} \in \mathrm{Proj}(\widehat{^*A})$ such that $\|(1 - \widehat{p})a\| < \delta$ and $\|\widehat{p}\,b\| < \delta$. By [8] [Theorem 3.22], we assume $p \in \mathrm{Proj}(^*A)$. Hence $\|(1 - p)\,^*a\| < \epsilon$ and $\|p\,^*b\| < \epsilon$. By Transfer, there exists $p \in \mathrm{Proj}(A)$ such that $\|(1 - p)a\| < \epsilon$ and $\|p\,b\| < \epsilon$. $\square$

Question 1. *In Proposition 3, does the converse implication hold for any internal C^*-algebra?*

Let A be an ordinary C^*-algebra and let $a, b \in A_+$. We write $a \ll b$ if $ba = a$ (equivalently: $ab = a$). In [11] [V.3.2.16], the author introduces an interpolation property for positive elements a, b in a C^*-algebra of real rank zero such that $a \ll b$. In [11] [V.3.2.17], he proves such property under the additional assumption that there is a positive element c such that $a \ll c \ll b$. Actually, the interpolation property holds, under no additional assumption, in all nonstandard hulls having real rank zero.

Proposition 5. *Let A be an internal C^*-algebra such that $\mathrm{RR}(\widehat{A}) = 0$ and let $\widehat{a}, \widehat{b} \in (\widehat{A})_+$, with $\widehat{a} \ll \widehat{b}$ and $\|\widehat{b}\| \leq 1$. Then there exists a projection $\widehat{p} \in \widehat{A}$ such that $\widehat{a} \ll \widehat{p} \ll \widehat{b}$. If $\|\widehat{a}\| \leq 1$ then $\widehat{p}$ also satisfies $\widehat{a} \leq \widehat{p} \leq \widehat{b}$.*

Proof. From $\widehat{a} \ll \widehat{b}$, we get $\widehat{a}(1 - \widehat{b}) = 0$. Since $\|\widehat{b}\| \leq 1$, from $\widehat{b} \leq \|\widehat{b}\|$ we get $0 \leq 1 - \widehat{b}$. By Proposition 2 there exists $\widehat{p} \in \mathrm{Proj}(\widehat{A})$ such that $(1 - \widehat{p})\,\widehat{a} = 0$ and $\widehat{p}\,(1 - \widehat{b}) = 0$, namely $\widehat{a} \ll \widehat{p} \ll \widehat{b}$.

Concerning the final claim, it is a general fact that if $c \ll d$ are positive elements in a C^*-algebra and $\|c\| \leq 1$ then $c \leq d$. To prove that, work in the commutative C^*-subalgebra generated by $\{c, d, 1\}$ and use the Gelfand transform. (See [11] [Theorem II.2.2.4]).

If follows that, assuming $\|\widehat{a}\| \leq 1$, we immediately get $\widehat{a} \leq \widehat{p} \leq \widehat{b}$ from $\widehat{a} \ll \widehat{p} \ll \widehat{b}$. $\square$

Next we recall the definition of P^*-algebra from [8] [§3.5.2]: a C^*-algebra A is a P^*-algebra if every self-adjoint element from A is the norm limit of real linear combinations of mutually orthogonal sequences of projections. Notice that the complex linear span of the projections is dense in a P^*-algebra.

Actually, the P^*-algebras are exactly the real rank zero algebras:

Proposition 6. *The following are equivalent for an ordinary C^*-algebra A:*

(1) $\mathrm{RR}(A) = 0$;
(2) *A is a P^*-algebra.*

Proof. $(1) \Rightarrow (2)$ We use the functional calculus (see [11] [Corollary II.2.3.1]). If $a \in A_{sa}$ has finite spectrum then $\mathrm{id}_{\sigma(a)}$ is a linear combination with real coefficients of mutually orthogonal projections in $C(\sigma(a))$ and the conclusion follows.
$(2) \Rightarrow (1)$ We verify (1) in the form of the equivalent condition [14] [Theorem 2.6 (ii)], simply by noticing that, if $(p_i)_{1 \leq i \leq n}$ is a tuple of mutually orthogonal projections and $(\lambda_i)_{1 \leq i \leq n} \in \mathbb{R}^n$, then $\sigma(\sum_{i=1}^n \lambda_i p_i) \subseteq \{\lambda_i : 1 \leq i \leq n\} \cup \{0\}$. Hence, by (2), the self-adjoints of finite spectra are dense in A_{sa}. $\square$

In light of the previous proposition, we may regard that of Proposition 3 as a simpler proof of [8] [Theorem 3.28]. One may object that the proof of Proposition 3 heavily relies on [14] [Theorem 2.6] and ask for a more direct proof of [8] [Theorem 3.28]. Here is one:

Proposition 7. *Let A be an internal C*-algebra. Then*

$$\mathrm{RR}(A) = 0 \Rightarrow \mathrm{RR}(\widehat{A}) = 0.$$

Proof. Let $\widehat{a} \in \widehat{A}_{sa}$. By [8] [Theorem 3.22], we assume $a \in A_{sa}$. Let $b \in A$ be an invertible element such that $\|b - a\| \approx 0$. By polar decomposition (see, for instance, [8] [Proposition 3.13]), let $u \in A$ be the unique unitary element such that $b = |b|u$. Let $0 < \epsilon \in \mathbb{R}$ and let $d = (|b| + \epsilon/2)u$. Since $|b|$ is invertible then d is invertible and $\|a - d\| < \epsilon$. It suffices to prove that $d^{-1} \in \mathrm{Fin}(A)$ to conclude that $\widehat{d}$ is invertible in $\widehat{A}$. By the functional calculus, $\left\| (|b| + \epsilon/2)^{-1} \right\| \le 2/\epsilon \in \mathrm{Fin}(^*\mathbb{R})$. Hence $\left\| d^{-1} \right\| \le 2/\epsilon$.

Summing up: For all $0 < \epsilon \in \mathbb{R}$ there exists an invertible $\widehat{d} \in \widehat{A}$ such that $\|\widehat{a} - \widehat{d}\| \le \epsilon$. Hence the conclusion. $\square$

Further preservation results that can be easily established are the following:

(1) An ordinary C^*-algebra is projectionless if it has no projection different from $0, 1$. It is easy to verify that, if p is a projection in an internal C^*-algebra, $p \approx 0$ implies $p = 0$ (hence $p \approx 1 \Rightarrow p = 1$). From [8] [Theorem 3.22(vi)] it then follows that the property of being projectionless is preserved and reflected by the nonstandard hull construction.

(2) An ordinary C^*-algebra has stable rank one if its invertible elements form a dense subset (see [11] [V.3.1.5]). The same proof as in Proposition 7 shows that the property of an internal C^*-algebra of having stable rank one is preserved by the nonstandard hull construction. Furthermore, an analogous of Proposition 4 can be proved with respect to the stable rank one property, by using [8] [Corollary 3.11].

3.3. Nonstandard Hulls of Internal Function Spaces

In this section, we extend the description given in [15] of the nonstandard hull of the internal Banach algebra of $^*\mathbb{R}$-valued continuous functions on some compact Hausdorff space to the case when A is the internal C^*-algebra $C(X)$ of $^*\mathbb{C}$-valued continuous functions on some compact Hausdorff space X. For $f \in \mathrm{Fin}(A)$, let $^\circ f : X \to \mathbb{C}$ be defined as follows: $(^\circ f)(x) = {}^\circ(f(x))$, for all $x \in X$. It is easy to verify that the nonstandard hull $\widehat{A}$ of A is formed by $\{ {}^\circ f : F \in \mathrm{Fin}(A) \}$, equipped with the operations inherited by A. In particular, $(^\circ f)(^\circ g) = {}^\circ(fg)$ and $(^\circ f)^* = {}^\circ(f^*)$. (In the latter equality, * denotes the adjoint.)

By the Gelfand-Naimark Theorem, the commutative C^*-algebra $\widehat{A}$ is isometrically isomorphic to the ordinary C^*-algebra $C(Y)$, where Y is the compact Hausdorff space of nonzero multiplicative linear functionals on $\widehat{A}$, equipped with the topology induced by the weak *-topology on the dual of $\widehat{A}$. The natural isomorphism $\Gamma : \widehat{A} \to C(Y)$, known as the Gelfand transform, is defined as follows: Let $^\circ f \in \widehat{A}$. Then

$$\Gamma(^\circ f): \quad \begin{array}{ccc} Y & \to & \mathbb{C} \\ \varphi & \mapsto & \varphi(^\circ f) \end{array}$$

(see [11] [II.2.2.4]).

To each $x \in X$ we associate the multiplicative linear functional

$$\widehat{x}: \quad \begin{array}{ccc} \widehat{A} & \to & \mathbb{C} \\ {}^\circ f & \mapsto & {}^\circ(f(x)) \end{array} \tag{1}$$

(In order to verify that $\widehat{x}$ satisfies the required properties, the assumption $f \in \mathrm{Fin}(A)$ is crucial.) Let $x \ne y$, $x, y \in X$. By Transfer of Urysohn's Lemma there exists an internal continuous function $f : X \to {}^*[0, 1]$ such that $f(x) = 0$ and $f(y) = 1$. It follows that $\widehat{x} \ne \widehat{y}$.

In general, the internal topology on X is not an ordinary topology, but forms a basis for an ordinary topology on X, that we denote by Q since it was named Q-topology by A. Robinson.

We notice that, for all $f \in \mathrm{Fin}(A)$, the map $^\circ f$ is continuous with respect to the Q-topology. Actually, let $B(z, r)$ be the open ball of radius r centered at $z \in \mathbb{C}$. Then

$$({}^\circ f)^{-1}(B(z, r)) = \bigcup_{n \in \mathbb{N}^+} \{ x \in X : |f(x) - z| < r - 1/n \},$$

and the latter is open in the Q-topology.

We let $\widehat{X} = \{ \widehat{x} : x \in X \}$ and we denote by τ the topology induced on $\widehat{X}$ by the weak *-topology on the dual of $\widehat{A}$. Keeping also in mind the notation previously introduced, we prove the following:

Proposition 8. *The function* $\widehat{} : (X, Q) \to (\widehat{X}, \tau)$ *that maps x to the multiplicative linear functional $\widehat{x}$ defined as in* ([1](#)) *is a homeomorphism. Moreover, the set $\widehat{X}$ is dense in Y.*

Proof. We have shown above that $\widehat{}$ is one-one.

Let $x \in X$. By definition of the weak*-topology, to prove the continuity of $\widehat{}$ it suffices to verify that, for all $f \in \mathrm{Fin}(A)$ and all $0 < r \in \mathbb{R}$, the set $\{ y \in X : |\widehat{y}({}^\circ f) - \widehat{x}({}^\circ f)| < r \}$ is Q-open: This is straightforward from the already established continuity of $^\circ f$.

To prove that $\widehat{}$ is an open map, it suffices to show that, for each internal open set $Z \subseteq X$, the set $\widehat{Z} = \{ \widehat{z} : z \in Z \}$ is τ-open. We fix $z \in Z$. By Transfer of Urysohn's Lemma, there exists a $^*[0, 1]$-valued $f \in A$ such that $f(z) = 0$ and $f(x) = 1$ for all $x \in X \setminus Z$. The set

$$U = \{ \widehat{x} \in \widehat{X} : |\widehat{x}({}^\circ f) - \widehat{z}({}^\circ f)| < 1/2 \}$$

is τ-open and contains $\widehat{z}$. Moreover, for all $x \in X$, $\widehat{x} \in U$ if and only if $| {}^\circ f(x)| < 1/2$. Hence $U \subseteq \widehat{Z}$. It follows that $\widehat{Z}$ is τ-open.

Concerning the second part of the statement, let us assume that there exists $\varphi \in Y$ which does not belong to the closure $(\widehat{X})^-$ of $\widehat{X}$. By Urysohn's Lemma, there exists a $[0, 1]$-valued $h \in C(Y)$ such that $h(\varphi) = 1$ and $h|_{(\widehat{X})^-} = 0$. Let $f \in C(X)$ be such that $\Gamma({}^\circ f) = h$, where Γ is the Gelfand transform defined above. Since

$$0 = \Gamma({}^\circ f)(\widehat{x}) = \widehat{x}({}^\circ f) = {}^\circ f(x) \quad \text{for all } x \in X,$$

then $^\circ f = 0$. Being Γ an isometry, we get a contradiction with $\|h\| = 1$. $\square$

4. Noncommutative Loeb Theory

At first reading, the title of this section may sound somewhat obscure. To clarify it, we recall that a Loeb probability measure is an ordinary probability measure that is obtained from an internal finitely-additive probability measure. See [5] or [6]. We recall that a C^*-probability space is a pair (A, ϕ), where A is a C^*-algebra and $\phi : A \to \mathbb{C}$ is a state, namely a positive linear functional with the property that $\phi(1) = 1$.

