STRICT INCLUSIONS OF HIGH RANK LOCI

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ABSTRACT. For a given projective variety X, the high rank loci are the closures of the sets of points whose X-rank is higher than the generic one. We show examples of strict inclusion arising from two consecutive high rank loci. Our first example comes from looking at the Veronese surface of plane quartics. Although Piene had already shown an example in which X is a curve, we construct infinitely many curves in $\mathbb{P}^4$ for which such strict inclusion appears. For space curves, we give two criteria to check whether the locus of points of maximum rank 3 is finite (possibly empty).

1. INTRODUCTION

Problem formulation. For any complex projective non-degenerate variety $X \subset \mathbb{P}^N$, there is a well-established notion of X-rank of a point $p \in \langle X \rangle = \mathbb{P}^N$:

$$\mathrm{rk}_X(p) = \min\{r \mid p \in \langle p_1, \ldots, p_r \rangle, \text{ where } p_i \in X\}.$$

The Zariski closure of the set of points of given rank k,

$$W_k = \overline{\{p \in \mathbb{P}^N \mid \mathrm{rk}_X(p) = k\}},$$

has been studied since decades when the integer k is smaller than or equal to the so-called *generic rank*. The generic rank is defined to be the smallest integer g such that $W_g = \langle X \rangle = \mathbb{P}^N$.

When $k \leq g$, the variety W_k coincides with $\sigma_k(X) := \overline{\cup_{p_1,\ldots,p_k \in X} \langle p_1, \ldots, p_k \rangle}$, the k-th secant variety of X. The cases when $k > g$ are by far less studied and yet of great interest. In [BHMT18], Buczyński, Han, Mella, and Teitler made the first systematic study of the loci W_k in the range $k > g$, i.e. the loci of points whose X-rank is higher than the generic one. It is worth noting that, in general, even the existence of such points is not known. In [BHMT18, Theorem 3.1], the authors show the following inclusion:

$$W_k + X \subseteq W_{k-1}, \tag{1}$$

where $W_k + X$ denotes the join between W_k and X, for all $g + 1 \leq k \leq m := \max\{\mathrm{rk}_X(p) \mid p \in \langle X \rangle\}$. This naturally gives rise to the following:

Problem 1. Do there exist instances where inclusion (1) is strict?

Related works and questions. The connection of the problem to secant varieties makes it extremely interesting from a purely geometric standpoint. In terms of applications, one of the most important cases where a solution to this problem will be of great benefit is when X is a variety parameterizing tensors, i.e. Segre and Veronese varieties,

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2010 *Mathematics Subject Classification.* (Primary) 14N07, 14N05, 15A72.

Key words and phrases. High tensor rank, Symmetric tensor rank, Zero-dimensional schemes, Cactus varieties, Projections of curves.

Grassmannians, along with their projections. Tensor ranks and symmetric tensor ranks have recently attracted a lot of attention, because of their natural appearance in several pure and applied contexts [Lan12, Lan17, KB09, CGLM08, CGO14, BBC$^+$19, BCC$^+$18]. One of the most intriguing open problems in the tensor decomposition literature is determining the maximum rank; see [BT15] for an elegant and updated exposition on this topic. Generally speaking, only bounds for the maximum rank are known to this date. Applied areas where Problem 1 is relevant are, for instance, complexity theory [Lan17], phylogenetics [KB09], and quantum physics [CZ19].

Tensors of high rank have been studied in several papers in the literature, e.g. [BT16, BGI11, De 17, LT10, Jel14, BB17]. However, the very first general results on high ranks have been pursued in [BHMT18], where the recursive containment (1) was proven.

Contributions of the paper. Let $X = \nu_3(\mathbb{P}^2) \subset \mathbb{P}^9$ be the Veronese surface of plane cubics, i.e. each point of X may be regarded as the third power of a linear form in three variables. The maximum X-rank is 5 (see e.g. [BGI11, LT10]) and it is only attained by all reducible cubics whose components are a smooth conic and a tangent line to it: those whose normal form is $y(x^2 + yz)$ (cf. [LT10]). In this case the X-generic rank is 4 (cf. [AH95]). Moreover, one can show that $\dim W_5 = 6$ [BHMT18, Example 4.11]. In this case, we have:

$$W_5 + X = W_4 = \sigma_4(X) = \mathbb{P}^9.$$

We will sketch an idea of the proof of this fact in Example 2.2.

One instance of strict inclusion $W_k + X \subsetneq W_{k-1}$ was already known in the literature, although stated in a different way. In 1981, Piene proved the existence of a smooth and non-degenerate degree 4 rational curve $X \subset \mathbb{P}^3$ with W_3 nonempty and finite ([Pie81, Case a_1 at p. 101]). Since $\sigma_2(X) = \mathbb{P}^3$ for any non-degenerate curve X, one has $W_2 = \mathbb{P}^3$ and $\dim(W_3 + X) = 2$. We elaborate more on this in Example 2.3.

In this article, our aim is to provide further examples where the containment in Problem 1 is strict. Our constructions are very classical and geometric.

Theorem 1.1. *Let $X = \nu_4(\mathbb{P}^2) \subset \mathbb{P}^{14}$ be the Veronese surface of plane quartics, and let W_7 be the maximum X-rank locus. Then:*

$$W_7 + X \subsetneq W_6 = \sigma_6(X) = \mathbb{P}^{14}.$$

Theorem 1.2. *Fix integers d, p_g such that $p_g \geq 0$ and $d \geq 3$. Let $X \subset \mathbb{P}^3$ be a smooth and non-degenerate curve of degree d and genus p_g. Assume either $23p_g < d^2 - 3d - 15$, or, when d is even, $p_g < (3d^2 - 18d + 16)/16$. Then W_3 is infinite if and only if $p_g = 0$ and X is a smooth rational curve projectively equivalent to the curve parametrized by $x_0 = z_0^d$, $x_1 = z_1 z_0^{d-1}$, $x_2 = a_2 z_1^{d-1} z_0$, $x_3 = a_3 z_1^d$ with $a_2 a_3 \neq 0$. Therefore, when W_3 is finite and nonempty,*

$$W_3 + X \subsetneq W_2 = \sigma_2(X) = \mathbb{P}^3.$$

See Example 4.1 for a discussion on the curves appearing in Theorem 1.2.

Theorem 1.3. *For every $d \geq 5$, there exists a degree d rational curve $X \subset \mathbb{P}^4$ such that $\dim W_4 = 1$. Thus*

$$W_4 + X \subsetneq W_3 = \sigma_3(X) = \mathbb{P}^4.$$

Besides the results, we believe that our contribution is also interesting from the perspective of the methods employed: we use a wide range of classical projective geometric tools (e.g. projections, singularities of plane and space curves, Hirzebruch surfaces) in order to show that certain points possessing high rank with respect to the given X exist. Moreover, the geometric behavior of these special points might give rise to further investigations and new research paths.

