POSITIVITY OF DIVISORS ON BLOWN-UP PROJECTIVE SPACES, I

OLIVIA DUMITRESCU AND ELISA POSTINGHEL

Abstract. We study l-very ample, ample and semi-ample divisors on the blown-up projective space $\mathbb{P}^n$ in a collection of points in general position. We establish Fujita's conjectures for all ample divisors with the number of points bounded above by $2n$ and for an infinite family of ample divisors with an arbitrary number of points.

Introduction

Ample line bundles are fundamental objects in Algebraic Geometry. From the geometric perspective, an ample line bundle is one such that some positive multiple of the underlying divisor moves in a linear system that is large enough to give a projective embedding. In numerical terms, a divisor is ample if and only if it lies in the interior of the real cone generated by nef divisors (Kleiman). Equivalently, a divisor is ample if it intersects positively every closed integral subscheme (Nakai-Moishezon). In cohomological terms, an ample line bundle is one such that a twist of any coherent sheaf by some power is generated by global sections (Serre). Over the complex numbers, ampleness of line bundles is also equivalent to the existence of a metric with positive curvature (Kodaira).

The very ampleness of divisors on blow-ups of projective spaces and other varieties was studied by several authors, e.g. Beltrametti and Sommese [8], Ballico and Coppens [3], Coppens [14, 15], Harbourne [22]. The notion of l-very ampleness of line bundles on surfaces was introduced by Beltrametti and Sommese [6] and l-very ample line bundles on del Pezzo surfaces were classified by Di Rocco [18]. Other notions of higher order embeddings were introduced in [5] by Beltrametti, Francia and Sommese.

This paper studies ampleness, l-very ampleness and further positivity properties of divisors on blow-ups of projective spaces of higher dimension in an arbitrary number of points in general position. The main tools used are the vanishing theorems for the higher cohomologies of divisors that were proved in [19]. Generalization of these results to the case of points in arbitrary position were studied in [4].

Vanishing theorems for divisors on blown-up spaces were firstly used in order to give a solution to the corresponding interpolation problem, namely to compute the dimension of the linear system of divisors on blown-up projective spaces in points in general position. The case of linear systems whose base locus consisted only of the union of the linear cycles spanned by the points with multiplicity was studied in [9] where, in particular, a formula for the dimension of all linear systems with $s \leq n+2$ points was given. The fact that the strict transform of these linear systems via a

2010 *Mathematics Subject Classification.* Primary: 14C20 Secondary: 14E25, 14C17, 14J70.
Key words and phrases. k-very ample line bundles, nef divisors, Fujita's conjecture.

resolution of the base locus is globally generated is proved in a subsequent paper by the authors [20]. Moreover, in [11] a conjectural formula for the dimension of all linear systems with $n+3$ points, that takes into account the contributions given by the presence in the base locus of the (unique) rational normal curve of degree n through the $n+3$ points and the joins of its secants with the linear subspaces spanned by the points, is given. In [10] linear systems in $\mathbb{P}^3$ that contain the unique quadric through nine points in their fixed locus are studied and, moreover, Nagata-type results are given for planar linear systems.

In this paper we employ the vanishing theorems to prove a a number of positivity properties. A first application of the vanishing theorems is the description of *l-very ample divisors*, in particular *globally generated* divisors and *very ample* divisors contained in Theorem 2.2.

Moreover we establish *Fujita's conjectures* for $\mathbb{P}^n$ blown-up in s points when $s \leq 2n$, Proposition 3.6, and for an infinite family of divisors for arbitrary s, with a bound on the coefficients, Proposition 3.7.

This paper is organized as follows. In Section 1 we introduce the general construction, notation and some preliminary facts. Section 2 contains one of the main results of this article, Theorem 2.2, that concerns l-very ampleness of line bundles on blown-up projective spaces in an arbitrary number of points in general position. In Section 3 we characterize other positivity properties of divisors on blown-up projective spaces at points such as nefness, ampleness, bigness, and we establish Fujita's conjecture.

Acknowledgments. The first author is supported by the NSF grant DMS - 1802082. The second author was partially supported by the EPSRC grant EP/S004130/1 and she is a member of GNSAGA-INDAM.

The authors would like to express their gratitude to the referee for several useful suggestions and to Edoardo Ballico for many interesting discussions.

1. Preliminary results and conjectures

Let K be an algebraically closed field of characteristic zero. Let $\mathcal{S} = \{p_1, \ldots, p_s\}$ be a collection of s distinct points in $\mathbb{P}^n_K$ and let S be the set of indices parametrizing $\mathcal{S}$, with $|S| = s$. Let

$$\mathcal{L} := \mathcal{L}_{n,d}(m_1, \ldots, m_s) \tag{1.1}$$

denote the linear system of degree-d hypersurfaces of $\mathbb{P}^n$ with multiplicity at least m_i at p_i, for $i = 1, \ldots, s$.

1.1. **The blow-up of $\mathbb{P}^n$.** Assume $\mathcal{S}$ consists of points in general position. We denote by X_s the blow-up of $\mathbb{P}^n$ in the points of $\mathcal{S}$ and by E_i the exceptional divisor of p_i, for all i. The Picard group of X_s is spanned by the class of a general hyperplane, H, and the classes of the exceptional divisors E_i, $i = 1, \ldots, s$.

Notation 1.1. *Fix non-negative integers $d, m_1, \ldots, m_s$ and define the following line bundle on X_s:*

$$dH - \sum_{i=1}^{s} m_i E_i. \tag{1.2}$$

In this paper we denote by D a general section of (1.2). Notice that the global sections of $\mathcal{O}_X(D)$ are in bijection with the elements of the linear system $\mathcal{L}$ defined in (1.1).

Remark 1.2. *It is proved in* [13, Proposition 2.3] *and* [12, Lemma 4.2] *that every general divisor D as in Notation 1.1 has multiplicity equal to m_i at the point p_i. It is important to mention here that even if it is often omitted in the framework of classical interpolation problems in $\mathbb{P}^n$, the generality hypothesis of the divisor D is always assumed.*

We denote by $s(d) := s_D(d)$ the number of points in $\mathcal{S}$ at which the multiplicity of D equals d; this number depends on $\mathcal{L}$ or, equivalently, on D. Let us introduce the following integer:

$$b = b(D) := \min\{n - s(d), s - n - 2\}. \tag{1.3}$$

Theorem 1.3 ([9, Theorem 5.3], [19, Theorem 5.12])**.** *Assume that $\mathcal{S} \subset \mathbb{P}^n$ is a set of points in general position. Let D be as in (1.2). Assume that*

$$\begin{aligned} &0 \leq m_i \leq d+1, \ \forall i \in \{1, \ldots, s\}, \\ &m_i + m_j \leq d+1, \ \forall i, j \in \{1, \ldots, s\}, \ i \neq j, \ (\textit{if } s > 1), \\ &\sum_{i=1}^{s} m_i \leq nd + \begin{cases} n & \textit{if } s \leq n+1 \textit{ and } d \geq 2 \\ 1 & \textit{if } s \leq n+1 \textit{ and } d = 1 \\ 1 & \textit{if } s = n+2 \\ b & \textit{if } s \geq n+3 \end{cases} \end{aligned} \tag{1.4}$$

Then $h^1(X_s, D) = 0$.