In short: States are the noncommutative counterparts of probability measures. In the following we deal with the problem of obtaining an ordinary weight from an internal one. Moreover, weights are closely related to states. Hence the title of this section.

We begin by recalling some notions and elementary facts relative to an ordinary C^*-algebra A.

A *weight* is an additive, positively homogeneous function $\phi : A_+ \to [0, \infty]$, i.e., $\phi(ra + b) = r\phi(a) + \phi(b)$, for all $a, b \in A_+$ and all $r \in [0, \infty)$, with the convention that $0 \cdot \infty = 0$ (so that $\phi(0) = 0$).

Let ϕ be a weight. From the inequality $a \leq \|a\|1$, $a \in A_+$ (see [11] [II.3.1.8]), it follows that $\phi(a) \leq \|a\|\phi(1)$. Therefore condition $\phi(1) < \infty$ is equivalent to $\phi(A_+) \subseteq [0, \infty)$. A weight is *finite* if it satisfies one of those two equivalent properties.

A finite weight ϕ extends uniquely to a positive linear functional on A, usually denoted by the same name. This is because each $a \in A$ can be uniquely written as $a = (a_1 - a_2) + i(a_3 - a_4)$, for some positive a_i, each of norm $\leq \|a\|$. (Recall that $a = (a + a^*)/2 + i[(a - a^*)/2i]$ and see, for instance, [8] [Corollary 3.21].) Conversely, every positive linear functional on A yields a finite weight.

A weight ϕ is *normal* if for any uniformly norm-bounded increasing net $F \subset A_+$, such that $\sup F$ exists in A_+, then

$$\phi(\sup F) = \sup_{a \in F} \phi(a).$$

Let κ be a cardinal. We say that a weight ϕ is κ-normal if the previous property holds for any uniformly norm-bounded directed family $F \subset A_+$ with $|F| < \kappa$.

For the rest of this section, if not otherwise stated, A is assumed to be an internal C^*-algebra.

Following nonstandard terminology we say that an internal weight $\phi : A_+ \to {}^*[0, \infty]$ is *S-continuous* if $\phi(a) \approx 0$ for all $0 \approx a \in A_+$. We recall the following (see [8] [Lemma 4.4]):

Lemma 2. *The following are equivalent for an internal weight $\phi : A_+ \to {}^*[0, \infty]$:*

(1) ϕ *is S-continuous;*
(2) $\phi(1) \in \mathrm{Fin}({}^*[0, \infty])$;
(3) *for all $a, b \in A_+$, if $a \approx b$ then $\phi(a) \approx \phi(b)$.*

For benefit of the reader who wants to check the proof of Lemma 2 given in [8], we point out that [8] [Proposition 3.12] lacks the crucial assumption $a - b \in \mathrm{Re}(\mathcal{M})$ (which is trivially satisfied if a, b are positive elements). Actually, as it stands, [8] [Proposition 3.12] is wrong, even for commutative internal algebras: Let $C(X)$ be the internal C^*-algebra of ${}^*\mathbb{C}$-valued functions on some compact space X an let $r, s \in {}^*\mathbb{R}$ be such that $r \approx s, r \neq s$. Let f, g be the constant functions $f(x) = ir$ and $g(x) = is$. Then $f \approx g$, but there is no $h \in C(X)$ that satisfies $f \approx h \approx g$ and $f \leq h, g \leq h$.

Let $\phi : A_+ \to {}^*[0, \infty]$ be an internal S-continuous weight. By Lemma 2, ϕ takes values in ${}^*[0, \infty)$ (we will say that it is a *finite weight). As previously noticed, we can extend ϕ to an internal positive linear functional defined on A, that we still denote by ϕ. By transfer of [16] [Theorem 4.3.2], we have $\|\phi\| = \phi(1)$. It follows from Lemma 2(2) that $\|\phi\| \in \mathrm{Fin}({}^*[0, \infty])$. Hence there is a one-to-one correspondence between the internal S-continuous weights and the internal positive linear functionals of (standard) finite norm.

By Lemma 2(3), from an internal S-continuous weight $\phi : A_+ \to {}^*[0, +\infty]$ we can define a map

$$
\begin{array}{cccc}
\widehat{\phi} : & (\widehat{A})_+ & \to & [0, +\infty) \\
& \widehat{a} & \mapsto & {}^\circ\phi(a)
\end{array}
\tag{2}
$$

Clearly $\widehat{\phi}$ is additive and positive homogeneous, hence a (finite) weight. It can be regarded as a noncommutative Loeb integral operator (see the discussion in [8] [§4.4]).

Here is an example of an internal weight which is not the nonstandard extension of any ordinary weight: Let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $M_N({}^*\mathbb{C})$ be the internal C^*-algebra of $N \times N$ matrices on ${}^*\mathbb{C}$. Let $\mathrm{tr} : M_N({}^*\mathbb{C})_+ \to {}^*[0, \infty)$ be the normalized trace defined by $\mathrm{tr}((a_{ij})) = \frac{1}{N} \sum_{i=1}^N a_{ii}$. By Lemma 2, tr is S-continuous. Notice that the non-normalized trace is not S-continuous.

Next we want to prove that every S-continuous internal weight in a κ-saturated nonstandard universe is κ-normal, thus strengthening [8] [Theorem 4.5] (see [8] [Question 11]). We point out that, in the following result, differently from [8] [Theorem 4.5], the internal weight is not required to be normal and the internal C^*-algebra is not necessarily commutative.

Let $r, s \in {}^*\mathbb{R}$. We write $r \gtrsim s$ if $r > s$ or $r \approx s$.

Theorem 1. *Let $\phi : A_+ \to {}^*\mathbb{C}$ be an internal S-continuous weight in a κ-saturated nonstandard universe. Then the weight $\widehat{\phi}$ defined in (2) is κ-normal.*

Proof. By Transfer of the Gelfand–Naimark Theorem ([11] [Corollary II.6.4.10]), we assume that A is a subalgebra of the internal C^*-algebra $B(H)$, for some internal Hilbert space H. As remarked at the end of Section 2, we regard $\widehat{A}$ as a subalgebra of $B(\widehat{H})$, where $\widehat{H}$ is the nonstandard hull of H.

We denote by H_1 the unit ball centered at the origin of H. By [11] [I.2.6.7], the following are equivalent for $\widehat{a}, \widehat{b} \in (\widehat{A})_{sa}$:

(1) $\widehat{a} \leq \widehat{b}$;
(2) for all $h \in H_1$, ${}^\circ\langle (b-a)h, h \rangle \geq 0$;
(3) for all $h \in H_1$, $\mathrm{Re}(\langle (b-a)h, h \rangle) \gtrsim 0$ and $\mathrm{Im}(\langle (b-a)h, h \rangle) \approx 0$.

Let $F \subseteq \widehat{A}_+$ be an infinite norm-bounded directed family with $|F| < \kappa$. Let L be a norm-bound for the elements of F. Let F_0 be formed by picking exactly one representative for each element in F, so that $F = \{\widehat{a} \,|\, a \in F_0\}$.

Let $R = \sup\{\widehat{\phi}(\widehat{a}) \,|\, \widehat{a} \in F\}$. Since F is norm-bounded, R is finite.

We claim that there exists $b \in \mathrm{Fin}(A)$ such that $\widehat{a} \leq \widehat{b}$ for all $a \in F_0$ and $\widehat{\phi}(\widehat{b}) = R$. To prove this, let $P_{<\omega}(F_0)$ be the set of finite subsets of F_0. Notice that $|P_{<\omega}(F_0)| < \kappa$. For each $C \in P_{<\omega}(F_0)$ and each $n \in \mathbb{N}^+$, let $F_{n,C}$ be the internal subset of A whose elements x satisfy the following properties:

(a) $\|x\| \leq L + 1$;
(b) for all $h \in H_1$ and all $a \in C$, $\mathrm{Re}(\langle (x-a)h, h \rangle) \geq -1/n$ and $-1/n \leq \mathrm{Im}(\langle (x-a)h, h \rangle) \leq 1/n$;
(c) $|\phi(x) - R| \leq 1/n$.

By directness of F, the equivalence (1) $\Leftrightarrow$ (3) above and the definition of R, the $F_{n,C}$ are nonempty. Moreover they have the finite intersection property, since $F_{\max(n,m),B \cup C} \subseteq F_{n,B} \cap F_{m,C}$.

By κ-saturation, we let $b \in \bigcap\{F_{n,C} : n \in \mathbb{N}^+ \text{and } C \in P_{<\omega}(F_0)\}$. Then $\widehat{b}$ satisfies the required conditions. It follows that $\widehat{\phi}(\widehat{a}) \leq \widehat{\phi}(\widehat{b})$ for all $\widehat{a} \in F$. Being $\widehat{b}$ a $\leq$-upper bound of F, if $\sup F$ exists in $(\widehat{A})_+$, then

$$\sup\{\widehat{\phi}(\widehat{a}) \,|\, \widehat{a} \in F\} \leq \widehat{\phi}(\sup F) \leq \widehat{\phi}(\widehat{b}) = \sup\{\widehat{\phi}(\widehat{a}) \,|\, \widehat{a} \in F\}.$$

Therefore $\widehat{\phi}$ is κ-normal. □

With reference to the previous theorem, it is straightforward to check that the weight $\widehat{\phi}$ is κ-completely additive, namely if I is a set of cardinality $< \kappa$ and $\{a_i\}_{i \in I}$ is a family of elements of $(\widehat{A})_+$ such that $\sum_{i \in I} a_i$ is defined, then $\widehat{\phi}(\sum_{i \in I} a_i) = \sum_{i \in I} \widehat{\phi}(a_i)$.

We briefly comment on [8] [Question 11]. If an internal weight $\phi : A_+ \to {}^*[0, \infty]$ is not S-continuous and $a \in \mathrm{Fin}(A_+)$ is such that $\phi(a) \neq {}^*\infty$, then there exists $a \approx b \in \mathrm{Fin}(A_+)$ such that $\phi(a) \not\approx \phi(b)$. Hence only when ϕ is the so-called degenerate weight (namely ϕ satisfies $\phi(0) = 0$ and $\phi(a) = {}^*\infty$, for $a \neq 0$), it is possible to define a weight $\widehat{\phi} : (\widehat{A})_+ \to [0, +\infty]$ as in (2). In such case, $\widehat{\phi}$ itself is the degenerate weight.

5. Nonstandard Noncommutative Probability

In this section we will be mostly concerned with an important part of noncommutative probability known as free probability. The latter was initiated by Voiculescu to attack a problem in the theory of von Neumann algebras. See [17].

In Section 4, we already recalled the definition of C^*-probability space (briefly: C^*ps). We recall the following definitions.

A state ϕ is *faithful* if

$$\phi(a^*a) = 0 \Rightarrow a = 0 \quad \text{for all } a \in A.$$

A state ϕ is *tracial* if
$$\phi(ab) = \phi(ba) \quad \text{for all } a, b \in A.$$

We notice that, by Lemma 2, the state ϕ in an internal C^*ps (A, ϕ) is S-continuous. Therefore, by defining $\widehat{\phi}$ as in (2) above, we have that $(\widehat{A}, \widehat{\phi})$ is an ordinary C^*ps. We will use this fact without further mention.

We say that an internal state $\phi : A \to {}^*\mathbb{C}$ is *S-faithful* if

$$\phi(a^*a) \approx 0 \Rightarrow a \approx 0 \quad \text{for all } a \in \text{Fin}(A).$$

We have the following characterization of faithfulness:

Proposition 9. *Let $\phi : A \to \mathbb{C}$ be an ordinary state. The following are equivalent:*

(1) *ϕ is faithful;*
(2) *${}^*\phi : {}^*A \to {}^*\mathbb{C}$ is S-faithful;*
(3) *$\widehat{\phi} : \widehat{A} \to \mathbb{C}$ is faithful.*

Proof.

$(1) \Rightarrow (2)$ We assume (1). Let $a \in {}^*A$ be such that $\phi(a^*a) \approx 0$. Then there exists some nonnegative infinitesimal $r \in \mathbb{R}$ such that $\phi(a^*a - r1) = 0$. Hence $\|a^*a\| \approx 0$. From the equality $\|a^*a\| = \|a\|^2$ we get $a \approx 0$.