Structure of the paper. In §2, we briefly discuss some preliminary examples: when $X = \nu_3(\mathbb{P}^2)$ and Piene's curve. In §3, we prove Theorem 1.1. In §4, we discuss space curves and prove Theorem 1.2. In §5, we construct rational curves of degree $d \geq 5$ in $\mathbb{P}^4$ with $\dim W_4 = 1$, thus proving Theorem 1.3.

Acknowledgements. We thank J. Buczyński, K. Han, M. Mella, and Z. Teitler for useful discussions. E. Ballico and A. Bernardi were partially supported by MIUR and GNSAGA of INdAM (Italy). E. Ventura would like to thank the Department of Mathematics of Università di Trento, where part of this project was conducted, for the warm hospitality. E. Ventura would like to thank all the participants of the secant varieties working group held at IMPAN in the Fall 2018, during the special semester *Varieties: Arithmetic and Transformations*, for the inspiring atmosphere. E. Ventura especially thanks J. Buczyński for leading this working group and acknowledges IMPAN for financial support. Finally, we thank the anonymous referees for their useful comments that substantially improved the exposition of the paper.

2. SOME PRELIMINARY EXAMPLES

For any projective scheme $Z \subset \mathbb{P}^N$, let $\langle Z \rangle$ denote its projective span. For any $X \subset \mathbb{P}^N$ and any point $q \in \mathbb{P}^N$, the *border rank* of q (with respect to X) is the minimal positive integer k such that $q \in \sigma_k(X)$.

Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{n} - 1$, be the Veronese embedding of the projective space $\mathbb{P}^n$ into the projective space of homogeneous polynomials of degree d in $n+1$ variables; the map ν_d sends every point of $\mathbb{P}^n$ (regarded as homogeneous linear polynomial in $n+1$ variables) to its d-th power. When $X = \nu_d(\mathbb{P}^n) \subset \mathbb{P}^N$, the border rank with respect to X of a homogeneous polynomial $q \in \mathbb{P}^N$ is often called the symmetric border rank of q.

Denote a zero-dimensional degree k scheme supported at a point $p \in \mathbb{P}^n$ with $Z(k, p)$.

Definition 2.1 ([BR13, RS11])**.** Let $f \in \mathbb{P}^N$ be a homogeneous polynomial of degree d in $n+1$ variables. The *cactus rank* of f is the minimal degree of a zero-dimensional scheme $Z \subset \mathbb{P}^n$ such that $f \in \langle \nu_d(Z) \rangle$. We say that such a Z *evinces* the cactus rank of f.

Example 2.2 ($X = \nu_3(\mathbb{P}^2)$)**.** We sketch a possible approach to see $W_5 + X = W_4 = \sigma_4(X) = \mathbb{P}^9$. First, recall that every scheme Z evincing the cactus rank of an element of W_5 is a scheme $Z(3, p)$ supported on a smooth conic ([BGI11, first part of the proof of Theorem 37]). Therefore, in order to show the equality it is enough to prove that a general cubic is in the span of a $Z(3, p)$ supported on a smooth conic and a simple point $Z(1, q)$.

Let $f \in \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(3)))$ be a general cubic. Its symmetric rank is 4. The variety parametrizing the schemes Z of degree 4 such that $f \in \langle Z \rangle$ is the so called *variety of sums of powers*, $\mathrm{VSP}(f,4)$ (see [RS00] for more on these classical varieties). For a general cubic f, it is a classical result that $\mathrm{VSP}(f,4) \cong \mathbb{P}^2$.

For a given form $f \in \mathbb{P}(\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^n}(d)))$, its *apolar ideal* $f^\perp$ is the homogeneous ideal in the ring of polynomial differential operators $T = \mathbb{C}[\partial_0, \ldots, \partial_n]$ generated by all $g \in T$ such that $g \circ f = 0$, where $\circ$ denotes the usual differentiation. For a general cubic f, its apolar ideal $f^\perp$ is generated in degree 2; the degree 2 homogeneous part $\left(f^\perp\right)_2$ is a net of conics with empty base locus.

The $\mathrm{VSP}(f,4)$ can be realized as the image of the regular map $\varphi_f : \mathbb{P}^2 \to \mathbb{P}^2$ defined by the net of conics $\left(f^\perp\right)_2$. The morphism φ_f is a generically $4:1$ map and its image is $\mathrm{VSP}(f,4)$; see [RS00, (2.5)].

The branch locus $\mathcal{B}_f$ of φ_f is a sextic curve, that is the dual curve of the Hessian $H(f)$ of the cubic f (cf. [MMSV17, §3]). Following the classical De Paolis algorithm (cf. [De 86]), fix the tangent line q to one of the nine flexes of $H(f)$. Then $\deg(q \cap H(f)) = 3$ and the intersection is supported at one point p. Under duality, the line q corresponds to a cusp $c_q \in \mathcal{B}_f$. Its fiber $\varphi_f^{-1}(c_q)$ is the scheme $Z = Z(3,p) + Z(1,q)$. By construction, Z is the intersection of two conics from the vector space $\langle \left(f^\perp\right)_2 \rangle \subset f^\perp$. By the apolarity lemma [IK99, Lemma 1.15], Z spans f.

Example 2.3 (Piene's curve [Pie81]**).** Consider a general projection of the rational normal curve $\nu_4(\mathbb{P}^1) \subset \mathbb{P}^4$ onto a rational curve $X \subset \mathbb{P}^3$. Recall that the maximum X-rank is 3 (see e.g. [Pie81, Case 3, p. 100], [LT10, Proposition 5.1], [BB11]). By [Pie81, Case a_1, p. 101], the only cuspidal planar projection of X is obtained by projecting X from the *unique* point $p \in \mathbb{P}^3$ of intersection of an ordinary tangent with a stall tangent. For the reader who is not familiar with this notation, a stall tangent is the tangent line at a *stall* point of X, i.e., a point whose local parametrization is:

$$\begin{aligned} x &= at + \cdots, \\ y &= b_2t^2 + b_3t^3 + \cdots, \\ z &= c_4t^4 + c_5t^5 + \cdots, \end{aligned} \tag{2}$$

with $ab_2c_4 \neq 0$ and $b_2c_5 \neq b_3c_4$ (see [Pie81, Lemma 1, p. 98]). The projected curve is a rational planar quartic curve with an ordinary cusp (arising from the ordinary tangent) and a ramphoid cusp of the 1st type [Pie81, Remark, p. 98] (arising from the stall tangent). Since a projection from a point $p \in \mathbb{P}^3$ is injective if and only if $\mathrm{rk}_X(p) > 2$, we have that $W_3 = \{p\}$. Therefore $W_3 + X \subsetneq \sigma_2(X) = \mathbb{P}^3$.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Here $X = \nu_4(\mathbb{P}^2)$ is the Veronese surface of plane quartics.