2. l-very ample divisors on X_s

Definition 2.1. *Let X be a smooth projective variety. For an integer $l \geq 0$, a line bundle $\mathcal{O}_X(D)$ on X is said to be l-very ample, if for every 0-dimensional subscheme $Z \subset X$ of lenght $h^0(Z, \mathcal{O}_Z) = l+1$, the restriction map $H^0(X, \mathcal{O}_X(D)) \to H^0(Z, \mathcal{O}_X(D)|_Z)$ is surjective.*

This notion was first introduced in [6] for surfaces.

We will now recall some of the results obtained in the study of positivity of blown-up surfaces and higher dimensional projective spaces. Di Rocco [18] classified l-very ample line bundles on del Pezzo surfaces, namely for $\mathbb{P}^2$ blown-up at $s \leq 8$ points in general position. For general surfaces, very ample divisors on rational surfaces were considered by Harbourne [22]. De Volder and Laface [17] classified l-very ample divisors, for $l = 0, 1$, on the blow-up of $\mathbb{P}^3$ at s points lying on a certain quartic curve. Ampleness and very ampleness properties of divisors on blow-ups at points of higher dimensional projective spaces in the case of points of multiplicity one were studied by Angelini [1], Ballico [2] and Coppens [15].

Positivity properties for blown-up $\mathbb{P}^n$ in general points were considered by Castravet and Laface. In particular, for small number of points in general position, $s \leq 2n$, the semi-ample and nef cones, that we describe in this paper in Theorem 3.2, were obtained via a different technique (private communication).

We can describe l-very ample line bundles over X_s, the blown-up projective space at s points in general position, of the form (1.2) as follows.

Take $l \geq 0$ and $n \geq 1$. For every $s \geq n+3$ and $d > l+2$ we introduce the following integer:

$$b_l := \begin{cases} \min\{n-1, s-n-2\} - l - 1 & \text{if } m_1 = d-l-1, m_i = 1, i \geq 2, \\ \min\{n, s-n-2\} - l - 1 & \text{otherwise,} \end{cases} \tag{2.1}$$

while for $s \leq n+2$ define $b_l := -l-1$. We remark that for $n = 1$, then $b_l \in \{-l, -l-1\}$.

Theorem 2.2 (l-very ample line bundles)**.** *Assume that $\mathcal{S} \subset \mathbb{P}^n$ is a collection of points in general position. Let l be a non-negative integer. Assume that either $s \leq 2n$ or $s \geq 2n+1$ and d is large enough, namely*

$$d > l+2, \quad \sum_{i=1}^{s} m_i - nd \leq b_l, \tag{2.2}$$

where b_l is defined as in (2.1). If $n = 1$, then D is l-very ample. If $n \geq 2$, the divisor D of the form (1.2) is l-very ample if and only if

$$\begin{aligned} &l \leq m_i, \ \forall i \in \{1, \ldots, s\}, \\ &l \leq d - m_i - m_j, \ \forall i, j \in \{1, \ldots, s\}, \ i \neq j. \end{aligned} \tag{2.3}$$

Remark 2.3. *In the notation of Theorem 2.2, when $n \geq 3$ and $n+3 \leq s \leq 2n$, then (2.3) implies (2.2). Indeed, the inequality $m_i + m_j - d \leq -l$ implies that*

$$\sum_{i=1}^{s} m_i - nd \leq -nl \leq \min\{n-1, s-n-2\} - l - 1.$$

The last inequality holds since $(n-1)l + \min\{n-1, s-n-2\} \geq 1$ holds.

Remark 2.4. *When $l = 0$ (resp. $l = 1$), l-very ampleness corresponds to* global generation*, or* spannedness *(resp. very ampleness).*

Notice that conditions (2.3) are equivalent to saying that the divisor D and the class of a line on E_i or, respectively, the strict transform on X of the line of $\mathbb{P}^n$ spanned by the points p_i, p_j, intersect at least l times.

Corollary 2.5 (Globally generated line bundles)**.** *In the same notation of Theorem 2.2, for $n \geq 2$, assume that either $s \leq 2n$ or $s \geq 2n+1$ and that*

$$d > 2, \quad \sum_{i=1}^{s} m_i - nd \leq b_0.$$

Then D is globally generated if and only if

$$\begin{aligned} &0 \leq m_i, \ \forall i \in \{1, \ldots, s\}, \\ &0 \leq d - m_i - m_j, \ \forall i, j \in \{1, \ldots, s\}, \ i \neq j. \end{aligned} \tag{2.4}$$

Corollary 2.6 (Very ample line bundles)**.** *In the same notation of Theorem 2.2, let $n \geq 2$ and assume that either $s \leq 2n$ or $s \geq 2n+1$ and that*

$$d > 3 \quad \sum_{i=1}^{s} m_i - nd \leq b_1.$$

Then D is very ample if and only if

$$\begin{aligned} &1 \leq m_i, \ \forall i \in \{1, \ldots, s\}, \\ &1 \leq d - m_i - m_j, \ \forall i, j \in \{1, \ldots, s\}, \ i \neq j. \end{aligned} \tag{2.5}$$

We now present three examples of not globally generated divisors for which the the bound (2.2) of Theorem 2.2, with $l = 0$, is not satisfied.

Example 2.7. *Consider the following divisor on the blow-up of $\mathbb{P}^3$ in 7 points:*

$$D := 2H - E_1 - \ldots - E_7.$$

It does not satisfy the hypotheses of Theorem 2.2. In fact, the linear system associated to D is generated by the strict transforms of three linearly independent quadrics that intersect in eight points. Therefore the divisor D is not globally generated.

Example 2.8. *Consider the following divisor on the blow-up of $\mathbb{P}^3$ in 8 points:*

$$D := 2H - E_1 - \ldots - E_8.$$

It is generated by two linearly independent quadrics that intersect along a quartic curve. Therefore D is not globally generated.

Example 2.9. *Let us consider the anticanonical divisor of the blown-up $\mathbb{P}^2$ in eight points in general position*

$$D := 3H - E_1 - \ldots - E_8.$$

It does not satisfy (2.2) for $l = 0$, since $b_l = 1$. Sections of D correspond to planar cubics passing through eight simple points. All such cubics meet in a ninth point, therefore D is not a globally generated divisor. However, D is nef.