$(2) \Rightarrow (3)$ We assume (2). Let $\widehat{a} \in \widehat{A}$. We get the following chain of implications:

$$\widehat{\phi}(\widehat{a}^* \widehat{a}) = 0 \Rightarrow \phi(a^*a) \approx 0 \Rightarrow a \approx 0 \Rightarrow \widehat{a} = 0.$$

$(3) \Rightarrow (1)$ Since we can assume without loss of generality that A is a subalgebra of $\widehat{A}$ and that $\widehat{\phi}$ extends ϕ, the result is straightforward. $\square$

We say that an internal state $\phi : A \to {}^*\mathbb{C}$ is *S-tracial* if

$$\phi(ab) \approx \phi(ba) \quad \text{for all } a, b \in \text{Fin}(A).$$

We leave the straightforward proof of the following to the reader.

Proposition 10. *Let $\phi : A \to \mathbb{C}$ be an ordinary state. The following are equivalent:*

(1) *ϕ is a tracial state;*
(2) *${}^*\phi : {}^*A \to {}^*\mathbb{C}$ is S-tracial;*
(3) *$\widehat{\phi}$ is a tracial state.*

To help the reader's intuition, we stress that, in a C^*ps (A, ϕ), the elements of A play the roles of random variables, whose expectation is given by ϕ.

Next we formulate the property of free independence (for short: freeness). See [17] [Proposition 3.5] for insights about such notion.

Definition 2. *Let (A, ϕ) be an ordinary C^*ps. A family $(B_j)_{j \in I}$ of C^*-subalgebras of A is free if for all $n \in \mathbb{N}$, all $i \in I^n$ and all $b \in \prod_{i=1}^n B_{i(j)}$ such that $i(1) \neq i(2) \neq \ldots \neq i(n)$ and $\phi(b_{i(1)}) = \cdots = \phi(b_{i(n)}) = 0$ it holds that $\phi(b_{i(1)} \cdots b_{i(n)}) = 0$.*

We stress that freeness depends on the state ϕ. Therefore, in the previous definition, it would be more appropriate to say that the family $(B_j)_{j \in I}$ is free with respect to ϕ. Usually it is the context that prevents any ambiguity.

Notice that Definition 2 makes sense also for a family of unital $*$-subalgebras of A.

Notational convention. A family $(B_j)_{j \in I}$ of C^*-algebras is actually a function B defined on I. Therefore we denote its nonstandard extension ${}^*B$, which is a function defined on ${}^*I$, by

$(\,^{*}B_j)_{j\in\,^{*}I}$. For notational simplicity we write $\widehat{B}_j$ for $\widehat{^{*}B_j}$. Without loss of generality, we can further assume that $I \subseteq\,^{*}I$.

The above notation and assumption are in force throughout this section.

The chain of equivalences in the following result provides a nonstandard characterization of freeness and proves that the latter is preserved and reflected by the nonstandard hull construction.

Proposition 11. *Let* (A, ϕ) *be an ordinary* $C^{*}ps$ *and let* $(B_j)_{j\in I}$ *be a family of* C^{*}*-subalgebras of* A. *The following are equivalent:*

(1) $(B_j)_{j\in I}$ *is free;*

(2) *there exists some* $N \in\,^{*}\mathbb{N} \setminus \mathbb{N}$ *for which the following holds: For all* $M \leq N$, *all internal* $i \in (\,^{*}I)^{M}$ *and all internal* $b \in \prod_{j=1}^{M}\,^{*}B_{i(j)}$ *such that* $i(1) \neq i(2) \neq \ldots \neq i(M)$ *if* $\phi(b_{i(1)}) = \cdots = \phi(b_{i(M)}) = 0$ *then* $\phi(b_{i(1)} \cdots b_{i(M)}) = 0$;

(3) $(\widehat{B}_j)_{j\in\,^{*}I}$ *is free with respect to* $\widehat{\phi}$.

Proof. (1) $\Rightarrow$ (2) is a consequence of Transfer.

Regarding (2) $\Rightarrow$ (3), we fix N as in (2). Let $0 < m \in \mathbb{N}$, $i \in (\,^{*}I)^{m}$ and $b \in \prod_{j=1}^{m} \mathrm{Fin}(\,^{*}B_{i(j)})$ be such that $i(1) \neq i(2) \neq \ldots \neq i(m)$ and $\widehat{\phi}(\widehat{b}_{i(1)}) = \cdots = \widehat{\phi}(\widehat{b}_{i(m)}) = 0$. Then $\phi(b_{i(j)}) \approx 0$ for all $1 \leq j \leq m$. Let $d_{i(j)} = b_{i(j)} - \phi(b_{i(j)})1$. Therefore $d_{i(j)} \approx b_{i(j)}$ and $\phi(d_{i(j)}) = 0$, for all $1 \leq j \leq m$. It follows by assumption that $\phi(d_{i(1)} \cdots d_{i(m)}) = 0$. Since $b_{i(1)} \cdots b_{i(m)} \approx d_{i(1)} \cdots d_{i(m)}$, we get $\phi(b_{i(1)} \cdots b_{i(m)}) \approx \phi(d_{i(1)} \cdots d_{i(m)})$. Therefore $0 = (\phi(b_{i(1)} \cdots b_{i(m)}))\widehat{} = \widehat{\phi}(\widehat{b}_{i(1)} \cdots \widehat{b}_{i(m)})$.

The proof of (3) $\Rightarrow$ (1) is straightforward from $B_j \subseteq \widehat{B}_j$, for all $j \in I$. $\quad\square$

The proof of the previous proposition naturally leads to formulating a nonstandard variant of the notion of freeness.

Definition 3. *Let* (A, ϕ) *be an internal* $C^{*}ps$. *A family* $(B_j)_{j\in I}$ *(not necessarily internal) of internal* C^{*}*-subalgebras of* A *is almost free if, for all* $n \in \mathbb{N}$, *all* $i \in I^{n}$ *and all* $b \in \prod_{j=1}^{n} \mathrm{Fin}(B_{i(j)})$, *whenever* $i(1) \neq i(2) \neq \ldots \neq i(n)$ *and* $\phi(b_{i(1)}) \approx 0, \ldots, \phi(b_{i(n)}) \approx 0$ *then* $\phi(b_{i(1)} \cdots b_{i(n)}) \approx 0$.

Proposition 12. *Let* (A, ϕ) *be an ordinary* $C^{*}ps$ *and let* $(B_j)_{j\in I}$ *be a family of subalgebras of* A. *The following are equivalent:*

(1) $(\,^{*}B_j)_{j\in\,^{*}I}$ *is free.*

(2) $(\,^{*}B_j)_{j\in\,^{*}I}$ *is almost free.*

Proof. (1) $\Rightarrow$ (2). Let $n \in \mathbb{N}$, $i \in (\,^{*}I)^{n}$ and $b \in \prod_{j=1}^{n} \mathrm{Fin}(\,^{*}B_{i(j)})$ be such that $i(1) \neq i(2) \neq \ldots \neq i(n)$ and $\phi(b_{i(1)}) \approx 0, \ldots, \phi(b_{i(n)}) \approx 0$. Since $\phi(b_{i(j)}) - \phi(b_{i(j)})1) = 0$, for all $1 \leq j \leq n$, then $\phi(\prod_{j=1}^{n}(b_{i(j)} - \phi(b_{i(j)})1)) = 0$. We notice that $\prod_{j=1}^{n}(b_{i(j)} - \phi(b_{i(j)})1)) = b_{i(1)} \cdots b_{i(n)} + S$, where S is a standard finite sum of terms each having infinitesimal norm. Therefore $\phi(b_{i(1)} \cdots b_{i(n)}) \approx 0$, as required.

(2) $\Rightarrow$ (1). The following chain of implications holds: $(\,^{*}B_j)_{j\in\,^{*}I}$ is almost free $\Rightarrow$ $(\widehat{B}_j)_{j\in\,^{*}I}$ is free $\Rightarrow$ $(B_j)_{j\in I}$ is free $\Rightarrow$ $(\,^{*}B_j)_{j\in\,^{*}I}$ is free. The leftmost implication is straightforward and the middle one holds by Proposition 11. The rightmost implication holds by Transfer. $\quad\square$

Corollary 2. *Let* (A, ϕ), $(B_j)_{j\in I}$ *be as in Proposition 12. Then* $(B_j)_{j\in I}$ *is free if and only if* $(\,^{*}B_j)_{j\in\,^{*}I}$ *is almost free.*

Let (A, ϕ) be an internal $C^{*}ps$ and let $(B_j)_{j\in I}$ be an internal free family of subalgebras of A. Notice that the same proof as (1) $\Rightarrow$ (2) in Proposition 12 shows that $(B_j)_{j\in I}$ is almost free.

Noncommutative probability has its own notion of convergence in distribution (see [17]):

Definition 4. *Let* $(A_m, \phi_m)_{m \in \mathbb{N}}$ *and* (A, ϕ) *be ordinary C*ps. For each $m \in \mathbb{N}$ let $a_m = (a_{m,j})_{j \in I}$ be a sequence in A_m and let $a = (a_j)_{j \in I}$ be a sequence in A. We say that*

(1) $(a_m)_{m \in \mathbb{N}}$ *converges in distribution to a if, for all* $n \in \mathbb{N}$ *and all* $i \in I^n$, $\lim_m \phi_m(a_{m,i(1)} \cdots a_{m,i(n)}) = \phi(a_{i(1)} \cdots a_{i(n)})$.

(2) $(a_m)_{m \in \mathbb{N}}$ *converges in $*$-distribution to a if for all* $n \in \mathbb{N}$, *all* $i \in I^n$ *and all* $(\epsilon_1, \ldots, \epsilon_n) \in \{1, *\}^n$,

$$\lim_m \phi_m(a_{m,i(1)}^{\epsilon_1} \cdots a_{m,i(n)}^{\epsilon_n}) = \phi(a_{i(1)}^{\epsilon_1} \cdots a_{i(n)}^{\epsilon_n}).$$

We stress that, in the previous definition, the "$*$" refers to the adjoint operator.

With the notation of Definition 4 in force, let $I' = I \cup \{k\}$, for some $k \notin I$, and let a'_m be the extension of a_m defined by $a_{m,k} = 1_{A_m}$, for all $m \in \mathbb{N}$. Similarly, let a' be the extension of a obtained by letting $a_k = 1_A$. We make the trivial observation that $(a_m)_{m \in \mathbb{N}}$ converges in $*$-distribution to a if and only if $(a'_m)_{m \in \mathbb{N}}$ converges in $*$-distribution to a'. From now on we assume that $(a_m)_{m \in \mathbb{N}}$ and a satisfy the following property:

$$\text{there exists } j \in I \text{ such that, for all } m \in \mathbb{N}, a_{m,j} = 1_{A_m} \text{ and } a_j = 1_A \quad (\circ)$$

Let $(^*A_m, {}^*\phi_m)_{m \in {}^*\mathbb{N}}$ be the nonstandard extension of $(A_m, \phi_m)_{m \in \mathbb{N}}$. Without loss of generality we assume $I \subseteq {}^*I$. We give the following nonstandard characterization of convergence in distribution. A similar characterization applies to convergence in $*$-distribution.

Proposition 13. *With the notation of Definition 4 in force, and under the subsequent assumptions, the following are equivalent:*

(1) $(a_m)_{m \in \mathbb{N}}$ *converges in distribution to a;*

(2) *there exists* $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ *such that the following holds for all internal N-tuples* $(i_1, \ldots, i_n)$ *in* $({}^*I)^N$:

$$\exists M \in {}^*\mathbb{N} \, \forall M < K \in {}^*\mathbb{N} ({}^*\phi_K({}^*a_{K,i(1)} \cdots {}^*a_{K,i(N)}) \approx {}^*\phi({}^*a_{i(1)} \cdots {}^*a_{i(N)})).$$

Proof. For $N \in {}^*\mathbb{N}$ we denote by $({}^*I)_\times^N$ the internal set formed by all internal tuples in $({}^*I)^N$.

$(1) \Rightarrow (2)$ From (1) we get by Transfer and Overspill that the internal set

$$\{N \in {}^*\mathbb{N} : \forall i \in ({}^*I)_\times^N \lim_{M \in {}^*\mathbb{N}} {}^*\phi_M({}^*a_{M,i(1)} \cdots {}^*a_{M,i(N)}) = {}^*\phi({}^*a_{i(1)} \cdots {}^*a_{i(N)})\}$$

properly contains $\mathbb{N}$. Any $N \in {}^*\mathbb{N}$ witnessing the proper inclusion satisfies the required property.

