Doctoral School in Mathematics

Variational methods for hyperbolic obstacle-type problems, k-harmonic maps with defects and optimal Steiner-type networks

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Introduction

In this thesis, we investigate the hyperbolic type obstacle problems, nonlinear waves in adhesive phenomena, harmonic maps with defects and their connection to Gilbert-Steiner problems.

On the obstacle problem for fractional semilinear wave equations:
The first problem that I have studied is the obstacle problem for fractional semilinear wave equations [17]. In [17], which is the joint work with M. Bonafini, M. Novaga, G. Orlandi, we aim to study both obstacle free-case (there is no obstacle), and the obstacle case (i.e. in the presence of an obstacle), and including also non-local operators (e.g. fractional Laplacian). In the obstacle-free case, the main motivation is that certain solutions of nonlinear wave equations (possibly non-local) giving rise to interfaces (or defects) evolving by curvature such as minimal surfaces in Minkowski space: for instance, consider the class of equations

$$u_{tt} - \Delta u + \frac{1}{\varepsilon^2}\nabla_u W(u) = 0 \tag{0.0.1}$$

for $u\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^m$, where W is a balanced double-well potential, $m = 1, 2$, and $\varepsilon > 0$ is a small parameter. This case can be seen as the parallel analogous results with the elliptic case:

$$-\Delta u + \frac{1}{\varepsilon^2}\nabla_u W(u) = 0 \tag{0.0.2}$$

where the defects are Euclidean minimal surfaces (see for instance [60, 44, 12]). And in the parabolic case:

$$u_t - \Delta u + \frac{1}{\varepsilon^2}\nabla_u W(u) = 0 \tag{0.0.3}$$

where the defects evolve according to motion by mean curvature. (see for example [45, 13]). In the hyperbolic scenario (0.0.1), this fact was observed in [61] by a formal analysis in the case $m = 1$. Then, a rigorous analysis was given in [47, 77, 69] where solutions of (0.0.1) having interfaces near a given timelike minimal surface were constructed. However, due to the onset of singularities during the evolution, those results are valid only for a short time. On the other hand, the analysis of singular limits of the hyperbolic Ginzburg-Landau equation possibly passing to the singularities was studied in [9] under conditional assumptions i.e. if those conditions are verified, then in the limits the Lagrangian density of the solutions will concentrate on timelike lorentzian minimal submanifolds of codimension m within the varifold framework developed in [10]. Therefore, our first goal is to propose a constructive time-discrete variational scheme in the spirit of minimizing movements to build a solution for (0.0.1), this method was also used to treat the fractional linear wave equation in [18](see also [42, 50]), and our results can be seen as the extension of those obtained in [18]. Moreover, our results embrace the vector-valued case as well as non-local operator:

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{0.0.4}$$

where $u : (0,T) \times \Omega \to \mathbb{R}^m$, $T > 0$, $m \in \mathbb{N}$, W a continuous potential with Lipschitz continuous derivative and $(-\Delta)^s$ stands for the fractional Laplacian having Fourier symbol $|\xi|^{2s}\mathcal{F}u$. We expect that those conditions in [9] could be relaxed for our constructive solutions thanks to the minimality of approximate solutions, and this will be the future investigation.

In the obstacle case, roughly speaking given an obstacle g, we look for a function u satisfying certain variational inequalities and at the same time lying above the obstacle g. In recent years, there has been a lot of works devoted to study the obstacle problems elliptic and parabolic setting (see [75, 28, 27, 63, 8] and reference therein), in the hyperbolic scenario there are still few works: for instance, the works of Schatzman and collaborators (see [71, 72, 73, 67]), Kikuchi dealing with the vibrating strings with an obstacle in the 1-dimensional case by using a time semi-discrete method [50]. By using the same approach in [50], the obstacle problem for fractional wave equations have been investigated in [18], more recently similar time semi-discrete methods have also been used to treat hyperbolic free boundary problems (see [1]). In [17], we have extended the results in [18] to the semilinear setting, more precisely we consider the following system:

$$\begin{cases} u_{tt} + (-\Delta)^s u + W'(u) \geq 0 & \text{in } (0,T) \times \Omega \\ u(t,\cdot) \geq g & \text{in } (0,T) \times \Omega \\ (u_{tt} + (-\Delta)^s u + W'(u))(u-g) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{0.0.5}$$

By using the scheme as the obstacle-free case, we provided energy estimates as well as compactness results which allow us to prove the existence of solutions of the system (0.0.5) in the suitable sense. Furthermore, we believe that the analysis developed in [17] could be used to analyse problems which are shared the same character. And our results in [17] have been published in the journal *Nonlinear Analysis. 210(2021), 112368.* This work will be presented in the Chapter 1.

Weak solutions for nonlinear waves in adhesive phenomena:
In a joint work with Mauro Bonafini [15], the second problem that I have dealt with is the dynamic of an elastic body (e.g. a string or a membrane) interacting with a rigid substrate through an adhesive layer. In recent years, there are many works devoted to study the adhesive and the debonding phenomenona (see for instance [46, 82, 39, 66] and references therein) which arise from the engineering and biophysics. The phenomenon depend heavily on the involved materials, and due to the complexity mechanism ranging from microscopic to macroscopic, mathematical models has been proposed in order to capture the essential features.

In [15], following [32], where the dynamic of an elastic body (e.g. a string) glued to a rigid substrate through an adhesive layer in 1-dimensional case can be modelled via a potential W describing the effect of the adhesive layer on the dynamic. More precisely, the Lagrangian governing the one dimensional dynamical system considered there is described by

$$\ell(u) = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + W(u), \tag{0.0.6}$$

for a scalar displacement field $u : [0,T] \times [0,L] \to \mathbb{R}$, the potential W has the following behavior:

$$W(u) = \begin{cases} u^2 & \text{if } |u| \le u^*, \\ (u^*)^2 & \text{if } |u| > u^*. \end{cases} \tag{0.0.7}$$

The lack of smoothness of W at points $\pm u^*$ gives rises the difficulties in showing the existence of solutions to the problem. And our goal is to address this issue, in particular our analysis extends to vector-valued case as well as non-local operator. In view of this, we consider the following system:

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T) \times \Omega, \\ u(x,t) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega), \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u_t(0,x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{0.0.8}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded domain with Lipschitz boundary, $u : [0,T] \times \Omega \to \mathbb{R}^m$ and $(-\Delta)^s$ stands for the fractional Laplacian ($s = 1$ provides the standard Laplacian). Furthermore, the consideration (0.0.8) includes also the other interaction which manifests the adhesive feature, more specifically in the case $s = 2$ i.e. bi-Laplacian, the equation models the elastic beam interacting with a substrate through an elastic-breakable forcing term, and it has been recently studied in [34]. In [15], we provide the existence results for the equation (0.0.8), and it depends on the regularity of the potential W. Whenever W is a non-negative potential in $C^1(\mathbb{R}^m)$ and having Lipschitz continuous gradient ∇W, the existence results was provided in [17]. In [15], by using the results in [17] and regularization methods of the considered potentials, we are able to prove the existence of solutions in case without the assumption Lipschitz-condition on the ∇W, and show difficulties in defining the notion of solutions to the problem. More precisely,

- (i) In first case we investigate the energetic contribution of adhesive layer, namely ∇W decays continuously to zero. We prove the existence of solutions with initial datum as in [17].
- (ii) In the other case where the adhesive layer discontinuously drop to zero as considered in [32, 33], we shall see the limitation of current method and the obstacles in defining the notion of solutions. However, we can provide the existence result for small initial datum combined with the condition $2s > d$. This work will be presented in the Chapter 2.

Energy minimizing maps with prescribed singularities and Gilbert-Steiner optimal networks:
The last problem that I have dealt with in this thesis is about the connection between k-harmonic valued maps with prescribed singularities and Gilbert-Steiner problems which is based on a joint work with S. Baldo, A. Massaccesi, G. Orlandi [7].
Steiner tree, Gilbert-Steiner (single sink) problems can be formulated as follows: given n distinct points $P_1, \ldots, P_n$ in $\mathbb{R}^d$, where $d, n \geq 2$, we are looking for an optimal connected transportation network, $L = \cup_{i=1}^{n-1} \lambda_i$, along which the unit masses located at $P_1, \ldots, P_{n-1}$ are transported to the target point P_n (single sink), here λ_i can be seen as the path of the mass at P_i flowing from P_i to P_n, and the cost of moving a mass m along a segment with length l is proportional to lm^α. Therefore, we are led to consider the problem

$$(I) \qquad \inf \left\{ I_\alpha(L) : \, L = \bigcup_{i=1}^{n-1} \lambda_i \text{ with } \{P_i, P_n\} \subset \lambda_i, \text{ for every } i = 1, \ldots, n-1 \right\}$$

where the energy I_α is computed as $I_\alpha(L) = \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x)$, with $\theta(x) = \sum_{i=1}^{n-1} \mathbf{1}_{\lambda_i}(x)$. Let us notice that θ stands for the mass density along the network. In recent years, there are a lot of works devoted to study the problem (I) from several aspects such as existence, regularity, stability and numerical feasibility. In particular, the recent profound works [54, 55] has shown the problem (I) as Plateau problem i.e. a mass-minimization problem of currents with coefficients in a norm group $(\mathbb{Z}^{n-1}, \psi)$ (see Chapter 3, Section 3.3 for more details on the norm ψ): roughly speaking, we can interpret the network $L = \bigcup_{i=1}^{n-1} \lambda_i$ as the superposition of $n-1$ paths λ_i connecting P_i to P_n labelled with multiplicity e_i, $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^{n-1}$, 1 is i-th position. This point of view requires a density function with values in $\mathbb{Z}^{n-1}$, which corresponds to the so-called 1-dimensional current with coefficients in the group $\mathbb{Z}^{n-1}$. Furthermore, by equipping $\mathbb{Z}^{n-1}$ with a certain norm (depending on the cost of the problem), we may define the notion of mass of those currents, and problem (I) turns out to be equivalent to

$$(M) \qquad \inf \left\{ \mathbb{M}(T), \;\; \partial T = \mu^+ - \mu^- \right\}$$

where T is a 1-dimensional current with coefficients in the group $\mathbb{Z}^{n-1}$, $\mu^+ = (e_1 + e_2 + \ldots + e_{n-1})\delta_{P_n}$, $\mu^- = e_1 \delta_{P_1} + \ldots + e_{n-1}\delta_{P_{n-1}}$, and we refer the reader to Section 3.2 for rigorous definitions. They also define the notion of calibration which are a useful tool to prove the minimality of minimizers. [54, 55] are also a starting point for the works [19, 20] where the authors provided a variational approximation functionals of the problem (I_α) in the sense of Γ-convergence.

Following the line [54, 55], [19, 20], in the present work motivated by the seminal work of Brezis, Coron and Lieb [25] who showed the relation between sphere-valued harmonic maps having prescribed topological singularities at given points in $\mathbb{R}^3$ with *minimal connections* between those points, i.e. optimal mass transportation networks (in the sense of Monge-Kantorovich) having those point singularities as marginals. More precisely, [25] showed that given $P_1, \ldots, P_m$, and $N_1, \ldots, N_m$ in $\mathbb{R}^d$, we have

$$\inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^{d-1} dx \mid u \in V \right\} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} L \tag{0.0.9}$$

where $V = \{u \in C^1(\mathbb{R}^d \setminus \{P_1, \ldots, P_m, N_1, \ldots, N_m\}; \mathbb{S}^{d-1}) \mid \deg(u, P_i) = 1, \; \deg(u, N_i) = -1\}$, α_{d-1} is the surface area of the unit ball in $\mathbb{R}^d$, and L is minimal connection between $P_1, \ldots, P_m$, and

$N_1, \ldots, N_m$ i.e. solutions of Monge-Kantorovich problems in which $P_1, \ldots, P_m$, and $N_1, \ldots, N_m$ play as marginals. And then this was later recast by Almgren, Browder and Lieb [4] by interpreting the minimal connection as a mass-minimization problem of classical 1-dimensional integral currents with the given topological singularities as prescribed boundary. Here, we aim to consider more general minimizing configurations of more general energies for maps with valued into manifolds and having prescribed topological singularities, and investigate their connection with Plateau problem in the context of currents or flat chains with coefficients in suitable groups which are linked to the topology of involved manifolds. In particular, we restrict our attention first to consider the manifold which is a product of unit spheres, and show an equivalent between an energy-minimizing configurations with Steiner tree, Steiner-Gilbert problems (I) under an appropriate condition, this can be done thanks to the works in [55, 54] by proving Steiner tree, Steiner-Gilbert problems as a mass-minimization problem of currents with coefficients in suitable norm group $(\mathbb{Z}^{n-1}, \psi)$. More precisely, let $P_1, \ldots, P_{n-1}, P_n$ in $\mathbb{R}^d$ be given, and consider the spaces H_i defined as the subsets of $W^{1,d-1}_{\text{loc}}(\mathbb{R}^d; \mathbb{S}^{d-1})$ where the functions are constant outside a neighbourhood of the segment joining P_i, P_n and have distributional Jacobian $\frac{\alpha_{d-1}}{d}(\delta_{P_i} - \delta_{P_n})$, respectively. Here α_{d-1} is the surface area of the unit ball in $\mathbb{R}^d$.

Let ψ be a norm on $\mathbb{R}^{n-1}$ which will be specified in Section 3.3 (see (3.4.9)), and set

$$\mathbb{H}(\mathbf{u}) = \int_{\mathbb{R}^d} \psi(|\nabla u_1|^{d-1}, |\nabla u_2|^{d-1}, \ldots, |\nabla u_{n-1}|^{d-1}) \, dx \tag{0.0.10}$$

where $\mathbf{u} = (u_1, \ldots, u_{n-1}) \in H_1 \times H_2 \times \ldots \times H_{n-1}$ is a 2-tensor. We investigate

$$(H) \qquad \inf \{\mathbb{H}(\mathbf{u}) : \mathbf{u} \in H_1 \times H_2 \times \ldots \times H_{n-1}\}.$$

Our main contribution is the following:
Assume that a minimizer of problem (M) admits a calibration (see Definition 3.2.8). Then, we have

$$\inf \mathbb{H} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf \mathbb{M} \tag{0.0.11}$$

or equivalently,

$$\inf \mathbb{H} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha. \tag{0.0.12}$$

This work will be presented in Chapter 3. In a companion paper in preparation [29] we will consider more general manifolds and state the results corresponding to more general situations.

Summary of research outcome. The thesis work led to the following publications and preprints.

[17] Mauro Bonafini, Van Phu Cuong Le, Matteo Novaga, and Giandomenico Orlandi. On the obstacle problem for fractional semilinear wave equations. *Nonlinear Analysis*, 210(112368), 2021

[15] Mauro Bonafini and Van Phu Cuong Le. Weak solutions for nonlinear waves in adhesive phenomena. *arXiv:2104.10437, submitted*, 2021

[7] Sisto Baldo, Van Phu Cuong Le, Annalisa Massaccessi, and Giandomenico Orlandi. Energy minimizing maps with prescribed singularities and Gilbert-Steiner optimal networks. *arXiv:2112.12511, submitted*, 2021

Contents

Chapter 1

On the obstacle problem for fractional semilinear wave equations

In this chapter we prove existence of weak solutions to the obstacle problem for semilinear wave equations (including the fractional case) by using a suitable approximating scheme in the spirit of minimizing movements. This extends the results in Bonafini et al [18], where the linear case was treated. In addition, we deduce some compactness properties of concentration sets (e.g. moving interfaces) when dealing with singular limits of certain nonlinear wave equations.

1.1 Introduction

Semilinear wave equations have been considered extensively in the mathematical literature with many dedicated contributions (see for example [71, 56, 78, 74, 61, 9, 47, 69] and references therein). Our main motivation is to study certain nonlinear wave equations (possibly non-local) giving rise to interfaces (or defects) evolving by curvature such as minimal surfaces in Minkowski space: for instance, consider the class of equations

$$\varepsilon^2(u_{tt} - \Delta u) + \nabla_u W(u) = 0 \tag{1.1.1}$$

for $u\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^m$, where W is a balanced double-well potential, $m \geq 1$, and $\varepsilon > 0$ is a small parameter (see for example [61, 9, 47, 77, 69]). This case is the hyperbolic version of the stationary Allen–Cahn equation where the defects are Euclidean minimal surfaces and the parabolic Ginzburg–Landau where defects evolve according to motion by mean curvature (see for instance [60, 45, 13] and references therein).

Obstacle problems in the elliptic and parabolic setting have attracted a lot of attention including both local and non-local operators (see for example [75, 28, 27, 63, 8] and references therein). In the hyperbolic scenario, we would like to mention works by Schatzman and collaborators (see for example [71, 72, 73, 67]) and more recently, a work by Kikuchi dealing with the vibrating strings with an obstacle in the 1-dimensional case by using a time semidiscrete method (see [50]). Notice that similar time semidiscrete methods have also been used to treat hyperbolic free boundary problems (see [42, 1]). By using the same approach as in [50], the obstacle problem for the fractional wave equation has been investigated in [18], in which the existence of suitably defined weak solutions is proved.

In this chapter, following [18], we implement a semidiscrete in time approximation scheme in order to prove existence of solutions to hyperbolic PDEs with possibly specific additional conditions. The scheme is closely related to the concept of minimizing movements introduced by De Giorgi, and it is also elsewhere known as the discrete Morse semiflow approach or Rothe's method [70]. Our main focus is to prove the existence of weak solutions to the following PDEs (including also the obstacle case):

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T) \times \Omega, \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega), \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u_t(0,x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1.2}$$

for $\Omega \subset \mathbb{R}^d$ an open bounded domain with Lipschitz boundary and W a continuous potential with Lipschitz continuous derivative. For $s > 0$ the operator $(-\Delta)^s$ stands for the fractional s-Laplacian. We prove a classical energy bound for the approximating trajectories in Proposition 1.3.4 and rely upon it to prove existence of a suitably defined weak solution of (1.1.2) in the obstacle-free case (Theorem 1.3.3) and in the obstacle case (Theorem 1.4.2). The approximation scheme allows us to deal with a variety of situations, including non-local fractional semilinear wave equations, and is valid in any dimension. This gives also some compactness results for concentration sets in the singular limit of (2.1.3).

Chapter 1 is organized as follows: in Section 1.2 we briefly review some properties of the fractional Sobolev spaces and fractional Laplace operator so as to fix notations. In Section 1.3 we introduce the approximating scheme and apply it to fractional semilinear wave equations by means of an appropriate variational problem, prove existence result Theorem 1.3.3 in the obstacle-free case, and the conservative property of the solutions, namely Proposition 1.3.10. In proposition 1.3.13 we prove compactness properties for the concentration sets in the singular limit of (2.1.3). In Section 1.4 we adapt the scheme to deal with the obstacle problem for fractional semilinear wave equations, and prove Theorem 1.4.2. Eventually, in Section 1.5 we present an example implementing a case related to moving interfaces in a relativistic setting.

1.2 Preliminaries

Let us fix $s \geq 0$ and $m \geq 1$. Following [62], we introduce fractional Sobolev spaces and the fractional Laplacian through Fourier transform. Consider the Schwartz space $\mathcal{S}$ of rapidly decaying C^∞ functions, namely $\mathcal{S}(\mathbb{R}^d; \mathbb{R}^m)$. For any $u \in \mathcal{S}(\mathbb{R}^d; \mathbb{R}^m)$ denote by

$$\mathcal{F}u(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\xi \cdot x} u(x)\, dx$$

the Fourier transform of u. The fractional Laplacian operator $(-\Delta)^s \colon \mathcal{S}(\mathbb{R}^d; \mathbb{R}^m) \to L^2(\mathbb{R}^d; \mathbb{R}^m)$ can then be defined, up to constants, as

$$(-\Delta)^s u = \mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F}u) \quad \text{for all } \xi \in \mathbb{R}^d.$$

Given $u, v \in L^2(\mathbb{R}^d; \mathbb{R}^m)$, we consider the bilinear form

$$[u,v]_s = \int_{\mathbb{R}^d} (-\Delta)^{s/2} u(x) \cdot (-\Delta)^{s/2} v(x)\, dx$$

and the corresponding semi-norm $[u]_s = \sqrt{[u,u]_s} = ||(-\Delta)^{s/2}u||_{L^2(\mathbb{R}^d;\mathbb{R}^m)}$. Given the semi-norm $[\cdot]_s$, we define the fractional Sobolev space of order s as

$$H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d;\mathbb{R}^m) \,:\, \int_{\mathbb{R}^d} (1+|\xi|^{2s})|\mathcal{F}u(\xi)|^2\, d\xi < +\infty \right\}$$

equipped with the norm $||u||_s = (||u||^2_{L^2(\mathbb{R}^d)} + [u]^2_s)^{1/2}$.

Fix now $\Omega \subset \mathbb{R}^d$ to be an open bounded set with Lipschitz boundary and define

$$\tilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^d;\mathbb{R}^m) \,:\, u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega\},$$

endowed with the $||\cdot||_s$ norm, and its dual $H^{-s}(\Omega) := (\tilde{H}^s(\Omega))^*$. One can prove, see e.g. [59], that $\tilde{H}^s(\Omega)$ corresponds to the closure of $C_c^\infty(\Omega)$ with respect to the $||\cdot||_s$ norm.

1.3 Weak solutions for the fractional semilinear wave equations

We prove in this section existence of weak solutions for the fractional semilinear wave equation. The proof, as in [18], is based on a constructive time-discrete variational scheme whose main ideas date back to [70] and which has since then been adapted to many instances of parabolic and hyperbolic equations.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary. For $u = u(t,x) : (0,T)\times\mathbb{R}^d \to \mathbb{R}^m$, let us consider the system

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T)\times\Omega \\ u(t,x) = 0 & \text{in } [0,T]\times(\mathbb{R}^d\setminus\Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.3.1}$$

with initial data $u_0 \in \tilde{H}^s(\Omega)$ and $v_0 \in L^2(\Omega) := L^2(\Omega;\mathbb{R}^m)$ (we conventionally intend that $v_0 = 0$ in $\mathbb{R}^d\setminus\Omega$), and a non-negative potential $W \in C^1(\mathbb{R}^m;\mathbb{R})$ having Lipschitz continuous derivative with Lipschitz constant $K > 0$, i.e.,

$$|\nabla W(x) - \nabla W(y)| \le K|x-y| \quad \text{for any } x,y \in \mathbb{R}^m. \tag{1.3.2}$$

As we are dealing with non-local operators, the boundary condition is imposed on the whole complement of Ω. We define a weak solution of (1.3.1) as follows:

Definition 1.3.1. *Let $T>0$. We say $u = u(t,x)$ is a weak solution of* (1.3.1) *in* $(0,T)$ *if*

1. $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ *and* $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$,
2. *for all* $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$

$$\int_0^T \langle u_{tt}(t),\varphi(t)\rangle dt + \int_0^T [u(t),\varphi(t)]_s\, dt + \int_0^T\int_\Omega \nabla_u W(u(t))\cdot\varphi(t)\, dxdt = 0 \tag{1.3.3}$$

with

$$u(0,x) = u_0 \quad \textit{and} \quad u_t(0,x) = v_0. \tag{1.3.4}$$

The energy of u is defined as

$$E(u(t)) = \frac{1}{2}||u_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]^2_s + ||W(u(t))||_{L^1(\Omega)}, \quad t \in [0,T].$$

Remark 1.3.2. In case $u_t \in L^\infty(0,T;\tilde{H}^s(\Omega))$, we observe that the following energy norms

$$\begin{aligned} \frac{1}{2}||u_t(\cdot)||^2_{L^2(\Omega)} \, &: \, [0,T] \to [0,\infty) \\ & t \mapsto \frac{1}{2}||u_t(t)||^2_{L^2(\Omega)} \end{aligned} \tag{1.3.5}$$

$$\begin{aligned} \frac{1}{2}[u(\cdot)]^2_s \, &: \, [0,T] \to [0,\infty) \\ & t \mapsto \frac{1}{2}[u(t)]^2_s \end{aligned} \tag{1.3.6}$$

$$\begin{aligned} ||W(u(\cdot))||_{L^1(\Omega)} \, &: \, [0,T] \to [0,\infty) \\ & t \mapsto ||W(u(t))||_{L^1(\Omega)} \end{aligned} \tag{1.3.7}$$

are absolutely continuous. Moreover, for a.e $t \in (0,T)$ one has:

$$\begin{aligned} \frac{1}{2}\frac{d||u_t(t)||^2_{L^2(\Omega)}}{dt} &=< u_{tt}(t), u_t(t) >, \; \frac{1}{2}\frac{d[u(t)]^2_s}{dt} = [u(t), u_t(t)]_s, \\ \text{and} \; \frac{d||W(u(t))||_{L^1(\Omega)}}{dt} &= \int_\Omega \nabla_u W(u(t)) \cdot u_t(t) dt. \end{aligned} \tag{1.3.8}$$

We refer the reader to [37] for these facts.