2.1. **Some technical lemmas.** In this section we prove a series of technical lemmas that will be useful in the proofs of the main theorem, Theorem 2.2. These will also justify the integer b_l appearing in (2.1) and show why we cannot obtain a better bound than (2.1). The next three lemmas consider the case $n \geq 2$.

Lemma 2.10. *Let D be the divisor defined in (1.2) and assume that (2.3) holds. Then $s_D(d) = 0$ unless $l = 0$, $s = 1$ and $m_1 = d$, in which case $s_D(d) = 1$.*

Proof. Assume $s_D(d) \neq 0$; in particular $m_1 = d$. Equations (2.3) imply $d \geq m_1 + m_i + l = d + m_i + l$, that gives $m_i = 0$ for all $i \geq 1$ and $l = 0$. In this case $D = dH - dE_1$ and $s_D(d) = 1$. □

Let F be the divisor obtained by subtracting a sum of $l + 1$ exceptional divisors E_i, with repetitions allowed, from D:

$$F := D - \sum_{i=1}^{s} \epsilon_i E_i, \quad \epsilon := \sum_{i=1}^{s} \epsilon_i = l + 1, \tag{2.6}$$

where the ϵ_i's are positive integers.

Lemma 2.11. *Let D and F be divisors defined respectively as in (1.2) and (2.6) and assume that D satisfies (2.2) and (2.3), then F satisfies (1.4).*

Proof. We first prove the following claim: $s_F(d) = 0$ unless $m_2 = \cdots = m_s = 1$, $m_1 = d - l - 1$ and $\epsilon = \epsilon_1 = l + 1$, in which case $s_F(d) = 1$.

In order to prove the claim, assume first that $\epsilon_i \leq l$ for all i (in particular $l \geq 1$). Then (2.3) implies $m_i \geq 1$ and $d - m_i \geq m_j + l$ for every i, j. Therefore $s_F(d) = 0$. Otherwise, after permuting the indices if necessary, assume that $\epsilon_1 = l + 1$. Then (2.3) gives $d - (m_1 + l + 1) \geq m_i - 1 \geq 0$. Notice that $s_F(d) = 0$, unless $m_1 + l + 1 = d$. In the latter case we have $m_i = 1$, for all $i \geq 2$, therefore we obtain

$$D = dH - (d - l - 1)E_1 - E_2 - \ldots - E_s$$

and

$$F = dH - dE_1 - E_2 - \ldots - E_s.$$

Finally, using the claim we compute the integer $b(F)$ defined in (1.3) for the divisor F: we obtain $b(F) = \min\{n, s-n-2\}$, unless the divisor D has $m_1 = d-l-1$ and $m_2 = 1$ and in this case it is $b(F) = \min\{n-1, s-n-2\}$. It is now a straightforward computation to prove that the statement holds. □

Let us now introduce the divisors G_j, $j = 0, 1$, where $\bar{s}_j := \min\{s, n-j\}$, obtained by subtracting $l+1$ times from D the strict transform on the blow-up of a hyperplane of $\mathbb{P}^n$ containing $p_1, \ldots, p_{\bar{s}_j}$:

$$G_j := (d-l-1)H - \sum_{i=1}^{\bar{s}_j}(m_i - l - 1)E_i - \sum_{i=\bar{s}_j+1}^{s} m_i E_i. \tag{2.7}$$

Lemma 2.12. *Let D and G_j be divisors defined respectively as in (1.2) and (2.7) and assume that D satisfies (2.2) and (2.3). Then G_j satisfies (1.4), for $j = 0, 1$.*

Proof. If $s_D(d) > 0$, then by Lemma 2.10 we have that $D = d(H - E_1)$. Therefore $\bar{s}_0 = \bar{s}_1 = s = 1$, $G_j = (d-l-1)(H - E_i)$, $s_{G_j}(d-l-1) = 1$ and G_j obviously satisfies (1.4), for $j = 0, 1$. Therefore we can assume that $s_D(d) = 0$.

If $m_i < d-l-1$ for all i's, then obviously $s_{G_j}(d-l-1) = 0$. Let us write

$$G_j = d'H - \sum_{i=1}^{d} m'_i E_i := (d-l-1)H - \sum_{i=1}^{\bar{s}_j}(m_i - l - 1)E_i - \sum_{i=\bar{s}_j+1}^{s} m_i E_i.$$

First of all take $j = 0$ and set $\bar{s} := \bar{s}_0$. We now verify that G_0 satisfies (1.4). Indeed, when $\bar{s} = s < n$ we have

$$\sum_{i=1}^{\bar{s}} m'_i - nd' = \sum_{i=1}^{\bar{s}}(m_i - l - 1) - n(d-l-1) \le 0,$$

because $m_i \le d$. Otherwise, if $\bar{s} = n \le s$, we compute

$$\sum_{i=1}^{s} m'_i - nd' = \sum_{i=1}^{\bar{s}}(m_i - l - 1) + \sum_{i=\bar{s}+1}^{s} m_i - n(d-l-1) = \sum_{i=1}^{s} m_i - nd.$$

The above number is bounded above by 0 whenever $s \le 2n$, and by $\min\{n, s-n-2\} - l - 1$ whenever $s \ge 2n+1$, by the hypotheses. Moreover, in all cases one has $m'_i + m'_j - d' \le 1$, for all $i \ne j$.

Now, take $j = 1$ and set $\bar{s} := \bar{s}_1$. We verify that G_1 satisfies (1.4) with a similar computation. Indeed, when $\bar{s} = s < n-1$, then it is the same computation as before. Whereas if $\bar{s} = n-1 \le s$, we have

$$\sum_{i=1}^{s} m'_i - nd' = \sum_{i=1}^{\bar{s}}(m_i - l - 1) + \sum_{i=\bar{s}+1}^{s} m_i - n(d-l-1) = \sum_{i=1}^{s} m_i - nd + l + 1.$$

The number on the right hand side of the above expression is bounded above by 0 if $s \le 2n$ and by $\min\{n, s-n-2\}$ if $s \ge 2n+1$.