$(2) \Rightarrow (1)$ Let n, l be positive natural numbers. From (2), recalling $(\circ)$, we get that $\forall i \in ({}^*I)^n \exists M \in {}^*\mathbb{N} \, \forall M < K \in {}^*\mathbb{N}$

$$| {}^*\phi_K({}^*a_{K,i(1)} \cdots {}^*a_{K,i(n)}) - {}^*\phi({}^*a_{i(1)} \cdots {}^*a_{i(n)})| < 1/l.$$

Hence, by Transfer and by arbitrariness of n, l, we get (1). □

Definition 5. *Let* (A, ϕ) *be an ordinary C*ps and let $(X_j)_{j \in I}$ be a family of subsets of A and let B_j be the unital C*-algebra generated by X_j, for $j \in I$. We say that $(X_j)_{j \in I}$ is $*$-free if $(B_j)_{j \in I}$ is free. A sequence $(a_i)_{i \in I}$ is $*$-free if so is $(\{a_i\})_{i \in I}$.*

We have already noticed that the notion of freeness can be formulated with reference to a family of $*$-subalgebras of a given C^*-algebra A in a C^*ps (A, ϕ). Actually the following holds:

Proposition 14. *Let (A, ϕ) be an ordinary C^*ps. Let $(A_j)_{j \in I}$ be a family of unital $*$-algebras of A and, for each $j \in I$, let B_j be the C^*-algebra generated by A_j. Then $(A_j)_{j \in I}$ is free if and only if so is $(B_j)_{j \in I}$.*

Proof. In order to establish the nontrivial implication we apply Corollary 2. Let $(^*A_j)_{j \in \,^*I}$ and $(^*B_j)_{j \in \,^*I}$ be the nonstandard extensions of the two families with the same names. Let $n \in \mathbb{N}$, $i \in (^*I)^n$ and $b \in \prod_{j=1}^n \mathrm{Fin}(^*B_{i(j)})$ be such that $i(1) \neq i(2) \neq \ldots \neq i(n)$ and $\phi(b_{i(1)}) \approx 0, \ldots, \phi(b_{i(n)}) \approx 0$. Since $b_{i(k)}$ is in the internal closure of $^*A_{i(k)}$, there exists some $a_{i(k)} \in \,^*A_{i(k)}$ such that $a_{i(k)} \approx b_{i(k)}$. Hence $\phi(a_{i(k)}) \approx \phi(b_{i(k)})$, for each $1 \leq k \leq n$. By almost freeness of $(^*A_j)_{j \in \,^*I}$ we get that $\phi(a_{i(1)} \cdots a_{i(n)}) \approx 0$. Since $a_{i(1)} \cdots a_{i(n)} \approx b_{i(1)} \cdots b_{i(n)}$, we finally get $\phi(b_{i(1)} \cdots b_{i(k)}) \approx 0$. Having established that $(^*B_j)_{j \in \,^*I}$ is almost free, we are done by Corollary 2. □

We apply the latter proposition and previous results to give an elementary nonstandard proof of the following known fact:

Proposition 15. *Let $(A_m, \phi_m)_{m \in \mathbb{N}}$ and (A, ϕ) be C^*ps. For each $m \in \mathbb{N}$ let $a_m = (a_{m,j})_{j \in I}$ be a $*$-free sequence in A_m. If $(a_m)_{m \in \mathbb{N}}$ converges in $*$-distribution to $a = (a_j)_{j \in I}$ then a is $*$-free.*

Proof. For notational simplicity let us consider the case when $|I| = 2$. For $m \in \mathbb{N}$ let $a_m = (b_m, c_m)$ and $a = (b, c)$.

Let $0 < k \in \mathbb{N}$ and let $u^1, \ldots, u^k$ and $v^1, \ldots, v^k$ be elements in the unital $*$-algebras generated by $\{1, b\}$ and $\{1, c\}$ respectively. Let us assume that $\phi(u^1) = \cdots = \phi(u^k) = 0$ and $\phi(v_1) = \cdots = \phi(v_k) = 0$. We claim that $\phi(u^1 v^1 \cdots u^k v^k) = 0$. Once more for the sake of simplicity, let us assume $k = 1$ and let $u = u^1$, $v = v^1$. (The argument below immediately extends to any positive k.)

Recalling how the $*$-algebra generated by $\{1, b\}$ is obtained, we associate to u a sequence $(u_m)_{m \in \mathbb{N}}$, where u_m belongs to the $*$-algebra generated by $\{1, b_m\}$ and u_m is defined from $1, b_m$ in the same way as u is defined from $1, b$. By assumption we have $\lim_m \phi_m(u_m) = \phi(u)$. We do the same with v.

Let us denote by $(^*A_M, \,^*\phi_M)_{M \in \,^*\mathbb{N}}$ and $(^*a_M)_{M \in \,^*\mathbb{N}}$ the nonstandard extensions of $(A_m, \phi_m)_{m \in \mathbb{N}}$ and $(a_m)_{m \in \mathbb{N}}$ respectively.

Next we use the nonstandard characterization of convergence of a sequence. Let $(^*u_M)_{M \in \,^*\mathbb{N}}$ and $(^*v_M)_{M \in \,^*\mathbb{N}}$ be the nonstandard extensions of $(u_M)_{M \in \mathbb{N}}$ and $(v_M)_{M \in \mathbb{N}}$ respectively. For all $M \in \,^*\mathbb{N} \setminus \mathbb{N}$ we have $^*\phi_M(^*u_M) \approx 0$ and $^*\phi_M(^*v_M) \approx 0$. By Corollary 2 we get that $^*\phi_M(^*u_M \,^*v_M) \approx 0$ for all $M \in \,^*\mathbb{N} \setminus \mathbb{N}$. Hence $\phi(uv) = 0$. □

Next we investigate the behaviour of the free product of C^*-probability spaces with respect to the nonstandard hull construction. We begin by recalling the definition of free product (see [18] [Definition 7.10]):

Definition 6. *Let $(A_i, \phi_i)_{i \in I}$ be a family of ordinary C^*ps such that the functionals $\phi_i : A_i \to \mathbb{C}$, $i \in I$, are faithful traces. A C^*ps (A, ϕ), with ϕ a faithful trace, is called a free product of the family $(A_i, \phi_i)_{i \in I}$ if there exists a family $(w_i : A_i \to A)_{i \in I}$ of norm-preserving unital homomorphisms with the following properties:*

(1) for all $i \in I$, $\phi \circ w_i = \phi_i$;
(2) the C^*-subalgebras $(w_i(A_i))_{i \in I}$ form a free family in (A, ϕ);
(3) $\bigcup_{i \in I} w_i(A_i)$ generates the C^*-algebra A.

It can be shown that a free product of the family $(A_i, \phi_i)_{i \in I}$ as in Definition 6 does exist. The assumption of faithfulness is just a technical simplification. Furthermore, (A, ϕ) and the family $(w_i : A_i \to A)_{i \in I}$ are unique up to isomorphism. See [18] [Theorem 7.9].

Theorem 2. *Let $(A_i, \phi_i)_{i \in I}$ be an ordinary family of C^*ps such that the functionals $\phi_i : A_i \to \mathbb{C}$, $i \in I$, are faithful traces. Let (A, ϕ) be the free product of the family with norm-preserving unital*

homomorphisms $(w_i : A_i \rightarrow A)_{i \in I}$ *as in Definition* 6. *If the C^*-algebra $\widehat{A}$ is generated by* $\bigcup_{i \in {}^*I} \widehat{w}_i(\widehat{A}_i)$ *then* $(\widehat{A}, \widehat{\phi})$ *is the free product of the family* $(\widehat{A}_i, \widehat{\phi}_i)_{i \in {}^*I}$, *with norm-preserving unital homomorphisms* $(\widehat{w}_i : \widehat{A}_i \rightarrow \widehat{A})_{i \in {}^*I}$.

Proof. At the beginning of Section 5 we have already observed that if (B, η) is an ordinary C^*ps then so is $(\widehat{B}, \widehat{\eta})$. Moreover, if η is faithful so is $\widehat{\eta}$, by Proposition 9. Furthermore, if η is tracial so is $\widehat{\eta}$: let $a, b \in \text{Fin}({}^*A)$. Then $\widehat{\eta}(\widehat{a}\,\widehat{b}) \approx {}^*\eta(ab) = {}^*\eta(ba) \approx \widehat{\eta}(\widehat{b}\,\widehat{a})$, where the middle equality holds by Transfer. Hence $\widehat{\eta}(\widehat{a}\,\widehat{b}) = \widehat{\eta}(\widehat{b}\,\widehat{a})$. We leave it to the reader to verify that if η is norm-preserving so is $\widehat{\eta}$.

In light of the previous considerations and of the assumption that $\widehat{A}$ is generated by $\bigcup_{i \in {}^*I} \widehat{w}_i(\widehat{A}_i)$, we are left to show that (1) and (2) of Definition 6 are satisfied by $(\widehat{A}_i, \widehat{\phi}_i)_{i \in {}^*I}$ and $(\widehat{w}_i : \widehat{A}_i \rightarrow \mathbb{C})_{i \in {}^*I}$. Condition $\widehat{\phi} \circ \widehat{w}_i$ holds for all $i \in {}^*I$ by Transfer and by definition of $\widehat{\phi}, \widehat{w}_i$.

Eventually, the family $(\widehat{w}_i(\widehat{A}_i))_{i \in {}^*I}$ is free with respect to $\widehat{\phi}$ by Proposition 11. $\square$

With reference to the proof of the previous theorem, we point out that we do not use the explicit construction of the free product outlined in [18] [Lecture 7]. We just make use of the universal property of that construction.

Regarding the assumption in the statement of Theorem 2 that the C^*-algebra $\widehat{A}$ is generated by $\bigcup_{i \in {}^*I} \widehat{w}_i(\widehat{A}_i)$, we notice that the other assumptions only ensure that the C^*-algebra generated by $\bigcup_{i \in {}^*I} \widehat{w}_i(\widehat{A}_i)$ is a subalgebra of $\widehat{A}$. Actually, if $(B_i)_{i \in I}$ is an internal family of C^*-subalgebras of the internal C^*-algebra A such that $\bigcup_{i \in {}^*I} B_i$ generates A, it might be that $\bigcup_{i \in I} \widehat{B}_i$ generates a proper C^*-subalgebra of $\widehat{A}$, as the following shows.

Example. Let $M \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $N = 2M$. Let us denote by ${}^*\mathbb{C}^N$ the internal C^*-algebra of internal functions $f : \{1, \ldots, N\} \rightarrow {}^*\mathbb{C}$, equipped with the supremum norm and with componentwise addition, multiplication and conjugation. Let $\mathbf{1}$ be the unit of ${}^*\mathbb{C}^N$ and, for $0 < i \leq N$, let e_i be the function in ${}^*\mathbb{C}^N$ that takes value 1 on i and 0 elsewhere. Clearly, ${}^*\mathbb{C}^N$ is internally generated by $\bigcup_{i \leq N} C_i$, where C_i is the C^*-algebra generated by $\{e_i, \mathbf{1}\}$, for $i \leq N$. Let $v \in \text{Fin}({}^*\mathbb{C}^N)$ be defined as follows:

$$v(i) = \begin{cases} 0 & \text{if } 1 \leq i < M; \\ 1 & \text{otherwise} \end{cases}$$

We observe that the ordinary C^*-algebra generated by $\bigcup_{i \leq N} \widehat{C}_i$ is just the C^*-algebra generated by $\{\widehat{e}_i : 1 \leq i \leq N\} \cup \{\widehat{1}\}$ and we show that $\widehat{v}$ does not belong to the latter.

First of all, we introduce a convenient presentation of $\widehat{{}^*\mathbb{C}^N}$. We associate to each f in $\text{Fin}({}^*\mathbb{C}^N)$ the map

$$\begin{array}{rcl} {}^\circ f : \{1, \ldots, N\} & \rightarrow & \mathbb{C} \\ i & \mapsto & {}^\circ f(i) \end{array}$$

Notice that ${}^\circ f$ is well-defined because $f(i) \in \text{Fin}({}^*\mathbb{C})$, for all $1 \leq i \leq N$. The set

$$A = \{{}^\circ f : F \in \text{Fin}({}^*\mathbb{C}^N)\}$$

is closed under componentwise addition, multiplication and conjugation. It can be easily verified that, equipped with the supremum norm, A becomes a C^*-subalgebra of the ordinary C^*-algebra $\mathbb{C}^N$ of complex valued functions defined on the discrete space $\{1, \ldots, N\}$. A little bit of work is only required to prove that A is closed. We sketch the relative proof to highlight the use of a fairly routine nonstandard argument.