Since $W_6 = \sigma_6(X) = \mathbb{P}^{14}$, in order to prove the inclusion between $W_7 + X$ and W_6 it is sufficient to show that $W_7 + X$ does not fill the ambient space.

Notice that plane quartics of border rank 5 are expected to fill the ambient space; however it is a classical result that they fail to do that, whereas $W_6 = \mathbb{P}^{14}$ (cf. [AH95]).

Recall that the *k-th cactus variety of X* is the closure of the union of the scheme-theoretic linear spans of all zero-dimensional subschemes of X of degree at most k (cf. [BB14, BR13, BJMMR18]); the k-th cactus variety contains the k-th secant variety of X. Thus the 6-th cactus variety of $\nu_4(\mathbb{P}^2)$ fills the ambient space as well. Now, Kleppe's classification [Kle99, Chapter 3] of quartics in W_7 shows that there is no $f \in W_7$ of symmetric border rank 6. All the normal forms of [Kle99] with symmetric rank 7 were also classified in [BGI11] and they all turn out to have cactus rank smaller than or equal to 5.

As in §2, we denote a zero-dimensional degree k scheme supported at a point $p \in \mathbb{P}^2$ with $Z(k,p)$. In [BGI11, Theorem 44] the stratification by symmetric rank of $\sigma_s(X) \setminus \sigma_{s-1}(X)$ for $s = 2,3,4,5$, is derived. Symmetric rank 7 arises in cactus ranks 3, 4 and 5.

- For cactus rank 3, there is one possible scheme:
  (I) $Z = Z(3,p)$ contained in a smooth conic.
- For cactus rank 4, there are the two possible schemes:
  (IIa) $Z = Z(2,p) + Z(2,o)$ (two 2-jets supported at p, o);
  (IIb) $Z = Z(2,p) + Z(1,o) + Z(1,q)$ (a 2-jet supported at p and two simple points o, q).
- For cactus rank 5, there are three possible schemes:
  (IIIa) $Z = Z(3,p) + Z(2,o)$ (a 3-jet supported at p and a 2-jet supported at o);
  (IIIb) $Z = Z(4,p) + Z(1,o)$ contained in a double line: its homogeneous ideal in $S = \bigoplus_{k\geq 0} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(k))$ is of the form $\mathcal{I}_Z = (Q, y^2) \cap (x-z, y)$, where Q is either a smooth conic whose tangent line at p coincides with $\{y=0\}$, or a reducible conic with vertex p;
  (IIIc) $Z = Z(2,p) + Z(2,o) + Z(1,q)$ contained in a double line: its homogeneous ideal in $S = \bigoplus_{k\geq 0} \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^2}(k))$ is of the form $\mathcal{I}_Z = (x,y) \cap (x^2 - z^2, y^2)$.

We first consider the case (IIIc), since it is easier to deal with it differently than all the other cases. Let $g \in W_7 + X$ be such that $g \in \langle \nu_4(Z), \nu_4(Z(1,t))\rangle$, where $Z = Z(2,p) + Z(2,o) + Z(1,q)$ and $Z(1,t) \in \mathbb{P}^2$. As shown at the end of the proof of [BGI11, Theorem 44], the scheme $\nu_4(Z(2,p) + Z(2,o))$ is contained in a rational normal curve $C_8 = \nu_8(\mathbb{P}^1)$ of degree 8. Since both $Z(2,p)$ and $Z(2,o)$ are 2-jets, one sees that $\langle \nu_4(Z(2,p)), \nu_4(Z(2,o))\rangle \subset \sigma_2(\tau(C_8))$, where $\tau(C_8)$ is the tangential surface of C_8.

Thus $g \in \sigma_2(\tau(C_8)) + \sigma_2(X)$. Now, $\dim(\sigma_2(\tau(C_8))) = 5$ [CGG, Proposition 3.1] and $\dim \sigma_2(X) = 5$. Therefore, one has $\dim(\sigma_2(\tau(C_8)) + \sigma_2(X)) \leq 11$. Let Υ_{IIIc} be the closure of the set of all $f \in W_7$ such that $f \in \langle \nu_4(Z)\rangle$, where Z is as in (IIIc). We have shown that $\dim(\Upsilon_{\mathrm{IIIc}} + X) \leq 11$.

Let $\mathrm{Hilb}_k(\mathbb{P}^2)$ be the Hilbert scheme parametrizing zero-dimensional schemes $Z \subset \mathbb{P}^2$ of degree k. Let Υ_7 be the subset of

$$\mathcal{H} = \bigcup_{3\leq k\leq 5} \mathrm{Hilb}_k(\mathbb{P}^2) \times \mathbb{P}^{14} = \left\{(Z,f),\ \text{where } Z \subset \mathbb{P}^2, \deg(Z) = k, f \in \langle \nu_4(Z)\rangle\right\}$$

consisting of all $(Z,f) \in \mathcal{H}$, such that $\mathrm{rk}_X(f) = 7$ and where Z is isomorphic to one of the schemes from (I) to (IIIb). The set Υ_7 is a constructible set by a theorem of Chevalley [Har77, II.Ex.3.18, II.Ex.3.19], i.e. it is a finite union of finitely many locally closed subsets of $\mathcal{H}$. The integer $\dim \Upsilon_7$ is by definition the maximum of the dimensions of these locally closed subsets (for any such a finite union giving Υ_7). With this definition, we have $\dim \Upsilon_7 = \dim \overline{\Upsilon_7}$.

Then W_7 is the union of the closure of the projection $\mathrm{pr}_2(\Upsilon_7)$ of Υ_7 to $\mathbb{P}^{14}$ and Υ_{IIIc}. For each one of the schemes (I)-(IIIb) above, we give an upper bound for the number of parameters involved:

- (I) $(Z, \langle \nu_4(Z)\rangle)$ is parametrized by the choice of a conic $C \in \mathbb{P}^5$ ($\mathbf{+5}$), a point $p \in C$ ($\mathbf{+1}$), and the span $\langle \nu_4(Z)\rangle$ ($\mathbf{+2}$). Thus this gives a parameter space of dimension at most 8.
- (IIa) $(Z, \langle \nu_4(Z)\rangle)$ is parametrized by the choice of two points $p, o \in \mathbb{P}^2$ ($\mathbf{+4}$), two lines L, R passing through each of them ($\mathbf{+2}$), and the span $\langle \nu_4(Z)\rangle$ ($\mathbf{+3}$). Thus this gives a parameter space of dimension at most 9.
- (IIb) $(Z, \langle \nu_4(Z)\rangle)$ is parametrized by the choice of three points $p, o, q \in \mathbb{P}^2$ ($\mathbf{+6}$), a line passing through one of them ($\mathbf{+1}$), and the span $\langle \nu_4(Z)\rangle$ ($\mathbf{+3}$). Thus this gives a parameter space of dimension at most 10.
- (IIIa) $(Z, \langle \nu_4(Z)\rangle)$ is parametrized by the choice of two lines ($\mathbf{+4}$), one point on each of them ($\mathbf{+2}$), and its span ($\mathbf{+4}$). Thus this gives a parameter space of dimension at most 10.
- (IIIb) Suppose Q is reducible. Since Q has vertex p, the parameter space for Q is $\mathbb{P}^2$. In this case, $(Z, \langle \nu_4(Z)\rangle)$ is parametrized by the choice of a line ($\mathbf{+2}$), two points p, o on it ($\mathbf{+2}$), a reducible quadric with vertex at p ($\mathbf{+2}$), and its span ($\mathbf{+4}$). Thus this gives a parameter space of dimension at most 10.