Finally, assume that $m_i = d-l-1$ for some i and assume, without loss of generality, that $i = 1$. In this case we have $m_i = 1$ for all $i > 0$, see the proof of Lemma 2.11, and $s_{G_j}(d-l-1) = 1$ provided that $d > l+2$. If $d = l+2$ then $G_j = H - \sum_{i=\bar{s}+1}^{s} E_i$. In both cases is easy to verify that G_j satisfies (1.4). □

2.2. **Proof of Theorem 2.2.** We will discuss the first induction case, $n = 1$, when $b_l \in \{-l, -l-1\}$ and we claim that if equation (2.2) holds, then D is *l-very ample.* Notice that on $\mathbb{P}^1$, we have

$$D = (d - \sum_{i=1}^{s} m_i)H.$$

In this case (2.2) becomes

$$d - \sum_{i=1}^{s} m_i \geq -b_l \geq l.$$

For every 0-dimensional subscheme $Z \subset \mathbb{P}^1$, of length $h^0(Z, \mathcal{O}_Z) = l+1$, we have

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D) \otimes \mathcal{I}_Z) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d - \sum_{i=1}^{s} m_i - l - 1)) = 0, \tag{2.8}$$

because condition (2.2) implies $d - \sum_{i=1}^{s} m_i - l - 1 \geq -1$. We remark that this proof holds for $b_l \in \{-l, -l-1\}$.

If $n \geq 2$, Theorem 2.2 states that a divisor D on X satisfying condition (2.2) is *l-very ample* if and only if (2.3) holds. The proof of this relies on the following vanishing theorem, that has its own intrinsic interest.

Let $\mathcal{I}_{\{q^{l+1}\}}$ denote the ideal sheaf of the fat point of multiplicity $l+1$ supported at $q \in \mathbb{P}^n$, where by fat point we mean the subscheme of $\mathbb{P}^n$ defined by the $(l+1)$th power of the ideal of q.

Theorem 2.13. *In the same notation as Theorem 2.2, fix integers $d, m_1, \ldots, m_s, l \geq 0$, $s \geq 1$. Assume that either $s \leq 2n$ or that $s \geq 2n+1$ and that (2.2) is satisfied. Moreover, assume that*

$$\begin{aligned} & l \leq m_i, \ \forall i \in \{1, \ldots, s\}, \\ & l \leq d - m_i - m_j, \ \forall i, j \in \{1, \ldots, s\}, \ i \neq j. \end{aligned} \tag{2.9}$$

Then $h^1(D \otimes \mathcal{I}_{\{q^{l+1}\}}) = 0$ for every $q \in X_s$.

Proof. If $n = 1$ the claim follows from (2.8). For the rest of the proof, we assume $n \geq 2$.

Case (1). Assume first of all that $q \in E_i$, for some $i \in \{1, \ldots, s\}$. We claim that

$$h^1(D \otimes \mathcal{I}_{\{q^{l+1}\}}) \leq h^1(D - (l+1)E_i). \tag{2.10}$$

Hence we conclude because the latter vanishes, by Theorem 1.3 and Lemma 2.11 with $F = D - (l+1)E_i$. We now prove that (2.10) holds. Let π be the blow-up of X_s at $q \in E_i$ with exceptional divisor E_q. By the projection formula we have that $\pi_*(\pi^*(D) - (l+1)E_q)) = D \otimes \mathcal{I}_{\{q^{l+1}\}}$. Since $R^i\pi_*(\pi^*(D) - (l+1)E_q) = 0$, for $i > 0$, invoking the Leray spectral sequence we obtain $H^i(D \otimes \mathcal{I}_{\{q^{l+1}\}}) \cong H^i(\pi^*(D) - (l+1)E_q)$, see [23, III.8]. For $l = 0$, consider the exact sequence of sheaves

$$0 \to \pi^*(D) - \pi^*(E_i) \to \pi^*(D) - E_q \to (\pi^*(D) - E_q)|_{\pi^*(E_i) - E_q} \to 0. \tag{2.11}$$

Notice that $\pi^*(E_i) - E_q$ is the blow-up of $E_i \cong \mathbb{P}^{n-1}$ at the point q: denote by h, e_q the generators of its Picard group. We have $(\pi^*(D) - E_q)|_{\pi^*(E_i) - E_q} \cong m_i h - e_q$, in particular it has vanishing first cohomology group. Hence, looking at the long exact sequence in cohomologies associated with (2.11), one gets that the map

$$H^1(\pi^*(D)-\pi^*(E_i))\to H^1(\pi^*(D)-E_q)$$

is surjective, therefore $h^1(\pi^*(D)-\pi^*(E_i))\geq h^1(\pi^*(D)-E_q)$. Finally, by the projection formula and using the Leray spectral sequence, as above, one has $H^i(\pi^*(D)-\pi^*(E_i))=H^i(D-E_i)$, so we conclude. For $l\geq 1$, one can iterate $l+1$ times the above argument, with $\pi^*(D)-E_q$ replaced by $\pi^*(D)-lE_q$ in (2.11); we conclude noticing that the restricted linear series is m_ih-le_q.

Case (2). Assume $q\in X_s\setminus\{E_1,\dots,E_s\}$. Hence q is the preimage of a point $q'\in\mathbb{P}^n\setminus\{p_1,\dots,p_s\}$.

We will prove the statement by induction on n. The case $n=1$ is obvious. Indeed, any such $D\otimes\mathcal{I}_{\{q^{l+1}\}}$ corresponds to a linear series on the projective line given by three points whose sum of the multiplicities is bounded above as follows $m_1+m_2+(l+1)\leq d+1$. Hence the first cohomology group vanishes. For $n\geq 2$, we will assume that the statement holds for $n-1$ and we prove it for n.

Recall that a set of points $\mathcal{S}$ of $\mathbb{P}^n$ is said to be in *linearly general position* if for each integer $r\leq n+1$ we have $\sharp(S\cap L)\leq r+1$, for all r-dimensional linear subspaces L in $\mathbb{P}^n$.

Case (2.a). Assume first that the points in $\mathcal{S}\cup\{q'\}$ are not in linearly general position in $\mathbb{P}^n$. If $s\geq n$, q' lies on a hyperplane H of $\mathbb{P}^n$ spanned by n points of $\mathcal{S}$. Reordering the points if necessary, assume that $q'\in H:=\langle p_1,\dots,p_n\rangle$. If $s<n$, let H be any hyperplane containing $\mathcal{S}\cup\{q'\}$. Let $\bar{H}$ denote the strict transform of H on X_s. Notice that $\bar{H}$ is isomorphic to the space $\mathbb{P}^{n-1}$ blown-up at $\bar{s}:=\min\{s,n\}$ distinct points in general position, so that we can write that $\bar{H}\cong X^{n-1}_{\bar{s}}$. Its Picard group is generated by $h:=H|_{\bar{H}}$, $e_i:=E_i|_{\bar{H}}$. The divisor class is $\bar{H}=H-\sum_{i=1}^{\bar{s}}E_i$. The restriction to $\bar{H}$ yields the short exact sequence

$$0\to D-\bar{H}\to D\to D|_{\bar{H}}\to 0.$$

Since $\bar{H}$ is a closed subvariety of X_s containing the point q, the following is a short exact sequence of sheaves

$$0\to(D-\bar{H})\otimes\mathcal{I}_{\{q^l\}}\to D\otimes\mathcal{I}_{\{q^{l+1}\}}\to(D\otimes\mathcal{I}_{\{q^{l+1}\}})|_{\bar{H}}\to 0,\tag{2.12}$$

that is commonly referred to as the *Castelnuovo sequence*. We iterate this restriction procedure $l+1$ times.