Let $({}^\circ f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in A. Let $r \in \mathbb{R}$ be such that $\|f_n\| < r$, for all $n \in \mathbb{N}$. Let $(n_m)_{0 < m \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers with the following properties:

(a) for all $0 < m$, $m < n_m$;
(b) for all $n_m < k, l$ it holds that $\|f_k - f_l\| < 1/m$.

For each positive natural number k, we let X_k be the internal set of (internal) sequences $(g_h)_{h \in {}^*\mathbb{N}}$ of elements of $\text{Fin}({}^*\mathbb{C}^N)$ with the following properties:

(1) for all $i \in {}^*\mathbb{N}$, $\|g_i\| < r$;
(2) for all $i \leq n_k$, $g_i = f_i$;
(3) for all $n_k < i, j \in {}^*\mathbb{N}$, $\|g_i - g_j\| < 1/k$.

Each X_k is internal, by the Internal Definition Principle. It is easily seen that the family $\{X_k\}_{0 < k \in \mathbb{N}}$ has the finite intersection property. By saturation, there exists $g \in \bigcap_{0 < k} X_k$. Let $N \in {}^*\mathbb{N} \setminus \mathbb{N}$. By definition of X_k, for all $0 < k$ and all $n_k < i \in \mathbb{N}$, $\|f_i - g_N\| < 1/k$. Then $({}^\circ f_n)_{n \in \mathbb{N}}$ converges to ${}^\circ g_N$.

Having established that A is a C^*-algebra, it is straightforward to verify that the map

$$\begin{array}{ccc} A & \to & \widehat{{}^*\mathbb{C}^N} \\ {}^\circ f & \mapsto & \widehat{f} \end{array}$$

is an isometric isomorphism. From now on we deal with A. We regard the maps v, **1** and the e_i's, $1 \leq i \leq N$, as elements of A. Finally, we prove that v does not belong to the C^*-algebra generated by $\{e_i : 1 \leq i \leq N\} \cup \{1\}$. First of all we notice that every element in the ordinary $*$-algebra B generated by $\{e_i : 1 \leq i \leq N\} \cup \{\mathbf{1}\}$ is a constant function on all but finitely many points. For the sake of contradiction, let $f \in B$ be such that $\|f - v\| < 1/2$. Let $1 \leq i < M$ and $M \leq j \leq N$ be such that $f(i) = f(j)$. From $\|f - v\| \geq |1 - f(j)| \geq 1 - |f(i)| \geq 1/2$ we get a contradiction. Hence v does not belong to the norm-closure of B.

Let $(I, <)$ be a directed partially ordered set. If for all $i, j \in I$ there exists $k \in I$ such that $i, j \leq k$ and $w_i(A_i) \cup w_i(A_j) \subseteq w_k(A_k)$, then the extra assumption in Theorem 2 is satisfied, as a consequence of the following:

Proposition 16. *Let $(J, \lhd)$ be an internal directed set. Let $(B_j)_{j \in J}$ be an internal family of subalgebras of an internal C^*-algebra B with the property that for all $i, j \in J$ there exists $k \in J$ such that $i, j \unlhd k$ and $B_i \cup B_j \subseteq B_k$.*

If B is generated by $\bigcup_{j \in J} B_j$ then $\widehat{B}$ is generated by $\bigcup_{j \in J} \widehat{B}_j$. Actually,

$$\widehat{B} = \bigcup_{j \in J} \widehat{B}_j.$$

Proof. Notice that $\bigcup_{j \in J} B_j$ is an internal $*$-algebra. From the assumption that B is generated by $\bigcup_{j \in J} B_j$ it follows that for each $b \in \text{Fin}(B)$ there exist $j \in J$ and $b' \in B_j$ such that $b \approx b'$. Hence $\widehat{b} \in \widehat{B}_j$ and so $\widehat{B} \subseteq \bigcup_{j \in J} \widehat{B}_j$. The converse inclusion is trivial. $\quad\square$

6. Nonstandard Noncommutative Stochastics

We begin with the definition of stochastic process over a C^*-algebra given in [9]:

Definition 7. *Let B be a C^*-algebra and let T be a set. An ordinary noncommutative stochastic process (briefly: nsp) over B indexed by T is a triple $\mathcal{A} = (A, (j_t : B \to A)_{t \in T}, \phi)$, where*

(a) *(A, ϕ) is a C^*ps;*
(b) *for each $t \in T$, j_t is a C^*-algebra homomorphism with the property that $j_t(1_B) = 1_A$;*

The stochastic process $\mathcal{A}$ is full if the C^*-algebra A is generated by $\bigcup_{t \in T} j_t(B)$.

Notice that, in [9], all nsp's are assumed to be full. Fullness is needed in the proof of [9] [Proposition 1.1].

Let us recall some notation and terminology from [9]: Let $\mathcal{A}$ be an ordinary nsp and, for all $0 < n \in \mathbb{N}$, let $\mathbf{t} = (t_1, \ldots, t_n) \in T^n$; $\mathbf{b} = (b_1, \ldots, b_n) \in B^n$. We define the map $j_{\mathbf{t}} : B^n \to A$ by letting $j_{\mathbf{t}}(\mathbf{b}) = j_{t_n}(b_n) \ldots j_{t_1}(b_1)$. The **t**-correlation kernel is the function

$$
\begin{array}{rcc}
w_{\mathbf{t}} : & B^n \times B^n & \to & \mathbb{C} \\
& (\mathbf{a}, \mathbf{b}) & \mapsto & \phi(j_{\mathbf{t}}(\mathbf{a})^* j_{\mathbf{t}}(\mathbf{b}))
\end{array}
$$

It is straightforward to verify that $w_{\mathbf{t}}$ is conjugate linear in each of the $\mathbf{a}$'s components and linear in each of the $\mathbf{b}$'s components. (This is the usual convention in Physics.)

We endow B^n with the supremum norm and we denote by B_1^n its unit ball. As is usual with sesquilinear forms, we define the norm of $w_{\mathbf{t}}$, for $\mathbf{t} \in T^n$, as follows:

$$
\|w_{\mathbf{t}}\| = \sup\{|w_{\mathbf{t}}(\mathbf{a}, \mathbf{b})| : \mathbf{a}, \mathbf{b} \in B_1^n\}.
$$

We recall the following definition from [9]:

Definition 8. *Let $\mathcal{A}_i = (A_i, (j_t^i : B \to A_i)_{t \in T}, \phi_i)$, $i = 1, 2$, be ordinary nsp's and let (H_i, π_i, ξ_i) be the GNS triples associated to (A_i, ϕ_i), for $i = 1, 2$ (see [11] [II.6.4]). The processes $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if there exists a unitary operator $u : H_1 \to H_2$ such that*

$$
u(\xi_1) = \xi_2 \text{ and, for all } b \in B \text{ and all } t \in T, \ u\pi_1 j_t^1(b) = \pi_2 j_t^2(b)u.
$$

The following is a characterization of equivalence between full nsp's (see [9] [Proposition 1.1]).

Proposition 17. *For $i = 1, 2$ let $\mathcal{A}_i = (A_i, (j_t^i : B \to A_i)_{t \in T}, \phi_i)$ be ordinary full stochastic processes. The two processes are equivalent if and only if, for all $0 < n \in \mathbb{N}$, all $\mathbf{a}, \mathbf{b} \in B^n$ and all $\mathbf{t} \in T^n$ it holds that*

$$
w_{\mathbf{t}}^1(\mathbf{a}, \mathbf{b}) = w_{\mathbf{t}}^2(\mathbf{a}, \mathbf{b}).
$$

We make use of Proposition 17 to give a nonstandard characterization of equivalence.

Theorem 3. *For $i = 1, 2$ let $\mathcal{A}_i = (A_i, (j_t^i : B \to A_i)_{t \in T}, \phi_i)$ be ordinary full nsp's. Let ${}^*\mathcal{A}_i$ be the nonstandard extension of $\mathcal{A}_i$, $i = 1, 2$. The following are equivalent:*

1. *$\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent;*
2. *there exists $N \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that, for all internal $\mathbf{t} \in ({}^*T)^N$ and all internal $\mathbf{a}, \mathbf{b} \in ({}^*B)^N$,*

$$
{}^*w_{\mathbf{t}}^1(\mathbf{a}, \mathbf{b}) = {}^*w_{\mathbf{t}}^2(\mathbf{a}, \mathbf{b}).
$$

Proof. $(1) \Rightarrow (2)$ is a straightforward consequence of Proposition 17 and of Transfer.

Concerning the converse implication, let N be as in (2). We fix $0 < n \in \mathbb{N}$. Let $\mathbf{t} \in ({}^*T)^n$; $\mathbf{a}, \mathbf{b} \in ({}^*B)^n$. We extend them to internal sequences of length N by letting, for instance, $\mathbf{t}' = (t_1, \ldots, t_n, t_n, t_n, \ldots)$, $\mathbf{a}' = (a_1, \ldots, a_n, 1_B, 1_B, \ldots)$, $\mathbf{b}' = (b_1, \ldots, b_n, 1_B, 1_B, \ldots)$. Then

$$
{}^*w_{\mathbf{t}}^1(\mathbf{a}, \mathbf{b}) = {}^*w_{\mathbf{t}'}^1(\mathbf{a}', \mathbf{b}') = {}^*w_{\mathbf{t}'}^2(\mathbf{a}', \mathbf{b}') = {}^*w_{\mathbf{t}}^2(\mathbf{a}, \mathbf{b}).
$$

Therefore

$$
\forall \mathbf{t} \in ({}^*T)^n \forall \mathbf{a} \in ({}^*B)^n \forall \mathbf{b} \in ({}^*B)^n ({}^*w_{\mathbf{t}}^1(\mathbf{a}, \mathbf{b}) = {}^*w_{\mathbf{t}}^2(\mathbf{a}, \mathbf{b})).
$$

By Transfer we get

$$
\forall \mathbf{t} \in T^n \forall \mathbf{a} \in B^n \forall \mathbf{b} \in B^n (w_{\mathbf{t}}^1(\mathbf{a}, \mathbf{b}) = w_{\mathbf{t}}^2(\mathbf{a}, \mathbf{b})).
$$

Being n arbitrary, by Proposition 17 we get that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. $\square$

The content of Theorem 3 is that a full nsp $\mathcal{A}$ is determined, up to equivalence, by the internal family of correlation kernels $\{\,^*w_{\mathbf{t}} : \mathbf{t} \in (\,^*T)^N\}$ of the process $\,^*\mathcal{A}$, for some infinite hyperatural N.

The reader who is familiar with the notion of stochastic process, as introduced for instance in [19], is invited to read the commentary on [9] [Section 1] to make sense of Definition 7. In short, let $\mathbf{X} = (X_t : \Omega \to S)_{t \in T}$ be an ordinary stochastic process, where the X_t's are measurable functions defined on a probability space $(\Omega, \mathcal{F}, \mu)$ with values in some measurable space $(S, \mathcal{G})$. Let $\phi : L_\infty(\Omega, \mathcal{F}) \to \mathbb{C}$ be defined by $\phi(g) = \int_\Omega g \, d\mu$, for all $g \in L_\infty(\Omega, \mathcal{F})$.

It can be shown that the triple

$$(L_\infty(\Omega, \mathcal{F}), (j_t : L_\infty(S, \mathcal{G}) \to L_\infty(\Omega, \mathcal{F}))_{t \in T}, \phi),$$

where $j_t(f) = f \circ X_t$ for all $t \in T$ and all $f \in L_\infty(S, \mathcal{G})$, forms a nsp in the sense of Definition 7. Furthermore, under additional assumptions on a nsp, one can associate to the latter an ordinary stochastic process.

Let $\mathcal{A} = (A, (j_t : B \to A)_{t \in T}, \phi)$ be an internal nsp. For all $t \in T$, the map $\widehat{j_t} : \widehat{B} \to \widehat{A}$ defined by $\widehat{j_t}(\widehat{b}) = \widehat{j_t(b)}$ is well-defined because C^*-algebra homomorphisms are norm contracting. It is straightforward to verify that the nonstandard hull $\widehat{\mathcal{A}} = (\widehat{A}, (\widehat{j_t} : \widehat{B} \to \widehat{A})_{t \in T}, \widehat{\phi})$ of $\mathcal{A}$ is an ordinary nsp. We point out that the C^*-algebra generated by $\bigcup_{t \in T} \widehat{j_t}(\widehat{B})$ is a subalgebra of $\widehat{A}$ but, in general, fullness of $\mathcal{A}$ is not inherited by $\widehat{\mathcal{A}}$. In this regard, see the Example in Section 5 and the discussion preceding it. The following is a sufficient condition for preservation of fullness.