  Suppose Q is a smooth conic and so $Q \in \mathbb{P}^5$. Choose a smooth conic Q and a point $p \in Q$. The tangent line $\{y = 0\}$ at p to Q is determined by $Z(4, p)$. So far we have 6 parameters. However, note that $h^0(\mathcal{I}_{Z(4,p)}(2)) = 2$ and hence there is a $\mathbb{P}^1$ of generically smooth conics providing the same scheme $Z(4, p)$. Thus the parameters are in fact 5 ($\mathbf{+5}$). Now choose a point $o \in \{y = 0\}$ ($\mathbf{+1}$), and the span of Z ($\mathbf{+4}$). Hence the parameter space for $(Z, \langle \nu_4(Z)\rangle)$ has dimension at most 10.

Consequently, for all these cases, we have $\dim \Upsilon_7 \leq 10$ and so $\dim(\mathrm{pr}_2(\Upsilon_7)) \leq 10$. Since the dimension of any join is at most the sum of the dimensions of the varieties involved plus one, one has $\dim(W_7 + X) = \max\{\dim(\mathrm{pr}_2(\Upsilon_7) + X), \dim(\Upsilon_{\mathrm{IIIc}} + X)\} \leq 13 < \dim W_6 = 14$.

4. Space curves

For a finite set A, let $\#A$ denote its cardinality. For a zero-dimensional scheme Z, let $\#Z$ denote the cardinality of its support Z_{red}. We start by discussing the rational curves appearing in Theorem 1.2.

Example 4.1. Let $X \subset \mathbb{P}^3$ be a smooth rational curve of degree $d \geq 4$. Assume the existence of a line $L \subset \mathbb{P}^3$ such that $\deg(X \cap L) \geq d - 1$ and $(X \cap L)_{\mathrm{red}} = \{p\}$, i.e., it is set-theoretically a single point. Therefore $\mathrm{rk}_X(q) = 3$ for any $q \in L \setminus \{p\}$.

To see this, first notice that $\mathrm{rk}_X(q) > 1$ since $q \in L \setminus X \cap L$. By [LT10, Proposition 5.1] we have $\mathrm{rk}_X(q) \leq 3$. Assume $\mathrm{rk}_X(q) = 2$ and take $A \subset X$ such that $\#A = 2$ and $q \in \langle A\rangle$. Since, by hypothesis, $\#(X \cap L) = 1$, L cannot coincide with $\langle A\rangle$, moreover $q \in \langle A\rangle \cap L$ therefore $\langle A\rangle \cup L$ spans a plane Π. If $p \notin A$, then $\deg(X \cap \Pi) \geq 2 + \deg(X \cap L) > d$, a contradiction with $\deg X = d$. If $p \in A$, then $\langle A\rangle \cap L = \{p\}$, because the lines are distinct, i.e., $L \neq \langle A\rangle$. Thus $q = p$, a contradiction.

Construction. All triples (X, L, p) as in the example above can be constructed as follows. The aim is to give a degree d embedding $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^3$ and $v \in \mathbb{P}^1$ such that the tangent line $L = T_pX$ has order of contact $d-1$ with X at $p = \varphi(v)$.

Let z_0, z_1 be homogeneous coordinates of $\mathbb{P}^1$, and x_0, x_1, x_2, x_3 be the ones of $\mathbb{P}^3$. Up to projective automorphisms of $\mathbb{P}^1$ and $\mathbb{P}^3$, we may assume $v = (1 : 0)$, $p = (1 : 0 : 0 : 0)$, $T_pX = \{x_2 = x_3 = 0\}$, and $O_pX = \{x_3 = 0\}$ be the tangent line and the osculating plane of X at p respectively. Up to the above actions, we may further assume the morphism φ is defined by $x_0 = z_0^d$, $x_1 = a_1z_1z_0^{d-1}$, $x_2 = a_2z_1^{d-1}z_0 + bz_1^d$, $x_3 = a_3z_1^d$ with $a_1a_2a_3 \neq 0$. Since $a_3 \neq 0$, we may further reduce to the case $b = 0$. Taking the automorphism $(z_0 : z_1) \mapsto (z_0 : tz_1)$, with $t = a_1^{-1}$, we may also assume $a_1 = 1$. Conversely, any parametrization defined by $x_0 = z_0^d$, $x_1 = z_1z_0^{d-1}$, $x_2 = a_2z_1^{d-1}z_0$, $x_3 = a_3z_1^d$ with $a_2a_3 \neq 0$ gives the desired rational curve.

Proof of Theorem 1.2: One direction is clear from Example 4.1. For the converse, assume W_3 is infinite. Since W_3 is a closed algebraic subset of $\mathbb{P}^3$, there is an irreducible curve $\Gamma \subset \mathbb{P}^3$ such that $\Gamma \subseteq W_3$ and $\mathrm{rk}_X(q) = 3$ for a general $q \in \Gamma$. Recall that $W_4 = \emptyset$ by [LT10, Proposition 5.1]. Let $\tau(X) \subset \mathbb{P}^3$ denote the tangential variety of X, i.e., (since X is smooth) the union of all tangent lines T_pX, $p \in X$. This is an integral surface containing X in its singular locus. Take an arbitrary point $a \in \mathbb{P}^3$ such that $\mathrm{rk}_X(a) > 2$. Since $\sigma_2(X) = \mathbb{P}^3$, one necessarily has $a \in \tau(X) \setminus X$; hence there exists $p \in X$ such that $a \in T_pX$.

Fix a general $q \in \Gamma \subseteq W_3$. Let $\pi_q : \mathbb{P}^3 \setminus \{q\} \longrightarrow \mathbb{P}^2$ denote the linear projection away from q. Since $\mathrm{rk}_X(q) > 1$, $q \notin X$ and so $\pi_{q|X}$ is a morphism. Set $D_q := \pi_q(X)$. Note that, since $\mathrm{rk}_X(q) > 2$, the map $\pi_{q|X}$ is injective. Thus D_q is a degree d plane curve with geometric genus p_g (its normalization is X).