For $0\leq\lambda\leq l$, set $D_\lambda:=(D-\lambda\bar{H})\otimes\mathcal{I}_{\{q^{l+1-\lambda}\}}$. The restricted series in the $(\lambda+1)$st exact sequence, $D_\lambda|_{\bar{H}}$, is the complete linear series on $X^{n-1}_{\bar{s}}$ given by

$$\left((d-\lambda)h-\sum_{i=1}^{\bar{s}}(m_i-\lambda)e_i\right)\otimes\mathcal{I}_{\{q^{l+1-\lambda}\}|_{\bar{H}}}.\tag{2.13}$$

We leave it to the reader to verify that it satisfies the hypotheses of the theorem, for every $0\leq\lambda\leq l$. Hence we conclude, by induction on n, that the first cohomology group vanishes.

The kernel of the $(\lambda+1)st$ sequence is $D-(\lambda+1)\bar{H}\otimes\mathcal{I}_{\{q^{l-\lambda}\}}$. In particular the kernel of the last sequence is the line bundle associated with the following divisor:

$$d'H-\sum_{i=1}^{d}m'_iE_i := (d-l-1)H-\sum_{i=1}^{\bar{s}}(m_i-l-1)E_i-\sum_{i=\bar{s}+1}^{s}m_iE_i, \tag{2.14}$$

that equals the divisor G_0 introduced in (2.7) for $i=0$. It satisfies the condition of Theorem 1.3 by Lemma 2.12, hence we conclude that $h^1(G_0)=0$. Putting everything together we obtain the series of inequalities $h^1(D)=h_1(D_0)\leq h^1(D_1)\leq \cdots \leq h^1(D_\lambda)\leq h^1(D_{\lambda+1})\leq\cdots\leq h^1(D_l)=h^1(G_0)=0$, that allows us to conclude in this case.

Case (2.b). Lastly, assume that $\mathcal{S}\cup\{q'\}$ is in linearly general position in $\mathbb{P}^n$. If $s\geq n-1$, let H denote the hyperplane $\langle p_1,\ldots,p_{n-1},q'\rangle$. If $s<n-1$, let H be any hyperplane containing $\mathcal{S}\cup\{q'\}$. In both cases such H exists by the assumption that points of $\mathcal{S}$ are in general position. As in the previous case, let $\bar{H}$ denote the strict transform of H on X_s. It is isomorphic to the space $\mathbb{P}^{n-1}$ blown-up at $\bar{s}:=\min\{s,n-1\}$ distinct points in general position, that we may denote by $\bar{H}\cong X^{n-1}_{\bar{s}}$.

We iterate the same restriction procedure shown in (2.12) $l+1$ times as in case (2.a). As before the restriction of the $(\lambda+1)$st exact sequence, that is of the form (2.13) with $\bar{s}$ differently defined here, verifies the hypotheses of the theorem, so it has vanishing first cohomology group by induction on n.

Furthermore, the kernel of the last sequence, that is in the shape (2.14), with $\bar{s},d',m'_i$ as defined here, is the divisor G_1 of (2.7) with $i=1$. It satisfies the condition of Theorem 1.3 by Lemma 2.12. Indeed, when $\bar{s}=s<n-1$, it is the same computation as before. While, if $\bar{s}=n-1\leq s$, we have

$$\sum_{i=1}^{s}m'_i-nd'=\sum_{i=1}^{\bar{s}}(m_i-l-1)+\sum_{i=\bar{s}+1}^{s}m_i-n(d-l-1)=\sum_{i=1}^{s}m_i-nd+l+1.$$

The number on the right hand side of the above expression is bounded above by 0 if $s\leq 2n$ and by b if $s\geq 2n+1$. This concludes the proof.

□

Theorem 2.13 shows that, under the assumptions, if we tensor the sheaf $\mathcal{O}_X(D)$ by the ideal sheaf of a fat point $\{q^{l+1}\}$ with support anywhere within X, we have vanishing of H^1. In the next two corollary, we generalise the statement to every 0-dimensional subscheme of X given as the union of a collection of fat points. In particular in Corollary 2.14 we cover the case where such points have support away from the exceptional divisors E_i's. Corollary 2.15 is the further extension of the above to the case where the fat points can lie both on or off the E_i's. To prove the corollaries we generalise the arguments used in case (2) and (1) respectively of the proof of the theorem.

Corollary 2.14. *Assume that D satisfies the same hypotheses as in Theorem 2.13. Then $h^1(D\otimes\bigotimes_j\mathcal{I}_{\{q_j^{\mu_j}\}})=0$, for every finite collection of points $\{q_j\}_j\subset X_s\setminus\bigcup_{i=1}^{s}E_i$ and integers $\mu_j\geq 0$ with $\sum_j\mu_j=l+1$.*

Proof. The statement is proved by iterating the procedure of Case (2) of the proof of Theorem 2.13. Let us consider an arbitrary collection of points, $\mathcal{S}_0=\{q_1,\ldots,q_{s_0}\}$, with $s_0 := |\mathcal{S}_0|$, all of them lying off the exceptional divisors of X_s. Consider the following union of fat points

$$Z_0 = q_1^{\mu_1} \cup \cdots \cup q_{s_0}^{\mu_{s_0}} \subset X_s \setminus \bigcup_{i=1} E_i.$$

Further, assume that $\sum_{j=1}^{s_0} \mu_i = l+1$. For every j, let $q'_j \in \mathbb{P}^n$ be the point whose pull-back is q_j.

The proof is by induction on n and the base step $n=1$ is trivial. We assume the statement holds for $n-1$ and we prove it for n. For every index $j=1,\ldots,s_0$, starting from $j=1$ and then increasing, we choose a suitable hyperplane of $\mathbb{P}^n$, $H(j)$, with $q'_j \in H(j)$ and, as in the proof of Theorem 2.13, we consider restriction sequences to that hyperplane. We will distinguish the two following cases, according to whether q'_j is or is not in linearly general position with respect to $\mathcal{S} = \{p_1,\ldots,p_s\}$. Now set $j=1$.

Case (a). Assume first that the points in $\mathcal{S} \cup \{q'_1\}$ are not in linearly general position in $\mathbb{P}^n$. Let $H = H(1)$ be a hyperplane containing q'_1 and $\bar{s} := \min\{s, n\}$ distinct points of $\mathcal{S}$. Notice that $\{q'_1\} \subseteq \mathcal{S}_0 \cap H$ might be a proper subset, namely there could be some integer $j > 1$ such that $q'_j \in \mathcal{S}_0 \cap H$ too. Let $\mu_H := \max\{\mu_j : q'_j \in \mathcal{S}_0 \cap H\}$.