This section is devoted to the proof of the following theorem.

Theorem 1.3.3.

(i) There exists a weak solution of the fractional semilinear wave equation (1.3.1) *such that it satisfies the energy inequality:*

$$E(u(t)) \leq E(u(0)) \tag{1.3.9}$$

for any $t \in [0,T]$.

(ii) if $u_0 \in \tilde{H}^{2s}(\Omega)$ and $v_0 \in \tilde{H}^s(\Omega)$, then there exists a solution u of the equation (1.3.1) *such that $u \in W^{1,\infty}(0,T;\tilde{H}^s(\Omega)), u_t \in W^{1,\infty}(0,T;L^2(\Omega))$. Moreover, for any $t \in [0,T]$*

$$E(u(t)) = E(u(0)), \tag{1.3.10}$$

i.e. the energy of u is conserved during the evolution.

(iii) The equation (1.3.1) *has unique solution in the class:*
$X = \{\, u \,|\, u$ is a weak solution of (1.3.1), *$u_t \in L^\infty(0,T;\tilde{H}^s(\Omega))\}$ in the sense that if $v, w \in X$, then for each $t \in [0,T]$*

$$v(t) = w(t) \text{ in } \tilde{H}^s(\Omega).$$

In particular the solution found in point (*ii*)*, since it belongs to X, it is unique.*

The proof relies on an extension of the approximating scheme already used in [18] in the linear case, where now one has to deal with the additional contribution of the (possibly non convex) potential term (the proof would simplify in case of a convex potential, as for example in [78]).

1.3.1 Approximating scheme

For $n \in \mathbb{N}$, set $\tau_n = T/n$ and define $t_i^n = i\tau_n$, $0 \le i \le n$. Let $u_{-1}^n = u_0 - \tau_n v_0$, $u_0^n = u_0$ and for every $i \ge 1$ let

$$u_i^n \in \arg\min_{u \in \tilde{H}^s(\Omega)} J_i^n(u) = \arg\min_{u \in \tilde{H}^s(\Omega)} \left[\int_\Omega \frac{|u - 2u_{i-1}^n + u_{i-2}^n|^2}{2\tau_n^2} \, dx + \frac{1}{2}[u]_s^2 + \int_\Omega W(u)dx \right]. \tag{1.3.11}$$

We can readily see, using the direct method of the calculus of variations, that each J_i^n admits a minimizer in $\tilde{H}^s(\Omega)$ so that u_i^n is indeed well defined (notice that we are not working under uniqueness assumptions, thus we may have to choose between multiple minimizers). For any fixed $i \in \{1, \dots, n\}$, by minimality we have

$$\frac{d}{d\varepsilon} J_i^n(u_i^n + \varepsilon\varphi)|_{\varepsilon=0} = 0 \quad \text{for every } \varphi \in \tilde{H}^s(\Omega)$$

or, equivalently,

$$\int_\Omega \left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}\right) \cdot \varphi \, dx + [u_i^n, \varphi]_s + \int_\Omega \nabla_u W(u_i^n) \cdot \varphi dx = 0 \quad \text{for every } \varphi \in \tilde{H}^s(\Omega). \tag{1.3.12}$$

We define the piecewise constant and piecewise linear interpolations over $[-\tau_n, T]$ as follows:

- piecewise constant interpolant

$$\bar{u}^n(t,x) = \begin{cases} u_{-1}^n(x) & t = -\tau_n \\ u_i^n(x) & t \in (t_{i-1}^n, t_i^n], \end{cases} \tag{1.3.13}$$

- piecewise linear interpolant

$$u^n(t,x) = \begin{cases} u_{-1}^n(x) & t = -\tau_n \\ \dfrac{t - t_{i-1}^n}{\tau_n} u_i^n(x) + \dfrac{t_i^n - t}{\tau_n} u_{i-1}^n(x) & t \in (t_{i-1}^n, t_i^n]. \end{cases} \tag{1.3.14}$$

At the same time, upon defining $v_i^n = (u_i^n - u_{i-1}^n)/\tau_n$, $0 \le i \le n$, let $\bar{v}^n$ be the piecewise constant interpolation and v^n be the piecewise linear interpolation over $[0,T]$ of the family $\{v_i^n\}_{i=0}^n$, defined similarly to (1.3.13), (1.3.14).

From (1.3.12), an integration over $[0,T]$ provides

$$\int_0^T \int_\Omega \left(\frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} \right) \cdot \varphi(t) \, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s \, dt + \int_0^T \int_\Omega \nabla_u W(\bar{u}^n(t)) \cdot \varphi(t) \, dxdt = 0$$

for all $\varphi \in L^1(0,T; \tilde{H}^s(\Omega))$, which is equivalent to

$$\int_0^T \int_\Omega v_t^n(t) \cdot \varphi(t) \, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s \, dt + \int_0^T \int_\Omega \nabla_u W(\bar{u}^n(t)) \cdot \varphi(t) \, dxdt = 0. \tag{1.3.15}$$

The strategy in proving Theorem 1.3.3 is then to consider (1.3.15), pass to the limit as $n \to \infty$ and prove that u^n and $\bar{u}^n$ converge to a weak solution of (1.3.1). In order to do so, we need the following energy estimate.

Proposition 1.3.4 (Key estimate)**.** *The approximate solutions* $\bar{u}^n$ *and* u^n *satisfy*

$$\frac{1}{2}\|u_t^n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}[\bar{u}^n(t)]_s^2 + ||W(\bar{u}^n(t))||_{L^1(\Omega)} \leq E(u(0)) + C\tau_n$$

for all $t \in [0,T]$*, with* $C = C(E(u(0)), K, T)$ *a constant independent of* n*.*

Proof. For fixed $i \in \{1,\dots,n\}$, we consider equation (1.3.12) with test function $\varphi = u_{i-1}^n - u_i^n = -\tau_n v_i^n$ to obtain

$$\begin{aligned} 0 &= \int_\Omega (v_{i-1}^n - v_i^n)\cdot v_i^n\,dx + [u_i^n, u_{i-1}^n - u_i^n]_s + \int_\Omega \nabla_u W(u_i^n)\cdot(u_{i-1}^n - u_i^n)dx \\ &\leq \frac{1}{2}\int_\Omega \left[|v_{i-1}^n|^2 - |v_i^n|^2\right]\,dx + \frac{1}{2}([u_{i-1}^n]_s^2 - [u_i^n]_s^2) - \int_{t_{i-1}^n}^{t_i^n}\int_\Omega \nabla_u W(u_i^n)\cdot v_i^n\,dxdt \end{aligned} \tag{1.3.16}$$

where we used the standard inequality $2a\cdot(b-a) \leq |b|^2 - |a|^2$, for $a, b \in \mathbb{R}^m$. Let us focus on the last term in the previous expression: for any $t \in (t_{i-1}^n, t_i^n]$, we write

$$\begin{aligned} &-\int_{t_{i-1}^n}^{t_i^n}\int_\Omega \nabla_u W(u_i^n)\cdot v_i^n\,dxdt = -\int_{t_{i-1}^n}^{t_i^n}\int_\Omega \nabla_u W(\bar{u}^n(t))\cdot\bar{v}^n(t)\,dxdt \\ &= -\int_{t_{i-1}^n}^{t_i^n}\int_\Omega \nabla_u W(u^n(t))\cdot\bar{v}^n(t)\,dxdt - \int_{t_{i-1}^n}^{t_i^n}\int_\Omega (\nabla_u W(\bar{u}^n(t)) - \nabla_u W(u^n(t))\cdot\bar{v}^n(t)\,dxdt \end{aligned}$$

We recognize in the first integral a derivative, so that

$$-\int_{t_{i-1}^n}^{t_i^n}\int_\Omega \nabla_u W(u^n(t))\cdot\bar{v}^n(t)\,dxdt = -\int_{t_{i-1}^n}^{t_i^n}\int_\Omega \frac{d}{dt}W(u^n(t))\,dxdt = \int_\Omega \left[W(u_{i-1}^n) - W(u_i^n)\right]\,dx$$

On the other hand, since $\bar{u}^n$ and u^n are just different interpolations of the same data and $\nabla_u W$ is Lipschitz continuous by assumption, the second integral can be estimated as

$$\begin{aligned} &-\int_{t_{i-1}^n}^{t_i^n}\int_\Omega (\nabla_u W(\bar{u}^n(t)) - \nabla_u W(u^n(t))\cdot\bar{v}^n(t)\,dxdt \leq K\int_{t_{i-1}^n}^{t_i^n}\int_\Omega |\bar{u}^n(t) - u^n(t)||\bar{v}^n(t)|\,dxdt \\ &= K\int_{t_{i-1}^n}^{t_i^n}\int_\Omega |u_i^n - (u_i^n + (t - t_i^n)v_i^n)|\cdot|v_i^n|\,dxdt = \frac{\tau_n^2}{2}K\int_\Omega |v_i^n|^2\,dx \end{aligned}$$

Hence, inequality (1.3.16) leads to

$$\begin{aligned} 0 \leq &\frac{1}{2}\left(||v_{i-1}^n||^2_{L^2(\Omega)} - ||v_i^n||^2_{L^2(\Omega)}\right) + \frac{1}{2}([u_{i-1}^n]_s^2 - [u_i^n]_s^2) \\ &+ \int_\Omega \left[W(u_{i-1}^n) - W(u_i^n))\right]\,dx + \frac{\tau_n^2}{2}K||v_i^n||^2_{L^2(\Omega)} \end{aligned}$$

Taking the sum for $i = 1,\dots,k$, with $1 \leq k \leq n$, we get

$$\begin{aligned} E_k^n &:= \frac{1}{2}||v_k^n||^2_{L^2(\Omega)} + \frac{1}{2}[u_k^n]_s^2 + \int_\Omega W(u_k^n)\,dx \\ &\leq \frac{1}{2}||v_0||^2_{L^2(\Omega)} + \frac{1}{2}[u_0]_s^2 + \int_\Omega W(u_0)\,dx + \frac{\tau_n^2}{2}K\sum_{i=1}^k ||v_i^n||^2_{L^2(\Omega)} \end{aligned} \tag{1.3.17}$$

In particular, we have

$$||v_k^n||^2_{L^2(\Omega)} \le 2E(u(0)) + \tau_n^2 K \sum_{i=1}^{k} ||v_i^n||^2_{L^2(\Omega)}$$

for any $k = 1, \dots, n$. For n large enough so that $(1 - \tau_n^2 K) > 1/2$, we write

$$||v_k^n||^2_{L^2(\Omega)} \le \frac{1}{(1-\tau_n^2 K)} \left(2E(u(0)) + \tau_n^2 K \sum_{i=1}^{k-1} ||v_i^n||^2_{L^2(\Omega)} \right) \tag{1.3.18}$$

Then, in view of the discrete Gronwall's inequality (cf. Proposition 1.6.1), we obtain that

$$||v_i^n||^2_{L^2(\Omega)} \le \bar{C} \quad \text{for every } i = 1, \dots, n \tag{1.3.19}$$

with $\bar{C} = \bar{C}(E(u(0)), K)$. Taking into account (1.3.19) into (1.3.17) we finally get

$$E_k^n = \frac{1}{2}||v_k^n||^2_{L^2(\Omega)} + \frac{1}{2}[u_k^n]_s^2 + \int_\Omega W(u_k^n)\, dx \le E(u(0)) + \frac{\tau_n^2}{2} K \sum_{i=1}^{k} \bar{C} \le E(u(0)) + \left(\frac{T}{2} K \bar{C}\right) \tau_n$$

for every $k = 1, \dots, n$, which is the sought for conclusion.

Thanks to the energy bound of Proposition 1.3.4 we can now provide a suitable uniform bound on $\nabla_u W(\bar{u}^n)$, which is one of the main ingredients to be able to pass to the limit in (1.3.15).

Proposition 1.3.5. *Let $\bar{u}^n$ be the piecewise constant interpolant constructed in* (1.3.13)*. Then, $\nabla_u W(\bar{u}^n(t))$ is bounded in $L^2(\Omega)$ uniformly in t and n.*

Proof. We first observe that $\bar{u}^n$ is bounded in $L^2(\Omega)$ uniformly in t and n. Indeed, one has

$$\begin{aligned} ||u^n(t_2,.) - u^n(t_1,.)||^2_{L^2(\Omega)} &= \int_\Omega \left| \int_{t_1}^{t_2} u_t^n(t,x) dt \right|^2 dx \le (t_2 - t_1) \int_\Omega \int_{t_1}^{t_2} |u_t^n(t,x)|^2 dt dx \\ &= (t_2 - t_1) \int_{t_1}^{t_2} \int_\Omega |u_t^n(t,x)|^2 dx dt \le C(t_2 - t_1)^2, \end{aligned} \tag{1.3.20}$$

for any $t_1 < t_2$ in $[0,T]$, where we made use of Jensen's inequality, Fubini's theorem and the uniform bound on u_t^n in $L^2(\Omega)$ provided by Proposition 1.3.4. That implies that u^n is bounded in $L^2(\Omega)$ uniformly in t and n, and so is $\bar{u}^n$ since $\lim_{n\to\infty} \sup_{t\in[0,T]} ||u^n(t,x) - \bar{u}^n(t,x)||^2_{L^2(\Omega)} = 0$. For every fixed $t \in [0,T]$, this uniform L^2-bound, combined with the Lipschitz continuity of $\nabla_u W$ and with boundedness of Ω, provides

$$\int_\Omega |\nabla_u W(\bar{u}^n(t))|^2\, dx \le C_1 \int_\Omega (|\bar{u}^n(t)| + 1)^2\, dx \le C_2 \tag{1.3.21}$$

uniformly in t and n.

We are now in the position to prove the convergence of u^n, $\bar{u}^n$, $W(\bar{u}^n)$ and $\nabla_u W(\bar{u}^n)$.

Proposition 1.3.6 (Convergence of u^n and v^n)**.** *There exist a subsequence of steps $\tau_n \to 0$ and a function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$, with $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$, such that*

(i) $u^n \to u$ *in* $C^0([0,T];L^2(\Omega))$,

(ii) $u^n_t \rightharpoonup^* u_t$ *in* $L^\infty(0,T;L^2(\Omega))$,

(iii) $u^n(t) \rightharpoonup u(t)$ *in* $\tilde{H}^s(\Omega)$ *for any* $t \in [0,T]$,

(iv) $v^n \to u_t$ *in* $C^0([0,T];H^{-s}(\Omega))$,

(v) $v^n_t \rightharpoonup^* u_{tt}$ *in* $L^\infty(0,T;H^{-s}(\Omega))$.

Proof. The existence of a limit function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and points (i), (ii) and (iii) follow from Proposition 1.3.4 combined with Ascoli-Arzelà's Theorem (for details see, e.g., [18, Proposition 6]).

To prove (iv) and (v), we observe that from (1.3.15), with the aid of Proposition 1.3.4 and Proposition 1.3.5, we have that $v^n_t(t)$ is bounded in $H^{-s}(\Omega)$ uniformly in t and n. Combining this with the L^2-bound on the velocities v^n_i, we have

$$v^n \text{ bounded in } L^\infty(0,T;L^2(\Omega)) \text{ and in } W^{1,\infty}(0,T;H^{-s}(\Omega)) \tag{1.3.22}$$

uniformly in t, n, and at the same time, for any given $\varphi \in H^s(\Omega)$ and for all $0 \le t_1 < t_2 \le T$, we have

$$\begin{aligned}\int_\Omega (v^n(t_2) - v^n(t_1)) \cdot \varphi dx &= \int_\Omega \int_{t_1}^{t_2} v^n_t dt \cdot \varphi dx = \int_\Omega \int_{t_1}^{t_2} v^n_t \cdot \varphi dt dx = \int_{t_1}^{t_2} \int_\Omega v^n_t \cdot \varphi dx dt \\ &\le \int_{t_1}^{t_2} ||v^n_t||_{H^{-s}} ||\varphi||_{H^s} dt \le C ||\varphi||_{H^s} (t_2 - t_1).\end{aligned}$$

Thus, there exists $v \in W^{1,\infty}(0,T;H^{-s}(\Omega))$ such that

$$v^n \to v \text{ in } C^0([0,T];H^{-s}(\Omega)) \quad \text{and} \quad v^n_t \rightharpoonup^* v_t \text{ in } L^\infty(0,T;H^{-s}(\Omega)).$$

Indeed, we have $v(t) = u_t(t)$ as elements of $L^2(\Omega)$ for a.e. $t \in [0,T]$: take $t \in (t^n_{i-1}, t^n_i]$ and $\varphi \in \tilde{H}^s(\Omega)$, we observe that $u^n_t(t) = v^n(t^n_i)$, so that

$$\begin{aligned}\int_\Omega (u^n_t(t) - v^n(t)) \cdot \varphi \, dx &= \int_\Omega (v^n(t^n_i) - v^n(t)) \cdot \varphi \, dx = \int_\Omega \left(\int_t^{t^n_i} v^n_t(s) \, ds \right) \cdot \varphi \, dx \\ &\le \tau_n ||v^n_t||_{L^\infty(0,T;H^{-s}(\Omega))} ||\varphi||_s\end{aligned}$$

which implies, for any $\psi(t,x) = \varphi(x)\eta(t)$ with $\varphi \in \tilde{H}^s(\Omega)$ and $\eta \in C^1_0([0,T])$, that

$$\begin{aligned}&\int_0^T \left[\int_\Omega (u_t(t) - v(t)) \cdot \varphi \, dx \right] \eta(t) \, dt = \int_0^T \int_\Omega (u_t(t) - v(t)) \cdot \psi \, dx dt \\ &= \lim_{n\to\infty} \int_0^T \int_\Omega (u^n_t(t) - v^n(t)) \cdot \psi \, dx dt = \lim_{n\to\infty} \int_0^T \left[\int_\Omega (u^n_t(t) - v^n(t)) \cdot \varphi \, dx \right] \eta(t) \, dt \\ &\le \lim_{n\to\infty} \tau_n T ||v^n_t||_{L^\infty(0,T;H^{-s}(\Omega))} ||\varphi||_s ||\eta||_\infty = 0.\end{aligned}$$

This implies

$$\int_\Omega (u_t(t) - v(t)) \cdot \varphi \, dx = 0 \quad \text{ for all } \varphi \in \tilde{H}^s(\Omega) \text{ and a.e. } t \in [0,T],$$

which yields $v(t) = u_t(t)$ for a.e. $t \in [0,T]$. Thus, $v_t = u_{tt}$ and $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$. □

Remark 1.3.7. From point (iv) in Proposition 1.3.6 we have that $v^n \to u_t$ in $C^0([0,T]; H^{-s}(\Omega))$. At the same time, due to Proposition 1.3.4, $v^n(t)$ is uniformly bounded in $L^2(\Omega)$. Thus, $v^n(t) \rightharpoonup u_t(t)$ in $L^2(\Omega)$, which in turn provides

$$u_t^n(t) \rightharpoonup u_t(t) \text{ in } L^2(\Omega) \quad \text{for any } t \in [0,T].$$

Proposition 1.3.8 (Convergence of $\bar{u}^n$ and $W(\bar{u}^n)$)**.** *Let u be the limit function obtained in Proposition 1.3.6. Then, up to a subsequence,*

(i) $\bar{u}^n \rightharpoonup^* u$ *in* $L^\infty(0,T; \tilde{H}^s(\Omega))$,

(ii) $\bar{u}^n(t) \rightharpoonup u(t)$ *in* $\tilde{H}^s(\Omega)$ *for any* $t \in [0,T]$,

(iii) $W(\bar{u}^n) \to W(u)$ *in* $C^0([0,T]; L^1(\Omega))$.

Proof. Regarding (i) and (ii) one can proceed as in [18, Proposition 7]. By construction, taking into account Proposition 1.3.4, we have

$$\sup_{t\in[0,T]} \int_\Omega |u^n(t,x) - \bar{u}^n(t,x)|^2 \, dx = \sum_{i=1}^n \sup_{t\in[t_{i-1}^n, t_i^n]} (t - t_i^n)^2 \int_\Omega |v_i^n|^2 \, dx \le \tau_n^2 \sum_{i=1}^n ||v_i^n||^2_{L^2(\Omega)} \le C\tau_n \tag{1.3.23}$$

which implies $\bar{u}^n \to u$ in $L^\infty(0,T; L^2(\Omega))$. Furthermore, again by Proposition 1.3.4, $\bar{u}^n(t)$ is bounded in $\tilde{H}^s(\Omega)$ uniformly in t and n, so that we have $\bar{u}^n \rightharpoonup^* u$ in $L^\infty(0,T; \tilde{H}^s(\Omega))$. Thanks to point (i) in Proposition 1.3.6, we also obtain pointwise convergence $\bar{u}^n(t) \rightharpoonup u(t)$ in $\tilde{H}^s(\Omega)$ for any $t \in [0,T]$, which is (ii).

For the convergence of $W(\bar{u}^n)$, we first observe a following property of W: there are positive constants C_1, C_2 such that

$$|W(x) - W(y)| \le (C_1(|x| + |y|) + C_2)(|x - y|) \tag{1.3.24}$$

for any $x, y \in \mathbb{R}^m$. Indeed, let $x, y \in \mathbb{R}^m$ be fixed, by the Mean Value Theorem there exists $c \in [x,y]$, here we denote $[x,y]$ the segment connecting x and y in $\mathbb{R}^m$, such that

$$W(x) - W(y) = \nabla W(c) \cdot (x - y). \tag{1.3.25}$$

Thus, from the Lipshitz continuity of ∇W we deduce that

$$\begin{aligned} |W(x) - W(y)| &\le |\nabla W(c)||x - y| \\ &\le (C_1|c| + C_2)|x - y| \\ &\le (C_1 \max\{|x|, |y|\} + C_2)|x - y| \\ &\le (C_1(|x| + |y|) + C_2)|x - y| \end{aligned} \tag{1.3.26}$$

where C_1, C_2 are positive constants independent of c, x, y.
Then, let $t \in [0,T]$ we have

$$\begin{aligned} \int_\Omega |W(\bar{u}^n(t)) - W(u(t))| dx &\le \int_\Omega (C_1(|\bar{u}^n(t)| + |u(t)| + C_2)|\bar{u}_n(t) - u(t)| dx \\ &\le \int_\Omega (C_1|\bar{u}^n(t) + u(t)| + C_2)|\bar{u}^n(t) - u(t)| dx \\ &\le ||(C_1|\bar{u}^n(t) + u(t)| + C_2)||_{L^2(\Omega)} ||\bar{u}^n(t) - u(t)||_{L^2(\Omega)} \\ &\le C_3 ||\bar{u}^n(t) - u(t)||_{L^2(\Omega)}, \end{aligned} \tag{1.3.27}$$

where C_3 is a constant independent of t, n due to the boundedness of $\bar{u}^n, u^n$ in $L^2(\Omega)$ uniformly in t, n and point (i) in Proposition 1.3.6. In addition, once again from point (i) in Proposition 5 combined with (1.3.23), it implies that $\bar{u}^n \to u$ in $C^0([0,T]; L^2(\Omega))$. So, we can conclude that $W(\bar{u}^n) \to W(u)$ in $C^0([0,T]; L^1(\Omega))$.