Call $\mathbb{T}(q)$ the set of all $p \in X$ such that $q \in T_pX$. Since $\pi_{q|X}$ is injective, we have $\mathrm{Sing}(D_q) = \pi_q(\mathbb{T}(q))$. Fix $p \in \mathbb{T}(q)$ (we do not claim that such a p is unique). Again, since $\pi_{q|X}$ is injective on points, we have that set-theoretically $\{p\} = T_pX \cap X$.

Let t be a uniformizing parameter of the complete local ring $\hat{\mathcal{O}}_{X,p} \cong \mathbb{K}[[t]]$. We may choose an affine coordinate system x, y, z around p such that X is locally given by the formal power series:

$$
\begin{aligned}
x &= at^{l_0+1} + \cdots ,\\
y &= bt^{l_1+2} + \cdots ,\\
z &= ct^{l_2+3} + \cdots ,
\end{aligned} \tag{3}
$$

where $abc \neq 0$ and $0 \leq l_0 \leq l_1 \leq l_2$; see, e.g., [Pie81, §2]. (The smoothness of X at p is equivalent to $l_0 = 0$.) In these coordinates, $T_pX = \{y = z = 0\}$. Two possibilities arise: either $\Gamma = T_pX$ or $\Gamma \neq T_pX$.

Assume $\Gamma = T_pX$. The linear projection $\pi_{T_pX} : \mathbb{P}^3 \setminus T_pX \longrightarrow \mathbb{P}^1$ induces a non-constant morphism $\psi : X\setminus T_pX \longrightarrow \mathbb{P}^1$. Since X is smooth, ψ extends to a non-constant morphism

$\psi' : X \longrightarrow \mathbb{P}^1$. Since $\{p\} = T_pX \cap X$ (set-theoretically), we have $\deg(T_pX \cap X) = l_1 + 2$ and hence $\deg(\psi') = d - l_1 - 2$. Since $X \neq T_pX$, we have $\deg(\psi') \geq 1$.

1. Assume $\deg(\psi') = 1$, i.e., $\psi' : X \longrightarrow \mathbb{P}^1$ is a birational morphism. Thus $g = 0$. We obtain $\deg(T_pX \cap X) = d-1$ and hence we are in the assumptions of Example 4.1.
2. Assume $\deg(\psi') \geq 2$. Since in characteristic zero the affine line $\mathbb{A}^1$ is algebraically simply connected (i.e., it does not admit any nontrivial étale covering), the morphism ψ' has at least two distinct ramification points. Thus, besides $\{p\} = X \cap T_pX$, there exists a tangent line R_q to X, such that $R_q \neq T_pX$ and $q \in R_q$. Thus $\{q\} = T_pX \cap R_q$. Varying q in T_pX, we derive that T_pX meets a general tangent line of X, i.e., the differential of ψ' is identically zero on X, a contradiction as we are in characteristic zero.

Thus, one concludes that $\Gamma \neq T_pX$. This means that a single T_pX cannot contain the curve $\Gamma \subseteq W_3$. Therefore, varying $q \in \Gamma$, the sets $\mathbb{T}(q)$ cover an open subset of X. Hence for a general $q \in \Gamma$, we find a point $p \in \mathbb{T}(q)$, whose sequence (l_0, l_1, l_2) as in (3) is $(0, 0, 0)$:

> *Claim.* For each $p \in \mathbb{T}(q)$, the sequence (l_0, l_1, l_2) is $(0, 0, 0)$.
>
> *Proof of the Claim.* Let $\mathcal{B}$ be the set of all $a \in X$ such that the sequence in (3) is not $(0, 0, 0)$, i.e., such that the osculating plane of X at a has order of contact > 3 with X at a (such a plane is also called *non-ordinary osculating plane*). The set $\mathcal{B}$ is finite. If it is empty, there is nothing to prove. Otherwise, we obtain the existence of $a \in \mathcal{B}$ such that $q \in T_aX$ for a general $q \in \Gamma$. Thus $T_aX = \Gamma$, which leads to a contradiction as above.

By the *Claim*, $\pi_q(p)$ is an ordinary cusp of the plane curve $D_q = \pi_q(X)$. The latter is then an integral degree d plane curve with *only* ordinary cusps as singularities. Since D_q has arithmetic genus $(d-1)(d-2)/2$ and geometric genus p_g, it has $(d-1)(d-2)/2 - p_g$ (ordinary) cusps. If $(d-1)(d-2)/2 - p_g > (21p_g + 17)/2$ (i.e., $23p_g < d^2 - 3d - 15$), we derive a contradiction with the results of Tono [Ton05]. (In fact, Tono's result is stronger, because it bounds the number of cusps in terms of the geometric genus p_g, without requiring the cusps being ordinary.)

Since our cusps are ordinary, there are other upper bounds for their number κ; see [Hir86, Ivi85]. (When the plane curve is rational, and one has the parameterization, there are algorithms to describe its singularities, see e.g. [BGI18].) In particular, if d is even, [Hir86, eq. 16] gives

$$\kappa \leq d(5d-6)/16. \tag{4}$$

Since in our case $\kappa = (d-1)(d-2)/2 - p_g$, for d even, we obtain $16p_g \geq 3d^2 - 18d + 16$, contradicting our assumption. □

By Castelnuovo's upper bound for non-degenerate curves [Har82, 3.12, 3.13, 3.14], the bounds on the arithmetic genus featured in Theorem 1.2 are quite good, although not optimal.

Remark 4.2. For $p_g = 1$, we cover all even integers ≥ 6, whereas we know that for $(d, p_g) = (4, 1)$ we have $\dim W_3 > 0$; see [BHMT18, Proposition 6.1] or [Pie81, Theorem 1]. For $p_g = 1$ and arbitrary d, we cover all $d \geq 9$.

Set $W_3^0 := \{q \in \mathbb{P}^3 \mid \mathrm{rk}_X(q) = 3\}$. As before, recall that for any $o \in \mathbb{P}^3$, $\pi_o : \mathbb{P}^3 \setminus \{o\} \longrightarrow \mathbb{P}^2$ denotes the linear projection away from o, and (since $o \notin X$), $\varphi_o := \pi_{o|X}$ is a morphism $\varphi_o : X \longrightarrow \mathbb{P}^2$.

Remark 4.3. As noticed in the proof of Theorem 1.2, the morphism φ_o is injective if and only if $o \in W_3^0$. Hence, if $o \in W_3^0$, $\varphi_o(X)$ is a plane curve of degree $\deg(X)$ and geometric genus $p_g(X)$ with only unibranch singularities.

Let $\Sigma(X)$ denote the set of all $o \in W_3^0$ such that $\varphi_o(X)$ has only ordinary cusps as singularities.

Remark 4.4. We give a description of the set $W_3^0 \setminus \Sigma(X)$ in terms of singularities of projections.