Case (b). Assume that the points in $\mathcal{S} \cup \{q'_1\}$ are in linearly general position in $\mathbb{P}^n$. Let $H = H(1)$ be a hyperplane containing q'_1 and $\bar{s} := \min\{s, n-1\}$ distinct points of $\mathcal{S}$. Let $\mu_H := \max\{\mu_j : q'_j \in \mathcal{S}_0 \cap H\}$.

We restrict $D \otimes \bigotimes_{j=1}^{s_0} \mathcal{I}_{\{q_j^{\mu_j}\}}$ to H iteratively μ_H times. In either cases (a) and (b), for each sequence, the restricted series satisfies (2.9) and therefore it has vanishing first cohomology group by induction on n, as we are assuming that the statement holds for $n-1$. Moreover the kernel obtained in each exact sequence still satisfies the assumptions (2.9).

Next, we consider the last kernel obtained with this procedure and the point q'_j, with $j \geq 2$ minimal with respect to the property that $q'_j \notin H(1)$. We proceed as above, point by point, using case (a) or (b) according to the position of q'_j with respect to $\mathcal{S}$, until we exhaust the set $\mathcal{S}_0$. The last kernel is:

$$G_2 = (d-\delta)H - \sum_{i=1}^{s} (m_i - c_i) E_i,$$

where δ is the total number of restrictions to hyperplanes performed and c_i is the number of restrictions to hyperplanes containing p_i. Notice that the following holds: $c_i \leq \delta \leq \sum_{j=1}^{s_0} \mu_j = l+1$. These inequalities imply that G_2 satisfies all the hypotheses of Theorem 1.3; this can be shown using computations similar to those in the proof of Lemma 2.12.

□

Corollary 2.15. *Assume that D satisfies the same hypotheses as Theorem 2.13. Then $h^1(D \otimes \bigotimes_i \mathcal{I}_{\{q_j^{\mu_j}\}}) = 0$, for every finite collection of points $\{q_j\}_i \subset X_s$ and integers $\mu_j \geq 0$ with $\sum_j \mu_j = l+1$.*

Proof. As in the proof of Corollary 2.14, let $\mathcal{S}_0 = \{q_{01}, \ldots, q_{0s_0}\}$ be an arbitrary collection of points away from the exceptional divisors, and consider the union of fat points

$$Z_0 = q_{01}^{\mu_{01}} \cup \cdots \cup q_{0s_0}^{\mu_{0s_0}} \subset X \setminus \bigcup_i E_i,$$

with sum of multiplicities $\mu_0 := \sum_{j=1}^{s_0} \mu_{0j}$. Similarly, for each $i = 1, \dots, s$, let us consider a union of infinitely near points (i.e. on E_i),

$$Z_i = q_{i1}^{\mu_{i1}} \cup \cdots \cup q_{is_i}^{\mu_{is_i}} \subset E_i,$$

whose sum of multiplicities is $\mu_i = \sum_{j=1}^{s_1} \mu_{ij} \geq 0$. Finally, assume that $\mu_0 + \sum_{i=1}^{s} \mu_i = l+1$.

Iterating the idea of Case (1) of the proof of Theorem 2.13, we can conclude that

$$h^1\left(\left(D \otimes \bigotimes_{i=1}^{s} \mathcal{I}_{Z_i}\right) \otimes \mathcal{I}_{Z_0}\right) = h^1\left(F \otimes \bigotimes_{j=1}^{s_0} \mathcal{I}_{\{q_{0j}^{\mu_{0j}}\}}\right) = 0,$$

where $F = D - \sum_{i=1}^{s} \mu_i E_i$. The latter vanishes by Corollary 2.14; we leave it to the reader to verify that F satisfies the hypotheses. □

Let $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_s)$ be the linear system of the form (1.1).

Corollary 2.16. *Assume that $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \dots, m_s)$ satisfies the conditions of Theorem 2.13. Then the linear sub-system of elements of $\mathcal{L}$ that vanish with multiplicity $l+1$ at an arbitrary extra point, $\mathcal{L}_{n,d}(m_1, \dots, m_s, l+1)$, is non-special.*

Proof. As in the proof of Theorem 2.13, the projection formula together with the Leray spectral sequence, implies that, for all $i \geq 0$, $H^i(X_s, D \otimes \mathcal{I}_{\{q^{l+1}\}}) \cong H^i(\mathbb{P}^n, \mathcal{L}_{n,d}(m_1, \dots, m_s, l+1))$. Therefore $\mathcal{L}_{n,d}(m_1, \dots, m_s, l+1)$ has the expected dimension. □

Before we proceed with the proof of Theorem 2.2, we need the following lemma.

Lemma 2.17. *Let X be a complex projective smooth variety and $\mathcal{O}_X(D)$ a line bundle. Let $Z_1 \subseteq Z_2$ be an inclusion of 0-dimensional schemes. Then $h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1}) \leq h^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_2})$.*

Proof. The proof of the statement is well known to experts. We include it here for the sake of completeness.

If $Z_1 = Z_2$, the statement is trivially holds. Assume $Z_1 \subsetneq Z_2$ and consider the short exact sequence:

$$0 \to \mathcal{O}_X(D) \otimes \mathcal{I}_{Z_2} \xrightarrow{\phi} \mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1} \to \operatorname{coker}(\phi) \to 0.$$

Consider the corresponding long exact sequences in cohomology. Since the coker(ϕ) has vanishing first cohomology because it has 0-dimensional support, the map $H^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_2}) \to H^1(\mathcal{O}_X(D) \otimes \mathcal{I}_{Z_1})$ is surjective and this concludes the proof. □

Lemma 2.17 will allow to reduce the proof of l-very ampleness for divisors D to the computation of vanishing theorems of the first cohomology group of $\mathcal{O}_X(D)$ tensored by the ideal sheaf of a union of fat points, Z_2, whose multiplicities sum up to $l+1$. In fact every 0-dimensional scheme Z_1, with $h^0(Z_1, \mathcal{O}_{Z_1}) = l+1$, is contained in a union of fat points.

Proof of Theorem 2.2. We first prove that (2.3) are sufficient conditions for D to be l-very ample. For every 0-dimensional scheme $Z \subset X_s$ of lenght $l+1$, consider the exact sequence of sheaves

$$0 \to \mathcal{O}_{X_s}(D) \otimes \mathcal{I}_Z \to \mathcal{O}_{X_s}(D) \to \mathcal{O}_{X_s}(D)|_Z \to 0. \tag{2.15}$$

We will prove that $h^1(X_s, \mathcal{O}_{X_s}(D) \otimes \mathcal{I}_Z) = 0$. This will imply the surjectivity of the map $H^0(X_s, \mathcal{O}_{X_s}(D)) \to H^0(Z, \mathcal{O}_{X_s}(D)|_Z)$, by taking the long exact sequence in cohomology associated with (2.15).