Proposition 18. *Let $(T, <)$ be an internal linearly ordered set and let $\mathcal{A} = (A, (j_t : B \to A)_{t \in T}, \phi)$ be an internal full nsp with the property that, for all $s < t$ in T, $j_s(B)$ is a subalgebra of $j_t(B)$. Then $\widehat{\mathcal{A}} = (\widehat{A}, (\widehat{j_t} : \widehat{B} \to \widehat{A})_{t \in T}, \widehat{\phi})$ is an ordinary full nsp.*

Proof. An immediate consequence of Proposition 16. □

Next we provide a nonstandard characterization of equivalence between nsp's of the form $\widehat{\mathcal{A}}$.

We make a preliminary remark. Let (A, ϕ) be an internal C^*ps and let $(H, \pi : A \to B(H), \xi)$ be the associated internal GNS triple, where ξ is the cyclic vector of the representation. As we already remarked at the end of Section 2, we can identify $\widehat{B(H)}$ with a C^*-subalgebra of $B(\widehat{H})$. It can be easily verified that

$$\begin{aligned} \widehat{\pi} : \quad \widehat{A} \quad &\to \quad B(\widehat{H}) \\ \widehat{a} \quad &\mapsto \quad \widehat{\pi(a)} \end{aligned}$$

is a $*$-homomorphism and that, for all $\widehat{a} \in \widehat{A}$, $\widehat{\phi}(\widehat{a}) = \langle \widehat{\pi}(\widehat{a})\widehat{\xi}, \widehat{\xi} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\widehat{H}$.

In order to conclude the verification that $(\widehat{H}, \widehat{\pi} : \widehat{A} \to B(\widehat{H}), \widehat{\xi})$ is a GNS triple for $(\widehat{A}, \widehat{\phi})$, we prove the following result, which is actually stronger than what we need:

Proposition 19. *Let $(A, \phi), (H, \pi : A \to B(H), \xi), (\widehat{H}, \widehat{\pi} : \widehat{A} \to B(\widehat{H}), \widehat{\xi})$ be as above. Then $\{\widehat{\phi}(\widehat{a})(\widehat{\xi}) : A \in Fin(A)\} = \widehat{H}$.*
Consequently, $(\widehat{H}, \widehat{\pi} : \widehat{A} \to B(\widehat{H}), \widehat{\xi})$ is a GNS triple for $(\widehat{A}, \widehat{\phi})$.

Proof. Following [11] [II.6.4], let $N_\phi = \{x \in A : \phi(x^*x) = 0\}$. By the GNS construction, we have that ξ is the image of the unit of A in A/N_ϕ and that $\pi(a)$ is the left multiplication operator by a on A/N_ϕ. Moreover A/N_ϕ is an inner product space with respect to $\langle \cdot, \cdot \rangle$ defined by

$$\langle x + N_\phi, y + N_\phi \rangle = \phi(y^*x), \quad x, y \in A.$$

Furthermore, H is the Hilbert space completion of A/N_ϕ and the set

$$\{\pi(a)(\xi) : A \in A\}$$

is dense in H.

Let $\widehat{h} \in \widehat{H}$. Then there exists $a \in A$ such that $a + N_\phi = \pi(a)(\xi) \approx h$ and, by definition of norm on a quotient space, there exists also $y \in N_\phi$ such that $\|a + y\| \approx \|a + N_\phi\|$. It follows that $\|a + y\|$ is finite. Furthermore

$$\pi(a + y)(\xi) = a + N_\phi \approx h.$$

Therefore there exists some $b \in \text{Fin}(A)$ such that $\pi(b)(\xi) \approx h$. For such a b it holds that $\widehat{\pi}(\widehat{b})(\widehat{\xi}) = \widehat{h}$. $\quad\square$

Theorem 4. *Let $\mathcal{A}_i = (A_i, (j^i_t : B \to A_i)_{t \in T}, \phi_i)$, $i = 1, 2$, be internal nsp's over the C*-algebra B such that $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ are full. Let (H_i, π_i, ξ_i) be the internal GNS triples associated to (A_i, ϕ_i), for $i = 1, 2$. The following are equivalent:*

(1)   the processes $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ are equivalent according to Definition 8;

(2)   there exists an infinite hypernatural N such that, for all $\mathbf{t} \in T^N$, the $\mathbf{t}$-correlation kernels $w^1_\mathbf{t}, w^2_\mathbf{t}$ relative to $\mathcal{A}_1, \mathcal{A}_2$ respectively satisfy the property $w^1_\mathbf{t} \approx w^2_\mathbf{t}$ (namely, $\|w^1_\mathbf{t} - w^2_\mathbf{t}\| \approx 0$).

Proof.

$(1)\Rightarrow(2)$ For $0 < k, m \in \mathbb{N}$, let C_{km} be the set

$$\{M \in {}^*\mathbb{N} : m \leq M \text{ and } \forall \mathbf{t} \in T^M \forall \mathbf{a}, \mathbf{b} \in B_1^M(|w^1_\mathbf{t}(\mathbf{a}, \mathbf{b}) - w^2_\mathbf{t}(\mathbf{a}, \mathbf{b})| \leq 1/k)\}.$$

By the Internal Definition Principle, each C_{km} is internal. It follows from [9] [Proposition 1.1] that the family $\{C_{km}\}_{0 < k, m \in \mathbb{N}}$ has the finite intersection property. By saturation, $\bigcap C_{km} \neq \varnothing$. Any N in the common intersection is an infinite hypernatural with the property that, for all $\mathbf{t} \in T^N$ and all $\mathbf{a}, \mathbf{b} \in B_1^N$, $w^1_\mathbf{t}(\mathbf{a}, \mathbf{b}) \approx w^2_\mathbf{t}(\mathbf{a}, \mathbf{b})$. Recalling that the supremum of an internal set of infinitesimals is itself an infinitesimal, we get that, for all $\mathbf{t} \in T^N$, $\|w^1_\mathbf{t} - w^2_\mathbf{t}\| \approx 0$, as required.

$(2)\Rightarrow(1)$ Let N be as in (2) and let $n \in \mathbb{N}$ be arbitrarily chosen. By [9] [Proposition 1.1] and by linearity it suffices to prove that

$$\text{for all } \mathbf{t} \in T^n \text{ and all } \mathbf{a}, \mathbf{b} \in B_1^n \quad \widehat{w}^1_\mathbf{t}(\widehat{\mathbf{a}}, \widehat{\mathbf{b}}) = \widehat{w}^2_\mathbf{t}(\widehat{\mathbf{a}}, \widehat{\mathbf{b}}).$$

Let $0 < n \in \mathbb{N}$ and let $\mathbf{t} \in T^n$; $\mathbf{a}, \mathbf{b} \in B_1^n$. We extend them to internal sequences of length N by letting $\mathbf{t}' = (t_1, \ldots, t_n, t_n, t_n, \ldots)$, $\mathbf{a}' = (a_1, \ldots, a_n, 1_B, 1_B, \ldots)$, $\mathbf{b}' = (b_1, \ldots, b_n, 1_B, 1_B, \ldots)$.

From $w^1_\mathbf{t}(\mathbf{a}, \mathbf{b}) = w^1_{\mathbf{t}'}(\mathbf{a}', \mathbf{b}') \approx w^2_{\mathbf{t}'}(\mathbf{a}', \mathbf{b}') = w^2_\mathbf{t}(\mathbf{a}, \mathbf{b})$ we get immediately that $\widehat{w}^1_\mathbf{t}(\widehat{\mathbf{a}}, \widehat{\mathbf{b}}) = \widehat{w}^2_\mathbf{t}(\widehat{\mathbf{a}}, \widehat{\mathbf{b}})$. $\quad\square$

The content of Theorem 4 is that the nonstandard hull $\widehat{\mathcal{A}}$ of an internal, full nsp $\mathcal{A}$ is determined, up to equivalence, by the internal family of correlation kernels $\{w_\mathbf{t} : \mathbf{t} \in T^N\}$ of $\mathcal{A}$, for some infinite hypernatural N.

When $\mathcal{A}$ is an ordinary nsp we write $\widehat{\mathcal{A}}$ for ${}^*\mathcal{A}$. (The context will prevent any ambiguity.) Notice that if $\mathcal{A}$ is indexed by set T, then $\widehat{\mathcal{A}}$ is indexed by ${}^*T$.

We prove that, under the additional assumption of fullness, equivalence of nsp's is preserved and reflected by the nonstandard hull construction.

Proposition 20. *Let $\mathcal{A}_i = (A_i, (j^i_t : B \to A_i)_{t \in T}, \phi_i)$, $i = 1, 2$, be ordinary full nsp's such that their nonstandard hulls are also full. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if and only if $\widehat{\mathcal{A}}_1$ and $\widehat{\mathcal{A}}_2$ are equivalent.*

Proof. Let us assume that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. Notice that if u is an internal unitary operator then $\widehat{u}$ is well-defined and unitary. Moreover equalities are preserved by the nonstandard hull construction. By Transfer of Definition 8 it is thus straightforward to prove that $\widehat{\mathcal{A}_1}$ and $\widehat{\mathcal{A}_2}$ are equivalent. Notice that we do not need the fullness property for this implication.

Regarding the converse implication, for all $0 < n \in \mathbb{N}$ and all $\mathbf{t} \in (\,{}^*T)^n$, let us write $\widehat{w_{\mathbf{t}}^i}$ for $\widehat{{}^*w_{\mathbf{t}}^i}$, $i = 1, 2$. Let us assume that $\widehat{\mathcal{A}_1}$ and $\widehat{\mathcal{A}_2}$ are full. By Proposition 17, we have that $\widehat{w_{\mathbf{t}}^1} = \widehat{w_{\mathbf{t}}^2}$. Then, for all $0 < k \in \mathbb{N}$,

$$\forall \mathbf{t} \in (\,{}^*T)^n (\left\| {}^*w_{\mathbf{t}}^1 - {}^*w_{\mathbf{t}}^2 \right\| < 1/k).$$

By Transfer we get

$$\forall \mathbf{t} \in T^n (w_{\mathbf{t}}^1 = w_{\mathbf{t}}^2).$$

Eventually, by applying Proposition 17 again, we get that $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. $\square$

Next we provide a nonstandard version of the Reconstruction Theorem ([9] [Theorem 1.3]). Let B be an internal C^*-algebra and T an internal set. We let $\mathbf{T} = \bigcup_{0 < N \in {}^*\mathbb{N}} T^N$. If $\mathbf{t} \in \mathbf{T}$ we let $\overset{\circ}{K}\mathbf{t}$ be the sequence obtained by removing the K-th component from the tuple $\mathbf{t}$. Same meaning for $\overset{\circ}{K}\mathbf{b}$, when $\mathbf{b} \in B^N$ and $1 \leq K \leq N$. Furthermore, we let $\overset{\bullet}{K}\mathbf{b} = (b_1, \ldots, b_{K-2}, b_K b_{K-1}, b_{K+1}, \ldots, b_N)$. If $\mathbf{t}, \mathbf{s} \in \mathbf{T}$, we let $\mathbf{t}u\mathbf{s}$ be the time sequence obtained by inserting the component $u \in T$ between $\mathbf{t}$ and $\mathbf{s}$. We denote by $\ell(\mathbf{t})$ the length of the sequence $\mathbf{t}$ and by $\mathbf{1}$ the element $(1, \ldots, 1)$ in B^N, for some $N \in {}^*\mathbb{N}$ (the context will prevent any ambiguity).