Fix $o \in W_3^0 \setminus \Sigma(X)$. By assumption, $\varphi_o(X)$ is a plane curve with degree $\deg(X)$, possessing only unibranch singularities, but with at least one non-ordinary cusp. Since X is smooth and φ_o is induced by a linear projection away from $o \notin X$, this non-ordinary singularity of $\varphi_o(X)$ corresponds to some $p \in X$ such that $o \in T_pX$ and the osculating plane O_pX to X at p has order of contact $\ell_2 + 3 > 3$ with X at p [Pie81, §2], i.e., O_pX is a non-ordinary osculating plane.

Theorem 4.5. *Let X be as in Theorem 1.2. Then $\Sigma(X)$ is finite. Moreover, $W_3^0 \setminus \Sigma(X)$ is infinite if and only if X is as in Example 4.1.*

Proof. First assume $\Sigma(X)$ is infinite. Since $\Sigma(X)$ is a constructible set and X has only finitely many non-ordinary osculating planes, we obtain the existence of $p \in X$ such that the tangent line $L := T_pX$ is contained in $\overline{\Sigma(X)}$ and $\Sigma(X)$ contains all but finitely many $o \in L$, i.e., there exists a finite set $A \subset L$ such that $W_3^0 \supseteq L \setminus A$. By definition of W_3^0, we also see that $L \cap (X \setminus \{p\}) = \emptyset$. Hence $\ell_1 + 2 := \deg(L \cap X)$ is the order of contact of L and X at p.

Now the proof proceeds exactly as the proof of Theorem 1.2: once we take the linear projection away from L, we derive a contradiction.

In conclusion, $\Sigma(X)$ is finite (possibly empty). Therefore $W_3^0 \setminus \Sigma(X)$ is infinite if and only if W_3^0 is infinite. Finally, apply Theorem 1.2 and conclude. □

5. Infinitely many curves in $\mathbb{P}^4$

This section is devoted to prove Theorem 1.3. For all integers $d \geq 5$, we construct a smooth, rational and non-degenerate degree d curve $X \subset \mathbb{P}^4$ such that $\dim W_4 = 1$. More precisely, we show $\dim W_4 < 2$ and that the locus W_4 contains a line.

Let $\mathbb{F}_1 \subset \mathbb{P}^4$ be the smooth rational ruled (cubic) surface, the first Hirzebruch surface [Har77, §V.2]; this is also the projection of the degree 4 Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ from a point on itself. Its Picard group is $\mathrm{Pic}(\mathbb{F}_1) \cong \mathbb{Z}^2$; we may take as a basis of $\mathrm{Pic}(\mathbb{F}_1)$, a fiber F of the ruling of $\mathbb{F}_1$ and the only integral curve $C_0 \subset \mathbb{F}_1$ with normal bundle of degree $C_0^2 = -1$. Note that $F^2 = 0$. The curve C_0 is a section of the ruling of $\mathbb{F}_1$ and so $C_0 \cdot F = 1$. The chosen embedding of this surface is the one given by the complete linear system $|C_0 + 2F|$. In this embedding, the section C_0 has degree one, i.e., it is a line. Moreover, all fibers $L \in |F|$ are lines and $\mathbb{F}_1$ contains no other line. Indeed, let $L \subset \mathbb{F}_1$ be an effective curve embedded as a line. Thus $L \in |aC_0 + bF|$ and so $1 = \deg(L) = (C_0 + 2F) \cdot (aC_0 + bF) = a + b$. If $L \neq F$ then $L \cdot F = 1$, and so $a = 1, b = 0$. However, since C_0 is a (-1)-curve and $L \in |C_0|$, one has $L = C_0$.

We will construct a curve $X \subset \mathbb{F}_1$ such that $W_4 \supseteq C_0$ and $\dim W_4 < 2$. Take any $Y \in |C_0+(d-1)F|$. We have $\deg(Y) = (C_0+(d-1)F)\cdot(C_0+2F) = d$. Notice that if C_0 is not a component of Y we have $\deg(C_0 \cap Y) = C_0 \cdot (C_0+(d-1)F) = d-2$. Furthermore, if $L \in |F|$ and L is not a component of Y, then $\deg(L \cap Y) = F \cdot (C_0 + (d-1)F) = 1$.

When Y is integral, it is a smooth rational curve of degree d. Indeed, the genus formula on the surface $\mathbb{F}_1$ implies $p_a(Y) = 1 + \frac{1}{2}(Y \cdot Y + Y \cdot K_{\mathbb{F}_1}) = 0 = p_g(Y)$.

For all $d \geq 4$, Y spans $\mathbb{P}^4$, because no element of $|C_0 + 2F|$ contains Y. We have $h^0(\mathcal{O}_{\mathbb{F}_1}(C_0 + (d-1)F)) = d + d - 1$, i.e., $\dim |C_0 + (d-1)F| = 2d - 2$.

Lemma 5.1. *Fix $o \in C_0$ and call $E \subset C_0$ the degree $(d-2)$ zero-dimensional subscheme whose support is $\{o\}$. We have $\dim |\mathcal{I}_E(C_0+(d-1)F)| = d$ and a general $X \in |\mathcal{I}_E(C_0+(d-1)F)|$ is smooth.*

Proof. The ideal sheaf exact sequence gives $\dim |\mathcal{I}_E(C_0 + (d-1)F)| \geq \dim |C_0 + (d-1)F| - \deg(E) = 2d - 2 - (d-2) = d$. Since $h^1(\mathbb{F}_1, \mathcal{O}_{\mathbb{F}_1}(C_0 + (d-1)F)) = 0$, we have $\dim |\mathcal{I}_E(C_0+(d-1)F)| = d$ if and only if $h^1(\mathbb{F}_1, \mathcal{I}_E(C_0+(d-1)F)) = 0$. Since $E \subset C_0$, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{F}_1}((d-1)F) \longrightarrow \mathcal{I}_E(C_0 + (d-1)F) \longrightarrow \mathcal{I}_{E,C_0}(C_0 + (d-1)F) \longrightarrow 0. \tag{5}$$

Using [Har77, Lemma 2.4, V], we have $h^1(\mathcal{O}_{\mathbb{F}_1}((d-1)F)) = 0$ because $d \geq 2$. Since $C_0 \cong \mathbb{P}^1$ and $\mathcal{O}_{C_0}(C_0+(d-1)F)$ has degree $d-2$, we have $h^1(C_0, \mathcal{I}_{E,C_0}(C_0+(d-1)F) = 0$, because $\deg(E) = d-2$. We conclude by using the cohomology exact sequence of (5).