Let

$$Z_0 \cup Z_1 \cup \cdots \cup Z_s$$

be the decomposition of Z in $s+1$ components (some of which might be the empty set) with the property that $\mathrm{Supp}(Z_0) \subset X_s \setminus \{E_1, \ldots, E_s\}$ and $\mathrm{Supp}(Z_i) \subset E_i$, for $i = 1, \ldots, s$. For every i, let $\mathrm{Supp}(Z_i) = \{q_{i1}, \ldots, q_{is_i}\}$, where q_{ij} are distint points. Let

$$Z_i = Z_{i1} \cup \cdots \cup Z_{is_i}$$

be the decomposition of Z_i in irreducicle components where we have $\mathrm{Supp}(Z_{ij}) = q_{ij}$. Set μ_{ij} to be the lenght of Z_{ij}, and $\mu_i := \sum_{j=1}^{s_i} \mu_{ij}$. We have $\mu_{ij} \geq 0$ and $\sum_{i=0}^{s} \mu_i = l+1$. We shall prove that

$$h^1\left(\mathcal{O}_X(D) \otimes \bigotimes_{i=0}^{s} \bigotimes_{j=1}^{s_i} \mathcal{I}_{Z_{ij}}\right) = 0.$$

Assume first of all that Z has support in a single point $q = q_{i1}$, for some i. As we have the following inclusion of schemes $Z \subset \{q^{l+1}\}$, the statement follows by Lemma 2.17 and Theorem 2.13.

Assume now that Z is supported in several points. To conclude, it is enough to prove that

$$h^i(X_s, D \otimes \mathcal{I}_Z) \leq h^i\left(X_s, \left(D - \sum_{i=1}^{s} \mu_i E_i\right) \otimes \bigotimes_{j=1}^{s_0} \mathcal{I}_{\{q_{0j}\}^{\mu_{0j}}}\right) = 0.$$

The first inequality follows from (2.10) and by Lemma 2.17. The equality follows from Corollary 2.15; we leave it to the reader to verify that the hypotheses on the coefficients are indeed satisfied.

We now prove that (2.3) are necessary conditions for D to be l-very ample, by induction on l.

Let us first assume $l = 0$, namely that D is base point free. If $m_i < 0$ then $m_i E_i$ would be contained in the base locus of D. If $m_i + m_j > d$ for some $i \neq j$, then the strict transform of the line $\langle p_i, p_j \rangle \subset \mathbb{P}^n$ would be contained in the base locus of D. In both cases we would obtain a contradiction.

Assume that $l = 1$, namely that D is very ample. If $m_i \leq 0$ (or $0 \leq d - m_i - m_j$ for some $i \neq j$), then E_i (resp. the strict transform of the line through p_i and p_j) would be contracted by D, a contradiction.

More generally, assume that D is l-very ample and $l \geq 2$. Then conditions (2.3) are satisfied. Indeed, if $m_i \leq l-1$ for some i, we can find a 0-dimensional scheme, Z, of lenght $l+1$ such that $h^1(D \otimes \mathcal{I}_Z) > 0$. Let $Z \subset E_i$ be an *l-jet scheme* centred at $q \in E_i$ (see [27]). Consider the restriction $D \otimes \mathcal{I}_Z|_{E_i} \cong m_1 h \otimes \mathcal{I}_Z$, where h is the hyperplane class of $E_i \cong \mathbb{P}^{n-1}$. We have $h^1(E_i, D \otimes \mathcal{I}_Z|_{E_i}) \geq 1$, hence $h^1(X_s, D\otimes\mathcal{I}_Z)\geq 1$. To see this, let $x_1\dots,x_{n-1}$ be affine coordinates for an affine chart $U\subset E_i$ and let Z be the jet-scheme with support $q=(0,\dots,0)\in U$ given by the tangent directions up to order l along x_1. The space of global sections of $D\otimes\mathcal{I}_Z|_{E_i}$ is isomorphic to the set of degree-m_i polynomials $f(x_1,\dots,x_{n-1})$, whose partial derivatives $\partial^\lambda f/\partial x_1^\lambda$ vanish at q, for $0\leq\lambda\leq l$. On the other hand, $H^1(E_i, D\otimes\mathcal{I}_Z|_{E_i})$ is the "space of linear dependencies" among the $l+1$ conditions imposed by the vanishing of the partial derivatives to the coefficients of f. Since $m_i\leq l-1$ then f is a polynomial of degree at most $l-1$, therefore $\partial^l f/\partial x_1^l\equiv 0$ for every such a polynomial, and we conclude.

Similarly, if $d-m_i-m_j\leq l-1$ for some $i,j,i\neq j$, then one finds a jet-scheme Z contained in the pull-back of the line through p_i and p_j, L, for which $h^1(X_s, D\otimes\mathcal{I}_Z)\geq 1$. Indeed, if Z is such a scheme, then the restriction is $D\otimes\mathcal{I}_Z|_L\cong(d-m_i-m_j)h\otimes\mathcal{I}_Z|_L$, where in this case h is the class of a point in L, and $Z|_L$ is a fat point of multiplicity l in L. One concludes by the Riemann-Roch Theorem that $h^1(L, D\otimes\mathcal{I}_Z|_L)\geq 1$ because $\chi(L, D\otimes\mathcal{I}_Z|_L)=(d-m_i-m_j)-l\leq -1$ and $h^0(L, D\otimes\mathcal{I}_Z|_L)=0$.

□

2.3. **l-jet ampleness.** From now on we will consider the case $n\geq 2$.

In Definition 2.1 we recalled a notion of higher order embedding, the j-very ampleness. In [5], Beltrametti, Francia and Sommese introduced other notions of higher order embeddings with the aim of studying the *adjoint bundle* on surfaces.

Definition 2.18. *In the same notation as Definition 2.1, if for every fat point $Z=\{q^{l+1}\}$, $q\in X$, the natural restriction map to Z, $H^0(X,\mathcal{O}_X(D))\to H^0(Z,\mathcal{O}_X(D)|_Z)$, is surjective, then D is said to be* l-jet spanned.

Moreover, if for every collection of fat points $Z=\{q_1^{\mu_1},\dots,q_\sigma^{\mu_\sigma}\}$ *such that $\sum_{i=1}^{\sigma}\mu_i=l+1$, the restriction map to Z is surjective, then D is said to be* l-jet ample.