Proposition 1.3.9 (Convergence of $\nabla_u W(\bar{u}^n)$)**.** *Let u be the limit function obtained in Proposition 1.3.6. Then, up to a subsequence, $\nabla_u W(\bar{u}^n) \rightharpoonup^* \nabla_u W(u)$ in $L^\infty(0,T; H^{-s}(\Omega))$.*

Proof. The same spirit of the analysis of the convergence $W(\bar{u}^n)$ in Proposition 1.3.8, one can check that, up to a subsequence,

$$\nabla_u W(\bar{u}^n) \to \nabla_u W(u) \text{ in } L^2((0,T) \times \Omega). \tag{1.3.28}$$

From Proposition 1.3.5, we observe that $\nabla_u W(\bar{u}^n)$ is bounded in $H^{-s}(\Omega)$ uniformly in t and n, this implies our conclusion.

1.3.2 Proof of Theorem 1.3.3

Proof of Theorem 1.3.3 (i)**.** Let u be the cluster point obtained in Proposition 1.3.6, we shall prove that u is a weak solution of (1.3.1). In fact, for each $n > 0$, from (1.3.15) one has

$$\int_0^T \int_\Omega v_t^n(t) \cdot \varphi(t)\, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s\, dt + \int_0^T \int_\Omega \nabla_u W(\bar{u}^n(t)) \cdot \varphi(t)\, dxdt = 0$$

for any $\varphi \in L^1(0,T; \tilde{H}^s(\Omega))$. Passing to the limit as $n \to \infty$, using Propositions 1.3.6, 1.3.8, 1.3.9, we immediately get

$$\int_0^T \langle u_{tt}(t), \varphi(t)\rangle dt + \int_0^T [u(t), \varphi(t)]_s\, dt + \int_0^T \int_\Omega \nabla_u W(u(t)) \cdot \varphi(t)\, dxdt = 0. \tag{1.3.29}$$

The fact that $u(0) = u_0$ and $u_t(0) = v_0$ follows observing that $u^n(0) = u_0$ and $v^n(0) = v_0$ for all n and that, thanks to Proposition 1.3.6, $u^n \to u$ in $C^0([0,T]; L^2(\Omega))$ and $v^n \to u_t$ in $C^0([0,T]; H^{-s}(\Omega))$. Finally, the verification of energy inequality is easily obtained by passing to the limit in energy estimate in Proposition 1.3.4.

In order to prove Theorem 1.3.3 (ii), i.e. energy conservation for the limiting solution u under more regular initial data, we actually have to slightly modify the approximating scheme, as precised in the following

Proposition 1.3.10. *Let $u_0 \in \tilde{H}^{2s}(\Omega), v_0 \in \tilde{H}^s(\Omega)$, and set $u_{-1}^n = u_0 - \tau_n v_0^n$ where $\{v_0^n\}_n \subset \tilde{H}^s(\Omega)$, $v_0^n \to v_0$ in $\tilde{H}^s(\Omega)$, and such that $||u_0 - \tau_n v_0^n||_{\tilde{H}^{2s}(\Omega)} \leq C$ with C independent of n. Then, let $u^n, \bar{u}^n$ be approximate solutions of* (1.3.1) *satisfying the equation* (1.3.15)*, and u be a limiting solution, we have that $u \in W^{1,\infty}(0,T; \tilde{H}^s(\Omega))$, $u_t \in W^{1,\infty}(0,T; L^2(\Omega))$. Moreover, for any $0 \leq t_1 < t_2 \leq T$, the energy $E(u)$ satisfies*

$$E(u(t_1)) = E(u(t_2)). \tag{1.3.30}$$

Remark 1.3.11. Observe that slightly changing the approximating scheme in the initialization step, by setting $u_{-1}^n = u_0 - \tau_n v_0^n$, doesn't affect the properties of the approximate solutions, namely the same energy estimate as in Proposition 1.3.4 holds true, and hence Proposition 1.3.6, 1.3.8, 1.3.9 remain valid.

Proposition 1.3.10 is a consequence of the following

Lemma 1.3.12. *Let* $u_0 \in \tilde{H}^{2s}(\Omega), v_0 \in \tilde{H}^s(\Omega)$ *and* $u_{-1}^n = u_0 - \tau_n v_0^n$ *be as in Proposition* 1.3.10. *Then, there exists a constant* C *independent of* n *such that:*

$$\int_\Omega |\frac{u_1^n - 2u_0^n + u_{-1}^n}{\tau_n^2}|^2 dx + [\frac{u_1^n - u_0^n}{\tau_n}]_s^2 \leq C. \tag{1.3.31}$$

Proof. By substituting the test function $\varphi = \frac{u_1^n - 2u_0^n + u_{-1}^n}{\tau_n^2}$ in the Euler's equation (1.3.12) with $i = 1$, we obtain that

$$\begin{aligned} &\int_\Omega |\frac{u_1^n - 2u_0^n + u_{-1}^n}{\tau_n^2}|^2\, dx + [u_1^n, \frac{u_1^n - 2u_0^n + u_{-1}^n}{\tau_n^2}]_s + \int_\Omega \nabla_u W(u_1^n) \cdot \frac{u_1^n - 2u_0^n + u_{-1}^n}{\tau_n^2} dx = 0 \\ \iff &\int_\Omega |a_1^n|^2 dx + [\frac{u_1^n - u_0^n}{\tau_n}]_s^2 - [\frac{u_0^n - u_{-1}^n}{\tau_n}]_s^2 + [u_{-1}^n, a_1^n]_s + \int_\Omega \nabla_u W(u_1^n).a_1^n dx = 0 \end{aligned} \tag{1.3.32}$$

where $a_1^n = \frac{u_1^n - 2u_0^n + u_{-1}^n}{\tau_n^2}$. It implies that

$$\int_\Omega |a_1^n|^2 dx + [\frac{u_1^n - u_0^n}{\tau_n}]_s^2 \leq [\frac{u_0^n - u_{-1}^n}{\tau_n}]_s^2 + ||\nabla_u W(u_1)||_{L^2(\Omega)} ||a_1^n||_{L^2(\Omega)} + |[u_{-1}^n, a_1^n]_s| \tag{1.3.33}$$

On the other hand, we observe that $[\frac{u_0^n - u_{-1}^n}{\tau_n}]_s^2 = [v_0^n]_s^2$ and

$$\begin{aligned} |[u_{-1}^n, a_1^n]_s| &= |\int_{\mathbb{R}^d} |\xi|^{2s} \mathcal{F}(u_{-1}^n)(\xi). \mathcal{F}(a_1^n)(\xi) d\xi| \\ &\leq \left(\int_{\mathbb{R}^d} |\xi|^{4s} |\mathcal{F}(u_{-1}^n)(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} |\mathcal{F}(a_1^n)(\xi)|^2 d\xi \right)^{\frac{1}{2}} \leq [u_{-1}^n]_{2s} ||a_1^n||_{L^2(\Omega)}. \end{aligned} \tag{1.3.34}$$

From this observation combined with the hypothesis, Proposition 1.3.5, and the inequality (1.3.33), we can deduce that there exist constants C_1, C_2 independent of n such that

$$\int_\Omega |a_1^n|^2 dx \leq \int_\Omega |a_1^n|^2 + [\frac{u_1^n - u_0^n}{\tau_n}]_s^2 \leq C_1 ||a_1^n||_{L^2(\Omega)} + C_2 \tag{1.3.35}$$

This gives rise to the uniform bound on $\int_\Omega |a_1^n|^2 dx$, and it also follows that $[\frac{u_1^n - u_0^n}{\tau_n}]_s^2$ is uniformly bounded, which is the sought conclusion.

We are now ready to prove Proposition 1.3.10 and hence Theorem 1.3.3(ii) :

Proof of Theorem 1.3.3 (ii)**.** For each n fixed, let the Euler's equation at the step i subtract the step $i - 1$ divided by τ_n, we obtain that

$$\int_\Omega (\frac{v_i^n - 2v_{i-1}^n + v_{i-2}^n}{\tau_n^2}) \cdot \varphi\, dx + [v_i^n, \varphi]_s + \int_\Omega \frac{\nabla_u W(u_i^n) - \nabla_u W(u_{i-1}^n)}{\tau_n} \cdot \varphi dx = 0 \quad \text{for every } \varphi \in \tilde{H}^s(\Omega). \tag{1.3.36}$$

for $i = 2, \ldots, n$, and $v_i^n = \frac{u_i^n - u_{i-1}^n}{\tau_n}$, $i = 0, \ldots, n$. Now, let $a_i^n = \frac{v_i^n - v_{i-1}^n}{\tau_n}$, and substituting the test function $\varphi = v_{i-1}^n - v_i^n$ into the equation (1.3.36). One has

$$\int_\Omega (a_{i-1}^n - a_i^n) \cdot a_i^n\, dx + [v_i^n, v_{i-1}^n - v_i^n]_s + \int_\Omega \frac{\nabla_u W(u_i^n) - \nabla_u W(u_{i-1}^n)}{\tau_n} \cdot (v_{i-1}^n - v_i^n) dx = 0. \tag{1.3.37}$$

with $i = 2, \ldots, n$. From the equation (1.3.37) and due to the Lipshitz continuity condition of ∇W, it follows that

$$
\begin{aligned}
0 &\le \int_\Omega (a_{i-1}^n - a_i^n) \cdot a_i^n \, dx + [v_i^n, v_{i-1}^n - v_i^n]_s + \tau_n \int_\Omega K|v_i^n| \cdot |a_i^n| dx \\
&\le \int_\Omega \frac{1}{2}(|a_{i-1}^n|^2 - |a_i^n|^2) \, dx + \frac{1}{2}([v_{i-1}^n]_s^2 - [v_i^n]_s^2) + \tau_n \int_\Omega K|v_i^n| \cdot |a_i^n| dx \\
&\le \int_\Omega \frac{1}{2}(|a_{i-1}^n|^2 - |a_i^n|^2) \, dx + \frac{1}{2}([v_{i-1}^n]_s^2 - [v_i^n]_s^2) + \frac{1}{2}\tau_n \int_\Omega K(|v_i^n|^2 + |a_i^n|^2) dx
\end{aligned}
\tag{1.3.38}
$$

Let's sum up the previous inequality for $i = 2, \ldots, k$ one has

$$
\begin{aligned}
\int_\Omega |a_k^n|^2 \, dx + [v_k^n]_s^2 &\le \int_\Omega |a_1^n|^2 \, dx + [v_1^n]_s^2 + \tau_n K \left(\Sigma_{i=2}^k \int_\Omega (|v_i^n|^2 + |a_i^n|^2) dx \right) \\
&\le ||a_1^n||_{L^2}^2 + [\frac{u_1 - u_0}{\tau_n}]_s^2 + \tau_n K \left(\Sigma_{i=2}^k \int_\Omega |a_i^n|^2 \right) + K' \tau_n (k-1). \\
&\le C + \tau_n K \left(\Sigma_{i=2}^k \int_\Omega |a_i^n|^2 \right) + K' T.
\end{aligned}
\tag{1.3.39}
$$

here we have made use of the Lemma 1.3.12, and the uniform bound in $L^2(\Omega)$ of v_i^n. From (1.3.39), we can deduce that

$$
\int_\Omega |a_k^n|^2 \, dx \le C + \tau_n K \left(\Sigma_{i=2}^k \int_\Omega |a_i^n|^2 \right) + K' T \tag{1.3.40}
$$

and, then in view of Gronwall's inequality Proposition 1.6.1, there exists a constant $C(T)$ such that

$$
\int_\Omega |a_k^n|^2 \, dx \le C(T) \tag{1.3.41}
$$

It also implies that $[v_i^n]_s^2$ is uniformly bounded i.e. there exists a constant $C_1(T)$ such that

$$
[v_i^n]_s^2 \le C_1(T). \tag{1.3.42}
$$

Due to uniform bounds (1.3.41), (1.3.42), and by the analysis as in the proof of Proposition 1.3.6, 1.3.8, one can show that $u \in W^{1,\infty}(0,T; \tilde{H}^s(\Omega))$, $u_t \in W^{1,\infty}(0,T; L^2(\Omega))$. Then, by substituting the test function $\varphi = I_{[t_1, t_2]} \times u_t$ in the weak equation of u, where $0 < t_1 < t_2 < T$, and $I_{[t_1, t_2]}$ is the indicator function on the time interval $[t_1, t_2]$, we obtain that

$$
\begin{aligned}
&\int_{t_1}^{t_2} < u_{tt}(t), u_t(t) > dt + \int_{t_1}^{t_2} [u(t), u_t(t)]_s dt + \int_{t_1}^{t_2} \int_\Omega \nabla_u W(u(t,x)) u_t(t,x) dx dt = 0 \\
&\Longleftrightarrow \int_{t_1}^{t_2} \frac{dE(u(t))}{dt} dt = 0 \\
&\Longleftrightarrow E(u(t_1)) = E(u(t_2))
\end{aligned}
\tag{1.3.43}
$$

i.e. E is constant inside the interval $(0,T)$, and we can extend the conservative property at endpoints by using the absolute continuity in time of u, u_t, and $W(u)$ in appropriate energy spaces.

□

Proof of Theorem 1.3.3, *(iii)***.** We are left to prove the uniqueness property: Indeed, let $v \in X$, and consider the following quantity

$$K(t) = \frac{1}{2}||u_t(t) - v_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t) - v(t)]^2_s.$$

From Remark 1.3.2, one has

$$\begin{aligned}
&\int_0^t \frac{dK(t')}{dt'} dt' = \int_0^t \int_\Omega < u_{t't'} - v_{t't'}, u_{t'} - v_{t'} > dt' + \int_0^t [u(t') - v(t'), u_{t'}(t') - v_{t'}(t')]_s dt' \\
&= -\int_0^t \left(\int_\Omega (\nabla_u W(u) - \nabla_v W(v))(u_{t'} - v_{t'}) dx \right) dt'
\end{aligned} \tag{1.3.44}$$

here we have made use of the weak equation of u with the test function $u_{t'} \times I_{[0,\,t]}$ subtracting the one of v with the test function $v_{t'} \times I_{[0,\,t]}$, $I_{[0,\,t]}$ is the indicator function on the time interval $[0, t]$. From the equation (2.2.1) combined with Lipshitz continuity property of ∇W and Poincaré-type inequality in Proposition 1.6.3, we obtain that

$$\begin{aligned}
K(t) &\le K \int_0^t \int_\Omega |u - v||u_t - v_t| dx \\
&\le \frac{1}{2} K \int_0^t \left(||u(t) - v(t)||^2_{L^2(\Omega)} + ||u_{t'}(t') - v_{t'}(t')||^2_{L^2(\Omega)} \right) dt' \\
&\le C_s \int_0^t \left(\frac{1}{2}[u(t') - v(t')]^2_s + \frac{1}{2}||u_{t'}(t') - v_{t'}(t')||_{L^2(\Omega)} \right) dt' \\
&\le C_s \int_0^t K(t') dt'
\end{aligned} \tag{1.3.45}$$

for some postive constants C_s, by Gronwall's inequality in Proposition 1.6.2, it implies that

$$K(t) = 0$$

for any $t \in [0, T]$ here we extend to the endpoint $t = T$ by using the absolute continuity in time of u, v. Then, it is easy to show that

$$u(t) = v(t) \text{ in } \tilde{H}^s(\Omega)$$

for any $t \in [0, T]$.

1.3.3 Singular limits of nonlinear wave equations

We turn our attention to the application of the results in the previous section to the singular limits of semilinear wave equation (2.1.3) related to topological defects (timelike minimal surfaces in Minkowski space). We consider the hyperbolic Ginzburg-Landau equation:

$$\begin{cases}
\varepsilon^2 \left(\dfrac{\partial^2 u_\varepsilon}{\partial^2 t} - \Delta u_\varepsilon \right) + \nabla_u W(u_\varepsilon) = 0 & \text{in } (0, T) \times \Omega, \\
u_\varepsilon(0, x) = u^0_\varepsilon(x) & \text{in } \Omega, \\
u_{\varepsilon t}(0, x) = v^0_\varepsilon(x) & \text{in } \Omega,
\end{cases} \tag{1.3.46}$$

where $\varepsilon > 0$ is a small parameter, Ω is a bounded domain in $\mathbb{R}^d$, for functions

$$u_\varepsilon : (0,T) \times \Omega \longrightarrow \mathbb{R}^m, \tag{1.3.47}$$

we will focus on the cases $m = 1$, $m = 2$, and W is a non-convex balanced double well potential of class C^2. So as to apply the results in Section 1.3, for simplicity we assume that the potential is given by

$$W(u) = \frac{(1 - |u|^2)^2}{1 + |u|^2}. \tag{1.3.48}$$

Let us now introduce relevant quantities when dealing with topological defects: the first one is the gradient ∇u_ε (for $m = 1$), and the second is the Jacobian 2-form, $Ju_\varepsilon = du^1_\varepsilon \wedge du^2_\varepsilon$ (in the case $m = 2$) defined on $(0,T) \times \Omega$. Both will be considered as distributions (concerning the distributional Jacobian, see for instance [48, 2]) . We can prove that under natural bounds on initial energy they enjoy compactness properties and concentrate on codimension m rectifiable sets in $(0,T) \times \Omega$ as $\varepsilon \to 0^+$. We have

Proposition 1.3.13. *Let* $(u_\varepsilon)_{0<\varepsilon<1}$ *be a sequence of solutions of* (1.3.46) *constructed by the approximating scheme in Section* 1.3 *for each* $0 < \varepsilon < 1$ *fixed such that* $\frac{E(u_\varepsilon(0))}{k_\varepsilon} \le C$ *where* C *is a constant independent of* ε, $k_\varepsilon = \frac{1}{\varepsilon}$ *for* $m = 1$ *and* $k_\varepsilon = |\log \varepsilon|$ *for* $m = 2$. *Then, up to a subsequence* $\varepsilon_n \to 0$,

- *In case* $m = 1$,
$$u_{\varepsilon_n} \to u \text{ in } L^1((0,T) \times \Omega),$$
where $u(t,x) \in \{-1, 1\}$ *for a.e.* $(t,x) \in (0,T) \times \Omega$, *and* $u \in BV((0,T) \times \Omega)$.

- *In case* $m = 2$,
$$Ju_{\varepsilon_n} \rightharpoonup J \text{ in } [C^{0,1}((0,T) \times \Omega)]^*,$$
where $\frac{1}{\pi} J$ *is a* $d - 1$ *dimensional integral current in* $(0,T) \times \Omega$.

Proof. In fact, for each ε, from Theorem 1.3.3 the solution u_ε which is constructed by the approximating scheme in Section 1.3 satisfies the energy inequality:

$$E(u_\varepsilon(t)) \le E(u_\varepsilon(0)) \tag{1.3.49}$$

for any $t \in [0,T]$. Recall that $E(u_\varepsilon(t))) = \frac{1}{2}||u_{\varepsilon t}(t)||^2_{L^2(\Omega)} + \frac{1}{2}||\nabla u_\varepsilon(t)||^2_{L^2(\Omega)} + \frac{1}{\varepsilon^2}||W(u_\varepsilon(t))||_{L^1}$. By assumption we have

$$\frac{E(u_\varepsilon(0))}{k_\varepsilon} \le C \tag{1.3.50}$$

where C is a constant independent of ε, $k_\varepsilon = \frac{1}{\varepsilon}$ for $m = 1$ and $k_\varepsilon = |\log \varepsilon|$ for $m = 2$.

Then,

- In the case $m = 1$, by integrating from 0 to T both side in (1.3.49) combined with (1.3.50) we obtain that
$$\int_{(0,T)\times\Omega} \varepsilon |\nabla_{t,x} u_\varepsilon(t,x)|^2 dxdt + \int_{(0,T)\times\Omega} \frac{1}{\varepsilon} W(u_\varepsilon(t,x)) dxdt \le TC \tag{1.3.51}$$
where $\nabla_{t,x}$ is the gradient in the space-time. In view of Modica-Mortola Theorem (see [60]), it follows that there exists a function $u \in BV((0,T) \times \Omega; \{-1,1\})$ such that u_ε converges

to u in $L^1((0,T)\times\Omega)$ up to a subsequence. Moreover, the reduced boundary of the set $\Sigma^1 = \{(t,x)\in(0,T)\times\Omega \,|\, u(t,x)=1\}$ denoted by $\partial^*\Sigma^1$ is a $d-$dimensional rectifiable set in $(0,T)\times\Omega$ (for the definition of reduced boundary, see [76]). The set $\partial^*\Sigma^1$ is said to be the jump set of u and it is a type of defects of the interfaces.

- In the complex case, following the results in [49], again from (1.3.49), up to a subsequence, we have that $Ju_\varepsilon \rightharpoonup J$ in $[C^{0,1}((0,T)\times\Omega)]^*$, where $\frac{1}{\pi}J$ is a $d-1$ dimensional integral current in $(0,T)\times\Omega$, which concentrates on $d-1$ dimensional rectifiable set Σ^2 so called the vorticity set.

□

To study the dynamics of jump and the vorticity sets one has to rely on the analysis of renormalized Lagrange density

$$\mu_\varepsilon = \frac{\ell(u_\varepsilon(t,x))}{k_\varepsilon}dxdt \tag{1.3.52}$$

where $\ell(u_\varepsilon) = \frac{-|u_{\varepsilon t}|^2+|\nabla u_\varepsilon|^2}{2} + \frac{W(u_\varepsilon)}{\varepsilon^2}$. In [61], Neu showed that certain solutions of (1.3.46) in case $m=1$ give rise to interfaces sweeping out a timelike lorentzian minimal surface of codimension 1. Further rigorous analysis were given in ([47, 77, 69]), where solutions of (1.3.46) having interfaces near a given timelike minimal surface were constructed. However, due to the presence of singularities, the validity of those results are only for short times. On the other hand, the limit behavior of hyperbolic Ginzburg-Landau equation (1.3.46) as $\varepsilon\to 0^+$ without restricting short times (i.e. also after the onset of singularities) has been treated in [9] under conditional assumptions that the measure μ_ε is shown to concentrate on a timelike lorentzian minimal submanifold of codimension m within the varifold framework developed in [10]. This has been proved by adapting the parabolic approach [6] to the hyperbolic setting through the analysis of the stress-energy tensor. We conjecture that the assumptions in [9] could be relaxed for the solutions constructed by our approximating scheme through exploiting the minimizing properties of our approximate solutions.