Let $1 < N \in {}^*\mathbb{N}$. Inspired by the notion of $\mathbf{t}$-correlation kernel previously introduced (see also [9] [Proposition 1.2]), we say that an internal family $\{w_{\mathbf{t}} : B^N \times B^N \to {}^*\mathbb{C} : \mathbf{t} \in T^N\}$ of maps is an N-system of correlation kernels over B if it satisfies the following properties (when not specified, quantifications refer to internal objects):

$\mathrm{CK0}_N$ for all $\mathbf{t} \in T^N$ and all $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2 \in \mathrm{Fin}(B^N)$ it holds that

- $w_{\mathbf{t}}(\mathbf{a}_1, \mathbf{a}_2) \in \mathrm{Fin}(\,{}^*\mathbb{C})$ and
- if $\mathbf{a}_1 \approx \mathbf{a}_2$ and $\mathbf{b}_1 \approx \mathbf{b}_2$ then $w_{\mathbf{t}}(\mathbf{a}_1, \mathbf{b}_1) \approx w_{\mathbf{t}}(\mathbf{a}_2, \mathbf{b}_2)$;

$\mathrm{CK1}_N$ for all $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{T}$, all $u, v \in T$, all norm-finite $\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2$ such that $\ell(\mathbf{a}_1) = \ell(\mathbf{b}_1) = \ell(\mathbf{t}_1)$, $\ell(\mathbf{a}_2) = \ell(\mathbf{b}_2) = \ell(\mathbf{t}_2)$ and $\ell(\mathbf{a}_1\mathbf{a}_2) = N - 1$ it holds that

$$w_{\mathbf{t}_1\mathbf{t}_2 u}(\mathbf{a}_1\mathbf{a}_2 1, \mathbf{b}_1\mathbf{b}_2 1) \approx w_{\mathbf{t}_1 v \mathbf{t}_2}(\mathbf{a}_1 1 \mathbf{a}_2, \mathbf{b}_1 1 \mathbf{b}_2);$$

$\mathrm{CK2}_N$ for all $\mathbf{t} \in T^N$, all $M \in {}^*\mathbb{N}$ and all internal sequences $\{c_r\}_{r \leq M} \subseteq \mathrm{Fin}(\,{}^*\mathbb{C})$ and $\{\mathbf{b}_r\}_{r \leq M} \subseteq \mathrm{Fin}(B^N)$ it holds that

$$\mathrm{Im}(\sum_{i,j} \overline{c}_i c_j w_{\mathbf{t}}(\mathbf{b}_i, \mathbf{b}_j)) \approx 0 \text{ and } \mathrm{Re}(\sum_{i,j} \overline{c}_i c_j w_{\mathbf{t}}(\mathbf{b}_i, \mathbf{b}_j)) \gtrsim 0.$$

$\mathrm{CK3}_N$ for all $\mathbf{t} \in T^N$

$$w_{\mathbf{t}}(\mathbf{1}, \mathbf{1}) \approx 1;$$

$\mathrm{CK4}_N$ for all $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{T}$ such that $\ell(\mathbf{t}_1\mathbf{t}_2) = N - 1$ and all $u \in T$ it holds that

- for all $\mathbf{b} \in \mathrm{Fin}(B^N)$, all norm-finite $\mathbf{a}_1, \mathbf{a}_2$ such that $\ell(\mathbf{a}_1) = \ell(\mathbf{t}_1)$, $\ell(\mathbf{a}_2) = \ell(\mathbf{t}_2)$ and $\ell(\mathbf{a}_1\mathbf{a}_2) = N - 1$, the map

$$a \mapsto w_{\mathbf{t}_1 u \mathbf{t}_2}(\mathbf{a}_1 a \mathbf{a}_2, \mathbf{b})$$

is approximately conjugate linear, namely: For all $r \in \text{Fin}(^*\mathbb{C})$ and all $a, b \in \text{Fin}(B)$

$$w_{\mathbf{t}_1 u \mathbf{t}_2}(\mathbf{a}_1(ra+b)\mathbf{a}_2, \mathbf{b}) \approx \bar{r} w_{\mathbf{t}_1 u \mathbf{t}_2}(\mathbf{a}_1 a \mathbf{a}_2, \mathbf{b}) + w_{\mathbf{t}_1 u \mathbf{t}_2}(\mathbf{a}_1 b \mathbf{a}_2, \mathbf{b});$$

- for all $\mathbf{a} \in \text{Fin}(B^N)$, all norm-finite $\mathbf{b}_1, \mathbf{b}_2$ such that $\ell(\mathbf{b}_1) = \ell(\mathbf{t}_1)$, $\ell(\mathbf{b}_2) = \ell(\mathbf{t}_2)$ and $\ell(\mathbf{b}_1 \mathbf{b}_2) = N - 1$, the map

$$b \mapsto w_{\mathbf{t}_1 u \mathbf{t}_2}(\mathbf{a}, \mathbf{b}_1 b \mathbf{b}_2)$$

is approximately linear (see above);

CK5$_N$ for all $\mathbf{t} \in T^N$ and all norm-finite $\mathbf{a}, \mathbf{b} \in B^{N-1}$, the map $w_{\mathbf{t}}^{\mathbf{a},\mathbf{b}} : B \times B \to {}^*\mathbb{C}$ defined by $(a, b) \mapsto w_{\mathbf{t}}(\mathbf{a}a, \mathbf{b}b)$ approximately factors through the map $\varphi : (a, b) \mapsto a^* b$, namely: There exists some internal map $\psi : B \to {}^*\mathbb{C}$, such that, for all $a, b \in \text{Fin}(B)$, $w_{\mathbf{t}}^{\mathbf{a},\mathbf{b}}(a, b) \approx \psi(\varphi(a, b))$;

CK6$_N$ for all $\mathbf{t} \in T^N$, all $u \in T$, all $1 < K \leq N$ and all $\mathbf{a}, \mathbf{b} \in \text{Fin}(B^N)$ if $t_{K-1} = t_K$ then

$$w_{\mathbf{t}}(\mathbf{a}, \mathbf{b}) \approx w_{\underset{(\mathring{K}\mathbf{t})u}{\circ}}((\mathring{K}\mathbf{a})1, (\mathring{K}\mathbf{b})1).$$

A 1-system of correlation kernels is a family $\{w_t : T \in T\}$ of maps that satisfies CK0$_1$ and CK2$_1$–CK5$_1$.

Notice that the definition of a system of correlation kernels given in [9], strict equalities are required. We do not impose that condition because we claim that an N-system, for some $N \in {}^*\mathbb{N} \setminus \mathbb{N}$, suffices to reconstruct an unique ordinary nsp. We prove that after a preliminary construction.

Let $0 < N \in {}^*\mathbb{N}$ and let $\{w_{\mathbf{t}} : T \in T^{N+1}\}$ be an internal $(N+1)$-system of correlation kernels over an internal C^*-algebra B. We define an N-system $\{w_{\mathbf{t}} : T \in T^N\}$ as follows: we fix $z \in T$ and, for each $\mathbf{t} \in T^N$, we let

$$w_{\mathbf{t}}(\mathbf{a}, \mathbf{b}) = w_{\mathbf{t}z}(\mathbf{a}1, \mathbf{b}1) \quad \text{for all } \mathbf{a}, \mathbf{b} \in B^N.$$

By CK1$_{N+1}$, a different choice of $z \in T$ amounts to an infinitesimal perturbation in the value of $w_{\mathbf{t}}(\mathbf{a}, \mathbf{b})$.

The verification that $\{w_{\mathbf{t}} : T \in T^N\}$ satisfies properties CK0$_N$–CK6$_N$ is straightforward. Thus we can repeat the construction and, by internal induction, we get a family W_K of K-systems of correlation kernels, one for each $1 \leq K \leq N+1$. Let $W = \bigcup_{0 < n \in \mathbb{N}} W_n$.

We notice that, for all $n \in \mathbb{N}$, $\widehat{B^n} = (\widehat{B})^n$ holds. By CK0$_n$, $0 < n \in \mathbb{N}$, the map

$$\widehat{w}_{\mathbf{t}} : \quad (\widehat{B})^n \times (\widehat{B})^n \quad \to \quad \mathbb{C}$$
$$(\widehat{\mathbf{a}}, \widehat{\mathbf{b}}) \quad \mapsto \quad {}^{\circ}w_{\mathbf{t}}(\mathbf{a}, \mathbf{b})$$

is well-defined for all $w_{\mathbf{t}} \in W$. We let $\widehat{W} = \{\widehat{w}_{\mathbf{t}} : w_{\mathbf{t}} \in W\}$.

The following holds:

Theorem 5. *Let N be an infinite hypernatural, T an internal set and let $\{w_{\mathbf{t}} : T \in T^{N+1}\}$ be an internal $(N+1)$-system of correlation kernels over some internal C^*-algebra B. There exists an ordinary nsp $\mathcal{A} = (A, (j_t : \widehat{B} \to A)_{t \in T}, \phi)$ whose family of correlation kernels is the family $\widehat{W}$ defined above. Moreover such $\mathcal{A}$ is unique up to equivalence.*

Proof. We verify that the family $\widehat{W}$ is a projective system of correlation kernels over $\widehat{B}$ indexed by T, according to [9]. Equalities up to an infinitesimal turn into equalities when taking the nonstandard part. First of all we notice that $\widehat{W}$ satisfies property CK1 as a consequence of the validity of CK1$_n$, $0 < n \in \mathbb{N}$. Concerning CK2, it suffices to keep in

mind that the standard part of the sum of finitely many finite addends is the sum of their standard parts.

The only property whose verification requires a little bit of work is CK5. We fix $0 < n \in \mathbb{N}$, $\mathbf{t} \in T^{n+1}$ and $\widehat{\mathbf{a}}, \widehat{\mathbf{b}} \in \widehat{B}^n$. We notice that the map $w_{\mathbf{t}}^{\widehat{\mathbf{a}}, \widehat{\mathbf{b}}} : \widehat{B} \times \widehat{B} \to \mathbb{C}$, $(\widehat{a}, \widehat{b}) \mapsto \widehat{w}_{\mathbf{t}}(\mathbf{a}\,\widehat{a}, \mathbf{b}\,\widehat{b})$ is well-defined by CK0$_{n+1}$. We prove that it factors through the map $\widehat{\varphi} : (\widehat{a}, \widehat{b}) \mapsto \widehat{a}^* \widehat{b}$. Let ψ be as in CK5$_{n+1}$ relative to $\mathbf{t}, \mathbf{a}, \mathbf{b}$. From $w_{\mathbf{t}}^{\mathbf{a}, \mathbf{b}}(a, b) \approx \psi(\varphi(a, b))$ and from CK0$_{n+1}$, we get $\psi[\mathrm{Fin}(B)] \subseteq \mathrm{Fin}(B)$ and $\psi(c) \approx \psi(d)$ whenever $c \approx d$. Hence $\widehat{\psi} : \widehat{B} \to \mathbb{C}$, $\widehat{b} \mapsto^{\circ} \psi(b)$ is well-defined.

Let $\widehat{a}, \widehat{b} \in \widehat{B}$. We have:

$$w_{\mathbf{t}}^{\widehat{\mathbf{a}}, \widehat{\mathbf{b}}}(\widehat{a}, \widehat{b}) = {}^{\circ}w_{\mathbf{t}}(\mathbf{a}a, \mathbf{b}b) = {}^{\circ}\psi(a^*b) = \widehat{\psi}(\widehat{a^*b}) = \widehat{\psi}(\widehat{\varphi}(\widehat{a}, \widehat{b})).$$

By arbitrariness of $\widehat{a}, \widehat{b}$, we get $w_{\mathbf{t}}^{\widehat{\mathbf{a}}, \widehat{\mathbf{b}}} = \widehat{\psi}\widehat{\varphi}$.

The remaining properties are easily verified.

Finally, we get the existence of an ordinary nsp $\mathcal{A}$ with the required properties from [9] [Theorem 1.3]. Notice also that the proof of the latter theorem ensures that $\mathcal{A}$ is full. □

Let N be an infinite hypernatural. As already anticipated, the content of Theorem 5 is that an N-system of correlation kernels contains enough information to uniquely reconstruct, up to equivalence, an ordinary nsp whose family of correlation kernels is determined by the N-system.

Let $\widehat{\mathcal{A}}$ be the nonstandard hull of some internal nsp $\mathcal{A}$. Admittedly, it is a limitation that the time set T of $\widehat{\mathcal{A}}$ is an internal set. This rules out many familiar sets. To overcome such restriction, we may suitably choose T.

One possibility is to fix some infinite hypernatural M and to let $T = \{K/M : 0 \leq K \leq M\}$. Then, for all $t \in [0, 1]$, we let $t_+ = \min\{0 \leq K \leq M : T \leq K/M\}$ and we define $j_t : \widehat{B} \to \widehat{A}$ as follows: $j_t(\widehat{b}) = \widehat{j_{t_+}(b)}$. In this way, the time set of $\widehat{\mathcal{A}}$ is the real unit interval. We may also make the additional assumption that the internal process $\mathcal{A}$ is S-continuous, namely that, for all $s, t \in T$, $s \approx t$ implies $j_s \approx j_t$. Under S-continuity, it follows that, for all $s, t \in T$ and all $b, c \in \mathrm{Fin}(B)$, if $s \approx t$ and $b \approx c$ then $j_t(b) \approx j_s(b) \approx j_s(c)$.