Remark 2.19. *Theorem 2.13 can be restated in terms of l-jet spannedness. Namely every divisor D satisfying the hypotheses is l-jet spanned.*

Proposition 2.20 ([7, Proposition 2.2])**.** *In the above notation, if D is l-jet ample, then D is l-very ample.*

The converse of Proposition 2.20 is true for the projective space $\mathbb{P}^n$ and for curves, but not in general. In this section we prove that the converse is true for lines bundle $\mathcal{O}_{X_s}(D)$ on X_s, that satisfy the hypotheses of Theorem 2.2.

Theorem 2.21. *Assume that $s\leq 2n$, or $s\geq 2n+1$ and (2.2). Assume that D is a line bundle on X_s of the form (1.2). The following are equivalent:*

(1) D satisfies (2.3);
(2) D is l-jet ample;
(3) D is l-very ample.

Proof. We proved that the natural restriction map of the global sections of D to any fat point of multiplicity $l+1$ is surjective in Theorem 2.13, see also Remark 2.19. We showed that the same is true in the case of arbitrary collections of fat points whose multiplicity sum up to $l+1$ in the first part of the proof of Theorem 2.2. This proves that (1) implies (2). Moreover, (2) implies (3) by Proposition 2.20. Finally, that (3) implies (1) was proved in the second part of the proof of Theorem 2.2. □

3. Other positivity properties of divisors on X_s

In this section we will apply Theorem 2.2 to establish further positivity properties of divisors on X_s. All results we prove in this section apply to $\mathbb{Q}-$divisors on the blown-up projective space.

3.1. **Semi-ampleness and ampleness.** A line bundle is ample if some positive power is very ample. It is known that for smooth toric varieties a divisor is ample if and only if it is very ample and nef if and only if it is globally generated. From Corollary 2.5 and Corollary 2.6, we obtain that this holds for a small number of points $s \leq 2n$ too, as well as for arbitrary s under a bound on the coefficients.

A line bundle is called *semi-ample*, or *eventually free*, if some positive power is globally generated. By (2.4), one can see that a divisor is semi-ample if and only if it is globally generated.

Theorem 3.1. *Let X_s be defined as in Section 1. Assume $s \leq 2n$.*

(1) The cone of semi-ample divisors in $\mathrm{N}^1(X_s)_{\mathbb{R}}$ is given by (2.4).
(2) The cone of ample divisors in $\mathrm{N}^1(X_s)_{\mathbb{R}}$ is given by (2.5).

Assume $s \geq 2n+1$.

(1) Divisors satisfying (2.2) with $l = 0$ are semi-ample if and only if (2.4).
(2) Divisors satisfying (2.2) with $l = 1$ are ample if and only if (2.5).

3.2. **Nefness.** For every projective variety, Kleiman [24] showed that a divisor is ample if and only if its numerical equivalence class lies in the interior of the nef cone (see also [25, Theorem 1.4.23]).

For a line bundle, being generated by the global sections implies being nef, but the opposite is not true in general, see e.g. Example 2.9. However for line bundles on X_s, with $s \leq 2n$, or with arbitrary s under a bound on the coefficients, these two properties are equivalent.

Theorem 3.2. *In the same notation as Theorem 2.2, assume that for D of the form (1.2) we have that either $s \leq 2n$ or $s \geq 2n+1$ and (2.2) with $l = 0$ is satisfied. Then D is nef if and only if is globally generated.*

Proof. If D is nef, then for effective 1-cycle C, $D \cdot C \geq 0$. In particular the divisor D intersects positively the classes of lines through two points and classes of lines in the exceptional divisors. This means inequalities (2.4) hold and therefore the divisor D is globally generated by Corollary 2.5. □

Remark 3.3. *If $s \leq 2^n$, the nef cone of X_s is given by (2.4), This follows from the description of the Mori cone of curves of X_s, see* [16, Prop. 4.1]*. Notice that for $s \leq 2n$, the description of the nef cone also follows from Theorem 3.2. Moreover the latter result shows that in this range every nef divisor is semi-ample, which does not hold in general.*

Corollary 3.4. *The nef cone and the cone of semi-ample divisors on X_s, for $s \leq 2n$, coincide.*

3.3. **Fujita's conjectures for the blown-up $\mathbb{P}^n$ in points.**

Conjecture 3.5 (Fujita's conjectures, [21])**.** *Let X be an n-dimensional projective algebraic variety, smooth or with mild singularities. Let K_X be the canonical divisor of X and D an ample divisor on X. Then the following holds.*

(1) For $m \geq n+1$, $mD + K_X$ is globally generated.
(2) For $m \geq n+2$, $mD + K_X$ is very ample.

Fujita's conjecture hold on every smooth variety where all nef divisors are semi-ample. In particular it holds on X_s with $s \leq 2n$.

Proposition 3.6. *Let X_s be the blown-up $\mathbb{P}^n$ at s points in general position with $s \leq 2n$. Conjecture 3.5 holds for X_s.*

Proof. For X_s, $s \leq 2n$, global generation (very ampleness) is equivalent to nefness (resp. ampleness), by Theorem 3.1 and Theorem 3.2. This concludes the proof. □

Using the results from this article, we can extend the above to an infinite family of divisors with arbitrary s.

Proposition 3.7. *Let X_s be the blown-up $\mathbb{P}^n$ in an arbitrary number of points in general position, s, and let D be a divisor on X_s such that*

$$\sum_{i=1}^{s} m_i \leq nd \tag{3.1}$$

Then Conjecture 3.5 holds for D.

Proof. It is enough to consider the case $s \geq 2n+1$. Write $X = X_s$. Notice that the divisor $mD + K_X$ has the following properties

$$\begin{aligned}\sum_{i=1}^{s}(mm_i - n + 1) - n(md - n - 1) &= m(\sum_{i=1}^{s} m_i - nd) + n(n+1) - s(n-1)\\ &\leq -m + n(n+1) - (2n+1)(n-1)\\ &= -m + (-n^2 + n) + (n+1)\\ &\leq -2 + n + 1\\ &= n - 1.\end{aligned}$$

Notice that $b_l(mD + K_X) = n - l - 2$, for $s = 2n+1$ and $b_l(mD + K_X) = n - l - 1$ for $s \geq 2n+2$, using the definition (1.3). Therefore $mD + K_X$ satisflies conditions of Theorem 3.1.

To prove that if D is ample then the divisor $mD + K_X$ satisfies conditions (2.4) and (2.5) is an easy computation that we leave to the reader. □

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1. University of North Carolina at Chapel Hill, 340 Phillips Hall CB 3250 NC 27599-3250 and 2. Simion Stoilow Institute of Mathematics Romanian Academy 21 Calea Grivitei Street 010702 Bucharest Romania

E-mail address: `dolivia@unc.edu`

Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14 I-38123 Povo di Trento (TN)

E-mail address: `elisa.postinghel@unitn.it`