1.4 The obstacle problem for fractional semilinear wave equations

In this section, following the pipeline of [18], we move on to study the obstacle problem for the fractional semilinear wave equation. From now on we assume $m = 1$ and work with real valued functions. Given an open bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a function $g \in C^0(\bar{\Omega})$, $g < 0$ on $\partial\Omega$, we are interested in the obstacle problem described by

$$\begin{cases} u_{tt} + (-\Delta)^s u + W'(u) \geq 0 & \text{in } (0,T) \times \Omega \\ u(t,\cdot) \geq g & \text{in } [0,T] \times \Omega \\ (u_{tt} + (-\Delta)^s u + W'(u))(u-g) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.4.1}$$

with $u_0 \in \tilde{H}^s(\Omega)$, $u_0 \geq g$ a.e. in Ω, and $v_0 \in L^2(\Omega)$ (with W as in Section 1.3 with $m = 1$). We define a weak solution of (1.4.1) as follows:

Definition 1.4.1. *Let $T > 0$. We say $u = u(t,x)$ is a weak solution of* (1.4.1) *in* $(0,T)$ *if*

1. *$u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and $u(t,x) \geq g(x)$ for a.e. $(t,x) \in (0,T) \times \Omega$;*
2. *there exist weak left and right derivatives $u_t^\pm$ on $[0,T]$ (with appropriate modifications at endpoints);*
3. *for all $\varphi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^1(0,T;\tilde{H}^s(\Omega))$ with $\varphi \geq 0$, $spt\varphi \subset [0,T)$, we have*
$$-\int_0^T \int_\Omega u_t \varphi_t \, dxdt + \int_0^T [u,\varphi]_s \, dt + \int_0^T \int_\Omega W'(u)\varphi dxdt - \int_\Omega v_0 \, \varphi(0) \, dx \geq 0$$
4. *the initial conditions are satisfied in the following sense*
$$u(0,\cdot) = u_0, \quad \int_\Omega (u_t^+(0) - v_0)(\varphi - u_0) \, dx \geq 0 \quad \forall \varphi \in \tilde{H}^s(\Omega), \varphi \geq g.$$

This section is then dedicated to prove the existence of such a weak solution, combining results from the previous section and extensions of arguments of [18, Section 4].

Theorem 1.4.2. *There exists a weak solution u of the obstacle problem for fractional semilinear wave equation* (1.4.1)*, and u satisfies*

$$\frac{1}{2}||u_t^\pm(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]^2_s + ||W(u(t))||_{L^1(\Omega)} \leq \frac{1}{2}||v_0||^2_{L^2(\Omega)} + \frac{1}{2}[u_0]^2_s + ||W(u_0)||_{L^1(\Omega)} \tag{1.4.2}$$

for a.e. $t \in [0,T]$.

Remark 1.4.3 (Non-uniqueness and energy behaviour)*.* The notion of weak solutions introduced in Definition 1.4.1 can be seen as the minimal requirement we can make, i.e., to control "upward" variations. This leaves us with less control on the behaviour of "downward" moving regions, which is intended in order to allow sudden adjustments when hitting the obstacle. However, these coarse requirements lead at the same time to non-uniqueness of solutions. Generally speaking, uniqueness and in particular existence of energy preserving solutions for (1.4.1) is still an open problem, with only partial results in specific one dimensional configurations hinging on purely one dimensional arguments (see, e.g., [72] for a specific 1d setting with local energy conservation at impacts). Within our framework a local (in space and time) energy conservation is expected whenever we are "away" from the obstacle (in the spirit of Proposition 1.4.8 below), but deducing/imposing any additional condition at impact times would require the use of more technical local arguments that need further specific investigations.

1.4.1 Approximating scheme

For $n \in \mathbb{N}$, set $\tau_n = T/n$ and define $t_i^n = i\tau_n$, $0 \le i \le n$. Let $u_{-1}^n = u_0 - \tau_n v_0$, $u_0^n = u_0$ and define

$$K_g := \{u \in \tilde{H}^s(\Omega) \,|\, u \ge g \text{ a.e. in } \Omega\}.$$

For every $0 < i \le n$, given u_{i-2}^n and u_{i-1}^n, we define u_i^n as

$$u_i^n \in \arg\min_{u \in K_g} J_i^n(u),$$

where J_i^n is defined as in (1.3.11). Existence of u_i^n can be obtained through the direct method of calculus of variations thanks to the convexity of K_g. In order to provide a variational characterization of each minimizer u_i^n, take $\varphi \in K_g$ and consider the function $(1-\varepsilon)u_i^n + \varepsilon\varphi$, which belongs to K_g for any sufficiently small positive ε. Thus, by minimality of u_i^n, we have

$$\frac{d}{d\varepsilon} J_i^n(u_i^n + \varepsilon(\varphi - u_i^n))|_{\varepsilon=0} \ge 0,$$

which is equivalent to

$$\int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}(\varphi - u_i^n)\, dx + [u_i^n, \varphi - u_i^n]_s + \int_\Omega W'(u_i^n)(\varphi - u_i^n) dx \ge 0 \quad \text{for all } \varphi \in K_g. \tag{1.4.3}$$

Moreover, because every $\varphi \ge u_i^n$ is also an admissible test function, we obtain that

$$\int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}\varphi\, dx + [u_i^n, \varphi]_s + \int_\Omega W'(u_i^n)\varphi dx \ge 0 \quad \text{for all } \varphi \in \tilde{H}^s(\Omega), \varphi \ge 0. \tag{1.4.4}$$

We define $\bar{u}^n$ and u^n to be the piecewise constant and the piecewise linear interpolations in terms of $\{u_i^n\}_i$, just as in (1.3.13) and(1.3.14); furthermore, let v^n be the piecewise linear interpolant of velocities $v_i^n = (u_i^n - u_{i-1}^n)/\tau_n$, $0 \le i \le n$. Taking into account (1.4.4), integrating from 0 to T, we obtain

$$\int_0^T \int_\Omega \left(\frac{u_t^n(t) - u_t^n(t - \tau_n)}{\tau_n} \right) \varphi(t)\, dx dt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s\, dt + \int_0^T \int_\Omega W'(\bar{u}^n(t))\varphi(t) dx dt \ge 0$$

for all $\varphi \in L^1(0,T; \tilde{H}^s(\Omega))$, $\varphi(t,x) \ge 0$ for a.e. $(t,x) \in (0,T) \times \Omega$.

Remark 1.4.4 (Extension of the key estimate). By choosing the test function $\varphi = u_{i-1}^n$ in (1.4.3), we have

$$0 \le \int_\Omega \frac{(u_i^n - 2u_{i-1}^n + u_{i-2}^n)(u_{i-1}^n - u_i^n)}{\tau_n^2}\, dx + [u_i^n, u_{i-1}^n - u_i^n]_s + \int_\Omega W'(u_i^n)(u_{i-1}^n - u_i^n) dx$$

and following the proof of Proposition 1.3.4, we obtain the same energy estimate

$$\frac{1}{2}\|u_t^n(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}[\bar{u}^n(t)]_s^2 + ||W(\bar{u}^n(t))||_{L^1(\Omega)} \leq E(u(0)) + C\tau_n$$

for all $t \in [0,T]$, with $C = C(E(u(0)), K, T)$ a constant independent of n.

Given that the main energy estimate is still true, we can largely repeat the convergence proofs presented in the previous section.

Proposition 1.4.5 (Convergence of u^n, $\bar{u}^n$, $W(\bar{u}^n)$, and $W'(\bar{u}^n)$, obstacle case)**.** *There exists a subsequence of steps $\tau_n \to 0$ and a function $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ such that*

$$u^n \to u \text{ in } C^0([0,T];L^2(\Omega)), \qquad u^n(t) \rightharpoonup u(t) \text{ in } \tilde{H}^s(\Omega) \text{ for any } t \in [0,T],$$
$$u_t^n \rightharpoonup^* u_t \text{ in } L^\infty(0,T;L^2(\Omega)), \qquad \bar{u}^n \rightharpoonup^* u \text{ in } L^\infty(0,T;\tilde{H}^s(\Omega)).$$

Furthermore, $u(t,x) \geq g(x)$ for a.e. $(t,x) \in [0,T] \times \Omega$. Also,

$$W(\bar{u}^n) \to W(u) \text{ in } C^0([0,T];L^1(\Omega)), \quad W'(\bar{u}^n) \rightharpoonup^* W'(u) \text{ in } L^\infty(0,T;H^{-s}(\Omega)).$$

Proof. See the proof of Propositions 1.3.6, 1.3.8 and 1.3.9. The fact that $u(t,x) \geq g(x)$ for a.e. $(t,x) \in [0,T] \times \Omega$ follows by the fact that $u_i^n \in K_g$ for all n and $0 \leq i \leq n$. □

Regarding the regularity of u_t, similar to what happens for the obstacle problem for the linear fractional wave equation, it is nearly impossible to expect u_t to posses the same regularity as the obstacle-free case, i.e. $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$, mainly due to dissipation of energy at the contact region with the obstacle. Nonetheless, extending the pipeline outlined in [18, Section 4], we are still able to provide some sort of higher regularity for u_t.

Proposition 1.4.6. *Let u be the function obtained in Proposition 1.4.5 and, for any fixed $0 \leq \varphi \in \tilde{H}^s(\Omega)$, let us define $F\colon [0,T] \to \mathbb{R}$ as follows*

$$F(t) = \int_\Omega u_t(t)\varphi\, dx. \tag{1.4.5}$$

Then $F \in BV(0,T)$. Moreover, $u_t^n(t) \rightharpoonup u_t(t)$ in $L^2(\Omega)$ for a.e. $t \in [0,T]$.

Proof. Consider the functions $F^n\colon [0,T] \to \mathbb{R}$ defined as

$$F^n(t) = \int_\Omega u_t^n(t)\varphi\, dx. \tag{1.4.6}$$

where φ is fixed in $\tilde{H}^s(\Omega)$ with $\varphi \geq 0$. The first observation is that because u_t^n is bounded in $L^2(\Omega)$ uniformly in n and t (see Remark 1.4.4), $||F^n||_{L^1(0,T)}$ is uniformly bounded. Moreover, $\{F^n\}_n$ is also uniformly bounded in $BV(0,T)$: indeed, for every fixed $n > 0$ and $0 \leq i \leq n$, from (1.4.4) taking into account Remark 1.4.4 and Proposition 1.3.5, we can deduce that

$$\begin{aligned}\left|\int_\Omega (v_i^n - v_{i-1}^n)\varphi\, dx\right| - \int_\Omega (v_i^n - v_{i-1}^n)\varphi\, dx &\leq 2\tau_n \left|[u_i^n,\varphi]_s + \int_\Omega W'(u_i^n)\varphi dx\right| \\ &\leq 2\tau_n|[u_i^n,\varphi]_s| + 2\tau_n\left|\int_\Omega W'(u_i^n)\varphi dx\right| \leq 4\tau_n C||\varphi||_s\end{aligned} \tag{1.4.7}$$

Summing over $i = 1, \ldots, n$, we obtain

$$\sum_{i=1}^{n} \left| \int_{\Omega} (v_i^n - v_{i-1}^n)\varphi \, dx \right| \leq \int_{\Omega} v_n^n \varphi \, dx - \int_{\Omega} v_0 \varphi \, dx + \sum_{i=1}^{n} 4\tau_n C ||\varphi||_s$$
$$\leq ||v_n^n||_{L^2(\Omega)} ||\varphi||_{L^2(\Omega)} + ||v_0||_{L^2(\Omega)} ||\varphi||_{L^2(\Omega)} + 4TC ||\varphi||_s \leq C||\varphi||_s,$$

with C independent of n, where we make use of the uniform bound on $||v_i^n||_{L^2(\Omega)}$. Thus, by Helly's selection theorem, there exists a function $\bar{F}$ of bounded variation such that $F^n(t) \to \bar{F}(t)$ for every $t \in (0, T)$ as $n \to \infty$. Taking into account that $u_t^n \rightharpoonup^* u_t$ in $L^\infty(0, T; L^2(\Omega))$, one can then prove that $F(t) = \bar{F}(t)$ and thus $u_t^n(t) \rightharpoonup u_t(t)$ for almost every $t \in (0, T)$ (we refer to [18, Proposition 11] for details). □

From now on we can select u_t to be

$$u_t(t) = \text{ weak-}L^2 \text{ limit of } u_t^n(t),$$

which is then defined for all $t \in [0, T]$.

Proposition 1.4.7. *Fix $0 \leq \varphi \in \tilde{H}^s(\Omega)$ and let F be defined as in* (1.4.5)*. Then, for any $t \in (0, T)$, we have*

$$\lim_{r \to t^-} F(r) \leq \lim_{s \to t^+} F(s).$$

Proof. Because $F \in BV(0, T)$, it has right and left limits at any point. Fix $t \in (0, T)$ and let $0 < r < t < s < T$. For each n, let us define r_n and s_n such that $r \in (t_{r_n - 1}^n, t_{r_n}^n]$ and $s \in (t_{s_n - 1}^n, t_{s_n}^n]$. From (1.4.6), proceeding as in (1.4.7), we see that

$$F^n(s) - F^n(r) = \int_{\Omega} (u_t^n(s) - u_t^n(r))\varphi \, dx = \int_{\Omega} (v_{s_n}^n - v_{r_n}^n)\varphi \, dx$$
$$= \sum_{i=r_n+1}^{s_n} \int_{\Omega} (v_i^n - v_{i-1}^n)\varphi \, dx \geq -2\tau_n \sum_{i=r_n+1}^{s_n} |[u_i^n, \varphi]_s| - 2\tau_n \sum_{i=r_n+1}^{s_n} \int_{\Omega} |W'(u_i^n)\varphi| dx$$
$$\geq -4C\tau_n(s_n - r_n)||\varphi||_s$$

for some positive constant C independent of n. Moreover, $|s - r| \geq |t_{s_n - 1}^n - t_{r_n}^n| = \tau_n(s_n - 1 - r_n)$, thus it implies that

$$F^n(s) - F^n(r) \geq -2C|s - r| \cdot ||\varphi||_s - 2C\tau_n ||\varphi||_s.$$

By passing to the limit $n \to \infty$ we obtain that $F(s) - F(r) \geq -2C|s - r| \cdot ||\varphi||_s$, this yields the conclusion. □

We are now ready to prove the existence result, namely Theorem 1.4.2.

Proof of Theorem 1.4.2. Let u be the cluster point obtained in Proposition 1.4.5. It is easy to see that the first condition in Definition 1.4.1 follows from Proposition 1.4.5. From Proposition 1.4.6, it implies that for any fix $\varphi \geq 0$, $\varphi \in \tilde{H}^s$, $F(t) = \int_{\Omega} u_t(t)\varphi dx$ has the left and right limits for any $t \in [0, T]$ since F is $BV(0, T)$, this in turn implies the second condition in Definition 1.4.1. Let us verify the third and the fourth conditions in Definition 1.4.1.

(3.) For $n > 0$ and any test function $\varphi \in W^{1,\infty}(0,T;L^2(\Omega)) \cap L^1(0,T;\tilde{H}^s(\Omega))$, with $\varphi \geq 0$, $\operatorname{spt}\varphi \subset [0,T)$, we recall that

$$\int_0^T \int_\Omega \left(\frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} \right) \varphi(t)\, dxdt + \int_0^T [\bar{u}^n(t), \varphi(t)]_s\, dt + \int_0^T \int_\Omega W'(\bar{u}^n)(t)\varphi(t) dxdt \geq 0 \tag{1.4.8}$$

From Proposition 1.4.5, we have

$$\int_0^T [\bar{u}^n(t), \varphi(t)]_s\, dt \to \int_0^T [u(t), \varphi(t)]_s\, dt \quad \text{as } n \to \infty,$$
$$\int_0^T \int_\Omega W'(\bar{u}^n(t))\varphi(t) dxdt \to \int_0^T \int_\Omega W'(u(t))\varphi(t) dxdt \quad \text{as } n \to \infty.$$

To deal with the first term of (1.4.8), we observe that

$$\begin{aligned}\int_0^T \int_\Omega \frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} \varphi(t)\, dxdt &= \int_0^{T-\tau_n} \int_\Omega u_t^n(t) \left(\frac{\varphi(t) - \varphi(t+\tau_n)}{\tau_n} \right) dxdt \\ &\quad - \int_0^{\tau_n} \int_\Omega \frac{v_0}{\tau_n} \varphi(t)\, dxdt + \int_{T-\tau_n}^T \int_\Omega \frac{u_t^n(t)}{\tau_n} \varphi(t)\, dxdt \\ &\to \int_0^T \int_\Omega u_t(t)(-\varphi_t(t))\, dxdt - \int_\Omega v_0\, \varphi(0)\, dx + 0 \quad \text{as } n \to \infty\end{aligned}$$

and this completes the proof of condition (3).

(4.) By the convergence of u^n to u in $C^0([0,T];L^2(\Omega))$ and $u^n(0) = u_0$, it implies that $u(0) = u_0$. So as to check the initial condition on velocity we assume, without loss of generality, that the sequence u^n is constructed by taking $n \in \{2^m : m > 0\}$ (each successive time grid is obtained dividing the previous one). Fix n and $\varphi \in K_g$, let $T^* = m\tau_n$ for $0 \leq m \leq n$ (i.e. T^* is a "grid point"). We have

$$\begin{aligned}&\int_0^{T^*} \int_\Omega \frac{u_t^n(t) - u_t^n(t-\tau_n)}{\tau_n} (\varphi - \bar{u}^n(t)) = \sum_{i=1}^m \int_{t_{i-1}^n}^{t_i^n} \int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2} (\varphi - u_i^n) \\ &= \int_\Omega \sum_{i=1}^m \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n} (\varphi - u_i^n) = \int_\Omega \sum_{i=1}^m (v_i^n - v_{i-1}^n)(\varphi - u_i^n) \\ &= - \int_\Omega v_0^n (\varphi - u_1^n)\, dx + \int_\Omega v_m^n (\varphi - u_m^n)\, dx + \tau_n \sum_{i=1}^{m-1} \int_\Omega v_i^n v_{i-1}^n\, dx \\ &= - \int_\Omega v_0 (\varphi - u_n(\tau_n))\, dx + \int_\Omega u_t^n(T^*)(\varphi - u^n(T^*))\, dx + \tau_n \sum_{i=1}^{m-1} \int_\Omega v_i^n v_{i-1}^n\, dx.\end{aligned}$$

which combined with (1.4.3) gives

$$\begin{aligned}&- \int_\Omega v_0 (\varphi - u_n(\tau_n))\, dx + \int_\Omega u_t^n(T^*)(\varphi - u^n(T^*))\, dx \geq -\tau_n \sum_{i=1}^{m-1} \int_\Omega v_i^n v_{i-1}^n\, dx \\ &- \tau_n \sum_{i=1}^m [u_i^n, \varphi - u_i^n]_s - \tau_n \sum_{i=1}^m \int_\Omega W'(u_i^n)(\varphi - u_i^n) \geq -CT^* - CT^* ||\varphi||_s\end{aligned}$$

thanks to the boundedness of $W'(u_i^n)$ in $L^2(\Omega)$. Passing to the limit as $n \to \infty$, using $u^n(\tau_n) \to u(0)$ and $u_t^n(T^*) \rightharpoonup u_t(T^*)$ (as noticed before we choose u_t being the weak-L^2 limit of u_t^n), we obtain that

$$-\int_\Omega v_0(\varphi - u(0))\,dx + \int_\Omega u_t(T^*)(\varphi - u(T^*))\,dx \geq -CT^* - C||\varphi||_s T^*.$$

Let T^* tend to 0 along a sequence of "grid points", we have that

$$\int_\Omega (u_t^+(0) - v_0)(\varphi - u(0))\,dx \geq 0.$$

To complete the proof, we observe that the energy estimate (1.4.2) is obtained by passing to the limit as $n \to \infty$ in

$$\frac{1}{2}\|u_t^n(t)\|^2_{L^2(\Omega)} + \frac{1}{2}[\bar{u}^n(t)]_s^2 + ||W(\bar{u}^n(t))||_{L^1(\Omega)} \leq E(u(0)) + C\tau_n$$

for all $t \in [0,T]$, with C a constant independent of n (cf. Remark 1.4.4).

We end this section by an observation that in the case $s = 1$ the solutions become more regular whenever the approximation u^n lies strictly above g.

Proposition 1.4.8 (Regions without contact). *Let $s = 1$ and, for $\delta > 0$, suppose there exists an open set $A_\delta \subset \Omega$ such that $u^n(t,x) > g(x) + \delta$ for a.e. $(t,x) \in (0,T) \times \Omega$ and for all $n > 0$. Then $u_{tt} \in L^\infty(0,T;H^{-1}(A_\delta))$ and u satisfies (2.2.6) for all $\varphi \in L^1(0,T;H_0^1(A_\delta))$.*

Proof. Fix $n > 0$ and $1 \leq i \leq n$. For every $\varphi \in H_0^1(\Omega)$ with spt $\varphi \subset A_\delta$ we have $u_i^n + \varepsilon\varphi \in K_g$ for ε sufficiently small. In particular, inequality (1.4.4) turns into

$$\int_\Omega \frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{\tau_n^2}\varphi\,dx + \int_\Omega \nabla u_i^n \cdot \nabla\varphi\,dx + \int_\Omega W'(u_i^n)\varphi\,dx = 0 \qquad (1.4.9)$$

The equality allows us to rescue the second part of the proof of Proposition 1.3.6: in the same notation, we can prove $v_t^n(t)$ to be bounded in $H^{-1}(A_\delta)$ uniformly in t and n by using the uniform bound on $||W'(u_i^n)||_{L^2(\Omega)}$ provided by Proposition 1.3.5. Thus, $v \in W^{1,\infty}(0,T;H^{-1}(A_\delta))$ and

$$v^n \rightharpoonup^* v \text{ in } L^\infty(0,T;L^2(A_\delta)) \quad \text{and} \quad v^n \rightharpoonup^* v \text{ in } W^{1,\infty}(0,T;H^{-1}(A_\delta)).$$

A localization on A_δ proves that $v_t = u_{tt}$ in A_δ so that

$$u_{tt} \in L^\infty(0,T;H^{-1}(A_\delta)).$$

To get (2.2.6) we pass to the limit in (1.4.9) as we have done in the proof of Theorem 1.3.3. □

1.5 A numerical example

We present in this section a simple example implementing the scheme of Section 1.3 for a two dimensional radially symmetric problem related to moving interfaces in the relativistic setting. We consider equation (1.3.1) with potential

$$W(u) = \frac{(1-u^2)^2}{1+u^2}$$

and a radially symmetric initial datum u_0 having a sharp transition at a given radius $R_0 > 0$ (the function transitioning form 1 inside to -1 outside). The initial velocity is assumed to be zero and the computational domain $\Omega = B(0, \bar{R})$ for $\bar{R} > R_0$. From results in [47], the solution $u(t, \cdot)$ is expected to keep the initial structure of a radially symmetric function with a sharp transition region, with said transition region evolving inwards: for $0 \leq t < R_0\pi/2$ the solution $u(t, \cdot)$ will display its transition region along the circle of radius

$$R(t) = R_0 \cos\left(\frac{t}{R_0}\right).$$

Thus, we incorporate the radial symmetry in the minimization of (1.3.11) and we translate the problem into a 1d optimization over $\bar{\Omega} = [0, \bar{R}]$ and assume Dirichlet boundary conditions ± 1 at $0, \bar{R}$. We employ the same discretization used in [18], based on classical piecewise linear finite elements. The finite dimensional optimization problem is then solved via a projected gradient descent method combined with a dynamic adaptation of the descent step size. We display the results in Figure 3.2: we can see how the solution evolves the transition region in time and how the position of the transition follows closely the expected radius.