Another possibility is to fix the factorial M of some infinite hypernatural number and to define T as above. Thus the set ${}^{\circ}T = \{{}^{\circ}t : T \in T\}$ contains all the rationals in the unit interval. Under the assumption of S-continuity, the map $j_{{}^{\circ}t} : \widehat{B} \to \widehat{A}$ defined by $j_{{}^{\circ}t}(\widehat{b}) = \widehat{j_t(b)}$ is a well-defined C^*-algebra homomorphism (see above). Therefore we get an ordinary nsp $(\widehat{A}, (\widehat{j_t} : \widehat{B} \to \widehat{A})_{t \in {}^{\circ}T}, \widehat{\phi})$ whose time set forms a dense subset of the real unit interval.

Alternatively, we may let $T = \{K \in {}^*A : K \leq M\}$, for some infinite hypernatural M or $T = {}^*\mathbb{N}$, and consider the ordinary nsp $(\widehat{A}, (\widehat{j_t})_{t \in \mathbb{N}}, \widehat{\phi})$.

Next we discuss the Markov property relative to a nsp and we formulate sufficient conditions for recovering an ordinary Markov nsp from an internal one.

We begin by recalling the definition of conditional expectation in the noncommutative framework. Let A be an ordinary C^*-algebra and let A_0 be a C^*-subalgebra of A. A mapping $E : A \to A_0$ is called a conditional expectation if

(1) E is a linear idempotent map onto A_0;
(2) $\|E\| = 1$.

It is straightforward to check that $E(1) = 1$ holds for a conditional expectation E. Moreover, the following hold (see [20]):

(a) $E(bac) = bE(a)c$, for all $a \in A$ and all $b, c \in A_0$;
(b) $E(a^*) = E(a)^*$, for all $a \in A$;
(c) E is positive.

Let T be a linearly ordered set. We say that a nsp $\mathcal{A} = (A, (j_t : B \to A)_{t \in T}, \phi)$ is adapted if, for all $s < t$ in T, $j_s(B)$ is a C^*-subalgebra of $j_t(B)$. By adopting this terminol-

ogy, the content of Proposition 18 is that fullness of an adapted nsp is preserved by the nonstandard hull construction.

Definition 9. *Let T be a linearly ordered set. The adapted process $\mathcal{A} = (A, (j_t : B \to A)_{t \in T}, \phi)$ is a Markov process with conditional expectations if there exists a family $\mathcal{E} = \{E_t : A \to j_t(B)\}_{t \in T}$ of conditional expectations such that, for all $s, t \in T$, the following hold:*

E2 $\phi = \phi|_{j_t(B)} \circ E_t$;

E3 $E_s E_t = E_{\min(s,t)}$.

Definition 9 is a restatement in the current setting of the definition of Markov nsp with conditional expectations in [9] [§2.2]. By property (a) above it follows immediately that property E1 in [9] [§2.2] holds and that, for all $s \in T$, $E_s|_{j_s(B)} = \mathrm{id}_{j_s(B)}$.

For all $s \in T$ let $A_{[s}$ be the C^*-algebra generated by $\bigcup_{s \leq t} j_t(B)$. It is straightforward to check that the Markov property

M' $E_s(A_{[s}) = j_s(B)$ for all $s \in T$,

introduced in [9] [§2.2] does hold for a Markov process as in Definition 9. Notice also that, for $t \leq s$, condition E3 always holds.

Let $\mathcal{A}$ be as in Definition 9. By letting $E_{s,t} = E_s|_{j_t(B)}$ for $s \leq t$ in T, we get a family $\mathcal{F} = \{E_{s,t} : j_t(B) \to j_s(B) : s, t \in T \text{ and } s \leq t\}$ of conditional expectations satisfying

(1) $E_{t,t} = \mathrm{id}_{j_t(B)}$ for all $t \in T$;

(2) $E_{s,t} E_{t,u} = E_{s,u}$ for all $s \leq t \leq u$ in T

as well as the Markov property M in [9]. It follows that the statement of [9] [Theorem 2.1] (with the exception of the normality property) and subsequent results do hold for $\mathcal{A}$ and $\mathcal{F}$. In particular the quantum regression theorem [9] [Corollary 2.2.1] does hold.

So far for the ordinary setting. Next we fix the factorial N of some infinite hypernatural number and we let $T = \{K/N : K \in {}^*\mathbb{N} \text{ and } 0 \leq K \leq N\}$. Let $\mathcal{A} = (A, (j_t : B \to A)_{t \in T}, \phi)$ be an internal S-continuous adapted Markov process with an internal family $\mathcal{E} = \{E_t : A \to j_t(B)\}_{t \in T}$ of conditional expectations.

We have previously remarked that the ordinary nsp $\widehat{\mathcal{A}} = (\widehat{A}, (\widehat{j}_t : \widehat{B} \to \widehat{A})_{t \in {}^{\circ}T}, \widehat{\phi})$ is well-defined and that $\mathbb{Q} \cap [0,1] \subseteq {}^{\circ}T \subseteq [0,1]$. Furthermore, $\widehat{j_t(B)} = \widehat{j}_t(\widehat{B})$ holds for all $t \in T$ and the map $\widehat{E}_t : \widehat{A} \to \widehat{j}_t(\widehat{B})$ given by $\widehat{E}_t(\widehat{a}) = \widehat{E_t(a)}$ is a well-defined conditional expectation. Under the assumption that the family $\mathcal{E}$ is S-continuous, namely $E_s \approx E_t$ whenever $s \approx t$, it follows that the map $\widehat{E}_{{}^{\circ}t} : \widehat{A} \to \widehat{j}_t(\widehat{B})$ defined by $\widehat{E}_{{}^{\circ}t}(\widehat{a}) = \widehat{E_t(a)}$ is well-defined. Moreover, the family $\{\widehat{E}_s : s \in {}^{\circ}T\}$ satisfies E2 and E3 of Definition 9 and the Markov property M'. Hence $\widehat{\mathcal{A}}$ is an ordinary adapted noncommutative Markov process with conditional expectations. It seems that the adaptedness property of the internal process $\mathcal{A}$ is needed in order to get the above conclusion, due to the already mentioned fact that the nonstandard hull construction, in general, does not behave well with respect to the operation of forming the C^*-algebra generated by family of subalgebras of a given algebra.

7. Nonstandard Fock Spaces

In most cases nonstandard universes are used to derive results about the standard universe. Some authors go beyond that. For instance, in [10], the author contends that "a nonstandard universe has physical significance in its own right" and, more specifically, "the states and observables of the nonstandard Fock space have physical significance". Admittedly, the author does not elaborate much on his statements in quotations.

In this short section we derive from standard results that each element of the nonstandard extension of the free Fock space is infinitely close to some "simple" element of a nonstandard free toy Fock space, in a sense that we make precise in the following.

Let H be a complex Hilbert space. We let

$$F(H) = \bigoplus_{n \in \mathbb{N}} H^{\otimes n},$$

where, for $0 < n$, $N^{\otimes n}$ is the n-fold tensor product of H and $H^{\otimes 0}$ is a one-dimensional space which is often denoted by $\mathbb{C}\Omega$. Here Ω is a distinguished unit vector, called the vacuum vector. Recall that the elements of $F(H)$ are of the form $(h_n)_{n \in \mathbb{N}}$, with $h_n \in H^{\otimes n}$ for all $n \in \mathbb{N}$ and $\sum_{n \in \mathbb{N}} \|h_n\|_n^2 < \infty$, where $\| \, \|_n$ is the norm on $H^{\otimes n}$. The space $F(H)$ is equipped with the norm $\| \, \|$ defined by

$$\|(h_n)_{n \in \mathbb{N}}\| = \big(\sum_{n \in \mathbb{N}} \|h_n\|_n^2 \big)^{\frac{1}{2}}.$$

In the following by the free Fock space we mean the space $F(L^2(\mathbb{R}_{\geq 0}, \mathbb{C}))$. We denote the latter by Φ. Regarding the physical import of Φ, we just say that it describes the quantum states of a number of identical particles from the single particle Hilbert space $L^2(\mathbb{R}_{\geq 0}, \mathbb{C})$.

We write an element $f \in \Phi$ as $f_0 \Omega + \sum_{0 < n \in \mathbb{N}} f_n$, where $f_0 \in \mathbb{C}$; Ω is the vacuum vector and, for all $0 < n \in \mathbb{N}$, $f_n \in L^2(\mathbb{R}_{\geq 0}, \mathbb{C})^{\otimes n}$.

Next we introduce the *free toy Fock space*. For each $i \in \mathbb{N}$, let $\mathbb{C}_i^2$ be an isomorphic copy of $\mathbb{C}^2$ and let $\{\Omega_i = (1,0)^\top, X_i = (0,1)^\top\}$ be the standard basis of $\mathbb{C}_i^2$. (Here and in the following we write vectors as column vectors.) The free toy Fock space $T\Phi$ is defined as

$$\mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \dots \neq i_n} (\mathbb{C}X_{i_1} \otimes \cdots \otimes \mathbb{C}X_{i_n}),$$

where Ω is the identification of the vacuum vectors Ω_i. As pointed out in [21], there is a one-to-one correspondence between the orthonormal basis of $T\Phi$ which is naturally associated to the construction of the latter and the set of all finite sequences $(i_1, \dots, i_n) \in \mathbb{N}^n$, $n \in \mathbb{N}$, and $i_1 \neq i_2 \neq \dots \neq i_n$.

It can be shown quite easily that $T\Phi$ can be embedded into Φ (see [21] [§5]). More interestingly for us, in [21] [§6] the authors construct a sequence of toy Fock spaces that approximate Φ. We recast the authors' main result in the framework of a nonstandard universe. First of all we notice that, by transfer, the nonstandard extension $^*\Phi$ of Φ is the internal norm closure of the internal direct sum of the Hilbert spaces $(^*L^2(\mathbb{R}_{\geq 0}, \mathbb{C}))^{\otimes N}$, $N \in {}^*\mathbb{N}$.

Let $K \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $0 = t_0 < t_1 < \cdots < t_N < \dots$, $N \in {}^*\mathbb{N}$, be an internal partition of $^*\mathbb{R}_{\geq 0}$ such that, for all $N \in {}^*\mathbb{N}$, $t_{N+1} - t_N = 1/K$. For each $N \in {}^*\mathbb{N}$ let X_N be the normalized characteristic function of the interval $[t_N, t_{N+1})$, namely the function

$$\frac{\mathbb{1}_{[t_N, t_{N+1})}}{\sqrt{t_{N+1} - t_N}}.$$

We form the internal toy Fock space

$$T_i\Phi = {}^*\mathbb{C}\Omega \oplus \bigoplus_{^*\mathbb{N} \ni N \geq 1} \bigoplus_{i_1 \neq \dots \neq i_N} ({}^*\mathbb{C}X_{i_1} \otimes \cdots \otimes {}^*\mathbb{C}X_{i_N}),$$

where the innermost direct sum is intended to range over all internal N-tuples $(i_1, i_2, \dots, i_N)$ of hypernaturals such that $i_1 \neq i_2 \neq \dots \neq i_N$.

Let $P : {}^*\Phi \to {}^*\Phi$ be the internal orthogonal projection onto $T_i\Phi$. We apply [21] [Theorem 1(1)] to the sequence of partitions $(\mathcal{S}_n)_{0 < n \in \mathbb{N}}$, where S_n has constant step $1/n$. By Transfer and by the nonstandard characterization of convergence of a sequence we get that $P(f) \approx f$, for all $f \in {}^*\Phi$. It follows that, up to an infinitesimal displacement, we can regard each $f \in {}^*\Phi$ as a hyperfinite (hence: A formally finite) sum of pairwise orthogonal elements, each belonging to some of the direct summands that occur in the definition of $T_i\Phi$.

Moreover, since the supremum of an internal set of infinitesimals is itself an infinitesimal, we also get $P \approx \mathrm{id}_{\Phi^*}$. Hence, by passing to nonstandard hulls and by writing $\widehat{\Phi}$ for $\widehat{{}^*\Phi}$ as is usual, the map $\widehat{P} : \widehat{\Phi} \to \widehat{\Phi}$ defined by $\widehat{f} \mapsto \widehat{P(f)}$, for $f \in \mathrm{Fin}({}^*\Phi)$, is just $\mathrm{id}_{\widehat{\Phi}}$. As a consequence we get that $\widehat{\Phi} = \widehat{T_i \Phi}$. Notice that the latter equality provides an equivalent definition of $\widehat{\Phi}$. In particular, every element of $\widehat{\Phi}$ can be lifted to some hyperfinite sum of the form described above.

By similar arguments, and in light of [21] [Theorem 1(2)], we can approximate up to an infinitesimal displacement the creation and the annihilation operator on ${}^*\Phi$ by means of hyperfinite sums involving the discrete counterparts of those operators defined on $T_i \Phi$. See [21] for details.

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