Now, let X be a general element of $|\mathcal{I}_E(C_0 + (d-1)F)|$. By the genus formula, to prove that X is smooth, it is sufficient to prove that it is irreducible, i.e., the set of all reducible $Y \in |\mathcal{I}_E(C_0 + (d-1)F)|$ has dimension $< d$. First we consider all the reducible $Y \in |\mathcal{I}_E(C_0 + (d-1)F)|$ containing C_0. They are of the form $Y = C_0 \cup B$ with $B \in |(d-1)F|$. Hence this set has dimension $d-1$. Next we consider the set of all reducible $Y \in |\mathcal{I}_E(C_0 + (d-1)F)|$ without C_0 as component. There is an integer $1 \leq b \leq d-2$ such that $Y = B \cup Y'$, with $B \in |bF|$ and Y' a curve in $|C_0+(d-1-b)F)|$. Since $\deg(Y' \cap C_0) = d-2-b$, $\deg(B \cap C_0) = b$ and $\deg(E) = d-2$, B has to contain the support of E (with multiplicity b) and so $B = G_o$, where G_o is the unique element of $|F|$ containing o. In other words, B is uniquely determined by o and hence it is sufficient to prove that the set of all Y' has dimension $\leq d-1$. The residual scheme of E with respect to B is the degree $(d-2-b)$ divisor E' of C_0 with support $\{o\}$. Thus $Y' \in |\mathcal{I}_{E'}(C_0 + (d-1-b)F)|$. However, the first paragraph of the proof shows $\dim |\mathcal{I}_{E'}(C_0 + (d-1-b)F)| = d - b \leq d-1$. □

Lemma 5.2. *Keeping the notation from above, fix any smooth $X \in |\mathcal{I}_E(C_0+(d-1)F|$. Then $\mathrm{rk}_X(q) = 4$, for all $q \in C_0 \setminus \{o\}$ and $W_4 \supseteq C_0$.*

Proof. By [LT10, Proposition 5.1] we have $\mathrm{rk}_X(q) \leq 4$. Assume $\mathrm{rk}_X(q) \leq 3$ and take a subscheme $S \subset X$ such that $\#S \leq 3$ and $q \in \langle S \rangle$, evincing its rank. Since $d \geq 5$ and C_0 is a line, one has $\langle E \rangle = C_0$.

First, assume $o \notin S$ and $\#S = 3$ (i.e., $\langle S\rangle$ is a plane). Since o is the only point of X contained in C_0, $\deg(E \cup S) = d + 1$. Since $\deg(X) = d$, we derive $\langle E \cup S\rangle = \mathbb{P}^4$. Hence the line $\langle E\rangle$ and the plane $\langle S\rangle$ must be disjoint, contradicting the assumption $q \in \langle S\rangle \cap C_0$.

Assume $o \notin S$ and $\#S = 2$ (i.e., $\langle S\rangle$ is a line). Since $\deg(E\cup S) = d$, the span $\langle E\cup S\rangle$ is either a hyperplane or $\mathbb{P}^4$, contradicting the assumption that the lines $\langle E\rangle$ and $\langle S\rangle$ contain q.

Assume $o \in S$ and $\#S = 3$. The span $\langle S\rangle$ is a plane containing o and q and hence containing C_0. Therefore $\deg(E \cup S) = d$ and hence $\dim\langle E \cup S\rangle \geq 3$, contradicting the inclusions $E \subset C_0 \subset \langle S\rangle$. The other cases are analogous and left to the reader. The second assertion follows from the first one, because C_0 is the closure of $C_0 \setminus \{o\}$. □

Remark 5.3. Since $\sigma_3(X) = \mathbb{P}^4$ for any non-degenerate curve $X \subset \mathbb{P}^4$, Lemma 5.1 and Lemma 5.2 show that in order to construct the desired example it is sufficient to prove the existence of a smooth $X \in |\mathcal{I}_E(C_0 + (d-1)F)|$ such that $\dim W_4 < 2$.

Henceforth we take a general $X \in |\mathcal{I}_E(C_0 + (d-1)F)|$. From now on, we assume $\dim W_4 \geq 2$ and take a projective irreducible surface $\Gamma \subseteq W_4$. Lemma 5.6 will provide a contradiction.

For any $a \in X \setminus \{o\}$, let $\pi_a : \mathbb{P}^4 \setminus \{a\} \longrightarrow \mathbb{P}^3$ denote the linear projection from a. Call $G \subset \mathbb{P}^3$ the closure in $\mathbb{P}^3$ of $\pi_a(\mathbb{F}_1 \setminus \{a\})$. Since $\deg(\mathbb{F}_1) = 3$ and $\mathbb{F}_1$ is non-degenerate and irreducible, G is an irreducible quadric surface. Note that π_a contracts the line $F_a \in |F|$ containing a. We see that $\pi_{a|\mathbb{F}_1\setminus F_a}$ is an embedding. Thus $\pi_{a|X\setminus\{a\}}$ is an embedding. Since a is a smooth point of X, one has $\deg(X') = d-1$. Moreover, $\pi_{a|X\setminus\{a\}}$ extends to a morphism $\varphi : X \longrightarrow \mathbb{P}^3$ with the image $a' := \varphi(a)$ corresponding to the tangent line T_aX of X at a.

Set $X' := \varphi(X)$. Note that $T_aX \neq F_a$, because $X \cdot F_a = 1$. We have $a' \notin \pi_a(X \setminus \{a\})$, because $\deg(T_aX \cap \mathbb{F}_1) = 2$, since $T_aX \neq F_a$ (and hence $T_aX \nsubseteq \mathbb{F}_1$) and $\mathbb{F}_1$ is scheme-theoretically cut out by quadrics. For a general $a \in X$ (it is in fact sufficient to assume that the osculating hyperplane to X at a has order of contact 3 with X at a), a' is a smooth point of X'. Thus X' is a smooth rational curve of degree $d-1 \geq 4$. By [Har77, Ex. V.2.9], G is a smooth quadric surface. Up to a choice of rulings of G, we have $X' \in |\mathcal{O}_G(1, d-2)|$.

The surface $\Gamma \subseteq \mathbb{P}^4$ is not a cone with vertex a for a general $a \in X$, because X spans $\mathbb{P}^4$, and the vertex of a cone is a linear subspace of the ambient space. Thus, for a general $a \in X$, $\Gamma_a = \overline{\pi_a(\Gamma \setminus \{a\})} \subset \mathbb{P}^3$ is a surface. Henceforth, assume $a \in X$ general.

Lemma 5.4. *We have* $\dim W_3(X') \leq 1$.

Proof. Since X' is a smooth non-degenerate curve, its first secant variety satisfies $\sigma_2(X') = \mathbb{P}^3$ and $\mathrm{rk}_X(q) = 2$ for all $q \in \mathbb{P}^3 \setminus \tau(X')$. Since $\tau(X')$ is an irreducible surface, it is sufficient to prove that $\mathrm{rk}_{X'}(q) \leq 2$ for a general $q \in \tau(X')$. Fix a general $p \in X'$. In particular, we assume that the order of contact of T_pX' with X' at p is 2: if T_pX' meets X' at another point, then all points of T_pX' have X'-rank at most 2. Thus we may assume $\deg(T_pX' \cap X') = 2$ and supported at p. Since X' is smooth at p and of degree $d-1$, the linear projection $\pi_{T_pX'} : \mathbb{P}^3 \setminus T_pX' \longrightarrow \mathbb{P}^1$ away from the line T_pX' extends to a degree $d-3$ morphism $\psi : X' \longrightarrow \mathbb{P}^1$. Since $d \geq 5$, a general fiber of ψ has cardinality at least two. This says that a general element of T_pX' has X'-rank ≤ 2.