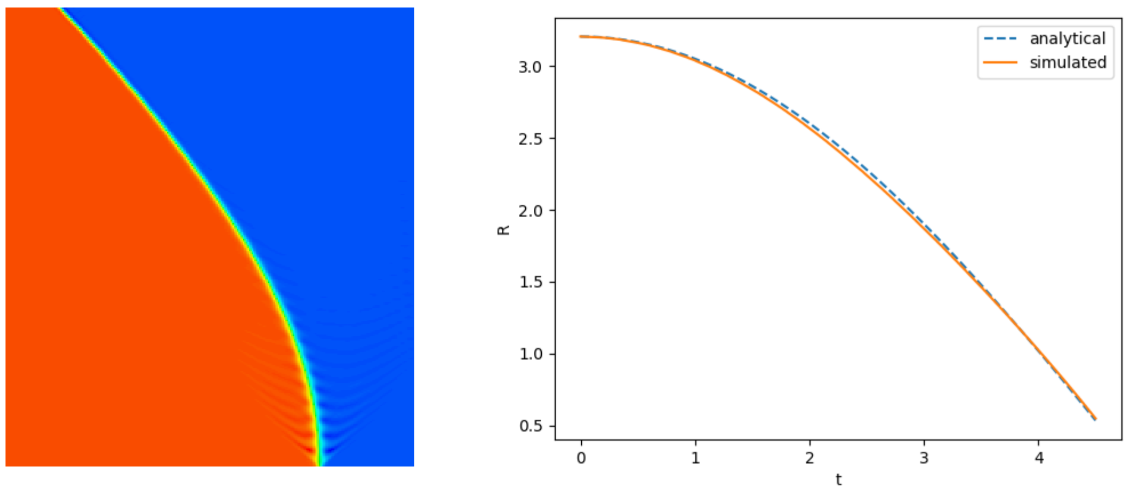

Figure 1.1: Left: space-time orthogonal view of the solution, red being $+1$ and blue being -1. Right: time evolution of the transition region (analytical vs. simulated).

1.6 Appendix

We recall the proof for the discrete Gronwall's inequality as used in the proof of Proposition 1.3.4.

Proposition 1.6.1. *(Discrete Gronwall inequality) Let* $\{y_n\}_{n=0}^N$ *be a sequence of non-negative numbers, and assume there exist two positive constants* $A, B > 0$ *such that*

$$y_0 = 0 \quad \text{and} \quad y_n \le A + \frac{B}{N}\sum_{j=0}^{n-1} y_j \quad \text{for all } n = 1, \dots, N.$$

Then,

$$y_i \le A\exp(B) \quad \text{for all } i = 1, \dots, N.$$

Proof. We first prove by induction that

$$y_i \le A\left(1 + \frac{B}{N}\right)^i \tag{1.6.1}$$

for all $0 \le i \le N$. The case $i = 0$ is obvious. Now suppose that (1.6.1) holds from 0 to k, then

$$\begin{aligned}
y_{k+1} &\le A + \frac{B}{N}\sum_{j=0}^{k} y_j \\
&\le A + \frac{B}{N}\left(A\left(1+\frac{B}{N}\right) + A\left(1+\frac{B}{N}\right)^2 + \ldots + A\left(1+\frac{B}{N}\right)^k\right) \\
&= A + \frac{AB}{N}\left(\frac{\left(1+\frac{B}{N}\right)^k - 1}{\frac{B}{N}}\right)\left(1+\frac{B}{N}\right) = A + A\left(\left(1+\frac{B}{N}\right)^k - 1\right)\left(1+\frac{B}{N}\right) \\
&= A\left(\left(1+\frac{B}{N}\right)^{k+1} - \frac{B}{N}\right) \le A\left(1+\frac{B}{N}\right)^{k+1}.
\end{aligned}$$

This yields (1.6.1), which in turn gives

$$y_i \le A\left(1+\frac{B}{N}\right)^i \le A\left[\exp\left(\frac{B}{N}\right)\right]^i = A\exp\left(\frac{i}{N}B\right) \le A\exp(B)$$

for all $i = 0, \dots, N$.

We also provide here the continuous version of Gronwall's inequality:

Proposition 1.6.2. *Let* $g : [0,1] \longrightarrow \mathbb{R}$ *be a non-negative continuous function, and it satisfies the following inequality:*

$$g(t) \le C\int_0^t g(s)ds$$

for any $t \in [0,1]$, *for some positive constants* C. *Then,* $g(t) = 0$ *for any* $t \in [0,1]$.

Proof. Let $m(t) = e^{-Ct} \int_0^t g(s)ds$. We observe that $\frac{dm(t)}{dt} \leq 0$ for any $t \in (0,1)$, and $m(0) = 0$. Therefore, we have

$$m(t) = 0$$

for any $t \in (0,1)$, this implies that $g(t) = 0$ for each $t \in (0,1)$, and we extend to the endpoints by continuity of g. □

The last inequality presented here which is the Poincaré-type inequality, is used in the proof of uniqueness.

Proposition 1.6.3. *Let $u \in \tilde{H}^s(\Omega)$. Then, there exists a postive constant C_s such that*

$$||u||_{L^2(\Omega)} \leq C_s[u]_s.$$

Proof. Let $u \in \tilde{H}^s(\Omega)$, by Heisenberg-Pauli-Weyl inequality, one has

$$\begin{aligned} ||u||^4_{L^2(\Omega)} &\leq C_s \left(\int_{\mathbb{R}^d} |x|^{2s} |u(x)|^2 dx \right) \left(\int_{\mathbb{R}^d} |\eta|^{2s} |\mathcal{F}(u)(\eta)|^2 d\eta \right) \\ &\leq D_s ||u||^2_{L^2(\Omega)} [u]^2_s. \end{aligned} \tag{1.6.2}$$

for some constants D_s, we have used that u is equal to 0 outside the bounded domain Ω. Thus, we obtain that

$$||u||_{L^2(\Omega)} \leq D_s[u]_s,$$

which is the conclusion.

Chapter 2

Weak solutions for nonlinear waves in adhesive phenomenona

In this chapter we discuss a notion of weak solution to a semilinear wave equation that models the interaction of an elastic body with a rigid substrate through an adhesive layer, relying on results in Chapter 1, Bonafini et al [17]. Our analysis embraces the vector-valued case in arbitrary dimension as well as the case of non-local operators (e.g. fractional Laplacian).

2.1 Introduction

In recent years, there have been many works devoted to adhesion phenomena arising from biophysics and engineering (see for instance [46, 82, 39, 66] and references therein). Because of the complex underlying mechanisms at both microscopic and macroscopic level, the rigorous mathematical description of such phenomena is quite challenging, and increasingly accurate mathematical models which are able to capture their essential features are being proposed (see for instance [26, 52, 53, 32, 1] and references therein).

Following [32], where the dynamic of an elastic body (e.g. a string or a membrane) glued to a rigid substrate through an adhesive layer can be modelled via a potential W describing the effect of the adhesive layer on the dynamic. More precisely, the Lagrangian governing the one dimensional dynamical system considered there is described by

$$\ell(u) = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + W(u), \tag{2.1.1}$$

for a scalar displacement field $u : [0,T] \times [0,L] \to \mathbb{R}$ (see, for instance, [33, Section 2] for the derivation of the model), the potential W has the following behavior:

$$W(u) = \begin{cases} u^2 & \text{if } |u| \le u^*, \\ (u^*)^2 & \text{if } |u| > u^*. \end{cases} \tag{2.1.2}$$

The lack of smoothness at points $\pm u^*$ gives rise to the difficulties in proving the existence of solutions to the problem. And our main focus is to address this issue, in particular our analysis extends also to vector-valued case as well as non-local operator. In view of this, the generalized model problem we will consider in this chapter is

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T)\times\Omega, \\ u(x,t) = 0 & \text{in } [0,T]\times(\mathbb{R}^d\setminus\Omega), \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ u_t(0,x) = v_0(x) & \text{in } \Omega, \end{cases} \tag{2.1.3}$$

where $\Omega\subset\mathbb{R}^d$ is an open bounded domain with Lipschitz boundary, $u:[0,T]\times\Omega\to\mathbb{R}^m$ and $(-\Delta)^s$ stands for the fractional Laplacian ($s=1$ provides the standard Laplacian). The equation considered in (2.1.3) also embraces the case $s=2$, i.e. the bi-Laplacian, recently studied in [34] to model an elastic beam interacting with a substrate through an elastic-breakable forcing term.

The key component in (2.1.3) is the potential W that models the energetic contribution of the glue layer and is responsible for the adhesive behaviour. In [32], for $d=1$, $s=1$ and $m=1$, and in [34] for $d=1$, $s=2$ and $m=1$, the authors propose to consider the potential W (2.1.2) for some critical state u^*. In this setting, we are assuming the stress of the glue layer to drop immediately to zero when the displacement of the string goes beyond the critical value. The derivative W' is then discontinuous at $\pm u^*$ and hence a suitable notion of weak solution is needed to handle such discontinuity in the equation.

Any notion of solution for (2.1.3) heavily depends on the regularity of the potential W. Whenever W is regular enough, i.e. W is non-negative and $W\in C^1(\mathbb{R}^m)$ with Lipschitz continuous gradient, a notion of weak solution and existence of it have been proved in Chapter 1. In this chapter, our aim is to explore how far such notion of weak solution can reach. We first investigate in Section 2.3 the case where the gradient of the potential is not Lipschitz continuous (but still continuous) and then tackle the discontinuous setting in Section 2.4. While in the former scenario we are able to prove existence of weak solutions via a limiting approach using smooth approximations of W, the latter scenario represents the boundary of the working setting we currently put us into: even in the one dimensional case described by (2.1.2), with W' discontinuous at $\pm u^*$, the general lack of informations about the distribution of the values of the approximate solutions around the critical states $-u^*$ and u^* prevents us from providing direct proofs by approximation, as shown in Examples 2.4.2 and 2.4.3. Hence, this is calling for increasingly weaker notions of solutions.

The plan of Chapter 2 is organized as follows: in Section 2.2 we briefly recall preliminary definitions and notations, together with the notion of weak solution. In Section 2.3 we deal with problem (2.1.3) in case of (non Lipschitz-) continuous ∇W and prove existence of weak solutions in Theorem 2.3.1. In Section 2.4, after discussing the limitations of the approximation approach in Examples 2.4.2 and 2.4.3, we extend the analysis to a particular class of potentials with discontinuous ∇W, proving existence of weak solutions under some very restrictive assumptions (Theorem 2.4.4).

2.2 Preliminaries and model problem

2.2.1 Notation.

Let $d,m\in\mathbb{N}$ and $s>0$. We define the fractional Laplacian operator $(-\Delta)^s$ as the operator whose Fourier symbol is $|\xi|^{2s}$, i.e., for any $u\in L^2(\mathbb{R}^d;\mathbb{R}^m)$

$$\mathcal{F}(-\Delta)^s u = |\xi|^{2s}\mathcal{F}u$$

where $\mathcal{F}$ denotes the Fourier transform. We denote by H^s the fractional Sobolev space of order s, which is defined as

$$H^s(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^d;\, \mathbb{R}^m) \,:\, \int_{\mathbb{R}^d} (1+|\xi|^{2s})|\mathcal{F}u(\xi)|^2 \, d\xi < +\infty \right\}$$

and equip it with the scalar product $[u,v]_s = \langle (-\Delta)^{s/2}u, (-\Delta)^{s/2}v\rangle$ (where $\langle\cdot,\cdot\rangle$ is the L^2 scalar product), the corresponding semi-norm $[u]_s = \sqrt{[u,u]_s} = ||(-\Delta)^{s/2}u||_{L^2(\mathbb{R}^d;\mathbb{R}^m)}$ and norm $||u||_s^2 = ||u||^2_{L^2(\mathbb{R}^d;\mathbb{R}^m)} + [u]_s^2$. For $\Omega \subset \mathbb{R}^d$ an open bounded set with Lipschitz boundary, we define

$$\tilde{H}^s(\Omega) = \{u \in H^s(\mathbb{R}^d;\mathbb{R}^m) \,:\, u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega\},$$

endowed with the $||\cdot||_s$ norm, and its dual $H^{-s}(\Omega) := (\tilde{H}^s(\Omega))^*$. One has $C_c^\infty(\mathbb{R}^d;\mathbb{R}^m)$ is dense in $H^s(\mathbb{R}^d)$ (see [79]). We recall the following embedding, and in order to be self-contained we provide a proof it (based on [79]).

Lemma 2.2.1. *Let $2s > d$ and $u \in H^s(\mathbb{R}^d)$. Then, $u \in C^0(\mathbb{R}^d;\mathbb{R}^m)$ and there exists a constant C independent of u such that*

$$||u||_{C^0(\mathbb{R}^d;\mathbb{R}^m)} \leq C||u||_{H^s(\mathbb{R}^d)}. \tag{2.2.1}$$

Proof. Let $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^m)$ be the Schwartz space of rapidly decaying functions. Let f be a function in $\mathcal{S}(\mathbb{R}^d;\mathbb{R}^m)$, we now prove the inequality (2.2.1) for f. One has

$$\begin{aligned}(2\pi)^{\frac{d}{2}}|f(x)| = \left|(2\pi)^{\frac{d}{2}}\mathcal{F}^{-1}(\hat{f})(x)\right| &= \left|\int_{\mathbb{R}^d} e^{ix\xi}\mathcal{F}f(\xi)d\xi\right| \leq \int_{\mathbb{R}^d} |\mathcal{F}f(\xi)|(1+|\xi|^s)\frac{1}{1+|\xi|^s}d\xi \\ &\leq \left(\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|^s)^2}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|\mathcal{F}f(\xi)|^2(1+|\xi|^s)^2 d\xi\right)^{\frac{1}{2}} \\ &\leq \sqrt{2}\left(\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|^s)^2}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|\mathcal{F}f(\xi)|^2(1+|\xi|^{2s})d\xi\right)^{\frac{1}{2}} \\ &\leq \sqrt{2}\left(\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|^s)^2}d\xi\right)^{\frac{1}{2}}||f||_{H^s(\mathbb{R}^d)}.\end{aligned}$$

Since we consider $2s > d$, $\int_{\mathbb{R}^d}\frac{1}{(1+|\xi|^s)^2}d\xi$ is finite. Thus, we obtain that

$$||f||_{C^0(\mathbb{R}^d;\mathbb{R}^m)} \leq C||f||_{H^s(\mathbb{R}^d)}. \tag{2.2.2}$$

Now let $u \in H^s(\mathbb{R}^d)$, and $\{f_n\}_n \subset \mathcal{S}(\mathbb{R}^d;\mathbb{R}^m)$ such that f_n converges to u in $H^s(\mathbb{R}^d)$. We first observe that $\{f_n\}$ is a Cauchy sequence in $H^s(\mathbb{R}^d)$, then combining this fact with (2.2.2),

$$||f_n - f_m||_{C^0(\mathbb{R}^d;\mathbb{R}^m)} \leq C||f_n - f_m||_{H^s(\mathbb{R}^d)}. \tag{2.2.3}$$

It follows that $\{f_n\}$ is a Cauchy sequence in $C^0(\mathbb{R}^d;\mathbb{R}^m)$, and finally by passing to the limit we obtain that

$$||u||_{C^0(\mathbb{R}^d;\mathbb{R}^m)} \leq C||u||_{H^s(\mathbb{R}^d)}. \tag{2.2.4}$$

□

2.2.2 Model problem.

For an open bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary and a potential $W\colon \mathbb{R}^m \to [0,\infty)$ (whose regularity we specify later on), we look for a solution $u = u(t,x)$, $u : [0,T] \times \Omega \to \mathbb{R}^m$, of

$$\begin{cases} u_{tt} + (-\Delta)^s u + \nabla_u W(u) = 0 & \text{in } (0,T) \times \Omega \\ u(t,x) = 0 & \text{in } [0,T] \times (\mathbb{R}^d \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u_t(0,x) = v_0(x) & \text{in } \Omega \end{cases} \tag{2.2.5}$$

with initial data $u_0 \in \tilde{H}^s(\Omega)$ and $v_0 \in L^2(\Omega;\mathbb{R}^m)$ (we intend that $v_0 = 0$ in $\mathbb{R}^d \setminus \Omega$). For $m = d$ one can conventionally interpret u as the displacement of an elastic body (see [33, Section 2]). A notion of weak solution for problem (2.2.5) can be given as follows.

Definition 2.2.2 (Weak solution and energy)**.** *Let $T > 0$. We say $u = u(t,x)$ is a weak solution of* (2.2.5) *in $(0,T)$ if*

1. *$u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ and $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$,*
2. *for all $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$*

$$\int_0^T \langle u_{tt}(t),\varphi(t)\rangle dt + \int_0^T [u(t),\varphi(t)]_s \, dt + \int_0^T \int_\Omega \nabla_u W(u(t))\varphi(t) \, dxdt = 0 \tag{2.2.6}$$

with

$$u(0,x) = u_0 \quad \text{and} \quad u_t(0,x) = v_0. \tag{2.2.7}$$

The energy of u is defined as

$$E(u(t)) = \frac{1}{2}||u_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]^2_s + ||W(u(t))||_{L^1(\Omega)} \quad \textit{for } t \in [0,T].$$

Existence of a weak solution in the sense of Definition 2.2.2 has been proved in Chapter 1 for non-negative potential $W \in C^1(\mathbb{R}^m)$ with Lipschitz continuous gradient. We prove in the next section the existence of weak solutions under the assumption $W \in C^1(\mathbb{R}^m)$, non-negative and dropping Lipschitz continuity of the gradient in favour of boundedness and uniform continuity assumptions. Less regular potentials are then partially addressed in Section 2.4, where limitations of the current approach are discussed in Examples 2.4.2 and 2.4.3, and existence of weak solutions is proved under the assumption of small initial data for $2s > d$.

2.3 The case of continuous ∇W

Taking into account prototypical potentials modelling adhesive behaviours, e.g. one expects constancy of W outside a bounded region, this section is devoted to the proof of the following theorem.

Theorem 2.3.1. *Let $W \in C^1(\mathbb{R}^m)$, and W be non-negative. Assume there exists $K > 0$ such that $0 \le W(y) \le K$ and $0 \le |\nabla W(y)| \le K$ for all $y \in \mathbb{R}^m$, with ∇W uniformly continuous. Then, there exists a weak solution of* (2.2.5) *satisfying the energy inequality*

$$E(u(t)) \le E(u(0)) \quad \textit{for any } t \in [0,T]. \tag{2.3.1}$$

Proof.
Step 1. Construction of regularized approximate weak solutions. Let us consider a family of non-negative potentials $(W_\varepsilon)_{\varepsilon>0}$ in $C^2(\mathbb{R}^m)$ such that:

(i) W_ε converges uniformly to W,

(ii) ∇W_ε converges uniformly to ∇W,

(iii) ∇W_ε is Lipschitz continuous for each ε.

Leveraging the existence result in [Theorem 1.3.3, Chapter 1], for each $\varepsilon > 0$ there exists a weak solution u^ε of

$$\begin{cases} u^\varepsilon_{tt} + (-\Delta)^s u^\varepsilon + \nabla_u W_\varepsilon(u^\varepsilon) = 0 & \text{in } (0,T)\times\Omega \\ u^\varepsilon(t,x) = 0 & \text{in } [0,T]\times(\mathbb{R}^d\setminus\Omega) \\ u^\varepsilon(0,x) = u_0(x) & \text{in } \Omega \\ u^\varepsilon_t(0,x) = v_0(x) & \text{in } \Omega \end{cases}$$

in the sense of Definition 2.2.2. This in particular means that

$$\int_0^T \langle u^\varepsilon_{tt}(t), \varphi(t)\rangle dt + \int_0^T [u^\varepsilon(t), \varphi(t)]_s\, dt + \int_0^T \int_\Omega \nabla_u W_\varepsilon(u^\varepsilon(t))\varphi(t)\, dxdt = 0 \tag{2.3.2}$$

for all $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$, and, for any $t\in[0,T]$, we have

$$\frac{1}{2}||u^\varepsilon_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u^\varepsilon(t)]^2_s + ||W_\varepsilon(u^\varepsilon(t))||_{L^1(\Omega)} \le \frac{1}{2}||v_0||^2_{L^2(\Omega)} + \frac{1}{2}[u_0]^2_s + ||W_\varepsilon(u_0)||_{L^1(\Omega)}. \tag{2.3.3}$$

Step 2. Existence of a cluster point. Since W_ε converges uniformly to W in $\mathbb{R}^m$ and W is bounded, for sufficiently small ε we have a uniform bound on W_ε. Thus, using (2.3.3), there exists a constant $C > 0$ such that for any $t\in[0,T]$

$$E(u^\varepsilon(t)) = \frac{1}{2}||u^\varepsilon_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u^\varepsilon(t)]^2_s + ||W_\varepsilon(u^\varepsilon(t))||_{L^1(\Omega)} \le C. \tag{2.3.4}$$

This energy bound, reasoning as in [Proposition 1.3.6, Chapter 1] or [18, Proposition 6], readily provides the existence of $u \in L^\infty(0,T;\tilde{H}^s(\Omega))\cap W^{1,\infty}(0,T;L^2(\Omega))$ such that, up to a subsequence,

(iv) $u^\varepsilon \to u$ in $C^0([0,T];L^2(\Omega))$,

(v) $u^\varepsilon_t \rightharpoonup^* u_t$ in $L^\infty(0,T;L^2(\Omega))$,

(vi) $u^\varepsilon(t) \rightharpoonup u(t)$ in $\tilde{H}^s(\Omega)$ for $t\in[0,T]$,

(vii) $u^\varepsilon \rightharpoonup^* u$ in $L^\infty(0,T;\tilde{H}^s(\Omega))$.

Step 3. Passage to the limit in the definition of weak solution. In order to prove that u is a weak solution we first has to pass to the limit in (2.3.2) as $\varepsilon\to 0$. To do so, observe that

- $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$ and $u^\varepsilon_{tt} \rightharpoonup^* u_{tt}$ in $L^\infty(0,T;H^{-s}(\Omega))$

  Indeed, from (2.3.4), (2.3.2), and the uniform bound on $|\nabla W_\varepsilon|$, we obtain that u^ε_{tt} is uniformly bounded in $L^\infty(0,T;H^{-s}(\Omega))$. This implies that $u^\varepsilon_{tt} \rightharpoonup^* u_{tt}$ in $L^\infty(0,T;H^{-s}(\Omega))$.

- $\nabla_u W_\varepsilon(u^\varepsilon) \rightharpoonup^* \nabla_u W(u)$ in $L^\infty(0,T;H^{-s}(\Omega))$

  Indeed, $u^\varepsilon \to u$ for a.e. $(x,t) \in (0,T)\times\Omega$ due to the convergence of u^ε to u in $C^0([0,T];L^2(\Omega))$. Thus, since ∇W_ε converges uniformly to ∇W in $\mathbb{R}^m$ and ∇W_ε is uniformly bounded, by the dominated convergence theorem we conclude that

  $$\nabla_u W_\varepsilon(u^\varepsilon) \to \nabla_u W(u) \text{ in } L^2((0,T)\times\Omega).$$

  On the other hand, $\nabla_u W_\varepsilon(u^\varepsilon)$ is uniformly bounded in $L^\infty(0,T;H^{-s}(\Omega))$, therefore we can conclude that $\nabla_u W_\varepsilon(u^\varepsilon) \rightharpoonup^* \nabla_u W(u)$ in $L^\infty(0,T;H^{-s}(\Omega))$.