To see the last sentence, fix a general $q \in \mathbb{P}^1$ and take two points $z_1, z_2 \in \psi^{-1}(q)$ such that $z_1 \neq z_2$. The line $L = \langle \{z_1, z_2\} \rangle$ is contained in the plane $\langle \{q\} \cup T_pX' \rangle$ and hence L meets the line T_pX' at some point z_q whose X'-rank is at most 2. Now, since ψ is not constant, the point z_q varies in T_pX'. Indeed, if z_q were fixed, the projection of X' from z_q would be *non-birational* onto its image. By [CC01, Theorem 1], there is only a finite set of points $\mathfrak{S} \subset \mathbb{P}^3$ inducing a non-birational projection of X'. However, in characteristic zero, for any fixed finite set $\mathfrak{S}$ and any non-degenerate curve X', the general tangent line of X' contains no point of $\mathfrak{S}$.

In conclusion, a general point of T_pX' has X'-rank equal to 2. □

By Lemma 5.4, we have $\mathrm{rk}_{X'}(b) \leq 2$ for a general $b \in \Gamma_a$. Thus $\mathrm{rk}_{X'}(\pi_a(q)) \leq 2$ for a general $q \in \Gamma$.

Fix a general $q \in \Gamma \subset W_4$ and recall that $a' \in \mathbb{P}^3$ is the image of a upon taking the closure of the image $\pi_a(X \setminus \{a\})$. Take $S' \subset X'$ such that $\#S' \leq 2$ and $\pi_a(q) \in \langle S' \rangle$. If $S' \subset X' \setminus \{a'\}$, then we may take $S'' \subset X \setminus \{a\}$ with $\pi_a(S'') = S'$ and hence $q \in \langle S'' \cup \{a\} \rangle$. Since $\#(S'' \cup \{a\}) \leq 3$, we derive $\mathrm{rk}_X(q) \leq 3$, a contradiction.

Now assume $a' \in S'$. We have $\#S' = 2$, because otherwise $q \in T_aX$, contradicting the generality of q. Set $\{b'\} := S' \setminus \{a'\}$. If $\#(\langle S' \rangle \cap X') \geq 3$, we may take $a'' \in X' \setminus \{a'\}$ such that $\pi_a(q) \in \langle \{a'', b'\} \rangle$ and derive $\mathrm{rk}_X(q) \leq 3$. Thus we may assume $\langle S' \rangle \cap X' = \{a', b'\}$ (set-theoretically).

Before we proceed we introduce another piece of notation.

Definition 5.5. For a given projective variety $X \subset \mathbb{P}^N$ and a point $p \in \mathbb{P}^N$, let $\mathcal{S}(X, p)$ denote the set of rank-decompositions of p with respect to X.

So $\mathcal{S}(X', \pi_a(q))$ denotes the set of rank-decompositions of $\pi_a(q)$ with respect to X'. Since any two different lines through a' meet only at a', S' is the only element of $\mathcal{S}(X, \pi_a(q))$ containing a', because otherwise another one should have contained $\pi_a(q) \neq a'$ as well. Thus, to conclude, it is enough to prove that the set $\mathcal{S}(X', \pi_a(q))$ contains at least another decomposition. As mentioned above, since Γ is not a cone with vertex a (for a general $a \in X$), Γ_a is a surface. So it is sufficient to prove that $\#\mathcal{S}(X', \pi_a(q)) > 1$ is satisfied by all points $\pi_a(q) \notin \Sigma$ (i.e., possibly outside some curve Σ) having at least a decomposition in $\mathcal{S}(X', \pi_a(q))$ containing a'. It is sufficient to apply the next lemma.

Lemma 5.6. *Assume $d \geq 5$. Let $C_{a'}(X') \subset \mathbb{P}^3$ be the cone with vertex a' and X' as its base. There exists a curve $\Sigma \subset C_{a'}(X')$ such that for all points $x \in C_{a'}(X') \setminus \Sigma$, with $\mathrm{rk}_{X'}(x) = 2$, we have $\#\mathcal{S}(X', x) > 1$.*

Proof. The surface $C_{a'}(X')$ is irreducible and $C_{a'}(X') \neq \tau(X')$. Therefore they intersect along a curve, $\dim(C_{a'}(X') \cap \tau(X')) = 1$.

Consider all $x \in C_{a'}(X') \setminus (\tau(X') \cap C_{a'}(X'))$, with $\mathrm{rk}_{X'}(x) = 2$, and suppose $\mathcal{S}(X', x)$ is not infinite.

Let $\pi_x : \mathbb{P}^3 \setminus \{x\} \longrightarrow \mathbb{P}^2$ denote the linear projection away from x. Since $x \notin \tau(X')$, then $x \notin X'$ and $\pi_{x|X'}$ is a local embedding. We are assuming $\mathcal{S}(X', x)$ is finite, i.e., $\pi_{x|X'}$ is birational onto its image. Note that if there are at least three different points of X' with the same image by π_x, then $\#\mathcal{S}(X', x) \geq 3 > 1$, and we are done. Likewise, if there are t singular points of $\pi_x(X')$, we have $\#\mathcal{S}(X', x) \geq t$, because $\pi_{x|X'}$ is a local embedding and so distinguishes tangent directions.

Thus we may assume that $\pi_x(X')$ has a unique singular point, α, which has exactly two branches, each of them smooth (the case with more branches was excluded above). Since $\deg(\pi_x(X')) = d-1$, one has $p_a(\pi_x(X')) = (d-2)(d-3)/2$. Thus α is a tacnode with delta-invariant $(d-2)(d-3)/2$, since the normalization of X is rational. Hence there exist distinct $a_1, a_2 \in X'$, such that the plane $\langle T_{a_1}X' \cup T_{a_2}X'\rangle$ contains x and the order of contact of $T_{a_i}X'$ with X' at a_i is at least three.

However, since X' is contained in a smooth quadric surface G, each tangent line of X' with order of contact at least three is contained in G. One of the rulings of G is formed by lines with degree of intersection $d-2 \geq 3$ with X'. Therefore we may take as Σ a finite union of lines of G and the curve $C_{a'}(X') \cap \tau(X')$. $\square$

We are finally ready to prove Theorem 1.3:

Proof of Theorem 1.3. Keeping the notation from above, let $X \in |\mathcal{I}_E(C_0 + (d-1)F)|$ be general and assume $\dim W_4 \geq 2$. Now, Lemma 5.4 and Lemma 5.6 altogether give a contradiction. Thus $\dim W_4 \leq 1$ and Lemma 5.2 shows that $\dim W_4 = 1$. This proves the statement. $\square$

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