Thus, letting $\varepsilon \to 0$ in (2.3.2) we obtain

$$\int_0^T \langle u_{tt}(t),\varphi(t)\rangle dt + \int_0^T [u(t),\varphi(t)]_s\,dt + \int_0^T\int_\Omega \nabla_u W(u(t))\varphi(t)\,dxdt = 0 \tag{2.3.5}$$

for all $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$. To finally conclude, observe that

- $E(u(t)) \le E(u(0))$ for each $t\in[0,T]$

  From the fact that u^ε_{tt} is uniformly bounded in $L^\infty(0,T;H^{-s}(\Omega))$, we deduce that $u^\varepsilon_t \to u_t$ in $C^0([0,T];H^{-s}(\Omega))$. On the other hand, each $u^\varepsilon_t(t)$ is uniformly bounded in $L^2(\Omega)$, thus we obtain that $u^\varepsilon_t(t) \rightharpoonup u_t(t)$ in $L^2(\Omega)$ for each $t\in[0,T]$. For the convergence of $W_\varepsilon(u^\varepsilon)$, let $t\in[0,T]$ and an arbitrary $\eta>0$, since W_ε converges uniformly to W, for sufficiently small ε we obtain that

  $$|W_\varepsilon(y) - W(y)| \le \eta \tag{2.3.6}$$

  for any $y\in\mathbb{R}^m$. Hence,

  $$\begin{aligned}\int_\Omega |W_\varepsilon(u^\varepsilon(t)) - W(u(t))|dx &\le \int_\Omega |W_\varepsilon(u^\varepsilon(t)) - W(u^\varepsilon(t))|dx + \int_\Omega |W(u^\varepsilon(t)) - W(u(t))|dx \\ &\le |\Omega|\eta + \max_{y\in\mathbb{R}^m}|\nabla W(y)|\cdot\max_{t\in[0,T]}||u^\varepsilon - u||_{L^2(\Omega)}|\Omega|^{\frac{1}{2}}\end{aligned}$$

  where we have made use of (2.3.6), Lipschitz continuity of W, and Hölder's inequality. Thus, from the fact that $u^\varepsilon \to u$ in $C^0([0,T];L^2(\Omega))$, we can deduce that up to a subsequence, $W_\varepsilon(u^\varepsilon) \to W(u)$ in $C^0([0,T];L^1(\Omega))$. The energy inequality for u follows passing to the limit in (2.3.3).

- $u(0,x) = u_0$, $u_t(0,x) = v_0$

  Since $u^\varepsilon \to u$ in $C^0([0,T];L^2(\Omega))$ and $u^\varepsilon_t \to u_t$ in $C^0([0,T];H^{-s}(\Omega))$, we have $u(0,x)=u_0$ and $u_t(0,x) = v_0$.

□

Remark 2.3.2. In case $u_0 \in \tilde{H}^{2s}(\Omega)$, $v_0 \in \tilde{H}^s(\Omega)$, the weak solution of (2.2.5) constructed in Theorem 2.3.1 is such that its energy $E(u)$ is conserved during the evolution. Indeed, for more regular initial data approximate solutions u^ε turns out to be more regular and energy preserving [Theorem 1.3.3, Chapter 1]. In particular we have a uniform $\tilde{H}^s$ bound on the velocity of the approximate solutions, namely $(u^\varepsilon_t)_\varepsilon$, due to the uniform bounds on W_ε and ∇W_ε. This allows us to obtain higher regularity for the velocity of the limiting solution, which in turn gives rise to the energy conservation by using suitable test functions (see Chapter 1).

2.4 The case of discontinuous ∇W

So as to express the manifestation of an adhesive phenomenon through a sharp discontinuity in the same spirit as [32], we consider here equation (2.2.5) with the prototypical continuous potential $W \in C(\mathbb{R}^m)$ defined as

$$W(y) = \begin{cases} |y|^2 & \text{if } y \in \overline{\mathbf{B}(0,1)} \\ 1 & \text{if } y \notin \mathbf{B}(0,1) \end{cases} \tag{2.4.1}$$

where $\mathbf{B}(0,1) = \{\, y \in \mathbb{R}^m \mid |y| < 1 \,\}$, $\overline{\mathbf{B}(0,1)} = \{\, y \in \mathbb{R}^m \mid |y| \leq 1 \,\}$, and we define

$$\nabla W(y) = \begin{cases} 2y & \text{if } y \in \overline{\mathbf{B}(0,1)} \\ 0 & \text{if } y \notin \mathbf{B}(0,1) \end{cases} \tag{2.4.2}$$

Then, we define weak solutions in this case as follows:

Definition 2.4.1 (Weak solution and energy for the discontinuous case)**.** *Let $T > 0$. We say $u = u(t,x)$ is a weak solution of* (2.2.5) *in* $(0,T)$ *if*

1. $u \in L^\infty(0,T;\tilde{H}^s(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$ *and* $u_{tt} \in L^\infty(0,T;H^{-s}(\Omega))$,
2. $\nabla_u W(u) \in L^\infty(0,T;H^{-s}(\Omega))$,
3. *for all* $\varphi \in L^1(0,T;\tilde{H}^s(\Omega))$,

$$\int_0^T \langle u_{tt}(t), \varphi(t)\rangle dt + \int_0^T [u(t),\varphi(t)]_s \, dt + \int_0^T \int_\Omega \nabla_u W(u(t))\varphi(t)\, dxdt = 0 \tag{2.4.3}$$

with

$$u(0,x) = u_0 \quad \textit{and} \quad u_t(0,x) = v_0. \tag{2.4.4}$$

The energy of u is defined as

$$E(u(t)) = \frac{1}{2}||u_t(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u(t)]_s^2 + ||W(u(t))||_{L^1(\Omega)} \quad \textit{for } t \in [0,T].$$

This potential designates $\partial\mathbf{B}(0,1)$ as the set of critical states serving as boundary of the adhesive dynamics. Trying now to prove existence of weak solutions as in the proof of Theorem 2.3.1, due to the discontinuous behaviour of ∇W, we cannot approximate the gradient uniformly and thus possibly lose any control on the behaviour of the approximating sequence, in particular in regions where approximate solutions approaches the discontinuity. It turns out that this approach is indeed not always successful when combined with Definition 2.4.1 of weak solutions, as it can be shown via some counterexamples for the one dimensional semilinear wave equation $u_{tt} - u_{xx} + W'(u) = 0$ (we consider here for simplicity a Neumann problem in order to be able to write explicitly some approximate solutions).

Example 2.4.2. Consider the 1-dimensional problem

$$\begin{cases} u_{tt} - u_{xx} + W'(u) = 0 & \text{in } (0,T) \times [0,L] \\ u_x(t,0) = u_x(t,L) = 0 & \text{in } [0,T] \\ u(0,x) = 1 & \text{in } [0,L] \\ u_t(0,x) = 0 & \text{in } [0,L] \end{cases} \tag{2.4.5}$$

for a potential $W \in C(\mathbb{R})$ defined as above

$$W(u) = \begin{cases} u^2 & \text{if } |u| \le 1 \\ 1 & \text{if } |u| > 1 \end{cases} \tag{2.4.6}$$

and

$$W'(u) = \begin{cases} 2u & \text{if } |u| \le 1 \\ 0 & \text{if } |u| > 1. \end{cases} \tag{2.4.7}$$

Consider now the sequence of approximate potentials W_ε with

$$W_\varepsilon'(u) = \begin{cases} (2-\varepsilon)u & \text{if } |u| \le 1 \\ \dfrac{2-\varepsilon}{\varepsilon}(1+\varepsilon-u) & \text{if } 1 \le u \le 1+\varepsilon \\ \dfrac{\varepsilon-2}{\varepsilon}(1+\varepsilon+u) & \text{if } -1-\varepsilon \le u \le 1 \\ 0 & \text{if } |u| \ge 1+\varepsilon. \end{cases}$$

One can easily show that $u^\varepsilon(t,x) = 1+\varepsilon$ solves the approximate problems

$$\begin{cases} u^\varepsilon_{tt} - u^\varepsilon_{xx} + W_\varepsilon'(u^\varepsilon) = 0 & \text{in } (0,T)\times[0,L] \\ u^\varepsilon_x(t,0) = u^\varepsilon_x(t,L) = 0 & \text{in } [0,T] \\ u^\varepsilon(0,x) = 1+\varepsilon & \text{in } [0,L] \\ u^\varepsilon_t(0,x) = 0 & \text{in } [0,L], \end{cases}$$

and such approximate solutions converge to the constant function 1 but

$$\lim_{\varepsilon\to 0} \int_\Omega W_\varepsilon'(u^\varepsilon(t))\varphi(t)\,dxdt = 0$$

while $W'(1) = 2$. Hence, we generally cannot expect to harmlessly pass to the limit in any definition of weak solution involving (2.2.6) (indeed, for general W, already the measurability of the integrand $\nabla_u W(u(t))$ may fail, breaking this notion of solution from the very beginning).

Example 2.4.3. Consider again problem (2.4.5). One can in general build non constant approximate solutions: consider potentials W_ε with

$$W_\varepsilon'(u) = \begin{cases} 2u-\varepsilon & \text{if } |u| \le 1 \\ \dfrac{2\varepsilon-2}{\varepsilon}u - \varepsilon + \dfrac{2}{\varepsilon} & \text{if } 1 \le u \le 1+\varepsilon, \\ \varepsilon & \text{if } u \ge 1+\varepsilon, \\ \dfrac{2}{\varepsilon}u - \dfrac{2}{\varepsilon} - 2 - \varepsilon & \text{if } -1-\varepsilon \le u \le -1 \\ -\varepsilon & \text{if } u \le -1-\varepsilon. \end{cases} \tag{2.4.8}$$

Then, $u^\varepsilon(t,x) = -\varepsilon\frac{t^2}{2} + \sqrt{\varepsilon} + 1$ solves

$$\begin{cases} u^\varepsilon_{tt} - u^\varepsilon_{xx} + W_\varepsilon'(u^\varepsilon) = 0 & \text{in } (0,T)\times[0,L] \\ u^\varepsilon_x(t,0) = u^\varepsilon_x(t,L) = 0 & \text{in } [0,T] \\ u^\varepsilon(0,x) = 1+\sqrt{\varepsilon} & \text{in } [0,L] \\ u^\varepsilon_t(0,x) = 0 & \text{in } [0,L] \end{cases} \tag{2.4.9}$$

and $u^\varepsilon \to 1$ and $W_\varepsilon'(u^\varepsilon) = \varepsilon$.

Observe that when approximate solutions do not approach the discontinuity region, one can indeed repeat the same exact steps as in the proof of Theorem 2.3.1. Within this very specific setting, one can prove the existence of weak solutions for small initial data (i.e., when the troublesome region is completely avoided).

Theorem 2.4.4. *Consider $2s > d$, W, $\nabla_u W$ as defined in* (2.4.1), (2.4.2) *respectively and assume that*

$$||u_0||_{\tilde{H}^s(\Omega)} \leq \varepsilon_1, \ ||v_0||_{L^2(\Omega)} \leq \varepsilon_2 \tag{2.4.10}$$

for sufficiently small ε_1, ε_2. Then, there exists a weak solution of problem (2.2.5) *in the sense of Definition* 2.4.1 *with*

$$|u(x,t)| < 1 \quad \text{for all } (t,x) \in [0,T] \times \Omega \tag{2.4.11}$$

and

$$E(u(t)) \leq E(u(0)) \quad \text{for any } t \in [0,T]. \tag{2.4.12}$$

Proof. We repeat the approach used in the proof of Theorem 2.3.1: construct a family of non-negative potentials $(W_\varepsilon)_{\varepsilon>0}$ in $C^2(\mathbb{R})$ such that:

(i) W_ε converges uniformly to W in $\mathbb{R}^m$,

(ii) ∇W_ε converges pointwise to ∇W in $\mathbb{R}^m \setminus \partial\mathbf{B}(0,1)$, ∇W_ε converges uniformly to ∇W in $\mathbf{B}(0,1)$, ∇W_ε is uniformly bounded in $\mathbb{R}^m$,

(iii) ∇W_ε is Lipschitz for each ε.

For each $\varepsilon > 0$ there exists a weak solution u_ε in the sense of Definition 2.2.2 corresponding to W_ε with initial data u_0, v_0 such that for any $t \in [0,T]$ one has

$$\frac{1}{2}||u_t^\varepsilon(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u^\varepsilon(t)]^2_s + ||W_\varepsilon(u^\varepsilon(t))||_{L^1(\Omega)} \leq \frac{1}{2}||v_0||^2_{L^2(\Omega)} + \frac{1}{2}[u_0]^2_s + ||W_\varepsilon(u_0)||_{L^1(\Omega)}. \tag{2.4.13}$$

Since W_ε converges uniformly to W in $\mathbb{R}^m$, for sufficiently small ε we have

$$|W_\varepsilon(y) - W(y)| \leq \varepsilon_3 \tag{2.4.14}$$

for any $y \in \mathbb{R}^m$ and $\varepsilon_3 > 0$ fixed. This fact combined with (2.4.10) implies that

$$||W_\varepsilon(u_0)||_{L^1(\Omega)} \leq ||W(u_0)||_{L^1(\Omega)} + \varepsilon_3|\Omega| \leq |\Omega|\varepsilon_1^2 + \varepsilon_3|\Omega|. \tag{2.4.15}$$

Thus, combining (2.4.13) with estimates in (2.4.10) and (2.4.15), we obtain that

$$E(u^\varepsilon(t)) = \frac{1}{2}||u_t^\varepsilon(t)||^2_{L^2(\Omega)} + \frac{1}{2}[u^\varepsilon(t)]^2_s + ||W_\varepsilon(u^\varepsilon(t))||_{L^1(\Omega)} \leq C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \Omega) \tag{2.4.16}$$

for any $t \in [0,T]$. On the other hand, we have

$$\begin{aligned} ||u^\varepsilon(t) - u^\varepsilon(0)||^2_{L^2(\Omega)} &= \int_\Omega \left| \int_0^t u_t^\varepsilon(s,x)ds \right|^2 dx \leq t \int_\Omega \int_0^t |u_t^\varepsilon(s,x)|^2 ds dx \\ &\leq T \int_0^t \int_\Omega |u_t^\varepsilon(s,x)|^2 dx ds \leq 2T^2 C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \Omega) \end{aligned} \tag{2.4.17}$$

where we have made use of Jensen's inequality and Fubini's theorem. Hence,

$$||u^\varepsilon(t)||_{L^2(\Omega)} \leq ||u^\varepsilon(0)||_{L^2(\Omega)} + T\sqrt{2C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \Omega)} \leq \varepsilon_1 + T\sqrt{2C(\varepsilon_1, \varepsilon_2, \varepsilon_3, \Omega)} \tag{2.4.18}$$

for all $t \in [0, T]$. So, from the estimates (2.4.16) and (2.4.18) we obtain that

$$||u^\varepsilon(t)||_{\tilde{H}^s(\Omega)} \leq C(\varepsilon_1, \varepsilon_2, \varepsilon_3, T, \Omega), \tag{2.4.19}$$

Since $2s > d$, by means of the Sobolev embedding from $\tilde{H}^s(\Omega)$ into the space $C^0(\mathbb{R}^d; \mathbb{R}^m)$ (see Lemma 2.2.1), we obtain

$$||u^\varepsilon(t)||_{C^0(\Omega;\mathbb{R}^m)} \leq C||u^\varepsilon(t)||_{\tilde{H}^s(\Omega)} \leq C(\varepsilon_1, \varepsilon_2, \varepsilon_3, T, \Omega), \tag{2.4.20}$$

for all $t \in [0, T]$, where $C(\varepsilon_1, \varepsilon_2, \varepsilon_3, T, \Omega)$ is decreasing as soon as $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are decreasing. Thus, for any small $\eta > 0$, by choosing $\varepsilon_1, \varepsilon_2, \varepsilon_3$ small enough one has

$$|u^\varepsilon(x, t)| \leq 1 - \eta \tag{2.4.21}$$

for any $(t, x) \in [0, T] \times \Omega$. Since approximate solutions never enter the discontinuity region of the gradient ∇W, one can then repeat the same steps as in the proof of Theorem 2.3.1 to pass to the limit along the sequence $(u^\varepsilon)_\varepsilon$ and obtain a weak solution satisfying (2.4.11) and (2.4.12). □

As the above discussion made clear, in order to be able to handle problems with a discontinuity of the adhesive glue layer, i.e. discontinuities in ∇W, a more robust notion of solution is needed. An immediate follow-up would be to consider for instance solutions in the sense of differential inclusions, e.g.

$$u_{tt} + (-\Delta)^s u \in -\partial W(u)$$

(see for example [31], and references therein) or in the sense of Young measures, but we postpone such discussion to future works.

Chapter 3

Energy minimizing maps with singularities and Gilbert-Steiner problems

In this chapter we investigate the relation between energy minimizing maps valued into spheres having topological singularities at given points and optimal networks connecting them (e.g. Steiner trees, Gilbert-Steiner irrigation networks). We show the equivalence of the corresponding variational problems, interpreting in particular the branched optimal transport problem as a homological Plateau problem for rectifiable currents with values in a suitable normed group. This generalizes the pioneering work by Brezis, Coron and Lieb [25].

3.1 Introduction

In their celebrated paper [25], Brezis, Coron and Lieb showed, in the context of harmonic maps and liquid crystals theory, the existence of a close relation between sphere-valued harmonic maps having prescribed topological singularities at given points in $\mathbb{R}^3$ and *minimal connections* between those points, i.e., optimal mass transportation networks (in the sense of Monge-Kantorovich) having those points as marginals. This relation was further enlightened by Almgren, Browder and Lieb in [4], who recovered the results in [25] by interpreting the (minimal connection) optimal transportation problem as a suitable Plateau problem for rectifiable currents having the given marginals as prescribed boundary.

Our aim is to consider minimizing configurations for maps valued into manifolds and with prescribed topological singularities when the energy is possibly more general than the Dirichlet energy, and investigate the connection with Plateau problems for currents (or flat chains) with coefficients in suitable groups. The choice of these groups is linked to the topology of the involved target manifolds.

In this chapter we will consider the particular case where the manifold is a product of spheres and the maps have assigned point singularities, and we will show, in Theorem 3.1.1 below, that energy minimizing configurations are related with Steiner-type optimal networks connecting the given points, i.e., solutions of the Steiner problem or solutions of the Gilbert-Steiner irrigation problem. In a companion paper in preparation [29] we will discuss and state the results which correspond to more general situations.

Steiner tree problems and Gilbert-Steiner (single sink) problems can be formulated as follows: given n distinct points $P_1, \dots, P_n$ in $\mathbb{R}^d$, where $d, n \geq 2$, we are looking for an optimal connected transportation network, $L = \cup_{i=1}^{n-1} \lambda_i$, along which the unit masses initially located at $P_1, \dots, P_{n-1}$ are transported to the target point P_n (single sink); here λ_i can be seen as the path of the i^{th} mass flowing from P_i to P_n, and the cost of moving a mass m along a segment with length l is proportional to lm^α, $\alpha \in [0, 1]$. Therefore, we are led to consider the problem

$$(I) \qquad \inf \left\{ I_\alpha(L) : \, L = \bigcup_{i=1}^{n-1} \lambda_i \text{ with } \{P_i, P_n\} \subset \lambda_i, \text{ for every } i = 1, \dots, n-1 \right\}$$

where the energy I_α is computed as $I_\alpha(L) = \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x)$, with $\theta(x) = \sum_{i=1}^{n-1} \mathbf{1}_{\lambda_i}(x)$. Let us notice that θ stands for the mass density along the network. In particular, we consider the range $\alpha \in [0, 1]$:

- when $\alpha = 0$ the problem is equivalent to optimize the total length of the graph L, as in the Steiner Tree Problem (STP);
- when $\alpha = 1$ the problem (I) becomes the well-known Monge-Kantorovich problem;
- and when $0 < \alpha < 1$ the problem is known as the Gilbert-Steiner problem, or, more generally, as a branched optimal transport problem, due to the fact that the cost is proportional to a concave function θ^α, which favours the clustering of the mass during the transportation, thus giving rise to the branched structures which characterize the solutions (we refer the reader to [11] for an overview on the topic).

In the last decade, the communities of Calculus of Variations and Geometric Measure Theory made some efforts to study (Gilbert-)Steiner problems in many aspects, such as existence, regularity, stability and numerical feasibility (see for example [81, 68, 54, 55, 35, 36, 65, 23, 58, 19, 21, 20] and references therein). Among all the significant results, we would like to mention recent works in [54, 55] and [19, 20], which are closely related to our work. To be more precise, in [54, 55] the authors turn the problem (I) into the problem of mass-minimization of integral currents with multiplicities in a suitable group. For the sake of readability we postpone proper definitions about currents to Section 3.2, in this introduction we only recall that a 1-dimensional integral current with coefficients in a group can be thought as a formal sum of finitely many curves and countably many loops with coefficients in a given normed abelian group. For instance, considering the group $\mathbb{Z}^{n-1}$ and assigning to the boundary datum $P_1, P_2, \dots, P_{n-1}, P_n$ the multiplicities $e_1, e_2, \dots, e_{n-1}, -(e_1 + \dots + e_{n-1})$, respectively (where $\{e_i\}_{1 \leq i \leq n-1}$ is the basis of $\mathbb{R}^{n-1}$), we recover the standard model in [54, 55].

In fact we can interpret the network $L = \bigcup_{i=1}^{n-1} \lambda_i$ as the superposition of $n-1$ paths λ_i connecting P_i to P_n labelled with multiplicity e_i. This point of view requires a density function with values in $\mathbb{Z}^{n-1}$, which corresponds to the so-called 1-dimensional current with coefficients in the group $\mathbb{Z}^{n-1}$. Furthermore, by equipping $\mathbb{Z}^{n-1}$ with a certain norm (depending on the cost of the problem), we may define the notion of mass of those currents, and problem (I) turns out to be equivalent to the Plateau problem.

$$(M) \qquad \inf \left\{ \mathbb{M}(T) : \, \partial T = e_1 \delta_{P_1} + e_2 \delta_{P_2} + \dots + e_{n-1} \delta_{P_{n-1}} - (e_1 + e_2 + \dots + e_{n-1}) \delta_{P_n} \right\}$$

where T is a 1-dimensional current with coefficients in the group $\mathbb{Z}^{n-1}$ (again, we refer the reader to the Section 3.2 for rigorous definitions). For mass minimization, there is the very useful notion of calibration (see section 3.3), that is, a tool to prove minimality when dealing with concrete configurations (see Example 3.3.1). To be precise, a calibration is a sufficient condition to prove minimality, see Definition 3.2.8 and the following remarks.

In [19, 20], by using [54, 55], a variational approximation of the problem (I) was provided through Modica-Mortola type energies in the planar case, and through Ginzburg-Landau type energies (see [3]) in higher dimensional ambient spaces via Γ-convergence. The corresponding numerical treatment is also shown there.

Following [54, 55], [19, 20], and the strategy outlined in [4] (relating the energy of harmonic maps with prescribed point singularities to the mass of 1-dimensional classical integral currents) we provide here a connection between k-harmonic manifold-valued maps with prescribed point singularities and (Gilbert-)Steiner problems (I). More precisely, let $P_1, \ldots, P_{n-1}, P_n$ in $\mathbb{R}^d$ be given, and consider the spaces H_i defined as the subsets of $W^{1,d-1}_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{S}^{d-1})$ where the functions are constant outside a neighbourhood of the segment joining P_i, P_n and have distributional Jacobian $\frac{\alpha_{d-1}}{d}(\delta_{P_i} - \delta_{P_n})$, respectively. Here α_{d-1} is the surface area of the unit ball in $\mathbb{R}^d$.

Let ψ be a norm on $\mathbb{R}^{n-1}$ which will be specified in Section 3.3 (see (3.4.9)), and set

$$\mathbb{H}(\mathbf{u}) = \int_{\mathbb{R}^d} \psi(|\nabla u_1|^{d-1}, |\nabla u_2|^{d-1}, \ldots, |\nabla u_{n-1}|^{d-1})\, dx \tag{3.1.1}$$

where $\mathbf{u} = (u_1, \ldots, u_{n-1}) \in H_1 \times H_2 \times \ldots \times H_{n-1}$ is a 2-tensor. We investigate

$$(H) \qquad \inf\left\{\mathbb{H}(\mathbf{u}) : \mathbf{u} \in H_1 \times H_2 \times \ldots \times H_{n-1}\right\}.$$

The main contribution of this chapter is the following

Theorem 3.1.1. *Assume that a minimizer of problem (M) admits a calibration (see Definition 3.2.8). Then, we have*

$$\inf \mathbb{H} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf \mathbb{M} \tag{3.1.2}$$

or equivalently, in view of paper [54, 55],

$$\inf \mathbb{H} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha\,. \tag{3.1.3}$$

Currently, we cannot evade the assumption on the existence of a calibration, because it is still not known if a calibration, or even a weak version of it, is not only sufficient but also a necessary condition for minimality (see Section 3.2). Nonetheless, dropping this assumption we can still state some partial result as follows.

Remark 3.1.2.
(i) If $\alpha = 1$, then we are able to prove that (3.1.3) still holds true, as a variant of the main result in Brezis, Coron, Lieb [25].

(ii) In case $0 \leq \alpha < 1$, we obtain the following inequality

$$(d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf \mathbb{M} = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha \geq \inf \mathbb{H}\,. \tag{3.1.4}$$

The investigation of equality in (3.1.4) when $0 \leq \alpha < 1$ is delicate and will be considered in forthcoming works.

Remark 3.1.3. We believe that the assumption of the existence of a calibration is not too restrictive. We actually conjecture that minimizing configurations for the problem (M) admit a calibration in case of uniqueness, which is somehow a generic property. We carry out in Example 3.3.1 the construction of configurations of n points in $\mathbb{R}^{n-1}$ with $n-2$ branching points which are generic in character and these configurations admit a calibration.

The organization of Chapter 3 is as follows: in Section 3.2, we briefly review some basic notions of Geometric Measure Theory which will be used in the chapter, in Section 3.3 we recall (Gilbert-) Steiner problems and briefly describe their connection with Plateau's problem for currents with coefficients in a group. Finally, in Section 3.4 we prove the Theorem 3.1.1.

3.2 Preliminaries and notations

3.2.1 Rectifiable currents with coefficients in a group G

In this section, we present the notion 1-dimensional currents with coefficients in the group $\mathbb{R}^{n-1}$ in the ambient space $\mathbb{R}^d$ with $n, d \geq 2$. We refer to [57] for a more detailed exposition of the subject.

Consider $\mathbb{R}^{n-1}$ equipped with a norm ψ and its dual norm ψ^*. Denote by $\Lambda_1(\mathbb{R}^d)$ the space of 1-dimensional vectors and by $\Lambda^1(\mathbb{R}^d)$ the space of 1-dimensional covectors in $\mathbb{R}^d$.

Definition 3.2.1. An $(\mathbb{R}^{n-1})^*$-valued 1-covector on $\mathbb{R}^d$ is a bilinear map

$$w : \Lambda_1(\mathbb{R}^d) \times \mathbb{R}^{n-1} \longrightarrow \mathbb{R}\,.$$

Let $\{e_1, e_2, \ldots, e_{n-1}\}$ be an orthonormal basis of $\mathbb{R}^{n-1}$, and let $\{e_1^*, e_2^*, \ldots, e_{n-1}^*\}$ be its dual. Then, each $(\mathbb{R}^{n-1})^*$-valued 1-covector on $\mathbb{R}^d$ can be represented as $w = w_1 e_1^* + \ldots + w_{n-1} e_{n-1}^*$, where w_i is a "classical" 1-dimensional covector in $\mathbb{R}^d$ for each $i = 1, \ldots, n-1$. To be precise, the action of w on a pair $(\tau, \theta) \in \Lambda_1(\mathbb{R}^d) \times \mathbb{R}^{n-1}$ can be computed as

$$\langle w; \tau, \theta \rangle = \sum_{i=1}^{n-1} \theta_i \langle w_i, \tau \rangle\,,$$

where the scalar product on the right hand side is the standard Euclidean scalar product in $\mathbb{R}^d$. We denote by $\Lambda^1_{(\mathbb{R}^{n-1},\psi)}(\mathbb{R}^d)$ the space of $(\mathbb{R}^{n-1})^*$-valued 1-covectors in $\mathbb{R}^d$, endowed with the following norm:

$$|w| := \sup\{\psi^*(\langle w; \tau, \cdot \rangle) \,:\, |\tau| \leq 1\}\,.$$

Definition 3.2.2. An $(\mathbb{R}^{n-1})^*$-valued 1-dimensional differential form defined on $\mathbb{R}^d$ is a map

$$\omega : \mathbb{R}^d \longrightarrow \Lambda^1_{(\mathbb{R}^{n-1},\psi)}(\mathbb{R}^d)\,.$$

Let us remark that the regularity of ω is inherited from the components ω_i, $i = 1, \ldots, n-1$. Let $\varphi = (\varphi_1, \ldots, \varphi_{n-1})$ be a function of class $C^1(\mathbb{R}^d; \mathbb{R}^{n-1})$. We denote

$$\mathrm{d}\varphi := \mathrm{d}\varphi_1 e_1^* + \ldots + \mathrm{d}\varphi_{n-1} e_{n-1}^*,$$

where $\mathrm{d}\varphi_i$ is the differential of φ_i. Thus $\mathrm{d}\varphi \in C(\mathbb{R}^d; \Lambda^1_{(\mathbb{R}^{n-1},\psi)}(\mathbb{R}^d))$.

Definition 3.2.3. A 1-dimensional current T with coefficients in $(\mathbb{R}^{n-1}, \psi)$ is a linear and continuous map

$$T : C_c^\infty\left(\mathbb{R}^d; \Lambda^1_{(\mathbb{R}^{n-1},\psi)}(\mathbb{R}^d)\right) \longrightarrow \mathbb{R}\,.$$

Here the continuity is meant with respect to the (locally convex) topology on $C_c^\infty(\mathbb{R}^d; \Lambda^1_{(\mathbb{R}^{n-1},\psi)}(\mathbb{R}^d))$ defined in analogy with the topology on $C_c^\infty(\mathbb{R}^d;\mathbb{R})$ which allows the definition of distributions. The mass of T is defined as

$$\mathbb{M}(T) := \sup\left\{T(\omega) : \sup_{x\in\mathbb{R}^d} |\omega| \le 1\right\}.$$

Moreover, if T is a 1-dimensional current with coefficients in $(\mathbb{R}^{n-1},\psi)$, we define the boundary ∂T of T as a distribution with coefficients in $(\mathbb{R}^{n-1},\psi)$, $\partial T : C_c^\infty(\mathbb{R}^d;(\mathbb{R}^{n-1},\psi)) \longrightarrow \mathbb{R}$, such that

$$\partial T(\varphi) := T(\mathrm{d}\varphi).$$

The mass of ∂T is the supremum norm

$$\mathbb{M}(\partial T) := \sup\left\{T(\mathrm{d}\varphi) : \sup_{x\in\mathbb{R}^d} \psi^*(\varphi) \le 1\right\}.$$

A current T is said to be normal if $\mathbb{M}(T) + \mathbb{M}(\partial T) < \infty$.

Definition 3.2.4. A 1-dimensional rectifiable current with coefficients in the normed (abelian) group $(\mathbb{Z}^{n-1},\psi)$ is a (1-dimensional) normal current (with coefficients in $(\mathbb{R}^{n-1},\psi)$) such that there exists a 1-dimensional rectifiable set $\Sigma \subset \mathbb{R}^d$, an approximate tangent vectorfield $\tau : \Sigma \longrightarrow \Lambda_1(\mathbb{R}^d)$, and a density function $\theta : \Sigma \longrightarrow \mathbb{Z}^{n-1}$ such that

$$T(\omega) = \int_\Sigma \langle \omega(x)\tau(x), \theta(x)\rangle \, d\mathcal{H}^1(x)$$

for every $\omega \in C_c^\infty\left(\mathbb{R}^d; \Lambda^1_{(\mathbb{R}^{n-1},\psi)}(\mathbb{R}^d)\right)$. We denote such a current T by the triple $[\![\Sigma,\tau,\theta]\!]$.

Remark 3.2.5. The mass of a rectifiable current $T = [\![\Sigma,\tau,\theta]\!]$ with coefficients in $(\mathbb{Z}^{n-1},\psi)$ can be computed as

$$\mathbb{M}(T) := \sup\left\{T(\omega) : \sup_{x\in\mathbb{R}^d} |\omega| \le 1\right\} = \int_\Sigma \psi(\theta(x))\, d\mathcal{H}^1(x).$$

Moreover, $\partial T : C_c^\infty(\mathbb{R}^d;(\mathbb{R}^{n-1},\psi)) \longrightarrow \mathbb{R}$ is a measure and there exist $x_1,\dots,x_m \in \mathbb{R}^d$, $p_1,\dots,p_m \in \mathbb{Z}^{n-1}$ such that

$$\partial T(\varphi) = \sum_{j=1}^m p_j \varphi(x_j).$$

Finally the mass of the boundary $\mathbb{M}(\partial T)$ coincides with $\sum_{j=1}^m \psi(p_j)$.

Remark 3.2.6. In the trivial case $n = 2$, we consider rectifiable currents with coefficients in the discrete group $\mathbb{Z}$ and we recover the classical definition of integral currents (see, for instance, [38]).

Finally, it is useful to define the components T with respect to the index $i \in \{1,\dots,n-1\}$: for every 1-dimensional test form $\tilde{\omega} \in C_c^\infty(\mathbb{R}^d;\Lambda^1(\mathbb{R}^d))$ we set

$$T^i(\tilde{\omega}) := T(\tilde{\omega} e_i^*).$$

Notice that T^i is a classical integral current (with coefficients in $\mathbb{Z}$). Roughly speaking, in some situations we are allowed to see a current with coefficients in $\mathbb{R}^{n-1}$ through its components $(T^1,\dots,T^{n-1})$.

When dealing with the Plateau problem in the setting of currents, it is important to remark a couple of critical features. For the sake of understandability, we recall them here for the particular case of 1-dimensional currents, but the matter does not depend on the dimension.

Remark 3.2.7. If a boundary $\{P_1, \ldots, P_n\} \subset \mathbb{R}^d$ is given, then the problem of the minimization of mass is well posed in the framework of rectifiable currents and in the framework of normal currents as well. In both cases the existence of minimizers is due to a direct method and, in particular, to the closure of both classes of currents. Obviously

$$\begin{aligned}
&\min\{\mathbb{M}(T) : T \text{ normal current with coefficients in } \mathbb{R}^{n-1} \text{ and boundary } \{P_1, \ldots, P_n\}\} \\
\leq &\min\{\mathbb{M}(T) : T \text{ rectifiable current with coefficients in } \mathbb{Z}^{n-1} \text{ and boundary } \{P_1, \ldots, P_n\}\},
\end{aligned}$$

but whether the inequality is actually an identity is not known for currents with coefficients in groups. The same question about the occurence of a Lavrentiev gap between normal and integral currents holds for classical currents of dimension bigger than 1 and it is closely related to the problem of the decomposition of a normal current in rectifiable ones (see [57] for a proper overview of this issue).

A formidable tool for proving the minimality of a certain current is to show the existence of a calibration.

Definition 3.2.8. Consider a rectifiable current $T = [\![\Sigma, \tau, \theta]\!]$ with coefficients in $\mathbb{Z}^n$, in the ambient space $\mathbb{R}^d$. A smooth $(\mathbb{R}^n)^*$-valued differential form ω in $\mathbb{R}^d$ is a calibration for T if the following conditions hold:

(i) for a.e $x \in \Sigma$ we have that $\langle \omega(x); \tau(x), \theta(x) \rangle = \psi(\theta(x))$;

(ii) the form is closed, i.e., $\mathrm{d}\omega = 0$;

(iii) for every $x \in \mathbb{R}^d$, for every unit vector $t \in \mathbb{R}^d$ and for every $h \in \mathbb{Z}^n$, we have that

$$\langle \omega(x); t, h \rangle \leq \psi(h)\,.$$

It is straightforward to prove that the existence of a calibration associated to a current implies the minimality of the current itself. Indeed, with the notation in Definition 3.2.8, if $T' = [\![\Sigma', \tau', \theta']\!]$ is a competitor, i.e., T' is a rectifiable current with coefficients in $\mathbb{Z}^n$ and $\partial T' = \partial T$, then

$$\mathbb{M}(T) = \int_\Sigma \psi(\theta) = \int_\Sigma \langle \omega; \tau, \theta \rangle = \int_{\Sigma'} \langle \omega; \tau', \theta' \rangle \leq \int_{\Sigma'} \psi(\theta') = \mathbb{M}(T')\,.$$

We stress that fact that the existence of a calibration is a sufficient condition for the minimality of a current, so it is always a wise attempt when a current is a good candidate for mass minimization. Nonetheless, it is also natural to wonder if every mass minimizing current has its own calibration and this problem can be tackled in two ways: for specific currents or classes of currents (such as holomorphic subvarieties) one has to face an extension problem with the (competing) constraints (ii) and (iii), since condition (i) already prescribes the behaviour of the form on the support of the current. In general, one may attempt to prove the existence of a calibration as a result of a functional argument, picking it in the dual space of normal currents, but this approach has two still unsolved problems:

- the calibration is merely an element of the dual space of normal currents, thus it is far to be smooth;

- this argument works in the space of normal currents and it is not known whether a minimizer in this class is rectifiable as well (see Remark 3.2.7).

Anyway, in this specific case of currents with coefficients in $\mathbb{Z}^n$ which match the energy minimizing networks of a branched optimal transport problem (with a subadditive cost), we think that the Lavrentiev phenomenon cannot occur, as explained in Remark 3.1.3.

3.2.2 Distributional Jacobian

We recall the notion of distributional Jacobian of a function $u \in W^{1,d-1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d) \cap L^\infty_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, see also [48, 2].

Let u be in $W^{1,d-1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d) \cap L^\infty_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$, we define an auxiliary map $\tilde u \in L^1_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ as

$$\tilde u := \left(\det(u, u_{x_2}, \dots, u_{x_d}), \det(u_{x_1}, u, \dots, u_{x_d}), \dots, \det(u_{x_1}, \dots, u_{x_{d-1}}, u)\right),$$

where u_{x_j} is a $L^{d-1}_{\rm loc}(\mathbb{R}^d;\mathbb{R}^d)$ representative of the partial derivative of u with respect to the $j^{\rm th}$ direction. Thus we define the Jacobian Ju of u as $\frac{1}{n}{\rm d}\tilde u$ in the sense of distributions. More explicitly, if $\varphi \in C^\infty_c(\mathbb{R}^d;\mathbb{R})$ is a test function, then one has

$$\int_{\mathbb{R}^d} \varphi\, Ju\, dx = -\frac{1}{n}\int_{\mathbb{R}^d} \nabla\varphi \cdot \tilde u\, dx\,. \tag{3.2.1}$$

The identity required in (3.2.1) is clearer if one notices that $\tilde u$ has been chosen in such a way that $\operatorname{div}(\varphi\tilde u) = \nabla\varphi\cdot\tilde u + n\varphi \det Du$ whenever u is smooth enough to allow the differential computation.

Once the singularities of the problem $p_1, \dots, p_n$ have been prescribed, we can also introduce the energy spaces H_i, for each $i = 1, \dots, n-1$. By definition a map $u \in W^{1,d-1}_{\rm loc}(\mathbb{R}^d;\mathbb{S}^{d-1})$ belongs to H_i if $Ju = \frac{\alpha_{d-1}}{d}(\delta_{P_i} - \delta_{P_n})$, and there exists a radius $r = r(u) > 0$ such that u is constant outside $B(0, r(u)) \ni P_i, P_n$, where $B(0,r)$ is the open ball of radius r centered at 0.

In order to prove Theorem 3.1.1 we consider the following problem:

$$(H) \qquad \inf\left\{\mathbb{H}(\mathbf{u}),\ \ \mathbf{u} = (u_1, \dots, u_{n-1}) \in H_1 \times H_2 \times \ldots \times H_{n-1}\right\}.$$

where

$$\mathbb{H}(\mathbf{u}) = \int_{\mathbb{R}^d} \psi(|\nabla u_1|^{d-1}, |\nabla u_2|^{d-1}, \dots, |\nabla u_{n-1}|^{d-1})\, dx\,. \tag{3.2.2}$$

As indicated in the introduction, the inspiration for considering the problem (H) and comparing it with the irrigation problem (I) is coming from the works [54, 55] and [4]. More precisely, [54, 55] provided a new framework for the problem (I) by proving it to be equivalent to the problem of mass-minimizing currents with coefficients in the group $\mathbb{Z}^{n-1}$ with a suitable norm. The point of view is to look at each irrigation network $L = \bigcup_{i=1}^{n-1} \lambda_i$ encoded in the current $T = (T^1, \dots, T^{n-1})$ where T^i is a classical current supported by λ_i, and the irrigation cost of L is the mass of the current T. Then, by combining this point of view with [4] (see also [25]), where the energy of harmonic maps with prescribed point singularities was related to 1-dimensional classical currents. This leads us to investigate the problem (H) in connection with problem (I).

3.3 (Gilbert-)Steiner problems and currents with coefficients in a group

Let us briefly recall the Gilbert-Steiner problem and the Steiner tree problem and see how it can be turned into a mass-minimization problem for integral currents in a suitable group.

Let n distinct points $P_1, \dots, P_n$ in $\mathbb{R}^d$ be given. Denote by $G(A)$ the set of all acyclic graphs $L = \bigcup_{i=1}^{n-1} \lambda_i$, along which the unit masses located at $P_1, \dots, P_{n-1}$ are transported to the target point P_n (single sink). Here λ_i is a simple rectifiable curve and represents the path of the mass at P_i flowing from P_i to P_n. In [54, 55], the occurrence of cycles in minimizers is ruled out, thus the problem (I) is proved to be equivalent to

$$(I) \qquad \inf\left\{\int_L |\theta(x)|^\alpha d\mathcal{H}^1(x), \;\; L \in G(A), \;\; \theta(x) = \sum_{i=1}^{n-1} \mathbf{1}_{\lambda_i}(x)\right\}$$

where θ is the mass density along the network L. Moreover, in [54, 55] the problem (I) can be turned into a mass-minimization problem for integral currents with coefficients in the group $\mathbb{Z}^{n-1}$: the idea is to label differently the masses located at $P_1, P_2 \dots, P_{n-1}$ (source points) and to associate the source points $P_1, \dots, P_{n-1}$ to the single sink P_n. Formally, we produce a 0-dimensional rectifiable current (a.k.a. a measure) with coefficients in $\mathbb{Z}^{n-1}$, given by the difference between

$$\mu^- = e_1\delta_{P_1} + e_2\delta_{P_2} + \dots + e_{n-1}\delta_{P_{n-1}} \text{ and } \mu^+ = (e_1 + \dots + e_n)\delta_{P_n}\,.$$

We recall that $\{e_1, e_2, \dots, e_n\}$ is the canonical basis of $\mathbb{R}^{n-1}$. The measures μ^-, μ^+ are the marginals of the problem (I). To any acyclic graph $L = \bigcup_{i=1}^{n-1} \lambda_i$ we associate a current T with coefficients in the group $\mathbb{Z}^{n-1}$ as follows: to each λ_i associate the current $T_i = [\![\lambda_i, \tau_i, e_i]\!]$, where τ_i is the tangent vector of λ_i. We associate to the graph $L = \bigcup_{i=1}^{n-1} \lambda_i$ the current $T = (T_1, \dots, T_{n-1})$ with coefficients in $\mathbb{Z}^{n-1}$. By construction we obtain

$$\partial T = \mu^+ - \mu^-\,.$$

Choosing the norm ψ on $\mathbb{Z}^{n-1}$ as

$$\psi(h) = \begin{cases} ||\cdot||_\alpha = \left(\sum_{j=1}^{n-1} |h_j|^{\frac{1}{\alpha}}\right)^\alpha & \text{in case } \alpha \in (0;1],\, h \in \mathbb{Z}^{n-1} \\ ||\cdot||_0 = \max\{h_1, \dots, h_{n-1}\} & \text{in case } \alpha = 0,\, h \in \mathbb{Z}^{n-1}\,, \end{cases} \tag{3.3.1}$$

in view of Remark 3.2.5, the problem (I) is equivalent to

$$(M) \qquad \inf\left\{\mathbb{M}(T), \;\; \partial T = \mu^+ - \mu^-\right\}\,.$$

We refer the reader to [54, 55] for more details. From now on we restrict our attention to the coefficients group $(\mathbb{Z}^{n-1}, ||\cdot||_\alpha)$, $0 \le \alpha \le 1$.

We remark that turning the problem (I) into a mass-minimization problem allows to rely on the (dual) notion of calibration, which is a useful tool to prove minimality, especially when dealing with concrete configurations. We also recall that the existence of a calibration (see Definition 3.2.8) associated with a current T implies that T is a mass-minimizing current for the boundary ∂T.

Example 3.3.1. Let us consider an irrigation problem with $\alpha = \frac{1}{2}$. We will consider a minimal network joining $n+1$ points in $\mathbb{R}^n$, the construction of the network is explained below. Let us stress that in this example the coincidence of the dimension of the ambient space with the dimension of the space of coefficients is needed.

Adopting the point of view of [43], we propose a calibration first, and only *a posteriori* we construct a current which fulfills the requirement (i) in Definition 3.2.8. We briefly remind that the problem (I) can be seen as the mass-minimization problem for currents with coefficients in $\mathbb{Z}^n$ with the norm $\|\cdot\|_{\frac{1}{2}}$.

Let $\{dx_1,\ldots,dx_n\}$ be the (dual) basis of covectors of $\mathbb{R}^n = \mathrm{span}(e_1,\ldots,e_n)$. We now prove that the differential form

$$\omega = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}$$

satisfies conditions (ii) and (iii) in Definition 3.2.8. Obviously $d\omega = 0$. Moreover, let $\tau = (\tau_1,\tau_2,\ldots,\tau_n) \in \mathbb{R}^n$ be a unit vector (with respect to the Euclidean norm). Thus, for our choice of the norm $\psi = \|\cdot\|_{\frac{1}{2}}$ we can compute $\|\langle\omega;\tau,\cdot\rangle\|^{\frac{1}{2}} = (\tau_1^2+\tau_2^2+\tau_3^2+\ldots+\tau_n^2)^{\frac{1}{2}} = 1$.

We will build now a configuration of $n+1$ points $P_1,P_2,\ldots,P_{n+1}$ in $\mathbb{R}^n$ calibrated by ω. Notice that the network has $n-1$ branching points and is somehow generic in character. More precisely, our strategy in building such a configuration is to choose end points, and branching points following the directions parallel to $e_1,e_2,e_3,\ldots,e_n,e_1+e_2,e_1+e_2+e_3,\ldots,e_1+e_2+\ldots+e_{n-1},e_1+e_2+\ldots+e_n$. We illustrate the construction in $\mathbb{R}^3,\mathbb{R}^4$. This process can be extended to any dimension.

- In $\mathbb{R}^3$, let us consider $P_1=(-1,0,0)$, $P_2=(0,-1,0)$, $P_3=(1,1,-1)$, $P_4=(2,2,1)$. Take, as branching points, $G_1=(0,0,0)$, $G_2=(1,1,0)$. Now consider the current $T = [\![\Sigma,\tau,\theta]\!]$ with support Σ obtained by the union of the segments $\overline{P_1G_1}, \overline{P_2G_1}, \overline{G_1G_2}, \overline{P_3G_2}, \overline{G_2P_4}$.

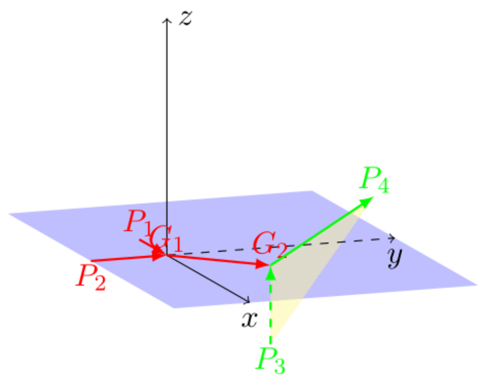

Figure 3.1: The picture illustrates the construction of T.

The multiplicity θ is set as

$$\theta(x) = \begin{cases} e_1 & \text{if } x \in \overline{P_1G_1} \\ e_2 & \text{if } x \in \overline{P_2G_1} \\ e_1+e_2 & \text{if } x \in \overline{G_1G_2} \\ e_3 & \text{if } x \in \overline{P_3G_2} \\ e_1+e_2+e_3 & \text{if } x \in \overline{G_2P_4} \\ 0 & \text{elsewhere.} \end{cases}$$

We observe that T is calibrated by ω, thus T is a minimal network for the irrigation problem with sources P_1,P_2 and P_3 and sink P_4. Notice that edges of the network meet at the branching points with the 90 degrees angles, as known for branched optimal structures with cost determined by $\alpha = 1/2$.

- In $\mathbb{R}^4$, we keep points $P_1 = (-1, 0, 0, 0)$, $P_2 = (0, -1, 0, 0)$, $P_3 = (1, 1, -1, 0)$ and, in general, the whole network of the example above as embedded in $\mathbb{R}^4$. We relabel $G_3 := (2, 2, 1, 0)$. We now pick P_4 and P_5 in such a way that $\overrightarrow{P_4G_3} = e_4$ and $\overrightarrow{G_3P_5} = e_1 + e_2 + e_3 + e_4$. For instance, we choose $P_4 = (2, 2, 1, -1)$ and $P_5 = (3, 3, 2, 1)$. As before, the marginals of the irrigation problem are P_1, P_2, P_3, P_4 as sources and P_5 as sink, while G_1, G_2, G_3 are branching points.

  Let us now consider the current $T = [\![\Sigma, \tau, \theta]\!]$ supported on the union of segments $\overline{P_1, G_1}, \overline{P_2G_1}, \overline{G_1G_2}, \overline{P_3G_2}, \overline{G_2G_3}, \overline{P_4G_3}, \overline{G_3P_5}$ and multiplicity θ given by

$$\theta(x) = \begin{cases} e_1 & \text{if } x \in \overline{P_1G_1} \\ e_2 & \text{if } x \in \overline{P_2G_1} \\ e_1 + e_2 & \text{if } x \in \overline{G_1G_2} \\ e_3 & \text{if } x \in \overline{P_3G_2} \\ e_1 + e_2 + e_3 & \text{if } x \in \overline{G_2G_3} \\ e_4 & \text{if } x \in \overline{P_4G_3} \\ e_1 + e_2 + e_3 + e_4 & \text{if } x \in \overline{G_3P_5} \\ 0 & \text{elsewhere.} \end{cases}$$

  It is easy to check that the orientation of each segment coincides with the multiplicity, therefore T is calibrated by ω.

- This procedure can be replicated to construct a configuration of $n+1$ points $P_1, P_2, \ldots, P_{n+1}$ in $\mathbb{R}^n$ calibrated by ω, always in the case $\alpha = 1/2$.

Example 3.3.2. We now consider a Steiner tree problem. As in the previous example, we aim to construct calibrated configurations joining $n + 1$ points in $\mathbb{R}^n$ (with $n - 1$ branching points). Consider the following differential form:

$$\omega = \begin{bmatrix} \frac{1}{2}\mathrm{d}x_1 + \frac{\sqrt{3}}{2}\mathrm{d}x_2 \\ \frac{1}{2}\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\mathrm{d}x_2 \\ \frac{-1}{2}\mathrm{d}x_1 - \frac{\sqrt{3}}{2}\mathrm{d}x_3 \\ \frac{-1}{4}\mathrm{d}x_1 + \frac{\sqrt{3}}{4}\mathrm{d}x_3 - \frac{\sqrt{3}}{2}\mathrm{d}x_4 \\ \frac{-1}{8}\mathrm{d}x_1 + \frac{\sqrt{3}}{8}\mathrm{d}x_3 + \frac{\sqrt{3}}{4}\mathrm{d}x_4 - \frac{\sqrt{3}}{2}\mathrm{d}x_5 \\ \vdots \\ \frac{-1}{2^{n-2}}\mathrm{d}x_1 + \frac{\sqrt{3}}{2^{n-2}}\mathrm{d}x_3 + \frac{\sqrt{3}}{2^{n-3}}\mathrm{d}x_4 + \ldots + \frac{\sqrt{3}}{2^{n-k}}\mathrm{d}x_{k+1} + \ldots + \frac{\sqrt{3}}{4}\mathrm{d}x_{n-1} - \frac{\sqrt{3}}{2}\mathrm{d}x_n \end{bmatrix}.$$

It is easy to check that the differential form ω is a calibration only among those currents having multiplicities $e_1, e_2, e_3, \ldots, e_n, e_1 + e_2, e_1 + e_2 + e_3, \ldots, e_1 + e_2 + \ldots + e_{n-1}, e_1 + e_2 + \ldots + e_n$ and hence it will allow to prove the minimality of configurations in the class of currents with those multiplicities (cf.[30] for the notion calibrations in families). Nevertheless, it is enough to prove the minimality of global minimizers in some configurations.

- Consider $n = 3$ and $P_1 = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0\right)$, $P_2 = \left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}, 0\right)$, $P_3 = \left(\frac{\sqrt{6}}{2} - \frac{1}{2}, 0, \frac{\sqrt{3}}{2}\right)$, $P_4 = \left(\frac{\sqrt{6}}{2} - \frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right)$ (see also the example in [20, Section 3]). Indeed, we observe that the lengths $|\overline{P_1P_2}| = |\overline{P_1P_3}| = |\overline{P_1P_4}| = |\overline{P_2P_3}| = |\overline{P_2P_4}| = |\overline{P_3P_4}| = \sqrt{3}$, meaning that the convex envelope of points P_1, P_2, P_3, P_4 is a tetrahedron: this observation allows us to restrict our investigation among all currents having multiplicities $e_1, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3$. More precisely, given any 1-dimensional integral current T with $\partial T = (e_1 + e_2 + e_3)\delta_{P_4} - e_1\delta_{P_1} - e_2\delta_{P_2} - \ldots - e_3\delta_{P_3}$ whose support is an acyclic graph with two additional Steiner points, we can always construct a corresponding current L with multiplicities e_1, e_2, $e_1 + e_2$, $e_1 + e_2 + e_3$ having the same boundary with T such that $\mathbb{M}(T) = \mathbb{M}(L)$ thanks to the symmetric configuration P_1, P_2, P_3, P_4 combined with the fact that any minimal configuration cannot have less than two Steiner points. Indeed, by contradiction, if a minimal configuration for the vertices of a tetrahedron had 1 Steiner point, then this configuration would violate the well-known property of the 120 degrees angles at Steiner points. Therefore, ω calibrates the current $T = [\![\Sigma, \tau, \theta]\!]$, where $S_1 = (0, 0, 0), S_2 = \left(\frac{\sqrt{6}}{2} - 1, 0, 0\right)$ are the Steiner points, $\Sigma = \overline{P_1S_1} \cup \overline{P_2S_1} \cup \overline{S_1S_2} \cup \overline{P_3S_2} \cup \overline{S_2P_4}$ and the multiplicity is given by

$$\theta(x) = \begin{cases} e_1 & \text{if } x \in \overline{P_1S_1} \\ e_2 & \text{if } x \in \overline{P_2S_1} \\ e_1 + e_2 & \text{if } x \in \overline{S_1S_2} \\ e_3 & \text{if } x \in \overline{P_3S_2} \\ e_1 + e_2 + e_3 & \text{if } x \in \overline{S_2P_4} \\ 0 & \text{elsewhere}\,. \end{cases}$$

- Using the same strategy of Example 3.3.1, we can build a configuration P_1, P_2, P_3, P_4, P_5 in $\mathbb{R}^4$ starting from the points P_1, P_2, P_3, P_4 above, in such a way that the new configuration is calibrated by ω among all currents with multiplicities $e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_2 + e_3, e_1 + e_2 + e_3 + e_4$. This construction can be extended to any dimension.

3.4 Proof of the main result

We devote this section to the proof of Theorem 3.1.1. The proof of Theorem 3.1.1 is much in the spirit of the dipole construction of [25] (in the version of [2]), and makes use of Coarea Formula to relate the harmonic energy to mass-minimization of classical integral currents, as in [4].

Proof. In the first steps we prove the inequality

$$\inf \mathbb{H} \leq (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha.$$

We briefly recall the dipole construction (see, for instance, [25, Theorem 3.1, Theorem 8.1]). Given a segment $\overline{AB} \subset \mathbb{R}^d$ and a pair of parameters $\beta, \gamma > 0$, we define

$$U := \{x \in \mathbb{R}^d : \operatorname{dist}(x, \overline{AB}) < \min\{\beta, \gamma \operatorname{dist}(\mathrm{x}, \{\mathrm{A}, \mathrm{B}\})\}\} \subset \mathbb{R}^d \tag{3.4.1}$$

to be a pencil-shaped neighbourhood with core $\overline{AB}$ and parameters β, γ. For any fixed $\varepsilon > 0$, the dipole construction produces a function $u \in W^{1,d-1}_{\text{loc}}(\mathbb{R}^d; \mathbb{S}^{d-1})$ with the following properties:

- $u \equiv (0, \ldots, 0, 1)$ in $\mathbb{R}^d \setminus U$;
- $Ju = \frac{\alpha_{d-1}}{d}(\delta_A - \delta_B)$;
- moreover the map u satisfies the following inequality

$$\frac{1}{(d-1)^{\frac{d-1}{2}}\alpha_{d-1}} \int_{\mathbb{R}^d} |\nabla u|^{d-1} dx \leq |AB| + \varepsilon \,, \tag{3.4.2}$$

Step 1. Let $L = \bigcup_{i=1}^{n-1} \lambda_i$ be an acyclic connected polyhedral graph, and T be the associated current with coefficients in $\mathbb{Z}^{n-1}$ corresponding to L. Since L is polyhedral, it can also be written as $L = \bigcup_{j=1}^{k} I_j$, where I_j are weighted segments. For each segment I_j we can find parameters $\delta_j, \gamma_j > 0$ such that the pencil-shaped neighbourhood $U_j = \left\{x \in \mathbb{R}^d : \operatorname{dist}(x, I_j) \leq \min\left\{\beta_j, \gamma_j \operatorname{dist}(x, \partial I_j)\right\}\right\}$ (modelled after (3.4.1)) is essentially disjoint from U_ℓ for every $\ell \neq j$. Then, for every $i = 1, \ldots, n-1$, let $V_i = \bigcup_{j \in K_i} U_j$ be a sharp covering of the path λ_i. To be precise, we choose $K_i \subset \{1, \ldots, k\}$ such that $V_i \cap U_\ell$ is at most an endpoint of the segment I_ℓ, if $\ell \notin K_i$.

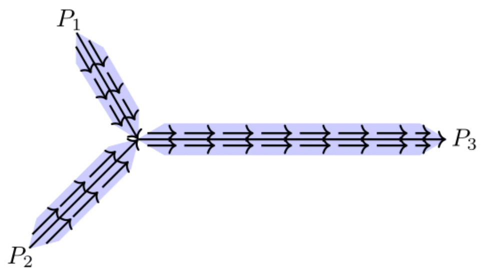

Figure 3.2: A dipole construction of a Y-shaped graph connecting 3 points.

For each path λ_i, $i = 1, \ldots, n-1$, we build the map $u_i \in H_i$ in such a way that it coincides with a dipole associated to the segment I_j in the neighbourhood U_j for each $j \in K_i$. We put $u_i \equiv (0, \ldots, 0, 1)$ in $\mathbb{R}^d \setminus V_i$.

We obtain that $u_i \in W^{1,d-1}_{\text{loc}}(\mathbb{R}^d; \mathbb{S}^{d-1})$ and satisfies $Ju_i = \frac{\alpha_{d-1}}{d}(\delta_{P_i} - \delta_{P_n})$. Moreover, summing up inequality (3.4.2) repeated for each segment I_j with $j \in K_i$, the following inequality holds

$$\frac{1}{(d-1)^{\frac{d-1}{2}}\alpha_{d-1}} \int_{\mathbb{R}^d} |\nabla u_i|^{d-1} dx \leq \mathbb{M}(T_i) + k\varepsilon \,,$$

where T_i is the (classical) integral current corresponding to the i^{th} component of T.

In particular, let us stress that the maps $u_1, \ldots, u_{n-1}$ have the following further property: if some paths $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_m}$ have a common segment I_j for some $j \in K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_m}$, then $u_{i_1}, \ldots, u_{i_m}$ agree in U_j. Furthermore, setting $h_{i_1,i_2,\ldots,i_m} = (0, \ldots, |\nabla u_{i_1}|^{d-1}, \ldots, |\nabla u_{i_m}|^{d-1}, \ldots, 0)$, we obtain

$$\frac{1}{(d-1)^{\frac{d-1}{2}}\alpha_{d-1}} \int_{U_j} ||h_{i_1,i_2,\ldots,i_m}||_\alpha dx \leq m^\alpha(|I_j| + k\varepsilon) \,,$$

where $h_{i_1,i_2,\ldots,i_m} = (0, \ldots, |\nabla u_{i_1}|^{d-1}, \ldots, |\nabla u_{i_m}|^{d-1}, \ldots, 0)$. This holds for every $\alpha \in [0, 1]$.

Combining all the previous observations, we can conclude that, given any $\tilde{\varepsilon} > 0$, there exist $u_i \in H_i$, $i = 1, \dots, n-1$ such that

$$\begin{aligned}\int_{\mathbb{R}^d} ||(|\nabla u_1|^{d-1}, |\nabla u_2|^{d-1}, \dots, |\nabla u_{n-1}|^{d-1})||_\alpha \, dx \leq &(d-1)^{\frac{d-1}{2}} \alpha_{d-1} \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x) + \tilde{\varepsilon} \\ = &(d-1)^{\frac{d-1}{2}} \alpha_{d-1} \mathbb{M}(T) + \tilde{\varepsilon} \,,\end{aligned}$$

where $\theta(x) = \sum_{i=1}^{n-1} \mathbf{1}_{\lambda_i}(x)$.

Step 2. Considering an arbitrary acyclic graph $L = \bigcup_{i=1}^{n-1} \lambda_i$, there is a sequence of acyclic polyhedral graphs $(L_m)_{m\geq 1}$, $L_m = \bigcup_{i=1}^{n-1} \lambda_i^m$ such that the Hausdorff distance $d_H(\lambda_i^m, \lambda_i) \leq \frac{1}{m}$, moreover (see [19, Lemma 3.10]) denoting by T and T_m the associated currents with coefficients in $\mathbb{Z}^{n-1}$ we also have that

$$\mathbb{M}(T_m) = \int_{L_m} |\theta_m(x)|^\alpha \, d\mathcal{H}^1(x) \leq \mathbb{M}(T) = \int_L |\theta(x)|^\alpha \, d\mathcal{H}^1(x) + \frac{1}{m}.$$

here $\theta_m(x) = \sum_{i=1}^{n-1} \mathbf{1}_{\lambda_i^m}(x)$. On the other hand, by previous construction there exists a sequence $\{\mathbf{u}_m\}_m$, $\mathbf{u}_m = (u_{1,m}, \dots, u_{n-1,m}) \in H_1 \times \ldots \times H_{n-1}$ such that

$$\begin{aligned}\int_{\mathbb{R}^d} ||(|\nabla u_{1,m}|^{d-1}, \dots, |\nabla u_{n-1,m}|^{d-1})||_\alpha \, dx &\leq (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \int_{L_m} |\theta_m(x)|^\alpha d\mathcal{H}^1(x) + \frac{1}{m} \\ &= (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \mathbb{M}(T_m) + \frac{1}{m} \\ &\leq (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \mathbb{M}(T) + \frac{C}{m} \\ &= (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \int_L |\theta(x)|^\alpha d\mathcal{H}^1(x) + \frac{C}{m} \,,\end{aligned}$$

where C is a constant depending on T. As m tends to infinity, we can conclude that

$$\inf \mathbb{H} \leq (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha. \tag{3.4.3}$$

Step 3: In this step, we are going to prove the reverse inequality, i.e.

$$\inf \mathbb{H} \geq (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha = (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf \mathbb{M} \,. \tag{3.4.4}$$

Let $\mathbf{u} = (u_1, \dots, u_{n-1}) \in H_1 \times \ldots \times H_{n-1}$. Take an array of non-negative measurable functions $\mathbf{g} = (g_1, \dots, g_{n-1})$. We distinguish two cases:

- if $\alpha \in [0, 1)$, then we assume $\left(\sum_{j=1}^{n-1} |g_j(x)|^{\frac{1}{1-\alpha}}\right)^{1-\alpha} \leq 1$ for every $x \in \mathbb{R}^d$;
- if $\alpha = 1$, then we assume $\max\{g_1(x), \dots, g_{n-1}(x)\} \leq 1$ for any $x \in \mathbb{R}^d$.

Thanks to this choice, we have

$$\int_{\mathbb{R}^d} ||(|\nabla u_1|^{d-1}, |\nabla u_2|^{d-1}, \dots, |\nabla u_{n-1}|^{d-1})||_\alpha \, dx \geq \int_{\mathbb{R}^d} \sum_{i=1}^{n-1} g_i(x) |\nabla u_i|^{d-1} \, dx \,. \tag{3.4.5}$$

By using a standard energy inequality (see [25]-page 64, [4]-A.1.3), we also obtain that

$$\int_{\mathbb{R}^d} \sum_{i=1}^{n-1} g_i(x)|\nabla u_i|^{d-1}\,dx \geq (d-1)^{\frac{d-1}{2}} \int_{\mathbb{R}^d} \sum_{i=1}^{n-1} g_i(x)|Ju_i(x)|\,dx\,, \tag{3.4.6}$$

where Ju_i is the $(d-1)$-dimensional distributional Jacobian of u_i. Then, using Coarea Formula (see, for instance, [2, Section 7.4 and Section 7.5]) under the further minor assumption that u_i is constant outside an open ball containing P_i, P_n, we have

$$\int_{\mathbb{R}^d} \sum_{i=1}^{n-1} g_i(x)|Ju_i(x)|\,dx = \sum_{i=1}^{n-1} \int_{\mathbb{S}^{d-1}} d\sigma(y) \left(\int_{u_i^{-1}(y)\setminus E_i} g_i(x)\,d\mathcal{H}^1(x) \right)\,, \tag{3.4.7}$$

where E_i is the set of all points where u_i is not approximately differentiable, $R^i_y := u_i^{-1}(y) \setminus E_i$ is a $\mathcal{H}^1$-rectifiable set connecting P_i to P_n for $\mathcal{H}^{d-1}$-a.e. $y \in \mathbb{S}^{d-1}$. For a.e $y \in \mathbb{S}^{d-1}$, let $\Sigma_y = \bigcup_{i=1}^{n-1} R^i_y$, we can canonically define the current with coefficients in $\mathbb{Z}^{n-1}$ corresponding to Σ_y, $T_y = [\![\Sigma_y, \tau_y, \theta_y]\!]$. Notice that each component of T_y is the 1-dimensional (classical) integral current associated to R^i_y, $M^i_y = [\![R^i_y, \tau^i_y, 1]\!]$, where τ^i_y is tangent to R^i_y a.e. (see [2, Theorem 3.8]). Moreover $\partial M^i_y = \delta_{P_n} - \delta_{P_i}$, in the sense of currents, for $\mathcal{H}^{d-1}$-a.e. $y \in \mathbb{S}^{d-1}$, $i = 1, \dots, n-1$.

Putting (3.4.5), (3.4.6) and (3.4.7) together, we deduce that

$$\mathbb{H}(\mathbf{u}) \geq (d-1)^{\frac{d-1}{2}} \sum_{i=1}^{n-1} \int_{\mathbb{S}^{d-1}} d\sigma(y) \left(\int_{R^i_y} g_i(x)\,d\mathcal{H}^1(x) \right)\,. \tag{3.4.8}$$

By assumption, the minimizer $T = [\![\Sigma, \tau, \theta]\!]$ of the problem $(\mathbb{M})$ is calibrated by a smooth differential form $\omega = (\omega_1, \dots, \omega_{n-1})$. Then, we choose g_i as follows:

$$g_i(x) = \begin{cases} |\langle \omega(x), \tau_y(x), e_i \rangle| & \text{in case } \exists\, y \in \mathbb{S}^{d-1},\ x \in R^i_y, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.9}$$

We observe that g_i is well-defined since for $\mathcal{H}^{d-1}$-a.e. $y_1, y_2 \in \mathbb{S}^{d-1}$, $y_1 \neq y_2$, $R^i_{y_1} \cap R^i_{y_2} = \emptyset$. One has

$$\begin{aligned} \mathbb{M}(T) = \int_\Sigma \langle \omega; \tau, \theta \rangle\, \mathcal{H}^1(x) &= \int_{\Sigma_y} \langle \omega; \tau_y, \theta_y \rangle\, \mathcal{H}^1(x) \\ &= \sum_{i=1}^{n-1} \int_{R^i_y} \langle \omega; \tau_y, e_i \rangle\, \mathcal{H}^1(x) \\ &\leq \sum_{i=1}^{n-1} \int_{R^i_y} g_i\, \mathcal{H}^1(x)\,. \end{aligned} \tag{3.4.10}$$

This implies that

$$\alpha_{d-1}\mathbb{M}(T) \leq \int_{\mathbb{S}^{d-1}} d\sigma(y) \left(\sum_{i=1}^{n-1} \int_{R^i_y} g_i\, \mathcal{H}^1(x) \right)\,. \tag{3.4.11}$$

From (3.4.8) and (3.4.11), since T is a minimizer we obtain that

$$\begin{aligned} \mathbb{H}(\mathbf{u}) &\geq (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf \mathbb{M} \\ &= (d-1)^{\frac{d-1}{2}} \alpha_{d-1} \inf I_\alpha. \end{aligned}$$

□

Remark 3.4.1. In the proof of Theorem 3.1.1, step 3, we must assume the existence of a calibration ω. Observe that, without this assumption, we still can deduce from (3.4.8) that

$$(d-1)^{\frac{d-1}{2}}\alpha_{d-1}\inf\mathbb{M} = (d-1)^{\frac{d-1}{2}}\alpha_{d-1}\inf I_\alpha \geq \inf\mathbb{H} \geq (d-1)^{\frac{d-1}{2}}\alpha_{d-1}\inf\mathbb{N}, \tag{3.4.12}$$

where $\inf\mathbb{N}$ is the infimum of the problem obtained measuring the mass among 1-dimensional normal currents with coefficients in $\mathbb{R}^{n-1}$ (compare with Remark 3.2.7).

Moreover, in case $\alpha = 1$, (I) turns out to coincide with the Monge-Kantorovich problem. Replicating the proof above and choosing $g_i = 1$ for every $i = 1, \ldots, n-1$ in step 3, then applying the Mean Value Theorem as before (combined with the fact that the minimizer of the problem (I) is obviously the weighted union of segments $\overline{P_iP_n}$) this implies that

$$\inf\mathbb{H} \geq (d-1)^{\frac{d-1}{2}}\alpha_{d-1}\inf I_\alpha = (d-1)^{\frac{d-1}{2}}\alpha_{d-1}\inf\mathbb{M}\,.$$

Another way to see this is to use the results of Brezis-Coron-Lieb [25] separately for each map u_i, $i = 1, \ldots, n-1$, for the energy

$$\mathbb{H}(\mathbf{u}) = \int_{\mathbb{R}^d} (|\nabla u_1|^{d-1} + |\nabla u_2|^{d-1} + \ldots + |\nabla u_{n-1}|^{d-1})\,dx\,,$$

where, again, $\mathbf{u} = (u_1, \ldots, u_{n-1}) \in H_1 \times \ldots \times H_{n-1}$. The investigation of equality cases in (3.4.12), when $0 \leq \alpha < 1$, will be considered in forthcoming works.

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